STUDIES ON AXIALLY AND CYLINDRICALLY SYMMETRIC EINSTEIN-MAXWELL EQUATIONS

BY

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LIST OF SYMBOLS

ε	mass energy density
p	pressure
u^μ	four-velocity of matter
ξ^μ	Killing vector
(ρ, z, ϕ)	space coordinates
f, k, ℓ, μ	metric functions for axially symmetric metric

$f_\rho \equiv \frac{\partial f}{\partial \rho}$, $f_z \equiv \frac{\partial f}{\partial z}$, etc. Subscripts ρ and z always indicate partial derivatives

$D = (\ell f + k^2)^{\frac{1}{2}}$

$\omega_{\mu\nu}$	rotation tensor
$\sigma_{\mu\nu}$	shear tensor
$R_{\lambda\mu\nu\gamma}$	Riemann tensor
$\tilde{R}_{\lambda\mu\nu\gamma}$	dual Riemann tensor
$F_{\mu\nu}$	electro-magnetic field tensor
$\tilde{F}_{\mu\nu}$	dual of the electro-magnetic field tensor

$W = f^{-1}k$

M	metric function
Σ	electro-magnetic potential
n	number density
m	mass of each particle
q	charge of each particle

$$\nabla^2 \equiv \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} + \rho^{-1} \frac{\partial}{\partial \rho}$$

$$\Delta \equiv \frac{\partial^2}{\partial \rho^2} + \frac{\partial}{\partial z^2} - \rho^{-1} \frac{\partial}{\partial \rho}$$

$\xi, \eta, \sigma, a_o, k_o, \rho_o$ context used in Islam (1983) solution (Section 4)

Ω angular velocity, dependent on ρ,z for differential rotation, but constant for rigid rotation

I_1, I_2, J_1, J_2 curvature invariants involving Riemann tensor

$g^q{}_{\alpha\dot{\beta}}$ in Section 5.2, those represent hermetian spinor matrices with q is space time index and $\alpha, \dot{\beta}$ spinor indices

$\varepsilon_{\alpha\beta}, \varepsilon^{\alpha\beta}, \varepsilon_{\dot{\alpha}\dot{\beta}}, \varepsilon^{\dot{\alpha}\dot{\beta}}$ antisymmetric fundamental spinor with $\varepsilon_{12} = 1$ etc.

$\phi^{\alpha\beta\mu\nu}, \psi^{\alpha\beta\mu\nu}$ etc. spinors corresponding to Riemann tensor, its dual, and Recci tensor

K_{AB} components of Riemann tensor in Pirani notation with A,B = 1, ... 6, etc.

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ABSTRACT

The first chapter is introductory.

The second chapter considers Einstien-Maxwell equations which admit a null Killing vector and a null electromagnetic field. I present certain solutions of these Einstien-Maxwell equations. This work is based on earlier work described in Kramer et al 1980 and Boachie and Islam 1983.

In the next chapter, I calculate the expansion, shear and rotation of certain axially symmetric solutions found by Islam (1977, 1983).

In Chapter 4, I find Killing vectors for the solution found by Islam (1983) mentioned earlier. I also apply to these Killing vectors the analysis applied by Bonnor (1980) to the Van Stockum solution (1937) to determine if there are any time-like hypersuperface-orthogonal Killing vectors, and show that Islam's solution is not static but stationary.

In Islam 1983, he found out exact global solution of Einstien-Maxwell equations. The solution thus obtained is regular and well behaved inside the matter. Such matched solutions are rare either for the Einstien or Einstien-Maxwell equations. Considering those solutions in Chapter 5, I have calculated out all nine curvature invariants. The invariants of the Riemann curvature tensor are first found in terms of its equivalent curvature invariants in terms of two-spinors given by Witten (1959) and Penros (1960). We consider briefly some properties of these invariants.

CHAPTER 1

ROTATING METRIC

1.1 INTRODUCTION

This thesis is concerned with stationary axially symmetric fields in general relativity. Before giving a summary of the following chapters we give a brief review of the relevant aspects of general relativity. This review is based mostly on the book by Islam (1985).

Einstein field equations are given by:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = 8\pi \; T_{\mu\nu} \tag{1.1}$$

where Newton's gravitational constant G and the velocity of light c are set equal to unity and where $T_{\mu\nu}$ is energy-momentum tensor of the source producing the gravitational field. The $T_{\mu\nu}$ for matter in the perfect fluid form is:

$$T^{\mu\nu} = (\varepsilon + p)u^{\mu}u^{\nu} - pg^{\mu\nu} \tag{1.2}$$

where ε is the mass-energy density, p is the pressure and u^{μ} is the four-velocity of matter given by:

$$u^{\mu} = \frac{dx^{\mu}}{ds} \tag{1.3}$$

where $x^{\mu}(s)$ describes the world-line of the matter in terms of the proper time s along the world-line.

The equations of motion of a particle in a gravitational field are given by the geodesic equations:

$$\frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\nu}}{ds} \frac{dx^{\lambda}}{ds} = 0 \tag{1.4}$$

Einstein's exterior field equations are given by $R_{\mu\nu} = 0$, putting $T_{\mu\nu} = 0$ and $R = 0$ in the Equation (1.1). These are a set of coupled non-linear partial differential equations for the ten unknown functions $g_{\mu\nu}$. In particular situations of physical interest, using space-time symmetries we can reduce the number of unknown functions. In Newtonian theory spherical symmetry is usually defined by a centre and the property that all points at any given distance from the centre are equivalent. When we are handling physical problems, symmetric systems have not only the advantage of a certain simplicity or even beauty, but also special physical effects require these symmetries. One can therefore expect in general relativity, too, that when a high degree of symmetry is present the field equations are easier to solve and that the resulting solutions possess special properties.

Our first problem is to define what we mean by a symmetry of a Riemannian Space. The mere impression of simplicity which a metric might give is not of course on its own sufficient, thus for example, the relatively complicated metric:

$$ds^2 = dx^2 - x \sin y \, dxdy + x^2(\tfrac{5}{4} + \cos y) \, dy^2 + x^2(\tfrac{5}{4} + \cos y - \tfrac{1}{4} \sin^2 y)\sin^2 y \, dt^2 - dz^2$$

in fact has more symmetries than the simple wave:

$$ds^2 = dx^2 + dy^2 + 2dudv + H(x,y,z)du^2$$

Rather, we must define a symmetry in a manner independent of the coordinate system. Here we shall restrict ourselves to continuous symmetries.

1.2 KILLING VECTORS

The symmetry of a system in Minkowski space or in three-dimensional (Euclidean) space is expressed through the fact that under translation along certain lines or on certain surfaces the physical variables do not change. One can carry over this intuitive idea to Riemannian spaces and ascribe a symmetry to the space if there exists an s-dimensional ($1 \leqslant s \leqslant 4$) manifold of points in it which are physically equivalent under a symmetry operation, that is, a motion which takes these points into one another, and the metric does not change. But in general relativity one has to find some coordinate independent and covariant manner of defining space time symmetries such as axial symmetry and stationarity. This is done with the help of Killing vectors, which we will now consider.

A metric $g_{\mu\nu}(x)$ is said to be form-invatiant under a given coordinate transformation $x \to x'$ when the transformed metric $g'_{\mu\nu}(x')$ is the same function of its argument x'^{μ} as the original metric $g_{\mu\nu}(x)$ was of its argument x^{μ}, that is:

$$g'_{\mu\nu}(y) = g_{\mu\nu}(y) \quad \text{for all } y \qquad (1.5)$$

At any given point the transformed metric is given by the relation:

$$g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(x)$$

or equivalently:

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g'_{\rho\sigma}(x')$$

therefore:

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g'_{\rho\sigma}(x') = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x')$$

or

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x') \qquad (1.6)$$

using the relation (1.5).

Any transformation $x \to x'$ that satisfies (1.6) is called an isometry.

Equation (1.6) can be simplified by going to the special case of an infinitesimal coordinate transformation:

$$x'^{\mu} = x^{\mu} + \varepsilon\xi^{\mu}(x) \text{ with } |\varepsilon| << 1 \qquad (1.7)$$

using (1.7) in Equation (1.6) we get:

$$g_{\mu\nu}(x) = \frac{\partial}{\partial x^{\mu}}(x^{\rho} + \varepsilon\xi^{\rho}) \cdot \frac{\partial}{\partial x^{\nu}}(x^{\sigma} + \varepsilon\xi^{\sigma}) \cdot g_{\rho\sigma}(x + \varepsilon\xi)$$

$$= (\delta^{\rho}_{\mu} + \varepsilon\frac{\partial\xi^{\sigma}}{\partial x^{\mu}})(\delta^{\sigma}_{\nu} + \varepsilon\frac{\partial\xi^{\sigma}}{\partial x^{\nu}})(g_{\rho\sigma}(x) + \varepsilon\xi^{\alpha}\frac{\partial g_{\rho\sigma}}{\partial x^{\alpha}})$$

$$= (\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} + \delta^{\rho}_{\mu}\varepsilon\frac{\partial\xi^{\sigma}}{\partial x^{\nu}} + \varepsilon\frac{\partial\xi^{\rho}}{\partial x^{\mu}}\delta^{\sigma}_{\nu})(g_{\rho\sigma}(x) + \varepsilon\xi^{\alpha}\frac{\partial g_{\rho\sigma}}{\partial x^{\alpha}})$$

neglecting terms involving ε^2.

$$= g_{\mu\nu}(x) + g_{\mu\sigma}\varepsilon \frac{\partial \xi^\sigma}{\partial x^\nu} + \varepsilon\, g_{\rho\nu} \frac{\partial \xi^\rho}{\partial x^\mu} + \varepsilon \xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha}$$

$$0 \quad = \left(g_{\mu\sigma} \frac{\partial \xi^\sigma}{\partial x^\nu} + g_{\rho\nu} \frac{\partial \xi^\rho}{\partial x^\mu} + \xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha}\right)\varepsilon$$

$$\text{or} \quad g_{\mu\sigma} \frac{\partial \xi^\sigma}{\partial x^\nu} + g_{\rho\nu} \frac{\partial \xi^\rho}{\partial x^\mu} + \xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} = 0$$

$$\text{or} \quad \frac{\partial}{\partial x^\nu}(g_{\mu\sigma}\xi^\sigma) - \xi^\sigma \frac{\partial g_{\mu\sigma}}{\partial x^\nu} + \frac{\partial (g_{\rho\nu}\xi^\rho)}{\partial x^\mu} - \xi^\rho \frac{\partial g_{\rho\nu}}{\partial x^\mu}$$

$$+ \xi^\alpha \frac{\partial g_{\mu\nu}}{\partial x^\alpha} = 0$$

$$\frac{\partial \xi_\mu}{\partial x^\nu} + \frac{\partial \xi_\nu}{\partial x^\mu} + \xi^\alpha \left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu} - \frac{\partial g_{\alpha\nu}}{\partial x^\mu}\right) = 0$$

$$\frac{\partial \xi_\mu}{\partial x^\nu} + \frac{\partial \xi_\nu}{\partial x^\mu} - 2\xi_\alpha \Gamma^\alpha_{\mu\nu} = 0$$

$$\frac{\partial \xi_\mu}{\partial x^\nu} - \xi_\alpha \Gamma^\alpha_{\mu\nu} + \frac{\partial \xi_\nu}{\partial x^\mu} - \xi_\alpha \Gamma^\alpha_{\mu\nu} = 0$$

$$\text{or} \quad \xi_{\mu;\nu} + \xi_{\nu;\mu} = 0 \qquad (1.8)$$

Equation (1.8) is Killing's equation and the vector field ξ^μ satisfying the Killing equation is called a Killing vector of the metric $g_{\mu\nu}$. If the Killing equation has a solution, then it represents an infinitesimal isometry of the metric $g_{\mu\nu}$ which implies that the metric has a certain symmetry. The above Killing equation is expressed in covariant manner, so it is a tensor equation, and if any metric has an isometry in a given coordinate system then it has also isometry in the transformed coordinate system. For an example of a Killing vector, we consider a situation in which the metric is independent of one of the four coordiantes. Let $x^o = \lambda$, be a time-like and x^i i=1,2,3 be space-like coordinates. In general $g_{\mu\nu}$ being independent of x^o means gravitational field is stationary i.e:

$$\frac{\partial g_{\mu\nu}}{\partial x^o} = g_{\mu\nu,o} = 0 \qquad (1.9)$$

Consider a vector ξ^μ given by:

$$(\xi^o, \xi^1, \xi^2, \xi^3) = (1, o, o, o) \qquad (1.10)$$

then

$$\xi_\mu = g_{\mu\nu}\xi^\nu = g_{\mu o}\xi^o = g_{\mu o}$$

We have:

$$\xi_{\mu;\nu} + \xi_{\nu;\mu} = \xi_{\mu,\nu} + \xi_{\nu,\mu} - g^{\lambda\sigma}(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})\xi_\lambda$$

$$= g_{\mu o,\nu} + g_{\nu o,\mu} - \xi^\sigma(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})$$

$$= g_{\mu o,\nu} + g_{\nu o,\mu} - g_{o\mu,\nu} - g_{o\nu,\mu} + g_{\mu\nu,o}$$

$$= g_{\mu\nu,o} \qquad (1.11)$$

Using (1.9) and (1.10). Thus if (1.9) is satisfied, the vector (1.10) gives a solution to Killing's equation.

Here we will derive some property of Killing vectors. Let $\xi^{(1)\mu}$ and $\xi^{(2)\mu}$ be two linearly independent solutions of the Killing Equation (1.8). We define the commutator of these two Killing vectors as the vector given by:

$$\xi^{\mu} = \xi^{(1)\mu}{}_{;\lambda}\xi^{(2)\lambda} - \xi^{(2)\mu}{}_{;\lambda}\xi^{(1)\lambda} \qquad (1.12)$$

In coordinate independent notation the commutator of $\xi^{(1)}$ and $\xi^{(2)}$ is written as $[\xi^{(1)}, \xi^{(2)}]$. Using symmetric relation of the Christoffel symbols i.e $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$, we can write covariant derivatives (1.12) as ordinary derivatives. So:

$$\xi^{\mu} = (\xi^{(1)\mu}{}_{,\lambda} - \Gamma^{\mu}_{\sigma\lambda}\xi^{(1)\sigma})\xi^{(2)\lambda} - (\xi^{(2)\mu}{}_{,\lambda} - \Gamma^{\mu}_{\sigma\lambda}\xi^{(2)\sigma})\xi^{(1)\lambda}$$

$$= \xi^{(1)\mu}{}_{,\lambda}\xi^{(2)\lambda} - \Gamma^{\mu}_{\sigma\lambda}\xi^{(1)\sigma}\xi^{(2)\lambda} - \xi^{(2)\mu}{}_{,\lambda}\xi^{(1)\lambda}$$

$$+ \Gamma^{\mu}_{\sigma\lambda}\xi^{(2)\sigma}\xi^{(1)\lambda}$$

$$= \xi^{(1)\mu}{}_{,\lambda}\xi^{(2)\lambda} - \Gamma^{\mu}_{\sigma\lambda}\xi^{(1)\sigma}\xi^{(2)\lambda} - \xi^{(2)\mu}{}_{,\lambda}\xi^{(1)\lambda}$$

$$+ \Gamma^{\mu}_{\lambda\sigma}\xi^{(2)\lambda}\xi^{(1)\sigma}$$

$$= \xi^{(1)\mu}{}_{,\lambda}\xi^{(2)\lambda} - \xi^{(2)\mu}{}_{,\lambda}\xi^{(1)\lambda}$$

It can be shown that ξ^{μ} is also a Killing vector (see e.g Islam 1985).

If we take n linearly independent, Killing vectors $\xi^{(i)\mu}$, i=1,2 ... n. Then the commutator of any two of these is a Killing vector and so must be a linear combination of some or all of the Killing vectors with constant coefficients, since there are no other solutions of the Killing equation.

So we have the result:

$$\xi^{(i)\mu}{}_{;\nu}\xi^{(j)\nu}-\xi^{(j)\mu}{}_{;\nu}\xi^{(i)\nu}=\sum_{k=1}^{n} a^{ij}_{k}\xi^{(k)\mu}, \quad i,j=1,2 \ldots n \tag{1.13}$$

In coordinate independent notation, we can write:

$$\left[\xi^{(i)}, \xi^{(j)}\right] = \sum_{k=1}^{n} a^{ij}_{k}\xi^{(k)}, \quad i,j=1,2 \ldots n \tag{1.14}$$

where a^{ij}_{k} are constants in both equations.

The result (1.14) can be obtained more elegantly with the use of Lie derivatives, but as this would require the explanation of Lie derivatives, we have preferred the longer and more elementary derivation given here.

1.3 AXIALLY SYMMETRIC STATIONARY METRICES

To derive the most general axially symmetric stationary metric with the use of Killing vectors, we need to consider suitable coordinate systems and make some reasonable physical assumptions. Consider the field to be generated by the steady rotation of a star made of perfect fluid, whose energy-momentum tensor is given by (1.2). The star and the field around it possesses axial symmetry about the axis of rotation which passes through the centre of the star, which we will consider the origin of the coordinate system. The axis of rotation is the z-axis. Because of the time independence and axial symmetry of the metric time-like $x^o = t$ and an angular coordinate $x^3 = \phi$ respectively of which the metric coefficients and all the matter variables are independent. So the coordinates are:

$(x^o, x^1, x^2, x^3) = (t, \rho, z, \phi)$ and we have:

$$g_{\mu\nu} = g_{\mu\nu}(\rho,z), \ \varepsilon = \varepsilon(\rho,z), \ p=p(\rho,z) \tag{1.15}$$

where ε is the total mass-energy density and p is the pressure. Since ϕ is the angular coordinate about the rotation axis, the coordinate values (t,ρ,z,ϕ) and $(t,\rho,z,\phi+2\pi)$ correspond to the same point in the space-time.

The star's matter rotates in the ϕ direction, so its four-velocity $u^\mu = \frac{dx^\mu}{ds}$ has the following form:

Since ρ, z are constants
and t, ϕ are variables, then:

$$u^o = \frac{dt}{ds}, \; u^1 = \frac{d\rho}{ds} = 0, \; u^2 = \frac{dz}{ds} = 0$$

$$u^3 = \frac{d\phi}{ds} = \frac{d\phi}{dt} \cdot \frac{dt}{ds} = \Omega u^o \qquad (1.16)$$

where $\Omega = \frac{d\phi}{dt}$ is the angular velocity measured in units of coordinate time t. Equation (1.16) reflects the fact that a material particle in the star corresponds to fixed values of the coordinates ρ and z and only its ϕ coordinate changes with time. For rigid rotation Ω is constant (independent of ρ and z) but for differential rotation Ω is in general a function of ρ and z.

Because the star rotates in the ϕ direction, the field generated by it is not invariant under time reversal $t \rightarrow -t$, since such a transformation would reverse the sense of rotation of the star, resulting in a different space-time geometry. Nor is the star's field invariant under the transformation $\phi \rightarrow -\phi$, for this would also reverse the sense of rotation of the star. However, the field of the star is invariant under a simultaneous reverse of t and ϕ: $(t,\phi \rightarrow -t,-\phi)$. In this case the motion would be exactly the same as before. Because of these properties the metric coefficients g_{o1}, g_{o2}, g_{13}, g_{23} must vanish because otherwise under $(t,\phi \rightarrow -t,-\phi)$, for example, the term $g_{o1}dtd\rho$ would change sign and so the metric would not be invariant, as it should be, under this transformation.

Thus we have:

$$g_{o1} = g_{o2} = g_{12} = g_{23} \doteq 0 \qquad (1.17)$$

So that the metric can be written as:

$$ds^2 = g_{oo}dt^2 + 2g_{o3}dtd\phi + g_{33}d\phi^2 + g_{AB}dx^A dx^B \qquad (1.18)$$

where A.B are to be summed over values 1, 2.

We can write the metric (1.18) as follows:

$$ds^2 = fdt^2 - 2k\ dtd\phi - \ell d\phi^2 - Ad\rho^2 - 2B\ d\rho dz - Cdz^2 \qquad (1.19)$$

where f,k,ℓ,A,B and C are all functions of ρ and z.

We carry out a coordinate transformation from (ρ,z) to (ρ',z') as follows:

$$\rho' = F(\rho,z),\ z' = G(\rho,z) \qquad (1.20)$$

$$\therefore\quad d\rho' = \frac{\partial F}{\partial \rho}d\rho + \frac{\partial F}{\partial z}dz = F_1 d\rho + F_2 dz$$

where:

$$F_1 \equiv \frac{\partial F}{\partial \rho}\ ,\ F_2 \equiv \frac{\partial F}{\partial z} \quad \text{etc} \tag{1.21}$$

$$dz' = \frac{\partial G}{\partial \rho}\, d\rho + \frac{\partial G}{\partial z}\, dz = G_1 d\rho + G_2 dz$$

where:

$$G_1 \equiv \frac{\partial G}{\partial \rho}\ ,\ G_2 \equiv \frac{\partial G}{\partial z} \tag{1.22}$$

$$\therefore\ d\rho' = F_1 d\rho + F_2 dz \tag{1.23}$$

$$dz' = G_1 d\rho + G_2 dz \tag{1.24}$$

Multiplying (1.23) by G_2 and (1.24) by F_2 and subtracting we get:

$$G_2 d\rho' - F_2 dz' = (F_1 G_2 - G_1 F_2)\, d\rho$$

$$\therefore\ J d\rho = G_2 d\rho' - F_2 dz' \quad \text{where} \quad J = F_1 G_2 - G_1 F_2$$

or $\qquad d\rho = J^{-1}(G_2 d\rho' - F_2 dz')$

(1.25)

similarly $\qquad dz = J^{-1}(- G_1 d\rho' + F_1 dz')$

where J can be written as:

$$J = \frac{\partial(F,G)}{\partial(\rho,z)} = \begin{vmatrix} \frac{\partial F}{\partial \rho} & \frac{\partial G}{\partial \rho} \\ \frac{\partial F}{\partial z} & \frac{\partial G}{\partial z} \end{vmatrix} = \begin{vmatrix} F_1 & G_1 \\ F_2 & G_2 \end{vmatrix} = F_1 G_2 - G_1 F_2 \tag{1.26}$$

and is assumed to be non-zero.

Substituting in (1.19) from (1.25) we get

$$ds^2 = fdt^2 - 2kdtd\phi - \ell d\phi^2 - J^{-2}\{(AG_2^2 - 2BG_1G_2 + CG_1^2)d\rho'^2$$

$$+ 2(-AG_2F_2 + BG_2F_1 + BG_1F_2 - CG_1F_1)d\rho' dz'$$

$$+ (AF_2^2 - 2BF_1F_2 + CF_1^2)dz'^2\} \tag{1.27}$$

The function F and G are so far arbitrary.

Assuming A, B, C are given function of ρ and z we now require F and G to satisfy the following two coupled non-linear partial differential equations of ρ and z.

$$AG_2^2 - 2BG_1G_2 + CG_1^2 = AF_2^2 - 2BF_1F_2 + CF_1^2 \tag{1.28a}$$

$$\text{or} \quad -AG_2F_2 + BG_2F_1 + BG_1F_2 - CG_1F_1 = 0 \tag{1.28b}$$

We assume that for the given A, B, C the system of Equation (1.28a, b) has a non-trivial solution with $J \neq 0$. Then the coordinates (ρ', z') in the metric (1.17) has its coefficients of $d\rho'^2$ equal to its coefficients of dz'^2 and the coefficient of $d\rho' dz'$ vanishes. We can now drop the primes from ρ' and z' and write the new metric as follows (the f, k, ℓ in the following are not the same functions as the f, k, ℓ in (1.19)):

$$ds^2 = fdt^2 - 2kdtd\phi - \ell d\phi^2 - e^{\mu}(d\rho^2 + dz^2) \tag{1.29}$$

where f, k, ℓ and μ are all functions of ρ and z.

1.4 CHRISTOFFEL SYMBOLS

Using the Christoffel symbols of the second kind:

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}(g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\nu\lambda,\sigma}) \qquad (1.30)$$

We can calculate all non-zero values of Γ's from the metric (1.29). The non-zero covariant and contravariant components of the metric tensor are as follows:

$$g_{oo} = f,\ g_{o3} = -k,\ g_{11} = g_{22} = -e^{\mu},\ g_{33} = -\ell \qquad (1.31a)$$

$$g^{oo} = D^{-2}\ell,\ g^{o3} = -D^{-2}k,\ g^{11} = g^{22} = -e^{-\mu},\ g^{33} = -D^{-2}f \qquad (1.31b)$$

where $D^2 \equiv \ell f + k^2$

The non-zero Christoffel symbols are as follows and

$(\frac{\partial f}{\partial \rho} \equiv f_{\rho}\ ,\ \frac{\partial f}{\partial z} \equiv f_z$ etc)

we have grouped them according to the values of the index λ.

$$\Gamma^{o}_{o1} = \frac{1}{2} D^{-2}(\ell f_{\rho} + kk_{\rho}) \quad \Gamma^{o}_{o2} = \frac{1}{2} D^{-2}(\ell f_z + kk_z)$$

$$\Gamma^{o}_{13} = \frac{1}{2} D^{-2}(k\ell_{\rho} - \ell k_{\rho}) \quad \Gamma^{o}_{23} = \frac{1}{2} D^{-2}(k\ell_z - \ell k_z) \qquad (1.32a)$$

$$\Gamma^{1}_{oo} = \frac{1}{2} e^{-\mu} f_{\rho},\ \Gamma^{1}_{o3} = -\frac{1}{2} e^{-\mu} k_{\rho},\ \Gamma^{1}_{11} = \frac{1}{2} \mu_{\rho}$$

$$\Gamma^{1}_{12} = \frac{1}{2} \mu_z,\ \Gamma^{1}_{22} = -\frac{1}{2} \mu_{\rho},\ \Gamma^{1}_{33} = -\frac{1}{2} e^{-\mu} \ell_{\rho} \qquad (1.32b)$$

$$\Gamma^2_{oo} = \frac{1}{2} e^{-\mu} f_z, \; \Gamma^2_{o3} = -\frac{1}{2} e^{-\mu} k_z, \; \Gamma^2_{11} = -\frac{1}{2} \mu_z$$

$$\Gamma^2_{12} = \frac{1}{2} \mu_\rho, \; \Gamma^2_{22} = \frac{1}{2} \mu_z, \; \Gamma^2_{33} = -\frac{1}{2} e^{-\mu} \ell_z \qquad (1.32c)$$

$$\Gamma^3_{o1} = \frac{1}{2} D^{-2} (fk_\rho - kf_\rho), \; \Gamma^3_{o2} = \frac{1}{2} D^{-2} (fk_z - kf_z)$$

$$\Gamma^3_{13} = \frac{1}{2} D^{-2} (f\ell_\rho + kk_\rho), \; \Gamma^3_{23} = \frac{1}{2} D^{-2} (f\ell_z + kk_z) \qquad (1.32d)$$

After this brief summary of some aspects of general relativity which will be needed in the subsequent chapters, we proceed to give a resume of the following chapters.

1.5 THE MAIN RESULTS OF THE THESIS

A class of exact and explicit solutions will be considered for the Einstein-Maxwell equations admitting a null Killing vector and a null electromagnetic field.

We begin with the solutions of the Einstein-Maxwell equations which admit a null Killing vector and a null electromagnetic field, as given in the book by Kramer et al (1973) (p 233, Equations (21.34), (21.35). The metric is given as follows:

$$ds^2 = 2\rho \, du(dv + Mdu) - \rho^{-1/2} (d\rho^2 + dz^2) \qquad (1.33)$$

We have changed the signature of the metric and written ρ, z for x, y respectively. M is a function of ρ and z satisfying the following equation (k_o is a constant).

$$\rho M_{\rho\rho} + M_{\rho} + \rho M_{zz} = k_o(\Sigma_{\rho}^2 + \Sigma_z^2) \qquad (1.34)$$

where Σ is related to the electromagnetic potential and satisfies

$$\Sigma_{\rho\rho} + \Sigma_{zz} = 0 \qquad (1.35)$$

We consider some motivation for finding exact solutions of the system (1.34) and 1.35). It is always useful to have exact and explicit solutions of the Einstein or the Einstein-Maxwell equation for studying their physical interpretation and for comparison with other known solutions. For the system given by (1.34) and (1.35), it is easy enough to find a solution of (1.35). One has only to take the real or imaginary part of any analytic function of $(\rho+iz)$. It is in general non-trivial to find explicit solutions of the coupled equations (1.34), (1.35) and we find some new solutions in Chapter 2.

In Chapter 3 we work out the expansion, shear and rotation of the differentially rotating interior solution for dust (Winicour 1985). Differential rotation implies that Ω is a function of ρ and z, in general. The covariant derivative of the covariant four-velocity of the fluid can in general be decomposed as follows:

$$\dot{u}_{\mu;\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}(g_{\mu\nu} - u_\mu u_\nu)u^{\sigma}{}_{;\sigma} + \dot{u}_\mu u_\nu \qquad (1.36)$$

where $u^{\sigma}{}_{;\sigma}$ is the expansion, $\dot{u}_\mu$ the covariant four-acceleration given by:

$$\dot{u}_\mu \equiv u_{\mu;\sigma}u^{\sigma} \qquad (1.37)$$

and $\omega_{\mu\nu}$ and $\sigma_{\mu\nu}$ are respectively the rotation and shear tensors given by:

$$\omega_{\mu\nu} = \frac{1}{2}(u_{\mu;\nu} - u_{\nu;\mu}) - \frac{1}{2}(\dot{u}_\mu u_\nu - \dot{u}_\nu u_\mu) \qquad (1.38)$$

$$\sigma_{\mu\nu} = \frac{1}{2}(u_{\mu;\nu} + u_{\nu;\mu}) - \frac{1}{2}(\dot{u}_\mu u_\nu + \dot{u}_\nu u_\mu)$$

$$- \frac{1}{3}(g_{\mu\nu} - u_\mu u_\nu)u^\sigma_{;\sigma} \qquad (1.39)$$

(See, for example, Misner et al (1973),(p 556) where, however, there are some differences in sign since for them $u_\mu u^\mu = -1$ while for us $u_\mu u^\mu = 1$). One of the objects of this chapter is to make explicit the fact that the shear is non-zero for differential rotation but that it vanishes identically when the rotation is rigid, i.e, when the angular velocity Ω is independent of ρ and z.

In Islam (1983), he found out exact cylindrically symmetric global solution of Einstien-Maxwell equations. The solution thus obtained is regular and well behaved inside the matter. Such matched solutions are rare either for the Einstien or the Einstien-Maxwell equations. Van Stockum (1937) found a rotating dust interior and three exterior metrics referring to different ranges of the mass per unit length. It has been stated in the literature (Frehland (1971)), that the exterior is static, but it was proved by Bonnor (1980) that this is so only in the low-mass case. In Chapter 4, an analytical argument is given following Bonnor (1980) to show that there is no time-like hypersurface orthogonal Killing vectors for the global solution found by Islam (1983), so that it is not static but stationary. The argumentsof this chapter are used partly on new Killing vectors which we find in addition to the usual ones representing the symmetry of the metric such as $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \phi}$. There is one such Killing vector for the interior solution and two for the exterior solution.

In Islam (1983), he found out exact global solution of Einstien-Maxwell equations. The solution thus obtained is regular and well behaved inside the matter. Such matched solutions are rare either for the Einstien or Einstien-Maxwell equations. Considering those solutions in Chapter 5, I have calculated out all nine curvature invariants as follows:

$$R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa}$$

$$R_{\lambda\mu\nu\kappa}R^{\nu\kappa\rho\sigma}R_{\rho\sigma}{}^{\lambda\mu}$$

$$\varepsilon^{\lambda\mu}{}_{\rho\sigma}R^{\rho\sigma\nu\kappa}R_{\lambda\mu\nu\kappa}$$

$$\varepsilon^{\tau\xi}{}_{\rho\sigma}(R_{\lambda\mu\nu\kappa}R^{\nu\kappa\rho\sigma}R_{\tau\xi}{}^{\lambda\mu})$$

$$F^{\mu\nu}F_{\mu\nu}$$

$$\tilde{F}^{\mu\nu}F_{\mu\nu}$$

$$R^{\lambda\mu\nu\sigma}F_{\lambda\mu}F_{\nu\sigma}$$

$$R^{\mu\nu\lambda\sigma}F_{\mu\nu}\tilde{F}_{\lambda\sigma}$$

$$R_{\lambda\mu\nu\kappa}R^{\nu\kappa\alpha\beta}F^{\lambda\mu}F_{\alpha\beta}$$

The invariants of the Riemann curvature tensor are first found in terms of its equivalent curvature invariants in terms of two-spinors given by Witten (1959) and Penrose (1960). We consider briefly some properties of these invariants, Pirani (1957).

CHAPTER 2

EXACT SOLUTIONS FOR EINSTEIN-MAXWELL EQUATIONS ADMITTING A NULL KILLING VECTOR

2.1 INTRODUCTION

In this chapter we consider the Einstein-Maxwell equations admitting a null Killing vector and a null electromagnetic field. These equations are given in Kramer et al (1980) but we have not found the derivation either in Kramer et al (1980) nor anywhere else. The derivation is given here.

The Einstein-Maxwell exterior equations in suitable units can be written as:

$$R_{\mu\nu} = 8\pi E_{\mu\nu} = -2F_{\mu}{}^{\alpha}F_{\nu\alpha} + \frac{1}{2} g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \qquad (2.1)$$

$$F_{\mu\nu;\sigma} + F_{\nu\sigma;\mu} + F_{\sigma\mu;\nu} = 0 \qquad (2.2)$$

$$F^{\mu\nu}{}_{;\nu} = -4\pi J^{\mu} = 0 \qquad F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \qquad (2.3)$$

where $E_{\mu\nu}$ is the electromagnetic energy-momentum tensor and J^{μ} the four-current which we put equal to zero since we are here considering only the exterior field. $F_{\mu\nu}$ is the electromagnetic field tensor, defined in terms of the four-vector potential A_{μ} by (2.3). A semicolon denotes covariant differentiation and a comma partial differentiation. Because of its definition in terms of A_{μ} the tensor $F_{\mu\nu}$ satisfies (2.2) identically. Equation (2.1) follows from the Einstein's equations given by:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi \, T_{\mu\nu} \qquad (2.4)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the source producing the gravitational field, if we interpret $T_{\mu\nu}$ as $E_{\mu\nu}$ and note that in this case the Ricci scalor R vanishes identically since $E^{\mu}_{\mu} \equiv 0$. It can also be shown, using standard procedure of Maxwell theory, the $E_{\mu\nu}$ has zero divergence:

$$E^{\mu\nu}_{;\nu} = 0 \qquad (2.5)$$

which represents the conservation of energy and momentum of the electromagnetic field.

With the use of the relations $D^2 = \ell f + k^2$ and $W = f^{-1}k$ the metric

$$ds^2 = fdt^2 - 2k\, dtd\phi - \ell d\phi^2 - e^{\mu}(d\rho^2 + dz^2)$$

can be written as:

$$ds^2 = f(dt - Wd\phi)^2 - \rho^2 f^{-1} d\phi^2 - e^{\mu}(d\rho^2 + dz^2) \qquad (2.6)$$

where f,W and μ are all functions of ρ and z.

Writing $(x^0, x^1, x^2, x^3) = ((t, \rho, z, \phi)$ the vector potential (A_0, A_1, A_2, A_3) can be written in terms of two scalar field ϕ and ϕ' and the metric functions f and W as follows (Ernst, (1968b), (the ϕ's used here are distinct to the azimuthal coordinate ϕ of the last chapter):

$A_o = \phi; \quad A_1 = A_2 = 0$

$$\frac{\partial A_3}{\partial \rho} = W\phi_\rho + \rho f^{-1}\phi'_z, \quad \frac{\partial A_3}{\partial z} = W\phi_z - \rho f^{-1}\phi'_\rho \qquad (2.7)$$

where $\phi_\rho \equiv \frac{\partial \phi}{\partial \rho}$ etc. The consistency of the last two relations in (2.7) is guaranteed by (2.11), (Islam 1985). The field Equations (2.1) and (2.3) in the metric (2.6) and for A_μ given by (2.7) yield, firstly, the following four equations:

$$f\nabla^2 f - f_\rho^2 - f_z^2 + \rho^{-2}f^4(W_\rho^2 + W_z^2)$$
$$= 2f(\phi_\rho^2 + \phi_z^2 + \phi'^2_\rho + \phi'^2_z) \qquad (2.8)$$

$$f\Delta W + 2f_\rho W_\rho + 2f_z W_z = 4\rho f^{-1}(\phi'_z\phi_\rho - \phi'_\rho\phi_z) \qquad (2.9)$$

$$f\nabla^2\phi = f_\rho\phi_\rho + f_z\phi_z + \rho^{-1}f^2(W_z\phi'_\rho - W_\rho\phi'_z) \qquad (2.10)$$

$$f\nabla^2\phi' = f_\rho\phi'_\rho + f_z\phi'_z + \rho^{-1}f^2(W_\rho\phi_z - W_z\phi_\rho) \qquad (2.11)$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} + \rho^{-1}\frac{\partial}{\partial \rho}$

$\Delta \equiv \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} - \rho^{-1}\frac{\partial}{\partial \rho}$

2.2 FIELD EQUATIONS FOR NULL KILLING VECTOR

Consider the possibility of one of the Killing vectors of the metric (2.6), namely $\frac{\partial}{\partial \phi}$, being null. In this case $\ell = 0$ and $W = \frac{\rho}{f}$, $f = 2\rho M$ it can be seen that the electromagnetic field in this case can be taken as null i.e:

$$\phi_\rho = A\phi'_z \ , \quad \phi_z = - A\phi'_\rho \ , \quad A \text{ is constant} \tag{2.12}$$

then

$$\omega_\rho = - \rho f^{-2} f_\rho + f^{-1} \ , \ \omega_z = - \rho f^{-2} f_z$$

$$f\left(\frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2} + \rho^{-1}\frac{\partial}{\partial\rho}\right) f - f_\rho^2 - f_z^2$$

$$+ \rho^{-2} f^4 \{(-\rho f^{-2} f_\rho + f^{-1})^2 + (-\rho f^{-2} f_z)^2\}$$

$$= 2f\{A^2 \phi_z'^2 + A^2 \phi_\rho'^2 + \phi_\rho'^2 + \phi_z'^2\}$$

$$f(f_{\rho\rho} + f_{zz} + \rho^{-1} f_\rho) - f_\rho^2 - f_z^2 + \rho^{-2} f^4 \{\rho^2 f^{-4} f_\rho^2$$

$$- 2\rho f^{-3} f_\rho + f^{-2} + \rho^2 f^{-4} f_z^2\}$$

$$= 2f\{(1 + A^2)\phi_\rho'^2 + (1 + A^2)\phi_z'^2\}$$

$$\therefore \ f_{\rho\rho} + f_{zz} - \rho^{-1} f_\rho + f\rho^{-2} = 2\{(1 + A^2)\phi_\rho'^2$$

$$+ (1 + A^2)\phi_z'^2\}$$

we have

$$f = 2\rho M \ , \ f_\rho = 2M + 2\rho M_\rho \ , \ f_{\rho\rho} = 2M_\rho + 2M_\rho + 2\rho M_{\rho\rho}$$

$$f_z = 2\rho M_z \ , \quad f_{zz} = 2\rho M_{zz}$$

$$\therefore \ 4M_\rho + 2\rho M_{\rho\rho} + 2\rho M_{zz} - \rho^{-1}(2M + 2\rho M_\rho) + 2\rho M \rho^{-2}$$

$$= 2\{(1 + A^2)\phi_\rho'^2 + (1 + A^2)\phi_z'^2\}$$

$$2(\rho M_{\rho\rho} + M_\rho + \rho M_{zz}) = 2(1 + A^2)\{\phi_\rho'^2 + \phi_z'^2\}$$

$$\therefore \ \rho M_{\rho\rho} + M_\rho + \rho M_{zz} = k\{\phi_\rho'^2 + \phi_z'^2\} \tag{2.13}$$

Similarly, putting the relation (2.12) in (2.9) we get the relation (2.13).

From Equation (2.10), using relation (2.12) and putting $A = -1$ we can write:

$$\phi_{\rho\rho} + \phi_{zz} = 0 \qquad (2.14)$$

Similarly, from Equation(2.11) we also get the same as (2.14). Secondly, the field equations yield the following two equations for μ, the consistency of which is guaranteed by (2.8 - 2.11):

$$\mu_\rho = - f^{-1} f_\rho + \frac{1}{2} f^{-2} \rho (f_\rho^2 - f_z^2) + 2\rho f^{-1} (\phi_z^2 - \phi_\rho^2 + \phi'^2_z - \phi'^2_\rho) + \frac{1}{2} \rho^{-1} f^2 (\omega_z^2 - \omega_\rho^2) \qquad (2.15)$$

$$\mu_z = - f^{-1} f_z + \rho f^{-2} f_\rho f_z - 4\rho f^{-1} (\phi_\rho \phi_z + \phi'_\rho \phi'_z) - \rho^{-1} f^2 \omega_\rho \omega_z \qquad (2.16)$$

From Equation (2.15) as $\phi_\rho = - \phi'_z$, $\phi_z = \phi'_\rho$ from (2.12) as $A = -1$:

$$\mu_\rho = - f^{-1} f_\rho + \frac{1}{2} \frac{\rho}{f^2} (f_\rho^2 - f_z^2) + \frac{1}{2} \frac{f^2}{\rho} \{ \frac{\rho^2}{f^4} f_z^2 - (- \frac{\rho}{f^2} f_\rho + \frac{1}{f})^2 \}$$

$$= - \frac{f_\rho}{f} + \frac{1}{2} \frac{\rho}{f^2} (f_\rho^2 - f_z^2) + \frac{1}{2} \frac{f^2}{\rho} \{ \frac{\rho^2}{f^4} f_z^2 - \frac{\rho^2}{f^4} f_\rho^2 + 2 \frac{\rho f_\rho}{f^3} - \frac{1}{f^2} \}$$

$$= - \frac{f_\rho}{f} + \frac{1}{2} \frac{\rho}{f^2} (f_\rho^2 - f_z^2) + \frac{1}{2} \frac{\rho}{f^2} (f_z^2 - f_\rho^2) + \frac{f_\rho}{f} - \frac{1}{2} \frac{1}{\rho}$$

$$\therefore \; M_\rho = - \frac{1}{2} \frac{1}{\rho} \qquad \therefore \; e^\mu = \rho^{-1/2}$$

In this way

$$\mu_z = -\frac{f_z}{f} + \rho\,\frac{f_\rho f_z}{f^2} - \frac{f^2}{\rho}\left(\rho^2\frac{f_\rho f_z}{f^4} - \frac{\rho f_z}{f^3}\right)$$

$$= -\frac{f_z}{f} + \frac{\rho}{f^2}\,f_\rho f_z - \frac{\rho f_\rho f_z}{f^2} + \frac{f_z}{f} = 0$$

The metric (2.6) can be reduced as:

$$ds^2 = f(dt - \frac{\rho}{f}\,d\phi)^2 - \frac{\rho^2}{f}\,d\phi^2 - e^{\mu}(d\rho^2 + dz^2)$$

$$= f(dt^2 - 2dt\,\frac{\rho}{f}\,d\phi + \frac{\rho^2}{f^2}\,d\phi^2) - \frac{\rho 2}{f}\,d\phi^2 - e^{\mu}(d\rho^2 + dz^2)$$

$$= fdt^2 - 2\,\rho\,dtd\phi - e^{\mu}(d\rho^2 + dz^2)$$

$$= fdt^2 - 2\rho\;dtd\phi\; - \;\rho^{-1/2}(d\rho^2 + dz^2) \qquad (2.17)$$

In this chapter I find a class of exact and explicit solutions of the Einstein-Maxwell equations admitting a null Killing vector and null electromagnetic field as given in the book by Kramer et al (1980), (p 233), Equations (21.34), (21.35); see also Boachie and Islam (1983). The metric is given as follows:

$$ds^2 = 2\rho\;du(dv + Mdu) - \rho^{-1/2}(d\rho^2 + dz^2) \qquad (2.18)$$

This is the same as (2.17) with $f = 2\rho M$, $u = t$ and $v = -\phi$ M is a function of ρ and z satisfying the following equation (α is a constant and a subscript represents differentiation with respect to the corresponding variable):

$$\rho M_{\rho\rho} + M_{\rho} + \rho M_{zz} = \alpha(\Sigma_{\rho}^{2} + \Sigma_{z}^{2}) \qquad (2.19)$$

where Σ is related to the electromagnetic potential and satisfies:

$$\Sigma_{\rho\rho} + \Sigma_{zz} = 0 \qquad (2.20)$$

Considering some motivation for finding exact solutions of the system (2.19) and (2.20), it is always useful to have exact and explicit solutions of the Einstein or the Einstein-Maxwell equations for studying their physical interpretation and for comparison with other known solutions. For the system given by (2.19) and (2.20), it is easy enough to find a solution of (2.20) - one has only to take the real or imaginary part of any analytic function of $(\rho + iz)$.

However, as one sees from the case:

$$\Sigma = \log(\rho^{2} + z^{2}) \qquad (2.21)$$

which satisfies (2.20), and yields for (2.19) the equation

$$\rho M_{\rho\rho} + M_{\rho} + \rho M_{zz} = 4\alpha(\rho^{2} + z^{2})^{-1} \qquad (2.22)$$

that a solution to (2.20) often leads to a form of (2.19) (such as (2.22)) which is difficult to solve explicitly. In this chapter we have found a class of solutions to (2.20) and some other solutions which lead to forms of (2.19) which we have solved exactly and explicitly.

Another motivation for studying this problem is connected with an earlier paper, Boachie and Islam (1983), in which was considered a solution of (2.19) and (2.20) independent of z which was called s. Then a cylindrically symmetric interior solution for rotating charged dust found earlier, Islam (1978) was considered which depended on a parameter α and which was called $\bar{s}(\alpha)$. It was then shown that as α tends to zero, the interior solution tends to s, i.e $\bar{s}(0) = s$. Although this does not imply matching of the interior and exterior solutions, it is nevertheless of some interest to have this kind of relation as the vast majority of exterior solutions of the Einstein and Einstein-Maxwell equations have no connection whatever to any known interior solutions. It is possible that a similar connection may exist between the explicit solution found in this chapter (which, unlike the solution considered in Boachie and Islam (1983) depend on both ρ and z) and some of the known ρ and z dependent interior solutions of the Einstein-Maxwell equations Islam (1977) . For this possible connection to be established it is useful to have the explicit solutions of this chapter.

2.3 THE NEW SOLUTIONS

If $u + iv = f(\rho + iz)$ real and imaginary parts satisfy:

$$u_{\rho\rho} + u_{zz} = 0 \qquad v_{\rho\rho} + v_{zz} = 0$$

and $u_\rho = v_z \, , \; u_z = - v_\rho$

the Cauchy-Riemann equations.

$$\therefore\; f = (\rho + iz)^2 = \rho^2 - z^2 + 2i\rho z, \; u = \rho^2 - z^2, \; v = 2\rho z$$

$$\therefore\; u_{\rho\rho} + u_{zz} = 0, \; v_{\rho\rho} + v_{zz} = 0, \; u_\rho = v_z \text{ and } u_z = - v_\rho$$

Consider solutions of (2.20) given by the following two expressions:

$$\Sigma = \rho^2 - z^2 \quad \text{or} \quad \Sigma = 2\rho z \tag{2.23}$$

both of which lead to the same of (2.18), namely:

$$\rho M_{\rho\rho} + M_\rho + \rho M_{zz} = \alpha(\Sigma_\rho^2 + \Sigma_z^2) = 4\alpha(\rho^2 + z^2) \tag{2.24}$$

By assuming M to be a cubic in ρ and z, as:

$$M = A\rho^3 + B\rho^2 z + C\rho z^2 + Dz^3 \tag{2.25}$$

$$M_\rho = 3A\rho^2 + 2B\rho z + C z^2 \,, \quad M_{\rho\rho} = 6A\rho + 2Bz$$

$$M_z = B\rho^2 + 2C\rho z + 3Dz^2 \,, \quad M_{zz} = 2C\rho + 6Dz$$

putting these values in (2.24):

$$\rho(6A\rho + 2Bz) + 3A\rho^2 + 2B\rho z + Cz^2 + \rho(2C\rho + 6Dz) = 4\alpha(\rho^2 + z^2)$$

or $\rho^2(9A + 2C) + \rho z(4B + 6D) + Cz^2 = 4\alpha(\rho^2 + z^2)$

Equating the coefficients, M can be written as

$$M = 4\alpha\rho(-\tfrac{1}{9}\rho^2 + z^2) + B(\rho^2 z - \tfrac{2}{3}z^3) \tag{2.26}$$

This can be written as

$$M = 4\alpha\rho(-\tfrac{1}{9}\rho^2 + z^2) \tag{2.27}$$

In (2.27) we have ignored a term, as we will do throughout, which does not vanish as α tends to zero, that is, we will not include solutions of (2.19) in which the right hand side is zero.

In fact when Σ is zero the solution reduces to the pure Einstein Van-Stockum exterior solution, Van-Stockum (1937), see also Islam (1985), Section (2.6) and Lewis (1932).

If $f = (\rho + iz)^3$, then real and imaginary parts are:

$\rho^3 - 3\rho z^2$ and $3\rho z - z^3$ respectively.

Considering solutions of (2.20) given by the following two expressions:

$$\Sigma = \rho^3 - 3\rho z^2 \quad \text{or} \quad \Sigma = 3\rho z - z^3 \qquad (2.28)$$

$$\Sigma_\rho = 3\rho^2 - 3z^2, \; \Sigma_{\rho\rho} = 6\rho, \; \Sigma_z = -6\rho z, \; \Sigma_{zz} = -6\rho$$

$$\therefore \; \Sigma_{\rho\rho} + \Sigma_{zz} = 0$$

$$\Sigma_\rho^2 + \Sigma_z^2 = (3\rho^2 - 3z^2)^2 + (-6\rho z)^2 = 9(\rho^2 + z^2)^2$$

both of Equations (2.28) which lead to the same form of (2.18) namely,

$$\rho M_{\rho\rho} + M_\rho + \rho M_{zz} = 9\alpha(\rho^2 + z^2)^2 \qquad (2.29)$$

By assuming M to be a fifth order of ρ and z, as:

$$M = A\rho^5 + B\rho^4 z + C\rho^3 z^2 + D\rho^2 z^3 + E\rho z^4 + Fz^5 \qquad (2.30)$$

$$M_\rho = 5A\rho^4 + 4B\rho^3 z + 3C\rho^2 z^2 + 2D\rho z^3 + Ez^4$$

$$M_{\rho\rho} = 20A\rho^3 + 12B\rho^2 z + 6C\rho z^2 + 2Dz^3$$

$$M_z = B\rho^4 + 2C\rho^3 z + 3D\rho^2 z^2 + 4E\rho z^3 + 5Fz^4$$

$$M_{zz} = 2C\rho^3 + 6D\rho^2 z + 12E\rho z^2 + 20Fz^3$$

putting these values in (2.29):

$$\therefore \rho(20A\rho^3 + 12B\rho^2 z + 6C\rho z^2 + 2Dz^3) + 5A\rho^4$$

$$+ 4B\rho^3 z + 3C\rho^2 z^2 + 2D\rho z^3 + Ez^4$$

$$+ \rho(2C\rho^3 + 6D\rho^2 z + 12E\rho z^2 + 20Fz^3) = 9\alpha(\rho^2 + z^2)^2$$

$$\rho^4(25A + 2C) + \rho^3 z(16B + 6D) + \rho^2 z^2(9C + 12E)$$

$$+ \rho z^3(4D + 20F) + Ez^4 = 9\alpha(\rho^4 + 2\rho^2 z^2 + z^4)$$

Equating the coefficients and putting the values of the constants.

$$M = \frac{29}{25}\alpha\rho^5 + Dz(-\frac{3}{8}\rho^4 + \rho^2 z^2 - \frac{1}{5}z^4) - 10\,\alpha\rho^3 z^2 + 9\alpha\rho z^4$$

$$= \alpha\rho(\frac{29}{25}\rho^4 - 10\,\rho^2 z^2 + 9z^4) + Dz(-\frac{3}{8}\rho^4 + \rho^2 z^2 - \frac{1}{5}z^4)$$

This can be written as:

$$M = \alpha\rho(\frac{29}{25}\rho^4 - 10\,\rho^2 z^2 + 9z^4) \qquad (2.31)$$

In (2.31) we have ignored a term, which does not vanish, as α tends to zero.

2.4 A CLASS OF SOLUTIONS

If we define functions Σ' and Σ'' by

$$\Sigma' + i\Sigma'' = f(\zeta), \quad \zeta = \rho + iz \tag{2.32}$$

where $f(\zeta)$ is an analytic function of ζ, then clearly Σ' and Σ'' satisfy (2.20), because of the Cauchy-Riemann equations. For these same equations it also follows that:

$$\left|\frac{df}{d\zeta}\right|^2 = \Sigma'^2_\rho + \Sigma'^2_z = \Sigma''^2_\rho + \Sigma''^2_z \tag{2.33}$$

where the left hand side stands for modulus squared. Clearly the solution given by (2.23) and (2.27) corresponds to $f(\zeta) = \zeta^2$. Consider now the solution given by:

$$f(\zeta) = \zeta^{n+1} = (\rho + iz)^{n+1} = \Sigma' + i\Sigma'' \tag{2.34}$$

where n is a positive integer. The exact expressions for Σ' and Σ'' can easily be obtained from (2.34) by binomial expansion. From (2.33) it follows that:

$$\Sigma'^2_\rho + \Sigma'^2_z = \Sigma''^2_\rho + \Sigma''^2_z = (n+1)^2 |(\rho + iz)^n|^2$$

$$= (n+1)^2 (\rho^2 + z^2)^n \tag{2.35}$$

The corresponding form of (2.19) is as follows:

$$\rho M_{\rho\rho} + M_\rho + \rho M_{zz} = \alpha(n+1)^2 (\rho^2 + z^2)^n \tag{2.36}$$

We proceed to solve (2.36). Assume M to be given by a polynomial in ρ and z of degree 2n+1, as follows:

$$M = \sum_{r=o}^{n} M_r \rho^{2n-2r+1} z^{2r} \qquad (2.37)$$

where the M_r are constants. Substituting in (2.36) and equating coefficients, we find the following recurrence relation for M_r:

$$(2n - 2r + 1)^2 M_r + (2r + 2)(2r + 1) M_{r+1}$$

$$= \alpha(n + 1)^2 ({}^n c_r) \qquad (2.38)$$

where ${}^n c_r = n!/r!(n - r)!$. One can obtain a solution to (2.38) by iterating to get the following expression:

$$M_{r+1} = \alpha(n+1)^2 \{ {}^n c_r / [(2r+2)(2r + 1)]$$

$$+ \sum_{k=1}^{r-1} (-1)^k \, {}^n c_{r-k} \Big[(2n - 2r + 1)^2$$

$$\ldots (2n - 2r + 2k - 1)^2 \Big] / [(2r + 2)$$

$$\ldots (2r - 2k + 1)] \} \qquad (2.39)$$

In the Appendix it is shown that (2.39) constitutes a solution of (2.38). Thus an exact solution of the system (2.19) and (2.20) is given by (2.34), (2.37) and (2.39). One may be able to get a closed expression for the right hand side of (2.39) in terms of n and r, but we have not looked for this.

2.5 SOME OTHER SOLUTIONS

Consider the form of $f(\zeta)$ given by:

$$f(\zeta) = a\zeta^3 + b\zeta^2 \tag{2.40}$$

where a and b are real constants. The corresponding Σ' and Σ'' are:

$$\Sigma' = a\rho^3 - 3a\rho z^2 + b\rho^2 - bz^2, \ \Sigma'' = 3a\rho^2 z - az^3 + 2b\rho z \tag{2.41}$$

and the related form of (2.19) is given as follows:

$$\rho M_{\rho\rho} + M_\rho + \rho M_{zz} = \alpha\{4(3a\rho z + bz)^2 + (3a\rho^2 - 3az^2 + 2b\rho)^2\} \tag{2.42}$$

Again assuming for M a suitable polynomial, one can get the following solution to (2.42):

$$M = a^2\alpha\rho(\frac{29}{25}\rho^4 - 10\rho^2 z^2 + 9z^4) + ab\alpha(\frac{3}{4}\rho^4 + z^4) + b^2\alpha\rho(-\frac{4}{9}\rho^2 + 4z^2) \tag{2.43}$$

This reduces to (2.27) when a = o, b = 1.

Consider now f given by ζe^ζ, leading to

$$\Sigma' = e^\rho(\rho \cos z - z \sin z), \ \Sigma'' = e^\rho(\rho \sin z + z \cos z) \tag{2.44}$$

and to the following form for (2.19):

$$\rho M_{\rho\rho} + M_\rho + \rho M_{zz} = \alpha\ e^{2\rho}\{(\rho + 1)^2 + z^2\} \tag{2.45}$$

we now assume M to be of the form $h(\rho) + g(\rho)z^2$. Then substitution into (2.45) yields coupled ordinary differential equations which can readily be solved to give the following expressions for h and g.

$$g(\rho) = -\frac{1}{2}\alpha\int^{\rho}\rho'^{-1}e^{2\rho'}d\rho', \quad h(\rho) = -\frac{1}{4}(2\rho^2 - 1)g + \frac{3}{16}\alpha e^{2\rho}\rho \tag{2.46}$$

APPENDIX

In this Appendix we establish (2.39). We write the expressions for M_{r+1} and M_r in detail, as follows:

$$\begin{aligned} M_{r+1} = \alpha(n+1)^2\{ & {}^nc_r / [(2r+2)\ (2r+1)] \\ & - {}^nc_{r-1}(2n-2r+1)^2/[(2r+2)\ \ldots\ (2r-1)] \\ & + {}^nc_{r-2}(2n-2r+1)^2(2n-2r+3)^2/[(2r+2)\ \ldots\ (2r-3)]\ \ldots \\ & + (-1)^{n-1}\ {}^nc_1(2n-2r+1)^2\ \ldots\ (2n-3)^2/[(2r+2)\ \ldots\ 3]\} \end{aligned} \tag{A1}$$

$$\begin{aligned} M_r = \alpha(n+1)^2\{ & {}^nc_{r-1} / [(2r)(2r-1)] \\ & - {}^nc_{r-2}(2n-2r+3)^2 / [2r\ \ldots\ (2r-3)] \\ & + {}^nc_{r-3}(2n-2r+3)^2(2n-2r+5)^2 / [2r\ \ldots\ (2r-5)]\ \ldots \\ & + (-1)^{r-2}\ {}^nc_1(2n-2r+3)^2\ \ldots\ (2n-3)^2/[2r\ \ldots\ 3]\} \end{aligned} \tag{A2}$$

If we now form the expression $(2r+2)(2r+1)M_{r+1} + (2n-2r+1)^2M_r$ with the use of (A1) and (A2), it is readily seen that only the first term in this expression, namely $\alpha\ {}^nc_r(n+1)^2$

survives while all the other terms cancel alternatively. For example, the term arising from the second term in (A1) cancels with that arising from the first term in (A2), and so on. This proves that (2.39) satisfies (2.38).

CHAPTER 3

SOME RESULTS ON DIFFERENTIAL ROTATION

3.1 INTRODUCTION

In this chapter we will calculate the shear and rotation for the interior solution of differentially rotating fluid. This section is based essentially on the work of Winicour (1975), (see also Chapter 4 of Islam (1985)). As mentioned in (1.2), when the pressure is zero there is no energy of the matter due to the random motion of the particles and so the mass-energy density consists of only the density of the particles which is mn, where m is the mass of each particle and n is the number density of the particles.

Thus in this case:

$$T^{\mu\nu} = mn\, u^{\mu}u^{\nu} \tag{3.1}$$

Einstein's Equations (1.35) are given by:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}R = 8\pi\, mn\, u^{\mu}u^{\nu} \tag{3.2}$$

Because the divergence of the left hand side vanishes, we get:

$$(nu^{\mu}u^{\nu})_{;\nu} = (nu^{\nu})_{;\nu}u^{\mu} + nu^{\mu}{}_{;\nu}u^{\nu} = 0 \tag{3.3}$$

Now $(nu^{\nu})_{;\nu} = 0$ is just the condition of the conservation of matter so that we get (with (1.3)):

$$u^{\mu}{}_{;\nu}u^{\nu} = \frac{d^2x^{\mu}}{ds^2} + \Gamma^{\mu}_{\nu\sigma} \frac{dx^{\sigma}}{ds} \frac{dx^{\nu}}{ds} = 0 \tag{3.4}$$

which is the geodesic equation so that particles of the dust follow geodesics.

The general rotating metric given by (1.29). Setting $(x^o, x^1, x^2, x^3) = (t, \rho, z, \phi)$, the components of the four-velocity of the rotating dust are:

$$u^o = \frac{dt}{ds} = (f - 2\Omega k - \Omega^2 \ell)^{-1/2}, \quad u' = \frac{d\rho}{ds} = 0$$
$$u^2 = \frac{dz}{ds} = 0, \quad u^3 = \frac{d\phi}{ds} = \frac{d\phi}{dt}\frac{dt}{ds} = \Omega u^o \tag{3.5}$$

We envisage that dust to be rotating about z-axis (the axis of symmetry) with angular velocity Ω which is in general a function of ρ and z.

It is more convenient to consider the following form of (3.2):

$$R_{\mu\nu} = 8\pi\, mn\, (u_\mu u_\nu - \frac{1}{2} g_{\mu\nu}) \tag{3.6}$$

In the metric (1.29) three of the field equations can be written as follows:

$$2e^\mu D^{-1} R_{oo} = (D^{-1} f_\rho)_\rho + (D^{-1} f_z)_z + D^{-3} f\Sigma$$
$$= 8\pi mn\, D^{-1} e^\mu (f - 2\Omega k - \Omega^2 \ell)^{-1} [2\Omega k(-f + \Omega k)$$
$$+ f(f + \Omega^2 \ell)] \tag{3.7a}$$

$$-\ 2e^\mu D^{-1} R_{o3} = (D^{-1} k_\rho)_\rho + (D^{-1} k_z)_z + D^{-3} k\Sigma$$
$$= 8\pi mn\, D^{-1} e^\mu (f - 2\Omega k - \Omega^2 \ell)^{-1} (fk + 2\Omega f\ell - \Omega^2 k\ell) \tag{3.7b}$$

$$- 2e^{\mu}D^{-1}R_{33} = (D^{-1}\ell_{\rho})_{\rho} + (D^{-1}\ell_{z})_{z} + D^{-3}\ell\Sigma$$

$$= - 8\pi mn\ D^{-1}e^{\mu}(f - 2\Omega k - \Omega^{2}\ell)^{-1}(f\ell + 2k^{2}$$

$$+ 2\Omega k\ell + \Omega^{2}\ell^{2}) \qquad (3.7c)$$

where $D^2 = \ell f + k^2$ and $\Sigma = \ell_\rho f_\rho + \ell_z f_z + k_\rho^2 + k_z^2$

An important combination of (3.7, a, b, c) is the following:

$$e^{\mu}D^{-1}(\ell R_{oo} - 2kR_{o3} - fR_{33}) = D_{\rho\rho} + D_{zz} = 0 \qquad (3.8)$$

we can therefore follow the usual procedure to derive the relation between f, k and ℓ:

$$\ell f + k^2 = \rho^2 \qquad (3.9)$$

The geodesic Equations (3.4) imply the following two equations:

$$f_\rho - 2\Omega k_\rho - \Omega^2\ell_\rho = 0 \qquad (3.10a)$$

$$f_z - 2\Omega k_z - \Omega^2\ell_z = 0 \qquad (3.10b)$$

The equations are valid for Ω as a function of ρ and z, so that we can have differential rotation. We transform to F, K, L given by:

$$L = \ell, \quad K = k + \Omega\ell, \quad F = f - 2\Omega k - \Omega^2\ell \qquad (3.11)$$

For constant Ω, this amounts to transforming to a rotating coordinate system.

In the case of axisymmetric differential rotation Ω is a function of ρ and z. Equations (3.7a, b, c), (3.8), (3.9) and (3.10a, b) are all valid in this case, with $\Omega = \Omega(\rho, z)$. We again transform to the new function L, K, F given as (3.11) but now these functions no longer refer to a rotating coordinate system since Ω is not constant. From (3.10a, b) we get:

$$F_\rho + 2\Omega_\rho k = 0 \tag{3.12a}$$

$$F_z + 2\Omega_z k = 0 \tag{3.12b}$$

from which it follows that:

$$\Omega_\rho F_z - \Omega_z F_\rho = 0 \tag{3.13}$$

which implies that F is a function of Ω,

$$F = F(\Omega) \tag{3.14}$$

Equations (3.12a, b) and (3.14) then imply that K is also a function of Ω given by:

$$F' = - 2K \tag{3.15}$$

where the prime denotes differentiation with respect to Ω. Only two of (3.7a, b, c) are independent. From these equations one can derive the following equation:

$$K\Delta F - F\Delta K + 4K(K_\rho \Omega_\rho + K_z \Omega_z) - 2KL(\Omega_\rho^2 + \Omega_z^2)$$
$$+ (2K^2 + LF)\Delta\Omega + 2F(L_\rho \Omega_\rho + L_z \Omega_z) = 0 \tag{3.16}$$

with the use of $FL + K^2 = \rho^2$ and identity

$$\Delta F = F'\Delta\Omega + F''(\Omega_\rho^2 + \Omega_z^2) \tag{3.17}$$

and similar identities for K, (3.16) can be written as follows:

$$(KF' - FK' + \rho^2 + K^2)\Delta\Omega$$
$$+ \left[KF'' - FK'' - F^{-1}F'(\rho^2 - K^2)\right](\Omega_\rho^2 + \Omega_z^2) + 4\rho\,\Omega_\rho = 0 \tag{3.18}$$

Let us now assume that the function Ω is given implicity by ξ satisfying $\Delta\xi = 0$ and ρ^2 as follows:

$$\xi = \xi(\rho,\Omega) \tag{3.19}$$

and let $\dot{\xi} = \left(\frac{\partial\xi}{\partial\rho}\right)_{\Omega = \text{constant}}$, $\xi' = \left(\frac{\partial\xi}{\partial\Omega}\right)_{\rho = \text{constant}}$ (3.20)

Then $\xi_\rho = \dot{\xi} + \xi'\Omega_\rho$ (3.21a)

$$\xi_{\rho\rho} = \ddot{\xi} + 2\dot{\xi}'\Omega_\rho + \xi''\Omega_\rho^2 + \xi'\Omega_{\rho\rho} \tag{3.21b}$$

$$\xi_{zz} = \xi''\Omega_z^2 + \xi'\Omega_{zz} \tag{3.21c}$$

The equation $\Delta\xi = 0$ can then be written as:

$$\Delta\xi = \xi'\Delta\Omega + \xi''(\Omega_\rho^2 + \Omega_z^2) + 2\dot{\xi}'\Omega_\rho + \ddot{\xi} - \rho^{-1}\dot{\xi} = 0 \tag{3.22}$$

Compare (3.22) with (3.18) multiplied by F^{-1}. For these two equations to coincide we must have:

$$\xi' = F^{-1}(KF' - FK' + \rho^2 + K^2) \tag{3.23a}$$

$$\xi'' = F^{-1}(KF'' - FK'' - F^{-1}F'(\rho^2 - K^2)) \qquad (3.23b)$$

$$2\dot{\xi}' = 4\rho F^{-1} \qquad (3.23c)$$

$$\ddot{\xi} - \xi^{-1}\dot{\xi} = 0 \qquad (3.23d)$$

It is readily verified that (3.23a, b, c, d) are satisfied by the following:

$$\xi = \int^{\Omega} F^{-1}(KF' - FK' + \rho^2 + K^2)\,d\Omega \qquad (3.24)$$

which gives Ω implicitly as a function of ρ^2 and ξ. The solution involves one arbitrary function, which can be taken as $F(\Omega)$ or $\Omega(F)$. The function K is then determined by (3.15) and Ω is then given as a function of ρ^2 and ξ by (3.24). This solution was first obtained by Winicour (1975). The corresponding function μ can be obtained from the following two relations:

$$R_{11} - R_{22} = \rho^{-1}\mu_\rho + \frac{1}{2}\rho^{-2}[F_\rho L_\rho + K_\rho^2 - F_z L_z - K_z^2$$

$$+ 2(KL_\rho - LK_\rho)\Omega_\rho - 2(KL_z - LK_z)\Omega_z + L^2(\Omega_\rho^2 - \Omega_z^2)] = 0 \qquad (3.25a)$$

$$R_{12} = \frac{1}{2}\rho^{-1}\mu_z + \frac{1}{4}\rho^{-2}[F_\rho L_z + F_z L_\rho + 2K_\rho K_z + 2(KL_z - LK_z)\Omega_\rho$$

$$+ 2(KL_\rho - LK_\rho)\Omega_z + 2L^2\Omega_\rho\Omega_z] = 0 \qquad (3.25b)$$

and the number density can be obtained from the following equation, which is derived from (3.7a, b, c):

$$8\pi mn\, e^{\mu} = F^{-1}\Delta F + 2F^{-1}(2K' - L)(\Omega_\rho^2 + \Omega_z^2)$$

$$+ 2F^{-1}K\Delta\Omega + \rho^{-2}\Sigma \qquad (3.26)$$

where $\Sigma = F_\rho L_\rho + F_z L_z + K_\rho^2 + K_z^2 + 2\Omega_\rho(KL_\rho - LK_\rho)$

$$+ 2\Omega_z(KL_z - LK_z) + L^2(\Omega_\rho^2 + \Omega_z^2) \qquad (3.27)$$

3.2 THE SHEAR TENSOR FOR DIFFERENTIAL ROTATION

The non-zero members of the Christoffel symbols can be calculated from the metric (1.29) using the relation (1.30).

The expansion is $u^\sigma_{;\sigma}$ where

$$u^\mu_{;\nu} = \frac{\partial u^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\sigma}u^\sigma \qquad (3.28)$$

setting $(x^o, x^1, x^2, x^3) = (t, \rho, z, \theta)$, the components of the four-velocity of the rotating dust are:

$$u^o = \frac{dt}{ds} = (f - 2\Omega k - \Omega^2 \ell)^{-1/2}, \quad u^1 = u^2 = 0,$$

$$u^3 = \frac{d\theta}{ds} = \Omega u^o \qquad (3.29)$$

where Ω is the angular velocity, a function of ρ and z. Here the expansion vanishes identically.

The rotation and shear tensors are given by:

$$\omega_{\mu\nu} = \frac{1}{2}(u_{\mu;\nu} - u_{\nu;\mu}) - \frac{1}{2}(\dot{u}_\mu u_\nu - \dot{u}_\nu u_\mu) \qquad (3.30)$$

$$\sigma_{\mu\nu} = \frac{1}{2}(u_{\mu;\nu} + u_{\nu;\mu}) - \frac{1}{2}(\dot{u}_\mu u_\nu + \dot{u}_\nu u_\mu)$$

$$- \frac{1}{3}(g_{\mu\nu} - u_\mu u_\nu)u^\sigma_{;\sigma} \qquad (3.31)$$

where $\dot{u}_\mu = u_{\mu;\sigma}u^\sigma$, $\quad u_{\mu;\nu} = \dfrac{\partial u_\mu}{\partial x^\nu} - \Gamma^\sigma_{\mu\nu}u_\sigma$

(Misner et al (1973), (p 566), where, however, there are some differences in sign since for them $u_\mu u^\mu = -1$ while for us $u_\mu u^\mu$). After a considerable amount of manipulation, it can be shown that the non-zero values are:

$$\omega_{o1} = \tfrac{1}{2}\Big[\{f_\rho - (k_\rho\Omega + k\Omega_\rho)\}(f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$- \tfrac{1}{2}(f - k\Omega)(f - 2\Omega k - \Omega^2\ell)^{-3/2}$$

$$\times \{f_\rho - (2\Omega_\rho + 2\Omega k_\rho) - (2\Omega\Omega_\rho\ell + \Omega^2\ell_\rho)\}\Big]$$

$$+ \tfrac{1}{2}\Big[\{-\tfrac{1}{2}\rho^{-2}(\ell f_\rho + kk_\rho)(f - k\Omega) + \tfrac{1}{2}\rho^{-2}(fk_\rho + kf_\rho)(k + \ell\Omega)\}$$

$$+ \{-\tfrac{1}{2}\rho^{-2}(k\ell_\rho - \ell k_\rho)(f - k\Omega)$$

$$+ \tfrac{1}{2}\rho^{-2}(f\ell_\rho + kk_\rho)(k + \ell\Omega)\}\Omega\Big] \times (f - k\Omega)(f - 2\Omega k - \Omega^2\ell)^{-3/2}$$

$$\omega_{o1} = \omega_{1o}$$

$$\omega_{o2} = \tfrac{1}{2}\Big[\{f_z - (k_z\Omega + k\Omega_z)\}(f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$- \tfrac{1}{2}(f - k\Omega)(f - 2\Omega k - \Omega^2\ell)^{-3/2} \times \{f_z - (2\Omega_z + 2\Omega k_z)$$

$$- (2\Omega\Omega_z\ell + \Omega^2\ell_z)\}\Big] + \tfrac{1}{2}\Big[\{-\tfrac{1}{2}\rho^{-2}(\ell f_z + kk_z)(f - k\Omega)$$

$$+ \tfrac{1}{2}\rho^{-2}(fk_z - kf_z)(k + \ell\Omega)\} + \{-\tfrac{1}{2}\rho^{-2}(k\ell_z - \ell k_z)(f - k\Omega)$$

$$+ \tfrac{1}{2}\rho^{-2}(f\ell_z + kk_z)(k + \ell\Omega)\}\Omega\Big] \times (f - k\Omega)(f - 2\Omega k - \Omega^2\ell)^{-3/2}$$

$$\omega_{o2} = -\omega_{2o}$$

$$\omega_{13} = -\frac{1}{2}\Big[-\{k_\rho + (\ell_\rho\Omega + \ell\Omega_\rho)\}(f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$+ \frac{1}{2}(k + \ell\Omega)(f - 2\Omega k - \Omega^2\ell)^{-3/2} \times \{f_\rho - (2\Omega_\rho k + 2\Omega k_\rho)$$

$$- (2\Omega\Omega_\rho\ell + \Omega^2\ell_\rho)\}\Big] - \frac{1}{2}\Big[\{-\frac{1}{2}\rho^{-2}(\ell f_\rho + kk_\rho)(f - k\Omega)$$

$$\times (f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$+ \frac{1}{2}\rho^{-2}(fk_\rho - kf_\rho)(k + \ell\Omega)(f - 2\Omega k - \Omega^2\ell)^{-1/2}\}$$

$$\times (f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$+ \{-\frac{1}{2}\rho^{-2}(k\ell_\rho - \ell k_\rho)(f - k\Omega)(f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$+ \frac{1}{2}\rho^{-2}(f\ell_\rho + kk_\rho)(k + \ell\Omega)(f - 2\Omega k - \Omega^2\ell)^{-1/2}\}$$

$$\times \Omega(f - 2\Omega k - \Omega^2\ell)^{-1/2}\Big]\{-(k + \ell\Omega)(f - 2\Omega k - \Omega^2\ell)^{-1/2}\}$$

$$\omega_{13} = -\omega_{31}$$

$$\omega_{23} = -\frac{1}{2}\Big[-\{k_z + (\ell_z\Omega + \ell\Omega_z)\}(f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$+ \frac{1}{2}(k + \ell\Omega)(f - 2\Omega k - \Omega^2\ell)^{-3/2} \times \{f_z - (2\Omega_z k + 2\Omega k_z)$$

$$- (2\Omega\Omega_z\ell + \Omega^2\ell_z)\}\Big] - \frac{1}{2}\Big[\{-\frac{1}{2}\rho^{-2}(\ell f_z + kk_z)(f - k\Omega)$$

$$\times (f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$+ \frac{1}{2}\rho^{-2}(f\ell_z - kf_z)(k + \ell\Omega)(f - 2\Omega k - \Omega^2\ell)^{-1/2}\}(f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$+ \{-\frac{1}{2}\rho^{-2}(k\ell_z - \ell k_z)(f - k\Omega)(f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$+ \frac{1}{2}\rho^{-2}(f\ell_z + kk_z)(k + \ell\Omega)$$

$$\times (f - 2\Omega k - \Omega^2 \ell)^{-1/2}\}\Omega(f - 2\Omega k - \Omega^2\ell)^{-1/2}]\{-(k$$

$$+ \ell\Omega)(f - 2\Omega k - \Omega^2\ell)^{-1/2}\}$$

$$\omega_{23} = - \omega_{32}$$

$$\sigma_{o1} = \frac{1}{2}\{(k + \ell\Omega)(f - k\Omega)(f - 2\Omega k - \Omega^2\ell)^{-3/2}$$

$$- k(f - 2\Omega k - \Omega^2\ell)^{-1/2}\}\Omega_\rho$$

$$\sigma_{o1} = \sigma_{1o}$$

$$\sigma_{o2} = \frac{1}{2}\{(k + \ell\Omega)(f - k\Omega)(f - 2\Omega k - \Omega^2\ell)^{-3/2}$$

$$- k(f - 2\Omega k - \Omega^2\ell)^{-1/2}\}\Omega_z$$

$$\sigma_{o2} = \sigma_{2o}$$

$$\sigma_{13} = \{- \frac{1}{2}\ell(f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$- \frac{1}{2}(k + \ell\Omega)^2(f - 2\Omega k - \Omega^2\ell)^{-3/2}\}\Omega_\rho$$

$$\sigma_{13} = \sigma_{31}$$

$$\sigma_{23} = \{- \frac{1}{2}\ell(f - 2\Omega k - \Omega^2\ell)^{-1/2}$$

$$- \frac{1}{2}(k + \ell\Omega)^2(f - 2\Omega k - \Omega^2\ell)^{-3/2}\}\Omega_z$$

$$\sigma_{23} = \sigma_{32}$$

The other components are zero. It is evident from these expressions that the shear vanishes when the rotation is rigid i.e when Ω = constant, as expected. One of the objects of this chapter is to derive expressions for the shear which makes this property evident. We have also calculated the rotation-tensor in detail but we do not give these here as these expressions are cumbersome and not instructive.

3.3 AN EXPLICIT SOLUTION

Substituting $F = a\Omega^2$ in the Equation (3.24) we have:

$F' = 2a\Omega = -2K \quad \therefore \; K = -a\Omega, \; K' = -a$

$$\therefore \; \xi = \int^{\Omega} F^{-1}(KF' - FK' + \rho^2 + K^2)\,d\Omega$$

$$= \int^{\Omega} \frac{1}{a}\,\Omega^{-2}(-a\Omega \times 2a\Omega + a\Omega^2 \times a + \rho^2 + a^2\Omega^2)\,d\Omega$$

$$= \frac{1}{a}\int \rho^2\,\Omega^{-2}d\Omega = -\frac{1}{a}\,\rho^2\,\Omega^{-1}$$

$$\therefore \; \xi = -\frac{1}{a}\,\rho^{-2}\,\Omega^{-1} \quad \text{or} \quad \Omega = -\frac{1}{a}\rho^{-2}\xi^{-1}$$

$$F = \frac{1}{a}\,\rho^4\xi^{-2}, \; K = \xi^{-1}\rho^2, \; L = a(\rho^{-2}\xi^2 - 1)$$

$$\Omega_\rho = -\frac{2}{a}\,\rho\xi^{-1} + \frac{1}{a}\,\rho^2\xi^{-2}\xi_\rho, \; \Omega_z = \frac{1}{a}\,\rho^2\xi^{-2}\xi_z \qquad \text{(A)}$$

$$F_\rho = \frac{1}{a}(4\,\rho^3\xi^{-2} - 2\rho^4\xi^{-3}\xi_\rho). \; F_z = -\frac{2}{a}\,\rho^4\xi^{-3}\xi_z$$

$$K_\rho = 2\rho\xi^{-1} - \xi^{-2}\rho^2\xi_\rho, \quad K_z = -\rho^2\xi^{-2}\xi_z$$

$$L_\rho = a(-2\rho^{-3}\xi^2 + 2\rho^{-2}\xi\xi_\rho), \; L_z = 2a\,\rho^{-2}\xi\xi_z$$

Using the above values we can calculate the non-zero Christoffel symbols, to calculate the expansion, shear and rotations. Here the expansion is zero. After a considerable amount of manipulation, it can be shown that:

$$\omega_{o1} = \frac{1}{a^{1/2}}\{- 2\rho\xi^{-1} + \rho^2\xi^{-2}\xi_\rho - \rho^{-1}\xi + \xi_\rho\}, \quad \omega_{o1} = - \omega_{10}$$

$$\omega_{o2} = - \frac{1}{a^{1/2}}\{\xi^{-2}\xi_z - 2\rho^2\xi^{-2}\xi_z - \xi_z\}, \quad \omega_{o2} = - \omega_{20}$$

$$\omega_{13} = a^{1/2}\{\frac{1}{2}\rho^{-3}\xi^2 - \frac{1}{2}\rho^{-2}\xi\xi_\rho - 2\rho^{-1} + \xi^{-1}\xi_\rho$$

$$+ 2\rho\xi^2 + \frac{1}{2}\rho^{-1}\xi^4 - \rho^2\xi\xi_\rho - \frac{1}{2}\xi^3\xi_\rho\}, \quad \omega_{13} = - \omega_{31}$$

$$\omega_{23} = a^{1/2}\{- \rho^{-2}\xi\xi_z + 2\rho^6\xi^{-4}\xi_z - \frac{1}{2}\rho^2\xi_z\}, \quad \omega_{23} = - \omega_{32}$$

$$\sigma_{o1} = \frac{1}{a^{1/2}}(\rho^{-1}\xi - \xi_\rho) \quad \sigma_{o1} = \sigma_{10}$$

$$\sigma_{o2} = \frac{1}{a^{1/2}}(- \xi^{-2}\xi_z + \frac{1}{2}\rho^2\xi^{-2}\xi_z - \xi_z), \quad \sigma_{o2} = \sigma_{20}$$

$$\sigma_{13} = a^{1/2}(\frac{5}{2}\rho^{-3}\xi^2 - \frac{5}{2}\rho^{-2}\xi\xi_\rho + 2\rho\xi^2$$

$$+ \frac{1}{2}\rho^{-1}\xi^4 - \rho^2\xi\xi_\rho - \frac{1}{2}\xi^3\xi_\rho), \quad \sigma_{13} = \sigma_{31}$$

$$\sigma_{23} = a^{1/2}(- \rho^{-2}\xi\xi_z + 2\rho^2\xi^{-4}\xi_z - \rho^4\xi^{-2} \quad \sigma_{23} = \sigma_{32}$$

$$+ \xi^{-2}\xi_z - \frac{1}{2}\rho^2\xi_z + \xi^{-1}\xi_z - \frac{1}{2}\rho^2\xi^{-3}\xi_z - \rho^{-2}\xi\xi_z),$$

We can work out an invariant measure of the rotation given as $\omega^2 = \omega^{\mu\nu}\omega_{\mu\nu}$. We will not present the details here.

CHAPTER 4

KILLING VECTORS OF A SOLUTION OF EINSTEIN-MAXWELL EQUATIONS

4.1 INTRODUCTION

Islam (1980, 1983) found an exact cylindrically symmetric stationary global solution of the Einstein-Maxwell equations representing rigidly rotating charged dust. By a global solution is meant a combination of interior and exterior solutions which match smoothly at the boundary of the matter distribution. Global solutions either of the pure Einstein or the Einstein-Maxwell equations are very rare and Islam's solution is of considerable importance. This solution Islam (1980, 1983) has been studied by Vanden Bergh and Wils (1985) who find that the interior solution is regular on the axis of symmetry. However, there are some peculiar features of the solution for which it is necessary to study the curvature invariants and Killing vectors. The curvature invariants will be discussed in Chapter 5; in this chapter we find Killing vectors for the global solution and give an argument, following Bonnor (1980) to show that there are no hypersurface-orthogonal (HSO) Killing vectors so that the solution is stationary and not static.

4.2 INTERIOR SOLUTION

We use the solutions of Islam (1980, 1983). The metric is given by:

$$ds^2 = fdt^2 - 2k\ d\theta dt - \ell d\theta^2 - e^{\mu}(d\rho^2 + dz^2) \qquad (4.1)$$

Here, f, k, ℓ and μ are functions of ρ only.

The interior solution is as follows (see Sections 3, 4 of Islam (1983):

$$\ell = \frac{(\rho^2 - \frac{a_o^2}{4\xi^2}\rho^4)}{\xi + \eta\rho^2}, \ K = -\frac{a_o}{2\xi}\rho^2, \ f = \xi + \eta\rho^2,$$

$$e^{\mu} = \alpha(\xi + \eta\rho^2)^{-m^2/q^2}$$

$$\alpha = e^{-2/3}\sigma^{4/3}\rho_o^{4/9}\lambda, \ 4q^2 = 3m^2, \ \xi = \frac{2}{3}\sigma\rho_o^{2/3}$$

$$\eta = \frac{1}{3}\sigma\rho_o^{-4/3} \qquad K_o = -\frac{3a_o}{4\sigma^2}\rho_o^{2/3}$$

$$8\sigma^2 = 27a_o^2\rho_o^{2/3}, \ a_o\kappa_o = -\frac{2}{9} \qquad (4.2)$$

Here a_o, k_o, ρ_o, λ are constants and $\rho = \rho_o$ is the boundary, with m, q the mass and charge (in relativistic units) respectively, of the particles. The non-zero Christoffel symbols are given by:

(with t, ρ, z, $\theta = x^o, x^1, x^2, x^3$)

$$\Gamma^o_{o1} = \frac{1}{2}\rho^{-2}(\ell f_\rho + kk_\rho) = \frac{\frac{4}{3}\rho}{2\rho_o^2 + \rho^2},$$

$$\Gamma^o_{13} = \frac{1}{2}\rho^{-2}(k\ell_\rho - \ell k_\rho) = -\frac{2\rho^3}{\frac{9}{2}a_o(2\rho_o^2 + \rho^2)^2}$$

$$\Gamma^1_{oo} = \frac{1}{2} e^{-\mu} f_\rho = \frac{\frac{1}{3}\sigma\rho\, e^{2/3}}{\rho_o^{6/9}\lambda}\left(\frac{2}{3}\rho_o^{2/3} + \frac{\rho^2}{\rho_o^{4/3}}\right)$$

$$\Gamma^1_{o3} = -\frac{1}{2} e^{-\mu} k_\rho = \frac{a_o\rho\, e^{2/3}\left(\frac{2}{3}\rho_o^{2/3} + \frac{1}{3}\frac{\rho^2}{\rho_o^{4/3}}\right)^{4/3}}{\frac{1}{3}\sigma\rho_o^{10/9}\lambda}$$

$$\Gamma^1_{11} = \frac{1}{2}\mu_\rho = -\frac{4\rho}{(2\rho_o^2 + \rho^2)}$$

Similarly

$$\Gamma^1_{22} = -\frac{1}{2}\mu_\rho = \frac{4\rho}{2\rho_o^2 + \rho^2}\ , \quad \Gamma^2_{12} = \frac{1}{2}\mu_\rho = -\frac{4\rho}{2\rho_o^2 + \rho^2}$$

$$\Gamma^3_{o1} = \frac{1}{2}\rho^{-2}(fk_\rho - kf_\rho) = -\frac{1}{2}\frac{a_o}{\rho}$$

$$\Gamma^1_{33} = -\frac{1}{2} e^{-\mu}\ell_\rho = \frac{\rho e^{2/3}(1 - 12\rho_o^4 - 4\rho_o^2)}{18\rho^{34/9}\lambda\sigma\left(\frac{2}{3}\rho_o^{2/3} + \frac{1}{3}\frac{\rho^2}{\rho_o^{4/3}}\right)^{2/3}}$$

$$\Gamma^3_{13} = \frac{1}{2}\rho^{-2}(kk_\rho + f\ell_\rho) = \frac{\rho^{-1}(2\rho_o^2 - \frac{1}{3}\rho^2)}{(2\rho_o^2 + \rho^2)}$$

4.3 KILLING VECTORS FOR THE SOLUTION

The Killing equation is $\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0$ (4.3)

where $\xi_\mu = (\xi_o, \xi_1, \xi_2, \xi_3)$ $\qquad \xi_{\mu;\nu} = \frac{\partial\xi_\mu}{\partial x^\nu} - \Gamma^\lambda_{\nu\mu}\xi_\lambda$

The non-zero and distinct equations from which we can find the values of ξ_μ are given below:

$$\mu = 0,\ \nu = 0 \quad \xi_{0;0} = \frac{\partial \xi_0}{\partial x^0} - (\Gamma^0_{00}\xi_0 + \Gamma^1_{00}\xi_1 + \Gamma^2_{00}\xi_2 + \Gamma^3_{00}\xi_3)$$

$$= 0 - (0 + \frac{1}{2} e^{-\mu} f_\rho + 0 + 0) = -\frac{1}{2} e^{-\mu} f_\rho \xi_1$$

$$\therefore\ \xi_{0;0} + \xi_{0;0} = - e^{-\mu} f_\rho \xi_1$$

$$= -\frac{2}{3} \rho\sigma e^{2/3} (\frac{2}{3} \rho_0^{2/3} + \frac{1}{3} \frac{\rho^2}{\rho_0^{4/3}}) \qquad (4.4)$$

leading to $\xi_1 = 0$ (4.5)

$$\mu = 1,\ \nu = 2,\ \xi_{1;2} = \frac{\partial \xi_1}{\partial x^2} - (\Gamma^0_{21}\xi_0 + \Gamma^1_{21}\xi_1 + \Gamma^2_{21}\xi_2 + \Gamma^3_{21}\xi_3)$$

$$= 0 - (0 + 0 + \frac{1}{2} \mu_\rho \xi_2 + 0) = -\frac{1}{2} \mu_\rho \xi_2$$

$$\xi_{2;1} = \frac{\partial \xi_2}{\partial x^1} - (\Gamma^0_{12}\xi_0 + \Gamma^1_{12}\xi_1 + \Gamma^2_{12}\xi_2 + \Gamma^3_{12}\xi_3)$$

$$= \frac{\partial \xi_2}{\partial \rho} - \frac{1}{2} \mu_\rho \xi_2$$

$$\therefore\ \xi_{1;2} + \xi_{2;1} = \frac{\partial \xi_2}{\partial \rho} - \mu_\rho \xi_2 = 0,\ \frac{\partial \xi_2}{\xi_2} = \mu_\rho d\rho = -\frac{\frac{8}{3}\rho}{2\rho_0^2 + \rho^2} d\rho \qquad (4.6)$$

$$\log \xi_2 = -\frac{8}{3} \int \frac{\rho}{2\rho_0^2 + \rho^2} d\rho = -\frac{8}{3} \cdot \frac{1}{2} \int \frac{2\rho}{2\rho_0^2 + \rho^2} d\rho$$

$$= -\frac{4}{3} \log(2\rho_0^2 + \rho^2) + \log c$$

$$= \log(2\rho_o^2 + \rho^2)^{-4/3} + \log c = \log c(2\rho_o^2 + \rho^2)^{-4/3} \qquad (4.7)$$

$$\therefore \xi_2 = c(2\rho_o^2 + \rho^2)^{-4/3} \qquad (4.8)$$

where C is any constant.

$$\mu = 0,\ \nu = 1,\ \xi_{o;1} = \frac{\partial \xi_o}{\partial x^1} - (\Gamma^o_{1o}\xi_o + \Gamma^1_{1o}\xi_1 + \Gamma^2_{1o}\xi_2 + \Gamma^3_{1o}\xi_3)$$

$$= \frac{\partial \xi_o}{\partial \rho} - (\Gamma^o_{1o}\xi_o + 0 + 0 + \Gamma^3_{1o}\xi_3) = \frac{\partial \xi_o}{\partial \rho} - (\Gamma^o_{1o}\xi_o + \Gamma^3_{1o}\xi_3)$$

$$\xi_{1;o} = \frac{\partial \xi_1}{\partial x^o} - (\Gamma^o_{o1}\xi_o + \Gamma^1_{o1}\xi_1 + \Gamma^2_{o1}\xi_2 + \Gamma^3_{o1}\xi_3)$$

$$= 0 - (\Gamma^o_{o1}\xi_o + 0 + 0 + \Gamma^3_{o1}\xi_3) = -(\Gamma^o_{o1}\xi_o + \Gamma^3_{o1}\xi_3)$$

$$\therefore \xi_{o;1} + \xi_{1;o} = \frac{\partial \xi_o}{\partial \rho} - (2\Gamma^o_{o1}\xi_o + 2\Gamma^3_{o1}\xi_3) = 0 \qquad (4.9)$$

$$\frac{\partial \xi_o}{\partial \rho} \quad \frac{2 \cdot \frac{4}{3}\rho}{2\rho_o^2 + \rho^2}\xi_o + \frac{a_o}{\rho}\xi_3 = 0, \quad a_o\xi_3 = \rho\left(-\frac{\partial \xi_o}{\partial \rho} + \frac{\frac{8}{3}\rho}{2\rho_o^2 + \rho^2}\xi_o\right)$$

$$= -\rho\frac{\partial \xi_o}{\partial \rho} + \frac{\frac{8}{3}\rho^2}{2\rho_o^2 + \rho^2}\xi_o$$

$$\therefore \xi_3 = \frac{1}{a_o}\left[-\rho\frac{\partial \xi_o}{\partial \rho} + \frac{\frac{8}{3}\rho^2}{2\rho_o^2 + \rho^2}\xi_o\right] \qquad (4.10)$$

$$\frac{\partial \xi_3}{\partial \rho} = \frac{1}{a_o}\left[-\frac{\partial \xi_o}{\partial \rho} - \rho\frac{\partial^2 \xi_o}{\partial \rho^2} + \frac{\frac{32}{3}\rho_o^2\rho^2}{(2\rho_o^2 + \rho^2)^2}\xi_o\right.$$

$$\left. + \frac{\frac{8}{3}\rho^2}{(2\rho_o^2 + \rho^2)}\right]\frac{\partial \xi_o}{\partial \rho} \qquad (4.11)$$

$\mu = 1, \nu = 3 \qquad \xi_{1;3} = \dfrac{\partial \xi_1}{\partial x^3} - (\Gamma^{o}_{31}\xi_o + \Gamma^{1}_{31}\xi_1$

$$+ \Gamma^{2}_{31}\xi_2 + \Gamma^{3}_{31}\xi_3)$$

$$= o - (\Gamma^{o}_{31}\xi_o + o + o + \Gamma^{3}_{31}\xi_3) = -(\Gamma^{o}_{31}\xi_o + \Gamma^{3}_{31}\xi_3)$$

$$\xi_{3;1} = \frac{\partial \xi_3}{\partial x^1} - (\Gamma^{o}_{13}\xi_o + \Gamma^{1}_{13}\xi_1 + \Gamma^{2}_{13}\xi_2 + \Gamma^{3}_{13}\xi_3)$$

$$= \frac{\partial \xi_3}{\partial \rho} - (\Gamma^{o}_{13}\xi_o + \Gamma^{3}_{13}\xi_3)$$

$$\therefore \xi_{1;3} + \xi_{3;1} = \frac{\partial \xi_3}{\partial \rho} - (2\Gamma^{o}_{13}\xi_o + 2\Gamma^{3}_{13}\xi_3) = o \qquad (4.12)$$

$$\frac{\partial \xi_3}{\partial \rho} + \frac{\frac{8}{9}\rho^3}{a_o(2\rho_o^2 + \rho^2)^2}\xi_o - 2 \cdot \frac{1}{\rho}\,\frac{(2\rho_o^2 - \frac{1}{3}\rho^2)}{(2\rho_o^2 + \rho^2)}\xi_3 = o \qquad (4.13)$$

Using the values of (4.10) and (4.11) in (4.13) we get:

$$\frac{1}{a_o}\left[-\frac{\partial \xi_o}{\partial \rho} - \rho\frac{\partial^2 \xi_o}{\partial \rho^2} + \frac{\frac{32}{3}\rho_o^2\rho}{(2\rho_o^2 + \rho^2)^2}\xi_o + \frac{\frac{8}{3}\rho^2}{2\rho_o^2 + \rho^2}\frac{\partial \xi_o}{\partial \rho}\right]$$

$$+ \frac{\frac{8}{9}\rho^3}{a_o(2\rho_o^2 + \rho^2)^2}\xi_o$$

$$- 2 \cdot \frac{1}{\rho}\,\frac{(2\rho_o^2 - \frac{1}{3}\rho^2)}{(2\rho_o^2 + \rho^2)} \times \frac{1}{a_o}\{-\rho\frac{\partial \xi_o}{\partial \rho} + \frac{\frac{8}{3}\rho^2}{2\rho_o^2 + \rho^2}\xi_o\} = o$$

$$-\rho\frac{\partial^2 \xi_o}{\partial \rho^2} + \frac{\partial \xi_o}{\partial \rho} + \frac{\frac{8}{2}\rho^3}{(2\rho_o^2 + \rho^2)^2}\xi_o = o \qquad (4.14)$$

$$\frac{\partial^2 \xi_o}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial \xi_o}{\partial \rho} - \frac{\frac{8}{3}\rho^2}{(2\rho_o^2 + \rho^2)^2}\xi_o = o \qquad (8) \quad (4.15)$$

putting $x = \rho^2$, $\sqrt{x} = \rho$, $\frac{\partial}{\partial \rho} = \frac{\partial}{\partial x} \frac{\partial x}{\partial \sqrt{x}} = \frac{\partial}{\partial x} / \frac{\partial \sqrt{x}}{\partial x}$

$$= \frac{\partial}{\partial x} / \frac{1}{2} \cdot \frac{1}{\sqrt{x}} = 2\sqrt{x} \quad \frac{\partial}{\partial x}$$

$$\frac{\partial^2}{\partial \rho^2} = \frac{\partial}{\partial \rho}\left(\frac{\partial}{\partial \rho}\right) = 2\sqrt{x} \quad \frac{\partial}{\partial x}\left(2\sqrt{x} \frac{\partial}{\partial x}\right) = 2\sqrt{x}\left\{\frac{2}{2\sqrt{x}} \frac{\partial}{\partial x} + 2\sqrt{x} \frac{\partial^2}{\partial x^2}\right\}$$

$$= 2 \frac{\partial}{\partial x} + 4x \frac{\partial^2}{\partial x^2}$$

putting the above values in Equation (4.15)

$$2 \frac{\partial \xi_o}{\partial x} + 4x \frac{\partial^2 \xi_o}{\partial x^2} - \frac{1}{\sqrt{x}} \cdot 2\sqrt{x} \frac{\partial \xi_o}{\partial x} - \frac{8x}{3(2\rho_o^2 + x)^2} \xi_o = 0$$

$$\frac{\partial^2 \xi_o}{\partial x^2} - \frac{2}{3(2\rho_o^2 + x)^2} \xi_o = 0 \qquad (4.16)$$

$$\text{or} \quad 3(2\rho_o^2 + x)^2 \frac{\partial^2 \xi_o}{\partial x^2} - 2\xi_o = 0 \qquad (4.17)$$

putting $z = 2\rho_o^2 + x$, $dz = dx$, $\frac{d\xi_o}{dx} = \frac{d\xi_o}{dz} \frac{dz}{dx} = \frac{d\xi_o}{dz}$

$$\therefore \frac{d^2 \xi_o}{dx^2} = \frac{d^2 \xi_o}{dz^2} \qquad (4.18)$$

So Equation (4.17) becomes

$$3z^2 \frac{\partial^2 \xi_o}{\partial z^2} - 2\xi_o = 0 \qquad (4.19)$$

Again putting $z = e^p$ $\therefore p = \log z$. $\frac{dp}{dz} = \frac{1}{z}$

$$\frac{d\xi_o}{dz} = \frac{d\xi_o}{d\rho} \frac{dp}{dz} = \frac{1}{z} \quad \frac{d\xi_o}{d\rho}$$

$$\frac{d^2\xi_o}{dz^2} = \frac{d}{dz}\left(\frac{1}{z}\frac{d\xi_o}{dp}\right) = -\frac{1}{z^2}\frac{d\xi_o}{dp} + \frac{1}{z}\frac{d\rho}{dz}\frac{d^2\xi_o}{dp^2}$$

$$= \frac{1}{z^2}\left(\frac{d^2\xi_o}{dp^2} - \frac{d\xi_o}{dp}\right) \qquad (4.20)$$

putting the above values in (4.19) we get:

$$3\frac{\partial^2\xi_o}{\partial p^2} - 3\frac{\partial\xi_o}{\partial p} - 2\xi_o = 0$$

$$\therefore \quad (3D^2 - 3D - 2)\xi_o = 0 \qquad \text{where D is operator} \qquad (4.21)$$

$$\xi_o = Ae^{\left(\frac{3+\sqrt{33}}{6}\right)p} + Be^{\left(\frac{3-\sqrt{33}}{6}\right)p}$$

$$= A\,e^{\left(\frac{3+\sqrt{33}}{6}\right)\log(2\rho_o^2+\rho^2)} + Be^{\left(\frac{3-\sqrt{33}}{6}\right)\log(2\rho_o^2+\rho^2)}$$

$$= A(2\rho_o^2+\rho^2)^{\frac{3+\sqrt{33}}{6}} + B(2\rho_o^2+\rho^2)^{\frac{3-\sqrt{33}}{6}} \qquad (4.22)$$

$$\frac{\partial\xi_o}{\partial\rho} = A\left(\frac{3+\sqrt{33}}{3}\right)(2\rho_o^2+\rho^2)^{\frac{3+\sqrt{33}}{6}-1}\rho$$

$$+ B\left(\frac{3-\sqrt{33}}{3}\right)(2\rho_o^2+\rho^2)^{\frac{3-\sqrt{33}}{6}-1}\rho \qquad (4.23)$$

putting the values of ξ_o and $\frac{\partial\xi_o}{\partial\rho}$ in Equation (4.10) we find

$$\zeta_3 = \frac{1}{a_o}\left[A\left(\frac{5-\sqrt{33}}{3}\right)(2\rho_o^2+\rho^2)^{\frac{3+\sqrt{33}}{6}}\rho^2\right.$$

$$\left.+ B\left(\frac{5+\sqrt{33}}{3}\right)(2\rho_o^2+\rho^2)^{\frac{3-\sqrt{33}}{6}}\rho^2\right] \qquad (4.24)$$

where A and B are constants.

4.4 PROOF OF THE STATIONARY NATURE OF THE SOLUTION

We have only found the above Killing vector and three others given by:

$$z^i = \delta^i_1 \ , \ \theta^i = \delta^i_3 \ , \ t^i = \delta^i_o \tag{4.25}$$

We assume that these are the only linearly independent Killing vectors. Following Bonnor (1980) we find the most general Killing vector, which can be written as:

$$\chi^i = p\delta^i_1 + q\delta^i_3 + s\delta^i_o + wY^i \tag{4.26}$$

where Y^i is given by $Y^i = (\xi_o, 0, \xi_2, \xi_3)$ with ξ_o, ξ_2, ξ_3 represented by Equations (4.22,(4.8) and (4.24) respectively where p, q, s and w are real constants.

A necessary condition for this to be HSO is that

$$\chi_{[i,j}\chi_{k]} = 0 \tag{4.27}$$

where the comma means partial differentiation and square brackets mean antisymmetrisation. Taking i, j, k respectively equal to 1, 2, 3 we at once find w = 0 in all cases. Again following Bonnor (1980), comparing i, j, k = 1, 2, 3 and i, j, k = i, 2, 0 we find that either P = 0 or

$$\frac{(e^\mu)'}{(e^\mu)} = \frac{q\ell' + sk'}{q\ell + sk} = \frac{qk' - sf'}{qk - sf} \tag{4.28}$$

where the prime means $\frac{d}{d\rho}$ which can be satisfied only if q = s = 0. In the latter cases (14) reduces to $\chi^i = p\delta^i_1$, which is orthogonal to the hypersurfaces z = constant, but space-like and so of no interest here. We therefore take $P = 0$

Putting i, j, k = 2, 3, 0 in (4.27) gives us:

$$q^2(\ell k' - \ell' k) + qs(\ell' f - \ell f') + s^2(fk - kf') = 0 \qquad (4.29)$$

since $\ell k' - \ell' k \neq 0$, $(s = 0, q \neq 0)$ does not lead to an HSO Killing vector, and we may write (4.29) as:

$$x^2(\ell k' - \ell' k) + x(\ell' f - \ell f') + (fk' - kf') = 0 \qquad (4.30)$$

where $x = q/s$.

The square of the magnitude of the Killing vector (4.26) (with $P = 0$) is:

$$g_{ik}\chi^i\chi^k = s^2(f - 2xk - x^2\ell) \qquad (4.31)$$

There exist HSO Killing vectors with $P = 0$ only if the roots of (4.30) are real and independent of P; and they are time-like, null or space-like according to whether (4.31) is positive, zero or negative. We examine the interior case putting the values of interior solution in (4.30) and find that the roots are not independent of ρ. So there exist no HSO Killing vectors.

4.5 EXTERIOR SOLUTION AND KILLING VECTORS FOR IT

From Islam (1983) Sections 2, 4 exterior solution is given as:

$$\ell = \left(\frac{5}{3}\,\frac{\rho^{4/3}}{\sigma} - \frac{9}{4}\,\frac{a_o^2}{\sigma^3}\,\rho^2 - k_o^2\sigma\rho^{2/3}\right),\quad k = -\frac{3a_o}{2\sigma}\,\rho^{4/3} - k_o\sigma\rho^{2/3}$$

$$f = \sigma\rho^{2/3},\quad e^{\mu} = \lambda\rho^{-4/9}\exp\left(-\frac{9}{2}\,b^2\sigma\rho^{2/3}\right),$$

$$\sigma = \left(\frac{a_o}{2b^2}\right)^{1/3},\quad \mu_\rho = -\frac{4}{9}\,\rho^{-1} - 3b^2\sigma\rho^{-1/3}$$

The non-zero Christoffel symbols are given by:

$$\Gamma^o_{o1} = \frac{1}{2}\,\rho^{-2}(\ell f_\rho + kk_\rho) = \left(\frac{2}{9}\,\rho^{-1} + \frac{3}{4}\,\frac{a_o^2}{\sigma^2}\,\rho^{-1/3}\right)$$

$$\Gamma^o_{13} = \frac{1}{2}\,\rho^{-2}(k\ell_\rho - \ell k_\rho) = \frac{1}{2}\,\rho^{-2}\Big\{-\frac{2}{3}\,\frac{a_o}{\sigma^2}\,\rho^{5/3}$$

$$-\frac{8}{9}\,k_o\rho + \frac{9}{4}\,\frac{a_o^3}{\sigma^4}\,\rho^{7/3}\Big\}$$

$$\Gamma^1_{oo} = \frac{1}{2}\,e^{-\mu}f_\rho = \frac{1}{2}\cdot\frac{\frac{2}{3}\,\sigma\rho^{-1/3}}{\lambda\rho^{-4/9}\exp\left(-\frac{9}{2}\,b^2\sigma\rho^{2/3}\right)}$$

$$\Gamma^1_{o3} = -\frac{1}{2}\,e^{-\mu}k_\rho = -\frac{1}{2}\,\frac{\frac{-2a_o}{\sigma}\,\rho^{1/3} - \frac{2}{3}\,k_o\sigma\rho^{-1/3}}{\lambda\rho^{-4/9}\exp\left(-\frac{9}{2}\,b^2\sigma\rho^{2/3}\right.}$$

$$\Gamma^1_{11} = \frac{1}{2}\,\mu_\rho = -\frac{2}{9}\,\rho^{-1} - \frac{3}{2}\,b^2\sigma\rho^{-1/3}$$

$$\Gamma^1_{22} = -\frac{1}{2}\,\mu_\rho = \frac{2}{9}\,\rho^{-1} + \frac{3}{2}\,b^2\sigma\rho^{-1/3}$$

$$\Gamma^1_{33} = -\frac{1}{2} e^{-\mu} \ell_\rho = -\frac{1}{2} \frac{(\frac{20}{9} \frac{\rho^{1/3}}{\sigma} - \frac{9}{2} \frac{a_o^2}{\sigma^3} \rho - \frac{2}{3} k_o^2 \sigma \rho^{-1/3})}{\lambda \rho^{-4/9} \exp(-\frac{9}{2} b^2 \sigma \rho^{2/3})}$$

$$\Gamma^2_{12} = \frac{1}{2} \mu_\rho = -\frac{2}{9} \rho^{-1} - \frac{3}{2} b^2 \sigma \rho^{-1/3}$$

$$\Gamma^3_{o1} = \frac{1}{2} \rho^{-2} (f k_\rho - k f_\rho) = -\frac{1}{2} \frac{a_o}{\rho}$$

$$\Gamma^3_{13} = \frac{1}{2} \rho^{-2} (f \ell_\rho + k k_\rho) = \frac{1}{2} \rho^{-2} (\frac{14}{9} \rho - \frac{3}{2} \frac{a_o^2}{\sigma^2} \rho^{5/3})$$

Putting the above non-zero values of Christoffel symbol to the Killing equations we can find the values of ξ_μ for exterior solution as given below:

$$\mu = 0,\ \nu = 0,\ \xi_{o;o} = \frac{\partial \xi_o}{\partial x^o} - (\Gamma^o_{oo}\xi_o + \Gamma^1_{oo}\xi_1 + \Gamma^2_{oo}\xi_2 + \Gamma^3_{oo}\xi_3)$$

$$= -\frac{1}{2} e^{-\mu} f_\rho \xi_1$$

$$\therefore\ \xi_{o;o} + \xi_{o;o} = -e^{-\mu} f_\rho \xi_1 = -\frac{\frac{2}{3} \sigma \rho^{-1/3}}{\lambda \rho^{-4/9} \exp(-\frac{9}{2} b^2 \sigma \rho^{2/3})} \xi_1 = 0$$

leading to $\xi_1 = 0$ (4.32)

$$\mu = 1,\ \nu = 2,\ \xi_{1;2} = \frac{\partial \xi_1}{\partial x^2} - (\Gamma^o_{21}\xi_o + \Gamma^1_{21}\xi_1 + \Gamma^2_{21}\xi_2$$

$$+ \Gamma^3_{21}\xi_3) = -\frac{1}{2} \mu_\rho \xi_2$$

$$\xi_{2;1} = \frac{\partial \xi_2}{\partial x^1} - (\Gamma^o_{12}\xi_o + \Gamma^1_{12}\xi_1 + \Gamma^2_{12}\xi_2 + \Gamma^3_{12}\xi_3) = \frac{\partial \xi_2}{\partial \rho} - \frac{1}{2} \mu_\rho \xi_2$$

$$\therefore\ \xi_{1;2} + \xi_{2;1} = \frac{\partial \xi_2}{\partial \rho} - \mu_\rho \xi_2 = 0 \qquad (4.33)$$

$$\mu_\rho = -\frac{4}{9}\rho^{-1} - 3b^2\sigma\rho^{-1/3} = -\frac{4}{9}\rho^{-1} - \frac{3}{2}\frac{a_o^2}{\sigma^3}\sigma\rho^{-1/3}$$

$$= -\frac{4}{9}\rho^{-1} - \frac{3}{2}\frac{a_o^2}{\sigma^2}\rho^{-1/3}$$

$$= -\frac{4}{9}\rho^{-1} - \frac{3}{2}\cdot\frac{8}{27}\cdot\frac{\rho^{-1/3}}{\rho_o^{2/3}} = -\frac{4}{9}\frac{1}{\rho}(1 + \frac{\rho^{2/3}}{\rho_o^{2/3}})$$

Equation (4.33) can be written as:

$$\frac{d\xi_2}{d\rho} + \frac{4}{9}\frac{1}{\rho}(1 + \frac{\rho^{2/3}}{\rho_o^{2/3}})\xi_2 = 0 \quad \text{or} \quad \frac{d\xi_2}{d\rho} = -\frac{4}{9}(\frac{1}{\rho} + \frac{\rho^{-1/3}}{\rho_o^{2/3}})\xi_2$$

$$\text{or,} \quad \frac{d\xi_2}{\xi_2} = -\frac{4}{9}(\frac{1}{\rho} + \frac{\rho^{-1/3}}{\rho_o^{2/3}})d\rho \quad \text{or} \quad \log\xi_2 = -\frac{4}{9}\{\log\rho + \frac{3}{2}\frac{\rho^{2/3}}{\rho_o^{2/3}}\} + C$$

$$\text{or} \quad \log\xi_2 + \frac{4}{9}\log\rho = -\frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}} + C$$

$$\text{or} \quad \log\xi_2 + \log\rho^{4/9} = -\frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}} + C$$

$$\log\xi_2\rho^{4/9} = -\frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}} + C$$

$$\therefore \xi_2\rho^{4/9} = e^{(-\frac{2}{3}\frac{\rho^{2/3}}{\rho^{2/3}} + C)}$$

$$\xi_2 = a\frac{1}{\rho^{4/9}}e^{-\frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}}} \tag{4.34}$$

where a is any constant.

$$\mu = 0,\ \nu = 1,\ \xi_{o;1} = \frac{\partial \xi_o}{\partial x^1} - (\Gamma^o_{1o}\xi_o + \Gamma^1_{1o}\xi_1 + \Gamma^2_{1o}\xi_2 + \Gamma^3_{1o}\xi_3)$$

$$= \frac{\partial \xi_o}{\partial \rho} - (\Gamma^o_{1o}\xi_o + \Gamma^3_{1o}\xi_3)$$

$$\xi_{1;o} = \frac{\partial \xi_1}{\partial x^o} - (\Gamma^o_{o1}\xi_o + \Gamma^1_{o1}\xi_1 + \Gamma^2_{o1}\xi_2 + \Gamma^3_{o1}\xi_3)$$

$$= - (\Gamma^o_{o1}\xi_o + \Gamma^3_{o1}\xi_3)$$

$$\therefore \xi_{o;1} + \xi_{1;o} = \frac{\partial \xi_o}{\partial \rho} - (2\Gamma^o_{o1}\xi_o + 2\Gamma^3_{o1}\xi_3) = 0$$

$$\text{or, } \frac{\partial \xi_o}{\partial \rho} - 2.\left(\frac{2}{9}\rho^{-1} + \frac{3}{4} \cdot \frac{a_o^2}{\sigma^2}\rho^{-1/3}\right)\xi_o - 2 \cdot \left(-\frac{1}{2}\frac{a_o}{\rho}\right)\xi_3 = 0$$

$$\frac{\partial \xi_o}{\partial \rho} - \frac{4}{9}\rho^{-1} - \frac{3}{2} \cdot \frac{a_o^2}{\frac{27}{8} a_o^2 \rho_o^{2/3}}\rho^{-1/3}\xi_o + \frac{a_o}{\rho}\xi_3 = 0$$

$$\text{or, } \frac{d\xi_o}{\partial \rho} - \frac{4}{9}\frac{1}{\rho}\left(1 + \frac{\rho^{2/3}}{\rho_o^{2/3}}\right)\xi_o + \frac{a_o}{\rho}\xi_3 = 0 \qquad (4.35)$$

$$\mu = 1,\ \nu = 3 \qquad \xi_{1;3} = \frac{\partial \xi_1}{\partial x^3} - (\Gamma^o_{31}\xi_o + \Gamma^1_{31}\xi_1 + \Gamma^2_{31}\xi_2 + \Gamma^3_{31}\xi_3)$$

$$= - (\Gamma^o_{31}\xi_o + \Gamma^3_{31}\xi_3)$$

$$\xi_{3;1} = \frac{\partial \xi_3}{\partial x^1} - (\Gamma^o_{13}\xi_o + \Gamma^1_{13}\xi_1 + \Gamma^2_{13}\xi_2 + \Gamma^3_{13}\xi_3)$$

$$= \frac{\partial \xi_3}{\partial \rho} - (\Gamma^o_{13}\xi_o + \Gamma^3_{13}\xi_3)$$

$$\therefore \xi_{1;3} + \xi_{3;1} = \frac{\partial \xi_3}{\partial \rho} - (2\Gamma^o_{13}\xi_o + 2\Gamma^3_{13}\xi_3) = 0 \qquad (4.36)$$

$$\frac{d\xi_3}{d\rho} - \frac{1}{\rho^2}\left(-\frac{2}{3}\frac{a_o}{\sigma^2}\rho^{5/3} - \frac{8}{9}k_o\rho + \frac{9}{4}\frac{a^3}{\sigma^4}\rho^{7/3}\right)\xi_o$$

$$- \frac{1}{\rho^2}\left(\frac{14}{9}\rho - \frac{3}{2}\frac{a_o^2}{\sigma^2}\rho^{5/3}\right)\xi_3 = 0$$

$$\text{or } \frac{d\xi_3}{d\rho} - \frac{1}{\rho^2}\left(-\frac{2}{3}\cdot\frac{8}{27}\frac{a_o}{a_o^2}\frac{\rho^{5/3}}{\rho_o^{2/3}} - \frac{8}{9}k_o\rho\right.$$

$$\left.+ \frac{9}{4}\cdot\frac{8\times 8}{27\times 27}\frac{a_o^3}{a_o^4}\frac{\rho^{7/3}}{\rho_o^{4/3}}\right)\xi_o - \frac{1}{\rho^2}\left(\frac{14}{9}\rho - \frac{3}{2}\cdot\frac{8}{27}\frac{a_o^2}{a_o^2}\frac{\rho^{5/3}}{\rho_o^{2/3}}\right)\xi_3 = 0$$

$$\text{or } \frac{d\xi_3}{d\rho} - \frac{1}{\rho^2}\left(-\frac{16}{81}\frac{\rho^{5/3}}{a_o\rho_o^{2/3}} - \frac{8}{9}k_o\rho + \frac{16}{81}\frac{\rho^{7/3}}{a_o\rho_o^{4/3}}\right)\xi_o$$

$$- \frac{1}{\rho^2}\left(\frac{14}{9}\rho - \frac{4}{9}\frac{\rho^{5/3}}{\rho_o^{2/3}}\right)\xi_3 = 0$$

$$\text{or, } \frac{d\xi_3}{d\rho} - \frac{1}{\rho}\left(-\frac{16}{81a_o}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{8}{9}k_o + \frac{16}{81a_o}\frac{\rho^{4/3}}{\rho_o^{4/3}}\right)\xi_o$$

$$- \frac{1}{\rho}\left(\frac{14}{9} - \frac{4}{9}\frac{\rho^{2/3}}{\rho_o^{2/3}}\right)\xi_3 = 0$$

$$\text{or, } \frac{d\xi_3}{d\rho} + \frac{1}{a_o\rho}\left(\frac{16}{81}\frac{\rho^{2/3}}{\rho_o^{2/3}} + a_ok_o\times\frac{8}{9} - \frac{16}{81}\frac{\rho^{4/3}}{\rho_o^{4/3}}\right)\xi_o$$

$$- \frac{1}{\rho}\left(\frac{14}{9} - \frac{4}{9}\frac{\rho^{2/3}}{\rho_o^{2/3}}\right)\xi_3 = 0 \qquad (4.37)$$

$$\frac{d\xi_3}{d\rho} + \frac{1}{a_o\rho}\left(\frac{16}{81}\frac{\rho^{2/3}}{\rho_o^{2/3}} + \left(-\frac{2}{9}\right)\times\frac{8}{9} - \frac{16}{81}\frac{\rho^{4/3}}{\rho_o^{4/3}}\right)\xi_o - \frac{1}{\rho}\left(\frac{14}{9} - \frac{4}{9}\frac{\rho^{2/3}}{\rho_o^{2/3}}\right)\xi_3 = 0$$

$$\frac{d\xi_3}{d\rho} + \frac{16}{81\,a_o\rho}\left(\frac{\rho^{2/3}}{\rho_o^{2/3}} - 1 - \frac{\rho^{4/3}}{\rho^{4/3}}\right)\xi_o - \frac{1}{\rho}\left(\frac{14}{9} - \frac{4}{9}\,\frac{\rho^{2/3}}{\rho_o^{2/3}}\right)\xi_3 = 0 \quad (4.38)$$

From Equation (4.35) $\dfrac{a_o}{\rho}\,\xi_3 + \dfrac{d\xi_o}{d\rho} - \dfrac{4}{9}\left(\dfrac{1}{\rho} + \dfrac{\rho^{-1/3}}{\rho_o^{2/3}}\right)\xi_o = 0$

or, $a_o\xi_3 = -\rho\,\dfrac{d\xi_o}{d\rho} + \dfrac{4}{9}\left(1 + \dfrac{\rho^{2/3}}{\rho_o^{2/3}}\right)\xi_o$

differentiating with respect to ρ

$$a_o\,\frac{d\xi_3}{d\rho} = -\frac{d\xi_o}{d\rho} - \rho\,\frac{d^2\xi_o}{d\rho^2} + \frac{4}{9}\left(0 + \frac{2}{3}\,\frac{\rho^{-1/3}}{\rho_o^{2/3}}\right)\xi_o + \frac{4}{9}\left(1 + \frac{\rho^{2/3}}{\rho_o^{2/3}}\right)\frac{d\xi_o}{d\rho}$$

or, $\dfrac{d\xi_3}{d\rho} = \dfrac{1}{a_o}\left\{-\dfrac{d\xi_o}{d\rho} - \rho\,\dfrac{d^2\xi_o}{d\rho^2} + \dfrac{8}{27}\,\dfrac{\rho^{-1/3}}{\rho_o^{2/3}}\,\xi_o + \dfrac{4}{9}\left(1 + \dfrac{\rho^{2/3}}{\rho_o^{2/3}}\right)\dfrac{d\xi_o}{d\rho}\right\}$ (4.39)

putting this equation in (4.39) we get:

$$\frac{1}{a_o}\left\{-\frac{d\xi_o}{d\rho} - \rho\,\frac{d^2\xi_o}{d\rho^2} + \frac{8}{27}\,\frac{\rho^{-1/3}}{\rho_o^{2/3}}\,\xi_o + \frac{4}{9}\left(1 + \frac{\rho^{2/3}}{\rho_o^{2/3}}\right)\frac{d\xi_o}{d\rho}\right\}$$

$$+ \frac{1}{a_o\rho}\,\frac{16}{81}\left(\frac{\rho^{2/3}}{\rho_o^{2/3}} - 1 - \frac{\rho^{4/3}}{\rho_o^{4/3}}\right)\xi_o$$

$$- \frac{1}{\rho}\left(\frac{14}{9} - \frac{4}{9}\,\frac{\rho^{2/3}}{\rho_o^{2/3}}\right) x \left\{-\frac{\rho}{a_o}\frac{d\xi_o}{d\rho} + \frac{4}{9}\cdot\frac{1}{a_o}\left(1 + \frac{\rho^{2/3}}{\rho_o^{2/3}}\right)\xi_o\right\} = 0$$

or, $\dfrac{1}{a_o}\Big[-\rho\,\dfrac{d^2\xi_o}{d\rho^2} + \dfrac{d\xi_o}{d\rho}\left\{-1 + \dfrac{4}{9}\left(1 + \dfrac{\rho^{2/3}}{\rho_o^{2/3}}\right) + \left(\dfrac{14}{9} - \dfrac{4}{9}\,\dfrac{\rho^{2/3}}{\rho_o^{2/3}}\right)\right\}$

$$+ \{\frac{8}{27} \frac{\rho^{-1/3}}{\rho_o^{2/3}} + \frac{1}{\rho} \frac{16}{81}(\frac{\rho^{2/3}}{\rho_o^{2/3}} - 1 - \frac{\rho^{4/3}}{\rho_o^{4/3}}) - \frac{1}{\rho}(\frac{14}{9} - \frac{4}{9} \frac{\rho^{2/3}}{\rho_o^{2/3}})$$

$$\times \frac{4}{9}(1 + \frac{\rho^{2/3}}{\rho_o^{2/3}})\}\xi_o] = 0$$

$$- \rho \frac{d^2\xi_o}{d\rho} + \frac{d\xi_o}{d\rho} + \{\frac{8}{27} \frac{\rho^{-1/3}}{\rho_o^{2/3}} + \frac{1}{\rho}(- \frac{8}{27} \frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{8}{9})\}\xi_o = 0$$

$$\text{or, } \frac{d^2\xi_o}{d\rho^2} - \frac{1}{\rho} \frac{d\xi_o}{d\xi} + \frac{1}{\rho^2} \frac{8}{9} \xi_o = 0$$

$$\text{or, } \rho^2 \frac{d^2\xi_o}{d\rho^2} - \rho \frac{d\xi_o}{d\rho} + \frac{8}{9} \xi_o = 0 \qquad (4.40)$$

Solving the Equation (4.40) we get

$$\xi_o^{(1)} = \rho^{4/3}, \quad \xi_o^{(2)} = \rho^{2/3}$$

Equation (4.38) can be written as:

$$\frac{d\xi_3}{d\rho} - \frac{1}{\rho}(\frac{14}{9} - \frac{4}{9} \frac{\rho^{2/3}}{\rho_o^{2/3}})\xi_3 = - \frac{16}{81 a_o\rho}(\frac{\rho^{2/3}}{\rho_o^{2/3}} - 1 - \frac{\rho^{4/3}}{\rho_o^{4/3}})\xi_o \qquad (4.41)$$

putting the values of ξ_o as $\rho^{4/3}$ in (4.41)

$$\therefore \frac{d\xi_3}{d\rho} - \frac{1}{\rho}(\frac{14}{9} - \frac{4}{9} \frac{\rho^{2/3}}{\rho_o^{2/3}})\xi_3 = - \frac{16}{81 a_o}(\frac{\rho^{2/3}}{\rho_o^{2/3}} - 1 - \frac{\rho^{4/3}}{\rho_o^{4/3}})\rho^{4/3} \qquad (4.42)$$

Integrating the above equation we can write:

$$\xi_3\, e^{(\log \rho^{-14/9} + \frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}})}$$

$$= \int \{-\frac{16}{81\, a_o}(\frac{\rho^{2/3}}{\rho_o^{2/3}} - 1 - \frac{\rho^{4/3}}{\rho_o^{4/3}})\rho^{1/3}\} e^{(\log \rho^{\frac{-14}{9}} + \frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}})} d\rho$$

$$\therefore \xi_3\{\rho^{\frac{-14}{9}} e^{\frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}}}\} = -\int \{\frac{16}{81\, a_o}(\frac{\rho^{2/3}}{\rho_o^{2/3}} - 1 - \frac{\rho^{4/3}}{\rho_o^{4/3}})\rho^{1/3}\}$$

$$\times\ \rho^{-14/9} e^{\frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}}} d\rho$$

$$= -\frac{16}{81\, a_o}\int \frac{\rho^{-5/9}}{\rho_o^{2/3}} - \rho^{-11/9} - \frac{\rho^{1/9}}{\rho_o^{4/3}}) e^{\frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}}} d\rho \qquad (4.43)$$

$$\therefore\ \xi_3 = -\frac{16}{81\, a_o}\ \frac{\rho^{14/9}}{e^{\frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}}}}\int (\frac{\rho^{-5/9}}{\rho_o^{2/3}} - \rho^{-11/9} - \frac{\rho^{1/9}}{\rho_o^{4/3}}) e^{\frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}}} d\rho$$

$$\therefore\ \xi_3^{(1)} = -\frac{16}{81\, a_o}\ \frac{\rho^{14/9}}{\exp(\frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}})}\int (\frac{\rho^{-5/9}}{\rho_o^{2/3}} - \rho^{-11/9}$$

$$- \frac{\rho^{1/9}}{\rho_o^{4/3}}) \exp(\frac{2}{3}\frac{\rho^{2/3}}{\rho_o^{2/3}}) d\rho \qquad (4.44a)$$

In a similar way putting the values of ξ_o as $\rho^{2/3}$ in Equation(4.41) we get:

$$\xi_3^{(2)} = -\frac{16}{81\,a_o}\,\frac{\rho^{14/9}}{\exp\left(\frac{2}{3}\,\frac{\rho^{2/3}}{\rho_o^{2/3}}\right)}\int\left(\frac{\rho^{-11/9}}{\rho_o^{2/3}} - \rho^{-17/9}\right.$$

$$\left. - \frac{\rho^{-5/9}}{\rho_o^{4/3}}\right)\exp\left(\frac{2}{3}\,\frac{\rho^{2/3}}{\rho_o^{2/3}}\right)d\rho \qquad (4.44b)$$

Here we have one pair of extra Killing vectors

$$Y_1^i = (\xi_o^{(1)},\ o,\ \xi_2\ ,\ \xi_{(3)}^{(1)}) \qquad (4.45)$$

$$Y_2^i = (\xi_o^{(2)},\ o,\ \xi_2\ ,\ \xi_{(3)}^{(2)}) \qquad (4.46)$$

The other Killing vectors are as in (4.47) so that the general Killing vector can be written as (4.48)

$$Z^i = \delta_1^i\ ,\ \theta^i = \delta_3^i\ ,\ t^i = \delta_o^i \qquad (4.47)$$

$$\chi^i = p'\delta_1^i + q'\delta_3^i + S'\delta_o^i + W_1Y_1^i + W_2Y_2^i \ \text{ etc.} \qquad (4.48)$$

where p', q', S', W_1 , W_2 are constants.

Using the necessary condition for this to be HSO $\chi_{[i,j}\chi_{k]} = 0$ in case of exterior solution we found that (just like Equation (4.30))roots of the equations are not independent of ρ; so there exist no HSO Killing vector.

Thus the global solution considered here is stationary and not static.

CHAPTER 5

CURVATURE INVARIANTS FOR A SOLUTION OF THE EINSTEIN-MAXWELL EQUATIONS

5.1 INTRODUCTION

In this chapter we shall work out all the independent curvature invariants of the global solution found by Islam (1983). As mentioned earlier, global solution of either the pure Einstein or Einstein-Maxwell equations are rare and it is worthwhile making a detailed study of global solutions as these give insight into the physical contents of general relativity.

As noted by Islam himself (1983), although the global solution is regular and well behaved for all finite values of the radial coordinate ρ, one of the curvature invariants goes to infinity as ρ tends to infinite and further, the spatial distance from any point with a finite value of ρ to $\rho = \infty$ along a line with constant angular coordinates is in fact finite. The physical significance of this is not clear. Partly with a view to clarifying the physical properties of this solution we work out all the curvature invariants here, of which there are nine; we will also consider Petrov classification of the solution using these invariants.

The curvature invariants for a Riemannian space have been worked out by Witten (1959) in terms of spinors. He finds that when $R_{\mu\nu} = 0$ in the pure gravitational field, there are four invariants, as follows:

$$I_1 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

$$I_2 = \sqrt{-g}\, R_{\mu\nu}{}^{\alpha\beta}\varepsilon_{\alpha\beta\rho\sigma}R^{\mu\nu\rho\sigma}$$

$$J_1 = R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\gamma\delta}R^{\gamma\delta}{}_{\mu\nu}$$

$$J_2 = \sqrt{-g}\, R^{\mu\nu\alpha\beta}\varepsilon_{\alpha\beta\rho\sigma}R^{\rho\sigma}{}_{\gamma\delta}R^{\gamma\delta}{}_{\mu\nu}$$

The invariants are mentioned explicitly by, e.g. Penrose (1960, p 333 (as reprinted by Kilmister 1973)), where his complex invariants I, J (with $\lambda = 0$) are related to the above as follows:

$$I = \frac{1}{2} I_1 + \frac{i}{4} I_2$$

$$J = \frac{1}{2} J_1 + \frac{i}{4} J_2$$

In the case of the electromagnetic field there are nine independent invariants which are all given in spinor form. We have not found a tensor form of these invariants in the literature so we devote the first part of this chapter to a brief discussion of spinors, with a view to using Witten's expressions in spinors form for the curvature invariants to derive their tensor form. This discussion of spinor is based mostly on Witten (1959) but the expressions for the curvature invariant of the electromagnetic field in tensor form, as far as we know, are new. There have been interesting discussion about spinors by Davis (1970), Penrose (1960) and Penrose and Rindler (1984).

5.2 CONNECTION BETWEEN TENSOR AND SPINOR

It is known that to each real tensor one can assign a spinor. In this section we assign a spinor to the Riemann curvature tensor and find out what the symmetries of the tensor require of the assigned spinor, which turns out to be a unique spinor. We use a representation in which the g matrices are Hermitian, $\bar{g}^{p}{}_{\dot{\alpha}\beta} = g^{p}{}_{\dot{\beta}\alpha}$ (bar denotes complex conjugate) and define the spin matrices by:

$$g^{p\dot{\alpha}\beta} g^{q}{}_{\dot{\alpha}\gamma} + g^{q\dot{\alpha}\beta} g^{p}{}_{\dot{\alpha}\gamma} = 2g^{pq}\delta^{\beta}{}_{\gamma} \tag{5.1}$$

The Latin index in $g^{p}{}_{\dot{\alpha}\beta}$, $g_{p\dot{\alpha}\beta}$ can be lowered or raised by g_{pq} or g^{pq} and Greek indices by $\varepsilon^{\dot{\alpha}\dot{\beta}}$, $\varepsilon_{\dot{\alpha}\dot{\beta}}$, $\varepsilon^{\alpha\beta}$, $\varepsilon_{\alpha\beta}$ which are numerical spinors invariant under spin transformation. The antisymmetric fundamental spinor is:

$$\left\|\varepsilon_{\alpha\beta}\right\| = \left\|\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}\right\| = \left\|\varepsilon^{\alpha\beta}\right\| \tag{5.2}$$

where

$$\varepsilon_{\alpha\beta} = \begin{matrix} 0, & \text{if} & \alpha = \beta \\ 1, & & \alpha = 1, \ \beta = 2 \\ -1 & & \alpha = 2, \ \beta = 1 \end{matrix}$$

Thus one can write

$$\phi^{\alpha} = \varepsilon^{\alpha\beta}\phi_{\beta},\ \psi_{\alpha} = \psi^{\beta}\varepsilon_{\beta\alpha}$$

So that

$$\phi^{\alpha}\psi_{\alpha} = \varepsilon^{\alpha\beta}\phi_{\beta}\ \psi^{\gamma}\varepsilon_{\gamma\alpha} = -\ \phi_{\alpha}\psi^{\alpha}\ \ \ldots \qquad (5.3)$$

(putting $\beta = \gamma = \alpha$)

Similarly it can be shown that

$$\phi^{\dot{\alpha}\beta}\psi_{\dot{\alpha}\beta} = \phi_{\dot{\alpha}\beta}\psi^{\dot{\alpha}\beta} \qquad (5.4a)$$

$$\psi^{\dot{\alpha}\beta\gamma}\phi_{\dot{\alpha}\beta\gamma} = -\ \psi_{\dot{\alpha}\beta\gamma}\phi^{\dot{\alpha}\beta\gamma} \qquad (5.4b)$$

$$\phi_{\dot{\alpha}\beta\gamma\delta}\psi^{\dot{\alpha}\beta\gamma\delta} = \psi_{\dot{\alpha}\beta\gamma\delta}\phi^{\dot{\alpha}\beta\gamma\delta} \qquad (5.4c)$$

The simplest way of seeing the connection between tensor in space-time and 2-spinors is through the observation that a second order Hermitian spinor $\phi_{\alpha\dot{\beta}} = \phi_{\beta\dot{\alpha}}$ and a vector field are both determined by specifying four real functions. Thus one can uniquely connect these two quantities by a relation of the form:

$$\phi_{\dot{\alpha}\beta} = g_{p\dot{\alpha}\beta}\ x^{p} \qquad (5.5)$$

Multiplying (5.5) by $g^{q\alpha\beta}$ and putting $p = q$ it can be shown that:

$$x^{q} = \frac{1}{2}\ g^{q}_{\dot{\alpha}\beta}\phi^{\dot{\alpha}\beta} \qquad (5.6)$$

The determinant $|\phi_{\dot{\alpha}\beta}| = \frac{1}{2} \phi_{\dot{\alpha}\beta}\phi^{\dot{\alpha}\beta}$ is invariant under unimodular spin transformations. In particular, if one defines $X_{\dot{\alpha}\beta}$ by $X_{\dot{1}1} = x^4 + ix^1$, $X_{\dot{1}2} = x^2 + ix^3$

$X_{\dot{2}1} = x^2 - ix^3$ and $X_{\dot{2}2} = x^4 - ix^1$

then $|X_{\dot{\alpha}\beta}| = -(x^1)^2 - (x^2)^2 - (x^3)^2 + (x^4)^2$ (5.7a)

where $|X'_{\dot{\alpha}\beta}| = -(\bar{x}^1)^2 - (\bar{x}^2)^2 - (\bar{x}^3)^2 + (\bar{x}^4)^2$ ... (5.7b)

Thus, the invariant interval takes the form

$$g_{ij}x^i x^j = \frac{1}{2} X^{\dot{\alpha}\beta}X_{\dot{\alpha}\beta} = \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\gamma\delta} g^i_{\dot{\alpha}\gamma} g^j_{\dot{\beta}\delta} x_i x_j \; ... \qquad (5.8)$$

which leads to Equation (5.1) if (5.8) is to be identically satisfied in x^i. It is already implicit in (5.7a) and (5.7b) that there exists a simple connection between unimodular spin transformations and Lorentz transformation in space-time.

The more complicated Riemann tensor can be transformed in the spinor form following Witten (1959) as:

$$R_{pqrs} = g_{p\dot{\alpha}k} g_{q\dot{\beta}\lambda} g_{r\dot{\mu}\rho} g_{s\dot{\nu}\sigma} \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma} \qquad (5.9)$$

The reality of R_{pqrs} requires $\bar{\phi}^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma} = \phi^{\dot{k}\dot{\lambda}\dot{\rho}\dot{\sigma}\alpha\beta\mu\nu}$

The antisymmetry relation $R_{pqrs} = - R_{pqrs}$ imposes the relation

$$\phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma} = - \phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}\lambda k\rho\sigma} \qquad (5.10)$$

So that

$$R_{pqrs} = \frac{1}{2}\, g_{p\dot{\alpha}k}\, g_{q\dot{\beta}\lambda}\, g_{r\dot{\mu}\rho}\, g_{s\dot{\nu}\sigma} (\phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma} - \phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}\lambda k\rho\sigma}) \qquad (5.11)$$

This can be written as:

$$R_{pqrs} = \frac{1}{2}\, g_{p\dot{\alpha}k}\, g_{q\dot{\beta}\lambda}\, g_{r\dot{\mu}\rho}\, g_{s\dot{\nu}\sigma} \{\varepsilon^{k\lambda}\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\rho\sigma} + \varepsilon^{\dot{\alpha}\dot{\beta}}\psi^{k\lambda\rho\sigma\dot{\mu}\dot{\nu}}\} \qquad (5.12)$$

where it is defined:

$$2\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\rho\sigma} = \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}{}_{\lambda}{}^{\lambda\rho\sigma} + \phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}}{}_{\lambda}{}^{\lambda\rho\sigma} \qquad (5.13)$$

ψ is symmetric in the indices $\dot{\alpha}\dot{\beta}$

So complex conjugate of Equation (5.13) is

$$2\psi^{\alpha\beta\mu\nu\dot{\rho}\dot{\sigma}} = \phi^{\alpha\beta\mu\nu}{}_{\dot{\lambda}}{}^{\dot{\lambda}\dot{\rho}\dot{\sigma}} + \phi^{\beta\alpha\mu\nu}{}_{\dot{\lambda}}{}^{\dot{\lambda}\dot{\rho}\dot{\sigma}} \qquad (5.14a)$$

and

$$2\psi^{k\lambda\sigma\rho\dot{\nu}\dot{\mu}} = \phi^{k\lambda\sigma\rho}{}_{\dot{\gamma}}{}^{\dot{\gamma}\dot{\nu}\dot{\mu}} + \phi^{\lambda k\sigma\rho}{}_{\dot{\gamma}}{}^{\dot{\gamma}\dot{\nu}\dot{\mu}} \qquad (5.14b)$$

From Equation (5.14a):

$$\phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}{}_{\lambda}{}^{\lambda\rho\sigma} = \delta^{\lambda}_{k}\, \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}{}_{\lambda}{}^{k\rho\sigma} = \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma}\varepsilon_{k\lambda} \qquad (5.15)$$

and $2\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\rho\sigma} = \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}{}_{\lambda}{}^{\lambda\rho\sigma} + \phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}}{}_{\lambda}{}^{\lambda\rho\sigma}$

$$= \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma}\varepsilon_{k\lambda} + \phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}k\lambda\rho\sigma}\varepsilon_{k\lambda}$$

similarly

$$2\psi^{k\lambda\rho\sigma\dot{\mu}\dot{\nu}} = \phi^{k\lambda\rho\sigma\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}\varepsilon_{\dot{\alpha}\dot{\beta}} + \phi^{\lambda k\rho\sigma\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}\varepsilon_{\dot{\alpha}\dot{\beta}}$$

we have $(R_{pqrs})^{*} = (R_{pqrs})$

From Equation (5.9)

$$(R_{pqrs})^* = (g_{p\dot{\alpha}k} g_{q\dot{\beta}\lambda} g_{r\dot{\mu}\rho} g_{s\dot{\nu}\sigma} \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma})^*$$

$$= g_{pk\dot{\alpha}} g_{q\lambda\dot{\beta}} g_{r\rho\dot{\mu}} g_{s\sigma\dot{\nu}} (\phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma})^*$$

$$= g_{pk\dot{\alpha}} g_{q\lambda\dot{\beta}} g_{r\rho\dot{\mu}} g_{s\sigma\dot{\nu}} (\phi^{\dot{k}\dot{\lambda}\dot{\rho}\dot{\sigma}\alpha\beta\mu\nu})$$

$$= g_{p\dot{\alpha}k} g_{q\dot{\beta}\lambda} g_{r\dot{\mu}\rho} g_{s\dot{\nu}\sigma} \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma}$$

$$= R_{pqrs}$$

$\therefore$ R_{pqrs} is real.

Similarly R_{qprs} can be written as:

$$R_{qprs} = g_{p\dot{\alpha}k} g_{p\dot{\beta}\lambda} g_{r\dot{\mu}\rho} g_{s\dot{\nu}\sigma} \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma}$$

$$= g_{q\dot{\beta}\lambda} g_{p\dot{\alpha}k} g_{r\dot{\mu}\rho} g_{s\dot{\nu}\sigma} \phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}\lambda k\rho\sigma} \tag{5.16}$$

$$= g_{p\dot{\alpha}k} g_{q\dot{\beta}\lambda} g_{r\dot{\mu}\rho} g_{s\dot{\nu}\sigma} (- \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma}) \tag{5.17}$$

$$= - R_{pqrs} \tag{5.18}$$

$\therefore$ From (5.16) and (5.17)

$$\phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma} = - \phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}\lambda k\rho\sigma} \tag{5.19}$$

From Equations (5.16), (5.17) and (5.18):

$$R_{pqrs} = \frac{1}{2} g_{p\dot{\alpha}k} g_{q\dot{\beta}\lambda} g_{r\dot{\mu}\rho} g_{s\dot{\nu}\sigma} (\phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma} - \phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}\lambda k\rho\sigma}) \qquad (5.20)$$

Using (5.14a), (5.14b) and (5.15) in (5.12) it can be written as:

$$R_{pqrs} = g_{p\dot{\alpha}k} g_{q\dot{\beta}\lambda} g_{r\dot{\mu}\rho} g_{s\dot{\nu}\sigma} \{\frac{1}{4} \varepsilon^{k\lambda} (\varepsilon_{\xi\nu} \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\xi\eta\rho\sigma} + \varepsilon_{\xi\eta} \phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}\xi\eta\rho\sigma}) + \frac{1}{4} \varepsilon^{\dot{\alpha}\dot{\beta}} (\varepsilon_{\dot{\phi}\dot{\theta}} \phi^{\dot{\phi}\dot{\theta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma} + \varepsilon_{\dot{\phi}\dot{\theta}} \phi^{\dot{\phi}\dot{\theta}\dot{\mu}\dot{\nu}\lambda k\rho\sigma})\} \qquad (5.21)$$

From Equations (5.20) and (5.21) it follows that:

$$\frac{1}{2} \{\varepsilon^{k\lambda} (\varepsilon_{\xi\eta} \phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\xi\eta\rho\sigma} + \varepsilon_{\xi\eta} \phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}\xi\eta\rho\sigma}) + \varepsilon^{\dot{\alpha}\dot{\beta}} (\varepsilon_{\dot{\phi}\dot{\theta}} \phi^{\dot{\phi}\dot{\theta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma} + \varepsilon_{\dot{\phi}\dot{\theta}} \phi^{\dot{\phi}\dot{\theta}\dot{\mu}\dot{\nu}\lambda k\rho\sigma})\} = (\phi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}k\lambda\rho\sigma} - \phi^{\dot{\beta}\dot{\alpha}\dot{\mu}\dot{\nu}\lambda k\rho\sigma}) \qquad (5.22)$$

Putting 1. $\dot{\alpha} = \dot{\beta}$, $k = \lambda$ 2. $\dot{\alpha} = \dot{\beta}$, $k \neq \lambda$ 3. $\dot{\alpha} \neq \dot{\beta}$, $k = \lambda$, 4. $\dot{\alpha} \neq \dot{\beta}$, $k \neq \lambda$ in all the four cases the above equation is satisfied so that Equation (5.12) holds.

The antisymmetry relation $R_{pqrs} = - R_{pqsr}$ imposes the relation:

$$\varepsilon^{k\lambda} \psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\rho\sigma} + \varepsilon^{\dot{\alpha}\dot{\beta}} \psi^{k\lambda\rho\sigma\dot{\mu}\dot{\nu}} = - \left[\varepsilon^{k\lambda} \psi^{\dot{\alpha}\dot{\beta}\dot{\nu}\dot{\mu}\sigma\rho} + \varepsilon^{\dot{\alpha}\dot{\beta}} \psi^{k\lambda\sigma\rho\dot{\nu}\dot{\mu}}\right] \qquad (5.23)$$

Using (5.23) in Equation (5.12) we can write:

$$R_{pqrs} = \frac{1}{4} g_{p\dot{\alpha}k} g_{q\dot{\beta}\lambda} g_{r\dot{\mu}\rho} g_{s\dot{\nu}\sigma} \Big[\varepsilon^{k\lambda} (\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}\rho\sigma} - \psi^{\dot{\alpha}\dot{\beta}\dot{\nu}\dot{\mu}\sigma\rho})$$

$$+ \varepsilon^{\dot{\alpha}\dot{\beta}} (\psi^{k\lambda\rho\sigma\dot{\mu}\dot{\nu}} - \psi^{k\lambda\sigma\rho\dot{\nu}\dot{\mu}}) \Big] \qquad (5.24)$$

Defining $2\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}} \equiv \psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}{}_{\rho}{}^{\rho} + \psi^{\dot{\alpha}\dot{\beta}\dot{\nu}\dot{\mu}}{}_{\rho}{}^{\rho}$

and $\quad 2\phi^{\dot{\alpha}\dot{\beta}\rho\sigma} \equiv \psi^{\dot{\alpha}\dot{\beta}}{}_{\lambda}{}^{\lambda\rho\sigma} + \psi^{\dot{\alpha}\dot{\beta}}{}_{\lambda}{}^{\lambda\sigma\rho}$

where $\psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}}$ is symmetric in $\dot{\alpha}\dot{\beta}$ and in $\dot{\mu}\dot{\nu}$; $\phi^{\dot{\alpha}\dot{\beta}\rho\sigma}$ is symmetric in $\dot{\alpha}\dot{\beta}$ and in $\rho\sigma$. The Equation (5.24) can be written as

$$R_{pqrs} = \frac{1}{4} g_{p\dot{\alpha}k} g_{q\dot{\beta}\lambda} g_{r\dot{\mu}\rho} g_{s\dot{\nu}\sigma} \Big[\varepsilon^{k\lambda} \varepsilon^{\rho\sigma} \psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}} + \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon^{\dot{\mu}\dot{\nu}} \psi^{K\lambda\rho\sigma}$$

$$+ \varepsilon^{k\lambda} \varepsilon^{\dot{\mu}\dot{\nu}} \phi^{\dot{\alpha}\dot{\beta}\rho\sigma} + \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon^{\rho\sigma} \phi^{k\lambda\dot{\mu}\dot{\nu}} \Big] \qquad (5.25)$$

Equality of (5.24) and (5.25) can be shown by putting $\dot{\mu} = \dot{\nu}$, $\rho = \sigma$; $\dot{\mu} = \dot{\nu}$, $\rho \neq \sigma$; $\dot{\mu} \neq \dot{\nu}$, $\rho = \sigma$; $\dot{\mu} \neq \dot{\nu}$, $\rho \neq \sigma$; in the (5.24) and (5.25) equations.

The symmetry requirement $R_{pqrs} = R_{rspq}$

i.e. $R_{pqrs} = - R_{prqs} = R_{prsq} = - R_{rpsq} = R_{rspq}$

So $R_{rspq} = \frac{1}{4} g_{r\dot{\mu}\rho} g_{s\dot{\nu}\sigma} g_{\rho\dot{\alpha}k} g_{q\dot{\beta}\lambda} \Big[\varepsilon^{\rho\sigma} \varepsilon^{k\lambda} \psi^{\dot{\mu}\dot{\nu}\dot{\alpha}\dot{\beta}} + \varepsilon^{\dot{\mu}\dot{\nu}} \varepsilon^{\dot{\alpha}\dot{\beta}} \psi^{\rho\sigma k\lambda}$

$$+ \varepsilon^{\rho\sigma} \varepsilon^{\dot{\alpha}\dot{\beta}} \phi^{\dot{\mu}\dot{\nu}k\lambda} + \varepsilon^{\dot{\mu}\dot{\nu}} \varepsilon^{k\lambda} \phi^{\rho\sigma\dot{\alpha}\dot{\beta}} \Big] \qquad (5.26)$$

$$\varepsilon^{k\lambda}\varepsilon^{\rho\sigma}\psi^{\dot\alpha\dot\beta\dot\mu\dot\nu} + \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\dot\mu\dot\nu}\psi^{k\lambda\rho\sigma} + \varepsilon^{k\lambda}\varepsilon^{\dot\mu\dot\nu}\phi^{\dot\alpha\dot\beta\rho\sigma} + \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\rho\sigma}\phi^{k\lambda\dot\mu\dot\nu}$$

$$\doteq \varepsilon^{\rho\sigma}\varepsilon^{k\lambda}\psi^{\dot\mu\dot\nu\dot\alpha\dot\beta} + \varepsilon^{\dot\mu\dot\nu}\varepsilon^{\dot\alpha\dot\beta}\psi^{\rho\sigma k\lambda} + \varepsilon^{\rho\sigma}\varepsilon^{\dot\alpha\dot\beta}\phi^{\dot\mu\dot\nu k\lambda}$$

$$+ \varepsilon^{\dot\mu\dot\nu}\varepsilon^{k\lambda}\phi^{\rho\sigma\dot\alpha\dot\beta} \qquad (5.27)$$

Equation (5.27) is satisfied if $\psi^{\alpha\beta\mu\nu} = \psi^{\mu\nu\alpha\beta}$ and $\bar\phi^{\dot\alpha\dot\beta\mu\nu} = \phi^{\alpha\beta\dot\mu\dot\nu} = \phi^{\dot\mu\dot\nu\alpha\beta}$ which conditions we now impose.

There is still one other restriction that the tensor R_{pqrs}, must satisfy before it can be considered a curvature tensor. This is the cyclic constraint:

$$R_{pqrs} + R_{prsq} + R_{psqr} = 0 \qquad (5.28)$$

Writing expression for R_{pqrs}, R_{prsq}, R_{psqr} and adding up we set:

$$R_{pqrs} + R_{prsq} + R_{psqr} = \frac{1}{4} g_{p\dot\alpha k} g_{q\dot\beta\lambda} g_{r\dot\mu\rho} g_{s\dot\nu\sigma} \Big[\varepsilon^{k\lambda}\varepsilon^{\rho\sigma}\psi^{\dot\alpha\dot\beta\dot\mu\dot\nu}$$

$$+ \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\dot\mu\dot\nu}\psi^{k\lambda\rho\sigma} + \varepsilon^{k\lambda}\varepsilon^{\dot\mu\dot\nu}\phi^{\dot\alpha\dot\beta\rho\sigma} + \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\rho\sigma}\phi^{k\lambda\dot\mu\dot\nu}$$

$$+ \varepsilon^{k\rho}\varepsilon^{\sigma\lambda}\psi^{\dot\alpha\dot\mu\dot\nu\dot\beta} + \varepsilon^{\dot\alpha\dot\mu}\varepsilon^{\dot\nu\dot\beta}\psi^{k\rho\sigma\lambda} + \varepsilon^{k\rho}\varepsilon^{\dot\nu\dot\beta}\phi^{\dot\alpha\dot\mu\sigma\lambda}$$

$$+ \varepsilon^{\dot\alpha\dot\mu}\varepsilon^{\sigma\lambda}\psi^{k\rho\dot\nu\dot\beta} + \varepsilon^{k\sigma}\varepsilon^{\lambda\rho}\psi^{\dot\alpha\dot\nu\dot\beta\dot\mu} + \varepsilon^{\dot\alpha\dot\nu}\varepsilon^{\dot\beta\dot\mu}\psi^{k\sigma\lambda\rho}$$

$$+ \varepsilon^{k\sigma}\varepsilon^{\dot\beta\dot\mu}\phi^{\dot\alpha\dot\nu\lambda\rho} + \varepsilon^{\dot\alpha\dot\nu}\varepsilon^{\lambda\rho}\phi^{k\sigma\dot\beta\dot\mu}\Big] \qquad (5.29)$$

Putting $k = 2$, $\lambda = 1$, $\rho = 2$, $\sigma = 1$, $\dot\alpha = 1$, $\dot\beta = 1$, $\dot\mu = 1$, $\dot\nu = 1$ in (5.29) we get Equation (5.28).

Write (5.25) in the R_{pqrs} forms i.e:

$$R^{p}{}_{qrs} = \frac{1}{4} g^{p}_{\dot{\alpha}k} g_{q\dot{\beta}\lambda} g_{r\hat{\mu}\rho} g_{s\hat{\nu}\sigma} \Big[\varepsilon^{k\lambda} \varepsilon^{\rho\sigma} \psi^{\dot{\alpha}\dot{\beta}\dot{\mu}\dot{\nu}} + \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon^{\dot{\mu}\dot{\nu}} \psi^{k\lambda\rho\sigma}$$

$$+ \varepsilon^{k\lambda} \varepsilon^{\dot{\mu}\dot{\nu}} \phi^{\dot{\alpha}\dot{\beta}\rho\sigma} + \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon^{\rho\sigma} \phi^{k\lambda\dot{\mu}\dot{\nu}} \Big]$$

Putting s = p and contracting, using also $\lambda=\rho=\sigma$ in the expression we get:

$$R_{qr} = \frac{1}{2} g_{q\dot{\beta}\rho} g_{r\dot{\mu}\sigma} \Big[\varepsilon^{\sigma\rho} \psi^{\dot{\alpha}\dot{\beta}\dot{\mu}}{}_{\dot{\alpha}} + \varepsilon^{\dot{\mu}\dot{\beta}} \psi^{k\rho\sigma}{}_{k} + 2\phi^{\dot{\mu}\dot{\beta}\sigma\rho} \Big] \tag{5.30}$$

Using the spinor relation:

$$g_{p}{}^{\dot{\beta}}{}_{\alpha} g_{q\dot{\beta}\mu} = g_{pq} \varepsilon_{\alpha\mu} + \frac{1}{2} g^{-1/2} \varepsilon_{pqrs} g^{r\dot{\beta}}{}_{\alpha} g^{s}{}_{\dot{\beta}\mu}$$

in the Equation (5.30) and putting r = q we get:

$$R_{pq} = g_{pq} \psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha} + g_{p\dot{\beta}\rho} g_{q\dot{\mu}\sigma} \phi^{\dot{\beta}\dot{\mu}\rho\sigma} \tag{5.31}$$

Contracting (5.31):

$$g^{pq} R_{pq} = g^{pq} (g_{pq} \psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha} + g_{p\dot{\beta}\rho} g_{q\dot{\mu}\sigma} \phi^{\dot{\beta}\dot{\mu}\rho\sigma})$$

$$\text{or} \quad R = 4\, \psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha} + \varepsilon_{\dot{\beta}\dot{\mu}} \varepsilon_{\rho\sigma} \phi^{\dot{\beta}\dot{\mu}\rho\sigma}$$

$\varepsilon_{\beta\mu}$, $\varepsilon_{\rho\sigma}$ is anti-symmetric and $\phi^{\dot{\beta}\dot{\mu}\rho\sigma}$ is symmetric so $\varepsilon_{\dot{\beta}\dot{\mu}}\, \varepsilon_{\rho\sigma} \phi^{\beta\mu\rho\sigma}$ vanishes.

$$\therefore \quad R = 4\, \psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha} \tag{5.32}$$

Let $s_{pq} = R_{pq} - g_{pq} \frac{R}{4}$ (5.33)

Using Equations(5.31) and (5.32) we get:

$$S_{pq} = g_{p\dot{\beta}\rho} g_{q\dot{\mu}\sigma} \phi^{\dot{\beta}\dot{\mu}\rho\sigma} \qquad (5.34)$$

Equations (5.33) and (5.34) can be written as:

$$R^{pq} = R^{pq} - \frac{1}{4} g^{pq} R \text{ and } S^{pq} = g^{p}{}_{\dot{\alpha}k} g^{q}{}_{\dot{r}\lambda} \phi^{\dot{\alpha}\dot{r}k\lambda}$$

$$\therefore R^{pq} - \frac{1}{4} g^{pq} R = g^{p}{}_{\dot{\alpha}k} g^{q}{}_{\dot{r}\lambda} \phi^{\dot{\alpha}\dot{r}k\lambda}$$

Multiplying both sides by $\frac{1}{4} g_{p}{}^{\dot{\beta}\rho} g_{q}{}^{\dot{\mu}\sigma}$ and contracting

$$\phi^{\dot{\beta}\dot{\mu}\rho\sigma} = \frac{1}{4} g_{p}{}^{\dot{\beta}\rho} g_{q}{}^{\dot{\mu}\sigma} (R^{pq} - \frac{1}{4} g^{pq} R) \qquad (5.35)$$

To obtain an expression for $\psi^{\alpha\beta\mu\nu}$ we have to define the double dual $\tilde{R}_{pqrs}$, of the Riemann curvature tensor

$$\tilde{R}_{pqrs} \equiv \frac{1}{4} g \varepsilon_{pqab} \varepsilon_{cdrs} R^{abcd} \qquad (5.36)$$

Now let $E_{pqrs} = R_{pqrs} + \tilde{R}_{pqrs}$

Defining the dual of it

$$\tilde{E}_{pqrs} = \frac{1}{2} g^{1/2} \varepsilon_{pqab} E^{ab}{}_{rs} \qquad (5.37)$$

$\tilde{E}_{pqrs}$ is purely imaginary if E_{pqrs} is real. From the above expression it can be readily seen that:

$$\psi^{k\lambda\rho\sigma} = \frac{1}{8} g^{p\dot{\beta}k} g^{q}{}_{\dot{\beta}}{}^{\lambda} g_{r}^{\dot{\alpha}\rho} g_{s\dot{\alpha}}{}^{\sigma} \left[E_{pq}{}^{rs} + \tilde{E}_{pq}{}^{rs} \right] \qquad (5.38)$$

Obviously any scalar invariant which can be constructed from $\psi^{\alpha\beta\mu\nu}$ and $\phi^{\dot{\alpha}\dot{\beta}\mu\nu}$ can be written, by use of (5.35) and (5.38), in terms of geometric tensors alone.

5.3 CURVATURE INVARIANTS IN TERMS OF SPINORS

The first and simplest curvature invariant corresponding to the scalar curvature R is given by:

$$\psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha} \qquad (5.39)$$

The following two independent complex invariants (four independent real invariants) can be constructed from $\psi^{\alpha\beta\mu\nu}$ alone:

$$\psi_{\alpha\beta\mu\nu}\psi^{\alpha\beta\mu\nu} \qquad (5.40)$$

and $$\psi_{\alpha\beta\mu\nu}\psi^{\mu\nu}{}_{\rho\sigma}\psi^{\rho\sigma\alpha\beta} \qquad (5.41)$$

From the $\phi^{\alpha\beta\mu\nu}$ one can construct the three real invariants:

$$\phi_{\dot{\alpha}\dot{\beta}\mu\nu}\phi^{\dot{\alpha}\dot{\beta}\mu\nu} \qquad (5.42)$$

$$\phi_{\dot{\alpha}\beta\mu\nu}\,\phi^{\dot{\beta}\dot{k}\nu\rho}\phi_{\dot{k}}{}^{\dot{\alpha}}{}_{\rho}{}^{\mu} \qquad (5.43)$$

and $$\phi_{\dot{\alpha}\dot{\beta}\mu\nu}\phi^{\dot{\beta}\dot{k}\nu\rho}\phi_{\dot{k}\dot{\lambda}\rho\sigma}\phi^{\dot{\lambda}\dot{\alpha}\sigma\mu} \qquad (5.44)$$

The remaining six independent real invariants must be constructed from $\psi^{\alpha\beta\mu\nu}$ and $\phi^{\dot{\alpha}\dot{\beta}\mu\nu}$ in combination. These invariants follow from the three complex independent invariants given by:

$$\phi_{\rho\sigma\dot{\alpha}\dot{\beta}}\psi^{\dot{\alpha}\dot{\beta}}{}_{\dot{\mu}\dot{\nu}}\ \phi^{\rho\sigma\dot{\mu}\dot{\nu}} \qquad (5.45)$$

$$\phi^{\gamma\delta}{}_{\dot{\alpha}\dot{\beta}}\psi^{\dot{\alpha}\dot{\beta}}{}_{\dot{\rho}\dot{\sigma}}\ \psi^{\dot{\rho}\dot{\sigma}\dot{k}\dot{\lambda}}\phi_{\gamma\delta\dot{k}\dot{\lambda}} \qquad (5.46)$$

and $\phi^{\gamma\delta\dot{\alpha}\dot{\beta}}\phi_{\rho\sigma\dot{\alpha}\dot{\beta}}\psi^{\rho\sigma k\lambda}\phi_{k\lambda\dot{\mu}\dot{\nu}}\phi_{\gamma\delta}{}^{\dot{\mu}\dot{\nu}}$ (5.47)

In the case of space filled with electromagnetic energy the spinors are constrained to satisfy conditions:

$$\psi^{\alpha}{}_{\beta}{}^{\beta}{}_{\alpha} = 0 \qquad \phi^{\dot{\alpha}\dot{\beta}\mu\nu} = -\ 2\phi^{\dot{\alpha}\dot{\beta}}\phi^{\mu\nu} \qquad (5.48)$$

For electromagnetic fields, invariant (5.39) vanishes. Of the three invariants (5.42), (5.43), 5.44), the first one survives in the form (neglecting in the following constant factors):

$$\phi_{\dot{\alpha}\dot{\beta}}\phi_{\mu\nu}\phi^{\dot{\alpha}\dot{\beta}}\phi^{\mu\nu} \qquad (5.49)$$

Using relation such as:

$$\phi^{\alpha\beta}\phi_{\beta}{}^{\gamma} = \frac{1}{2}\,\varepsilon^{\alpha\gamma}\phi^{\mu\nu}\phi_{\mu\nu} \qquad (5.50)$$

One sees that the invariant (5.43) vanishes and (5.44) is expressed as the square of (5.49). The four real invariants of (5.40), (5.41) are unaffected by the change in the controlling equation for R_{pq}. Considering the three expressions (5.45), (5.46), (5.47), the first two remain independent and are expressed by:

$$\phi_{\alpha\beta}\phi_{\dot{\mu}\dot{\nu}}\psi^{\dot{\mu}\dot{\nu}}{}_{\dot{\rho}\dot{\sigma}}\phi^{\alpha\beta}\phi^{\dot{\rho}\dot{\sigma}} \qquad (5.51)$$

$$\phi^{\alpha\beta}\phi_{\dot{\mu}\dot{\nu}}\psi^{\dot{\mu}\dot{\nu}}{}_{\dot{\rho}\dot{\sigma}}\psi^{\dot{\rho}\dot{\sigma}\dot{k}\dot{\lambda}}\psi_{\alpha\beta}\psi_{\dot{k}\dot{\lambda}} \tag{5.52}$$

The third expression (5.47) becomes:

$$\phi^{\alpha\beta}\phi^{\dot{\mu}\dot{\nu}}\phi_{\gamma\delta}\phi_{\dot{\mu}\dot{\nu}}\psi^{\gamma\delta\eta\xi}\phi_{\eta\xi}\phi_{\dot{\rho}\dot{\sigma}}\phi_{\alpha\beta}\phi^{\dot{\rho}\dot{\sigma}} \tag{5.53}$$

This is not independent, being the product of (5.49) and the complex conjugate of (5.51). Thus in the case of electromagnetic radiation there may in general be nine independent non-vanishing invariants; only one invariant must vanish; only nine of the remaining thirteen are independent.

5.4 CURVATURE INVARIANTS INVOLVING $F_{\mu\nu}$ IN TENSOR FORMS

The curvature invariants have been worked out in terms of spinors following the work of Witten. We have to work out the corresponding expressions in terms of tensors. As far as we are aware the invariants for the gravitational and electromagnetic fields have not been written down in terms of tensors. We will work out these tensor invariants expressions here. We will give a brief derivation for a typical case; the other cases are similar and we will simply write down the result. We start with the expression for the electromagnetic field spinor $\phi^{\mu\nu}$ in terms of the self-dual tensor ω_{pq} which is defined as

$$\omega_{pq} = F_{pq} + \tilde{F}_{pq} \tag{5.54}$$

Here we have used Latin indices for tensors and Greek indices for spinors:

$$\phi^{\mu\nu} = \frac{1}{8}\, g_p{}^{\dot{\alpha}\mu} g_{q\dot{\alpha}}{}^{\nu} \omega^{pq} \tag{5.55}$$

$$\phi_{\mu\nu} = \frac{1}{8}\, g_p{}^{\dot{\alpha}}{}_{\mu} g_{q\dot{\alpha}\nu} \omega^{pq} \tag{5.56}$$

Thus $$\phi^{\mu\nu}\phi_{\mu\nu} = \frac{1}{64}\, g_p{}^{\dot{\alpha}\mu} g_{q\dot{\alpha}}{}^{\nu} g_r{}^{\dot{\beta}}{}_{\mu} g_{s\dot{\beta}\nu} \omega^{pq} \omega^{rs} \tag{5.57}$$

With the use of the relations

$$g_p{}^{\dot{\alpha}\beta} g_{q\dot{\alpha}\gamma} + g_q{}^{\dot{\alpha}\beta} g_{p\dot{\alpha}\gamma} = 2g_{pq}\delta^{\beta}{}_{\gamma} \tag{5.58}$$

and $g_{p\ \alpha}^{\ \dot\beta} g_{q\dot\beta\mu} = g_{pq}\varepsilon_{\alpha\mu} + \frac{1}{2} i\sqrt{-g}\ \varepsilon_{pqrs}\ g^{r\dot\beta}_{\ \ \alpha} g^{s}_{\ \dot\beta\mu}$ (5.59)

One can reduce (5.57) to a scalar product of tensors $\omega^{pq}\omega_{pq}$. This process is rather long and a much simpler method is to note that because of the invariant nature of the left hand side of (5.57), the result must be a linear combination of

$$k_1 = g_{pq}g_{rs}\omega^{pq}\omega^{rs} \tag{5.60}$$

$$k_2 = g_{pr}g_{qs}\omega^{pq}\omega^{rs} \tag{5.61}$$

$$k_3 = g_{ps}g_{qr}\omega^{pq}\omega^{rs} \tag{5.62}$$

Because of the antisymmetry of ω^{pq}, k_1 vanishes and k_2, k_3 are equal to within a sign. Thus in fact $\phi^{\mu\nu}\phi_{\mu\nu}$ is proportional to $\omega^{pq}\omega_{pq}$. One can apply a similar reasoning to all the other invariants worked out in terms of spinors and we finally obtain the following tensor expressions for the nine independent invariants in this case: the four invariants I_1, J_1, I_2, J_2 have already been written down earlier. The remaining five are as follows (we revert to Greek indices for tensors)

$$F^{\mu\nu}F_{\mu\nu} \tag{5.63}$$

$$\tilde{F}^{\mu\nu}F_{\mu\nu} \tag{5.64}$$

$$L_1 = R^{\lambda\mu\nu\sigma}F_{\lambda\mu}F_{\nu\sigma} \tag{5.65}$$

$$L_2 = R^{\mu\nu\lambda\sigma}F_{\mu\nu}\tilde{F}_{\lambda\sigma} \tag{5.66}$$

$$M = R_{\lambda\mu\nu\kappa}R^{\nu\kappa\alpha\beta}F^{\lambda\mu}F_{\alpha\beta} \tag{5.67}$$

As is well know, in flat space the pure electromagnetic invariants (5.63,5.64) are proportional to ($\underline{E}^2 - \underline{B}^2$) and $\underline{E}.\underline{B}$, where $\underline{E}$ and $\underline{B}$ are respectively the electric and magnetic fields. Thus a null field is one in which these invariants vanish, so that the electric and magnetic fields are equal in magnitude and perpendicular, as in a plane wave. The interpretation of the other curvature invariants is not so straightforward.

5.5 CALCULATION OF THE RIEMANN TENSOR AND CURVATURE INVARIANTS

5.5.1 INVARIANTS INVOLVING THE RIEMANN TENSOR ALONE (I_1, I_2, J_1, J_2)

The non-zero members of the Christoffel symbols for the metric:

$$ds^2 = f dt^2 - 2k d\theta dt - \ell d\rho^2 - e^{\mu}(d\rho^2 + dz^2)$$

with the functions dependent on ρ only are:

$$\Gamma^{o}_{o1} = \frac{1}{2}\bar{\rho}^{2}(\ell f_{\rho} + k k_{\rho}) \quad \Gamma^{o}_{13} = \frac{1}{2}\bar{\rho}^{2}(k\ell_{\rho} - \ell k_{\rho})$$

$$\Gamma^{1}_{oo} = \frac{1}{2}\bar{e}^{\mu} f_{\rho} \qquad \Gamma^{1}_{o3} = -\frac{1}{2}\bar{e}^{\mu} k_{\rho}, \; \Gamma^{1}_{11} = \frac{1}{2}\mu_{\rho}$$

$$\Gamma^{1}_{22} = -\frac{1}{2}\mu_{\rho}, \quad \Gamma^{1}_{33} = -\frac{1}{2}\bar{e}^{\mu}\ell_{\rho}, \quad \Gamma^{2}_{12} = \frac{1}{2}\mu_{\rho}$$

$$\Gamma^{3}_{o1} = \frac{1}{2}\bar{\rho}^{2}(f k_{\rho} - k f_{\rho}), \; \Gamma^{3}_{13} = \frac{1}{2}\bar{\rho}^{2}(k k_{\rho} + f\ell_{\rho})$$

The algebraic properties of the curvature tensor are greatly clarified if we consider, instead of $R^{\lambda}{}_{\mu\nu k}$, its fully covariant form:

$$R_{\lambda\mu\nu k} = g_{\lambda\sigma} R^{\sigma}{}_{\mu\nu k}$$

$$\text{where } R^{\lambda}{}_{\mu\nu k} \equiv \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{k}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\nu\eta} \qquad (5.68)$$

Considering all the values of $(\lambda,\mu,\nu,k = 0,1,2,3)$ the non-zero values of Riemann-Christoffel curvature tensor are (apart from sign):

R_{0131}, R_{0113}, R_{1031}, R_{1013}, R_{1301}, R_{3101}, R_{3110}, R_{1310}

R_{0223}, R_{0232}, R_{2302}, R_{2032}, R_{2023}, R_{2320}, R_{3220}, R_{3202}

R_{0101}, R_{0110}, R_{1001}, R_{1010}

R_{0330}, R_{0303}, R_{3030}, R_{3003}

R_{0220}, R_{0202}, R_{2002}, R_{2020}

R_{1221}, R_{1212}, R_{2121}, R_{2112}

R_{1331}, R_{1313}, R_{3131}, R_{3113}

R_{2332}, R_{2323}, R_{3223}, R_{3232}

If we have one value of each group, using symmetry properties of Riemann-Christoffel curvature tensor we can calculate the rest of the values.

A representative value from each of the group:

$$R_{0131} = -\frac{k_{\rho\rho}}{2} + \frac{1}{4}\mu_{\rho}k_{\rho} + \frac{1}{4\rho^2}(\ell f_{\rho}k_{\rho} + kk^2_{\rho} - k\ell_{\rho}f_{\rho} + f\ell_{\rho}k_{\rho})$$

$$R_{0223} = \frac{\mu_{\rho}k_{\rho}}{4}$$

$$R_{o1o1} = \frac{f_{\rho\rho}}{2} - \frac{\mu_\rho f_\rho}{4} = \frac{1}{4\rho^2}(\ell f_\rho^2 + 2kf_\rho k_\rho - fk_\rho^2)$$

$$R_{o33o} = -\frac{1}{4e^\mu}(\ell_\rho f_\rho + k_\rho^2) \qquad \text{(A)}$$

$$R_{o22o} = -\frac{\mu_\rho f_\rho}{4}$$

$$R_{1221} = \frac{1}{2} e^\mu \mu_{\rho\rho}$$

$$R_{1331} = \frac{\ell_{\rho\rho}}{2} - \frac{\mu_\rho \ell_\rho}{4} - \frac{1}{4\rho^2}(-\ell k_\rho^2 + 2k\ell_\rho k_\rho + f\ell_\rho^2)$$

$$R_{2332} = \frac{\ell_\rho \mu_\rho}{4}$$

Using the covariant values of $R_{\lambda\mu\nu\kappa}$ we can calculate the contravariant values of $R^{\lambda\mu\nu\kappa}$. So one value of each group is (using $\Delta = \rho^2$)i.e $R^{\lambda\mu\nu\kappa} = g^{\mu\sigma} g^{\nu\delta} g^{\kappa\theta} R^{\lambda}{}_{\sigma\delta\theta}$ (5.69)

$$\therefore R^{o131} = \frac{1}{e^{2\mu}\Delta^2}\Big[\frac{\mu_\rho}{4}(\ell k\ell_\rho + k^2 k_\rho \quad \ell fk_\rho + fk\ell_\rho)$$

$$+ \frac{1}{2}(-\ell kf_{\rho\rho} - k^2 k_{\rho\rho} + \ell fk_{\rho\rho} - fk\ell_{\rho\rho})$$

$$+ \frac{1}{4\rho^2}(\ell^2 kf_\rho^2 + 3\ell k^2 f_\rho k_\rho - 3\ell fkk_\rho^2 - k^3 \ell_\rho f_\rho + k^3 k_\rho^2$$

$$+ 3fk^2 \ell_\rho k_\rho - \ell^2 ff_\rho k_\rho + \ell fk\ell_\rho f_\rho - \ell f^2 \ell_\rho k_\rho$$

$$+ f^2 k\ell_\rho^2)\Big]$$

$$R^{o223} = \frac{e^{-2\mu}\mu_\rho}{4\Delta^2}(\ell kf_\rho + k^2 k_\rho - \ell fk_\rho + fk\ell_\rho)$$

$$R^{o1o1} = \frac{1}{e^{2\mu}\Delta^2}\Big[\frac{\mu_\rho}{4}(-\ \ell^2 f_\rho - 2\ell k k_\rho + k^2 \ell_\rho) + \frac{1}{2}$$

$$+ \frac{1}{2}(\ell^2 f_{\rho\rho} + 2\ell k k_{\rho\rho} - k^2 \ell_{\rho\rho})$$

$$+ \frac{1}{4\rho^2}(-\ \ell^3 f_\rho^2 - 4\ell^2 k f_\rho k_\rho + \ell^2 f k_\rho^2 + 2\ell k^2 f_\rho \ell_\rho - 3\ell k^2 k_\rho$$

$$- 2\ell f k \ell_\rho k_\rho + 2k^3 \ell_\rho k_\rho + f k^2 \ell_\rho^2)\Big]$$

$$R^{o33o} = -\frac{1}{4e^{\mu}\Delta^2}(\ell_\rho f_\rho + k_\rho^2) \qquad \text{(B)}$$

$$R^{o22o} = -\frac{e^{-2\mu}\mu_\rho}{4\Delta^2}(\ell^2 f_\rho + 2\ell k k_\rho - k^2 \ell_\rho)$$

$$R^{1221} = \frac{1}{2}\, e^{-3\mu}\mu_{\rho\rho}$$

$$R^{1331} = -\frac{1}{e^{2\mu}\Delta^2}\Big[\frac{\mu_\rho}{4}(-\ k^2 f_\rho + f k k_\rho + f k k_\rho)$$

$$+ \frac{1}{2}(k^2 f_{\rho\rho} - f k k_{\rho\rho} - f^2 \ell_{\rho\rho} - f k k_{\rho\rho})$$

$$+ \frac{1}{4\rho^2}(-\ \ell k^2 f_\rho - 2k^3 f_\rho k_\rho + 3 f k^2 k_\rho^2 + 2\ell f k f_\rho k_\rho$$

$$- 2 f k^2 \ell_\rho f_\rho + 4 f^2 k \ell_\rho k_\rho - \ell f^2 k_\rho^2 + f^3 \ell_\rho^2)\Big]$$

$$R^{2332} = \frac{e^{-2\mu}\mu_\rho}{4\Delta^2}(f^2 \ell_\rho + 2 f k k_\rho - k^2 f_\rho)$$

Using the symmetric and antisymmetric properties rest of the values can be found out.

We can now proceed to evaluate the invariants I_1, I_2, J_1, J_2 we first rewrite $R_{\lambda\mu\nu\kappa}$ as K_{AB} with the following correspondence Pirani (1957)

01	02	03	23	31	12	$(\lambda\mu)$	$(\mu\kappa)$
1	2	3	4	5	6	A	B

So that $R_{0101} \rightarrow K_{11}$, $R_{0102} \rightarrow K_{12}$ and so on. Note that because of the symmetry property $R_{\lambda\mu\nu k} = R_{\nu k\lambda\mu}$, we have $K_{AB} = K_{BA}$

$$K_{11} = R_{0101},\ K_{12} = R_{0102},\ K_{13} = R_{0103},\ K_{14} = R_{0123}$$

$$K_{15} = R_{0131},\ K_{16} = R_{0112},\ K_{22} = R_{0202},\ K_{23} = R_{0203}$$

$$K_{24} = R_{0223},\ K_{25} = R_{0231},\ K_{26} = R_{0212},\ K_{33} = R_{0303}$$

$$K_{34} = R_{0323},\ K_{35} = R_{0331},\ K_{36} = R_{0312},\ K_{44} = R_{2323}$$

$$K_{45} = R_{2331},\ K_{46} = R_{2312},\ K_{55} = R_{3131},\ K_{56} = R_{3112}$$

$$K_{66} = R_{1212}$$

The K_{AB} can be arranged in a matrix form, as follows:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix}$$

The non-zero part of the matrix is

$$\begin{bmatrix} K_{11} & 0 & 0 & 0 & K_{15} & 0 \\ 0 & K_{22} & 0 & K_{24} & 0 & 0 \\ 0 & 0 & K_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{66} \end{bmatrix}$$

In terms of K_{AB}, I_1 can be expressed as follows:

$$I_1 = R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa} = 4(K_{11}K^{11} + 2K_{15}K^{15} + K_{22}K^{22} + 2K_{24}K^{24} + K_{33}K^{33} + K_{44}K^{44} + K_{55}K^{55} + K_{66}K^{66}) \qquad (5.70)$$

Using values from Islam (1983), Section 2, we calculated all the values of K_{AB}, K^{AB} and $K_B{}^A$.

$$K_{11} = -\frac{4}{27}\sigma\bar{\rho}^{4/3} + \frac{1}{2}\frac{a_o^2}{\sigma}\bar{\rho}^{2/3}, \quad k^{11} = \frac{1}{e^{2\mu}\Delta^2}\left[\frac{71}{81}\frac{\rho^{4/3}}{\sigma}\right. \tag{5.71}$$

$$\left. + \frac{2}{27}k_o^2\sigma\rho^{2/3} + \frac{2}{3}\frac{a_o^2}{\sigma^3}\rho^2 + \frac{27}{16}\frac{a_o^4}{16}\rho^{8/3}\right]$$

$$K_{11}K^{11} = \frac{1}{e^{2\mu}\Delta^2}\left(-\frac{280}{2187} - \frac{8}{729}K_o^2\sigma^2\bar{\rho}^{2/3} + \frac{55}{162}\frac{a_o^4}{\sigma^4}\rho^{4/3}\right.$$

$$\left. + \frac{27}{32}\frac{a_o^6}{\sigma^6}\rho^2\right) \tag{5.72}$$

$$K_{15} = -\frac{2}{9}\frac{a_o}{\sigma}\bar{\rho}^{2/3} - \frac{4}{27}K_o\sigma\bar{\rho}^{4/3} + \frac{9}{8}\frac{a_o^3}{\sigma^3} \tag{5.73}$$

$$K^{15} = -\frac{1}{e^{2\mu}\Delta^2}\left[-\frac{13}{6}\frac{a_o}{\sigma}\rho^{4/3} - \frac{28}{27}K_o\sigma\rho^{2/3} - \frac{3}{2}\frac{a_o^3}{\sigma^3}\rho^2\right] \tag{5.74}$$

$$K_{15}K^{15} = -\frac{1}{e^{2\mu}\Delta^2}\left[\frac{16}{27}\frac{a_o^2}{\sigma^2}\rho^{2/3} - \frac{268}{2187} - \frac{101}{48}\frac{a_o^4}{\sigma^4}\rho^{4/3}\right.$$

$$\left. + \frac{108}{729}K_o^2\sigma^2\bar{\rho}^{2/3} - \frac{27}{16}\frac{a_o^6}{\sigma^6}\rho^2\right] \tag{5.75}$$

$$K_{22} = -\frac{2}{27}\sigma\bar{\rho}^{4/3} - \frac{1}{4}\frac{a_o^2}{\sigma}\bar{\rho}^{2/3}, \quad K^{22} = \frac{1}{e^{2\mu}\Delta^2}\left[-\frac{8}{81}\frac{\rho^{4/3}}{\sigma}\right.$$

$$\left. + \frac{4}{27}K_o^2\sigma\rho^{2/3} - \frac{5}{12}\frac{a_o^2}{\sigma^3}\rho^2\right] \tag{5.76}$$

$$K_{22}K^{22} = \frac{1}{e^{2\mu}\Delta^2}\left(\frac{20}{2187} - \frac{8}{729}K_o^2\sigma^2\bar{\rho}^{2/3} + \frac{1}{18}\frac{a_o^2}{\sigma^2}\rho^{2/3}\right.$$

$$\left. + \frac{5}{48}\frac{a_o^4}{\sigma^4}\rho^{4/3}\right) \tag{5.77}$$

$$K_{33} = \frac{1}{4e^{\mu}}\left(\frac{4}{81} + \frac{1}{9}\frac{a_o^2}{\sigma^2}\rho^{2/3}\right), \quad K^{33} = \frac{1}{4e^{\mu}\Delta^2}\left(\frac{8}{9} + \frac{a_o^2}{\sigma^2}\rho^{2/3}\right) \tag{5.78}$$

$$K_{33}K^{33} = \frac{1}{e^{2\mu_\Delta 2}}\left(\frac{4}{81} + \frac{1}{9}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{1}{16}\frac{a_o^4}{\sigma^4}\rho^{4/3}\right) \quad (5.79)$$

$$K_{44} = -\frac{\ell_\rho \mu_\rho}{4}\ ,\ K^{44} = -\frac{\bar{e}^{2\mu}\mu_\rho}{4\rho^4}(f^2\ell_\rho + 2fkk_\rho - k^2 f_\rho) \quad (5.80)$$

$$K_{44}K^{44} = \frac{1}{e^{2\mu_\Delta 2}}\left[-\frac{48}{2187} + \frac{6}{81}\frac{a_o^2}{\sigma^2}\rho^{2/3} - \frac{1}{12}\frac{a_o^4}{\sigma^4}\rho^{4/3}\right.$$

$$\left. - \frac{27}{32}\frac{a_o^6}{\sigma^6}\rho^2 - \frac{8}{729}K_o^2\sigma^2\bar{\rho}^{2/3}\right] \quad (5.81)$$

$$K_{55} = \frac{4}{27}\frac{\bar{\rho}^{2/3}}{\sigma} - \frac{57}{12}\frac{a_o^2}{\sigma^3} + \frac{2}{27}K_o^2\sigma\bar{\rho}^{4/3} + \frac{9}{4}\frac{a_o^4}{\sigma^5}\rho^{2/3} \quad (5.82)$$

$$K^{55} = \frac{1}{e^{2\mu_\Delta 2}}\left[\frac{2}{27}\sigma\rho^{2/3} + \frac{1}{4}\frac{a_o^2}{\sigma}\rho^{4/3}\right] \quad (5.83)$$

$$K_{55}K^{55} = \frac{1}{e^{2\mu_\Delta 2}}\left[\frac{26}{2187} - \frac{51}{162}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{4}{729}K_o^2\sigma^2\bar{\rho}^{2/3}\right.$$

$$\left. + \frac{49}{48}\frac{a_o^4}{\sigma^4}\rho^{4/3} + \frac{9}{16}\frac{a_o^6}{\sigma^6}\rho^2\right] \quad (5.84)$$

$$K_{24} = \frac{1}{6}\frac{a_o}{\sigma}\bar{\rho}^{2/3} + \frac{2}{27}K_o\sigma\bar{\rho}^{4/3} + \frac{3}{4}\frac{a_o^3}{\sigma^3}\ , \quad (5.85)$$

$$K^{24} = \frac{1}{e^{2\mu_\Delta 2}}\left[\frac{4}{27}K_o\sigma\rho^{2/3} + \frac{3}{8}\frac{a_o^3}{\sigma^3}\rho^2\right] \quad (5.86)$$

$$K_{24}K^{24} = \frac{1}{e^{2\mu_\Delta 2}}\left[-\frac{4}{729} + \frac{8}{729}K_o^2\sigma^2\bar{\rho}^{2/3}\right.$$

$$\left. - \frac{2}{81}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{1}{16}\frac{a_o^4}{\sigma^4}\rho^{4/3} - \frac{1}{162}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{9}{32}\frac{a_o^6}{\sigma^6}\rho^2\right] \quad (5.87)$$

$$K_{66} = -\frac{1}{2} e^{\mu}\left(\frac{4}{9}\rho^{-2} + \frac{a_o^2}{2\sigma^2}\rho^{-4/3}\right), \tag{5.88}$$

$$K^{66} = -\frac{1}{2} e^{-3\mu}\left(\frac{4}{9}\rho^{-2} + \frac{1}{2}\frac{a_o^2}{\sigma^2}\rho^{-4/3}\right) \tag{5.89}$$

$$K_{66}K^{66} = \frac{1}{e^{2\mu}\Delta^2}\left[\frac{4}{81} + \frac{1}{9}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{1}{16}\frac{a_o^4}{\sigma^4}\rho^{4/3}\right] \tag{5.90}$$

Putting these values in Equation (5.70)

$$I_1 = R_{\lambda\mu\nu\kappa}R^{\lambda\mu\nu\kappa} = \frac{4}{e^{2\mu}\Delta^2}\Big[\left(-\frac{280}{2187} - \frac{8}{729}K_o^2\sigma^2\rho^{-2/3}\right.$$

$$\left. + \frac{55}{162}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{1}{12}\frac{a_o^4}{\sigma^4}\rho^{4/3} + \frac{27}{32}\frac{a_o^6}{\sigma^6}\rho^2\right)$$

$$+ 2\left(\frac{16}{27}\frac{a_o^2}{\sigma^2}\rho^{2/3} - \frac{268}{2187} - \frac{101}{48}\frac{a_o^4}{\sigma^4}\rho^{4/3} + \frac{108}{729}K_o^2\sigma^2\rho^{-2/3}\right.$$

$$\left. - \frac{27}{16}\frac{a_o^6}{\sigma^6}\rho^2\right) + \left(\frac{20}{2187} - \frac{8}{729}K_o^2\sigma^2\rho^{-2/3}\right.$$

$$\left. + \frac{1}{18}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{5}{48}\frac{a_o^4}{\sigma^4}\rho^{4/3}\right) + 2\left(-\frac{4}{729} + \frac{8}{729}K_o^2\sigma^2\rho^{-2/3}\right.$$

$$- \frac{2}{81}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{1}{16}\frac{a_o^4}{\sigma^4}\rho^{4/3}$$

$$\left. - \frac{1}{162}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{9}{32}\frac{a_o^6}{\sigma^6}\rho^2\right) + \left(\frac{4}{81} + \frac{1}{9}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{1}{16}\frac{a_o^4}{\sigma^4}\rho^{4/3}\right)$$

$$+ \left(-\frac{48}{2187} + \frac{6}{81}\frac{a_o^2}{\sigma^2}\rho^{2/3}\right.$$

$$- \frac{1}{12}\frac{a_o^4}{\sigma^4}\rho^{4/3} - \frac{27}{32}\frac{a_o^6}{\sigma^6}\rho^2 - \frac{8}{729}K_o^2\sigma\,\rho^{-2/3})$$

$$+ (\frac{26}{2187} - \frac{51}{162}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{4}{729}K_o^2\sigma^2\rho^{-2/3} - \frac{49}{48}\frac{a_o^4}{\sigma^4}\rho^{4/3}$$

$$+ \frac{9}{16}\frac{a_o^6}{\sigma^6}\rho^2) + (\frac{4}{81} + \frac{1}{9}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{1}{16}\frac{a_o^4}{\sigma^4}\rho^{4/3})\Big]$$

$$\doteq \frac{4}{e^{2\mu}{}_{\Delta}{}^{2}}\Big[- \frac{626}{2187} + \frac{106}{2187}\frac{\rho_o^{2/3}}{\rho^{2/3}} + \frac{4}{9}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{312}{729}\frac{\rho^{4/3}}{\rho_o^{4/3}}$$

$$- \frac{128}{6561}\frac{\rho^2}{\rho_o^2}\Big] \qquad (5.91)$$

We next evaluate J_1 using group A or B, we can calculate all non-zero values of $R_{\nu\kappa}{}^{\lambda\mu}$

$$\therefore K_1{}^1 = - \frac{1}{e^{2\mu}\rho^2}\Big[- \frac{14}{81} + \frac{19}{81}\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{4}{81}\frac{\rho^{4/3}}{\rho_o^{4/3}}\Big] \qquad (5.92)$$

Multiplying (5.72) and (5.92) we get

$$K_{11}K^{11}K_1{}^1 = - \frac{1}{e^{3\mu}{}_{\Delta}{}^{3}}\Big[\frac{56}{177147}\frac{\rho_o^{2/3}}{\rho^{2/3}} + \frac{3844}{177147} - \frac{6106}{177147}\frac{\rho^{2/3}}{\rho_o^{2/3}}$$

$$- \frac{299}{177147}\frac{\rho^{4/3}}{\rho_o^{4/3}} - \frac{148}{177147}\frac{\rho^2}{\rho_o^2} + \frac{976}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}}$$

$$+ \frac{192}{177147}\frac{\rho^{10/3}}{\rho_o^{10/3}}\Big] \qquad (5.93)$$

Using the values of K_{11} from (5.71) and Equation (5.74) and calculating the values of $K_5{}^1$ we get

$$K_5{}^1 = \frac{1}{e^{\mu}\rho^2}\left[\frac{4}{54}\,\frac{a_o}{\sigma^2}\,\rho^{2/3} + \frac{1}{9}\,K_o - \frac{5}{2}\,\frac{a_o^3}{\sigma^4}\,\rho^{4/3} + \frac{27}{32}\,\frac{a_o^5}{\sigma^6}\,\rho^2\right.$$

$$\left. - \frac{1}{9}\,K_o^3\sigma^2\rho^{-2/3}\right] \tag{5.94}$$

Multiplying K_{11}, K^{15} and $K_5{}^1$ with factor 3 we get

$$3K_{11}K^{15}K_5{}^1 = -\frac{3}{e^{3\mu_\Delta 3}}\left[\frac{4228}{177147}\,\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{2170}{177147}\right.$$

$$- \frac{18112}{177147}\,\frac{\rho^{4/3}}{\rho_o^{4/3}} - \frac{8480}{177147}\,\frac{\rho^2}{\rho_o^2} + \frac{274}{59049}\,\frac{\rho_o^{2/3}}{\rho^{2/3}}$$

$$\left.- \frac{28}{59049}\,\frac{\rho_o^{4/3}}{\rho^{4/3}} + \frac{1568}{177147}\,\frac{\rho^{8/3}}{\rho_o^{8/3}} - \frac{256}{177147}\,\frac{\rho^{10/3}}{\rho_o^{10/3}}\right] \tag{5.95}$$

Multiplying Equation (5.73), (5.83) and (5.94) with factor 3 we get

$$3K_{15}K^{55}K_5{}^1 = \frac{3}{e^{3\mu_\Delta 3}}\left[-\frac{370}{177147}\,\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{212}{177147}\,\frac{\rho^{4/3}}{\rho_o^{4/3}}\right.$$

$$+ \frac{74}{177147} - \frac{256}{177147}\,\frac{\rho^2}{\rho_o^2} - \frac{14}{59049}\,\frac{\rho_o^{2/3}}{\rho^{2/3}}$$

$$\left.- \frac{928}{177147}\,\frac{\rho^{8/3}}{\rho_o^{8/3}} - \frac{32}{59049}\,\frac{\rho^{10/3}}{\rho_o^{10/3}} + \frac{2}{59049}\,\frac{\rho_o^{4/3}}{\rho^{4/3}}\right] \tag{5.96}$$

Multiplying (5.77) with $K_2^{\,2}$ as below:

$$K_2^{\,2} = \frac{1}{e^{\mu_\Delta}}\left(\frac{4}{81} + \frac{1}{3}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{9}{16}\frac{a_o^4}{\sigma^4}\rho^{4/3}\right) \tag{5.97}$$

$$\therefore\ K_{22}K^{22}K_2^{\,2} = \frac{1}{e^{3\mu_\Delta 3}}\left[\frac{76}{177147} - \frac{16}{177147}\frac{\rho_o^{2/3}}{\rho^{2/3}} - \frac{896}{531441}\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{448}{177147}\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{304}{177147}\frac{\rho^2}{\rho_o^2} + \frac{80}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}}\right] \tag{5.98}$$

$$K_4^{\,2} = \frac{1}{e^{\mu_\Delta}}\left(\frac{8}{81}k_o - \frac{27}{32}\frac{a_o^5}{\sigma^6}\rho^2\right) \tag{5.99}$$

Using K_{22} from Equation (5.76), and multiplying with (5.86) and (5.99) with factor we get

$$\therefore\ 3K_{22}K^{24}K_4^{\,2} = \frac{3}{e^{3\mu_\Delta 3}}\left[-\frac{32}{177147} - \frac{3}{177147}\frac{\rho_o^{2/3}}{\rho^{2/3}} + \frac{32}{177147}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{2}{19683}\frac{\rho^2}{\rho_o^2} + \frac{16}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}} + \frac{32}{177147}\frac{\rho^{10/3}}{\rho_o^{10/3}}\right] \tag{5.100}$$

$$K_4{}^2 = e^{-\mu}\{\frac{\mu_\rho}{4\rho^2}(-\ell k_\rho + k\ell_\rho)\} \tag{5.101}$$

Multiplying K^{44} from (5.80) with Equation (5.85) and (5.101) with factor 3, we get as follows

$$3K_{24}K^{44}K_4{}^2 = \frac{1}{e^{3\mu_\Delta 3}}\left[\frac{2048}{531441}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{608}{531441}\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{16}{59049}\frac{\rho^2}{\rho_o^2} + \frac{400}{531441}\frac{\rho^{8/3}}{\rho_o^{8/3}} - \frac{64}{59049} + \frac{32}{59049}\frac{\rho_o^{2/3}}{\rho^{2/3}}\right] \tag{5.102}$$

$$K_3^3 = -\frac{1}{4\rho^2 e^{\mu}}(\frac{8}{9} + \frac{a_o^2}{\sigma^2}\rho^{2/3}) \tag{5.103}$$

Multiplying (5.79) with (5.103) we get

$$K_{33}K^{33}K_3^3 = -\frac{1}{e^{3\mu_\Delta 3}}\left[\frac{8}{729} + \frac{8}{729}\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{8}{2187}\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{8}{19683}\frac{\rho^2}{\rho_o^2}\right] \tag{5.104}$$

$$K_4{}^4 = \frac{1}{4e^{\mu_\Delta}}\left(\frac{56}{81} + \frac{5}{3}\frac{a_o^2}{\sigma^2}\rho^{2/3} - \frac{9}{4}\frac{a_o^4}{\sigma^4}\rho^{4/3}\right) \qquad (5.105)$$

Multiplying (5.105) with (5.81) we get

$$K_{44}K^{44}K_4{}^4 = \frac{1}{e^{3\mu_\Delta 3}}\left[-\frac{712}{177147} + \frac{208}{177147}\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{448}{177147}\frac{\rho^{4/3}}{\rho_o^{4/3}} - \frac{1024}{177147}\frac{\rho^2}{\rho_o^2} - \frac{1664}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}} + \frac{64}{177147}\frac{\rho^{10/3}}{\rho_o^{10/3}} - \frac{56}{177147}\frac{\rho_o^{2/3}}{\rho^{2/3}}\right] \qquad (5.106)$$

Multiplying (5.84) with (5.107) below we get Equation (5.108)

$$K_5{}^5 = -\frac{1}{e^{\mu_\Delta}}\left(-\frac{1}{3}\frac{a_o^2}{\sigma^2}\rho^{2/3} - \frac{4}{81}\frac{9}{16}\frac{a_o^4}{\sigma^4}\rho^{4/3} + \frac{3}{27}K_o^2\sigma^2\rho^{-2/3}\right) \qquad (5.107)$$

$$\therefore\ K_{55}K^{55}K_5{}^5 = -\frac{1}{e^{3\mu_\Delta 3}}\left[\frac{322}{177147}\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{2360}{177147}\frac{\rho^{4/3}}{\rho_o^{4/3}} - \frac{426}{177147} + \frac{2256}{177147}\frac{\rho^2}{\rho^2} + \frac{528}{177147}\frac{\rho^2}{\rho_o^2} + \frac{528}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}} + \frac{31}{177147}\frac{\rho_o^{2/3}}{\rho^{2/3}} - \frac{256}{177147}\frac{\rho^{10/3}}{\rho_o^{10/3}} + \frac{1}{59049}\frac{\rho_o^{4/3}}{\rho^{4/4}}\right] \qquad (5.108)$$

$$K_6{}^6 = \frac{1}{e^{\mu_\Delta}}\left(-\frac{2}{9} - \frac{1}{4}\frac{a_o^2}{\sigma^2}\rho^{2/3}\right) \tag{5.109}$$

Multiplying (5.109) with (5.90) we get

$$K_{66}K^{66}K_6{}^6 = \frac{1}{e^{e\mu_\Delta 3}}\left[-\frac{8}{729} - \frac{8}{729}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{8}{2187}\frac{\rho^{4/3}}{\rho_o^{4/3}} - \frac{8}{19683}\frac{\rho^2}{\rho_o^2}\right] \tag{5.110}$$

In terms of K_{AB}, J_1 can be written as

$$J_1 \doteq R_{\lambda\mu\nu k}R^{\nu k\rho\sigma}R_{\rho\sigma}{}^{\lambda\mu} = 8(K_{11}K^{11}K_1{}^1 + 3K_{11}K^{15}K_5{}^1 + 3K_{15}K^{55}K_5{}^1 + K_{22}K^{22}K_2{}^2$$

$$+ 3K_{22}K^{24}K_4{}^2 + 3K_{24}K^{44}K_4{}^2 + K_{33}K^{33}K_3{}^3 + K_{44}K^{44}K_4{}^4$$

$$+ K_{55}K^{55}K_5{}^5 + K_{66}K^{66}K_6{}^6)$$

$$= \frac{8}{e^{3\mu_\Delta 3}}\Big[\Big(-\frac{56}{177147}\frac{\rho_o^{2/3}}{\rho_o^{2/3}} - \frac{3844}{177147} + \frac{6106}{177147}\frac{\rho^{2/3}}{\rho_o^{2/3}}$$

$$+ \frac{299}{177147}\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{148}{177147}\frac{\rho^2}{\rho_o^2}$$

$$- \frac{976}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}} - \frac{192}{177147}\frac{\rho^{10/3}}{\rho_o^{10/3}}\Big) + \Big(-\frac{12648}{177147}\frac{\rho^{2/3}}{\rho_o^{2/3}}$$

$$+ \frac{6510}{177147} + \frac{54336}{177147}\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{25440}{177147}\frac{\rho^2}{\rho_o^2}$$

$$- \frac{822}{59049}\frac{\rho_o^{2/3}}{\rho^{2/3}} + \frac{84}{59049}\frac{\rho_o^{4/3}}{\rho^{4/3}} - \frac{4704}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}} + \frac{768}{177147}\frac{\rho^{10/3}}{\rho_o^{10/3}}\Big)$$

$$+ \Big(-\frac{1110}{177147}\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{636}{177147}\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{222}{177147} -$$

$$- \frac{768}{177147}\frac{\rho^2}{\rho_o^2} - \frac{42}{59049}\frac{\rho_o^{2/3}}{\rho^{2/3}} - \frac{2784}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}}$$

$$- \frac{96}{59049}\frac{\rho^{10/3}}{\rho_o^{10/3}} + \frac{6}{59049}\frac{\rho_o^{4/3}}{\rho^{4/3}}\Big) + \Big(\frac{76}{177147} - \frac{16}{177147}\frac{\rho_o^{2/3}}{\rho^{2/3}}$$

$$- \frac{896}{531441}\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{448}{177147}\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{304}{177147}\frac{\rho^2}{\rho_o^2} + \frac{80}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}}\Big)$$

$$+ \Big(-\frac{96}{177147} - \frac{9}{177147}\frac{\rho_o^{2/3}}{\rho^{2/3}} + \frac{96}{177147}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{6}{19683}\frac{\rho^2}{\rho_o^2}$$

$$+ \frac{48}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}} + \frac{96}{177147}\frac{\rho^{10/3}}{\rho_o^{10/3}}\Big) + \Big(\frac{2048}{531441}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{608}{531441}\frac{\rho^{4/3}}{\rho_o^{4/3}}$$

$$+ \frac{16}{59049}\frac{\rho^2}{\rho_o^2} + \frac{400}{531441}\frac{\rho^{8/3}}{\rho_o^{8/3}} - \frac{64}{59049} + \frac{32}{59049}\frac{\rho_o^{2/3}}{\rho^{2/3}}\Big)$$

$$+ \left(-\frac{8}{729} - \frac{8}{729}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{8}{2187}\frac{\rho^{4/3}}{\rho_o^{4/3}} - \frac{8}{19683}\frac{\rho^2}{\rho_o^2}\right)$$

$$+ \left(-\frac{712}{177147} + \frac{208}{177147}\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{448}{177147}\frac{\rho^{4/3}}{\rho_o^{4/3}}\right.$$

$$\left. - \frac{1024}{177147}\frac{\rho^2}{\rho_o^2} - \frac{1664}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}} + \frac{64}{177147}\frac{\rho^{10/3}}{\rho_o^{10/3}} - \frac{56}{177147}\frac{\rho_o^{2/3}}{\rho^{2/3}}\right)$$

$$+ \left(-\frac{322}{177147}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{2360}{177147}\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{426}{177147} - \frac{2256}{177147}\frac{\rho^2}{\rho_o^2}\right.$$

$$\left. - \frac{528}{177147}\frac{\rho^{8/3}}{\rho_o^{8/3}} - \frac{31}{177147}\frac{\rho_o^{2/3}}{\rho^{2/3}} + \frac{256}{177147}\frac{\rho^{10/3}}{\rho_o^{10/3}} - \frac{1}{59049}\frac{\rho_o^{4/3}}{\rho^{4/3}}\right)$$

$$\left. + \left(-\frac{8}{729} - \frac{8}{729}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{8}{2187}\frac{\rho^{4/3}}{\rho_o^{4/3}} - \frac{8}{19683}\frac{\rho^2}{\rho_o^2}\right)\right]$$

$$= \frac{8}{e^{3\mu}{}_\Delta{}^3}\left[\frac{89}{59049}\frac{\rho_o^{4/3}}{\rho^{4/3}} + \frac{2664}{177147}\frac{\rho_o^{2/3}}{\rho^{2/3}} + \frac{1498}{177147} - \frac{33636}{531441}\frac{\rho^{2/3}}{\rho_o^{2/3}}\right.$$

$$\left. + \frac{156925}{531441}\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{21694}{177147}\frac{\rho^2}{\rho_o^2} + \frac{31184}{531441}\frac{\rho^{8/3}}{\rho_o^{8/3}} + \frac{704}{177147}\frac{\rho^{10/3}}{\rho_o^{10/3}}\right] \quad (5.111)$$

Similarly, one can carry out the evaluation of the invariants I_2 and J_2. After a detailed calculation, which we will not reproduce here, it can be shown that these two invariants vanish for the metric under consideration.

5.5.2 INVARIANTS THAT INVOLVE THE ELECTROMAGNETIC FIELD TENSOR

The electromagnetic field tensor $F_{\mu\nu}$ can be expressed in terms of the four-vector potential A_μ as

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \tag{5.112}$$

A comma denotes partial differentiation.

The electromagnetic four-potential A_μ is given by

$$(A_o, A_1, A_2, A_3) = (\phi, 0, 0, \chi)$$

where ϕ and χ are respectively the electric and magnetic potentials. We have numbered the co-ordinants as follows

$$(x^o, x^1, x^2, x^3) = (t, \rho, z, \theta)$$

From Equation (5.112), and (5.113) below

$$F^{\mu\nu} = g^{\mu\rho}g^{\nu\sigma}F_{\rho\sigma} \tag{5.113}$$

the covariant and contravariant electromagnetic tensor can be calculated and non-zero components of the covariants electromagnetic fields are Islam (1983)

$$F_{o1} = - F_{1o} = \phi_\rho$$

$$F_{13} = - F_{31} = - \chi_\rho$$

and the contravariants electromagnetic components are

$$F^{o1} = - F^{1O} = \bar{\rho}^{2}\, \bar{e}^{\mu}(k\chi_{\rho} - \ell\phi_{\rho})$$

$$F^{13} = - F^{31} = - \bar{\rho}^{2}\, \bar{e}^{\mu}(k\phi_{\rho} + f\chi_{\rho}) = 0$$

the other components being zero.

So that the curvature invariants of exterior solution

$$F^{\mu\nu}F_{\mu\nu} = - (2b^{2}\sigma/\lambda)\rho^{-8/9} \exp(\tfrac{9}{2}\, b^{2}\sigma\rho^{2/3}) \qquad (5.114)$$

See Islam (1983).

We have $\tilde{F}^{\mu\nu} = \frac{1}{2\sqrt{-g}}\, \varepsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$

So curvature invariants $\tilde{F}^{\mu\nu}F_{\mu\nu}$ can be calculated as

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = (\tilde{F}^{o\nu}F_{o\nu} + \tilde{F}^{1\nu}F_{1\nu} + \tilde{F}^{2\nu}F_{2\nu} + \tilde{F}^{3\nu}F_{3\nu})$$

$$= 2(\tilde{F}^{o1}F_{o1} + \tilde{F}^{13}F_{12}) = 0 \qquad (5.115)$$

where $\tilde{F}^{13} = \frac{1}{2}\,\frac{\varepsilon^{13\lambda\sigma}}{\sqrt{-g}}\, F_{\lambda\sigma} = \frac{1}{2\sqrt{-g}}(\varepsilon^{1302}F_{o2} + \varepsilon^{1320}F_{2O})$

$$= - \frac{1}{\sqrt{-g}}\, F_{o2} = 0$$

Similarly $\tilde{F}^{o1} = \frac{1}{\sqrt{-g}}\, F_{23} = 0$

where $F_{o2} = \phi_z = 0 \qquad F_{23} = -\chi_z = 0$

we do not need other $\tilde{F}^{\mu\nu}$ for the above calculation.

Next we consider the invariant L_1.

After some manipulation, it can be shown that

$$L_1 = R^{\lambda\mu\nu\sigma}F_{\lambda\mu}F_{\nu\sigma}$$

$$= F_{o1}(k^{11}F_{o1} + k^{11}F_{o1} + k^{15}F_{31} + k^{15}F_{31})$$

$$+ F_{o1}(k^{11}F_{o1} + k^{11}F_{o1} + k^{15}F_{31} + K^{15}F_{31})$$

$$+ F_{31}(k^{15}F_{o1} + k^{15}F_{o1} + k^{55}F_{13} + k^{55}F_{13})$$

$$+ F_{31}(k^{15}F_{o1} + k^{15}F_{o1} + k^{55}F_{31} + k^{55}F_{31})$$

$$= 4F_{o1}(k^{11}F_{o1} + k^{15}F_{31}) + 4F_{31}(k^{15}F_{o1} + k^{55}F_{13})$$

$$= 4\{\phi_\rho(k^{11}\phi_\rho + k^{15}\chi_\rho) + \chi_\rho(k^{15}\phi_\rho - k^{55}\chi_\rho)\}$$

$$= 4\{k^{11}\phi_\rho^2 + 2k^{15}\phi_\rho\chi_\rho - k^{55}\chi_\rho^2\}$$

$$= 4\{k^{11}(b^2\sigma^2\bar{\rho}^{-2/3}) + 2k^{15}(-b\sigma\bar{\rho}^{-1/3})(-\tfrac{3}{2}\frac{a_o b}{\sigma}\rho^{1/3}$$

$$- k_o b\sigma\bar{\rho}^{-1/3}) - k^{55}(\tfrac{9}{8}\frac{a_o^4}{\sigma^5}\rho^{2/3} - \tfrac{1}{3}\frac{a_o^2}{\sigma^3} + \tfrac{2}{81}\frac{\bar{\rho}^{-2/3}}{\sigma})\}$$

$$= 4\{k^{11}(b^2\sigma^2\bar{\rho}^{-2/3}) + 2k^{15}(\frac{3}{2}\,a_o b^2 + k_o b^2\sigma^2\bar{\rho}^{-2/3})$$

$$- k^{55}(\frac{9}{8}\,\frac{a_o^4}{\sigma^5}\,\rho^{2/3} - \frac{1}{3}\,\frac{a_o^2}{\sigma^3}\,\frac{2}{81}\,\frac{\bar{\rho}^{-2/3}}{\sigma})\} \qquad (5.116)$$

Calculating three factors individually using Islam (1983) and putting in (5.116) we get the result

$$(1)\quad k^{11}(\frac{a_o^2}{2\sigma}\,\bar{\rho}^{-2/3}) = \frac{1}{e^{2\mu_\Delta 2}}\Big[\frac{71}{81}\,\frac{\rho^{4/3}}{\sigma} + \frac{2}{27}\,k_o^2\sigma\rho^{2/3}$$

$$+ \frac{2}{3}\,\frac{a_o^2}{\sigma^3}\,\rho^2\,\frac{27}{16}\,\frac{a_o^4}{\sigma^5}\,\rho^{8/3}\Big] \quad \frac{a_o^2}{2\sigma}\,\bar{\rho}^{-2/3}$$

$$= \frac{1}{e^{2\mu_\Delta 2}}\Big[\frac{71}{162}\,\frac{a_o^2}{\sigma^2}\,\rho^{2/3} + \frac{1}{27}\,a_o^2 k_o^2 + \frac{1}{3}\,\frac{a_o^4}{\sigma^4}\,\rho^{4/3}$$

$$+ \frac{27}{32}\,\frac{a_o^6}{\sigma^6}\,\rho^2\Big]$$

$$= \frac{1}{e^{2\mu_\Delta 2}}\Big[\frac{284}{2187}\,\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{4}{2187}\,\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{16}{729}\,\frac{\rho^2}{\rho_o^2}\Big]$$

$$(2)\quad 2k^{15}(\frac{3}{4}\,\frac{a_o^3}{\sigma^3} - \frac{1}{9}\,\frac{a_o}{\sigma}\,\bar{\rho}^{-2/3}) = -\frac{2}{e^{2\mu_\Delta 2}}\Big[-\frac{13}{6}\,\frac{a_o}{\rho}\,\rho^{4/3}$$

$$- \frac{28}{27}\,k_o\sigma\rho^{2/3} - \frac{3}{2}\,\frac{a_o^3}{\sigma^3}\,\rho^2\Big](\frac{3}{4}\,\frac{a_o^3}{\sigma^3} - \frac{1}{9}\,\frac{a_o}{\sigma}\,\bar{\rho}^{-2/3})$$

$$= -\frac{2}{e^{2\mu_\Delta 2}}\Big[-\frac{13}{8}\,\frac{a_o^4}{\sigma^4}\,\rho^{4/3} - \frac{7}{9}\,\frac{a_o^3 k_o}{\sigma^2}\,\rho^{2/3}$$

$$- \frac{9}{8}\,\frac{a_o^6}{\sigma^6}\,\rho^2 + \frac{13}{54}\,\frac{a_o^2}{\sigma^2}\,\rho^{2/3} + \frac{28}{27 \times 9}\,a_o k_o + \frac{1}{6}\,\frac{a_o^4}{\sigma^4}\,\rho^{4/3}\Big]$$

$$= -\frac{2}{e^{2\mu_\Delta 2}}\Big[-\frac{13}{8}\frac{a_o^4}{\sigma^4}\rho^{4/3} + \frac{14}{81}\frac{a_o^2}{\sigma^2}\rho^{2/3} - \frac{9}{8}\frac{a_o^6}{\sigma^6}\rho^2$$

$$+ \frac{13}{54}\frac{a_o^2}{\sigma^2}\rho^{2/3} - \frac{56}{2187} + \frac{1}{6}\frac{a_o^4}{\sigma^4}\rho^{4/3}\Big]$$

$$= -\frac{2}{e^{2\mu_\Delta 2}}\Big[\frac{104}{729}\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{680}{6561}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{64}{6561}\frac{\rho^2}{\rho_o^2} - \frac{56}{6561} + \frac{32}{6561}\frac{\rho^{4/3}}{\rho_o^{4/3}}\Big]$$

(3) $$k^{55}\Big(\frac{9}{8}\frac{a_o^4}{\sigma^5}\rho^{2/3} - \frac{1}{3}\frac{a_o^2}{\sigma^3} + \frac{2}{81}\frac{\rho^{-2/3}}{\sigma}\Big) = \frac{1}{e^{2\mu_\Delta 2}}\Big(\frac{2}{27}\sigma\rho^{2/3}$$

$$+ \frac{1}{4}\frac{a_o^2}{\sigma}\rho^{4/3}\Big)\Big(\frac{9}{8}\frac{a_o^4}{\sigma^5}\rho^{2/3} - \frac{1}{3}\frac{a_o^2}{\sigma^3} + \frac{2}{81}\frac{\rho^{-2/3}}{\sigma}\Big)$$

$$= \frac{1}{e^{2\mu_\Delta 2}}\Big(\frac{1}{12}\frac{a_o^4}{\sigma^4}\rho^{4/3} - \frac{2}{81}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{4}{729}$$

$$+ \frac{9}{32}\frac{a_o^6}{\sigma^6}\rho^2 - \frac{1}{12}\frac{a_o^4}{\sigma^4}\rho^{4/3} + \frac{1}{162}\frac{a_o^2}{\sigma^2}\rho^{2/3}\Big)$$

$$= \frac{1}{e^{2\mu_\Delta 2}}\Big(-\frac{3}{162}\frac{a_o^2}{\sigma^2}\rho^{2/3} + \frac{4}{2187} + \frac{9}{32}\frac{a_o^6}{\sigma^6}\rho^2\Big)$$

$$= \frac{1}{e^{2\mu_\Delta 2}}\Big(-\frac{4}{729}\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{4}{2187} + \frac{16}{2187}\frac{\rho^2}{\rho_o^2}\Big)$$

Putting the above expression (1), (2) and (3) in Equation (5.116) we get the following

$$R^{\lambda\mu\nu\sigma}F_{\lambda\mu}F_{\nu\sigma} = \frac{4}{e^{2\mu_\Delta 2}}\Big[\frac{112}{2187} - \frac{240}{2187}\frac{\rho^{2/3}}{\rho_o^{2/3}} + \frac{624}{2187}\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{160}{2187}\frac{\rho^2}{\rho_o^2}\Big]$$

(5.117)

Similarly after a detailed calculation it can be shown that the invariant L_2 vanishes.

So

$$L_2 = R^{\mu\nu\lambda\sigma} F_{\mu\nu} \tilde{F}_{\lambda\sigma} = 0 \tag{5.118}$$

Lastly we consider M. It can be shown after detailed calculation that

$$M = R_{\lambda\mu\nu k} R^{\nu k\alpha\beta} F^{\lambda\mu} F_{\alpha\beta}$$

$$= 8(R_{o1o1} R^{o1o1} F^{o1} F_{o1} + R_{o113} R^{13o1} F^{o1} F_{o1}$$

$$+ R_{o113} R^{13o1} F^{13} F_{13}$$

$$+ R_{o1o1} R^{o113} F^{o1} F_{13} + R_{o113} R^{1313} F^{o1} F_{13} R_{13o1} R^{o1o1} F^{13} F_{o1}$$

$$+ R_{1313} R^{13o1} F^{13} F_{o1}$$

$$+ R_{1313} R^{1313} F^{13} F_{13})$$

$$= 8\{(K_{11} K^{11} + K_{15} K^{15}) F^{o1} F_{o1} + (K_{11} K^{15}$$

$$+ K_{15} K^{55}) F^{o1} F_{13}\}$$

$$= 8\Big[\{\frac{1}{e^{2}\mu_{\Delta}{}^{2}}(-\frac{280}{2187} - \frac{8}{729}k_o^2\sigma^2\bar{\rho}^{-2/3} + \frac{55}{162}\frac{a_o^2}{\sigma^2}\rho^{2/3}$$

$$+ \frac{1}{12}\frac{a_o^4}{\sigma^4}\rho^{4/3} + \frac{27}{32}\frac{a_o^6}{\sigma^6}\rho^2 - \frac{16}{27}\frac{a_o^2}{\sigma^2}\rho^{2/3}$$

$$+ \frac{268}{2187} + \frac{101}{48}\frac{a_o^4}{\sigma^4}\rho^{4/3} - \frac{108}{729}k_o^2\sigma^2\bar{\rho}^{-2/3} + \frac{27}{16}\frac{a_o^6}{\sigma^6}\rho^2)\}F^{ol}F_{ol}$$

$$+ \{(-\frac{4}{27}\sigma\bar{\rho}^{-4/3} + \frac{1}{2}\frac{a_o^2}{\sigma}\bar{\rho}^{-2/3}) \times \frac{1}{e^{2}\mu_{\Delta}{}^{2}}(\frac{13}{6}\frac{a_o}{\sigma}\rho^{4/3}$$

$$+ \frac{28}{27}k_o\sigma\rho^{2/3} + \frac{3}{2}\frac{a_o^3}{\sigma^3}\rho^2)$$

$$\div (-\frac{2}{9}\frac{a_o}{\sigma}\bar{\rho}^{-2/3} - \frac{4}{27}k_o\sigma\bar{\rho}^{-4/3} + \frac{9}{8}\frac{a_o^3}{\sigma^3})$$

$$\times \frac{1}{e^{2}\mu_{\Delta}{}^{2}}(\frac{2}{27}\sigma\rho^{2/3} + \frac{1}{4}\frac{a_o^2}{\sigma}\rho^{4/3})\}F^{ol}F_{13}\Big]$$

$$= \frac{8}{e^{2}\mu_{\Delta}{}^{2}}\Big[\{-\frac{12}{2187} - \frac{116}{729}k_o^2\sigma^2\bar{\rho}^{-2/3} - \frac{41}{162}\frac{a_o^2}{\sigma^2}\rho^{2/3}$$

$$+ \frac{105}{48}\frac{a_o^4}{\sigma^4}\rho^{4/3} + \frac{81}{32}\frac{a_o^6}{\sigma^6}\rho^2\}$$

$$\times \frac{1}{e^{\mu}\Delta}(-\frac{4}{27}\frac{\rho^{2/3}}{\rho_o^{2/3}}) + \{-\frac{108}{243}a_o - \frac{120}{27\times 27}k_o\sigma^2\bar{\rho}^{-2/3}$$

$$+ \frac{8}{9}\frac{a_o^3}{\sigma^2}\rho^{2/3} + \frac{33}{32}\frac{a_o^5}{\sigma^4}\rho^{4/3}\}$$

$$\times \frac{8}{243}\frac{1}{e^{\mu}\Delta}\frac{1}{a_o}(2\frac{\rho^{4/3}}{\rho_o^{4/3}} + \frac{\rho^{2/3}}{\rho_o^{2/3}})\Big]$$

$$= \frac{8}{e^{3\mu_{\Delta}3}}\left[\frac{472}{59049} - \frac{336}{59049}\frac{\rho^{2/3}}{\rho_o^{2/3}} - \frac{560}{59049}\frac{\rho^{4/3}}{\rho_o^{4/3}} - \frac{480}{59049}\frac{\rho^{2}}{\rho_o^{2}} - \frac{224}{59049}\frac{\rho^{8/3}}{\rho_o^{8/3}}\right] \qquad (5.119)$$

5.6 PROPERTIES OF INVARIANTS

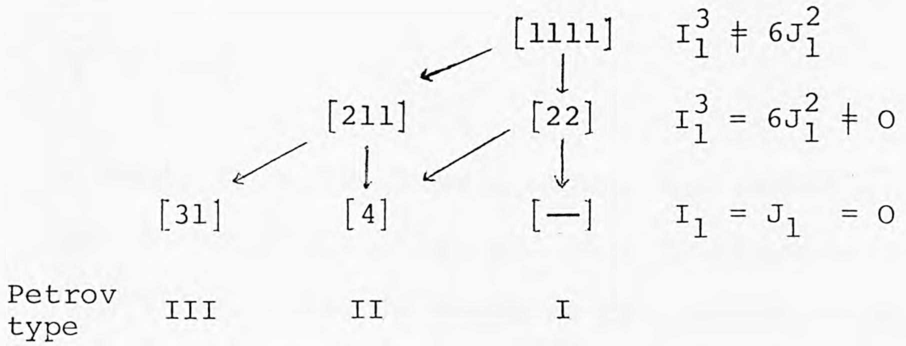
As noticed by Islam (1983) the invariant $F_{\mu\nu}F^{\mu\nu}$ tends to infinity as ρ tends to infinity. However, as noticed by Islam (1983), the spatial distance from any finite value of ρ to $\rho = \infty$ along the curve θ = constant, z = constant is finite. Whatever the situation of the surface (or point) $\rho = \infty$, the fact that $F_{\mu\nu}F^{\mu\nu}$ tends to infinity, indicates the presence of sources there.

We note that all the other invariants have the form of a polynomial in $\rho^{1/3}$ or $\rho^{-1/3}$ multiplied by a negative power of $\rho^2 e^{\mu}$. Recall the expression for e^{μ}, which

$$e^{\mu} = \lambda\rho^{-4/9} \exp(- \frac{9}{2} b^2\sigma\rho^{2/3}) \qquad (5.120)$$

Thus a negative power of e^{μ} dominates the asymptotic behaviour as $\rho \to \infty$ and so <u>all</u> the invariants tend to infinity as $\rho \to \infty$. This confirms Islam's analysis that these may be present sources at $\rho = \infty$.

As regards Petrov classification, Petrov (1969), Pirani (1957), consider the following diagram (Penrose (1960)):



(5.121)

The brackets [22] etc refer to the coincidence of the principal directions of the Riemann tensor, which are defined in terms of the intersections of certain planes which are determined by 'eigenbivectors' of $R_{\mu\nu\rho\sigma}$, i.e, from the non-zero (complex) skew tensors $x^{\mu\nu}$ which satisfy a relation

$$R^{\mu\nu}{}_{\rho\sigma}x^{\rho\sigma} = \alpha x^{\mu\nu} \tag{5.122}$$

(Penrose (1960)). [22], for example means that the four principal directions coincide in pairs. I_1, J_1 refer to the invariants given in page 69.

Coming to the invariants calculated in this chater, we see that

$$I_1^3 = \frac{64}{e^{6\mu}\Delta^6}\{(\frac{106}{2187})^3 \frac{\rho_o^2}{\rho^2} + \ldots\} \tag{5.123}$$

$$J_1^2 = \frac{64}{e^{6\mu}\Delta^6}\{\frac{(89)^2}{(59049)^2}\frac{\rho_o^{8/3}}{\rho^{8/3}} + \ldots\} \tag{5.124}$$

The dots indicate terms which are of higher powers. Thus we see that in this case we have

$$I_1^3 \neq 6J_1^2 \tag{5.125}$$

So that, from the above diagram, the solution here is type I and [1111], that is, with distinct principal directions. This is known as the general type.

It is hoped that the curvature invariants calculated here can be used for other purposes in studying the properties of Islam's solution and indeed, as an example of curvature invariants of an Einstein-Maxwell solution.

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