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This article examines how excess verdicts affect the insurance industry and studies insurance contract design from the policyholder's perspective, focusing on cases where court awards exceed policy limits. Excess verdicts refer to court decisions that grant compensation higher than the maximum coverage stated in an insurance policy. They are increasingly common in severe liability cases such as wrongful death claims and create both financial and legal risks for insurers and policyholders. These risks lead to uncertainty in premiums, solvency management, and overall risk control within the insurance market. To address these issues, we develop a mathematical framework that models excess verdicts by separating loss levels, legal outcomes, and contractual terms that specify coverage beyond standard policy limits. The framework applies value-at-risk (VaR) and conditional value-at-risk (CVaR) within a premium principle to capture the trade-off between risk exposure and cost in a manageable form. This approach provides a structured way to study how insurers and policyholders can share risks more efficiently when facing large and unpredictable legal awards. The results show that insurance contracts with multiple layers of indemnity can improve financial stability and fairness by distributing losses across different levels of coverage. Layered contracts reduce legal disputes, support balanced cost-sharing between insurers and policyholders, and give both sides clearer expectations about loss coverage. In practice, this structure helps insurers maintain solvency under extreme outcomes while offering policyholders more certainty about compensation in severe claim situations. The study provides a quantitative basis for designing more stable and transparent insurance products that can handle the growing problem of excess verdicts in modern markets.

1. INTRODUCTION

Excess verdicts, also called "nuclear verdicts," happen when a court awards damages far beyond an insurance policy's limit. For example, if a policy covers $1 million but the court awards $10 million, the extra $9 million is an excess verdict. These large awards, common in serious injury or wrongful death cases, create financial strain for both insurers and policyholders. Though they may seem similar to operational risks like system failures or internal mistakes, the two differ in cause and how they are managed. Operational risks come from within the organization and are usually predictable through models and scenario analysis (Power 2005; Amin 2016). Excess verdicts, however, arise from external legal actions and depend on unpredictable factors such as jury decisions and court interpretation. This makes them especially challenging in layered insurance, where excess coverage only starts after the primary policy is used up. Unlike primary coverage, excess insurance often lacks clear rules for handling large claims (Richmond 2000), and sometimes, settling below the primary limit shifts full liability to the excess layer, increasing financial risk and solvency concerns (O'Connor 2003).

In recent years, various legal doctrines, social trends, and plaintiff strategies have driven both the frequency and size of excess verdicts upward. The landmark 1977 case of *Bates v. State Bar of Arizona* raised awareness of litigation rights and expanded noneconomic damages such as pain and suffering (Supreme Court Of The United States 1976; Sharma 2023), while tort reform efforts have not succeeded in curbing these amounts (Heaton and Lucas 2000). Evolving tort doctrines, notably those involving "bad faith" and "negligence," have increased insurer exposure. "Bad faith" typically refers to fraudulent or dishonest conduct by the insurer (Epps and Chappell 1958), while "negligence" captures failures in reasonable claims handling. Courts may find bad faith where insurer actions are objectively unreasonable (Gallogly 2006), often leading to elevated

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settlements and penalties (Asmat and Tennyson 2014). On the plaintiff side, strategies such as the reptile theory (Murray et al. 2020), which appeals to jurors' concern for community safety (Silverman and Appel 2023), and the anchoring effect of high initial damage demands (Chang et al. 2015) have contributed to rising award levels. Broader economic and social factors also play a role; social inflation, where claims costs outpace general inflation, has placed additional pressure on settlements and verdicts (Pain 2020). In some jurisdictions, courts have shown increasing willingness to impose insurer liability beyond policy limits in cases involving severe harm or delayed settlements, and litigation involving bad faith claims continues to grow (see Appendix Table D.1; Deng and Zanjani 2018). High litigation costs and uncertain outcomes frequently push parties to settle early to avoid lengthy disputes, high legal fees, and delayed recovery (Shavell 1982; Cooter and Rubinfeld 1989).

To address the risks posed by excess verdicts, this study proposes a contract framework that makes excess coverage explicit and easier to manage. We consider contracts where both the trigger and payment structure are agreed in advance, reducing disputes and clarifying the responsibilities of each party. Inspired by parametric insurance designs (Broberg 2020) and the trigger-based approach of Asimit et al. (2021), our framework introduces two sequential triggers. The first activates when the court award exceeds the primary policy limit, and the second determines the excess payment based on both the total award and the insurer's conduct. We derive the optimal indemnity function by minimizing a combination of the policyholder's capital requirement and the premium, using risk measures such as value-at-risk (VaR) and conditional value-at-risk (CVaR) and a general premium principle. The solution offers a simple and effective structure combining a fixed deductible with a cap tailored to the risk environment. This contract design enhances pricing transparency, mitigates moral hazard by clarifying each party's obligations, and facilitates policyholders' better understanding and comparison of coverage.

This contract design is part of a class of trigger-based indemnity contracts that operate in settings with multiple risks, where coverage depends on externally defined events rather than realized losses. Such triggers appear in index-linked insurance, catastrophe bonds, and other risk-linked securities. For example, Miranda and Vedenov (2001) showed that index-linked insurance can reduce weather-related crop losses and smooth income in developing countries. The foundations of optimal contract design were set by Borch (1960) with the expected value principle for reinsurance and by Arrow (1963) with stop loss contracts for risk averse insurers. Later work by Raviv (1979) on deductibles and coinsurance and by Young (1999) using Wang's premium principle extended these ideas. To measure risk more precisely, Cai and Tan (2007) introduced VaR and CVaR, and follow-up studies by Cai et al. (2008), Chi and Tan (2011, 2013), Asimit et al. (2013a, 2013b), and Cheung et al. (2015) found that layered contracts remain optimal under these measures. Frees and Valdez (1998) used copulas to build multivariate risk models, while Cummins et al. (2004) and Goodwin (1993) emphasized the importance of clear triggers in catastrophe and crop insurance. In more complex situations, coverage may depend on multiple events occurring in sequence, as often happens with excess verdicts.

While the proposed contract structure addresses cost allocation, behavioral factors such as background risk also influence risk preferences and insurance decisions. Gollier and Pratt (1996) introduced the concept of risk vulnerability, showing that external risks can heighten aversion to independent risks. Eeckhoudt et al. (1996) demonstrated that background wealth changes may increase risk aversion under certain conditions. Heaton and Lucas (2000) applied these ideas to portfolio decisions, finding that labor and business income risks affect asset allocations. More recently, Strobl (2022) showed that healthcare costs as background risk lead individuals to prefer safer investments, while low insurance literacy and reluctance to pay premiums also matter. These behavioral effects alter optimal contract design. For example, Lu et al. (2018) showed that when background risk becomes large relative to insurable risk, deductible contracts become optimal, a result confirmed by Chi and Wei (2018) under various correlation structures. Chi and Tan (2021) found that background risk reduces opportunities for inflated claims by affecting incentive compatibility, and Hinck and Steinorth (2023) showed that risk vulnerability, and loss-dependent background risk can increase insurance demand. This aligns with Hofmann et al. (2019), who found that limited liability and background risk can explain demand for excess coverage under negative correlations.

The rest of this article is organized as follows. Section 2 discusses key issues in optimal insurance contracts and presents our model. Section 3 explores excess verdict modeling within our framework. Section 4 presents numerical simulations that support the model. Finally, Section 5 summarizes our findings and suggests directions for future research. Additional details are in Appendix A, with full proofs in Appendix B.

2. OPTIMAL INSURANCE WITH MULTIPLE INDEMNITY ENVIRONMENTS

The concept of excess verdict insurance can be represented with four distinct and mutually exclusive environments, each of which depends on the progress of the legal proceedings and the conduct of the insurer. The first situation is where there is no loss. The second scenario corresponds to the case in which the damages awarded remain within the prescribed insurance limit. In the third scenario, the damages awarded exceed the limit specified in the insurance policy. However, the subsequent litigation does not reveal any bad faith or misconduct on the part of the insurer. The last scenario describes a situation where, after the compensation awarded exceeds the insurance limit, a further lawsuit also reveals bad faith from the insurer. This categorization, pertinent to excess verdict insurance, motivates the study of the broader conceptual framework of "multiple indemnity environments," a notion rigorously examined through the lens of Pareto optimal risk-sharing in the work by Asimit et al. (2021). In the following sections, we state and solve an optimal insurance problem where the indemnity function depends on the prevailing environment. This discussion encompasses the problem's definition, optimization using VaR and CVaR, and further analysis through the proportional hazard transform.

2.1. Problem Definition

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which all random variables are defined. We consider a one-period economy where a primary risk holder is endowed with a nonnegative loss X that is payable at a fixed future time $T > 0$. We denote by $\mathbb{E}$ the expectation under $\mathbb{P}$.

The primary risk holder, or (insurance) buyer, intends to share the loss at time T with another party, or (insurance) seller, and accepts to pay a premium at time 0. Both parties agree to achieve optimality in terms of their risk positions by choosing appropriate amounts of indemnity and premium. However, unlike classical risk-sharing problems, this article considers a setting such that the indemnity level depends upon an external factor, which cannot be influenced by either party, yet can be precisely observed at time T.

To this end, let Y be the trigger characterizing the exogenous environment so that the sample space Ω is partitioned into $K+1$ disjoint events $\{\omega \in \Omega : Y(\omega) = k\}$, for $k = 0, 1, ..., K$, all with positive probability. Moreover, if $Y = 0$ then $X = 0$, implying that under the environment $Y = 0$ there is no loss. For each remaining environment $k = 1, ..., K$, the loss is risky, in the sense that $\mathbb{P}(X > 0 | Y = k) > 0$. Thus, we explicitly assume that the random variables X and Y are *not* independent.

If the realized environment is nonrisky, that is, $Y = 0$, no indemnity transfer is required. Moreover, if the prevailing environment is $Y = k$, for some $k = 1, ..., K$, the buyer will transfer the amount $I_k(X)$ to the seller at time T and retain the amount $R_k(X) = X - I_k(X)$, where $I_k : [0, \infty) \to \mathbb{R}$ is called an indemnity function and $R_k : [0, \infty) \to \mathbb{R}$ is called a retention function. Note that both parties have to agree at time 0 on a profile of indemnity functions $\boldsymbol{I} = (I_1, ..., I_K)$ since the exogenous environment is not revealed until time T.

A profile of indemnity functions is admissible if it belongs to the set

$$\mathcal{I} = \{\boldsymbol{I} : 0 \leq I_k \leq Id,\ R_k = Id - I_k,\ I_k \text{ and } R_k \text{ are non-decreasing for all } k = 1, ..., K\},$$

where Id denotes the identity function. Hence, under each environment, the indemnity is at most the loss, and misrepresentation of the loss is disincentivized, precluding ex post moral hazard from both parties, as suggested by Huberman et al. (1983). Note that the functions I_k and R_k are 1-Lipschitz continuous. We refer to a tuple $\boldsymbol{I} \in \mathcal{I}$ as a *contract*.

For each contract $\boldsymbol{I} \in \mathcal{I}$, we let $R_Y(X) = \sum_{k=1}^{K} R_k(X) \mathbb{1}_{\{Y=k\}}$ and $I_Y(X) = \sum_{k=1}^{K} I_k(X) \mathbb{1}_{\{Y=k\}}$. The realized risk position of the buyer is given by

$$\mathbf{B}(\boldsymbol{I}) = R_Y(X) + (1+\rho) P_g(I_Y(X)), \tag{2.1}$$

where $\mathbb{1}_A$ is the indicator function of an event $A \subset \Omega$. On the right-hand side of Equation (2.1), the first component is the loss retained by the buyer, which depends on the prevailing environment. The second term is the seller's premium, calculated with Wang's (risk-adjusted) premium principle P_g and inflated by the explicit safety load $\rho \geq 0$. For any loss $Z \geq 0$, $P_g(Z)$ is defined as

$$P_g(Z) = \int_0^\infty g(S_Z(z)) dz, \tag{2.2}$$

where $g : [0, 1] \to [0, 1]$ is a distortion function, that is, a nondecreasing concave function with $g(0) = 0$, $g(1) = 1$, and S_Z is the survival function of Z.

Let φ denote the buyer's risk measure, designed to rank their risk preferences at time $t = 0$. Formally, φ is a real function defined on a linear space of losses containing the constants. We assume φ to be translation invariant and monotone, therefore ensuring consistency in the evaluation of risk positions with respect to capital injections. With this in mind, the buyer's risk position at $t = 0$ corresponding to Equation (2.1) can be expressed as

$$\mathcal{R}_\varphi(\boldsymbol{I}) = \varphi(\mathbf{B}(\boldsymbol{I})) = \varphi(R_Y(X)) + (1+\rho) P_g(I_Y(X)), \tag{2.3}$$

2.2. Optimality with VaR and CVaR Preferences

In this section, we assume that the risk preferences φ of the buyer are represented by either VaR or CVaR. Then $\varphi = \text{VaR}$ or $\varphi = \text{CVaR}$.

Recall that for a loss Z, the VaR at level $\alpha \in (0, 1)$ is

$$\text{VaR}_\alpha(Z) = \inf\{z \in \mathbb{R} : \mathbb{P}(Z > z) \leq 1 - \alpha\}.$$

The solvency probability α is associated with the buyer's risk tolerance level.

The conditional VaR at level $\alpha \in (0, 1)$ is

$$\text{CVaR}_\alpha(Z) = \frac{1}{1-\alpha}\int_\alpha^1 \text{VaR}_s(Z)\ ds.$$

The CVaR is alternatively called the expected shortfall and has gained practitioners' interest since the introduction of Basel III regulations; see McNeil et al. (2015) for further discussion.

The buyer seeks to minimize his/her risk position at time $t = 0$, given by Equation (2.3), over all admissible indemnity profiles. The buyer's minimization problem is then given by

$$\min_{\boldsymbol{I} \in \mathcal{I}} \mathcal{R}_\varphi(\boldsymbol{I}). \tag{2.4}$$

Consider now the following subset of admissible indemnity profiles

$$\begin{aligned}\mathcal{I}^* = \{&\mathbf{I} \in \mathcal{I} : \text{for each } k = 1, ..., K, \text{ there exist } m_k \in [0, \text{ess sup}(X)], \\ &\text{and } n_k \in [m_k, \text{ess sup}(X)], \text{ such that } I_k(x) = (x - m_k)_+ - (x - n_k)_+\}.\end{aligned}$$

where $(x)_+ = \max\{x, 0\}$ and ess $\sup(X)$ denotes the essential supremum of X. Each profile $\tilde{\boldsymbol{I}} = (\tilde{I}_1, ..., \tilde{I}_K) \in \mathcal{I}^*$ represents a layer-type contract that provides indemnity for losses between a deductible m_k and an upper limit n_k, specific to each environment k.

The buyer may wish to restrict attention to the subclass $\mathcal{I}^*$, reducing the infinite-dimensional optimization problem (2.4) to a finite-dimensional one. In general, such a restriction may lead to suboptimal solutions. However, the next result shows that, under specific risk preferences, this restriction is without loss of generality.

Theorem 2.2.1. *Let $\varphi = \text{VaR}_\alpha$ or $\varphi = \text{CVaR}_\alpha$. For any $\rho \geq 0$ and any indemnity profile $\boldsymbol{I} \in \mathcal{I}$, there exists a layer-type profile $\tilde{\boldsymbol{I}} \in \mathcal{I}^*$ such that $\mathcal{R}_\varphi(\tilde{\boldsymbol{I}}) \leq \mathcal{R}_\varphi(\boldsymbol{I})$. Furthermore, $\tilde{\boldsymbol{I}}$ can be chosen so that the deductible levels are the same across environments.*

The proof of Theorem 2.2.1 is provided in Appendix B.1 for the case $\varphi = \text{VaR}_\alpha$, and in Appendix B.2 for the case $\varphi = \text{CVaR}_\alpha$.

Remark 2.2.1. *Theorem 2.2.1 establishes that for widely used risk measures such as VaR and CVaR, the class of layer-type indemnity profiles $\mathcal{I}^*$ is sufficient to attain optimality. That is, any admissible indemnity profile is dominated by one in $\mathcal{I}^*$. This structural insight allows one to replace the infinite-dimensional space of feasible contracts with a finite-dimensional subset parametrized by the deductible and upper limit in each environment.*

Theorem 2.2.1 also shows, as a by-product, that the deductible m_k can be taken to be the same in each environment, that is, $m_1 = \cdots = m_K$. This means that there is no particular benefit in having an environment-specific cutoff level above which losses are transferred to the seller. The left tails of the conditional loss distribution jointly concur in determining a unique optimal deductible level. As the buyer is concerned mostly with large losses, it is, however, essential to have the freedom of setting upper limits contingent on the prevailing scenario. Note that the inclusion of a deductible in the optimal insurance contract is consistent with the existing related literature, see for instance, Arrow (1974) *and* Ghossoub (2017).

Theorem 2.2.1 also has practical value. It justifies restricting the search for an optimal indemnity to the subclass $\mathcal{I}^*$ when performing numerical optimization. Although the objective function $\mathcal{R}_\varphi$ is generally nonconvex, the reduced dimensionality of the feasible set makes computational approaches tractable. We illustrate this in Section 4.

The following result strengthens Theorem 2.2.1 by establishing that a strict improvement can be achieved whenever the initial indemnity profile is not of layer type. In particular, if the minimum of Equation (2.4) exists, it is attained in $\mathcal{I}^*$, and optimal contracts may be selected from the class of layer-type profiles.

Corollary 2.2.1. *Let $\varphi = \mathrm{VaR}_\alpha$ or $\varphi = \mathrm{CVaR}_\alpha$, and suppose $\boldsymbol{I} \in \mathcal{I} - \mathcal{I}^*$. If the support of X is an interval in every environment, then there exists a layer-type profile $\tilde{\boldsymbol{I}} \in \mathcal{I}^*$ such that $\mathcal{R}_\varphi(\tilde{\boldsymbol{I}}) < \mathcal{R}_\varphi(\boldsymbol{I})$.*

Appendix B.3 provides a proof of Corollary 2.2.1 for $\varphi = \mathrm{VaR}_\alpha$ based on the argument in Theorem 2.2.1 from Appendix B.1. The case $\varphi = \mathrm{CVaR}_\alpha$ follows by the same reasoning, using the results in Appendix B.2. Details of the proof are omitted for brevity.

Following from the structural result of Theorem 2.2.1 that the optimal contract can be chosen to be of a layer type, we now examine the conditions that characterize the optimal deductible and upper limits. We restrict ourselves, for the sake of simplicity, to the case of CVaR_α risk preferences and of expected value premium principle.

Proposition 2.2.1. *Let $\varphi = \mathrm{CVaR}_\alpha$, and $g(x) = x,\ x \in [0, 1]$ in Equation (2.2). The optimal indemnity profile $I^* \in \mathcal{I}^*$ is characterized by a deductible $m^* \geq 0$ and upper limits $n_k^* \geq m^*$ for $k = 1, ..., K$.*

i. *(No insurance) If $\rho > \frac{\alpha}{1-\alpha}$, then $m^* = n_k^*$ for all k.*
ii. *If $\mathbb{P}(X > 0) > \frac{1}{1+\rho}$, then $m^* > 0$ or $n_k^* = m^* = 0$ for at least one k.*

Remark 2.2.2. *Case (i) implies that if the loading rate is excessively high (insurance too expensive), or the solvency probability is not too high (lenient capital requirement), then the buyer will react by retaining all the risk in every environment.*

Case (ii) holds when the (unconditional) probability of no loss is limited. In this situation, the optimal contract features either a positive deductible or at least one environment with no insurance. Note that this case holds whenever $X > 0$ almost surely (this requires both $\mathbb{P}(Y = 0) = 0$ and $X > 0 | Y = k$ almost surely in every environment).

2.3. Pareto Optimal Contracts

The problem considered in Equation (2.3) focuses on minimizing the buyer's risk, accounting for the premium determined by the seller. We show in this section that the solution of Equation (2.3) can be framed as a Pareto optimal contract. The proof is omitted, as similar arguments are developed in the Pareto insurance literature, for example, in Ghossoub et al. (2022).

A contract is a pair $C = (\boldsymbol{I}, \pi) \in \mathcal{I} \times \mathbb{R}$, where π denotes the premium paid by the buyer. The buyer's risk position at time $t = 0$ is given by

$$\Phi(C) := \varphi(R_Y(X) + \pi), \tag{2.5}$$

where φ is as before. The seller evaluates their risk exposure using a distortion risk measure

$$\Psi_g(C) := P_g(c(I_Y(X)) - \pi), \tag{2.6}$$

where g is a distortion function, see Equation (2.2), and $c : [0, +\infty) \to [0, +\infty)$ is an increasing and convex cost function.

A contract $C = (\boldsymbol{I}, \pi) \in \mathcal{I} \times \mathbb{R}$ is Pareto optimal if it cannot be strictly improved by another contract, when the targets of the buyer and seller are given by Φ and Ψ_g, respectively. It can be shown that a contract is Pareto optimal if and only if it solves, for some $\lambda \geq 0$, the constrained optimization problem

$$\min_{C \in \mathcal{I} \times \mathbb{R}} \Phi(C) \quad \text{subject to} \quad \Psi_g(C) = \lambda. \tag{2.7}$$

This formulation seeks to minimize the buyer's risk in Equation (2.5) subject to a fixed level of seller exposure in Equation (2.6), where λ reflects the seller's required profit or surplus. In particular, when $c(x) = (1 + \rho)x$ and $\lambda = 0$, that is, the seller is exactly compensated for their risk exposure, the constraint in Equation (2.7) reduces to $\pi = (1 + \rho)P_g(I_Y(X))$ and the Pareto optimization problem in Equation (2.7) reduces to the buyer's problem in Equation (2.3).

2.4. Optimality with the Proportional Hazard Transform

In this section, we consider a modification of the premium principle employed in Equation (2.1). Specifically, we allow for the distortion function used to calculate the premium to be dependent on the prevailing risk environment, which can therefore be assessed assuming a different degree of risk aversion. Of particular interest is the case of *proportional hazard (PH) transform* (see Wang (1995)), which features a power distortion function with the power coefficient depending on the prevailing environment and featuring a scenario-specific buyer's risk aversion. Thus, for any given indemnity profile $\boldsymbol{I} \in \mathcal{I}$, the realized risk position of the buyer can be expressed as

$$\mathbf{B}(\boldsymbol{I}) = R_Y(X) + \sum_{k=1}^{K} P_{g_k}\big(I_k(X)\mathbb{1}_{\{Y=k\}}\big), \tag{2.8}$$

where the second term on the right-hand side is the seller's risk-adjusted premium. Analogously, with respect to Equation (2.8), the buyer's risk position at $t = 0$ is articulated as

$$\varphi(\mathbf{B}(\boldsymbol{I})) = \varphi(R_Y(X)) + \sum_{k=1}^{K} P_{g_k}\big(I_k(X)\mathbb{1}_{\{Y=k\}}\big) \tag{2.9}$$

The main theorem in Section 2.2 still works if the objective function in Equation (2.4) is replaced by Equation (2.9). The proof is omitted as it is similar to that in the main Theorem (2.2.1). The Pareto interpretation of the optimal buyer's indemnity seen in Section 2.3 can also be extended to this setting with a suitable modification of the seller's distortion risk measure.

3. EXCESS VERDICTS

This section builds on the theoretical framework in Section 2 by applying it to the problem of excess verdicts. We aim to model how these outcomes arise by examining the joint decisions made by insurance buyers and sellers, particularly in the presence of gaps between court awarded damages and policy coverage. The focus is on identifying the main triggering factors and understanding how external conditions influence these outcomes, particularly in legal systems where insurance contracts may fall short in covering unexpected losses.

Suppose a loss, denoted by X, occurs due to an external event, such as injury or property damage. Let L be the loss amount that triggers the policy limit. When $X \le L$, the loss is handled according to the contract terms, with both buyer and seller sharing the cost. When $X > L$, the amount above L is generally the buyer's responsibility.

As in Section 2, let Y be a variable that represents how the legal process unfolds, especially in relation to excess verdicts and the seller's conduct. The flowchart in Figure C.1 helps illustrate the different cases: $Y = 0$ means no loss; $Y = 1$ means a legal claim is made, but the awarded damages stay within the insurance limit ($X \le L$), so excess verdict does not exist; $Y = 2$ means $X > L$, so there is excess verdict, but no finding of bad faith by the seller; and $Y = 3$ means there is both an excess verdict and a court decision that the seller acted in bad faith. Clearly, in both cases $Y = 2$ and $Y = 3$, the loss exceeds the policy limit.

Note that we do not consider the possibility of bankruptcy or solvency constraints for either the buyer or seller. Also, the legal process works in two steps: first, whether there is an excess verdict, and second, whether the seller is found to have acted in bad faith. In Sections 3.1 and 3.2, we look at contracts with and without provisions that depend on the legal outcome and compare their effects.

3.1. Contract Without Environment Contingent Provisions

Let $\hat{I}$ be the indemnity function and define the retention as $\hat{R}(X) = X - \hat{I}(X)$. We consider a contract that does not include any special rules for how to handle excess verdicts. The contract terms apply in scenarios $Y = 1$ and $Y = 2$. In case $Y = 2$, where the loss exceeds the policy limit ($X > L$), the seller pays only up to the limit, so $\hat{I}(X) = \hat{I}(L)$. The buyer then covers the rest, with total liability equal to $\hat{R}(X) = \hat{R}(L) + (X - L)$.

In case $Y = 3$, when the loss exceeds L, the court will decide how to split the payment between the buyer and the seller, which may require a lengthy legal process. Let $\hat{I}^c(X)$ be the seller's payment as decided by the court, and $\hat{R}^c(X) = X - \hat{I}^c(X)$ be the buyer's share. These court-ordered amounts differ from the original contract terms. Also, the longer the legal process, the greater is the financial and emotional stress faced by the plaintiff. Therefore, the loss X in case $Y = 3$ tends to be larger than in cases $Y = 1$ or $Y = 2$, and the seller's actual obligation $\hat{I}^c(X)$ is usually much higher than the agreed indemnity $\hat{I}(X) = \hat{I}(L)$.

3.2. Contract With Environment Contingent Provisions

Now consider a contract that includes special terms for handling excess verdicts, especially when bad faith by the seller is confirmed after the verdict. The goal of such a provision is to reduce the uncertainty and length of legal disputes related to excess losses.

For cases $Y=1$ and $Y=2$, the contract works similarly to the one in Section 3.1, with indemnity functions I_1, I_2 and retention functions R_1, R_2. In case $Y=2$, where $X>L$, the buyer pays the excess: $R_2(X)=R_2(L)+(X-L)$. In the excess verdict case $Y=3$, where the seller's bad faith is found and $X>L$, the contract uses a new indemnity rule I_3 and retention R_3 that were agreed in advance. Including these provisions affects how losses are handled in all three cases ($Y=1,2,3$), and the limit L may be different from that in Section 3.1.

Based on the results in Section 2.2, the optimal contract in each case will share a common deductible m and have an upper limit n_i for each scenario $i=1,2,3$. In normal situations ($Y=1$ or $Y=2$), the indemnity can follow a common structure, but with different ranges of X. The limit L used in these cases corresponds to $n_1=n_2$. When $Y=3$, the seller agrees to pay for all losses up to a higher limit $\tilde{L}=n_3$, where $\tilde{L}>L$. Then the seller's indemnity is given by $I_3(X)=X-R_3(\tilde{L})$ if $X\le\tilde{L}$, and $I_3(X)=\tilde{L}-R_3(\tilde{L})$ if $X>\tilde{L}$. The buyer's retention is $R_3(X)=R_3(\tilde{L})$ when $X\le\tilde{L}$, and $R_3(X)=R_3(\tilde{L})+(X-\tilde{L})$ when $X>\tilde{L}$. Table 1 shows how the buyer and seller payments differ depending on whether the contract includes these environment-contingent provisions.

4. NUMERICAL OPTIMIZATION ANALYSIS

In this section, we provide some numerical examples, focusing on the role played by risk aversion.

4.1. Model Parameters Setting

We consider three risk environments, each representing a different legal outcome. In each case, the loss (in thousands of monetary units) follows a Type II Pareto distribution. Table 2 summarizes the scenario probabilities and distribution parameters. The expected loss and standard deviation increase across the scenarios, with the third environment representing the highest and most uncertain losses, such as those from excess verdicts.

TABLE 1
Payments of the Buyer and Seller in the Contract with/without Environment Contingent Provisions across Different Environments

Environment	Party	Without provisions	With provisions
$Y=1$	Buyer	$\hat{R}(X)$	$R_1(X)$
	Seller	$\hat{I}(X)$	$I_1(X)$
$Y=2$	Buyer	$\hat{R}(X)=\hat{R}(L)+(X-L)$	$R_2(X)=R_2(L)+(X-L)$
	Seller	$\hat{I}(X)=\hat{I}(L)$	$I_2(X)=I_2(L)$
$Y=3$	Buyer	$\hat{R}^c(X)$	$R_3(\tilde{L})+(X-\tilde{L})_+$
	Seller	$\hat{I}^c(X)$	$I_3(X)=X-R_3(\tilde{L})-(X-\tilde{L})_+$

TABLE 2
Risk Environment Parameters and Their Statistical Properties

Risk environment	$\mathbb{P}(Y=k)$	λ	α	$\mathbb{E}[X\|Y=k]$	$SD[X\|Y=k]$
$Y=1$	60%	40	5	10	12.91
$Y=2$	30%	200	3	100	173.21
$Y=3$	10%	1,500	2.5	1,000	2236.07

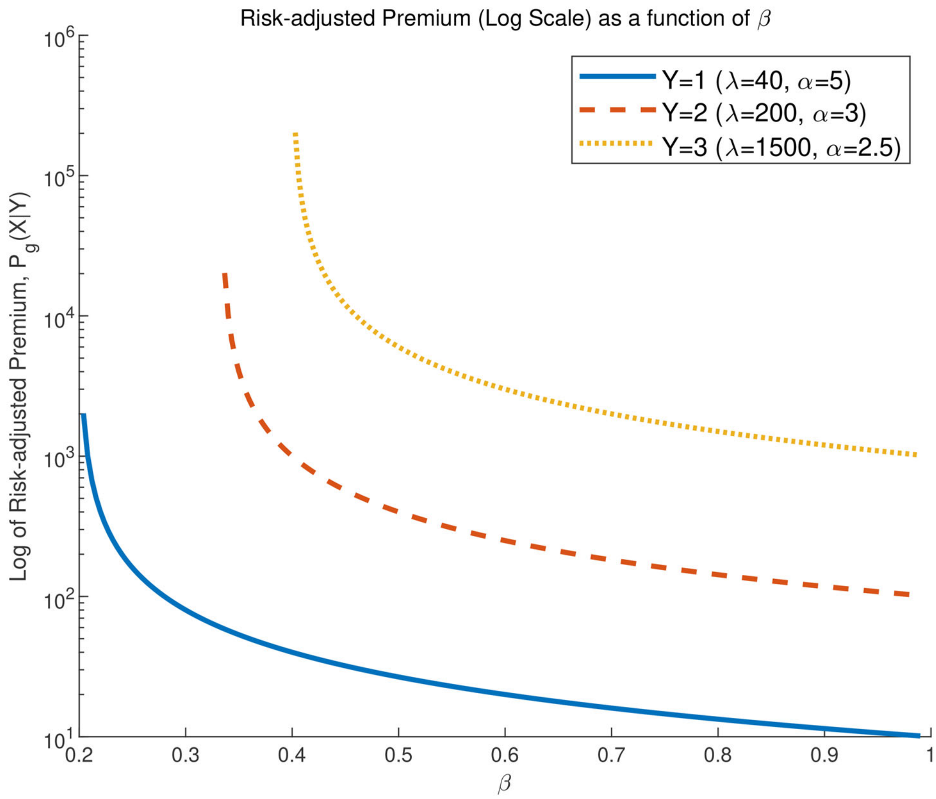

FIGURE 1. Risk-Adjusted Premium $P_g(X)$, on a Log-Scale, with $g(z) = z^\beta$ for Different Value of β and Environment.

We use $\mathrm{CVaR}_{95\%}$ as the risk measure and apply the PH transformation (introduced in Section 2.4) to adjust risk premiums in the presence of heavy-tailed losses. Figure 1 shows how the premium for full insurance changes with the distortion parameter β in the function $g(z) = z^\beta$. Lower values of β make the distortion more concave, leading to higher premiums.

4.2. Results Analysis

This section presents numerical results for the optimal contracts described in Section 2, and in particular Section 2.4. According to Theorem 2.2.1, these contracts can be chosen to feature a common deductible $m = m_1 = m_2 = m_3$ and environment-specific limits n_1, n_2, n_3. We exploit the finite-dimensional nature of the problem to find the optimal contract by minimizing, using standard numerical optimization tools, the objective function over the parameters m, n_1, n_2, n_3. We analyze the interplay between the distortion parameters $\beta_1, \beta_2, \beta_3$ and the policyholder coverage preferences across different risk scenarios.

We use as baseline values for the distortion coefficients $\beta_1 = 0.65$, $\beta_2 = 0.55$, and $\beta_3 = 0.45$, so that the risk aversion and the loading increase with the riskiness of the scenario. Tables 3, 4, and 5, respectively, show the loss quantiles, conditional on each environment, corresponding to the deductible m and the limits n_1, n_2, n_3, as each of the distortion coefficients is separately stressed. These values show the chance of not exceeding the deductible (the loss is fully paid by the buyer) and of exceeding the scenario-specific limit (the coverage is exhausted).

When β_k increases, the risk aversion in scenario k decreases, making insurance cheaper and leading to more generous coverage, that is, lower deductible and higher upper limit. However, changes in β_k mostly affect coverage in scenario k and have limited impact on the limits in the other scenarios. In scenario k, the limit rises quickly with β_k, and eventually full insurance is offered above the deductible, similar to classic optimal insurance results. This pattern also appears in the high-risk case $Y = 3$, although the pricing rule must embody a much-limited distortion before full insurance is attained.

The coverage structure depends strongly on the scenario. The common deductible is high enough that, in the low-risk case $Y = 1$, insurance only starts to pay for large losses, but almost all losses beyond that are covered. In contrast, in the high-risk case $Y = 3$, very extreme losses are not covered unless the buyer has low risk aversion in that scenario.

TABLE 3
CDF at the Deductible and Survival Function at the Upper Limit, Conditional on Each Scenario, for Different Values of β_1

	Risk environment Y_1		Risk environment Y_2		Risk environment Y_3	
β_1	$F_X(m)$	$S_X(n_1)$	$F_X(m)$	$S_X(n_2)$	$F_X(m)$	$S_X(n_3)$
0.45	93.13%	0.72%	32.80%	0.43%	4.57%	4.31%
0.55	89.44%	0.21%	27.58%	0.43%	3.69%	4.31%
0.65	85.40%	0.03%	23.60%	0.43%	3.06%	4.31%
0.75	81.19%	0.00%	20.48%	0.43%	2.60%	4.31%
0.85	76.93%	0.00%	17.95%	0.43%	2.24%	4.31%
0.95	72.71%	0.00%	15.87%	0.43%	1.95%	4.31%

TABLE 4
CDF at the Deductible and Survival Function at the Upper Limit, Conditional on Each Scenario, for Different Values of β_2

	Risk environment Y_1		Risk environment Y_2		Risk environment Y_3	
β_2	$F_X(m)$	$S_X(n_1)$	$F_X(m)$	$S_X(n_2)$	$F_X(m)$	$S_X(n_3)$
0.45	90.69%	0.032%	29.12%	1.44%	3.94%	4.31%
0.55	85.40%	0.032%	23.60%	0.43%	3.06%	4.31%
0.65	79.93%	0.032%	19.67%	0.06%	2.48%	4.31%
0.75	74.51%	0.032%	16.72%	0.00%	2.07%	4.31%
0.85	69.28%	0.00%	14.41%	0.00%	1.75%	4.31%
0.95	64.32%	0.00%	12.57%	0.00%	1.51%	4.31%

TABLE 5
CDF at the Deductible and Survival Function at the Upper Limit, Conditional on Each Scenario, for Different Values of β_3

	Risk environment Y_1		Risk environment Y_2		Risk environment Y_3	
β_3	$F_X(m)$	$S_X(n_1)$	$F_X(m)$	$S_X(n_2)$	$F_X(m)$	$S_X(n_3)$
0.45	85.40%	0.032%	23.60%	0.43%	3.06%	4.31%
0.55	78.72%	0.032%	18.95%	0.43%	2.38%	1.28%
0.65	72.53%	0.00%	15.80%	0.43%	1.94%	0.19%
0.75	67.09%	0.00%	13.56%	0.43%	1.64%	0.006%
0.85	62.45%	0.00%	11.94%	0.43%	1.43%	0.00002%
0.95	58.57%	0.00%	10.73%	0.43%	1.27%	0.00%

5. CONCLUSIONS AND FUTURE RESEARCH

In this article, we study the optimal insurance problem from the buyer's perspective in multiple indemnity environments, with a focus on the legal and financial effects of excess verdicts. Our model examines risk sharing between policyholders and insurers, especially when legal judgments lead to damages far beyond policy limits. The occurrence of excess verdicts, where court-mandated payments exceed the policyholder's coverage, shows the practical importance of our framework. Our analysis demonstrates that the optimal contract structure is a layered indemnity for each risk environment while keeping consistent deductibles across all environments. By using risk measures such as VaR and CVaR, we simplify complex optimization problems and improve the efficiency of numerical optimization and decision-making. This approach may improve risk sharing among parties and provide a practical framework for managing excess liability in real-world insurance cases.

While VaR and CVaR are widely used in the insurance industry, future research may consider alternative risk measures to gain further insight into risk sharing in different indemnity settings. A possible line of future research may explore whether

Theorem 2.2.1 holds true under a wide class of risk measures, including VaR and CVaR. In addition, evaluating the enforceability of anticipatory clauses across jurisdictions could enhance the legal strength of the excess verdict model. Empirical studies using real-world insurance data, particularly in cases involving excess verdicts, are important for validating our theoretical framework. These topics offer promising directions for future work.

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Supplemental material for this article can be accessed on the publisher's website at www.tandfonline.com/uaaj.

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APPENDIX A. ANCILLARY RESULTS

A.1. Left and Right Continuous Inverses

Given the role of left and right continuous inverse functions in the proof of the main result of this article, Theorem 2.2.1, we provide in this section their definitions and some of their properties.

Definition A.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. The right-continuous inverse of f is given by*

$$f^{-1+}(y) = \inf\{x \in \mathbb{R} : f(x) > y\},\ \ y \in \mathbb{R}.$$

The left-continuous inverse of $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f^{-1}(y) = \inf\{x \in \mathbb{R} : f(x) \geq y\},\ \ y \in \mathbb{R}.$$

In this definition, we use the convention that $\inf\emptyset = +\infty$.

The next result summarizes key properties of right-continuous functions and their right-continuous inverses. For detailed discussions and proofs related to left- and right-continuous inverses, see Embrechts and Hofert (2013).

Proposition A.1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Then, for a given $y \in \mathbb{R}$, the following properties hold:*

a. $f^{-1+}(y) = -\infty$ *if and only if* $f(x) > y$ *for all* $x \in \mathbb{R}$ *; further,* $f^{-1+}(y) = +\infty$ *if and only if* $f(x) \le y$ *for all* $x \in \mathbb{R}$.
b. *Assume f is right-continuous and* $f^{-1+}(y) < \infty$*, then* $f(f^{-1+}(y)) \ge y$ *; further, if f is continuous, then* $f(f^{-1+}(y)) = y$.
c. $x > f^{-1+}(y)$ *implies that* $f(x) > y$*, and the reverse implication holds if f is left-continuous; further,* $f(x) \le y$ *implies that* $x \le f^{-1+}(y)$*, and the reverse implication holds if f is left-continuous.*
d. *Let f be nondecreasing and right-continuous. Then* $f^{-1}(y) \le x$ *if and only if* $f(x) \ge y$*, for any* $(x, y) \in \mathbb{R}^2$.

APPENDIX B. PROOFS OF THE MAIN RESULTS

We now present the proof of our main results. For clarification, we focus on using Wang's premium principle. Although our provided numerical optimization is based on the risk-adjusted premium of the PH transform, the proof approaches are similar. Therefore, we do not provide a separate proof for the PH transform method.

Since our proof relies on some properties related to stochastic ordering, we recall the formal definition of stop-loss order and apply some related results; see Denuit et al. (2006), Rolski et al. (1999) and Shaked and Shanthikumar (2007).

B.1. Proof of Theorem 2.2.1 with VaR Preferences

Fix $\boldsymbol{I} \in \mathcal{I}$ and let $R_k = Id - I_k$. Define

$$b := \mathrm{VaR}_\alpha(R_Y). \tag{B.1}$$

For each $k = 1, ..., K$, let R_k^{-1+} denote the right-continuous inverse of R_k and note that $R_k(R_k^{-1+}(b)) = b$ provided $R_k^{-1+}(b) < +\infty$; see Proposition A.1.1b. In this case, it holds $b \le R_k^{-1+}(b)$ since $R_k + I_k = Id$. The same inequality holds if $R_k^{-1+}(b) = +\infty$. Consequently, define $m_k = b$, $n_k = R_k^{-1+}(b)$ and $\tilde{I}_k$ by

$$\tilde{I}_k(x) = (x - b)_+ - \left(x - R_k^{-1+}(b)\right)_+ = \begin{cases} 0 & \text{if } 0 \le x < b, \\ x - b & \text{if } b \le x \le R_k^{-1+}(b), \\ R_k^{-1+}(b) - b & \text{if } R_k^{-1+}(b) < x. \end{cases} \tag{B.2}$$

Note that the deductible $m_k = b$ is independent of the environment. It follows that

$$\tilde{R}_k(x) = Id(x) - \tilde{I}_k(x) = \begin{cases} x & \text{if } 0 \le x < b, \\ b & \text{if } b \le x \le R_k^{-1+}(b), \\ x - R_k^{-1+}(b) + b & \text{if } R_k^{-1+}(b) < x. \end{cases} \tag{B.3}$$

It is understood that in Equations (B.2) and (B.3), only the first two cases apply when $R_k^{-1+}(b) = +\infty$. The first step is to demonstrate that

$$\left\{R_k(X) > b\right\} = \left\{\tilde{R}_k(X) > b\right\} \quad \text{for any} \quad k = 1, ..., K. \tag{B.4}$$

According to Proposition A.1.1a, if $R_k^{-1+}(b) = +\infty$ then $R_k(x) \le b$ for all x. But it is seen from Equation (B.3) that the latter is equivalent to $\tilde{R}_k(x) \le b$ for all x. Therefore, we only need to consider the case when $R_k^{-1+}(b) < +\infty$. Suppose $R_k(x) > b$. According to Proposition A.1.1c, we deduce that $x > R_k^{-1+}(b)$ and from Equation (B.3), we obtain $\tilde{R}_k(x) > b$. Conversely, suppose $\tilde{R}_k(x) > b$. From Equation (B.3) it follows that $x > R_k^{-1+}(b)$ and Proposition A.1.1c implies that $R_k(x) > b$. Therefore, Equation (B.4) holds.

In the second step, we aim to prove that $\tilde{I}_k(x) \leq I_k(x)$ for all x, from which

$$\tilde{I}_k(X) \leq I_k(X) \quad \text{for any} \quad k = 1, ..., K. \tag{B.5}$$

We proceed by cases on the value of x, exploiting the form of $\tilde{I}_k(x)$ in Equation (B.2). For $0 \leq x < b$, we have $\tilde{I}_k(x) = 0 \leq I_k(x)$. For $b \leq x \leq R_k^{-1+}(b)$, we find $\tilde{I}_k(x) = x - b$. Therefore, $\tilde{I}_k(x) \leq I_k(x)$ if and only if $R_k(x) \leq b$. If $R_k^{-1+}(b) = +\infty$, this follows from Proposition A.1.1a. If instead $R_k^{-1+}(b) < +\infty$, then $R_k(x) \leq R_k(R_k^{-1+}(b)) = b$ by Proposition A.1.1b. Finally, assume $R_k^{-1+}(b) < x$, so that $R_k^{-1+}(b) < +\infty$ and we have $\tilde{I}_k(x) = R_k^{-1+}(b) - b$. Therefore, $\tilde{I}_k(x) \leq I_k(x)$ if and only if $R_k(x) \leq x - R_k^{-1+}(b) + b$. By the 1-Lipschitz-continuity of R_k, we have $0 \leq R_k(x) - R_k(R_k^{-1+}(b)) \leq x - R_k^{-1+}(b)$, from which the conclusion follows since $R_k(R_k^{-1+}(b)) = b$ by Proposition A.1.1b. Thus, Equation (B.5) is obtained.

Letting $\tilde{R}_Y(X) = \sum_{k=1}^K \tilde{R}_k(X)\mathbb{1}_{\{Y=k\}}$, the third step is to demonstrate

$$\mathrm{VaR}_\alpha(R_Y(X)) = \mathrm{VaR}_\alpha\big(\tilde{R}_Y(X)\big). \tag{B.6}$$

From Equation (B.5), we have $\tilde{R}_k(X) \geq R_k(X)$ for all $k = 1, ..., K$, which implies $\mathrm{VaR}_\alpha(\tilde{R}_Y(X)) \geq \mathrm{VaR}_\alpha(R_Y(X))$ by stochastic dominance. From Equation (B.4) we get $\mathbb{P}(R_k(X) > b) = \mathbb{P}(\tilde{R}_k(X) > b)$ for $k = 1, ..., K$. Consequently, we deduce $\mathbb{P}(\tilde{R}_Y(X) > b) = \mathbb{P}(R_Y(X) > b) \leq 1 - \alpha$, since, by definition, $b = \mathrm{VaR}_\alpha(R_Y(X))$. By Proposition A.1.1d, it follows that $\mathrm{VaR}_\alpha(\tilde{R}_Y(X)) \leq \mathrm{VaR}_\alpha(R_Y(X))$. Therefore, Equation (B.6) holds.

Let $\tilde{I}_Y(X) = \sum_{k=1}^K \tilde{I}_k(X)\mathbb{1}_{\{Y=k\}}$, and the last step establishes the inequality

$$P_g\big(\tilde{I}_Y\big) \leq P_g(I_Y). \tag{B.7}$$

Recall that $\tilde{I}_k(X) \leq I_k(X)$ for all $k = 1, ..., K$, which implies that $\tilde{I}_Y(X) \leq I_Y(X)$. Equation (B.7) follows as the distortion premium principle P_g with stochastic dominance. From this, $P_g(\tilde{I}_Y(X)) \leq P_g(I_Y(X))$ follows and Equation (B.7) holds, which completes the last step.

Finally, Equation (B.6), together with Equation (B.7) shows that, for any $\rho \geq 0$,

$$\mathcal{R}(\tilde{\boldsymbol{I}}) = \ \mathrm{VaR}_\alpha\big(\tilde{R}_Y\big) + (1+\rho)P_g\big(\tilde{I}_Y\big) \leq \mathrm{VaR}_\alpha(R_Y) + (1+\rho)P_g(I_Y) = \mathcal{R}(\boldsymbol{I}).$$

B.2. Proof of Theorem 2.2.1 with CVaR Preferences

Fix $\boldsymbol{I} \in \mathcal{I}$ and define b as in the Appendix B.1. For $k = 1, ..., K$, define $m_k = b$, and n_k should be a value which satisfies $n_k \geq R_k^{-1+}(b)$. Further, define $\tilde{I}_k$ by

$$\tilde{I}_k(x) = (x-b)_+ - (x-n_k)_+ = \begin{cases} 0 & \text{if } 0 \leq x < b, \\ x - b & \text{if } b \leq x \leq n_k, \\ n_k - b & \text{if } n_k < x, \end{cases} \tag{B.8}$$

and

$$\tilde{R}_k(x) = Id(x) - \tilde{I}_k(x) = \begin{cases} x & \text{if } 0 \leq x < b, \\ b & \text{if } b \leq x \leq n_k, \\ x - n_k + b & \text{if } n_k < x. \end{cases} \tag{B.9}$$

In the initial step, we confirm that there exists an $n_k \geq R_k^{-1+}(b)$ for which

$$\mathbb{E}\big[\big(\tilde{R}_k(X) - b\big)+\big] = \mathbb{E}\big[(R_k(X) - b)_+\big] \quad \text{holds for every} \quad k = 1, ..., K. \tag{B.10}$$

According to Proposition A.1.1a, if $n_k = R_k^{-1+}(b) = +\infty$, then $R_k(x) \leq b$ for all x, and also $\tilde{R}_k(x) \leq b$ for all x from Equation (B.9). Consequently, both sides of Equation (B.10) equate to zero. Given this, we restrict our attention only to the case $R_k^{-1+}(b) < +\infty$.

For $x \leq R_k^{-1+}(b)$, it follows from Proposition A.1.1c that $R_k(x) \leq b$, and from Equation (B.9), this leads to $\tilde{R}_k(x) \leq b$, which further means that

$$\mathbb{E}\left[\left(\tilde{R}_k(X)-b\right)_+ \mathbb{1}_{\{X \leq R_k^{-1+}(b)\}}\right] = \mathbb{E}\left[\left(R_k(X)-b\right)_+ 1_{\{X \leq R_k^{-1+}(b)\}}\right] = 0. \tag{B.11}$$

For $x > R_k^{-1+}(b)$, using the 1-Lipschitz-continuity of R_k, it follows that $0 \leq R_k(x) - R_k(R_k^{-1+}(b)) \leq x - R_k^{-1+}(b)$, and then by Proposition A.1.1b, we have $R_k(x) \leq x - R_k^{-1+}(b) + b$ for all $x > R_k^{-1+}(b)$. If we consider the case $R_k(x) = x - R_k^{-1+}(b) + b$ for all $x > R_k^{-1+}(b)$, which can be visualized in Figure 2, then we can choose $n_k = R_k^{-1+}(b)$ exactly, so $\tilde{R}_k(x) = x - R_k^{-1+}(b) + b$ holds for $x > n_k = R_k^{-1+}(b)$ from Equation (B.9), which implies that $R_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}} = \tilde{R}_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}$. This means $\mathbb{E}\left[(\tilde{R}_k(X)-b)_+ \mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right] = \mathbb{E}\left[(R_k(X)-b)_+ \mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right]$, which can be combined with Equation (B.11) to obtain that Equation (B.10) holds. If we consider the case $R_k(c_k) < c_k - R_k^{-1+}(b) + b$ for some $c_k > R_k^{-1+}(b)$, which can be visualized in Figure 3, then we can have $R_k^{-1+}(b) < c_k - R_k(c_k) + b$. Furthermore, we can suppose there exists a $n_k > R_k^{-1+}(b)$ such that $R_k(c_k) = c_k - n_k + b$ holds, which implies that $n_k = c_k - R_k(c_k) + b < c_k$ since $c_k > R_k^{-1+}(b)$ implies $R_k(c_k) > b$ from Proposition A.1.1c. Besides, we also have $\tilde{R}_k(c_k) = c_k - n_k + b$ from Equation (B.9) since $c_k > n_k$, so $R_k(c_k) = \tilde{R}_k(c_k)$ means that for $x > R_k^{-1+}(b)$, $R_k(x)$ and $\tilde{R}_k(x)$ can always intersect at a point with $x = c_k > R_k^{-1+}(b)$, and then there exist a $n_k > R_k^{-1+}(b)$ satisfying $n_k = c_k - R_k(c_k) + b$ such that, for $x \in (R_k^{-1+}(b), c_k]$, $\tilde{R}_k(x) \leq R_k(x) < x - R_k^{-1+}(b) + b$ is established, which further yields that

$$\mathbb{E}\left[R_k(X)\mathbb{1}_{\{R_k^{-1+}(b) < X \leq c_k\}}\right] \geq \mathbb{E}\left[\tilde{R}_k(X)\mathbb{1}_{\{R_k^{-1+}(b) < X \leq c_k\}}\right]. \tag{B.12}$$

Also, for $x > c_k$, the inequalities $R_k(x) \leq \tilde{R}_k(x) < x - R_k^{-1+}(b) + b$ are held, which further yields that

$$\mathbb{E}\left[R_k(X)\mathbb{1}_{\{X > c_k\}}\right] \leq \mathbb{E}\left[\tilde{R}_k(X)\mathbb{1}_{\{X > c_k\}}\right]. \tag{B.13}$$

Then we can define a function as $f(c_k) = \mathbb{E}\left[\tilde{R}_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right] - \mathbb{E}\left[R_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right]$ for $c_k > R_k^{-1+}(b)$. If $c_k \uparrow +\infty$, then $n_{k_{\max}} \uparrow c_k - R_k(c_k) + b$, and we can further obtain that $0 \leq \mathbb{E}\left[\tilde{R}_k(X)\mathbb{1}_{\{X > c_k\}}\right] - \mathbb{E}\left[R_k(X)\mathbb{1}_{\{X > c_k\}}\right] \to 0$, then combined with the inequality from Equation (B.12), this further means that

$$f(c_k) = \mathbb{E}\left[\tilde{R}_k(X)\mathbb{1}_{\{R_k^{-1+}(b) < X \leq c_k\}}\right] - \mathbb{E}\left[R_k(X)\mathbb{1}_{\{R_k^{-1+}(b) < X \leq c_k\}}\right] \leq 0 \tag{B.14}$$

If $c_k \downarrow R_k^{-1+}(b)$ then $n_k \downarrow R_k^{-1+}(b)$, and we can further deduce that $0 \leq \mathbb{E}\left[R_k(X)\mathbb{1}_{\{R_k^{-1+}(b) < X \leq c_k\}}\right] - \mathbb{E}\left[\tilde{R}_k(X)\mathbb{1}_{\{R_k^{-1+}(b) < X \leq c_k\}}\right] \to 0$; then combining this with the inequality from Equation (B.13) further implies that

$$f(c_k) = \mathbb{E}\left[\tilde{R}_k(X)\mathbb{1}_{\{X > c_k\}}\right] - \mathbb{E}\left[R_k(X)\mathbb{1}_{\{X > c_k\}}\right] \geq 0. \tag{B.15}$$

Furthermore, we know that $f(c_k)$ is nondecreasing and continuous for $c_k > R_k^{-1+}(b)$, so from Equations (B.14) and (B.15), it is evident that finding a suitable $c_{k_0} > R_k^{-1+}(b)$ to satisfy that $R_k(c_{k_0}) < c_{k_0} - R_k^{-1+}(b) + b$, then for $n_k > R_k^{-1+}(b)$, we can have that

$$f(c_{k_0}) = \mathbb{E}\left[\tilde{R}_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right] - \mathbb{E}\left[R_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right] = 0. \tag{B.16}$$

This further indicates that $\mathbb{E}\left[(\tilde{R}_k(X)-b)_+ \mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right] = \mathbb{E}\left[(R_k(X)-b)_+ \mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right]$ since $x > R_k^{-1+}(b)$, $R_k(x) > b$ from Proposition A.1.1c and $\tilde{R}_k(x) \geq b$ from Equation (B.9), which can be combined with Equation (B.11) to obtain that Equation (B.10) holds.

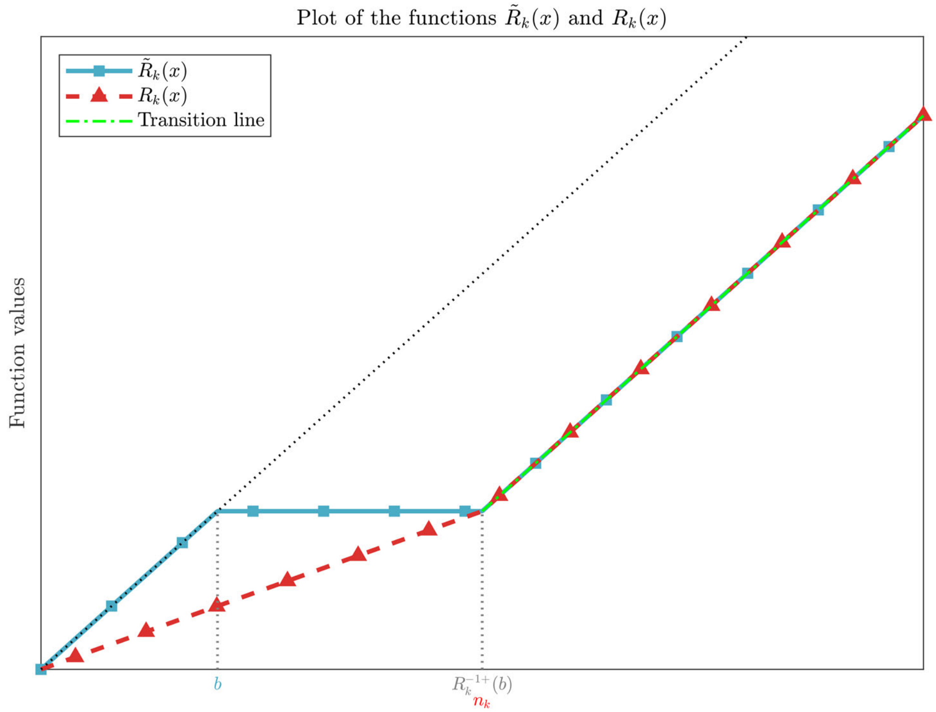

FIGURE 2. Construction of R_k under the Conditions for CVaR Risk Preference: $R_k(x) = x - R_k^{-1+}(b) + b$ for $x > R_k^{-1+}(b)$. A Linear R_k is Chosen for Graphical Convenience.

In our second step, we aim to demonstrate that

$$\tilde{I}_k(X) \le_{sl} I_k(X) \quad \text{for any} \quad k = 1, ..., K. \tag{B.17}$$

which means that $\tilde{I}_k(X)$ is smaller than $I_k(X)$ in stop-loss order based on Equation (B.10). We proceed by cases on the value of x.

For $0 \le x \le R_k^{-1+}(b) \le n_k$, by cross-referencing with our earlier derivations in Appendix B.1, we can confirm that $\tilde{I}_k(x) \le I_k(x)$ for all $x \le R_k^{-1+}(b)$. Taking this a step further, define $P(X) = \tilde{I}_k(X)\mathbb{1}_{\{X \le R_k^{-1+}(b)\}}$ and $Q(X) = I_k(X)\mathbb{1}_{\{X \le R_k^{-1+}(b)\}}$. So for any realization of X, we have $P(X) \le Q(X)$, which further leads to $P(X) \le_{sl} Q(X)$ from Theorem 3.2.1 in Rolski et al. (1999).

For $x > R_k^{-1+}(b)$, define $M(X) = \tilde{I}_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}$ and $N(X) = I_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}$. For the case $R_k(x) = x - R_k^{-1+}(b) + b$, we have $\tilde{R}_k(x) = R_k(x)$ for $x > R_k^{-1+}(b)$ from the first step, which mean that $\tilde{I}_k(x) = I_k(x)$ for $x > R_k^{-1+}(b)$ by using the identity $I_k = Id - R_k$; this further leads to $M(X) = N(X)$, and then we can get

$$\begin{aligned} \tilde{I}_k(X) = {} & \tilde{I}_k(X)\mathbb{1}_{\{X \le R_k^{-1+}(b)\}} + \tilde{I}_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}} = P(X) + M(X) = P(X) + N(X) \\ & \le Q(X) + N(X) = I_k(X)\mathbb{1}_{\{X \le R_k^{-1+}(b)\}} + I_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}} I_k(X) \end{aligned}$$

This implies that Equation (B.17) holds by applying Theorem 3.2.1 in Rolski et al. (1999). For the case $R_k(x) < x - R_k^{-1+}(b) + b$, then focusing on $x \in (R_k^{-1+}(b), c_k]$, we have $\tilde{R}_k(x) \le R_k(x)$ from the first step, which guarantees that $\tilde{I}_k(x) \ge I_k(x)$. This further implies that $\tilde{I}_k(X)\mathbb{1}_{\{R_k^{-1+}(b) < X \le c_k\}} \ge I_k(X)\mathbb{1}_{\{R_k^{-1+}(b) < X \le c_k\}}$, which can be $\tilde{I}_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}\mathbb{1}_{\{X \le c_k\}} \ge I_k(X)1_{\{X > R_k^{-1+}(b)\}}\mathbb{1}_{\{X \le c_k\}}$. So for any realization of X, we have $M(X) \cdot \mathbb{1}_{\{X \le c_k\}} \ge N(X) \cdot \mathbb{1}_{\{X \le c_k\}}$. Then this means that

$$\mathbb{P}(M(X) \le t, X \le c_k) \le \mathbb{P}(N(X) \le t, X \le c_k) \tag{B.18}$$

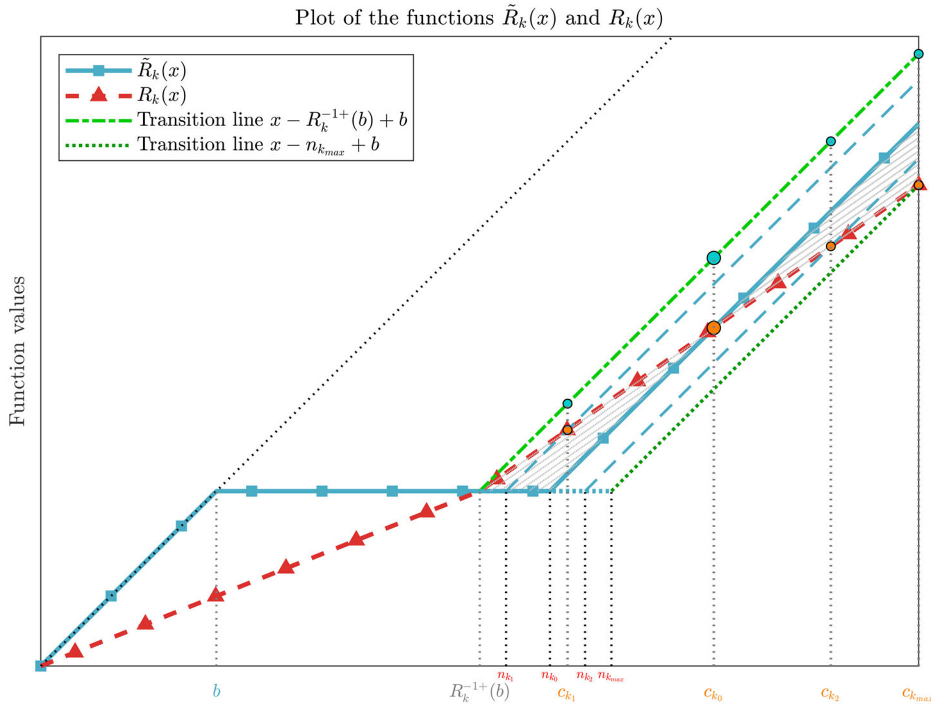

FIGURE 3. Construction of $\tilde{R}_k$ under the Conditions for CVaR Risk Preference: $\tilde{R}_k(x) < x - R_k^{-1+}(b) + b$ for $x > R_k^{-1+}(b)$. The Increase in c_k Leads to an Increase in the Lower Part of the Shaded Region, While the Upper Part of the Shaded Region Decreases, Which is Consistent with Our Proof. A Linear R_k is Chosen for Graphical Convenience.

Then for $x > c_k$, $R_k(x) \le \tilde{R}_k(x)$ in the initial step can lead to $I_k(x) \ge \tilde{I}_k(x)$. This further implies that $I_k(X)\mathbb{1}_{\{X>c_k\}} \ge \tilde{I}_k(X)\mathbb{1}_{\{X>c_k\}}$, which can be $I_k(X)\mathbb{1}_{\{X>R_k^{-1+}(b)\}}\mathbb{1}_{\{X>c_k\}} \ge \tilde{I}_k(X)\mathbb{1}_{\{X>R_k^{-1+}(b)\}}\mathbb{1}_{\{X>c_k\}}$. So for any realization of X, $N(X)\cdot\mathbb{1}_{\{X>c_k\}} \ge M(X)\cdot\mathbb{1}_{\{X>c_k\}}$ holds. Then this leads to

$$\mathbb{P}(N(X) \le t, X > c_k) \le \mathbb{P}(M(X) \le t, X > c_k). \tag{B.19}$$

Besides, from the definition of M and N, it is clear that $M(x) = N(x) = 0$ for $x \le R_k^{-1+}(b)$. For $x \in (R_k^{-1+}(b), c_k]$, we know that $\tilde{I}_k(x) \in (R_k^{-1+}(b) - b, n_k - b]$ from Equation (B.8), which implies that $M(x) \in (R_k^{-1+}(b) - b, n_k - b]$. Furthermore, we have $I_k(R_k^{-1+}(b)) = R_k^{-1+}(b) - R_k(R_k^{-1+}(b) = R_k^{-1+}(b) - b$ by using Proposition A.1.1b, and $I_k(c_k) = \tilde{I}_k(c_k) = n_k - b$ since $R_k(c_k) = \tilde{R}_k(c_k)$ from the first step with the identity $I_k = Id - R_k$, which implies that $I_k(x) \in (R_k^{-1+}(b) - b, n_k - b]$ for $x \in (R_k^{-1+}(b), c_k]$. So we obtain that $N(x) \in (R_k^{-1+}(b) - b, n_k - b]$ for $x \in (R_k^{-1+}(b), c_k]$. For $x > c_k$, we have $\tilde{I}_k(x) = n_k - b$ from Equation (B.8), which means that $M(x) = n_k - b$. We also have that $I_k(x) \ge \tilde{I}_k(x)$ for $x > c_k$, which means that $N(x) \ge n_k - b$ for $x > c_k$. Therefore, we can summarize M as

$$M(x) = \begin{cases} 0 & \text{if } x \le R_k^{-1+}(b), \\ \in \left(R_k^{-1+}(b) - b, n_k - b\right] & \text{if } x \in \left(R_k^{-1+}(b), c_k\right], \\ n_k - b & \text{if } x > c_k, \end{cases} \tag{B.20}$$

and N as

$$N(x) = \begin{cases} 0 & \text{if } x \le R_k^{-1+}(b), \\ \in \left(R_k^{-1+}(b) - b, n_k - b\right] & \text{if } x \in \left(R_k^{-1+}(b), c_k\right], \\ \ge n_k - b & \text{if } x > c_k. \end{cases} \tag{B.21}$$

Now, for $t < 0$, we have $F_{M(X)}(t) = F_{N(X)}(t) = 0$ from Equations (B.20) and (B.21). For $t \in [0, n_k - b)$, from Equations (B.20) and (B.21), we know that for $x > c_k$, $N(x) \geq n_k - b = M(x)$, this further leads to $\mathbb{P}(M(X) \leq t, X > c_k) = \mathbb{P}(N(X) \leq t, X > c_k) = 0$. Therefore, we have

$$\begin{aligned} F_{M(X)}(t) &= \mathbb{P}(M(X) \leq t, X \leq c_k) + \mathbb{P}(M(X) \leq t, X > c_k) = \mathbb{P}(M(X) \leq t, X \leq c_k) \\ &\leq \mathbb{P}(N(X) \leq t, X \leq c_k) \\ &= \mathbb{P}(N(X) \leq t, X \leq c_k) + \mathbb{P}(N(X) \leq t, X > c_k) = F_{N(X)}(t). \end{aligned}$$

The third inequality comes from Equation (B.18). For $t \geq n_k - b$, we have $F_{M(X)}(t) = 1$ since $M(x) \leq n_k - b$ from Equation (B.20), and $F_{N(X)}(t) \leq 1$ because from Equation (B.21), we have $N \geq n_k - b$ when $x > c_k$. Consequently, $F_N(t) \leq F_M(t)$ for $t \geq n_k - b$. So denoting $t_0 = n_k - b$, when $t < t_0$, we have the condition $F_{M(X)}(t) \leq F_{N(X)}(t)$, when $t \geq t_0$, $F_{N(X)}(t) \leq F_{M(X)}(t)$. Besides, Equation (B.16) and the identity $I_k = Id - R_k$ give that $\mathbb{E}\left[I_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right] = \mathbb{E}\left[\tilde{I}_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right]$, which means that $\mathbb{E}[M(X)] = \mathbb{E}[N(X)]$, and applying Theorem 3.2.4 in Rolski et al. (1999), we can assert that $M(X) \leq_{sl} N(X)$. Now, for any $d \geq 0$, we have

$$\begin{aligned} \mathbb{E}\left[\left(\tilde{I}_k(X) - d\right)_+\right] &= \mathbb{E}\left[\left(\tilde{I}_k(X) - d\right)_+ \mathbb{1}_{\{X \leq R_k^{-1+}(b)\}}\right] + \mathbb{E}\left[\left(\tilde{I}_k(X) - d\right)_+ \mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right] \\ &= \mathbb{E}\left[\left(\tilde{I}_k(X)\mathbb{1}_{\{X \leq R_k^{-1+}(b)\}} - d\right)_+\right] + \mathbb{E}\left[\left(\tilde{I}_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}} - d\right)_+\right] \\ &= \mathbb{E}\left[(P(X) - d)_+\right] + \mathbb{E}\left[(M(X) - d)_+\right] \\ &\leq \mathbb{E}\left[(Q(X) - d)_+\right] + \mathbb{E}\left[(N(X) - d)_+\right] \\ &= \mathbb{E}\left[\left(I_k(X)\mathbb{1}_{\{X \leq R_k^{-1+}(b)\}} - d\right)_+\right] + \mathbb{E}\left[\left(I_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}} - d\right)_+\right] \\ &= \mathbb{E}\left[(I_k(X) - d)_+ \mathbb{1}_{\{X \leq R_k^{-1+}(b)\}}\right] + \mathbb{E}\left[(I_k(X) - d)_+ \mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right] \\ &= \mathbb{E}\left[(I_k(X) - d)_+\right]. \end{aligned}$$

The justification for the fourth inequality is grounded in the established stop-loss orders $P(X) \leq_{sl} Q(X)$ and $M(X) \leq_{sl} N(X)$, coupled with the application of Theorem 3.2.2 in Rolski et al. (1999). Besides, $\tilde{I}_k(x) \leq I_k(x)$ for all $x \leq R_k^{-1+}(b)$ means $\mathbb{E}\left[\tilde{I}_k(X)\mathbb{1}_{\{X \leq R_k^{-1+}(b)\}}\right] \leq \mathbb{E}\left[I_k(X)\mathbb{1}_{\{X \leq R_k^{-1+}(b)\}}\right]$, and combined with the fact that $\mathbb{E}\left[\tilde{I}_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right] = \mathbb{E}\left[I_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right]$, we have for any $d < 0$ that

$$\begin{aligned} \mathbb{E}\left[\left(\tilde{I}_k(X) - d\right)_+\right] &= \mathbb{E}\left[\tilde{I}_k(X) - d\right] = \mathbb{E}\left[\tilde{I}_k(X)\right] - d \\ &= \left(\mathbb{E}\left[\tilde{I}_k(X)\mathbb{1}_{\{X \leq R_k^{-1+}(b)\}}\right] + \mathbb{E}\left[\tilde{I}_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right]\right) - d \\ &\leq \left(\mathbb{E}\left[I_k(X)\mathbb{1}_{\{X \leq R_k^{-1+}(b)\}}\right] + \mathbb{E}\left[I_k(X)\mathbb{1}_{\{X > R_k^{-1+}(b)\}}\right]\right) - d \\ &= \mathbb{E}[I_k(X)] - d = \mathbb{E}[I_k(X) - d] \\ &= \mathbb{E}\left[(I_k(X) - d)_+\right] \end{aligned}$$

Therefore, for all $d \in \mathbb{R}$, we have $\mathbb{E}\left[\left(\tilde{I}_k(X) - d\right)_+\right] \leq \mathbb{E}\left[(I_k(X) - d)_+\right]$. Then, by Theorem 3.2.2 in Rolski et al. (1999), Equation (B.17) follows, which completes the second part of the proof.

The third step demonstrates that

$$\mathrm{CVaR}_\alpha\left(\tilde{R}_Y\right) = \mathrm{CVaR}_\alpha(R_Y). \tag{B.22}$$

From Equation (B.17), we have $R_k(X) \leq_{sl} \tilde{R}_k(X)$ for all $k = 1, \ldots, K$, which implies $\mathrm{CVaR}_\alpha(\tilde{R}_Y(X)) \geq \mathrm{CVaR}_\alpha(R_Y(X))$ by stochastic dominance. Recalling Equation (B.10), we can deduce $\mathbb{E}\left[\left(\tilde{R}_Y(X) - b\right)_+\right] = \mathbb{E}\left[(R_Y(X) - b)_+\right]$. Utilizing the dual representation of CVaR by Rockafellar and Uryasev (2000), we have

$$\mathrm{CVaR}_\alpha(R_Y(X)) = \inf_{t\in\mathbb{R}}\left\{t + \frac{1}{1-\alpha}\mathbb{E}\left[(R_Y(X)-t)_+\right]\right\}.$$

where the infimum is achieved at $t^* = b$, yielding $\mathrm{CVaR}_\alpha(R_Y(X)) = b + \frac{1}{1-\alpha}\mathbb{E}\left[(R_Y(X)-b)_+\right]$. Therefore, we have

$$\begin{aligned}\mathrm{CVaR}_\alpha\left(\tilde{R}_Y(X)\right) &= \inf_{t\in\mathbb{R}}\left\{t + \frac{1}{1-\alpha}\mathbb{E}\left[\left(\tilde{R}_Y(X)-t\right)_+\right]\right\} \\ &\le b + \frac{1}{1-\alpha}\mathbb{E}\left[\left(\tilde{R}_Y(X)-b\right)_+\right] \\ &= b + \frac{1}{1-\alpha}\mathbb{E}\left[(R_Y(X)-b)_+\right] = \mathrm{CVaR}_\alpha(R_Y(X)).\end{aligned}$$

Thus, we establish Equation (B.22) and the third part of the proof is done.

In the final step, we aim to show that

$$P_g\left(\tilde{I}_Y\right) \le P_g(I_Y), \tag{B.23}$$

Recall Equation (B.17), which further implies $\tilde{I}_Y(X) \le_{sl} I_Y(X)$. Applying Theorem 1 in Wang (1996), we obtain $P_g(\tilde{I}_Y(X)) \le P_g(I_Y(X))$. Therefore, Equation (B.23) holds.

Thus, combining Equations (B.22) and (B.23) demonstrates that, for $\rho \ge 0$,

$$\mathcal{R}(\tilde{\boldsymbol{I}}) = \mathrm{CVaR}_\alpha\left(\tilde{R}_Y\right) + (1+\rho)P_g\left(\tilde{I}_Y\right) \le \mathrm{CVaR}_\alpha(R_Y) + (1+\rho)P_g(I_Y) = \mathcal{R}(\boldsymbol{I}).$$

B.3. Proof of Corollary 2.2.1 with VaR Preferences

Let $(I_1, ..., I_m) \in \mathcal{I} - \mathcal{I}^*$, and define $\tilde{I}_k$ as in Equation (B.8) for all k. Since $(I_1, ..., I_m)$ is not of layer type, there exists at least one index $k_0 \in \{1, ..., m\}$ such that $\mathbb{P}(\tilde{I}_{k_0}(X) < I_{k_0}(X)) > 0$. Since $\mathbb{P}(Y = k_0) > 0$ and $\tilde{I}_{k_0}(X) < I_{k_0}(X)$ on a set of positive probability, it follows that $\mathbb{P}(\tilde{I}_Y(X) < I_Y(X)) > 0$ and $\tilde{I}_Y(X) \le I_Y(X)$ almost surely. As a result, $I_Y(X)$ strictly dominates $\tilde{I}_Y(X)$ in the usual stochastic dominance order. Since the function g is non-decreasing and strictly concave on $(0, 1)$, we obtain

$$P_g\left(\tilde{I}_Y(X)\right) < P_g(I_Y(X)). \tag{B.24}$$

Combining Equations (B.6) and (B.24), we have for any $\rho > 0$,

$$\mathcal{R}_\varphi(\tilde{\boldsymbol{I}}) = \mathrm{VaR}_\alpha\left(\tilde{R}_Y(X)\right) + (1+\rho)P_g\left(\tilde{I}_Y(X)\right) < \mathrm{VaR}_\alpha(R_Y(X)) + (1+\rho)P_g(I_Y(X)) = \mathcal{R}_\varphi(\boldsymbol{I}),$$

B.4. Proof of Proposition 2.2.1.

As established in Theorem 2.2.1, the optimal contract can be found within the class $\mathcal{I}^*$. Under CVaR risk preferences and the expected premium principle, and exploiting the dual representation of CVaR by Rockafellar and Uryasev (2000), the optimal contract can be found by solving the problem

$$\min_{t, m, \{n_k\}}\left(t + \frac{1}{1-\alpha}\mathbb{E}\left[(R_Y(X)-t)_+\right] + (1+\rho)\mathbb{E}[I_Y(X)]\right),$$

subject to $m \ge 0$, $n_k \ge m$ for all k and $t \in \mathbb{R}$. The KKT first-order necessary conditions for an optimum $(t^*, m^*, \{n_k^*\})$ require the existence of multipliers μ_k for $k = 0, ..., K$, such that

$$
\begin{cases}
\mathbb{P}(R_Y^*(X) > t^*) = 1 - \alpha & \text{(B.25)} \\
-\dfrac{1}{1-\alpha}\mathbb{P}(R_Y^*(X) > t^*, X > m^*) + (1+\rho)\mathbb{P}(X > m^*) = \sum_{k=1}^{K}\mu_k - \mu_0 & \text{(B.26)} \\
-\dfrac{1}{1-\alpha}\mathbb{P}(R_k^*(X) > t^*, X > n_k^*, Y = k) + (1+\rho)\mathbb{P}(X > n_k^*, Y = k) = \mu_k, & \text{(B.27)} \\
\quad \text{for} \quad k = 1, ..., K & \\
\mu_0 m^* = 0 \quad \text{and} \quad \mu_k(m^* - n_k^*) = 0, \quad \text{for} \quad k = 1, ..., K & \text{(B.28)}
\end{cases}
$$

Proof of (i). Assume by contradiction that $n_j^* > m^*$ for at least one environment $1 \le k \le K$, so that $\mu_k = 0$, and Equation (B.27) for environment j gives, after rearranging,

$$\mathbb{P}(R_k^*(X) > t^* | X > n_k^*, Y = j) = (1+\rho)(1-\alpha), \tag{B.29}$$

a contradiction as $(1+\rho)(1-\alpha) > 1$.

Proof of (ii). Assume by contradiction that $m^* = 0 < n_k^*$ for all k. This requires that $\mu_k = 0$ for all k, so that Equation (B.26) becomes

$$\frac{1}{1-\alpha}\mathbb{P}(R_Y^*(X) > t^*, X > 0) - (1+\rho)\mathbb{P}(X > 0) = \mu_0.$$

Noting that, from Equation (B.25), $t^* \ge 0$, the latter equation can be simplified, again using Equation (B.25), into

$$1 - (1+\rho)\mathbb{P}(X > 0) = \mu_0 \ge 0,$$

which rearranged gives $\mathbb{P}(X > 0) \le \frac{1}{1+\rho}$, a contradiction.

APPENDIX C. FLOW CHARTS: COURT PROCESS

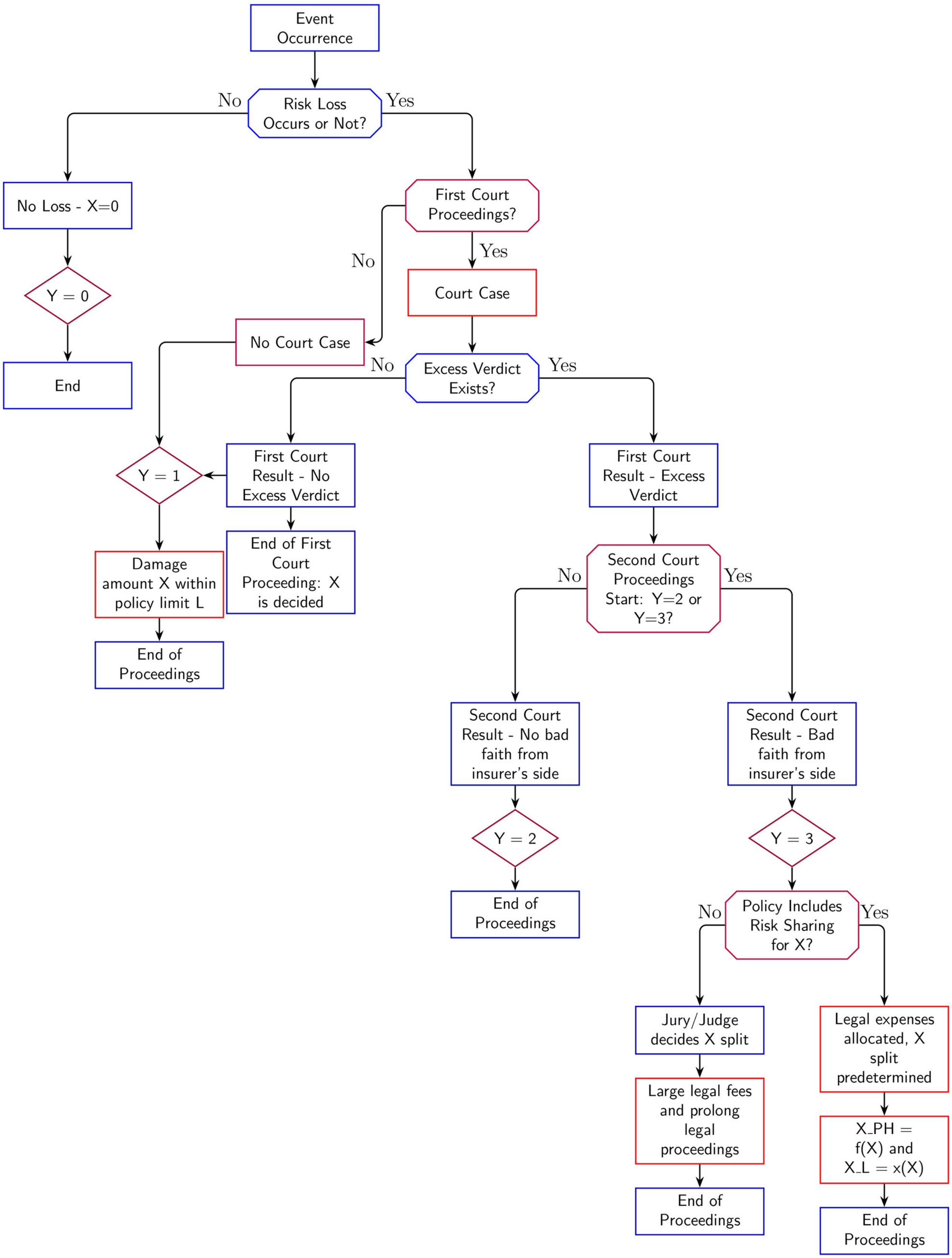

FIGURE C.1. Flowchart Illustrating the Stages of Legal Proceedings Concerning Insurance Claims and the Subsequent Apportionment of Liabilities Based on the Loss Threshold and Seller Conduct.

APPENDIX D. EXCESS VERDICT IN AUTO & LIABILITY INSURANCE

TABLE D.1
Wrongful Deaths Cases: Auto & Liability Insurance

Case	Event time	Court start time	Verdict time	Compensatory damage cost	Punitive damage cost	Total cost	Excess verdict trigger	Type of policy
Dock vs. McLendon et al.	Jan. 26, 2019	July 27, 2021	July 30, 2021	$66.5 million	N/A	$66.5 million	Motor Vehicle	Auto Insurance
Cargal vs. Forehand & FedEx	Sep. 8, 2018	Oct. 15, 2021	Oct. 24, 2021	$30,000,000	N/A	$30,000,000	Motor Vehicle	Auto Insurance
Godwin vs. Carroll & Eaton Asphalt Paving Co., Inc.	Jan. 9, 2019	July 12, 2021	N/A	$24,000,000	$50,000,000	$74,000,000	Motor Vehicle	Auto Insurance
Leslie vs. Rodriguez	May 1, 2017	March 6, 2020	N/A	1.82 million & $2.8 million	N/A	$4.62 million	Motor Vehicle	Auto Insurance
Pedro Pasillas-Sanchez vs. Consolidated Materials, Inc. & Lee	March 26, 2018	Nov. 13, 2020	N/A	$9,000,000	N/A	$9,000,000	Motor Vehicle	Auto Insurance
Ware vs. Home Opportunity, LLC, Ewing & Marchman	Oct. 2, 2016	Jan. 22, 2020	N/A	$9,689,948.18	N/A	$9,689,948.18	Premises Liability	Liability Insurance
Church & Austin vs. Case New Holland Industrial of America, LLC	March 2, 2016	Nov. 12, 2020	N/A	$3,000,000	$10,000,000	$13,000,000	Products Liability	Liability Insurance
Madere & Thomas vs. Greenwich Insurance Company et al.	July 18, 2016	Aug. 23, 2019	Aug. 28, 2019	$180,065,000	$100,000,000	$280,065,000	Negligence	Auto Insurance
Mayfield & Phillips vs. Kennison	April 10, 2006	Feb. 26, 2019	March 2, 2019	$33,413,000	N/A	$32,412,610 (after comparative negligence adjustment)	Motor Vehicle	Auto Insurance
Garmon vs. Jenkins and Atlas Excavating/Atlas Trucking	Sept. 7, 2012	Oct. 3, 2019	Oct. 10, 2019	$22,144,971.88	$10,000,000	$32,144,971.88	Negligent Hiring	Auto Insurance
Plascencia & Trujillo vs. Newcomb etc.	April 19, 2014	March 25, 2019	N/A	$30,000,000	N/A	$12,000,000 (after apportionment)	U-Turn	Auto Insurance
Willoughby vs. Ellison & 21st Century Centennial Insurance Company	Nov. 2, 2012	March 15, 2019	March 22, 2019	$30,101,599	N/A	$34,668,619	Passenger	Auto Insurance
Thornton vs. Ralston GA LLC d/b/a The Ralston etc.	July 6, 2017	July 1, 2019	July 7, 2019	$35,000,000	$50,000,000	$125,000,000	Negligent Repair	Liability Insurance
Enriquez, Jr., Martinez & Irene Gonzalez vs. Lasko Products, Inc.	Jan. 3, 2016	Nov. 21, 2019	Nov. 28, 2019	$36,240,000	N/A	$36,240,000	Manufacturing Defect	Liability Insurance
Johnson vs. Lee & Corrugated Replacements, Inc.	July 1, 2011	Sep. 14, 2018	Sep. 21, 2018	$128,813,522	N/A	$128,813,522	Motor Vehicle	Auto Insurance
Herrera & Sweeting vs. Extended Stay America, Inc., etc.	Nov. 12, 2014	Nov. 12, 2018	Nov. 20, 2018	$46,000,000	N/A	$41,400,000 (after apportionment)	Negligence	Liability Insurance
Barron vs. B & G Crane Service etc.	May 11, 2016	Sep. 13, 2018	N/A	$44,370,000	N/A	$20,791,235.34	Negligence	Liability Insurance
The Estate of Kari Dunn vs. OM Lodging LLC etc.	Dec. 1, 2013	June 22, 2018	June 26, 2018	$41,550,000	N/A	$2,400,000	Negligence	Liability Insurance

(*Continued*)

TABLE D.1
(*Continued*).

Case	Event time	Court start time	Verdict time	Compensatory damage cost	Punitive damage cost	Total cost	Excess verdict trigger	Type of policy
Anaya vs. Superior Industries Inc. et al.	Oct. 7, 2013	March 19, 2018	March 26, 2018	$30,000,000	N/A	$30,000,000	Negligence	Auto Insurance
Sitton et al. vs. Ceeda Enterprises, Inc.	March 28, 2016	July 17, 2018	July 19, 2018	$27,091,054	N/A	$27,091,054	Negligence	Auto Insurance
Dougherty & Forester vs. WCA of Florida, LLC	Oct. 28, 2016	Oct. 5, 2018	Oct. 10, 2018	$25,000,000	N/A	$20,000,000 (after 20% comparative negligence reduction)	Right Turn Motor Vehicle	Auto Insurance
Braswell vs. The Brickman Group Ltd, LLC. et al.	May 16, 2014	May 3, 2017	May 9, 2017	$39,960,000	N/A	$27,172,800 (after the reduction for comparative fault)	Motor Vehicle	Liability Insurance
Jester vs. Utilimap Corporation & Duke Energy Ohio, Inc.	Feb. 27, 2014	Jun. 7, 2017	Jun. 28, 2017	$27,871,944	N/A	$27,871,944	Negligent Training	Liability Insurance
Cruz et al. vs. Methenge et al.	Aug. 29, 2012	Jul. 21, 2017	Aug. 10, 2017	$24,931,109	N/A	$24,931,109	Design Defect	Auto Insurance
Angulo & Lopez vs. J. Calero et al.	May 28, 2015	Oct. 26, 2017	N/A	$20,000,000	$25,005,000	$45,005,000	Negligence	Liability Insurance
Debra Morris et al. vs. AirCon Corporation, et al.	April 26, 2014	Nov. 1, 2017	Nov. 10, 2017	$18,460,279	N/A	$923,014	Negligence	Liability Insurance
Stolowski et al. vs. 234 East 178th Street LLC & City N.Y	Jan. 23, 2005	Feb. 22, 2016	June, 2016	$140,100,000	N/A	$183,261,737	Negligence	Liability Insurance
Garcia vs. Manhattan Vaughn JVP et al.	Dec. 4, 2013	Feb. 10, 2016	April 29, 2016	$53,852,558	N/A	$55,834,971.47 (final judgment)	Worker/Workplace Negligence	Liability Insurance
Garcia, et al. vs. O'Reilly Auto Enterprises, LLC & Shoots	Feb. 28, 2015	Jul. 19, 2016	Jul. 25, 2016	$37,945,000	N/A	$9,000,000 (reduced due to high/low agreement)	Motor Vehicle	Auto Insurance
Swenson et al. vs. Troy et al.	May 22, 2012	April 18, 2016	May 1, 2016	$35,029,371	$100,000	$35,129,371	Motor Vehicle	Auto Insurance
Dubuque vs. Cumberland Farms, Inc. & V.S.H. Realty, Inc.	Nov. 28, 2008	Feb. 23, 2016	March 8, 2016	$32,369,024.30	$10	$32,369,034.30	Negligence	Auto insurance
Gonzalez et al. vs. Atlas Construction Supply Inc. et al.	Aug. 2, 2011	July 27, 2016	Aug. 8, 2016	$26,920,170	N/A	$16,345,170 (no jointly liability)	Negligence	Liability Insurance
Jacobs Engineering Group Inc. vs. ConAgra Foods Inc	2013	March 25, 2016	April 22, 2016	$108,913,520.89	N/A	$108,913,520.89	Exposition	Liability Insurance
Hinson et al. vs. Dorel Juvenile Group, Inc. et al.	May 15, 2013	June 17, 2016	June 21, 2016	$24,438,000	$10,000,000	$34,438,000	Failure to Warn	Liability Insurance

Source: Case details from Report 1, Report 2, Report 3, Report 4, Report 5, and Report 6.