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Citation: Liu, P., Tsanakas, A. & Wei, Y. (2026). Risk sharing with Lambda Value-at-Risk under heterogeneous beliefs. *Finance and Stochastics*, 30(4), pp. 1179-1223. doi: 10.1007/s00780-026-00604-9

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Risk sharing with lambda value-at-risk under heterogeneous beliefs

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Received: 26 September 2024 / Accepted: 8 August 2025
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Abstract

In this paper, we study the risk sharing problem among multiple agents using lambda value-at-risk as their preference functional, under heterogeneous beliefs, where each agent's belief is represented by a probability measure. We obtain semi-explicit formulas for the inf-convolution of multiple lambda value-at-risk measures under heterogeneous beliefs and the explicit forms of the corresponding optimal allocations. To show the impact of belief heterogeneity, we consider three cases: homogeneous beliefs, conditional beliefs, and general beliefs with two agents. For those cases, we find more explicit expressions for the inf-convolution, showing the influence of the relation of the beliefs on the inf-convolution. Moreover, we consider in a two-agent setting the inf-convolution of one lambda value-at-risk and a general risk measure, including expected utility, distortion risk measures and lambda value-at-risk as special cases, with differing beliefs. The expression of the inf-convolution and the form of an optimal allocation are obtained. In all above cases, we demonstrate that trivial outcomes arise when both belief inconsistency and risk tolerance are high. Finally, we discuss risk sharing for an alternative definition of lambda value-at-risk.

Keywords Lambda value-at-risk · Value-at-risk · Risk sharing · Inf-convolution · Distortion risk measure · Expected shortfall · CoVaR · CoES

Mathematics Subject Classification 91B05 · 91G70

JEL Classification C71 · C69

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1 Introduction

The Pareto-optimal risk sharing problem, with the preferences of agents represented by risk measures, has been studied extensively since the introduction of convex and coherent risk measures by Artzner et al. [4], Föllmer and Schied [26] and Frittelli and Rosazza Gianin [28]. For instance, convex risk measures were used in Barrieu and El Karoui [7], Jouini et al. [36] and Filipović and Svindland [25] to investigate the risk sharing problem, showing the existence of an optimal allocation which is *comonotonic* under the assumption of law-invariance. Recently, more focus has been placed on the risk sharing problem for non-convex risk measures, such as quantile-based risk measures including value-at-risk (VaR) and expected shortfall (ES) as special cases (e.g. Embrechts et al. [23], Liu et al. [40] and Weber [55]) and distortion riskmetrics including the inter-quantile range as an example (e.g. Laugier et al. [38]). In those papers, it is shown that an optimal allocation is pairwise countermonotonic, as defined in Laugier et al. [37].

In the above papers, the agents are assumed to share the same beliefs on the distribution of the future risk. However, in the current regulatory frameworks (e.g. BCBS Standards [8, Chap. C]), internal models are extensively used, leading to heterogeneous views of different agents regarding the same future risk. Moreover, the belief heterogeneity may also stem from the asymmetric information accessed by the agents. We refer to Embrechts [22] for a discussion of applications of internal models in banking and insurance and Xiong [57] for a relevant discussion in finance. The model/belief heterogeneity in the risk sharing problem has been considered by several papers in the literature. For instance, under model heterogeneity, risk sharing between two agents endowed with concave monetary utilities was studied in Acciaio and Svindland [1], where the random variables were constrained to a finite σ -algebra and the existence of Pareto optimal allocation was discussed. Risk sharing for VaR or ES under heterogeneous beliefs was studied in Embrechts et al. [24], who showed that optimal risk allocations are pairwise countermonotonic with model heterogeneity. Liu [42] investigated comonotonic risk sharing with heterogeneous beliefs for distortion risk measures, and Liebrich [39] studied risk sharing for consistent risk measures with model heterogeneity, obtaining a sufficient condition for the existence of optimal allocations under some assumptions on the relation of the beliefs. Model heterogeneity was also studied in the context of optimal insurance and reinsurance design; see Amarante et al. [3], Boonen [13], Chi [18], Boonen and Ghossoub [15], Asimit et al. [5] and the references therein.

In this paper, we use lambda value-at-risk (Λ VaR) to represent the agents' preferences. As an extension of VaR, Λ VaR was introduced by Frittelli et al. [29] by changing the fixed probability level to a probability/loss function $1 - \Lambda$. The Λ -function can be chosen either increasing (relatively risk-averse) or decreasing (relatively risk-seeking), representing decision makers' individual risk appetite, as shown in [29]. The choice of a Λ -function based on data was studied in Hitaj et al. [33]. Compared with VaR, one advantage of lambda value-at-risk is its ability to distinguish the tail risk for decreasing Λ -functions, in the same spirit as the Loss VaR proposed in Bignozzi et al. [11]. For increasing Λ -functions, Λ VaR may incorporate some additional requirement such as a risk manager's judgement in the process

of risk management; see [Bellini and Peri \[9\]](#). Moreover, the recent literature shows that Λ VaR satisfies other desirable properties. [Han et al. \[32\]](#) showed that Λ VaR with an increasing Λ -function satisfies quasi-star-shapedness, which is a property weaker than quasi-convexity and penalising a kind of risk concentration. In addition, Λ VaR with an increasing Λ -function also satisfies cash-subadditivity, which is useful to measure a future financial loss with stochastic interest rates; see [El Karoui and Ravanelli \[21\]](#) and [Han et al. \[32\]](#). We refer to [Bellini and Peri \[9\]](#) for the monotonicity, locality and other properties, and [Burzoni et al. \[16\]](#) for robustness, elicibility and consistency. The risk sharing problem for Λ VaR under homogeneous beliefs was studied in [Liu \[43\]](#) and [Xia and Hu \[56\]](#), where expressions for the inf-convolution and the forms of an optimal allocation were derived. The Λ VaR was also applied to optimal reinsurance design with model homogeneity in [Balbás et al. \[6\]](#) and [Boonen et al. \[14\]](#). Moreover, the application of Λ VaR to robust portfolio selection and sensitivity analysis can be found in [Han and Liu \[31\]](#) and [Ince et al. \[34\]](#).

In our study, we put our focus on the risk sharing problem with Λ VaR and belief heterogeneity, extending many results in the literature. In Sect. 3, we study the inf-convolution of multiple Λ VaR with heterogeneous beliefs, where the beliefs are represented by a set of probability measures. We obtain a semi-explicit formula for the inf-convolution and explicit forms for optimal risk allocations, which show that agents take contingent losses on disjoint sets of the sample space.

The relation between the beliefs plays a crucial role in the risk sharing problem; see e.g. [Chi \[18\]](#). To study the impact of belief heterogeneity, we consider three cases in Sect. 3: homogeneous beliefs, conditional beliefs, and general beliefs with two agents. For homogeneous beliefs, we show that the inf-convolution of Λ VaR is still a Λ VaR for general Λ -functions, extending the results in [Liu \[43\]](#) and [Xia and Hu \[56\]](#) where the monotonicity of the Λ -functions is typically required. In particular, our results include the case that some Λ VaR have increasing Λ -functions and others have decreasing Λ -functions, demonstrating that the agents may have different risk appetites. The conditional beliefs reflect the asymmetric information available to the agents, with different agents conditioning on different events or information sets. Under this setup, we also obtain an explicit formula for the inf-convolution of Λ VaR, showing that the inf-convolution of Λ VaR under conditional beliefs is also a Λ VaR under a new conditional belief. Conditional beliefs are closely related to the conditional risk measures proposed in the literature to quantify systemic risk, such as CoVaR ([Adrian and Brunnermeier \[2\]](#) and [Girardi and Tolga Ergün \[30\]](#)) and CoES ([Mainik and Schaanning \[45\]](#)). The risk sharing problem in this case can be interpreted as risk sharing with some co-risk measures (Co Λ VaR, a new conditional risk measure proposed in Sect. 3). For the third case, we consider risk sharing between two agents. With the aid of the Lebesgue decomposition theorem, we find a more explicit formula for the inf-convolution of Λ VaR with general heterogeneous beliefs. Note that we also give an explicit expression for the inf-convolution of VaR with heterogeneous beliefs, which is new to the literature.

In Sect. 4, we study the inf-convolution of one Λ VaR and a general monotone risk measure with belief heterogeneity. The expression of the inf-convolution and the form of an optimal allocation are derived. Then we consider concrete examples. For conditional beliefs, we obtain explicit expressions for the inf-convolution of one Λ VaR and

one distortion risk measure or ΛVaR^+ . Moreover, for general heterogeneous beliefs, we find a formula for the inf-convolution of one ΛVaR and one ES or expected utility. In Sect. 5, we consider the inf-convolution of one ΛVaR^+ (an alternative definition of lambda value-at-risk) and a general monotone risk measure with belief heterogeneity. We obtain an expression for the inf-convolution, which is complicated, and the form of an optimal allocation. Our results include the inf-convolution of two ΛVaR^+ under heterogeneous beliefs as a special case. The results here are not explicit and very different from the previous sections, as they involve the shape of the Λ -function and the best case of risk aggregation under dependence uncertainty. This means that ΛVaR^+ is very different from ΛVaR for its application within the risk sharing problem. Some special cases are considered to illustrate our theory in Sect. 5.

Throughout the paper, the stated results provide conditions for trivial, i.e., infinite, inf-convolutions. We consistently show that these conditions arise when, broadly speaking, agents' beliefs are strongly inconsistent and risk tolerances are high. Hence our contribution illuminates how the entanglement of beliefs and preferences impacts key features of the risk sharing problem with ΛVaR .

The notation and key definitions are given in Sect. 2, and all proofs are relegated to the [Appendix](#).

2 Notation and definitions

For a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{L}^\infty$ denote the collection of all bounded random variables. Moreover, let $\mathcal{X}$ be a set of random variables containing $\mathcal{L}^\infty$. We call $\mathcal{X}$ unbounded if it contains unbounded random variables. We suppose $\mathcal{X}$ has good enough properties to conduct our study, such as $\mathcal{X} = \mathcal{L}^p$ for some $p \geq 0$, where $\mathcal{L}^p$ represents the set of all random variables with finite p th moments. **Throughout this paper**, we consider the random variables themselves, not the corresponding equivalence classes, because as seen in the sequel, several probability measures are considered simultaneously.

For any $X \in \mathcal{X}$, a positive value of X represents a financial loss. For a risk measure, that is, a mapping $\rho : \mathcal{X} \rightarrow \mathbb{R}$, we call ρ *monotone* if $X \leq Y$ implies $\rho(X) \leq \rho(Y)$, and *cash-additive* if $\rho(X + c) = \rho(X) + c$ for $X \in \mathcal{X}$ and $c \in \mathbb{R}$. We call ρ a *monetary* risk measure if it is monotone and cash-additive. Furthermore, ρ is *law-invariant* under a probability measure $\mathbb{Q}$ if for all $X, Y \in \mathcal{X}$,

$$X \stackrel{\mathbb{Q}}{=} Y \quad \text{implies} \quad \rho(X) = \rho(Y),$$

where $\stackrel{\mathbb{Q}}{=}$ stands for equality in distribution under $\mathbb{Q}$. Note that $\mathbb{Q}$ may differ from $\mathbb{P}$. **Throughout this paper**, all risk measures used are law-invariant with respect to certain probability measures, which will be specified later. Hence when evaluating the risk measure ρ on X , with respect to the measure $\mathbb{Q}$, we consistently use the notation $\rho^{\mathbb{Q}}(X)$ for the resulting value. For more details on risk measures, one can refer to Föllmer and Schied [27, Chap. 4].

For a probability measure $\mathbb{Q}$, the distribution function of X under $\mathbb{Q}$ is denoted by $F_X^{\mathbb{Q}}$. For $X \in \mathcal{X}$, $F_X^{\mathbb{Q}, -1}$ represents its left quantile under $\mathbb{Q}$, which is defined by

$$F_X^{\mathbb{Q}, -1}(p) = \inf\{x : F_X^{\mathbb{Q}}(x) \geq p\}, \quad p \in (0, 1],$$

with the convention that $\inf \emptyset = \infty$. For any $X \in \mathcal{X}$, if $\mathbb{Q}$ is atomless, we denote by $U_X^{\mathbb{Q}}$ a uniform random variable on $[0, 1]$ under $\mathbb{Q}$ such that $X = F_X^{\mathbb{Q}, -1}(U_X^{\mathbb{Q}})$ a.s. under $\mathbb{Q}$. The existence of such $U_X^{\mathbb{Q}}$ for any random variable X is guaranteed by e.g. Föllmer and Schied [27, Lemma A.32].

We next define tail risk measures, which are important for our results later. For a random variable $X \in \mathcal{X}$, $p \in (0, 1]$ and an atomless probability measure $\mathbb{Q}$, we call

$$X_p = (F_X^{\mathbb{Q}})^{-1}(1 - p + pU_X^{\mathbb{Q}})$$

the tail risk of X beyond its $(1 - p)$ -quantile under $\mathbb{Q}$. The distribution of X_p under $\mathbb{Q}$ is given by

$$\mathbb{Q}[X_p \leq x] = \frac{(F_X^{\mathbb{Q}}(x) - (1 - p))^+}{p}, \quad x \in \mathbb{R},$$

where $x^+ = \max(x, 0)$. We call ρ a p -tail risk measure under $\mathbb{Q}$ for some $p \in (0, 1)$ if $X_p \stackrel{\mathbb{Q}}{=} Y_p$ implies $\rho^{\mathbb{Q}}(X) = \rho^{\mathbb{Q}}(Y)$ for all $X, Y \in \mathcal{X}$. One can refer to Liu and Wang [41] and Liu et al. [40] for more details on the definition, properties and applications of tail risk measures.

Next, we define the inf-convolution. The random variable $X \in \mathcal{X}$ represents a risk to be shared between n agents. Then define the set of *allocations* of X as

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}.$$

Throughout this paper, we assume that the probability measures $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ live on $(\Omega, \mathcal{F})$ and with the exception of Sect. 3.1, the corresponding probability spaces are atomless. We suppose the agents may have heterogeneous beliefs, represented by the probability measures $\mathbb{Q}_1, \dots, \mathbb{Q}_n$. The *inf-convolution* of the risk measures $\rho_1, \dots, \rho_n$ under the respective probability measures $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ is the mapping $\square_{i=1}^n \rho_i^{\mathbb{Q}_i} : \mathcal{X} \rightarrow [-\infty, \infty)$ defined as

$$\square_{i=1}^n \rho_i^{\mathbb{Q}_i}(X) = \inf \left\{ \sum_{i=1}^n \rho_i^{\mathbb{Q}_i}(X_i) : (X_1, \dots, X_n) \in \mathbb{A}_n(X) \right\}.$$

Note that ρ_i is law-invariant under $\mathbb{Q}_i$, where $\mathbb{Q}_i$ represents the belief of the i th agent. In case of belief heterogeneity, the inf-convolution need not be law-invariant, i.e., if $X \stackrel{\mathbb{Q}_i}{=} Y$ for all $i = 1, \dots, n$, then $\square_{i=1}^n \rho_i^{\mathbb{Q}_i}(X) \neq \square_{i=1}^n \rho_i^{\mathbb{Q}_i}(Y)$ may hold; see more details in Remark 3.9 below.

An n -tuple $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is called an *optimal allocation* of X for $\rho_1, \dots, \rho_n$ under $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ if $\sum_{i=1}^n \rho_i^{\mathbb{Q}_i}(X_i) = \square_{i=1}^n \rho_i^{\mathbb{Q}_i}(X)$. The inf-convolution

$\square_{i=1}^n \rho_i^{\mathbb{Q}_i}(X)$ can be interpreted as the smallest possible aggregate capital for the total risk X in the financial system if $\rho_i^{\mathbb{Q}_i}(X_i)$ represents the capital charge for the i th financial institution to hold the risky position X_i . More economic interpretations on the inf-convolution can be found in e.g. Delbaen [19, Chap. 9], Rüschendorf [50, Chap. 10] and Embrechts et al. [23].

For risk measures $(\rho_1, \dots, \rho_n)$ under $(\mathbb{Q}_1, \dots, \mathbb{Q}_n)$ and a total risk X , an allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is *Pareto-optimal* if for any allocation $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$, $\rho_i^{\mathbb{Q}_i}(Y_i) \leq \rho_i^{\mathbb{Q}_i}(X_i)$ for all $i = 1, \dots, n$ implies $\rho_i^{\mathbb{Q}_i}(Y_i) = \rho_i^{\mathbb{Q}_i}(X_i)$ for all $i = 1, \dots, n$. For finite-valued monetary risk measures, it is shown in [23] that an allocation is optimal if and only if it is Pareto-optimal. Note that ΛVaR (defined in (2.1)) does not satisfy cash-additivity; see Frittelli et al. [29] and Bellini and Peri [9]. Hence it is not a monetary risk measure. However, one can still show that an optimal allocation of the inf-convolution of ΛVaR is Pareto-optimal.

Finally, we define the lambda value-at-risk. For $\Lambda : \mathbb{R} \rightarrow [0, 1]$ and a probability measure $\mathbb{Q}$, the lambda value-at-risk is given by

$$\begin{aligned}\Lambda \text{VaR}^{\mathbb{Q}}(X) &= \inf\{x \in \mathbb{R} : F_X^{\mathbb{Q}}(x) \geq 1 - \Lambda(x)\}, \\ \Lambda \text{VaR}^{+, \mathbb{Q}}(X) &= \sup\{x \in \mathbb{R} : F_X^{\mathbb{Q}}(x) < 1 - \Lambda(x)\},\end{aligned}\quad (2.1)$$

where $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. Note that $\Lambda \text{VaR}^{\mathbb{Q}}(X) = \Lambda \text{VaR}^{+, \mathbb{Q}}(X)$ if Λ is increasing; otherwise, this need not be true, see [9, Proposition 6]. We refer to [9] for two other definitions of ΛVaR . If $\Lambda \equiv p$ is constant, then $\Lambda \text{VaR}^{\mathbb{Q}}$ boils down to $\text{VaR}^{\mathbb{Q}}$, i.e., $\text{VaR}^{\mathbb{Q}}$ at level $p \in [0, 1)$ which is given by

$$\text{VaR}_p^{\mathbb{Q}}(X) = \Lambda \text{VaR}^{\mathbb{Q}}(X) = F_X^{\mathbb{Q}, -1}(1 - p), \quad X \in \mathcal{X}.$$

Moreover, by definition, $\text{VaR}_1^{\mathbb{Q}}(X) = -\infty$. Although VaR has been criticised from different angles, it has been widely applied in practice for risk management due to its simplicity and some nice properties; see McNeil et al. [47, Sect. 2.3] and the references therein for more detailed discussions on VaR. Compared to VaR, ΛVaR is more flexible in the choice of Λ -functions; it satisfies cash-subadditivity and quasi-star-shapedness for increasing Λ , and ΛVaR^+ is able to capture the tail risk for decreasing Λ -functions (see e.g. [29], Hitaj et al. [33] and Han et al. [32]).

We denote by $\mathcal{H}$ the collection of all right-continuous functions $\Lambda : \mathbb{R} \rightarrow [0, 1]$. Hereafter, for any Λ , we set $\lambda^- = \inf_{x \in \mathbb{R}} \Lambda(x)$ and $\lambda^+ = \sup_{x \in \mathbb{R}} \Lambda(x)$. We say that a constant λ is *attainable* for Λ if there exists $x \in \mathbb{R}$ such that $\Lambda(x) = \lambda$.

The interplay among $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ plays an important role in the inf-convolution, which is our main concern in this paper. This can be seen from the following result. We call a mapping $\rho : \mathcal{X} \rightarrow \mathbb{R}$ *constant-preserving* if $\rho(c) = c$ for all $c \in \mathbb{R}$. Note that both $\Lambda \text{VaR}^{\mathbb{Q}}$ and $\Lambda \text{VaR}^{+, \mathbb{Q}}$ satisfy this.

Proposition 2.1 *Suppose $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are mutually singular and ρ_2 is constant-preserving. Then for $X \in \mathcal{X}$,*

$$\rho_1^{\mathbb{Q}_1} \square \rho_2^{\mathbb{Q}_2}(X) = -\infty.$$

The conclusion in Proposition 2.1 implies that for any two monetary risk measures ρ_1 and ρ_2 satisfying $\rho_1(0) = \rho_2(0) = 0$, we have $\rho_1^{\mathbb{Q}_1} \square \rho_2^{\mathbb{Q}_2}(X) = -\infty$ if $\mathbb{Q}_1$ and $\mathbb{Q}_2$ are mutually singular. This means that if the beliefs of agents are too divergent, then each can get an unreasonably good deal by transferring risks between them. Then the fact that the inf-convolution is $-\infty$ reflects that agents can always make their position better and hence there is no optimal allocation. In contrast, if $\mathbb{Q}_1 = \mathbb{Q}_2$, then $\rho_1^{\mathbb{Q}_1} \square \rho_2^{\mathbb{Q}_2}(X) > -\infty$ for many monetary risk measures ρ_1 and ρ_2 ; see e.g. Embrechts et al. [23] for quantile-based risk measures and Filipović and Svindland [25] for convex risk measures. Note that the conclusion in Proposition 2.1 can be easily extended to the case with n agents for $n \geq 3$.

3 Inf-convolution of multiple Δ VaR

In this section, we investigate the inf-convolution of n Δ VaRs with heterogeneous beliefs represented by $\mathbb{Q}_1, \dots, \mathbb{Q}_n$, respectively. Suppose $\mathcal{X} \supseteq \mathcal{L}^\infty$. We find an expression for the inf-convolution and also a corresponding optimal allocation, covering the results of the inf-convolution of Δ VaR or VaR with homogeneous or heterogeneous beliefs in Liu [43], Embrechts et al. [24] and Embrechts et al. [23]. We first consider general beliefs and then discuss three different cases about the relation of the beliefs: homogeneous beliefs, conditional beliefs, and general beliefs with two agents.

3.1 General beliefs

In this subsection only, we assume that $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ are general probability measures that are not required to be atomless. We first introduce the following notation. Let $\Pi_n(\Omega) = \{(A_1, \dots, A_n) : \bigcup_{i=1}^n A_i = \Omega, A_i \cap A_j = \emptyset, i \neq j\}$. For $\Lambda_1, \dots, \Lambda_n \in \mathcal{H}$ with $0 < \lambda_i^- \leq \lambda_i^+ < 1$, let

$$\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) = \inf \left\{ \sum_{j=1}^n y_j : \mathbb{Q}_i \left[\left\{ X > \sum_{j=1}^n y_j \right\} \cap A_i \right] \leq \Lambda_i(y_i) \right. \\ \left. \text{for some } (A_1, \dots, A_n) \in \Pi_n(\Omega) \right\}.$$

Note that $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) \in [-\infty, \infty)$.

Theorem 3.1 For $\Lambda_i \in \mathcal{H}$ with $0 < \lambda_i^- \leq \lambda_i^+ < 1$, we have

$$\square_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X) = \Gamma_{\Lambda_1, \dots, \Lambda_n}(X).$$

Moreover, if a minimiser $(y_1^*, \dots, y_n^*, A_1^*, \dots, A_n^*)$ of $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X)$ exists, then an optimal allocation is given by

$$X_i = \left(X - \sum_{j=1}^n y_j^* \right) \mathbb{1}_{A_i^*} + y_i^*, \quad i = 1, \dots, n. \quad (3.1)$$

The allocation (3.1) can be understood as follows. The i th agent is allocated a constant loss y_i^* plus a contingent loss, which is the excess of the total risk over the aggregate cash allocation, restricted to the set A_i^* . The key feature of those sets is seen by the definition of $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X)$: for each agent, their joint probability of (i) the set A_i^* on which they suffer the contingent loss and (ii) the total loss exceeding the aggregate cash allocation, is restricted by their Λ_i -function.

Note that Theorem 3.1 is very general, covering [43, Theorems 1, 2, 4] and [24, Theorem 4] as special cases. Compared with the results in [43], our results in Theorem 3.1 offer semi-explicit formulas for the inf-convolution. Later, we specify some special relations of $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ to obtain more explicit formulas for the inf-convolution. However, it is unclear what the inf-convolution of multiple Λ VaR under general heterogeneous beliefs is, leaving it as an open problem deserving further study in the future. Moreover, Theorem 3.1 does not explicitly show when the inf-convolution is $-\infty$. The discussion on finiteness of $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X)$ is complicated as it heavily relies on the properties of Λ_i -functions and the relationship between the $\mathbb{Q}_i$; see [43] and Proposition 2.1. This will be discussed in the next two results. Finally, note that if all Λ_i are constant, Theorem 3.1 boils down to [24, Theorem 4] where the inf-convolution of VaR has been studied with belief heterogeneity.

Next, let us discuss the finiteness of $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X)$. Let $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$,

$$\bigwedge_{i=1}^n x_i = \min_{i=1, \dots, n} x_i, \quad \bigvee_{i=1}^n x_i = \max_{i=1, \dots, n} x_i.$$

Proposition 3.2 *For $\Lambda_i \in \mathcal{H}$ with $0 < \lambda_i^- \leq \lambda_i^+ < 1$, the following hold:*

- (i) *If $\bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^-} \leq 1$ for some partition $(A_1, \dots, A_n) \in \Pi_n(\Omega)$, then we have $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) = -\infty$.*
- (ii) *If $\inf_{(A_1, \dots, A_n) \in \Pi_n(\Omega)} \bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^+} > 1$, then $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) > -\infty$.*

Note that if all Λ_i are constant and $\mathbb{Q}_1 = \mathbb{Q}_2 = \dots = \mathbb{Q}_n$, then Proposition 3.2 offers a sufficient and necessary condition for the finiteness of the inf-convolution. In general, due to the complexity of the Λ_i -functions and the heterogeneity of probability measures, those conditions are sufficient but not necessary.

As we can see from Proposition 3.2, cases (i) and (ii) cannot cover all scenarios. The discussion of other cases may require more information on the Λ_i -functions. We next consider all monotone Λ_i -functions in the same direction.

Proposition 3.3 *Suppose all $\Lambda_i \in \mathcal{H}$ with $0 < \lambda_i^- \leq \lambda_i^+ < 1$ are decreasing. Then we have the following conclusion:*

- (i) *If $\inf_{(A_1, \dots, A_n) \in \Pi_n(\Omega)} \bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^+} < 1$, then $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) = -\infty$.*
- (ii) *If $\inf_{(A_1, \dots, A_n) \in \Pi_n(\Omega)} \bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^+} > 1$, then $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) > -\infty$.*

Suppose all $\Lambda_i \in \mathcal{H}$ with $0 < \lambda_i^- \leq \lambda_i^+ < 1$ are increasing. Then we have the following conclusion:

- (i) If $\inf_{(A_1, \dots, A_n) \in \Pi_n(\Omega)} \bigwedge_{i=1}^n \left(\frac{\mathbb{Q}_i[A_i]}{\lambda_i^-} \vee \bigvee_{j \neq i} \frac{\mathbb{Q}_j[A_j]}{\lambda_j^+} \right) < 1$, then we obtain $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) = -\infty$.
- (ii) If $\inf_{(A_1, \dots, A_n) \in \Pi_n(\Omega)} \bigwedge_{i=1}^n \left(\frac{\mathbb{Q}_i[A_i]}{\lambda_i^-} \vee \bigvee_{j \neq i} \frac{\mathbb{Q}_j[A_j]}{\lambda_j^+} \right) > 1$, then we obtain $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) > -\infty$.

In this more refined result, we see that $\bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^+}$ plays a pivotal role when the Λ_i -functions are decreasing. We see that in order to guarantee a finite inf-convolution for decreasing Λ_i , it is necessary for this quantity to exceed 1. For simplicity, assume that for all i , λ_i^+ is attained by Λ_i . Then the condition in case (ii) of Proposition 3.3 for decreasing Λ_i is violated when one can find a partition of Ω such that $\mathbb{Q}_i[A_i] \leq \lambda_i^+$ for all agents. This means that for each agent i , we can find a set A_i on which the risk is suitably bounded by the largest acceptable probability to the agent of an adverse tail event. We can consider $\inf_{(A_1, \dots, A_n) \in \Pi_n(\Omega)} \bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^+}$ as a measure of belief homogeneity, which also reflects preferences. Consider two extreme cases. First, if sets $A_1, \dots, A_n$ exist for which we have $\mathbb{Q}_i[A_i] = 0$ for all i , then $\bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^+} = 0$. The ability to find such sets is of course a case of extremely diverging beliefs. On the other hand, we can consider belief homogeneity, i.e., $\mathbb{Q}_1 = \dots = \mathbb{Q}_n = \mathbb{P}$. If $\mathbb{P}$ is atomless, then $\inf_{(A_1, \dots, A_n) \in \Pi_n(\Omega)} \bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^+} > 1$ if and only if $\sum_{i=1}^n \lambda_i^+ < 1$, which reflects the impact of preferences. Clearly, when $\sum_{i=1}^n \lambda_i^+ < 1$, indicating that agents accept lower total probabilities of adverse events, the inf-convolution is finite. In contrast, if $\sum_{i=1}^n \lambda_i^+ > 1$, the increased aggregate tolerance for tail events makes it possible to construct arbitrarily favourable allocations, leading to an infinite value of the inf-convolution.

When the λ_i functions are increasing, the picture is more complicated. Assuming attainability of all λ_i^- , λ_i^+ , the condition of part (i) is satisfied if we can find a partition for which, for some i , $\mathbb{Q}_i[A_i] < \lambda_i^-$ and $\mathbb{Q}_j[A_j] < \lambda_j^+$, $j \neq i$. Hence the conditions of Proposition 3.3 compared to those in Proposition 3.2 are weaker for part (i) and essentially weaker for part (ii), due to the monotonicity assumption on the functions $\Lambda_1, \dots, \Lambda_n$.

3.2 Homogeneous beliefs

In this section, we focus on the case $\mathbb{Q}_1 = \dots = \mathbb{Q}_n = \mathbb{P}$, where all the corresponding probability spaces are assumed to be **atomless**. We find a more explicit expression for the inf-convolution. For $x \in \mathbb{R}$, let

$$\Lambda^*(x) = \sup_{y_1 + \dots + y_n = x} \left(1 \wedge \sum_{i=1}^n \Lambda_i(y_i) \right).$$

We call Λ^* *attainable* if for each $x \in \mathbb{R}$, there exists $(y_1, \dots, y_n) \in \mathbb{R}^n$ satisfying $\sum_{i=1}^n y_i = x$ such that $1 \wedge \sum_{i=1}^n \Lambda_i(y_i) = \Lambda^*(x)$. For homogeneous beliefs, $\Lambda^*(x)$ represents the largest aggregate risk tolerance of the agents to control the tail event $\{X > x\}$ in the definition of $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X)$.

Theorem 3.4 For $\Lambda_i \in \mathcal{H}$ with $0 < \lambda_i^- \leq \lambda_i^+ < 1$, if Λ^* is attainable, then we have

$$\bigcap_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{P}}(X) = \Lambda^* \text{VaR}^{\mathbb{P}}(X).$$

Moreover, if additionally $x^* := \Lambda^* \text{VaR}^{\mathbb{P}}(X) > -\infty$ and Λ^* is right-continuous at x^* , then an optimal allocation is given by (3.1) with $(y_1^*, \dots, y_n^*)$ and $(A_1^*, \dots, A_n^*)$ satisfying $1 \wedge \sum_{i=1}^n \Lambda_i(y_i^*) = \Lambda^*(x^*)$ and $\mathbb{P}[\{X > x^*\} \cap A_i^*] \leq \Lambda_i(y_i^*)$ for all i .

Theorem 3.4 shows that the optimal sets $(A_1^*, \dots, A_n^*)$ need not be unique. Under the assumption of Theorem 3.4, if $(y_1^*, \dots, y_n^*)$ satisfies $1 \wedge \sum_{i=1}^n \Lambda_i(y_i^*) = \Lambda^*(x^*)$, optimal sets can be chosen as

$$A_i^* = \left\{ 1 - \sum_{j=1}^i \Lambda_j(y_j^*) < U_X^{\mathbb{P}} \leq 1 - \sum_{j=1}^{i-1} \Lambda_j(y_j^*) \right\}, \quad i = 1, \dots, n-1,$$

$$A_n^* = \left\{ \left(1 - \sum_{j=1}^n \Lambda_j(y_j^*) \right) \vee 0 < U_X^{\mathbb{P}} \leq 1 - \sum_{j=1}^{n-1} \Lambda_j(y_j^*) \right\},$$

with the convention that an empty sum is 0. Moreover, the right-continuity of Λ^* at x^* can be removed if all the Λ_i are decreasing. Given the non-uniqueness of the allocation and the particular form of the A_i^* , it is clear that agents' preferences do not impact which specific part of (the tail of) X they are exposed to at the optimum. What is, however, impacted is the probability of the contingent events; specifically, for $\Lambda^*(x^*) < 1$, we have $\mathbb{P}[A_i^*] = \Lambda_i(y_i^*)$, $i = 1, \dots, n-1$, and $\mathbb{P}[A_n^*] = 1 - \sum_{j=1}^{n-1} \Lambda_j(y_j^*)$.

Note that in Theorem 3.4, we do not require any monotonic properties of the Λ_i -functions. They are very general with only one requirement on the attainability of Λ^* . Hence it extends the results of [43, Theorems 1, 2, 4] which investigate the inf-convolution of multiple Λ VaR with monotone Λ_i -functions for homogeneous beliefs. In particular, Theorem 3.4 covers the case that some Λ VaR have increasing Λ -functions and others have decreasing Λ -functions, demonstrating that the agents may have different risk appetites.

We can also remove the assumption on the attainability of Λ^* in Theorem 3.4 to obtain an alternative expression. For $n \geq 2$, let

$$\Lambda^{y_{n-1}}(x) = \left(\Lambda_n(x - y_{n-1}) + \sum_{i=1}^{n-1} \Lambda_i(y_i - y_{i-1}) \right) \wedge 1,$$

where $y_{n-1} = (y_1, \dots, y_{n-1})$ and $y_0 = 0$.

Proposition 3.5 For $\Lambda_i \in \mathcal{H}$ with $0 < \lambda_i^- \leq \lambda_i^+ < 1$, we have

$$\bigcap_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{P}}(X) = \inf_{y_{n-1} \in \mathbb{R}^{n-1}} \Lambda^{y_{n-1}} \text{VaR}^{\mathbb{P}}(X).$$

Differently from Theorem 3.1, the assumption of the measures being atomless cannot be removed in Theorem 3.4. This can be seen from the following example.

Example 3.6 For $m \geq 2$, suppose there exist distinct $\omega_1, \omega_2, \dots, \omega_m \in \Omega$ such that $\{\omega_1\}, \dots, \{\omega_m\} \in \mathcal{F}$ and $\mathbb{P}[\{\omega_1\}] = \dots = \mathbb{P}[\{\omega_m\}] = 1/m$. Define a mapping X with $X(\omega_i) = i, i = 1, \dots, m$, and $X(\omega) = 0$ if $\omega \notin \{\omega_1, \dots, \omega_m\}$. Clearly X is measurable on $(\Omega, \mathcal{F})$. Moreover, suppose $\Lambda_1 = \Lambda_2 \equiv 1/(2m)$. It follows from Theorem 3.1 that

$$\begin{aligned} \Lambda_1 \text{VaR}^{\mathbb{P}} \square \Lambda_2 \text{VaR}^{\mathbb{P}}(X) \\ = \inf \{y \in \mathbb{R} : \mathbb{P}[\{X > y\} \cap A] \vee \mathbb{P}[\{X > y\} \cap A^c] \leq 1/(2m) \text{ for some } A \in \mathcal{F}\} \\ = m. \end{aligned}$$

On the other hand,

$$\Lambda^* \text{VaR}^{\mathbb{P}}(X) = \text{VaR}_{1/m}^{\mathbb{P}}(X) = m - 1.$$

Hence $\Lambda_1 \text{VaR}^{\mathbb{P}} \square \Lambda_2 \text{VaR}^{\mathbb{P}}(X) \neq \Lambda^* \text{VaR}^{\mathbb{P}}(X)$. This shows that the conclusion in Theorem 3.4 may fail if the atomless assumption is removed.

3.3 Conditional beliefs

In this section, we study further how the relationship among $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ can affect the inf-convolution. In particular, we consider the case where the agents operate with different information. To model this, consider the case $\mathbb{Q}_i[\cdot] = \mathbb{P}[\cdot | B_i]$ for some $B_i \in \mathcal{F}$ with $\mathbb{P}[B_i] > 0$. If $\mathbb{P}$ represents the physical probability and $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ are the beliefs of the agents, then those $\mathbb{Q}_i$ demonstrate that the agents' may have different assessments of the occurrence of events due to asymmetric information or being subject to different regulating agencies with heterogeneous stressing events; see e.g. Acciaio and Svindland [1]. For $B_1, \dots, B_n \in \mathcal{F}$ and $\Lambda_1, \dots, \Lambda_n \in \mathcal{H}$, if $\mathbb{P}[\bigcap_{i=1}^n B_i] > 0$, let

$$\Lambda^\diamond(x) = \sup_{y_1 + \dots + y_n = x} \left(1 \wedge \sum_{i=1}^n \frac{\mathbb{P}[B_i] \Lambda_i(y_i)}{\mathbb{P}[\bigcap_{j=1}^n B_j]} \right)$$

for all $x \in \mathbb{R}$. Note that $\Lambda^\diamond(x)$ can be viewed as the largest weighted aggregate risk tolerance of the agents to control the tail event $\{X > x\}$.

Theorem 3.7 Suppose $\mathbb{Q}_i[\cdot] = \mathbb{P}[\cdot | B_i]$ for some $B_i \subseteq \Omega$ with $\mathbb{P}[B_i] > 0$ for $i = 1, \dots, n$. For $\Lambda_i \in \mathcal{H}$ with $0 < \lambda_i^- \leq \lambda_i^+ < 1$, if $\mathbb{P}[\bigcap_{i=1}^n B_i] = 0$, then

$$\square_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X) = -\infty.$$

If $\mathbb{P}[\bigcap_{i=1}^n B_i] > 0$ and $\Lambda^\diamond$ is attainable, then

$$\square_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X) = \Lambda^\diamond \text{VaR}^{\mathbb{Q}}(X),$$

where $\mathbb{Q}[\cdot] = \mathbb{P}[\cdot | \bigcap_{i=1}^n B_i]$. Moreover, if additionally $x^* = \Lambda^\diamond \text{VaR}^\mathbb{Q}(X) > -\infty$ and $\Lambda^\diamond$ is right-continuous at x^* , then an optimal allocation is given by (3.1) with $(y_1^*, \dots, y_n^*)$ satisfying $\sum_{i=1}^n \frac{\mathbb{P}[B_i]\Lambda_i(y_i^*)}{\mathbb{P}[\bigcap_{j=1}^n B_j]} = \Lambda^\diamond(x^*)$ and $(A_1^*, \dots, A_n^*)$ satisfying

$$\mathbb{P}\left[\left\{X > \sum_{i=1}^n y_i^*\right\} \cap \left(\bigcap_{i=1}^n B_i\right) \cap A_j^*\right] \leq \mathbb{P}[B_j]\Lambda_j(y_j^*).$$

Theorem 3.7 shows that the optimal sets $(A_1^*, \dots, A_n^*)$ need not be unique. Under the assumption of Theorem 3.7, for $(y_1^*, \dots, y_n^*)$ with $\sum_{i=1}^n \frac{\mathbb{P}[B_i]\Lambda_i(y_i^*)}{\mathbb{P}[\bigcap_{j=1}^n B_j]} = \Lambda^\diamond(x^*)$, optimal sets can be chosen as

$$\begin{aligned} A_i^* &= \left\{1 - \sum_{j=1}^i \frac{\mathbb{P}[B_j]\Lambda_j(y_j^*)}{\mathbb{P}[\bigcap_{k=1}^n B_k]} < U_X^\mathbb{Q} \leq 1 - \sum_{j=1}^{i-1} \frac{\mathbb{P}[B_j]\Lambda_j(y_j^*)}{\mathbb{P}[\bigcap_{k=1}^n B_k]}\right\}, \\ i &= 1, \dots, n-1, \\ A_n^* &= \left\{\left(1 - \sum_{j=1}^n \frac{\mathbb{P}[B_j]\Lambda_j(y_j^*)}{\mathbb{P}[\bigcap_{k=1}^n B_k]}\right) \vee 0 < U_X^\mathbb{Q} \leq 1 - \sum_{j=1}^{n-1} \frac{\mathbb{P}[B_j]\Lambda_j(y_j^*)}{\mathbb{P}[\bigcap_{k=1}^n B_k]}\right\} \end{aligned}$$

with $\mathbb{Q}[\cdot] = \mathbb{P}[\cdot | \bigcap_{i=1}^n B_i]$. Moreover, the right-continuity of $\Lambda^\diamond$ at x^* can be removed if all the Λ_i are decreasing. Now if $\Lambda^\diamond(x^*) < 1$, the probabilities of the optimal sets under the probability $\mathbb{Q}$ representing evaluations with respect to the information set that all agents agree on are given by

$$\mathbb{Q}[A_i^*] = \frac{\mathbb{P}[B_i]\Lambda_i(y_i^*)}{\mathbb{P}[\bigcap_{k=1}^n B_k]}, \quad i = 1, \dots, n-1, \quad \mathbb{Q}[A_n^*] = 1 - \sum_{j=1}^{n-1} \frac{\mathbb{P}[B_j]\Lambda_j(y_j^*)}{\mathbb{P}[\bigcap_{k=1}^n B_k]},$$

or equivalently

$$\begin{aligned} \mathbb{P}\left[\left(\bigcap_{k=1}^n B_k\right) \cap A_i^*\right] &= \mathbb{P}[B_i]\Lambda_i(y_i^*), \quad i = 1, \dots, n-1, \\ \mathbb{P}[A_n^*] &= \mathbb{P}\left[\bigcap_{k=1}^n B_k\right] - \sum_{j=1}^{n-1} \mathbb{P}[B_j]\Lambda_j(y_j^*). \end{aligned}$$

In this special case, the interplay among $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ is characterised by the relationship between the sets $B_1, \dots, B_n$. Following the conclusion in Theorem 3.7, those sets $B_1, \dots, B_n$ appear in the expressions of the inf-convolution and an optimal allocation, suggesting the influence of the relation of $\mathbb{Q}_1, \dots, \mathbb{Q}_n$ on the inf-convolution. In particular, if the agents have a strong disagreement on the interesting events so that $\mathbb{P}[\bigcap_{i=1}^n B_i] = 0$, it leads to a trivial case. Note that the crucial difference to the homogeneous case is that here, in characterising the allocations, we use the set $(\bigcap_{k=1}^n B_k) \cap A_i^*$ which includes both the set A_i^* on which the contingent loss

is paid and the conditioning set (information) on which all agents can agree. If all Λ_i -functions in Theorem 3.7 are assumed to be constant, we immediately derive the explicit formulas for the inf-convolution of multiple VaR under conditional beliefs.

Corollary 3.8 *Suppose $\mathbb{Q}_i[\cdot] = \mathbb{P}[\cdot | B_i]$ for some $B_i \subseteq \Omega$ with $\mathbb{P}[B_i] > 0$ for $i = 1, \dots, n$. For $\alpha_i \in (0, 1)$, if $\mathbb{P}[\bigcap_{i=1}^n B_i] = 0$, then*

$$\bigcap_{i=1}^n \text{VaR}_{\alpha_i}^{\mathbb{Q}_i}(X) = -\infty.$$

If $\mathbb{P}[\bigcap_{i=1}^n B_i] > 0$, then

$$\bigcap_{i=1}^n \text{VaR}_{\alpha_i}^{\mathbb{Q}_i}(X) = \text{VaR}_{1 \wedge \sum_{i=1}^n \frac{\alpha_i \mathbb{P}[B_i]}{\mathbb{P}[\bigcap_{j=1}^n B_j]}}^{\mathbb{Q}}(X),$$

where $\mathbb{Q}[\cdot] = \mathbb{P}[\cdot | \bigcap_{i=1}^n B_i]$.

The results in Corollary 3.8 extend the elegant formula for the inf-convolution of VaR under homogeneous beliefs to the case with conditional beliefs; see Embrechts et al. [23, Corollary 2]. The simplicity of the expressions for VaR allows some additional insight. We can see that the trivial outcome $\bigcap_{i=1}^n \text{VaR}_{\alpha_i}^{\mathbb{Q}_i}(X) = -\infty$ is not only achieved in the case of full disagreement between agents, i.e., $\mathbb{P}[\bigcap_{i=1}^n B_i] = 0$. It also occurs under weaker disagreement when $\sum_{i=1}^n \frac{\alpha_i \mathbb{P}[B_i]}{\mathbb{P}[\bigcap_{j=1}^n B_j]} \geq 1$. This inequality may be satisfied when $\mathbb{P}[\bigcap_{j=1}^n B_j]$ is sufficiently small, indicating high – but partial – disagreement in conditional beliefs, and the levels α_i are sufficiently large, indicating a high level of risk tolerance.

It is worth mentioning that the results in Proposition 3.7 can be explained from the perspective of co-risk measures. Recently, co-risk measures were introduced in the context of systemic risk measurement, with reference to conditional VaR (Adrian and Brunnermeier [2], Girardi and Tolga Ergün [30]) and conditional ES (Mainik and Schaanning [45]). Let $B \in \mathcal{F}$ satisfy $\mathbb{P}[B] > 0$. For $\alpha \in (0, 1)$, the conditional VaR (CoVaR) and conditional ES (CoES) are defined respectively as

$$\text{VaR}_{\alpha}^{\mathbb{P}}(X|B) = \inf\{x : \mathbb{P}[X \leq x | B] \geq 1 - \alpha\},$$

$$\text{ES}_{\alpha}^{\mathbb{P}}(X|B) = \frac{1}{\alpha} \int_0^{\alpha} \text{VaR}_t^{\mathbb{P}}(X|B) dt.$$

Note that if $B = \Omega$, conditional VaR and ES boil down to classical VaR and ES, i.e., $\text{VaR}_{\alpha}^{\mathbb{P}}(X) = \text{VaR}_{\alpha}^{\mathbb{P}}(X|\Omega)$ and $\text{ES}_{\alpha}^{\mathbb{P}}(X) = \text{ES}_{\alpha}^{\mathbb{P}}(X|\Omega)$. Analogously, the conditional Λ VaR (Co Λ VaR) can be defined as

$$\Lambda \text{VaR}^{\mathbb{P}}(X|B) = \inf\{x : \mathbb{P}[X \leq x | B] \geq \Lambda(x)\}.$$

Note that $\Lambda_i \text{VaR}^{\mathbb{Q}_i}(X) = \Lambda_i \text{VaR}^{\mathbb{P}}(X|B_i)$ if $\mathbb{Q}_i[\cdot] = \mathbb{P}[\cdot | B_i]$. Hence the results in Theorem 3.7 can be interpreted as a risk sharing using multiple Co Λ VaR to represent the preference of the agents. In the regulatory context, those B_i could be stressing test events given by different supervising agencies.

Remark 3.9 The law-invariance of the inf-convolution for some risk functionals was discussed in Liu et al. [44], showing that under homogeneous beliefs, it is guaranteed by some mild condition on the risk functionals if they are law-invariant. Proposition 3.5 shows that the inf-convolution of ΛVaR under homogeneous beliefs is law-invariant. However, if the beliefs are heterogeneous, the conclusion need not hold, which can be seen from Corollary 3.8. Suppose $\mathbb{Q}_i[\cdot] = \mathbb{P}[\cdot | B_i]$ for some $B_i \subseteq \Omega$ with $\mathbb{P}[\bigcap_{i=1}^n B_i] > 0$ and $\alpha_i \in (0, 1)$ satisfying $\sum_{i=1}^n \frac{\alpha_i \mathbb{P}[B_i]}{\mathbb{P}[\bigcap_{j=1}^n B_j]} < 1$. By Corollary 3.8, we have

$$\square_{i=1}^n \text{VaR}_{\alpha_i}^{\mathbb{Q}_i}(X) = \text{VaR}_{\sum_{i=1}^n \frac{\alpha_i \mathbb{P}[B_i]}{\mathbb{P}[\bigcap_{j=1}^n B_j]}}^{\mathbb{Q}}(X),$$

where $\mathbb{Q}[\cdot] = \mathbb{P}[\cdot | \bigcap_{i=1}^n B_i]$. One can easily check that $X \stackrel{\mathbb{Q}_i}{=} Y$ for all $i = 1, \dots, n$ does not imply $X \stackrel{\mathbb{Q}}{=} Y$. Hence $\square_{i=1}^n \text{VaR}_{\alpha_i}^{\mathbb{Q}_i}(X) \neq \square_{i=1}^n \text{VaR}_{\alpha_i}^{\mathbb{Q}_i}(Y)$ may hold.

3.4 General beliefs with $n = 2$

Next we consider the case $n = 2$. By Föllmer and Schied [27, Theorem A.17], for probability measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$, we have

$$\mathbb{Q}_2[B] = \mathbb{E}^{\mathbb{Q}_1}[\eta \mathbb{1}_B] + \mathbb{Q}_2[B \cap N] \quad \text{for all } B \in \mathcal{F}, \quad (3.2)$$

where $N \in \mathcal{F}$ satisfies $\mathbb{Q}_1[N] = 0$ and $\eta = \frac{d\mathbb{Q}_2}{d\mathbb{Q}_1} \mathbb{1}_{N^c}$. If $\mathbb{Q}_2[N] = 0$, then $\mathbb{Q}_2 \ll \mathbb{Q}_1$ on Ω ; if $\mathbb{Q}_2[N] > 0$, then $\mathbb{Q}_2 \ll \mathbb{Q}_1$ on N^c but not on N . **In this section**, the relation of $\mathbb{Q}_1$ and $\mathbb{Q}_2$ is represented by (3.2).

Theorem 3.10 For $\Lambda_i \in \mathcal{H}$ with $0 < \lambda_i^- \leq \lambda_i^+ < 1$, we have

$$\begin{aligned} & \Lambda_1 \text{VaR}^{\mathbb{Q}_1} \square \Lambda_2 \text{VaR}^{\mathbb{Q}_2}(X) \\ &= \inf \left\{ x \in \mathbb{R} : g_x \left((1 - F_X^{\mathbb{Q}_1}(x) - \Lambda_1(y))^+ \right) \leq \Lambda_2(x - y) \text{ for some } y \in \mathbb{R} \right\}, \end{aligned}$$

where

$$g_x(t) = \mathbb{E}^{\mathbb{Q}_1}[\eta \mathbb{1}_{\{X > x\} \cap \{U_\eta^{\mathbb{Q}_1} \leq x_t\}}], \quad t \in [0, 1 - F_X^{\mathbb{Q}_1}(x)],$$

with x_t satisfying $\mathbb{Q}_1[\{X > x\} \cap \{U_\eta^{\mathbb{Q}_1} \leq x_t\}] = t$. Moreover, if

$$x^* = \Lambda_1 \text{VaR}^{\mathbb{Q}_1} \square \Lambda_2 \text{VaR}^{\mathbb{Q}_2}(X) > -\infty$$

and

$$g_{x^*} \left((1 - F_X^{\mathbb{Q}_1}(x^*) - \Lambda_1(y^*))^+ \right) \leq \Lambda_2(x^* - y^*) \quad \text{for some } y^* \in \mathbb{R},$$

then an optimal allocation is given by

$$X_1 = (X - x^*) \mathbb{1}_{\Omega \setminus A^*} + y^*, \quad X_2 = (X - x^*) \mathbb{1}_{A^*} + x^* - y^*, \quad (3.3)$$

where $A^* = \{X > x^*\} \cap \{U_\eta^{\mathbb{Q}_1} \leq z^*\} \cap N^c$ with z^* satisfying

$$\mathbb{Q}_1[\{X > x^*\} \cap \{U_\eta^{\mathbb{Q}_1} \leq z^*\}] = (1 - F_X^{\mathbb{Q}_1}(x^*) - \Lambda_1(y^*))^+.$$

An immediate conclusion from Theorem 3.10 is the following result, showing that the inf-convolution is $-\infty$ if $\mathbb{Q}_2[N]$ is large enough.

Corollary 3.11 *Under the assumptions of Theorem 3.10, if $\mathbb{Q}_2[N] \geq 1 - \lambda_2^-$, then*

$$\Lambda_1 \text{VaR}^{\mathbb{Q}_1} \square \Lambda_2 \text{VaR}^{\mathbb{Q}_2}(X) = -\infty.$$

We point out that the idea used in the proof of Theorem 3.10 is not valid for the case $n \geq 3$. That case may require a new method to simplify the main result obtained in Theorem 3.1.

In Theorem 3.10, we do not obtain explicit formulas as in the previous results. This is due to the complexity of the relation between $\mathbb{Q}_1$ and $\mathbb{Q}_2$, which can be observed from the expression of g_x depending on the joint distribution of (X, η) under $\mathbb{Q}_1$. Let us next see some examples of g_x and the corresponding A^* in (3.3).

(i) If X and η are comonotonic, then

$$g_x(t) = \int_{F_X^{\mathbb{Q}_1}(x)}^{F_X^{\mathbb{Q}_1}(x)+t} F_\eta^{\mathbb{Q}_1, -1}(s) ds, \quad t \in [0, 1 - F_X^{\mathbb{Q}_1}(x)],$$

and $A^* = \{X > x^*\} \cap \{U_\eta^{\mathbb{Q}_1} \leq (1 - \Lambda_1(y^*)) \vee F_X^{\mathbb{Q}_1}(x^*)\} \cap N^c$.

(ii) If X and η are countermonotonic, then

$$g_x(t) = \int_0^t F_\eta^{\mathbb{Q}_1, -1}(s) ds, \quad t \in [0, 1 - F_X^{\mathbb{Q}_1}(x)],$$

and $A^* = \{X > x^*\} \cap \{U_\eta^{\mathbb{Q}_1} \leq (1 - F_X^{\mathbb{Q}_1}(x^*) - \Lambda_1(y^*))^+\} \cap N^c$.

(iii) If X and η are independent under $\mathbb{Q}_1$, then

$$g_x(t) = (1 - F_X^{\mathbb{Q}_1}(x)) \int_0^{t/(1-F_X^{\mathbb{Q}_1}(x))} F_\eta^{\mathbb{Q}_1, -1}(s) ds, \quad t \in [0, 1 - F_X^{\mathbb{Q}_1}(x)]$$

with the convention that $0/0 = 0$, and

$$A^* = \{X > x^*\} \cap \left\{ U_\eta^{\mathbb{Q}_1} \leq \frac{(1 - F_X^{\mathbb{Q}_1}(x^*) - \Lambda_1(y^*))^+}{1 - F_X^{\mathbb{Q}_1}(x^*)} \right\} \cap N^c$$

with $U_\eta^{\mathbb{Q}_1}$ being independent of X under $\mathbb{Q}_1$.

Note that the optimal sets A^* in (i) and (ii) have very different structures. To see this, suppose both X and η are continuous random variables under $\mathbb{Q}_1$ and $N = \emptyset$. In (i), A^* can be rewritten as $A^* = \{F_X^{\mathbb{Q}_1}(x^*) < U_X^{\mathbb{Q}_1} < (1 - \Lambda_1(y^*)) \vee F_X^{\mathbb{Q}_1}(x^*)\}$. Hence the contingent loss of agent 2 takes place on a set that does not include the full right tail of X . This is understandable because the comonotonicity of X and η means

that agent 2 considers the distribution of aggregate losses more risky in first-order stochastic dominance, i.e., $F_X^{\mathbb{Q}_2} \succeq_{\text{st}} F_X^{\mathbb{Q}_1}$ (see e.g. [Pesenti et al. \[48\]](#)). In the alternative case (ii) of comonotonicity, we can write $A^* = \{U_X^{\mathbb{Q}_1} > F_X^{\mathbb{Q}_1}(x^*) + \Lambda_1(y^*)\}$ so that agent 2 is exposed to the full tail of X , consistently with $F_X^{\mathbb{Q}_1} \succeq_{\text{st}} F_X^{\mathbb{Q}_2}$. Hence by (3.3), we see that if X and η are comonotonic, a lower contingent loss is allocated to the second agent; if X and η are countermonotonic, a higher contingent loss is allocated to the second agent.

Note that Theorem 3.10 can be simplified when the Λ -functions are constants.

Corollary 3.12 *For $\alpha_1, \alpha_2 \in (0, 1)$, we have*

$$\text{VaR}_{\alpha_1}^{\mathbb{Q}_1} \square \text{VaR}_{\alpha_2}^{\mathbb{Q}_2}(X) = \inf \left\{ x \in \mathbb{R} : g_x \left((1 - F_X^{\mathbb{Q}_1}(x) - \alpha_1)^+ \right) \leq \alpha_2 \right\},$$

where

$$g_x(t) = \mathbb{E}^{\mathbb{Q}_1} [\eta \mathbb{1}_{\{X > x, U_{\eta}^{\mathbb{Q}_1} \leq x_t\}}], \quad t \in [0, 1 - F_X^{\mathbb{Q}_1}(x)],$$

with x_t satisfying $\mathbb{Q}_1[\{X > x\} \cap \{U_{\eta}^{\mathbb{Q}_1} \leq x_t\}] = t$.

Applying Corollary 3.12, we have the following results.

Corollary 3.13 (i) *If X and η are comonotonic, then*

$$\text{VaR}_{\alpha_1}^{\mathbb{Q}_1} \square \text{VaR}_{\alpha_2}^{\mathbb{Q}_2}(X) = \inf \left\{ x \in \mathbb{R} : \int_{F_X^{\mathbb{Q}_1}(x)}^{F_X^{\mathbb{Q}_1}(x) \vee (1 - \alpha_1)} F_{\eta}^{\mathbb{Q}_1, -1}(s) ds \leq \alpha_2 \right\}.$$

(ii) *If X and η are countermonotonic, then*

$$\text{VaR}_{\alpha_1}^{\mathbb{Q}_1} \square \text{VaR}_{\alpha_2}^{\mathbb{Q}_2}(X) = \inf \left\{ x \in \mathbb{R} : \int_0^{(1 - F_X^{\mathbb{Q}_1}(x) - \alpha_1)^+} F_{\eta}^{\mathbb{Q}_1, -1}(s) ds \leq \alpha_2 \right\}.$$

(iii) *If X and η are independent under $\mathbb{Q}_1$, then*

$$\begin{aligned} & \text{VaR}_{\alpha_1}^{\mathbb{Q}_1} \square \text{VaR}_{\alpha_2}^{\mathbb{Q}_2}(X) \\ &= \inf \left\{ x \in \mathbb{R} : \int_0^{(1 - F_X^{\mathbb{Q}_1}(x) - \alpha_1)^+ / (1 - F_X^{\mathbb{Q}_1}(x))} F_{\eta}^{\mathbb{Q}_1, -1}(s) ds (1 - F_X^{\mathbb{Q}_1}(x)) \leq \alpha_2 \right\}, \end{aligned}$$

with the convention that $0/0 = 0$.

4 Inf-convolution of Λ VaR and one monotone risk measure with heterogeneous beliefs

In this section, we investigate the risk sharing problem between two agents, considering the inf-convolution of Λ VaR and a monotone risk measure ρ under the probability

measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$, where $\mathbb{Q}_2$ may be different from $\mathbb{Q}_1$. Note that ρ here is very general, including many commonly used risk measures as special cases. In our results below, we consider both the cases $\mathcal{X} = \mathcal{L}^\infty$ and unbounded $\mathcal{X}$, each with different assumptions on ρ .

Theorem 4.1 *Suppose $\Lambda \in \mathcal{H}$ with $0 < \lambda^- \leq \lambda^+ < 1$ and ρ is monotone. If either (i) $\mathcal{X} = \mathcal{L}^\infty$ and ρ is law-invariant under $\mathbb{Q}_2$ or (ii) $\mathcal{X}$ is unbounded and ρ is an ε -tail risk measure under $\mathbb{Q}_2$ for some $\varepsilon \in (0, 1)$, then we have*

$$\Lambda \text{VaR}^{\mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X) = \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B] = 1 - \Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \right). \quad (4.1)$$

Moreover, the existence of an optimal allocation for the inf-convolution is equivalent to the existence of a minimiser of (4.1). If (x^*, y^*, B^*) is a minimiser, then one optimal allocation is given by

$$X_1^* = (X - x^* - y^*)\mathbb{1}_{(B^*)^c} + x^*, \quad X_2^* = (X - x^* - y^*)\mathbb{1}_{B^*} + y^*. \quad (4.2)$$

The optimal allocation in (4.2) has the same form as in Theorem 3.1: each agent is allocated a constant risk and a contingent risk, which is the excess of the total risk over the aggregate cash allocation restricted to some contingent event. Note that the expression of the inf-convolution in Theorem 4.1 involves an optimisation problem with three parameters (x, y, B) , where B is a set in $\mathcal{F}$. The set B makes this optimisation problem complicated. The optimal allocation in (4.2) need not be pairwise countermonotonic.

Our conclusion in Theorem 4.1 extends the results of Embrechts et al. [24] for the case $n = 2$, where the inf-convolution of VaR or ES with heterogeneous beliefs is considered. If $\mathbb{Q}_1 = \mathbb{Q}_2$, Theorem 4.1 coincides with Liu [43, Theorem 3]. We notice that for the scenario that $\mathbb{Q}_1 = \mathbb{Q}_2$ and Λ is a constant, Theorem 4.1 extends Liu et al. [40, Theorem 2] (ρ is there considered to be a monetary tail risk measure) and Wang and Wei [52, Theorem 5.3] (ρ is there a distortion risk measure). The result of Theorem 4.1 is still new even if ΛVaR is reduced to VaR. Let us now consider the case $\mathbb{Q}_i[\cdot] = \mathbb{P}[\cdot | B_i]$ for some $B_i \in \mathcal{F}$ with $\mathbb{P}[B_i] > 0$, where $\mathbb{P}$ is atomless. Under this setup, the contingent set B in (4.1) is given explicitly.

Proposition 4.2 *Suppose $\mathbb{Q}_i[\cdot] = \mathbb{P}[\cdot | B_i]$ for some $B_i \subseteq \Omega$ with $\mathbb{P}[B_i] > 0$, $i = 1, 2$. Moreover, suppose $\Lambda \in \mathcal{H}$ with $0 < \lambda^- \leq \lambda^+ < 1$ and ρ is monotone. If either (i) $\mathcal{X} = \mathcal{L}^\infty$ and ρ is law-invariant under $\mathbb{Q}_2$ or (ii) $\mathcal{X}$ is unbounded and ρ is an ε -tail risk measure under $\mathbb{Q}_2$ for some $\varepsilon \in (0, 1)$, then we have*

$$\Lambda \text{VaR}^{\mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X) = \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \left(x + \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_{A_x} + y\mathbb{1}_{A_x^c}) \right), \quad (4.3)$$

where $A_x = \emptyset$ if $\mathbb{P}[B_1 \cap B_2^c] \geq (1 - \Lambda(x))\mathbb{P}[B_1]$ and

$$A_x = (B_1 \cap B_2^c) \cup (B_1 \cap B_2 \cap \{U_X^{\mathbb{P}} < \alpha_x\})$$

if $\mathbb{P}[B_1 \cap B_2^c] < (1 - \Lambda(x))\mathbb{P}[B_1]$, with α_x satisfying

$$\mathbb{P}[B_1 \cap B_2 \cap \{U_X^{\mathbb{P}} < \alpha_x\}] = (1 - \Lambda(x))\mathbb{P}[B_1] - \mathbb{P}[B_1 \cap B_2^c].$$

Moreover, the existence of an optimal allocation of the inf-convolution is equivalent to the existence of a minimiser of (4.3). If (x^*, y^*) is a minimiser, then one optimal allocation is given by

$$X_1^* = (X - x^* - y^*)\mathbb{1}_{A_{x^*}^c} + x^*, \quad X_2^* = (X - x^* - y^*)\mathbb{1}_{A_{x^*}} + y^*. \quad (4.4)$$

Note that in an optimal allocation given by (4.4), the contingent event A_{x^*} is a fixed event, making an optimal allocation more concrete than that of Theorem 4.1. We next interpret an optimal allocation given by (4.4) by focusing on the case $A_{x^*} = \emptyset$. Note that the condition for $A_{x^*} = \emptyset$ can be reformulated as the condition

$$\mathbb{P}[B_1 \cap B_2^c] \geq (1 - \Lambda(x^*))\mathbb{P}[B_1], \quad \text{i.e., } \mathbb{Q}_1[B_2] \leq \Lambda(x^*).$$

Clearly, $A_{x^*} = \emptyset$ means that all the risk stays with agent 1. Observe that the condition $\mathbb{Q}_1[B_2] \leq \Lambda(x^*)$ is satisfied when $\mathbb{Q}_1[B_2]$ is small (this means that agent 1 dismisses the view of agent 2, so that we have more information asymmetry), or when $\Lambda(x^*)$ is large (this means that agent 1 has a higher risk tolerance.)

Let $\mathcal{G}$ be the set of all increasing and left-continuous functions $g : [0, 1] \rightarrow [0, 1]$ satisfying $g(0) = 0$ and $g(1) = 1$. For $g \in \mathcal{G}$, the distortion risk measure ρ_g under $\mathbb{Q}$ is defined as

$$\rho_g^{\mathbb{Q}}[X] = \int_0^1 \text{VaR}_t^{\mathbb{Q}}(X) dg(t).$$

Distortion risk measures form a popular class of risk measures applied in insurance pricing, performance evaluation, decision theory and many other topics; see e.g. Föllmer and Schied [27, Chap. 4.6], Wang [53, 54], Cherny and Madan [17] and Yaari [58]. Note that if $\mathbb{Q}[\cdot] = \mathbb{P}[\cdot|B]$ for some $B \in \mathcal{F}$ satisfying $\mathbb{P}[B] > 0$, then $\rho_g^{\mathbb{Q}}(X)$ corresponds to the co-distortion risk measures introduced in Dhaene et al. [20].

In the next result, let $\mathbb{Q}_i[\cdot] = \mathbb{P}[\cdot|B_i]$ for some $B_i \subseteq \Omega$ with $\mathbb{P}[B_1 \cap B_2] > 0$, indicating that beliefs are not totally incompatible, and $\mathbb{Q}[\cdot] = \mathbb{P}[\cdot|B_1 \cap B_2]$, which in a sense represents the consensus view of the two agents. Applying Proposition 4.2, we obtain the inf-convolution of ΛVaR and ρ_g , as well as that of ΛVaR and $\Lambda_1 \text{VaR}^+$, under heterogeneous beliefs. In order to simplify the expression, we suppose λ^+ is attainable for Λ , i.e., there is $x \in \mathbb{R}$ such that $\Lambda(x) = \lambda^+$.

Corollary 4.3 Suppose $\Lambda \in \mathcal{H}$ with $0 < \lambda^- \leq \lambda^+ < 1$ and λ^+ is attainable. Then the following conclusion holds with $\beta = \frac{(1-\lambda^+)\mathbb{P}[B_1] - \mathbb{P}[B_1 \cap B_2^c]}{\mathbb{P}[B_2]}$:

(i) For $g \in \mathcal{G}$ satisfying $g(\varepsilon) = 1$ for some $\varepsilon \in (0, 1)$, if $\mathbb{Q}_1[B_2] \leq \lambda^+$, then

$$\Lambda \text{VaR}^{\mathbb{Q}_1} \square \rho_g^{\mathbb{Q}_2}(X) = -\infty;$$

if $\mathbb{Q}_1[B_2] > \lambda^+$, then

$$\Lambda \text{VaR}^{\mathbb{Q}_1} \square \rho_g^{\mathbb{Q}_2}(X) = \begin{cases} \int_{[0, \beta)} \text{VaR}_{1 - \frac{\mathbb{P}[B_2](\beta-t)}{\mathbb{P}[B_1 \cap B_2]}}^{\mathbb{Q}}(X) dg(t), & g(\beta) = 1, \\ -\infty, & g(\beta) < 1. \end{cases}$$

(ii) For $\Lambda_1 \in \mathcal{H}$ with $0 < \lambda_1^- \leq \lambda_1^+ < 1$, if $\mathbb{Q}_1[B_2] \leq \lambda^+$, then

$$\Lambda \text{VaR}^{\mathbb{Q}_1} \square \Lambda_1 \text{VaR}^{+, \mathbb{Q}_2}(X) = -\infty;$$

if $\mathbb{Q}_1[B_2] > \lambda^+$, then

$$\Lambda \text{VaR}^{\mathbb{Q}_1} \square \Lambda_1 \text{VaR}^{+, \mathbb{Q}_2}(X) = \inf_{x \in \mathbb{R}} \bar{\Lambda}^x \text{VaR}^{+, \mathbb{Q}}(X),$$

where $\bar{\Lambda}^x(z) = \frac{\mathbb{P}[B_1]\Lambda(x) + \mathbb{P}[B_2]\Lambda_1(z-x)}{\mathbb{P}[B_1 \cap B_2]} \wedge 1$.

Note that if we let $g(t) = (t/\alpha) \wedge 1$ for $\alpha \in (0, 1)$, then the distortion risk measure is the expected shortfall, i.e., $\rho_g = \text{ES}_\alpha$. In that case, part (i) of Corollary 4.3 simplifies to stating that if $\mathbb{Q}_1[B_2] > \lambda^+$, then

$$\Lambda \text{VaR}^{\mathbb{Q}_1} \square \text{ES}_\alpha^{\mathbb{Q}_2}(X) = \begin{cases} \frac{1}{\alpha} \int_0^\alpha \text{VaR}_{1 - \frac{\mathbb{P}[B_2](\beta-t)}{\mathbb{P}[B_1 \cap B_2]}}^{\mathbb{Q}}(X) dt, & \alpha \leq \beta, \\ -\infty, & \alpha > \beta. \end{cases}$$

For all three cases in Corollary 4.3, the condition $\mathbb{Q}_1[B_2] \leq \lambda^+$, indicating high levels of belief heterogeneity or a high level of risk tolerance for agent 1, gives a sufficient condition for the inf-convolution to be infinite. For part (i), we also get an infinite allocation as long as the quantity β is small enough so that $g(\beta) < 1$. Note that in the special case where ρ_g is expected shortfall at level α , this means $\beta < \alpha$. To interpret small values of β , we can reformulate this quantity as

$$\beta = \frac{\mathbb{P}[B_1]}{\mathbb{P}[B_2]}(\mathbb{Q}_1[B_2] - \lambda^+).$$

It is seen that β becomes small if

- λ^+ is high, so that agent 1 has a high maximal risk tolerance;
- $\mathbb{Q}_1[B_2]$ is low, so we have more belief heterogeneity;
- $\frac{\mathbb{P}[B_1]}{\mathbb{P}[B_2]}$ is low, since we are in the case that $\mathbb{Q}_1[B_2] > \lambda^+$. This fraction can be understood, in some sense, as a comparison of the subjective informativeness of the conditioning done by the two agents. If a low probability event is observed, one could claim that more has been learned about the underlying risk. Hence if the ratio is small, agent 1 has subjectively learned more than agent 2 and is in that sense more confident.

These interpretations indicate that an infinite inf-convolution may arise when beliefs are heterogeneous, agent 1 is more risk tolerant, and the observed event is sufficiently informative to agent 1.

We mention that the results in Corollary 4.3 extend the results in the literature in the sense that Λ is not constant and the second risk measure is not law-invariant under $\mathbb{Q}_1$. For instance, (i) of Corollary 4.3 exactly extends Wang and Wei [52, Theorem 5.3] (ρ is a distortion risk measure which is law-invariant under $\mathbb{Q}_1$) in these two directions. It is worth mentioning that (ii) of Corollary 4.3 is a new result even for the case $\mathbb{Q}_1 = \mathbb{Q}_2$.

Next, suppose $\mathbb{Q}_1$ and $\mathbb{Q}_2$ have the relation (3.2). For an increasing function u , the expected utility is defined as $\mathbb{E}_u^{\mathbb{Q}} : X \mapsto \mathbb{E}^{\mathbb{Q}}[u(X)]$. Instead of finding a simplified version of Theorem 4.1 for this case, we apply Theorem 4.1 directly to some

specific ρ . For $\alpha \in (0, 1)$, let $\text{LES}_\alpha^{\mathbb{Q}}(X) = \frac{1}{\alpha} \int_{1-\alpha}^1 \text{VaR}_t^{\mathbb{Q}}(X) dt$, which represents the lower conditional tail expectation.

Corollary 4.4 For $\Lambda \in \mathcal{H}$ with $0 < \lambda^- \leq \lambda^+ < 1$, we have the following:

(i) For $\alpha \in (0, 1)$,

$$\Lambda \text{VaR}^{\mathbb{Q}_1} \square \text{ES}_\alpha^{\mathbb{Q}_2}(X) = \inf_{t \in \mathbb{R}} \left(t + \frac{1 - \lambda^+}{\alpha} \text{LES}_{1-\lambda^+}^{\mathbb{Q}_1}(\eta(X - t)^+) \right).$$

(ii) For $\mathcal{X} = \mathcal{L}^\infty$ and an increasing function u , we have

$$\begin{aligned} \Lambda \text{VaR}^{\mathbb{Q}_1} \square \mathbb{E}_u^{\mathbb{Q}_2}(X) = \inf_{x \in \mathbb{R}} \left(x + u(-\infty) (\mathbb{Q}_2[N] + \Lambda(x) \text{ES}_{\Lambda(x)}^{\mathbb{Q}_1}(\eta)) \right. \\ \left. + \mathbb{E}^{\mathbb{Q}_1} [\eta u(X - x) \mathbb{1}_{\{U_\eta^{\mathbb{Q}_1} < 1 - \Lambda(x)\}}] \right), \end{aligned}$$

where $u(-\infty) = \lim_{y \rightarrow -\infty} u(y)$.

Note that in (i) of Corollary 4.4, if Λ is constant, the result coincides with Embrechts et al. [24, Theorem 5] for $n = 2$, where a semi-explicit formula is offered. Using (3.2), it might be difficult to find a more explicit expression for the inf-convolution of $\Lambda \text{VaR}^{\mathbb{Q}_1}$ and $\rho_g^{\mathbb{Q}_2}$, as well as that of $\Lambda \text{VaR}^{\mathbb{Q}_1}$ and $\Lambda_1 \text{VaR}^{+, \mathbb{Q}_2}$.

5 Inf-convolution of ΛVaR^+ and a monotone risk measure

The inf-convolution of multiple ΛVaR^+ under heterogeneous beliefs is a very challenging problem even if we suppose the beliefs are homogeneous; see the discussion for homogeneous beliefs in Liu [43]. One important feature of ΛVaR^+ is to distinguish the tail risk for decreasing Λ ; see Frittelli et al. [29] and Hitaj et al. [33]. In this section, we focus on the inf-convolution of ΛVaR^+ and a monotone risk measure ρ under $\mathbb{Q}_1$ and $\mathbb{Q}_2$, which includes the inf-convolution of two ΛVaR^+ as a special case. For any $\Lambda \in \mathcal{H}$, we set $\Lambda_x(z) = \inf_{x \leq t \leq z} \Lambda(t)$, $z \geq x$. Then $\Lambda_x(z)$ is decreasing and right-continuous for $z \geq x$. If Λ is decreasing and right-continuous, then $\Lambda_x(z) = \Lambda(z)$, $z \geq x$, and if Λ is increasing, then $\Lambda_x(z) = \Lambda(x)$, $z \geq x$. The following result is valid for general $\Lambda \in \mathcal{H}$. We call ρ continuous from above on $\mathcal{X}$ if $X_n \downarrow X$ implies $\lim_{n \rightarrow \infty} \rho(X_n) = \rho(X)$ for $X_n, X \in \mathcal{X}$.

Theorem 5.1 Suppose $\Lambda \in \mathcal{H}$ with $0 < \lambda^- \leq \lambda^+ < 1$ and ρ is monotone and law-invariant under $\mathbb{Q}_2$. If either (i) $\mathcal{X} = \mathcal{L}^\infty$ or (ii) $\mathcal{X}$ is unbounded and ρ is continuous from above, then we have

$$\Lambda \text{VaR}^{+, \mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X) = \inf_{x \in \mathbb{R}} \inf_{y \geq x} \inf_{U \sim \mathcal{U}[0, 1]} \left(x + \rho^{\mathbb{Q}_2}(X - \Lambda_{x, y}^{-1}(U)) \right), \quad (5.1)$$

where

$$\Lambda_{x,y}(z) = \begin{cases} 0, & z < x, \\ 1 - \Lambda_x(z), & x \leq z < y, \\ 1, & z \geq y. \end{cases}$$

Moreover, for case (i), an optimal allocation of the inf-convolution exists if and only if a minimiser of (5.1) exists. For both cases (i) and (ii), if (x^*, y^*, U^*) is a minimiser, then one optimal allocation is given by

$$X_1^* = \Lambda_{x^*,y^*}^{-1}(U^*), \quad X_2^* = X - \Lambda_{x^*,y^*}^{-1}(U^*). \quad (5.2)$$

In the case of $\rho = \Lambda_1 \text{VaR}^+$ for $\Lambda_1 \in \mathcal{H}$ with $0 < \lambda_1^- \leq \lambda_1^+ < 1$, it follows from Theorem 5.1 that an optimal risk allocation for risk sharing between ΛVaR^+ and $\Lambda_1 \text{VaR}^+$ under $\mathbb{Q}_1$ and $\mathbb{Q}_2$ has the form (5.2), which is neither comonotonic nor pairwise countermonotonic. This means that an optimal risk sharing for multiple ΛVaR^+ under heterogeneous beliefs is qualitatively different from the cases with convex risk measures or quantile-based risk measures in the literature. The expression of the inf-convolution in (5.1) is an optimisation problem involving three parameters (x, y, U) , where $U \sim \mathcal{U}[0, 1]$ under $\mathbb{Q}_1$. Note that the distribution of U under $\mathbb{Q}_2$ depends on the relation between $\mathbb{Q}_1$ and $\mathbb{Q}_2$. Moreover, the expression in (5.1) and also the optimal allocation in (5.2) show that the shape of the Λ -function plays an important role in both the expression of the inf-convolution and the form of optimal allocations, which is very different from the results of ΛVaR displayed in Sects. 3 and 4.

For case (ii) of Theorem 5.1, we suppose that ρ is continuous from above. In fact, many risk functionals satisfy this assumption, such as distortion risk measures ρ_g with continuous g (including ES_α as a special case), expected utility $\mathbb{E}_u$ for continuous and increasing u , and right quantiles. Later, we define an alternative ΛVaR measure that also satisfies this assumption.

We are also aware that the conclusion in Theorem 5.1 is complicated in the sense that it involves $\inf_{U \sim \mathbb{Q}_1 \mathcal{U}[0,1]} \rho^{\mathbb{Q}_2}(X - \Lambda_{x,y}^{-1}(U))$, which is a very difficult problem even if $\mathbb{Q}_1 = \mathbb{Q}_2$. For $\mathbb{Q}_1 = \mathbb{Q}_2$, $\inf_{U \sim \mathbb{Q}_1 \mathcal{U}[0,1]} \rho^{\mathbb{Q}_2}(X - \Lambda_{x,y}^{-1}(U))$ corresponds to the problem of robust risk aggregation with dependence uncertainty, i.e., finding the value of $\inf_{X \sim F, Y \sim G} \rho(X + Y)$, where the marginal distributions are known but the dependence structure is completely unknown. We refer to Wang and Wang [51], Jakobsen et al. [35], Blanchet et al. [12], Han and Liu [31] and the references therein for discussions on robust risk aggregation.

We write $X \leq_{\text{icx}} Y$ if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all increasing and convex functions f for which the expectations are finite. A mapping $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is called SSD-consistent if $X \leq_{\text{icx}} Y$ implies $\rho(X) \leq \rho(Y)$. Note that here $\mathcal{X} \subseteq \mathcal{L}^1$.

Corollary 5.2 Suppose $\mathcal{X} \subseteq \mathcal{L}^1$. If $\mathbb{Q}_1 = \mathbb{Q}_2 = \mathbb{Q}$ and ρ is an SSD-consistent risk measure, then under the assumptions of Theorem 5.1, we have

$$\Lambda \text{VaR}^{+, \mathbb{Q}} \square \rho^{\mathbb{Q}}(X) = \inf_{x \in \mathbb{R}} \inf_{y \geq x} \left(x + \rho^{\mathbb{Q}}(X - \Lambda_{x,y}^{-1}(U_X^{\mathbb{Q}})) \right).$$

Note that this result coincides with Liu [43, Theorem 5]. In the following corollary, we consider a risk measure ρ that is not SSD-consistent.

For $\Lambda : \mathbb{R} \rightarrow [0, 1]$ and an increasing $f : \mathbb{R} \rightarrow [0, 1]$, define

$$q_{\Lambda}^{+}(f) = \inf\{x \in \mathbb{R} : f(x) > 1 - \Lambda(x)\},$$

where $\inf \emptyset = \infty$. Note that if Λ is constant and f is a distribution function, then $q_{\Lambda}^{+}(f)$ is the right quantile. We refer to Bellini and Peri [9] and Han and Liu [31] for properties and applications of q_{Λ}^{+} .

Define an alternative version of ΛVaR by $\overline{\Lambda \text{VaR}}(X) = q_{\Lambda}^{+}(F_X)$ for $X \in \mathcal{X}$. One can easily show that $\overline{\Lambda \text{VaR}}$ is continuous from above if $\Lambda \in \mathcal{H}$. Then applying Theorem 5.1, we have the following result.

Corollary 5.3 Suppose $\mathbb{Q}_1 = \mathbb{Q}_2 = \mathbb{Q}$, $\Lambda \in \mathcal{H}$ with $0 < \lambda^{-} \leq \lambda^{+} < 1$ is strictly decreasing and $\Lambda_1 \in \mathcal{H}$ with $0 < \lambda_1^{-} \leq \lambda_1^{+} < 1$. Then we have for $X \in \mathcal{X}$ with continuous $F_X^{\mathbb{Q}, -1}$ over $(0, 1)$ that

$$\Lambda \text{VaR}^{+, \mathbb{Q}} \square \overline{\Lambda_1 \text{VaR}}^{\mathbb{Q}}(X) = \inf_{x \in \mathbb{R}} \inf_{y \geq x} (x + q_{\Lambda_1}^{+}(\ell_{x,y}^{-1})),$$

where

$$\ell_{x,y}(t) = \sup_{s \in (0,1]} (F_X^{\mathbb{Q}, -1}(s) - \Lambda_{x,y}^{-1}(1 - t + s)), \quad t \in (0, 1)$$

and $\ell_{x,y}^{-1}(z) = \inf\{w \in (0, 1) : \ell_{x,y}(w) \geq z\}$, $z \in \mathbb{R}$, with $\inf \emptyset = 1$.

Corollary 5.3 requires that $F_X^{\mathbb{Q}, -1}$ is continuous over $(0, 1)$, which is a technical assumption for the computation of $\inf_{U \sim \mathcal{U}[0,1]} \overline{\Lambda_1 \text{VaR}}^{\mathbb{Q}}(X - \Lambda_{x,y}^{-1}(U))$. Our computation need not be correct if this assumption is violated.

To simplify (5.1), we consider a special case of Λ -functions. This makes it easier to find more explicit expressions for the inf-convolution in concrete examples of ρ .

Proposition 5.4 For $x_1 \in \mathbb{R}$ and $0 < \lambda_1 < \lambda_2 < 1$, let

$$\Lambda(z) = (1 - \lambda_1)\mathbb{1}_{\{z < x_1\}} + (1 - \lambda_2)\mathbb{1}_{\{z \geq x_1\}}.$$

Under the assumptions of Theorem 5.1, we have

$$\begin{aligned} & \Lambda \text{VaR}^{+, \mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X) \\ &= \inf_{x \in \mathbb{R}} \inf_{y \geq x \vee x_1} \inf_{(B_1, B_2) \in \mathcal{B}_{\lambda_1, \lambda_2}} \left(x + \rho^{\mathbb{Q}_2}(X - x\mathbb{1}_{B_1} - x \vee x_1\mathbb{1}_{B_2} - y\mathbb{1}_{(B_1 \cup B_2)^c}) \right), \end{aligned} \quad (5.3)$$

where $\mathcal{B}_{\lambda_1, \lambda_2} = \{(B_1, B_2) : \mathbb{Q}_1[B_1] = \lambda_1, \mathbb{Q}_1[B_2] = \lambda_2 - \lambda_1, B_1 \cap B_2 = \emptyset\}$.

Note that the optimisation problem in (5.3) is much easier than that in (5.1) because it only has parameters (x, y, B_1, B_2) . To illustrate Proposition 5.4, we next consider the example of $\rho^{\mathbb{Q}_2} = \text{ES}_{\alpha}^{\mathbb{Q}_2}$. In the following result, we suppose $\mathbb{Q}_i[\cdot] = \mathbb{P}[\cdot | D_i]$ for some $D_i \in \mathcal{F}$, $i = 1, 2$, and $\mathbb{Q}[\cdot] = \mathbb{P}[\cdot | D_1 \cap D_2]$ with $\mathbb{P}[D_1 \cap D_2] > 0$.

Corollary 5.5 *For $x_1 \in \mathbb{R}$ and $0 < \lambda_1 < \lambda_2 < 1$, let*

$$\Lambda(z) = (1 - \lambda_1)\mathbb{1}_{\{z < x_1\}} + (1 - \lambda_2)\mathbb{1}_{\{z \geq x_1\}}.$$

Then we have

$$\Lambda \text{VaR}^{+, \mathbb{Q}_1} \square \text{ES}_{\alpha}^{\mathbb{Q}_2}(X) = \begin{cases} \frac{1}{\alpha} \frac{\mathbb{P}[D_1 \cap D_2]}{\mathbb{P}[D_2]} \int_{\alpha_0}^{\alpha_1} \text{VaR}_t^{\mathbb{Q}}(X) dt, & \alpha_2 \geq \alpha, \\ -\infty, & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \alpha_0 &= 1 - \frac{\lambda_1 \mathbb{P}[D_1] - \mathbb{P}[D_1 \cap D_2^c]}{\mathbb{P}[D_1 \cap D_2]}, \\ \alpha_1 &= 1 - \frac{\lambda_1 \mathbb{P}[D_1] - \mathbb{P}[D_1 \cap D_2^c] - \alpha \mathbb{P}[D_2]}{\mathbb{P}[D_1 \cap D_2]}, \\ \alpha_2 &= \frac{\lambda_1 \mathbb{P}[D_1] - \mathbb{P}[D_1 \cap D_2^c]}{\mathbb{P}[D_2]}. \end{aligned}$$

6 Conclusion

In this paper, we consider the inf-convolution of multiple ΛVaR under heterogeneous beliefs. We obtain a general expression for the inf-convolution and the corresponding optimal allocations. To study the impact of the relation of the beliefs on the inf-convolution, we discuss three different cases: i) homogeneous beliefs; ii) conditional beliefs; iii) general beliefs with two agents. For all these cases, we obtain more explicit expressions for the inf-convolution. In all cases, we demonstrate that trivial outcomes arise when both belief inconsistency and risk tolerance are high. The inf-convolution of one ΛVaR or ΛVaR^+ and a general risk measure with belief heterogeneity are also discussed. There are still some unsolved problems which deserve further study, such as what the inf-convolution of multiple ΛVaR under general heterogeneous beliefs is, and what the inf-convolution of multiple ΛVaR^+ under heterogeneous beliefs is.

Appendix: Proofs

Proof of Proposition 2.1 Note that there exists $A \in \mathcal{F}$ such that $\mathbb{Q}_1[A] = 0$ and $\mathbb{Q}_2[A] = 1$. It follows from the law-invariance of ρ_1 and ρ_2 and the constant-

preservation of ρ_2 that

$$\begin{aligned}\rho_1^{\mathbb{Q}_1} \square \rho_2^{\mathbb{Q}_2}(X) &\leq \rho_1^{\mathbb{Q}_1}(X - c\mathbb{1}_A) + \rho_2^{\mathbb{Q}_2}(c\mathbb{1}_A) \\ &= \rho_1^{\mathbb{Q}_1}(X) + \rho_2^{\mathbb{Q}_2}(c) = \rho_1^{\mathbb{Q}_1}(X) + c \longrightarrow -\infty\end{aligned}$$

as $c \rightarrow -\infty$. Hence $\rho_1^{\mathbb{Q}_1} \square \rho_2^{\mathbb{Q}_2}(X) = -\infty$. $\square$

Proof of Theorem 3.1 We first show $\square_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X) \geq \Gamma_{\Lambda_1, \dots, \Lambda_n}(X)$. For fixed $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$, let $x = \sum_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X_i)$ and $y_i = \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X_i)$. By definition, $0 < \lambda_i^- \leq \lambda_i^+ < 1$ implies $y_i \in \mathbb{R}$ and $x \in \mathbb{R}$. It follows from the definition of $\Lambda_i \text{VaR}$ and the right-continuity of Λ_i that $\mathbb{Q}_i[C_i] \leq \Lambda_i(y_i)$ with $C_i = \{X_i > y_i\}$. We denote $C_i \cup (\bigcup_{i=1}^n C_i)^c$ by D_i . Then it follows that

$$\{X > x\} = \left\{ \sum_{i=1}^n X_i > \sum_{i=1}^n y_i \right\} \subseteq \bigcup_{i=1}^n C_i.$$

For $(A_1, \dots, A_n) \in \Pi_n(\Omega)$, let $A_i \subseteq D_i$. Then we have

$$\{X > x\} \cap A_i \subseteq \{X > x\} \cap D_i = \{X > x\} \cap C_i \subseteq C_i.$$

Using $\mathbb{Q}_i[C_i] \leq \Lambda_i(y_i)$, we have $\mathbb{Q}_i[\{X > x\} \cap A_i] \leq \Lambda_i(y_i)$. This implies $\sum_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X_i) = x \geq \Gamma_{\Lambda_1, \dots, \Lambda_n}(X)$. Because $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is arbitrary, we have $\square_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X) \geq \Gamma_{\Lambda_1, \dots, \Lambda_n}(X)$.

Next we show the converse inequality. Suppose there exist $(A_1, \dots, A_n) \in \Pi_n(\Omega)$ and $y_i \in \mathbb{R}$ such that $\mathbb{Q}_i[\{X > \sum_{j=1}^n y_j\} \cap A_i] \leq \Lambda_i(y_i)$. Let

$$X_i = (X - x)\mathbb{1}_{A_i} + y_i, \quad i = 1, \dots, n,$$

where $x = \sum_{i=1}^n y_i$. Then $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$. Direct computation gives

$$\Lambda_i \text{VaR}^{\mathbb{Q}_i}(X_i) = y_i + \Lambda_i^{y_i} \text{VaR}^{\mathbb{Q}_i}((X - x)\mathbb{1}_{A_i}),$$

where $\Lambda_i^{y_i}(z) = \Lambda_i(z + y_i)$ for $z \in \mathbb{R}$. Note that

$$\mathbb{Q}_i[(X - x)\mathbb{1}_{A_i} > 0] = \mathbb{Q}_i[\{X > x\} \cap A_i] \leq \Lambda_i(y_i) = \Lambda_i^{y_i}(0).$$

By definition, we have $\Lambda_i^{y_i} \text{VaR}^{\mathbb{Q}_i}((X - x)\mathbb{1}_{A_i}) \leq 0$. Hence

$$\sum_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X_i) \leq \sum_{i=1}^n y_i = x.$$

This implies $\square_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X) \leq \sum_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X_i) \leq \sum_{i=1}^n y_i$. Because the y_i are arbitrary, we have $\square_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X) \leq \Gamma_{\Lambda_1, \dots, \Lambda_n}(X)$. One can directly check that X_i^* , $i = 1, \dots, n$, is an optimal allocation. This completes the proof. $\square$

Proof of Proposition 3.2 (i) Note that the property that $\bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^-} \leq 1$ for some $(A_1, \dots, A_n) \in \Pi_n(\Omega)$ implies $\mathbb{Q}_i[A_i] \leq \lambda_i^-$ for all $i = 1, \dots, n$. Hence for $(y_1, \dots, y_n) \in \mathbb{R}^n$, it follows that

$$\mathbb{Q}_i \left[\left\{ X > \sum_{j=1}^n y_j \right\} \cap A_i \right] \leq \mathbb{Q}_i[A_i] \leq \lambda_i^- \leq \Lambda_i(y_i).$$

Consequently, $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) = -\infty$.

(ii) We assume by way of contradiction that $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) = -\infty$. By definition, there exist $(y_1^{(m)}, \dots, y_n^{(m)})$ which satisfy $\lim_{m \rightarrow \infty} \sum_{i=1}^n y_i^{(m)} = -\infty$ and there exist $(A_1^{(m)}, \dots, A_n^{(m)}) \in \Pi_n(\Omega)$ such that for $i = 1, \dots, n$,

$$\mathbb{Q}_i \left[\left\{ X > \sum_{j=1}^n y_j^{(m)} \right\} \cap A_i^{(m)} \right] \leq \Lambda_i(y_i^{(m)}).$$

Moreover, using that $\inf_{(A_1, \dots, A_n) \in \Pi_n(\Omega)} \bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^+} > 1$, there exists $\varepsilon > 0$ such that $\bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^+} > 1 + \varepsilon$ for any $(A_1, \dots, A_n) \in \Pi_n(\Omega)$. Therefore there exists $i_m \in \{1, 2, \dots, n\}$ such that $\mathbb{Q}_{i_m}[A_{i_m}^{(m)}] > (1 + \varepsilon)\lambda_{i_m}^+$. For $(i_m)_{m \in \mathbb{N}}$, we can find a constant subsequence. Without loss of generality, we assume $i_m = 1$ for all $m \in \mathbb{N}$. This means $\mathbb{Q}_1[A_1^{(m)}] > (1 + \varepsilon)\lambda_1^+$. This together with the fact that $\lim_{m \rightarrow \infty} \sum_{i=1}^n y_i^{(m)} = -\infty$ implies for m large enough that

$$\begin{aligned} \mathbb{Q}_1 \left[\left\{ X > \sum_{j=1}^n y_j^{(m)} \right\} \cap A_1^{(m)} \right] &= \mathbb{Q}_1[A_1^{(m)}] - \mathbb{Q}_1 \left[\left\{ X \leq \sum_{j=1}^n y_j^{(m)} \right\} \cap A_1^{(m)} \right] \\ &> (1 + \varepsilon)\lambda_1^+ - \mathbb{Q}_1 \left[X \leq \sum_{j=1}^n y_j^{(m)} \right] \\ &\geq (1 + \varepsilon/2)\lambda_1^+, \end{aligned}$$

which contradicts $\mathbb{Q}_1[\{X > \sum_{j=1}^n y_j^{(m)}\} \cap A_1^{(m)}] \leq \Lambda_1(y_1^{(m)}) \leq \lambda_1^+$. Hence we have $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) > -\infty$. $\square$

Proof of Proposition 3.3 We first assume $\Lambda_i \in \mathcal{H}$ is decreasing. If $\bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^+} < 1$ for some $(A_1, \dots, A_n) \in \Pi_n(\Omega)$, then $\mathbb{Q}_i[A_i] < \lambda_i^+$, $i = 1, \dots, n$. Note that $\lim_{y_i \rightarrow -\infty} \Lambda(y_i) = \lambda_i^+$. So there is y_i^0 with $\Lambda(y_i) \geq \mathbb{Q}_i[A_i]$ for $y_i < y_i^0$. It follows that $\mathbb{Q}_i[\{X > \sum_{j=1}^n y_j\} \cap A_i] \leq \mathbb{Q}_i[A_i] \leq \Lambda_i(y_i)$ if $y_i \leq y_i^0$, $i = 1, \dots, n$. This implies $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) = -\infty$. If $\inf_{(A_1, \dots, A_n) \in \Pi_n(\Omega)} \bigvee_{i=1}^n \frac{\mathbb{Q}_i[A_i]}{\lambda_i^+} > 1$, the conclusion follows from part (ii) of Proposition 3.2.

Next we consider increasing $\Lambda_i \in \mathcal{H}$. For some $(A_1, \dots, A_n) \in \Pi_n(\Omega)$, if

$$\bigwedge_{i=1}^n \left(\frac{\mathbb{Q}_i[A_i]}{\lambda_i^-} \vee \bigvee_{j \neq i} \frac{\mathbb{Q}_j[A_j]}{\lambda_j^+} \right) < 1,$$

there exists $i \in \{1, \dots, n\}$ such that $\mathbb{Q}_i[A_i] < \lambda_i^-$ and $\mathbb{Q}_j[A_j] < \lambda_j^+$ for $j \neq i$. Note that there exists $y_j^0 \in \mathbb{R}$ such that $\mathbb{Q}_j[A_j] \leq \Lambda_j(y_j^0)$ for all $j \neq i$. Moreover, for any $y_i \in \mathbb{R}$, it follows that $\mathbb{Q}_j[\{X > y_i + \sum_{k \neq i} y_k^0\} \cap A_j] \leq \mathbb{Q}_j[A_j] \leq \Lambda_j(y_j^0)$ for $j \neq i$, and $\mathbb{Q}_i[\{X > y_i + \sum_{k \neq i} y_k^0\} \cap A_i] \leq \Lambda_i(y_i)$. It follows from the arbitrariness of y_i that $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) = -\infty$.

Finally, suppose $\inf_{(A_1, \dots, A_n) \in \Pi_n(\Omega)} \bigwedge_{i=1}^n \left(\frac{\mathbb{Q}_i[A_i]}{\lambda_i^-} \vee \bigvee_{j \neq i} \frac{\mathbb{Q}_j[A_j]}{\lambda_j^+} \right) > 1$. Let

$$1 + \varepsilon = \inf_{(A_1, \dots, A_n) \in \Pi_n(\Omega)} \bigwedge_{i=1}^n \left(\frac{\mathbb{Q}_i[A_i]}{\lambda_i^-} \vee \bigvee_{j \neq i} \frac{\mathbb{Q}_j[A_j]}{\lambda_j^+} \right)$$

for some $\varepsilon > 0$. Suppose by way of contradiction that $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) = -\infty$. Then there exist $y_1^{(m)}, \dots, y_n^{(m)} \in \mathbb{R}$ satisfying $\lim_{m \rightarrow \infty} \sum_{i=1}^n y_i^{(m)} = -\infty$, and $(A_1^{(m)}, \dots, A_n^{(m)}) \in \Pi_n(\Omega)$ such that $\mathbb{Q}_i[\{X > \sum_{j=1}^n y_j^{(m)}\} \cap A_i^{(m)}] \leq \Lambda_i(y_i^{(m)})$. Without loss of generality, we can assume $\lim_{m \rightarrow \infty} y_i^{(m)} = y_i$. Let

$$E_1 = \{i \in \{1, \dots, n\} : y_i = -\infty\}, \quad E_2 = \{i \in \{1, \dots, n\} : y_i > -\infty\}.$$

Note that $\lim_{m \rightarrow \infty} \sum_{i=1}^n y_i^{(m)} = -\infty$ implies $E_1 \neq \emptyset$. Moreover, for any $0 < \eta < \varepsilon$, there exists $m_0 > 0$ such that $\mathbb{Q}_i[X \leq \sum_{j=1}^n y_j^{(m)}] < \lambda_i^- \eta/2$ for all $m \geq m_0$. Hence we have for all $m \geq m_0$ that

$$\mathbb{Q}_i[A_i^{(m)}] \leq (1 + \eta/2) \Lambda_i(y_i^{(m)}).$$

Note that $\lim_{m \rightarrow \infty} \Lambda_i(y_i^{(m)}) = \lambda_i^-$ for $i \in E_1$. Hence for any $0 < \eta < \varepsilon$, there exists $m_1 \geq m_0$ such that

$$\mathbb{Q}_i[A_i^{(m)}] \leq \lambda_i^- (1 + \eta) \tag{A.1}$$

for all $m \geq m_1$ and $i \in E_1$. Moreover,

$$\mathbb{Q}_i[A_i^{(m)}] \leq \lambda_i^+ (1 + \eta) \tag{A.2}$$

for all $m \geq m_1$ and $i \in E_2$. Combining (A.1) and (A.2) leads to

$$\bigwedge_{i=1}^n \left(\frac{\mathbb{Q}_i[A_i^{(m)}]}{\lambda_i^-} \vee \bigvee_{j \neq i} \frac{\mathbb{Q}_j[A_j^{(m)}]}{\lambda_j^+} \right) \leq 1 + \eta < 1 + \varepsilon$$

for $m > m_1$, which is a contradiction. Hence $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X) > -\infty$. $\square$

Proof of Theorem 3.4 By Theorem 3.1, we have

$$\begin{aligned} \bigcap_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{P}}(X) &= \inf \left\{ \sum_{j=1}^n y_j : \mathbb{P} \left[\left\{ X > \sum_{j=1}^n y_j \right\} \cap A_i \right] \leq \Lambda_i(y_i) \right. \\ &\quad \left. \text{for some } (A_1, \dots, A_n) \in \Pi_n(\Omega) \right\}. \end{aligned}$$

Note that $\mathbb{P}[\{X > \sum_{j=1}^n y_j\} \cap A_i] \leq \Lambda_i(y_i)$ for some $(A_1, \dots, A_n) \in \Pi_n(\Omega)$ if and only if $\mathbb{P}[X > \sum_{i=1}^n y_i] \leq \sum_{i=1}^n \Lambda_i(y_i)$. Hence we have

$$\begin{aligned} \bigcap_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{P}}(X) &= \inf \left\{ \sum_{i=1}^n y_i : (y_1, \dots, y_n) \in \mathbb{R}^n, \mathbb{P} \left[X > \sum_{i=1}^n y_i \right] \leq \sum_{i=1}^n \Lambda_i(y_i) \right\} \\ &= \inf \left\{ x \in \mathbb{R} : \sum_{i=1}^n y_i = x, \mathbb{P}[X > x] \leq \sum_{i=1}^n \Lambda_i(y_i) \right\} \\ &= \inf \left\{ x \in \mathbb{R} : \mathbb{P}[X > x] \leq \Lambda^*(x) \right\} = \Lambda^* \text{VaR}^{\mathbb{P}}(X). \end{aligned} \quad (\text{A.3})$$

Next we focus on optimal allocations. Note that the right-continuity of Λ^* at x^* implies $\mathbb{P}[X > x^*] \leq \Lambda^*(x^*)$. Moreover, there exist $(y_1^*, \dots, y_n^*)$ and $(A_1^*, \dots, A_n^*)$ with

$$1 \bigwedge \sum_{i=1}^n \Lambda_i(y_i^*) = \Lambda^*(x^*) \quad \text{and} \quad \mathbb{P}[\{X > x^*\} \cap A_i^*] \leq \Lambda_i(y_i^*).$$

Hence $(y_1^*, \dots, y_n^*, A_1^*, \dots, A_n^*)$ is a minimiser of $\Gamma_{\Lambda_1, \dots, \Lambda_n}(X)$. Using Theorem 3.1, the claimed allocation is an optimal allocation. $\square$

Proof of Proposition 3.5 In view of (A.3), we have

$$\begin{aligned} \bigcap_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{P}}(X) &= \inf \left\{ x \in \mathbb{R} : \sum_{i=1}^n y_i = x, \mathbb{P}[X > x] \leq \sum_{i=1}^n \Lambda_i(y_i) \right\} \\ &= \inf_{\mathbf{y}_{n-1} \in \mathbb{R}^{n-1}} \Lambda^{\mathbf{y}_{n-1}} \text{VaR}^{\mathbb{P}}(X). \end{aligned} \quad \square$$

Proof of Theorem 3.7 By Theorem 3.1, we have

$$\begin{aligned} \bigcap_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X) &= \inf \left\{ \sum_{j=1}^n y_j : \mathbb{P} \left[\left\{ X > \sum_{j=1}^n y_j \right\} \cap B_i \cap A_i \right] \leq \mathbb{P}[B_i] \Lambda_i(y_i) \right. \\ &\quad \left. \text{for some } (A_1, \dots, A_n) \in \Pi_n(\Omega) \right\}. \end{aligned}$$

Let $\mathcal{N}_n = \{\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\} : m = 1, 2, \dots, n-1\}$. For any $\{i_1, \dots, i_m\} \in \mathcal{N}_n$, we set $C_{\{i_1, \dots, i_m\}} = (\bigcap_{i \in \{i_1, \dots, i_m\}} B_i) \setminus (\bigcup_{i \in \{1, \dots, n\} \setminus \{i_1, \dots, i_m\}} B_i)$.

Then for two sets $\{i_1, \dots, i_m\}, \{i'_1, \dots, i'_{m'}\} \in \mathcal{N}_n$, we have $C_{\{i_1, \dots, i_m\}} \cap C_{\{i'_1, \dots, i'_{m'}\}} = \emptyset$ if $\{i_1, \dots, i_m\} \neq \{i'_1, \dots, i'_{m'}\}$. Moreover, it follows that

$$\bigcup_{\{i_1, \dots, i_m\} \in \mathcal{N}_n} C_{\{i_1, \dots, i_m\}} = \left(\bigcup_{i=1}^n B_i \right) \setminus \left(\bigcap_{i=1}^n B_i \right).$$

Next, for each $(A_1, \dots, A_n) \in \Pi_n(\Omega)$, we do some operations on it. We fix $\{i_1, \dots, i_m\} \in \mathcal{N}_n$. For each $i \in \{i_1, \dots, i_m\}$, we do the operations

$$A_i \mapsto A_i \setminus C_{\{i_1, \dots, i_m\}}, \quad A_j \mapsto A_j \cup (A_i \cap C_{\{i_1, \dots, i_m\}}),$$

where $j = \min(\{1, \dots, n\} \setminus \{i_1, \dots, i_m\})$. The new sets after the operations are denoted by $(A'_1, \dots, A'_n)$. Clearly, $(A'_1, \dots, A'_n) \in \Pi_n(\Omega)$ and $B_i \cap A'_i \subseteq B_i \cap A_i$ for all $i = 1, \dots, n$, and $A'_i \cap C_{\{i_1, \dots, i_m\}} = \emptyset$ for all $i \in \{i_1, \dots, i_m\}$. Those operations are done for all $\{i_1, \dots, i_m\} \in \mathcal{N}_n$. The final sets are denoted by $(A''_1, \dots, A''_n)$. Clearly, $(A''_1, \dots, A''_n) \in \Pi_n(\Omega)$ and $B_i \cap A''_i \subseteq B_i \cap A_i$. Moreover, we have $A''_i \cap C_{\{i_1, \dots, i_m\}} = \emptyset$ for all $\{i_1, \dots, i_m\} \in \mathcal{N}_n$ satisfying $i \in \{i_1, \dots, i_m\}$. This implies $B_i \cap A''_i = (\bigcap_{i=1}^n B_i) \cap A''_i$ for all $i = 1, \dots, n$. By the construction of A''_i , it follows that $(\bigcap_{i=1}^n B_i) \cap A_i \subseteq B_i \cap A''_i$, which together with the fact that $B_i \cap A''_i \subseteq B_i \cap A_i$ implies

$$\left(\bigcap_{i=1}^n B_i \right) \cap A_i = \left(\bigcap_{i=1}^n B_i \right) \cap A''_i = B_i \cap A''_i.$$

Hence we have

$$\begin{aligned} & \inf \left\{ \sum_{j=1}^n y_j : \mathbb{P} \left[\left\{ X > \sum_{j=1}^n y_j \right\} \cap B_i \cap A_i \right] \leq \mathbb{P}[B_i] \Lambda_i(y_i) \right. \\ & \quad \left. \text{for some } (A_1, \dots, A_n) \in \Pi_n(\Omega) \right\} \\ & \leq \inf \left\{ \sum_{j=1}^n y_j : \mathbb{P} \left[\left\{ X > \sum_{j=1}^n y_j \right\} \cap \left(\bigcap_{j=1}^n B_j \right) \cap A_i \right] \leq \mathbb{P}[B_i] \Lambda_i(y_i) \right. \\ & \quad \left. \text{for some } (A_1, \dots, A_n) \in \Pi_n(\Omega) \right\}. \end{aligned}$$

Note that the converse inequality is trivial. Hence we have

$$\begin{aligned} & \bigcap_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}[X] \\ & = \inf \left\{ \sum_{j=1}^n y_j : \mathbb{P} \left[\left\{ X > \sum_{j=1}^n y_j \right\} \cap \left(\bigcap_{j=1}^n B_j \right) \cap A_i \right] \leq \mathbb{P}[B_i] \Lambda_i(y_i) \right. \\ & \quad \left. \text{for some } (A_1, \dots, A_n) \in \Pi_n(\Omega) \right\}. \end{aligned}$$

Clearly, $\bigcap_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X) = -\infty$ if $\mathbb{P}[\bigcap_{i=1}^n B_i] = 0$.

Next we focus on the case $\mathbb{P}[\bigcap_{i=1}^n B_i] > 0$. Clearly, having

$$\mathbb{P}\left[\left\{X > \sum_{j=1}^n y_j\right\} \cap \left(\bigcap_{j=1}^n B_j\right) \cap A_i\right] \leq \mathbb{P}[B_i] \Lambda_i(y_i)$$

for some $(A_1, \dots, A_n) \in \Pi_n(\Omega)$ and some $(y_1, \dots, y_n) \in \mathbb{R}^n$ is equivalent to

$$\mathbb{P}\left[\left\{X > \sum_{j=1}^n y_j\right\} \cap \left(\bigcap_{j=1}^n B_j\right)\right] \leq \sum_{i=1}^n \mathbb{P}[B_i] \Lambda_i(y_i)$$

for some $(y_1, \dots, y_n) \in \mathbb{R}^n$. Using this and the attainability of $\Lambda^\diamond$, we have

$$\begin{aligned} & \square_{i=1}^n \Lambda_i \text{VaR}^{\mathbb{Q}_i}(X) \\ &= \inf \left\{ x : x = \sum_{i=1}^n y_i, \mathbb{P}\left[\left\{X > x\right\} \cap \left(\bigcap_{j=1}^n B_j\right)\right] \leq \sum_{i=1}^n \mathbb{P}[B_i] \Lambda_i(y_i) \right\} \\ &= \inf \left\{ x : x = \sum_{i=1}^n y_i, \mathbb{Q}[X > x] \leq \sum_{i=1}^n \frac{\mathbb{P}[B_i] \Lambda_i(y_i)}{\mathbb{P}[\bigcap_{j=1}^n B_j]} \right\} \\ &= \inf \left\{ x : \mathbb{Q}[X > x] \leq \Lambda^\diamond(x) \right\} = \Lambda^\diamond \text{VaR}^{\mathbb{Q}}(X). \end{aligned}$$

One can easily check that the claimed allocation is optimal. $\square$

Proof of Theorem 3.10 Using Theorem 3.1, we have

$$\begin{aligned} & \Lambda_1 \text{VaR}^{\mathbb{Q}_1} \square \Lambda_2 \text{VaR}^{\mathbb{Q}_2}(X) \\ &= \inf \left\{ x \in \mathbb{R} : \mathbb{Q}_1[\{X > x\} \cap A^c] \leq \Lambda_1(y), \mathbb{Q}_2[\{X > x\} \cap A] \leq \Lambda_2(x - y) \right. \\ & \quad \left. \text{for some } y \in \mathbb{R}, A \in \mathcal{F} \right\} \\ &= \inf \left\{ x \in \mathbb{R} : \mathbb{Q}_1[\{X > x\} \cap A^c] \leq \Lambda_1(y) \text{ and} \right. \\ & \quad \left. \mathbb{E}^{\mathbb{Q}_1}[\eta \mathbb{1}_{\{X > x\} \cap A}] + \mathbb{Q}_2[\{X > x\} \cap A \cap N] \leq \Lambda_2(x - y) \right. \\ & \quad \left. \text{for some } y \in \mathbb{R}, A \in \mathcal{F} \right\}. \end{aligned}$$

Let $A' = A \cap N^c$. Using that $\mathbb{Q}_1[N] = 0$, we have

$$\begin{aligned} & \mathbb{Q}_1[\{X > x\} \cap (A')^c] = \mathbb{Q}_1[\{X > x\} \cap A^c], \\ & \mathbb{E}^{\mathbb{Q}_1}[\eta \mathbb{1}_{\{X > x\} \cap A'}] = \mathbb{E}^{\mathbb{Q}_1}[\eta \mathbb{1}_{\{X > x\} \cap A}], \\ & \mathbb{Q}_2[\{X > x\} \cap A' \cap N] = 0. \end{aligned}$$

Hence the inf-convolution can be simplified to

$$\begin{aligned} & \Lambda_1 \text{VaR}^{\mathbb{Q}_1} \square \Lambda_2 \text{VaR}^{\mathbb{Q}_2}(X) \\ &= \inf \left\{ x \in \mathbb{R} : \mathbb{Q}_1[\{X > x\} \cap A^c] \leq \Lambda_1(y) \text{ and } \mathbb{E}^{\mathbb{Q}_1}[\eta \mathbb{1}_{\{X > x\} \cap A}] \leq \Lambda_2(x - y) \right. \\ & \quad \left. \text{for some } y \in \mathbb{R}, A \in \mathcal{F} \right\}. \end{aligned}$$

Let $f(t) = \mathbb{Q}_1[\{X > x\} \cap \{U_\eta^{\mathbb{Q}_1} \leq t\}]$, $t \in [0, 1]$. Clearly f is increasing and continuous. Let f^{-1} denote its left quantile. Define $A_{x,t} = \{U_\eta^{\mathbb{Q}_1} \leq f^{-1}(t)\}$ for $t \in [0, 1 - F_X^{\mathbb{Q}_1}(x)]$. Let $t_0 = \mathbb{Q}_1[\{X > x\} \cap A]$. Then it follows that

$$\begin{aligned} & \mathbb{Q}_1[\{X > x\} \cap A_{x,t_0}] = \mathbb{Q}_1[\{X > x\} \cap A], \\ & \mathbb{E}^{\mathbb{Q}_1}[\eta \mathbb{1}_{\{X > x\} \cap A_{x,t_0}}] = \mathbb{E}^{\mathbb{Q}_1}[F_\eta^{\mathbb{Q}_1, -1}(U_\eta^{\mathbb{Q}_1}) \mathbb{1}_{\{X > x\} \cap A_{x,t_0}}] \leq \mathbb{E}^{\mathbb{Q}_1}[\eta \mathbb{1}_{\{X > x\} \cap A}]. \end{aligned}$$

Consequently, we have

$$\begin{aligned} & \Lambda_1 \text{VaR}^{\mathbb{Q}_1} \square \Lambda_2 \text{VaR}^{\mathbb{Q}_2}(X) \\ &= \inf \left\{ x \in \mathbb{R} : \mathbb{Q}_1[\{X > x\} \cap A_{x,t}^c] \leq \Lambda_1(y), \mathbb{E}^{\mathbb{Q}_1}[\eta \mathbb{1}_{\{X > x\} \cap A_{x,t}}] \leq \Lambda_2(x - y) \right. \\ & \quad \left. \text{for some } y \in \mathbb{R}, t \in [0, 1 - F_X^{\mathbb{Q}_1}(x)] \right\} \\ &= \inf \{ x \in \mathbb{R} : 1 - F_X^{\mathbb{Q}_1}(x) - t \leq \Lambda_1(y), g_x(t) \leq \Lambda_2(x - y) \\ & \quad \text{for some } y \in \mathbb{R}, t \in [0, 1 - F_X^{\mathbb{Q}_1}(x)] \} \\ &= \inf \left\{ x \in \mathbb{R} : g_x \left((1 - F_X^{\mathbb{Q}_1}(x) - \Lambda_1(y))^+ \right) \leq \Lambda_2(x - y) \text{ for some } y \in \mathbb{R} \right\}. \end{aligned}$$

Next we show that (X_1, X_2) given by (3.3) is an optimal allocation. Note that

$$\begin{aligned} & \mathbb{Q}_1[X_1 > y^*] = \mathbb{Q}_1[\{X > x^*\} \cap (\Omega \setminus A^*)] \\ &= \mathbb{Q}_1[X > x^*] - \mathbb{Q}_1[\{X > x^*\} \cap A^*] \\ &= 1 - F_X^{\mathbb{Q}_1}(x^*) - (1 - F_X^{\mathbb{Q}_1}(x^*) - \Lambda_1(y^*))^+ \leq \Lambda_1(y^*), \\ & \mathbb{Q}_2[X_2 > x^* - y^*] = \mathbb{Q}_2[\{X > x^*\} \cap A^*] = \mathbb{E}^{\mathbb{Q}_1}[\eta \mathbb{1}_{\{X > x^*\} \cap \{U_\eta^{\mathbb{Q}_1} \leq z^*\}}] \\ &= g_{x^*} \left((1 - F_X^{\mathbb{Q}_1}(x^*) - \Lambda_1(y^*))^+ \right) \leq \Lambda_2(x^* - y^*). \end{aligned}$$

Hence we have $\Lambda_1 \text{VaR}^{\mathbb{Q}_1}(X_1) \leq y^*$ and $\Lambda_2 \text{VaR}^{\mathbb{Q}_2}(X_2) \leq x^* - y^*$, which implies

$$\Lambda_1 \text{VaR}^{\mathbb{Q}_1}(X_1) + \Lambda_2 \text{VaR}^{\mathbb{Q}_2}(X_2) \leq \Lambda_1 \text{VaR}^{\mathbb{Q}_1} \square \Lambda_2 \text{VaR}^{\mathbb{Q}_2}(X).$$

By the definition of the inf-convolution, (X_1, X_2) is an optimal allocation. This completes the proof. $\square$

Proof of Corollary 3.11 Note that $\mathbb{Q}_2[N] \geq 1 - \lambda_2^-$ implies

$$g_x(t) \leq 1 - \mathbb{Q}_2[N] \leq \lambda_2^- \leq \Lambda_2(x - y)$$

for all $x, y \in \mathbb{R}$. Hence the inf-convolution is $-\infty$. $\square$

Proof of Theorem 4.1 For any x, y and B satisfying $\mathbb{Q}_1[B] = 1 - \Lambda(x)$, let

$$X_1 = x\mathbb{1}_B + (X - y)\mathbb{1}_{B^c}.$$

Direct calculation gives

$$\Lambda \text{VaR}^{\mathbb{Q}_1}(X_1) + \rho^{\mathbb{Q}_2}(X - X_1) \leq x + \rho^{\mathbb{Q}_2}(X - X_1) = x + \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}).$$

This implies $\Lambda \text{VaR}^{\mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X) \leq x + \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c})$. By the arbitrariness of x, y and B satisfying $\mathbb{Q}_1[B] = 1 - \Lambda(x)$, we have

$$\Lambda \text{VaR}^{\mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X) \leq \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \right).$$

To show the converse inequality, for any $X_1 \in \mathcal{X}$, let $x_1 = \Lambda \text{VaR}^{\mathbb{Q}_1}(X_1)$. Note that $X_1 = F_{X_1}^{\mathbb{Q}_1, -1}(U_{X_1}^{\mathbb{Q}_1})$ a.s. under $\mathbb{Q}_1$. For $m > m_0$ (m_0 to be specified later), set

$$Y_1 = x_1 \mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} \leq 1 - \Lambda(x_1)\}} + (X + m) \mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} > 1 - \Lambda(x_1)\}}.$$

Note that $U_{X_1}^{\mathbb{Q}_1}$ can be chosen such that $\{U_{X_1}^{\mathbb{Q}_1} \leq t\} \subseteq \{X_1 \leq F_{X_1}^{\mathbb{Q}_1, -1}(t)\}$ holds for all $t \in (0, 1)$. Moreover, by the definition of ΛVaR and the right-continuity of Λ , we have $F_{X_1}^{\mathbb{Q}_1}(x_1) \geq 1 - \Lambda(x_1)$, which implies $x_1 \geq F_{X_1}^{\mathbb{Q}_1, -1}(1 - \Lambda(x_1))$. Hence $X_1(\omega) \leq x_1$ for all $\omega \in \{U_{X_1}^{\mathbb{Q}_1} \leq 1 - \Lambda(x_1)\}$.

For the case $\mathcal{X} = \mathcal{L}^\infty$, let

$$m_0 = (x_1 - \text{ess inf}^{\mathbb{Q}_1} X) \vee (\text{ess sup}^{\mathbb{Q}_1} |X| + \text{ess sup}^{\mathbb{Q}_2} |X_1| + |x_1|),$$

where $\text{ess inf}^{\mathbb{Q}_i}$ and $\text{ess sup}^{\mathbb{Q}_i}$ represent the ess inf and ess sup under $\mathbb{Q}_i$, $i = 1, 2$. Then we have $X_1 \leq Y_1$ a.s. under $\mathbb{Q}_2$, which further implies $X - X_1 \geq X - Y_1$ a.s. under $\mathbb{Q}_2$. By the monotonicity of ρ , we have

$$\begin{aligned} & \Lambda \text{VaR}^{\mathbb{Q}_1}(X_1) + \rho^{\mathbb{Q}_2}(X - X_1) \\ & \geq \rho^{\mathbb{Q}_2}(X - Y_1) + x_1 \\ & = \rho^{\mathbb{Q}_2}((X - x_1)\mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} \leq 1 - \Lambda(x_1)\}} - m\mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} > 1 - \Lambda(x_1)\}}) + x_1 \\ & \geq \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \right). \end{aligned}$$

For the case that ρ is an ε -tail risk measure, let $m_0 = -\text{VaR}_\varepsilon^{\mathbb{Q}_2}(X - X_1)$. Then we have for $x \geq \text{VaR}_\varepsilon^{\mathbb{Q}_2}(X - X_1)$ that

$$\begin{aligned} & \mathbb{Q}_2[X - X_1 \leq x] \\ &= \mathbb{Q}_2[\{X - X_1 \leq x\} \cap \{U_{X_1}^{\mathbb{Q}_1} \leq 1 - \Lambda(x_1)\}] \\ & \quad + \mathbb{Q}_2[\{X - X_1 \leq x\} \cap \{U_{X_1}^{\mathbb{Q}_1} > 1 - \Lambda(x_1)\}] \\ &\leq \mathbb{Q}_2[\{X - x_1 \leq x\} \cap \{U_{X_1}^{\mathbb{Q}_1} \leq 1 - \Lambda(x_1)\}] + \mathbb{Q}_2[U_{X_1}^{\mathbb{Q}_1} > 1 - \Lambda(x_1)] \\ &= \mathbb{Q}_2[X - Y_1 \leq x], \end{aligned}$$

which implies

$$\frac{(F_{X-X_1}^{\mathbb{Q}_2}(x) - (1 - \varepsilon))^+}{\varepsilon} \leq \frac{(F_{X-Y_1}^{\mathbb{Q}_2}(x) - (1 - \varepsilon))^+}{\varepsilon}, \quad x \in \mathbb{R}.$$

By the monotonicity of ρ and the fact that ρ is an ε -tail risk measure, we have

$$\begin{aligned} & \Lambda \text{VaR}^{\mathbb{Q}_1}(X_1) + \rho^{\mathbb{Q}_2}(X - X_1) \\ &\geq \rho^{\mathbb{Q}_2}(X - Y_1) + x_1 \\ &= \rho^{\mathbb{Q}_2}((X - x_1)\mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} \leq 1 - \Lambda(x_1)\}} - m\mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} > 1 - \Lambda(x_1)\}}) + x_1 \\ &\geq \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \right). \end{aligned}$$

By the arbitrariness of X_1 , we obtain

$$\Lambda \text{VaR}^{\mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X) \geq \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \right).$$

Combining the above gives (4.1).

We next show that existence of an optimal allocation implies existence of a minimiser of (4.1). Suppose there exists $X_1 \in \mathcal{X}$ such that

$$\Lambda \text{VaR}^{\mathbb{Q}_1}(X_1) + \rho^{\mathbb{Q}_2}(X - X_1) = \Lambda \text{VaR}^{\mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X).$$

Following the same argument as above, let $x_1 = \Lambda \text{VaR}^{\mathbb{Q}_1}(X_1)$ and

$$Y_1 = x_1 \mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} \leq 1 - \Lambda(x_1)\}} + (X + m) \mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} > 1 - \Lambda(x_1)\}}$$

with some $m > m_0$ such that $X_1 \leq Y_1$ a.s. under $\mathbb{Q}_2$ if $\mathcal{X} = \mathcal{L}^\infty$, and

$$(F_{X-X_1}^{\mathbb{Q}_2}(x) - (1 - \varepsilon))^+ \leq (F_{X-Y_1}^{\mathbb{Q}_2}(x) - (1 - \varepsilon))^+, \quad x \in \mathbb{R},$$

if $\mathcal{X}$ is unbounded and ρ is an ε -tail risk measure. We have

$$\begin{aligned} & \Delta \text{VaR}^{\mathbb{Q}_1}(X_1) + \rho^{\mathbb{Q}_2}(X - X_1) \\ & \geq \rho^{\mathbb{Q}_2}(X - Y_1) + \Delta \text{VaR}^{\mathbb{Q}_1}(Y_1) \\ & = \rho^{\mathbb{Q}_2}\left((X - x_1)\mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} \leq 1 - \Lambda(x_1)\}} - m\mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} > 1 - \Lambda(x_1)\}}\right) + x_1 \\ & \geq \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B] = 1 - \Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}\left((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}\right)\right). \end{aligned}$$

Using $\Delta \text{VaR}^{\mathbb{Q}_1}(X_1) + \rho^{\mathbb{Q}_2}(X - X_1) = \Delta \text{VaR}^{\mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X)$ and (4.1) gives

$$\begin{aligned} & x_1 + \rho^{\mathbb{Q}_2}\left((X - x_1)\mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} \leq 1 - \Lambda(x_1)\}} - m\mathbb{1}_{\{U_{X_1}^{\mathbb{Q}_1} > 1 - \Lambda(x_1)\}}\right) \\ & = \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B] = 1 - \Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}\left((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}\right)\right). \end{aligned}$$

This implies that $(x_1, -m, \{U_{X_1}^{\mathbb{Q}_1} \leq 1 - \Lambda(x_1)\})$ is a minimiser of (4.1).

Now suppose (x^*, y^*, B^*) is a minimiser of (4.1). We check that (X_1^*, X_2^*) given in (4.2) is an optimal allocation. Indeed,

$$\begin{aligned} \Delta \text{VaR}^{\mathbb{Q}_1}(X_1^*) + \rho^{\mathbb{Q}_2}(X_2^*) & \leq x^* + \rho^{\mathbb{Q}_2}\left((X - x^*)\mathbb{1}_{B^*} + y^*\mathbb{1}_{(B^*)^c}\right) \\ & = \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B] = 1 - \Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}\left((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}\right)\right) \\ & = \Delta \text{VaR}^{\mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X). \end{aligned}$$

Consequently $\Delta \text{VaR}^{\mathbb{Q}_1}(X_1^*) + \rho^{\mathbb{Q}_2}(X_2^*) = \Delta \text{VaR}^{\mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X)$ and (X_1^*, X_2^*) is an optimal allocation. This completes the proof. $\square$

Proof of Proposition 4.2 In view of Theorem 4.1, we have

$$\Delta \text{VaR}^{\mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X) = \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B] = 1 - \Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}\left((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}\right)\right).$$

If $\mathbb{P}[B_1 \cap B_2^c] \geq (1 - \Lambda(x))\mathbb{P}[B_1]$, there exists $D \in \mathcal{F}$ such that $D \subseteq B_1 \cap B_2^c$ and $\mathbb{Q}_1[D] = 1 - \Lambda(x)$. Using the law-invariance of ρ under $\mathbb{Q}_2$, we have

$$\begin{aligned} & \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B] = 1 - \Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}\left((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}\right)\right) \\ & \leq \inf_{y \in \mathbb{R}} \left(x + \rho^{\mathbb{Q}_2}\left((X - x)\mathbb{1}_D + y\mathbb{1}_{D^c}\right)\right) \\ & = \inf_{y \in \mathbb{R}} \{x + \rho^{\mathbb{Q}_2}(y)\}. \end{aligned}$$

If $\mathcal{X} = \mathcal{L}^\infty$, for $y < \text{ess inf } \rho^{\mathbb{Q}_2}(X - x)$, we have $(X - x)\mathbb{1}_B + y\mathbb{1}_{B^c} \geq y$ a.s. under $\mathbb{Q}_2$. It follows from the monotonicity and law-invariance of ρ that

$$\rho^{\mathbb{Q}_2}\left((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}\right) \geq \rho^{\mathbb{Q}_2}(y).$$

Using the monotonicity of ρ again, we have

$$\inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}((X-x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \right) \geq \inf_{y \in \mathbb{R}} (x + \rho^{\mathbb{Q}_2}(y)).$$

If ρ is an ε -tail risk measure, for $y < \text{VaR}_{\varepsilon}^{\mathbb{Q}_2}(X-x)$, we have for $z \geq y$ that

$$\mathbb{Q}_2[(X-x)\mathbb{1}_B + y\mathbb{1}_{B^c} \leq z] = \mathbb{Q}_2[\{X-x \leq z\} \cap B] + \mathbb{Q}_2[B^c] \leq 1 = \mathbb{Q}_2[y \leq z].$$

Moreover, $\mathbb{Q}_2[(X-x)\mathbb{1}_B + y\mathbb{1}_{B^c} < y] = \mathbb{Q}_2[\{X-x < y\} \cap B] < 1 - \varepsilon$. Hence we have $\text{VaR}_{\varepsilon}^{\mathbb{Q}_2}((X-x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \geq y$ and

$$\frac{(\mathbb{Q}_2[(X-x)\mathbb{1}_B + y\mathbb{1}_{B^c} \leq z] - (1 - \varepsilon))^+}{\varepsilon} \leq \frac{(\mathbb{1}_{[y, \infty)}(z) - (1 - \varepsilon))^+}{\varepsilon}, \quad z \in \mathbb{R}.$$

Using the monotonicity of ρ and that ρ is an ε -tail risk measure under $\mathbb{Q}_2$ gives

$$\inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}((X-x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \right) \geq \inf_{y \in \mathbb{R}} (x + \rho^{\mathbb{Q}_2}(y)).$$

Consequently,

$$\inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}((X-x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \right) = \inf_{y \in \mathbb{R}} (x + \rho^{\mathbb{Q}_2}(y)).$$

Next we consider the case $\mathbb{P}[B_1 \cap B_2^c] < (1 - \Lambda(x))\mathbb{P}[B_1]$. Let

$$A_x = (B_1 \cap B_2^c) \cup (B_1 \cap B_2 \cap \{U_X^{\mathbb{P}} < \alpha_x\})$$

with α_x satisfying $\mathbb{P}[B_1 \cap B_2 \cap \{U_X^{\mathbb{P}} < \alpha_x\}] = (1 - \Lambda(x))\mathbb{P}[B_1] - \mathbb{P}[B_1 \cap B_2^c]$. For any $B \in \mathcal{F}$ with $\mathbb{Q}_1[B] = 1 - \Lambda(x)$ and $z \geq y$, we have

$$\begin{aligned} \mathbb{Q}_2[(X-x)\mathbb{1}_B + y\mathbb{1}_{B^c} \leq z] &= \frac{\mathbb{P}[\{X \leq z+x\} \cap B \cap B_2] + \mathbb{P}[B^c \cap B_2]}{\mathbb{P}[B_2]} \\ &= 1 - \frac{\mathbb{P}[\{X > z+x\} \cap B \cap B_2]}{\mathbb{P}[B_2]}. \end{aligned}$$

Observe that

$$\begin{aligned} \mathbb{P}[\{X > z+x\} \cap A_x \cap B_2] &= \mathbb{P}[\{X > z+x\} \cap B_1 \cap B_2 \cap \{U_X^{\mathbb{P}} \leq \alpha_x\}] \\ &\leq \mathbb{P}[\{X > z+x\} \cap B_1 \cap B_2 \cap B_3] \end{aligned}$$

for any $B_3 \in \mathcal{F}$ such that $\mathbb{P}[B_1 \cap B_2 \cap B_3] \geq (1 - \Lambda(x))\mathbb{P}[B_1] - \mathbb{P}[B_1 \cap B_2^c]$. Note that $\mathbb{Q}_1[B] = 1 - \Lambda(x)$ implies $\mathbb{P}[B_1 \cap B_2 \cap B] \geq (1 - \Lambda(x))\mathbb{P}[B_1] - \mathbb{P}[B_1 \cap B_2^c]$. Hence we have

$$\mathbb{P}[\{X > z+x\} \cap A_x \cap B_2] \leq \mathbb{P}[\{X > z+x\} \cap B_1 \cap B_2 \cap B] \leq \mathbb{P}[\{X > z+x\} \cap B \cap B_2].$$

This implies

$$\mathbb{Q}_2[(X-x)\mathbb{1}_B + y\mathbb{1}_{B^c} \leq z] \leq \mathbb{Q}_2[(X-x)\mathbb{1}_{A_x} + y\mathbb{1}_{A_x^c} \leq z], \quad z \geq y.$$

For $\mathcal{X} = \mathcal{L}^\infty$, let $y < \text{ess inf}^{\mathbb{Q}_2}(X - x)$. Then for $z < y$, we have

$$\mathbb{Q}_2[(X - x)\mathbb{1}_B + y\mathbb{1}_{B^c} \leq z] = \mathbb{Q}_2[(X - x)\mathbb{1}_{A_x} + y\mathbb{1}_{A_x^c} \leq z] = 0.$$

By the law-invariance and monotonicity of ρ , for $B \in \mathcal{F}$ satisfying $\mathbb{Q}_1[B] = 1 - \Lambda(x)$ and $y < \text{ess inf}^{\mathbb{Q}_2}(X - x)$, we have

$$\rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_{A_x} + y\mathbb{1}_{A_x^c}) \leq \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}).$$

If ρ is an ε -tail risk measure, let $y < \text{VaR}_\varepsilon^{\mathbb{Q}_2}(X - x)$. Then for $z < y$,

$$\mathbb{Q}_2[(X - x)\mathbb{1}_B + y\mathbb{1}_{B^c} \leq z] < 1 - \varepsilon,$$

which implies $\text{VaR}_\varepsilon^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \geq y$. Combining the above yields, for $z \in \mathbb{R}$,

$$\begin{aligned} & \frac{(\mathbb{Q}_2[(X - x)\mathbb{1}_B + y\mathbb{1}_{B^c} \leq z] - (1 - \varepsilon))^+}{\varepsilon} \\ & \leq \frac{(\mathbb{Q}_2[(X - x)\mathbb{1}_{A_x} + y\mathbb{1}_{A_x^c} \leq z] - (1 - \varepsilon))^+}{\varepsilon}. \end{aligned}$$

Using the monotonicity of ρ and the fact that ρ is an ε -tail risk measure, we have for $B \in \mathcal{F}$ satisfying $\mathbb{Q}_1[B] = 1 - \Lambda(x)$ and $y < \text{VaR}_\varepsilon^{\mathbb{Q}_2}(X - x)$ that

$$\rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_{A_x} + y\mathbb{1}_{A_x^c}) \leq \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}).$$

Consequently

$$\begin{aligned} & \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \left(x + \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \right) \\ & = \inf_{y \in \mathbb{R}} \left(x + \rho^{\mathbb{Q}_2}((X - x)\mathbb{1}_{A_x} + y\mathbb{1}_{A_x^c}) \right). \end{aligned}$$

Hence (4.3) holds. The existence and characterisation of an optimal allocation follow from the same arguments as in the proof of Theorem 4.1. The details are omitted. $\square$

Proof of Corollary 4.3 (i) Note that

$$x + \rho_g^{\mathbb{Q}_2}((X - x)\mathbb{1}_{A_x} + y\mathbb{1}_{A_x^c}) = \rho_g^{\mathbb{Q}_2}(X\mathbb{1}_{A_x} + (x + y)\mathbb{1}_{A_x^c}).$$

If $\mathbb{Q}_1[B_2] \leq \lambda^+$, there exists $x \in \mathbb{R}$ such that $\mathbb{P}[B_1 \cap B_2^c] \geq (1 - \Lambda(x))\mathbb{P}[B_1]$. Hence $A_x = \emptyset$. By Proposition 4.2, we have

$$\Lambda \text{VaR}^{\mathbb{Q}_1} \square \rho_g^{\mathbb{Q}_2}(X) = \inf_{x, y \in \mathbb{R}} \rho_g^{\mathbb{Q}_2}(x + y) = -\infty.$$

Next we consider the case $\mathbb{Q}_1[B_2] > \lambda^+$. Suppose $y < \text{VaR}_\varepsilon^{\mathbb{Q}_2}(X) - x$. Then for $z \geq x + y$, we have

$$\begin{aligned} & \mathbb{Q}_2[X\mathbb{1}_{A_x} + (x+y)\mathbb{1}_{A_x^c} \leq z] \\ &= \frac{\mathbb{P}[A_x^c \cap B_2] + \mathbb{P}[\{X \leq z\} \cap A_x \cap B_2]}{\mathbb{P}[B_2]} \\ &= 1 - \beta_x + \frac{\mathbb{P}[\{X \leq z\} \cap B_1 \cap B_2 \cap \{U_X^{\mathbb{P}} \leq \alpha_x\}]}{\mathbb{P}[B_2]}, \end{aligned} \quad (\text{A.4})$$

where $\beta_x = \frac{(1-\Lambda(x))\mathbb{P}[B_1] - \mathbb{P}[B_1 \cap B_2^c]}{\mathbb{P}[B_2]}$. Moreover, $\mathbb{Q}_2[X\mathbb{1}_{A_x} + (x+y)\mathbb{1}_{A_x^c} \leq z] < \varepsilon$ for all $z < x + y$. If $g(\beta) < 1$, there exists $x \in \mathbb{R}$ such that $g(\beta_x) < 1$. By (A.4), we have $\mathbb{Q}_2[X\mathbb{1}_{A_x} + (x+y)\mathbb{1}_{A_x^c} \leq z] \geq 1 - \beta_x$ for all $z \geq x + y$, which implies

$$\text{VaR}_t^{\mathbb{Q}_2}(X\mathbb{1}_{A_x} + (x+y)\mathbb{1}_{A_x^c}) \leq x + y \longrightarrow -\infty \quad \text{as } y \rightarrow -\infty$$

for all $t \in [\beta_x, 1)$. Hence for $x \in \mathbb{R}$ with $\beta_x = \beta$, we have

$$\inf_{y \in \mathbb{R}} \rho_g^{\mathbb{Q}_2}(X\mathbb{1}_{A_x} + (x+y)\mathbb{1}_{A_x^c}) \leq \lim_{y \rightarrow -\infty} \int_0^1 \text{VaR}_t^{\mathbb{Q}_2}(X\mathbb{1}_{A_x} + (x+y)\mathbb{1}_{A_x^c}) dg(t) = -\infty.$$

Now we assume $g(\beta) = 1$. Direct computation shows that for $0 < t < \beta_x \wedge \varepsilon$,

$$\begin{aligned} & \text{VaR}_t^{\mathbb{Q}_2}(X\mathbb{1}_{A_x} + (x+y)\mathbb{1}_{A_x^c}) \\ &= \inf \left\{ z \geq x + y : 1 - \beta_x + \frac{\mathbb{P}[\{X \leq z\} \cap B_1 \cap B_2 \cap \{U_X^{\mathbb{P}} \leq \alpha_x\}]}{\mathbb{P}[B_2]} \geq 1 - t \right\} \\ &= \text{VaR}_{1 - \frac{\mathbb{P}[B_2](\beta_x - t)}{\mathbb{P}[B_1 \cap B_2]}}^{\mathbb{Q}}(X). \end{aligned}$$

Hence we have

$$\begin{aligned} \Lambda \text{VaR}^{\mathbb{Q}_1} \square \rho_g^{\mathbb{Q}_2}(X) &= \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \int_{[0, \beta_x \wedge \varepsilon)} \text{VaR}_t^{\mathbb{Q}_2}(X\mathbb{1}_{A_x} + (x+y)\mathbb{1}_{A_x^c}) dg(t) \\ &= \inf_{x \in \mathbb{R}} \int_{[0, \beta_x \wedge \varepsilon)} \text{VaR}_{1 - \frac{\mathbb{P}[B_2](\beta_x - t)}{\mathbb{P}[B_1 \cap B_2]}}^{\mathbb{Q}}(X) dg(t) \\ &= \int_{[0, \beta)} \text{VaR}_{1 - \frac{\mathbb{P}[B_2](\beta - t)}{\mathbb{P}[B_1 \cap B_2]}}^{\mathbb{Q}}(X) dg(t). \end{aligned}$$

(ii) Similarly as above, if $\mathbb{Q}_1[B_2] \leq \lambda^+$, then by Proposition 4.2, we have $\Lambda \text{VaR}^{\mathbb{Q}_1} \square \Lambda_1 \text{VaR}^{+, \mathbb{Q}_2}(X) = \inf_{x, y \in \mathbb{R}} (x + y) = -\infty$.

Now consider the case $\mathbb{Q}_1[B_2] > \lambda^+$. In view of Proposition 4.2 and (A.4), we have

$$\begin{aligned}
 \Lambda \text{VaR}^{\mathbb{Q}_1} \square \Lambda_1 \text{VaR}^{+, \mathbb{Q}_2}(X) &= \inf_{x \in \mathbb{R}} \lim_{y \rightarrow -\infty} \left(x + \sup \left\{ z \geq y : 1 - \beta_x + \frac{\mathbb{P}[\{X \leq x + z\} \cap B_1 \cap B_2 \cap \{U_X^{\mathbb{P}} \leq \alpha_x\}]]}{\mathbb{P}[B_2]} \right. \right. \\
 &\quad \left. \left. < 1 - \Lambda_1(z) \right\} \right) \\
 &= \inf_{x \in \mathbb{R}} \lim_{y \rightarrow -\infty} \sup \left\{ z \geq y + x : 1 - \beta_x + \frac{\mathbb{P}[\{X \leq z\} \cap B_1 \cap B_2 \cap \{U_X^{\mathbb{P}} < \alpha_x\}]]}{\mathbb{P}[B_2]} \right. \\
 &\quad \left. < 1 - \Lambda_1(z - x) \right\} \\
 &= \inf_{x \in \mathbb{R}} \sup \left\{ z \in \mathbb{R} : 1 - \beta_x + \frac{\mathbb{P}[\{X \leq z\} \cap B_1 \cap B_2]}{\mathbb{P}[B_2]} < 1 - \Lambda_1(z - x) \right\} \\
 &= \inf_{x \in \mathbb{R}} \sup \left\{ z \in \mathbb{R} : \frac{\mathbb{P}[\{X \leq z\} \cap B_1 \cap B_2]}{\mathbb{P}[B_2]} < \beta_x - \Lambda_1(z - x) \right\} \\
 &= \inf_{x \in \mathbb{R}} \sup \left\{ z \in \mathbb{R} : \frac{\mathbb{P}[\{X \leq z\} \cap B_1 \cap B_2]}{\mathbb{P}[B_1 \cap B_2]} \right. \\
 &\quad \left. < 1 - \frac{\mathbb{P}[B_1]\Lambda(x) + \mathbb{P}[B_2]\Lambda_1(z - x)}{\mathbb{P}[B_1 \cap B_2]} \right\} \\
 &= \inf_{x \in \mathbb{R}} \sup \left\{ z \in \mathbb{R} : F_X^{\mathbb{Q}}(z) < 1 - \frac{\mathbb{P}[B_1]\Lambda(x) + \mathbb{P}[B_2]\Lambda_1(z - x)}{\mathbb{P}[B_1 \cap B_2]} \right\} \\
 &= \inf_{x \in \mathbb{R}} \overline{\Lambda}^x \text{VaR}^{+, \mathbb{Q}}(X),
 \end{aligned}$$

where $\overline{\Lambda}^x(z) = \frac{\mathbb{P}[B_1]\Lambda(x) + \mathbb{P}[B_2]\Lambda_1(z - x)}{\mathbb{P}[B_1 \cap B_2]} \wedge 1$. This completes the proof. $\square$

Proof of Corollary 4.4 (i) In view of Theorem 4.1, we have

$$\begin{aligned}
 \Lambda \text{VaR}^{\mathbb{Q}_1} \square \text{ES}_{\alpha}^{\mathbb{Q}_2}(X) &= \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B] = 1 - \Lambda(x)} \left(x + \text{ES}_{\alpha}^{\mathbb{Q}_2}((X - x)\mathbb{1}_B + y\mathbb{1}_{B^c}) \right) \\
 &= \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B] = 1 - \Lambda(x)} \left(\text{ES}_{\alpha}^{\mathbb{Q}_2}(X\mathbb{1}_B + (x + y)\mathbb{1}_{B^c}) \right).
 \end{aligned}$$

It is shown in Rockafellar and Uryasev [49] that

$$\text{ES}_{\alpha}^{\mathbb{Q}_2}(X) = \inf_{t \in \mathbb{R}} \left(t + \frac{1}{\alpha} \mathbb{E}^{\mathbb{Q}_2}[(X - t)^+] \right).$$

Hence we have

$$\begin{aligned}
& \Lambda \text{VaR}^{\mathbb{Q}_1} \square \text{ES}_\alpha^{\mathbb{Q}_2}(X) \\
&= \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \left(\text{ES}_\alpha^{\mathbb{Q}_2}(X \mathbb{1}_B + (x+y) \mathbb{1}_{B^c}) \right) \\
&= \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \inf_{t \in \mathbb{R}} \left(t + \frac{1}{\alpha} \mathbb{E}^{\mathbb{Q}_2}[(X-t)^+ \mathbb{1}_B] + \frac{1}{\alpha} \mathbb{E}^{\mathbb{Q}_2}[(x+y-t)^+ \mathbb{1}_{B^c}] \right) \\
&= \inf_{x \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \inf_{t \in \mathbb{R}} \left(t + \frac{1}{\alpha} \mathbb{E}^{\mathbb{Q}_2}[(X-t)^+ \mathbb{1}_B] \right) \\
&= \inf_{x \in \mathbb{R}} \inf_{t \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \left(t + \frac{1}{\alpha} \mathbb{E}^{\mathbb{Q}_1}[\eta(X-t)^+ \mathbb{1}_B] + \frac{1}{\alpha} \mathbb{E}^{\mathbb{Q}_2}[(X-t)^+ \mathbb{1}_{B \cap N}] \right) \\
&= \inf_{x \in \mathbb{R}} \inf_{t \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} \left(t + \frac{1}{\alpha} \mathbb{E}^{\mathbb{Q}_1}[\eta(X-t)^+ \mathbb{1}_B] \right) \\
&= \inf_{x \in \mathbb{R}} \inf_{t \in \mathbb{R}} \left(t + \frac{1}{\alpha} \int_{\Lambda(x)}^1 \text{VaR}_s^{\mathbb{Q}_1}(\eta(X-t)^+) ds \right) \\
&= \inf_{t \in \mathbb{R}} \left(t + \frac{1}{\alpha} \int_{\lambda^+}^1 \text{VaR}_s^{\mathbb{Q}_1}(\eta(X-t)^+) ds \right).
\end{aligned}$$

(ii) It follows from Theorem 4.1 that

$$\begin{aligned}
\Lambda \text{VaR}^{\mathbb{Q}_1} \square \mathbb{E}_u^{\mathbb{Q}_2}[X] &= \inf_{x \in \mathbb{R}} \inf_{y \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} (x + \mathbb{E}^{\mathbb{Q}_2}[u(X-x) \mathbb{1}_B + u(y) \mathbb{1}_{B^c}]) \\
&= \inf_{x \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} (x + \mathbb{E}^{\mathbb{Q}_2}[u(X-x) \mathbb{1}_B + u(-\infty) \mathbb{1}_{B^c}]).
\end{aligned}$$

Note that $u(-\infty) \leq u(X-x)$. Hence we have

$$\begin{aligned}
& \Lambda \text{VaR}^{\mathbb{Q}_1} \square \mathbb{E}_u^{\mathbb{Q}_2}[X] \\
&= \inf_{x \in \mathbb{R}} \inf_{\mathbb{Q}_1[B]=1-\Lambda(x)} (x + \mathbb{E}^{\mathbb{Q}_1}[\eta u(X-x) \mathbb{1}_B] + u(-\infty) \mathbb{E}^{\mathbb{Q}_1}[\eta \mathbb{1}_{B^c}] \\
&\quad + \mathbb{E}^{\mathbb{Q}_2}[u(X-x) \mathbb{1}_{B \cap N}] + u(-\infty) \mathbb{E}^{\mathbb{Q}_2}[\mathbb{1}_{B^c \cap N}]) \\
&= \inf_{x \in \mathbb{R}} \left(x + \mathbb{E}^{\mathbb{Q}_1}[\eta u(X-x) \mathbb{1}_{\{U_\eta^{\mathbb{Q}_1} < 1-\Lambda(x)\}}] \right. \\
&\quad \left. + u(-\infty) \left(\mathbb{Q}_2[N] + \int_0^{\Lambda(x)} \text{VaR}_t^{\mathbb{Q}_1}(\eta) dt \right) \right). \quad \square
\end{aligned}$$

Proof of Theorem 5.1 For $x \in \mathbb{R}$, $y \geq x$ and $U \stackrel{\mathbb{Q}_1}{\sim} \mathcal{U}[0, 1]$, we have

$$\Lambda \text{VaR}^{+, \mathbb{Q}_1}(\Lambda_{x,y}^{-1}(U)) = x.$$

Hence we have

$$\begin{aligned}\Lambda \text{VaR}^{+, \mathbb{Q}_1} \square \rho(X) &\leq \Lambda \text{VaR}^{+, \mathbb{Q}_1}(\Lambda_{x,y}^{-1}(U)) + \rho^{\mathbb{Q}_2}(X - \Lambda_{x,y}^{-1}(U)) \\ &= x + \rho^{\mathbb{Q}_2}(X - \Lambda_{x,y}^{-1}(U)).\end{aligned}$$

Using the arbitrariness of $x \in \mathbb{R}$, $y \geq x$ and $U \stackrel{\mathbb{Q}_1}{\sim} \mathcal{U}[0, 1]$ gives

$$\Lambda \text{VaR}^{+, \mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X) \leq \inf_{x \in \mathbb{R}} \inf_{y \geq x} \inf_{U \stackrel{\mathbb{Q}_1}{\sim} \mathcal{U}[0, 1]} \left(x + \rho(X - \Lambda_{x,y}^{-1}(U)) \right).$$

To show the converse inequality, for $X_1 \in \mathcal{X}$, we let $x_1 = \Lambda \text{VaR}^{+, \mathbb{Q}_1}(X_1)$. We choose $U_{X_1}^{\mathbb{Q}_1}$ such that $\{U_{X_1}^{\mathbb{Q}_1} \leq t\} \subseteq \{X_1 \leq F_{X_1}^{\mathbb{Q}_1, -1}(t)\}$ for all $t \in (0, 1)$. This implies

$$\{F_{X_1}^{\mathbb{Q}_1, -1}(U_{X_1}^{\mathbb{Q}_1}) \leq x\} = \{U_{X_1}^{\mathbb{Q}_1} \leq F_{X_1}^{\mathbb{Q}_1}(x)\} \subseteq \{X_1 \leq F_{X_1}^{\mathbb{Q}_1, -1}(F_{X_1}^{\mathbb{Q}_1}(x))\} \subseteq \{X_1 \leq x\}$$

for all $x \in \mathbb{R}$. Hence $X_1(\omega) \leq F_{X_1}^{\mathbb{Q}_1, -1}(U_{X_1}^{\mathbb{Q}_1})$ for all $\omega \in \Omega$. By definition, $F_{X_1}^{\mathbb{Q}_1}(x) \geq 1 - \Lambda(x)$ for all $x \geq x_1$. This implies $F_{X_1}^{\mathbb{Q}_1}(x) \geq \Lambda_{x_1, y}(x)$ for $x_1 \leq x \leq y$. Suppose $\text{ess sup}^{\mathbb{Q}_1} X_1 < \infty$. If $y \geq \text{ess sup}^{\mathbb{Q}_1} X_1$, then $F_{X_1}^{\mathbb{Q}_1}(x) = 1 \geq \Lambda_{x_1, y}(x)$ for $x \geq y$. Hence if $y \geq \text{ess sup}^{\mathbb{Q}_1} X_1$, then for all $x \in \mathbb{R}$, we have $F_{X_1}^{\mathbb{Q}_1}(x) \geq \Lambda_{x_1, y}(x)$. Combining all the above results, we have $X_1 \leq F_{X_1}^{\mathbb{Q}_1, -1}(U_{X_1}^{\mathbb{Q}_1}) \leq \Lambda_{x_1, y}^{-1}(U_{X_1}^{\mathbb{Q}_1})$ if $y \geq \text{ess sup}^{\mathbb{Q}_1} X_1$. Therefore for $y \geq \text{ess sup}^{\mathbb{Q}_1} X_1$, we have

$$\begin{aligned}\Lambda \text{VaR}^{+, \mathbb{Q}_1}(X_1) + \rho^{\mathbb{Q}_2}(X - X_1) &\geq x_1 + \rho^{\mathbb{Q}_2}(X - \Lambda_{x_1, y}^{-1}(U_{X_1}^{\mathbb{Q}_1})) \\ &\geq \inf_{x \in \mathbb{R}} \inf_{y \geq \hat{x}} \inf_{U \stackrel{\mathbb{Q}_1}{\sim} \mathcal{U}[0, 1]} \left(x + \rho^{\mathbb{Q}_2}(X - \Lambda_{x, y}^{-1}(U)) \right) \\ &= \inf_{x \in \mathbb{R}} \inf_{y \geq x} \inf_{U \stackrel{\mathbb{Q}_1}{\sim} \mathcal{U}[0, 1]} \left(x + \rho^{\mathbb{Q}_2}(X - \Lambda_{x, y}^{-1}(U)) \right),\end{aligned}$$

where $\hat{x} = x \vee \text{ess sup}^{\mathbb{Q}_1} X_1$. If $\text{ess sup}^{\mathbb{Q}_1} X_1 = \infty$, then let $X_m = X_1 \wedge m$ for $m \geq 2$. Using the above conclusion, we have

$$\Lambda \text{VaR}^{+, \mathbb{Q}_1}(X_m) + \rho^{\mathbb{Q}_2}(X - X_m) \geq \inf_{x \in \mathbb{R}} \inf_{y \geq x} \inf_{U \stackrel{\mathbb{Q}_1}{\sim} \mathcal{U}[0, 1]} \left(x + \rho^{\mathbb{Q}_2}(X - \Lambda_{x, y}^{-1}(U)) \right).$$

Direct computation gives $\lim_{m \rightarrow \infty} \Lambda \text{VaR}^{+, \mathbb{Q}_1}(X_m) = \Lambda \text{VaR}^{+, \mathbb{Q}_1}(X_1)$. Note that

$$X - X_m \downarrow X - X_1 \quad \text{as } m \rightarrow \infty.$$

Using the fact that ρ is continuous from above, we have

$$\lim_{m \rightarrow \infty} \rho^{\mathbb{Q}_2}(X - X_m) = \rho^{\mathbb{Q}_2}(X - X_1).$$

Hence letting $m \rightarrow \infty$, we obtain

$$\Lambda \text{VaR}^{+, \mathbb{Q}_1}(X_1) + \rho^{\mathbb{Q}_2}(X - X_1) \geq \inf_{x \in \mathbb{R}} \inf_{y \geq x} \inf_{U \sim \mathbb{Q}_1 \mathcal{U}[0,1]} \left(x + \rho^{\mathbb{Q}_2}(X - \Lambda_{x,y}^{-1}(U)) \right).$$

By the arbitrariness of X_1 , we have

$$\Lambda \text{VaR}^{+, \mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X) \geq \inf_{x \in \mathbb{R}} \inf_{y \geq x} \inf_{U \sim \mathbb{Q}_1 \mathcal{U}[0,1]} \left(x + \rho^{\mathbb{Q}_2}(X - \Lambda_{x,y}^{-1}(U)) \right).$$

Hence we obtain (5.1).

Next we show for $\mathcal{X} = \mathcal{L}^\infty$ that an optimal allocation of the inf-convolution exists if and only if a minimiser of (5.1) exists. Suppose $(X_1, X - X_1)$ is an optimal allocation of the inf-convolution. Then we have

$$\Lambda \text{VaR}^{+, \mathbb{Q}_1}(X_1) + \rho^{\mathbb{Q}_2}(X - X_1) = \Lambda \text{VaR}^{+, \mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X).$$

Let $x_1 = \Lambda \text{VaR}^{+, \mathbb{Q}_1}(X_1)$ and $y_1 > x_1 \vee \text{ess sup}^{\mathbb{Q}_1} X_1$. Moreover, we choose $U_{X_1}^{\mathbb{Q}_1}$ such that $X_1(\omega) \leq F_{X_1}^{\mathbb{Q}_1, -1}(U_{X_1}^{\mathbb{Q}_1})$ for all $\omega \in \Omega$. Using the above argument, we can conclude that $F_{X_1}^{\mathbb{Q}_1, -1}(U_{X_1}^{\mathbb{Q}_1}) \leq \Lambda_{x_1, y_1}^{-1}(U_{X_1}^{\mathbb{Q}_1})$. By the monotonicity of ρ , we have

$$\Lambda \text{VaR}^{+, \mathbb{Q}_1}(X_1) + \rho^{\mathbb{Q}_2}(X - X_1) \geq x_1 + \rho^{\mathbb{Q}_2}(X - \Lambda_{x_1, y_1}^{-1}(U_{X_1}^{\mathbb{Q}_1})).$$

This implies

$$x_1 + \rho^{\mathbb{Q}_2}(X - \Lambda_{x_1, y_1}^{-1}(U_{X_1}^{\mathbb{Q}_1})) = \inf_{x \in \mathbb{R}} \inf_{y \geq x} \inf_{U \sim \mathbb{Q}_1 \mathcal{U}[0,1]} \left(x + \rho^{\mathbb{Q}_2}(X - \Lambda_{x,y}^{-1}(U)) \right).$$

Hence $(x_1, y_1, U_{X_1}^{\mathbb{Q}_1})$ is a minimiser of (5.1). Conversely, if (x_1, y_1, U) is a minimiser of (5.1), one can easily check that $(\Lambda_{x_1, y_1}^{-1}(U), X - \Lambda_{x_1, y_1}^{-1}(U))$ is an optimal allocation of the inf-convolution. This completes the proof. $\square$

Proof of Corollary 5.3 In view of Theorem 5.1, we have

$$\Lambda \text{VaR}^{+, \mathbb{Q}} \square \overline{\Lambda_1 \text{VaR}}^{\mathbb{Q}}(X) = \inf_{x \in \mathbb{R}} \inf_{y \geq x} \inf_{U \sim \mathbb{Q} \mathcal{U}[0,1]} \left(x + \overline{\Lambda_1 \text{VaR}}^{\mathbb{Q}}(X - \Lambda_{x,y}^{-1}(U)) \right).$$

Let $\hat{F}_{x,y}(z) = \{F_{X - \Lambda_{x,y}^{-1}(U)}^{\mathbb{Q}}(z) : U \sim \mathbb{Q} \mathcal{U}[0,1]\}$ for $z \in \mathbb{R}$ and $\overline{\text{VaR}}_\beta^{\mathbb{Q}}$ represent $\overline{\Lambda_1 \text{VaR}}^{\mathbb{Q}}$ for $\Lambda_1 = \beta \in (0, 1)$. This means that $\overline{\text{VaR}}_\beta^{\mathbb{Q}}$ is the right VaR. By Han and Liu [31, Theorem 1], we have

$$\inf_{U \sim \mathbb{Q} \mathcal{U}[0,1]} \overline{\Lambda_1 \text{VaR}}^{\mathbb{Q}}(X - \Lambda_{x,y}^{-1}(U)) = q_{\Lambda_1}^+(\hat{F}_{x,y}).$$

It follows from [31, Corollary 1] that for $\alpha \in (0, 1)$,

$$q_\alpha^+(\hat{F}_{x,y}) = \inf_{U \sim \mathbb{Q} \mathcal{U}[0,1]} \overline{\text{VaR}}_\alpha^{\mathbb{Q}}(X - \Lambda_{x,y}^{-1}(U)),$$

where q_{α}^{+} is $q_{\Lambda_1}^{+}$ with $\Lambda_1 = \alpha$. Direct computation shows that

$$\inf_{U \sim \mathcal{U}[0,1]} \overline{\text{VaR}}_{\alpha}^{\mathbb{Q}}(X - \Lambda_{x,y}^{-1}(U)) = - \sup_{U \sim \mathcal{U}[0,1]} \text{VaR}_{1-\alpha}^{\mathbb{Q}}(-X + \Lambda_{x,y}^{-1}(U)).$$

In view of Bernard et al. [10, Lemma 4.3] and the result in Makarov [46], we have

$$\begin{aligned} \sup_{Y \sim F_X^{\mathbb{Q}}, U \sim \mathcal{U}[0,1]} \overline{\text{VaR}}_{1-\alpha}^{\mathbb{Q}}(-Y + \Lambda_{x,y}^{-1}(U)) &= \overline{\text{VaR}}_{1-\alpha}^{\mathbb{Q}}(-X + \Lambda_{x,y}^{-1}(V)) \\ &\leq \sup_{U \sim \mathcal{U}[0,1]} \overline{\text{VaR}}_{1-\alpha}^{\mathbb{Q}}(-X + \Lambda_{x,y}^{-1}(U)), \end{aligned}$$

where $V = U_{-X}^{\mathbb{Q}} \mathbb{1}_{\{U_{-X}^{\mathbb{Q}} \leq \alpha\}} + (1 + \alpha - U_{-X}^{\mathbb{Q}}) \mathbb{1}_{\{U_{-X}^{\mathbb{Q}} > \alpha\}}$, which implies

$$\begin{aligned} \sup_{U \sim \mathcal{U}[0,1]} \overline{\text{VaR}}_{1-\alpha}^{\mathbb{Q}}(-X + \Lambda_{x,y}^{-1}(U)) &= \sup_{Y \sim F_X^{\mathbb{Q}}, U \sim \mathcal{U}[0,1]} \overline{\text{VaR}}_{1-\alpha}^{\mathbb{Q}}(-Y + \Lambda_{x,y}^{-1}(U)) \\ &= \overline{\text{VaR}}_{1-\alpha}^{\mathbb{Q}}(-X + \Lambda_{x,y}^{-1}(V)). \end{aligned}$$

Using the above conclusion and the continuity of $\Lambda_{x,y}^{-1}$ and $F_X^{\mathbb{Q},-1}$, applying [10, Lemma 4.4] gives

$$\sup_{U \sim \mathcal{U}[0,1]} \text{VaR}_{1-\alpha}^{\mathbb{Q}}(-X + \Lambda_{x,y}^{-1}(U)) = \sup_{U \sim \mathcal{U}[0,1]} \overline{\text{VaR}}_{1-\alpha}^{\mathbb{Q}}(-X + \Lambda_{x,y}^{-1}(U)).$$

Hence

$$q_{\alpha}^{+}(\hat{F}_{x,y}) = -\overline{\text{VaR}}_{1-\alpha}^{\mathbb{Q}}(-X + \Lambda_{x,y}^{-1}(V)) = \ell_{x,y}(1 - \alpha),$$

where $\ell_{x,y}(t) = \sup_{s \in (0,t]} (F_X^{\mathbb{Q},-1}(s) - \Lambda_{x,y}^{-1}(1-t+s))$, $t \in (0, 1)$. Direct computation shows that $\hat{F}_{x,y}(z) = \ell_{x,y}^{-1}(z)$ on $\mathbb{R}$ except in a countable number of points.

Next we show that $q_{\Lambda_1}^{+}(\ell_{x,y}^{-1}) = q_{\Lambda_1}^{+}(\hat{F}_{x,y})$. Let $z_0 = q_{\Lambda_1}^{+}(\hat{F}_{x,y})$. Clearly $z_0 \in \mathbb{R}$. By definition, there exist $z_n \downarrow z_0$ such that $\hat{F}_{x,y}(z_n) > 1 - \Lambda_1(z_n)$. It follows from the right-continuity of Λ_1 and the monotonicity of $\hat{F}_{x,y}$ that there exist $0 < \varepsilon_n < 1/n$ such that $\hat{F}_{x,y}(z) > 1 - \Lambda_1(z)$ for $z \in (z_n, z_n + \varepsilon_n)$. Then there exist $w_n \in (z_n, z_n + \varepsilon_n)$ such that $\ell_{x,y}^{-1}(w_n) = \hat{F}_{x,y}(w_n) > 1 - \Lambda_1(w_n)$. Hence $q_{\Lambda_1}^{+}(\ell_{x,y}^{-1}) \leq w_n \rightarrow z_0 = q_{\Lambda_1}^{+}(\hat{F}_{x,y})$. We can similarly show the converse inequality. Hence $q_{\Lambda_1}^{+}(\ell_{x,y}^{-1}) = q_{\Lambda_1}^{+}(\hat{F}_{x,y})$. This completes the proof. $\square$

Proof of Proposition 5.4 Direct calculation shows that if $x < x_1$, then

$$\Lambda_{x,y}(z) = \lambda_1 \mathbb{1}_{\{x \leq z < x_1\}} + \lambda_2 \mathbb{1}_{\{x_1 \leq z < y\}} + \mathbb{1}_{\{z \geq y\}},$$

and if $x \geq x_1$, then

$$\Lambda_{x,y}(z) = \lambda_2 \mathbb{1}_{\{x \leq z < y\}} + \mathbb{1}_{\{z \geq y\}}.$$

Hence we have $\Lambda_{x,y}^{-1}(t) = x \mathbb{1}_{\{0 < t \leq \lambda_1\}} + x \vee x_1 \mathbb{1}_{\{\lambda_1 < t \leq \lambda_2\}} + y \mathbb{1}_{\{\lambda_2 < t < 1\}}$. For any uniform random variable $U \stackrel{\mathbb{Q}_1}{\sim} \mathcal{U}[0, 1]$, it follows that

$$\Lambda_{x,y}^{-1}(U) = x \mathbb{1}_{\{0 < U \leq \lambda_1\}} + x \vee x_1 \mathbb{1}_{\{\lambda_1 < U \leq \lambda_2\}} + y \mathbb{1}_{\{\lambda_2 < U < 1\}}.$$

Note that $\mathbb{Q}_1[0 < U \leq \lambda_1] = \lambda_1$, $\mathbb{Q}_1[\lambda_1 < U \leq \lambda_2] = \lambda_2 - \lambda_1$ and $\mathbb{Q}_1[\lambda_2 < U < 1] = 1 - \lambda_2$, and those three sets are disjoint. Consequently, in view of Theorem 5.1, we have

$$\begin{aligned} \Lambda \text{VaR}^{+, \mathbb{Q}_1} \square \rho^{\mathbb{Q}_2}(X) \\ &= \inf_{x \in \mathbb{R}} \inf_{y \geq x} \inf_{U \stackrel{\mathbb{Q}_1}{\sim} \mathcal{U}[0, 1]} \left(x + \rho^{\mathbb{Q}_2}(X - \Lambda_{x,y}^{-1}(U)) \right) \\ &= \inf_{x \in \mathbb{R}} \inf_{y \geq x \vee x_1} \inf_{(B_1, B_2) \in \mathcal{B}_{\lambda_1, \lambda_2}} \left(x + \rho^{\mathbb{Q}_2}(X - x \mathbb{1}_{B_1} - x \vee x_1 \mathbb{1}_{B_2} - y \mathbb{1}_{(B_1 \cup B_2)^c}) \right). \quad \square \end{aligned}$$

Proof of Corollary 5.5 In view of Proposition 5.4, we have

$$\begin{aligned} \Lambda \text{VaR}^{+, \mathbb{Q}_1} \square \text{ES}_\alpha^{\mathbb{Q}_2}(X) \\ &= \inf_{x \in \mathbb{R}} \inf_{y \geq x \vee x_1} \inf_{(B_1, B_2) \in \mathcal{B}_{\lambda_1, \lambda_2}} \left(x + \text{ES}_\alpha^{\mathbb{Q}_2}(X - x \mathbb{1}_{B_1} - x \vee x_1 \mathbb{1}_{B_2} - y \mathbb{1}_{(B_1 \cup B_2)^c}) \right) \\ &= \inf_{x \in \mathbb{R}} \inf_{y \geq x \vee x_1} \inf_{(B_1, B_2) \in \mathcal{B}_{\lambda_1, \lambda_2}} \text{ES}_\alpha^{\mathbb{Q}_2}(X + 0 \wedge (x - x_1) \mathbb{1}_{B_2} + (x - y) \mathbb{1}_{(B_1 \cup B_2)^c}) \\ &= \inf_{(B_1, B_2) \in \mathcal{B}_{\lambda_1, \lambda_2}} \lim_{y \rightarrow \infty} \text{ES}_\alpha^{\mathbb{Q}_2}(X - y \mathbb{1}_{B_1^c}). \end{aligned}$$

From the above expression, we should choose B_1 to get $\mathbb{Q}_2[B_1]$ as small as possible. Hence if $\lambda_1 \mathbb{P}[D_1] - \mathbb{P}[D_1 \cap D_2^c] \geq \alpha \mathbb{P}[D_2]$, the optimal way to choose B_1 is to set

$$B_1 = (D_1 \cap D_2^c) \cup (\{U_X^{\mathbb{P}} < t^+\} \cap D_1 \cap D_2)$$

with $\mathbb{P}[\{U_X^{\mathbb{P}} < t^+\} \cap D_1 \cap D_2] = \lambda_1 \mathbb{P}[D_1] - \mathbb{P}[D_1 \cap D_2^c]$. Then we have

$$\Lambda \text{VaR}^{+, \mathbb{Q}_1} \square \text{ES}_\alpha^{\mathbb{Q}_2}(X) = \lim_{y \rightarrow \infty} \text{ES}_\alpha^{\mathbb{Q}_2}(X - y \mathbb{1}_{B_1^c}) = \frac{1}{\alpha} \mathbb{E}^{\mathbb{Q}_2}[X \mathbb{1}_{\{t^- < U_X^{\mathbb{P}} < t^+\} \cap D_1 \cap D_2}]$$

with $\mathbb{P}[\{t^- < U_X^{\mathbb{P}} < t^+\} \cap D_1 \cap D_2] = \alpha \mathbb{P}[D_2]$. Note that $\frac{d\mathbb{Q}_2}{d\mathbb{Q}} = \frac{\mathbb{P}[D_1 \cap D_2]}{\mathbb{P}[D_2]}$ on $D_1 \cap D_2$. Hence we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}_2}[X \mathbb{1}_{\{t^- < U_X^{\mathbb{P}} < t^+\} \cap D_1 \cap D_2}] &= \frac{\mathbb{P}[D_1 \cap D_2]}{\mathbb{P}[D_2]} \mathbb{E}^{\mathbb{Q}}[X \mathbb{1}_{\{t^- < U_X^{\mathbb{P}} < t^+\} \cap D_1 \cap D_2}] \\ &= \frac{\mathbb{P}[D_1 \cap D_2]}{\mathbb{P}[D_2]} \int_{\alpha_0}^{\alpha_1} \text{VaR}_t^{\mathbb{Q}}(X) dt \end{aligned}$$

with $\alpha_0 = 1 - \frac{\lambda_1 \mathbb{P}[D_1] - \mathbb{P}[D_1 \cap D_2^c]}{\mathbb{P}[D_1 \cap D_2]}$ and $\alpha_1 = 1 - \frac{\lambda_1 \mathbb{P}[D_1] - \mathbb{P}[D_1 \cap D_2^c] - \alpha \mathbb{P}[D_2]}{\mathbb{P}[D_1 \cap D_2]}$, which implies the desired result.

If $\lambda_1 \mathbb{P}[D_1] - \mathbb{P}[D_1 \cap D_2^c] \leq 0$, we choose $B_1 \subseteq D_1 \cap D_2^c$. Then $\mathbb{Q}_2[B_1^c] = 1$, which implies $\Delta \text{VaR}^{\mathbb{Q}_1} \square \text{ES}_\alpha^{\mathbb{Q}_2}(X) = -\infty$.

If $0 < \lambda_1 \mathbb{P}[D_1] - \mathbb{P}[D_1 \cap D_2^c] < \alpha \mathbb{P}[D_2]$, then we choose

$$B_1 = (D_1 \cap D_2^c) \cup (\{U_X^{\mathbb{P}} < t^+\} \cap D_1 \cap D_2),$$

where t^+ is defined above. Direct computation shows $\mathbb{Q}_2[B_1] < \alpha$, which implies

$$\Delta \text{VaR}^{+, \mathbb{Q}_1} \square \text{ES}_\alpha^{\mathbb{Q}_2}(X) = \lim_{y \rightarrow \infty} \text{ES}_\alpha^{\mathbb{Q}_2}(X - y \mathbb{1}_{B_1^c}) = -\infty.$$

This completes the proof. $\square$

Acknowledgements The authors thank the Editor, an Associate Editor and two anonymous referees for their helpful comments. YW acknowledges financial support from the Natural Sciences and Engineering Research Council of Canada (RGPIN-2023-04674, DGEGR-2023-00454) and a start-up fund at Carleton University.

Declarations

Competing interests The authors declare no competing interests.

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