585

Subgroup Theoretical Properties and Some Classes of Generalized Nilpotent and Soluble Groups

by

P. R. Cox

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Contents

Page No.

Acknowledgements and Declaration 4

Abstract 5

Notation 7

Introduction 10

1. Subgroup Theoretical Properties 17
2. $\mathfrak{X}$-Centrality and $\mathfrak{X}_D$-Centrality 44
3. $\mathfrak{X}$- and $\mathfrak{X}_D$ - Centrality for Varieties 59
4. The class of groups $[\mathfrak{V}]$ 78
5. The class of groups $\mathfrak{X}(1)$ 89
6. Generalized Soluble Groups 105
7. FC-groups 117

References 128

List of Figures

Figure No.	Page No.
Figure 1	51
Figure 2	61
Figure 3	84
Figure 4	85
Figure 5	92
Figure 6	113

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P. Cox

Abstract

This thesis studies subgroup theoretical properties and classes of groups defined in terms of subgroup theoretical properties. After an introductory chapter, containing some definitions and some general results, two particular subgroup theoretical properties are examined. These two represent the concepts of $\mathfrak{X}$ - and $\mathfrak{X}_D$ - centrality, for a class of groups $\mathfrak{X}$, after Stanley and Arrell. They are generalizations of the more familiar subgroup theoretical property "is a central subgroup of".

Some connections are established between classes of groups defined in terms of $\mathfrak{X}$ - and $\mathfrak{X}_D$ - centrality, notably that the classes of groups with $\mathfrak{X}$-central and $\mathfrak{X}_D$-central series coincide, and that when $\mathfrak{X}$ is restricted to be a variety, similar results hold for ascending and descending series. Some results of Petty are used to prove certain closure properties for some of the classes under discussion. In particular, the local closure of the class of groups with a $\mathfrak{V}$-central series, for a variety $\mathfrak{V}$, is proved.

A generalization of the class of residually commutable groups is introduced. Some of Ayoub's results about residually commutable groups, including her local theorem, are generalized accordingly.

When $\mathfrak{X}$ is the class of nilpotent groups, $\mathfrak{X}$-centrality is shown to yield a class of hypercentral groups. The relationships between this class and other classes of generalized nilpotent groups are studied. Also, when $\mathfrak{X}$ is the class of soluble

groups, $\mathfrak{X}$-centrality gives rise to several new classes of generalized soluble groups. Taking $\mathfrak{X}$ to be the class of finite groups provides a characterization of the class of FC-groups. This characterization is used to investigate the effect of the condition of being an FC-group on generalized soluble and nilpotent groups. In particular, it is proved that residually central FC-groups are hypercentral, with hypercentral length not exceeding ω.

Notation

If α is an ordinal number we write $\zeta_\alpha(G)$ for the α-th term of the upper central series of G and $\gamma_\alpha(G)$ for the α-th term of the lower central series of G.

If Λ is any set and $\{G_\lambda ; \lambda \varepsilon \Lambda\}$ is a family of groups, $\underset{\lambda\varepsilon\Lambda}{\mathrm{Dr}}\ G_\lambda$ and $\underset{\lambda\varepsilon\Lambda}{\mathrm{Cr}}\ G_\lambda$ are the direct product and cartesian product respectively of these groups. Cartesian product is defined by :

$G = \underset{\lambda\varepsilon\Lambda}{\mathrm{Cr}}\ G_\lambda$ if G is the set of all functions

$x : \Lambda \to \underset{\lambda\varepsilon\Lambda}{\cup} G_\lambda$ such that $x(\lambda) \varepsilon G_\lambda$ and $(xy)(\lambda) = x(\lambda)y(\lambda)$.

If L is a normal subgroup of G which can be generated by finitely many elements of L, together with their conjugates in G, we write $d_G(L)$ for the least number of such elements.

In this section, α is taken to be a non-zero ordinal number. We denote classes of groups as follows :

1 = the trivial class.

$\mathfrak{F}$ = the class of finite groups.

$\mathfrak{G}$ = the class of finitely generated groups.

$\hat{\mathfrak{M}}$ = the class of groups satisfying the maximal condition on subgroups.

$\check{\mathfrak{M}}_n$ = the class of groups satisfying the minimal condition on normal subgroups.

$\mathfrak{C}$ = the class of cyclic groups.

$\mathfrak{A}$ = the class of abelian groups.

$\mathfrak{N}_c$ = the class of nilpotent groups of nilpotency class not exceeding c, for a postive integer c.

$\mathfrak{N}$ = $\bigcup_{c\geq 1} \mathfrak{N}_c$.

ZA_α = the class of hypercentral groups of hypercentral length less than or equal to α .

ZA = $\bigcup ZA_\alpha$.

ZD_α = the class of hypocentral groups of hypocentral length less than or equal to α.

ZD = $\bigcup ZD_\alpha$.

N, N_1 = the classes of groups all of whose subgroups are ascendant, subnormal respectively.

Z = the class of groups with a central series.

$\mathfrak{A}^d$ = the class of soluble groups of derived length not exceeding d, for a positive integer d.

$\mathfrak{S}$ = $\bigcup_{d\geq 1} \mathfrak{A}^d$.

SI^*_α = the class of groups with ascending normal abelian series of length less than or equal to α .

SI* = $\bigcup SI^*_\alpha$.

SJ* = the class of subsoluble groups.

SN* = the class of groups with ascending abelian series.

SD_α = the class of hypoabelian group with derived length less than or equal to α .

SD = $\bigcup SD_\alpha$.

SI,SN = the classes of groups with normal abelian series, abelian series repsectively.

$\overline{\overline{SI}}, \overline{\overline{SN}}$ = the largest subclasses of SI,SN respectively which are closed with respect to forming homomorphic images and subgroups.

We use the closure operations defined as follows for a class of groups $\mathfrak{X}$:

$G \in S\mathfrak{X}$ if G is isomorphic to a subgroup of an $\mathfrak{X}$-group.

$G \in S_n\mathfrak{X}$ if G is isomorphic to a normal subgroup of an $\mathfrak{X}$-group.

$G \in Q\mathfrak{X}$ if G is a homomorphic image of an $\mathfrak{X}$-group.

$G \in R\mathfrak{X}$ if, for each non-trivial element x of G, there exists a normal subgroup N, depending on x, such that $x \notin N$ and $G/N \in \mathfrak{X}$.

$G \in C\mathfrak{X}$ if G is isomorphic to a cartesian product of $\mathfrak{X}$-groups.

$G \in D\mathfrak{X}$ if G is isomorphic to a direct product of $\mathfrak{X}$-groups.

$G \in D_0\mathfrak{X}$ if G is isomorphic to the direct product of finitely many $\mathfrak{X}$-groups.

$G \in N\mathfrak{X}$ if G can be generated by its subnormal $\mathfrak{X}$-subgroups.

$G \in \acute{N}\mathfrak{X}$ if G can be generated by its ascendant $\mathfrak{X}$-subgroups.

$\mathfrak{X} = N_0\mathfrak{X}$ if the product of any pair of normal $\mathfrak{X}$-subgroups of any group is an $\mathfrak{X}$-group.

$G \in P\mathfrak{X}$ if G has a series of finite length in which every factor is an $\mathfrak{X}$-group.

$G \in \hat{P}\mathfrak{X}, \acute{P}\mathfrak{X}, \grave{P}\mathfrak{X}$ if G has a series, an ascending series, a descending series respectively in which every factor is an $\mathfrak{X}$-group.

$G \in \acute{P}_{sn}\mathfrak{X}$ if G has an ascending subnormal series in which every factor is an $\mathfrak{X}$-group.

$G \in L\mathfrak{X}$ if every finite subset of G is contained in an $\mathfrak{X}$-subgroup of G.

In addition, we use the following operations :

$G \in \hat{P}_n\mathfrak{X}, \acute{P}_n\mathfrak{X}$ if G has a normal $\mathfrak{X}$-series, an ascending normal $\mathfrak{X}$-series respectively.

Introduction

Classes of groups can be defined in various ways. For although all classes are ultimately defined in terms of relationships between elements of a group, many familiar classes of groups are best defined in terms of a relationship between some or all of the subgroups of a group and the group itself. An example is the class N_1 of groups all of whose subgroups are subnormal. Some classes are also defined by saying that a group G is a member of the class if it has a normal series $\{H_\sigma, K_\sigma \ ; \sigma \in \Sigma\}$ of subgroups such that there is a relationship between each factor group H_σ/K_σ and the corresponding group G/K_σ. An example is the class $\mathfrak{N}$ of nilpotent groups. Here we say that $G \in \mathfrak{N}$ if there exists a finite normal series $\{H_i, 0 \leq i \leq n\}$, for some integer n, such that H_{i+1}/H_i is a central subgroup of G/H_i for $i = 0,1,\ldots,n-1$.

The first part of this thesis is an attempt to begin to standardize and study the defining of classes of groups by means of the concept of a subgroup theoretical property. A subgroup theoretical property χ is a property pertaining to subgroups when regarded as subgroups of a given group such that $1 \chi G$ is true for all groups G and whenever $H \chi G$ and θ is an isomorphism of G with some other group then it follows that $\theta(H)\chi\theta(G)$.

Robinson [1] observes that it is possible to develop a theory of closure properties on subgroup theoretical properties. However, we have found no such theory in the literature and so have begun to do so from scratch. Robinson does mention certain properties which apply to some subgroup theoretical properties. Some of these are, in fact, expressible in terms of such closure properties. An example of this is the property of being inherited by homomorphic images. We use some of the properties of Robinson and introduce several more in order to prove some of our results.

If X is a subgroup theoretical property, Robinson defines hyper-X groups and hypo-X groups as groups having ascending and descending normal series, respectively, each of whose factors satisfy X in relation to the corresponding factor group of the group. We extend this to introduce other group theoretical classes defined in terms of subgroup theoretical properties. As may be expected, we are able to show that if a subgroup theoretical property satisfies certain conditions then certain group theoretical classes defined by it must also have certain properties. This again extends an idea of Robinson.

Chapter I closes with some examples of subgroup theoretical properties and results concerning some corresponding group theoretical classes. In particular, we consider the subgroup theoretical property "is a marginal subgroup of" and use it to provide an alternative characterization of a class of Hulse and Lennox [2].

Stanley [3] made the following generalization of the concept of the centre of a group : For any $\langle S, D_0 \rangle$ -closed class of groups $\mathfrak{X}$, define $H_1(G : \mathfrak{X})$ to be the set of all elements x of G for which there exists a normal subgroup N of G,

depending on x, such that $[x,N] = 1$ and $G/N \in \mathfrak{X}$.
Stanley calls $H_1(G:\mathfrak{X})$ the $\mathfrak{X}$-centre of G and proves it to be a characteristic subgroup of G. If $\mathfrak{X}$ is also Q-closed, he defines the upper $\mathfrak{X}$-central series of a group by analogy with the upper central series and denotes the limit of this series by $\overline{H}(G:\mathfrak{X})$. It is easy to see that $H_1(G:1) = \zeta_1(G)$ and that $H_1(G:\mathfrak{F})$ is the FC-subgroup of G. Then if $G = \overline{H}(G:1)$ then G is a hypercentral group and if $G = \overline{H}(G:\mathfrak{F})$ then G is an FC-hypercentral group.

The idea of a generalization of descending central series was introduced by Arrell [4] . He defined a descending series $\{W_\alpha ; \alpha < \gamma\}$ to be $\mathfrak{X}_D$-central, for a class of groups $\mathfrak{X}$, if for each α , $\cap\{[W_\alpha,N] \; ; N \triangleleft G, G/N \in \mathfrak{X}\} \leq W_{\alpha+1}$.
For a group G, he defined the following subgroups ; $D_1(G:\mathfrak{X}) = G$,
$D_\alpha(G:\mathfrak{X}) = \bigcap_{\beta<\alpha} D_\beta(G:\mathfrak{X})$ if α is a limit ordinal, and
$D_\alpha(G:\mathfrak{X}) = \cap\{[D_{\alpha-1}(G:\mathfrak{X}),N] \; ; N \triangleleft G, G/N \in \mathfrak{X}\}$ otherwise.
Arrell observed that $\{D_\alpha(G:\mathfrak{X}) \; ; \alpha$ an ordinal$\}$ is the lower $\mathfrak{X}_D$-central series of a group G, in the sense that if $\{G_\alpha ; \alpha$ an ordinal$\}$ is a descending $\mathfrak{X}_D$-central series of G then $G_\alpha \geq D_\alpha(G:\mathfrak{X})$ for all α .

As observed by Stanley and Bhattacharyya [5] , these notions of $\mathfrak{X}$-centrality and $\mathfrak{X}_D$-centrality lend themselves to the definition of subgroup theoretical properties, as follows :
For any $\langle S,Q,D_0\rangle$ -closed class of groups $\mathfrak{X}$, define the subgroup theoretical properties $\phi_{\mathfrak{X}}$ and $\psi_{\mathfrak{X}}$ by

$$H \; \phi_{\mathfrak{X}} \; G \quad \text{if} \quad H \leq H_1(G:\mathfrak{X})$$

$$H \; \psi_{\mathfrak{X}} \; G \quad \text{if} \quad \cap\{[H,N] \; ; \; N \triangleleft G, G/N \in \mathfrak{X}\} = 1 .$$

We omit the extra condition required in [5] that H be a normal subgroup of G in each definition. It is easy to see that if $G = \bar{H}(G:\mathfrak{X})$ then G is a hyper-$\phi_{\mathfrak{X}}$ group and if $D_{\alpha}(G:\mathfrak{X}) = 1$ for some ordinal α then G is a hypo-$\psi_{\mathfrak{X}}$ group.

We discuss $\phi_{\mathfrak{X}}$ and $\psi_{\mathfrak{X}}$ in Chapter II and use some results from Chapter I to obtain properties of various classes of groups determined by $\phi_{\mathfrak{X}}$ and $\psi_{\mathfrak{X}}$. We also prove the equality of several of these classes, notably that a group has a $\phi_{\mathfrak{X}}$-series if and only if it has a $\psi_{\mathfrak{X}}$-series.

In Chapter III we restrict our attention to $\mathfrak{V}$-centrality and $\mathfrak{V}_D$-centrality for varieties $\mathfrak{V}$. We prove the equality of the class of hyper-$\phi_{\mathfrak{V}}$ groups and hyper $\psi_{\mathfrak{V}}$-groups, of hypo-$\phi_{\mathfrak{V}}$ groups and hypo-$\psi_{\mathfrak{V}}$ groups and of groups with $\phi_{\mathfrak{V}}$-series of finite length and groups with $\psi_{\mathfrak{V}}$-series of finite length.

We use the operations of Petty [6] and a technique of Mal'cev (see [1], part II, from page 93) to obtain some of what Petty calls weak homomorphic image closure properties of some classes defined in terms of $\phi_{\mathfrak{V}}$ and $\psi_{\mathfrak{V}}$. We also prove a local theorem for the class of groups with a $\phi_{\mathfrak{V}}$-series and give an alternative proof of the local theorem for the class of groups denoted in [7] by $\mathfrak{V}^*$, the class of groups G for which no non-trivial element x of G belongs to the subgroup $[x^G, \mathfrak{V}(G)]$ of G, where $\mathfrak{V}(G)$ denotes the verbal subgroup of G. Notice that in this thesis we denote this class by $\overset{*}{\phi}_{\mathfrak{V}}$, reserving the notation $\mathfrak{V}^*$ for the subgroup theoretical property "is a marginal subgroup of".

Stanley [3] proved that, for any variety $\mathfrak{V}$, the class $\mathfrak{V}(c)$ of groups G such that $G = H_c(G:\mathfrak{V})$, for an integer c, is a variety. In particular, $\mathfrak{V}(1)$ is a variety. It is easy to prove that for any variety $\mathfrak{V}$, the $\mathfrak{V}$-centre of a group is just the centralizer of its verbal subgroup. We use this fact to prove that the $\mathfrak{V}(1)$-verbal subgroup of a group G is exactly the commutator subgroup $[\mathfrak{V}(G),G]$. We define a condition (X) on a variety $\mathfrak{V}$ by saying that $\mathfrak{V}$ satisfies (X) if, for all groups G, $\mathfrak{V}(\mathfrak{V}(G)) \leq [\mathfrak{V}(G),G]$ and prove that if $\mathfrak{V}$ satisfies (X) then all groups with a normal $\mathfrak{V}(1)$-factor cover, in the sense of Durbin [8], have a normal $\mathfrak{V}$-factor cover. We also show that, for any variety $\mathfrak{V}$, the variety $\mathfrak{V}(1)$ satisfies (X). This, together with the easy result that $\overset{*}{\phi}_{\mathfrak{V}}$-groups have a normal $\mathfrak{V}(1)$-factor cover, enables us to prove that all $\overset{*}{\phi}_{\mathfrak{V}(1)}$-groups have a normal $\mathfrak{V}(1)$-factor cover and that if $\mathfrak{V}$ satisfies (X) then all $\overset{*}{\phi}_{\mathfrak{V}(1)}$-groups have a normal $\mathfrak{V}$-factor cover. Varieties satisfying (X) include the class of nilpotent groups of nilpotency class less than or equal to c and the class of soluble groups of derived length less than or equal to d for any positive integers c and d.

Ayoub [9] introduced the class of residually commutable groups and proved that every SI-group is residually commutable. In fact, every group with an abelian factor cover is residually commutable. In Chapter IV, we generalize this idea to the class $[\mathfrak{V}]$ for any variety $\mathfrak{V}$. We say $G \in [\mathfrak{V}]$ if, for any word $\omega(x_1,\ldots,x_n)$ in the set of words determining $\mathfrak{V}$ and any elements $g_1,\ldots,g_n$ of G, not all of which are the identity element, there exists a normal subgroup of G containing $\omega(g_1,\ldots,g_n)$ but not containing all the g_i's. Then if $\mathfrak{A}$ is the class of abelian groups, $[\mathfrak{A}]$ is just the class of residually commutable groups. Ayoub proved the local closure of $[\mathfrak{A}]$ and Robinson [1] proves that a minimal normal subgroup of an $[\mathfrak{A}]$-group is abelian. We generalize these results to prove that, for any variety $\mathfrak{V}$, $[\mathfrak{V}] = L[\mathfrak{V}]$ and if M is a minimal normal subgroup of a $[\mathfrak{V}]$-group G then $M \in \mathfrak{V}$ and $G/M \in [\mathfrak{V}]$. It follows that a $[\mathfrak{V}]$-group satisfying the minimal condition on normal subgroups possesses an ascending normal $\mathfrak{V}$-series, generalizing Robinson's theorem [1] that a residually commutable group satisfying the minimal condition on normal subgroups is hyperabelian. We also prove that $[\mathfrak{V}]$ is $\langle S,R \rangle$-closed and that every group with a $\mathfrak{V}$-factor cover is a $[\mathfrak{V}]$-group.

We end Chapter IV with another generalization. For varieties $\mathfrak{V}_1,\ldots,\mathfrak{V}_n$ we define the class $(\mathfrak{V}_1,\ldots,\mathfrak{V}_n)$ by saying that G belongs to $(\mathfrak{V}_1,\ldots,\mathfrak{V}_n)$ if no non-trivial element x of G belongs to the corresponding subgroup $[x^G, \mathfrak{V}_1(G),\ldots,\mathfrak{V}_n(G)]$. Then $(\mathfrak{V})$ is just the class $\overset{*}{\phi}_{\mathfrak{V}}$. We are able to prove the local closure of any of the classes $(\mathfrak{V}_1,\ldots,\mathfrak{V}_n)$.

Two classes which are of particular interest are $\mathfrak{X}(1)$ and $\mathfrak{S}(1)$, where $\mathfrak{S}$ denotes the class of soluble groups. We prove $\mathfrak{X}(1)$ to be a class of generalized nilpotent groups. In fact, if we denote by ZA_γ the class of hypercentral groups of hypercentral length less than or equal to γ, we are able to prove that $ZA_\omega \leq \mathfrak{X}(1) < ZA_{\omega+1}$. We also prove that $\mathfrak{X}(1)$ is contained in the class of Fitting groups, the class of all groups in which every subgroup is descendant and the class of hypocentral groups of hypocentral length not exceeding $\omega+1$. This is in contrast to the class $ZA_{\omega+1}$, which is contained in none of these classes.

After studying $\mathfrak{N}(1)$ in Chapter V, we turn our attention to generalized soluble groups in Chapter VI. In contrast to the fact that $\mathfrak{N}(2)$ is not a class of generalized nilpotent groups, we prove that several new classes of generalized soluble groups can be defined using the subgroup theoretical property ϕ_{σ}. After proving that $\sigma(1)$ properly contains σ and is properly contained in each of the classes of hyperabelian groups, locally soluble groups and hypoabelian groups of length less than or equal to $\omega+1$, we prove that hyper-ϕ_{σ} groups are subsoluble and groups with finite ϕ_{σ}-series are hypoabelian. We also prove that groups satisfying these conditions locally are SI- and SN - groups, respectively. We include various results concerning related classes, including the fact that $\overset{*}{\phi}_{\sigma}$ is a class of generalized soluble groups.

Finally, in Chapter VII, we restrict our discussion to the class of FC-groups. This is just the class $\mathfrak{F}(1)$, where $\mathfrak{F}$ denotes the class of finite groups. We prove that if $\mathfrak{X}$ is a Q-closed class of generalized nilpotent groups then $\mathfrak{X} \cap \mathfrak{F}(1) \leq \mathfrak{N}(1)$ and if $\mathfrak{X}$ is a Q- or S_n-closed class of generalized soluble groups then $\mathfrak{X} \cap \mathfrak{F}(1) \leq \sigma(1)$. We also prove that for any variety $\mathfrak{V}$, $\overset{*}{\phi}_{\mathfrak{V}} \cap \mathfrak{F}(1)$ is contained in the class of hyper-$\phi_{\mathfrak{V}}$ groups. It follows that residually central FC-groups are ZA_{ω}-groups.

I Subgroup Theoretical Properties

The following definition is from [1] .

Definition Let χ be a property pertaining to subgroups. If H has the property χ when regarded as a subgroup of a group G, we write $H \chi G$. χ is a subgroup theoretical property if $1 \chi G$ is always valid and if $\theta(H) \chi \theta(G)$ follows from $H \chi G$ whenever θ is an isomorphism from G to some other group.

In this Chapter we present some operations and conditions on subgroup theoretical properties and obtain a number of results about certain classes of groups defined in terms of subgroup theoretical properties. We will need the following proposition for any subgroup theoretical property χ .

Proposition 1·1 If B and N are normal subgroups of a group G and if $N \leq B \leq A \leq G$ then $(A/B) \chi (G/B)$ if and only if $\left(\frac{A/N}{B/N}\right) \chi \left(\frac{G/N}{B/N}\right)$.

Proof In the isomorphism $\theta : \frac{G/N}{B/N} \longrightarrow G/B$ defined by $\theta(gN(B/N)) = gB$, it is easy to see that $\theta\left(\frac{A/N}{B/N}\right) = A/B$.

Thus $\left(\frac{A/N}{B/N}\right) \chi \left(\frac{G/N}{B/N}\right)$ if and only if $(A/B) \chi (G/B)$.

We define a partial ordering on subgroup theoretical properties as follows :

Definition Let χ and χ' be subgroup theoretical properties. Then $\chi \leq \chi'$ if $H \chi G$ always implies that $H \chi' G$.

We define our first two subgroup theoretical properties as follows :

Definition $H \,\iota\, G$ if $H = 1$.

Definition $H \,\delta\, G$ if H is a direct factor of G.

Following the concepts of operation and closure operation on group theoretical classes, we define operations and closure operations on subgroup theoretical properties as follows :

Definition A function $\underline{A}$ assigning to each subgroup theoretical property χ a subgroup theoretical property $\underline{A}\chi$ is an operation on subgroup theoretical properties if $\underline{A}\iota = \iota$ is always true and if $\chi \leq \underline{A}\chi \leq \underline{A}\chi'$ whenever $\chi \leq \chi'$. $\underline{A}$ is a closure operation if, in addition, $\underline{A}\chi = \underline{A}^2\chi$ is always true.

Notice that if $\chi = \underline{A}\chi$ we may say that χ is $\underline{A}$-closed.

We define two operations on subgroup theoretical properties.

Definition $H \,\underline{S}\chi\, G$ if $H \leq K \,\chi\, G$ for some subgroup K of G.

Definition $K \,\underline{H}\chi\, G$ if there exist groups K_1 and G_1 and a homomorphism θ from G_1 onto G such that $K_1 \,\chi\, G_1$ and $\theta(K_1) = K$.

Notice that $\chi = \underline{H}\chi$ is equivalent to the statement, as found in [1] and elsewhere, that χ is inherited by homomorphic images. We may also say that χ is inherited by subgroups if $\chi = \underline{S}\chi$.

Proposition 1·2 $\underline{S}$ and $\underline{H}$ are closure operations on subgroup theoretical properties.

Proof If $H \, \underline{S}^2 \chi \, G$ then $H \leq K \leq L \, \chi \, G$ for some subgroups K and L of G. Thus $H \leq L \, \chi \, G$. If $K \, \underline{H}^2 \chi \, G$ then there exist groups K_1, G_1, K_2 and G_2 and homomorphisms θ_1 and θ_2 such that $\theta_2(G_2) = G_1$, $\theta_1(G_1) = G$, $\theta_2(K_2) = K_1$, $\theta_1(K_1) = K$ and $K_2 \, \chi \, G_2$. By letting $\theta = \theta_1\theta_2$ we see that $K \, \underline{H} \, \chi \, G$.

Following Robinson [1] we define the following condition on subgroup theoretical properties :

Definition χ satisfies (*) if, given normal subgroups X, Y and N of a group G such that $Y \leq X$ and $(X/Y) \, \chi \, (G/Y)$, it always follows that $((N \cap X)/(N \cap Y)) \, \chi (G/N \cap Y)$.

We define two more such conditions as follows :

Definition χ satisfies (†) if, whenever $A \, \chi \, H \, \delta \, G$, it follows that $A \, \chi \, G$.

Definition χ is persistent if, whenever $H \, \chi \, G$ and $H \leq K \leq G$, then $H \, \chi \, K$.

The following result will be of use later.

Lemma 1·3 If χ satisfies (†) and B is a normal subgroup of G with $B \leq A \leq H \, \delta \, G$, such that $(A/B) \, \chi \, (H/B)$, then $(A/B) \, \chi \, (G/B)$.

Proof Since $H \, \delta \, G$ there exists a normal subgroup K of G such that $G = H \times K$. So $G/B = (H \times K)/B = (H/B) \times (KB/B)$ which shows that $(A/B)\ \chi\ (H/B)\ \delta\ (G/B)$. Therefore $(A/B)\ \chi\ (G/B)$.

We observe that δ is persistent and satisfies (†). Two obvious instances of subgroup theoretical properties are "is a normal subgroup of" and "is a central subgroup of". It is easy to prove that both of these properties satisfy (*) and (†) and are persistent and inherited by homomorphic images and that the second is also inherited by subgroups.

The following definition is due to Robinson [1] .

Definition A normal series $\{H_\sigma, K_\sigma\ ;\ \sigma \in \Sigma\}$ in a group G is called a *χ-series* if, for each $\sigma \in \Sigma$, $(H_\sigma/K_\sigma)\ \chi\ (G/K_\sigma)$.

We denote the classes of groups with χ-series, ascending χ-series, descending χ-series and χ-series of finite length by $\hat{\chi}$, $\acute{\chi}$, $\grave{\chi}$ and $\overset{o}{\chi}$, respectively. The classes $\acute{\chi}$ and $\grave{\chi}$ are sometimes called the classes of *hyper-χ groups* and *hypo-χ groups*, respectively.

We use the concept of a subgroup theoretical property to define two more classes of groups :

Definition Suppose that for each non-trivial element x of a group G there exists a normal subgroup N of G, depending on x , such that $x \notin N$ and $(x^G N/N)\ \chi\ (G/N)$. Then we say that G is a *residually-χ group* and we write $G \in \overset{*}{\chi}$.

Definition Suppose that for each non-trivial element x of a group G there exist normal subgroups M and N of G, depending on x, such that $N \leq M$, $x \in M - N$ and $(M/N)\ \chi\ (G/N)$.

Then we say that G has a <u>X-factor covering</u> and we write

$G \in \underline{X}$.

We establish some connections between $\hat{\chi}$ and these two classes.

<u>Theorem 1·4</u> (i) $\hat{X} \leq \underline{X}$.

(ii) $\overset{*}{X} \leq \underline{X}$ and if χ satisfies $(*)$ or if $\chi = \underline{S}\chi$ then $\overset{*}{X} = \underline{X}$.

(iii) $\overset{*}{X} < \underline{X}$ in general and $\hat{\chi} < \underline{\chi}$ in general.

<u>Proof</u> (i) This part is trivial.

(ii) Let $G \in \overset{*}{X}$ and let x be a non-trivial element of G. Then there exists a normal subgroup N of G such that $x \notin N$ and $(x^G N/N)\ \chi\ (G/N)$. But $N \lhd x^G N \lhd G$ and $x \in x^G N - N$. Therefore $G \in \underline{X}$.

Let $G \in \underline{X}$ and let x be a non-trivial element of G. Then there exist normal subgroups K and H of G with $K \leq H$ such that $x \in H - K$ and $(H/K)\ \chi\ (G/K)$.

If χ satisfies $(*)$ then

$(x^G K \cap H)/(x^G K \cap K)\ \chi\ (G/(x^G K \cap K))$.

But $x^G K \leq H$ and so

$(x^G K/K)\ \chi\ (G/K)$.

Therefore $G \in \overset{*}{X}$.

If, on the other hand, $\chi = \underline{S}\chi$ then, since

$(x^G K/K) \leq (H/K)\ \chi\ (G/K)$

we have

$(x^G K/K)\ \chi\ (G/K)$.

Again, $G \in \overset{*}{X}$.

(iii) Let $\mathfrak{X}$ be the subgroup theoretical property "is the trivial subgroup of or is not a finitely generated subgroup of" and let G be any FC-group which is not finitely generated. Clearly G is a $\underline{X}$-group because for any non-trivial element x of G, we have $x \in G - 1$ and $G/1 \notin \mathfrak{F}$. Suppose $G \in \overset{*}{X}$. If x is a non-trivial element of G then there exists a normal subgroup N of G with $x \notin N$ and $(x^G N/N)\ \chi\ (G/N)$.
Since $x \notin N$ we have $x^G N/N \notin \mathfrak{F}$. However $x^G \in \mathfrak{F}$ because G is an FC-group, which is a contradiction. Therefore $G \notin \overset{*}{X}$.

Let χ be the subgroup theoretical property "is a central subgroup of". An example of Phillips and Roseblade [10] is a $\underline{X}$-group but not a $\hat{\chi}$-group.

We now turn our attention to the subgroup closure of these classes of groups. We require the following lemma.

<u>Lemma 1·5</u> Suppose $\chi = \underline{S}\chi$ and χ is persistent.

If B is a normal subgroup of a group G such that $B \leq A \leq G$ and $(A/B)\ \chi\ (G/B)$, then $((H \cap A)/(H \cap B))\ \chi\ (H/(H \cap B))$ for any subgroup H of G.

<u>Proof</u> Let $M = H \cap A$ and $N = H \cap B$. Now $MB/B \leq A/B$ and so $(MB/B)\ \chi\ (G/B)$. Also $MB/B \leq HB/B$ and so $(MB/B)\ \chi\ (HB/B)$. Define $\phi : HB/B \to H/N$ by $\phi(hB) = hN$ for all $h \in H$. Then ϕ is an isomorphism and $\phi(MB/B) = MN/N = M/N$. Therefore $(M/N)\ \chi\ (H/N)$, as required.

When $\mathfrak{X}$ is the subgroup theoretical property "is a central subgroup of", the classes $\overset{*}{\mathfrak{X}}, \hat{\mathfrak{X}}$, $\acute{\mathfrak{X}}$, $\grave{\mathfrak{X}}$ and $\mathring{\mathfrak{X}}$ are the class of residually central groups, Z , ZA , ZD and $\mathfrak{H}$, respectively. Also, by the $\underline{S}$-closure of $\mathfrak{X}$ and by 1·4 (ii), $\underline{\mathfrak{X}} = \overset{*}{\mathfrak{X}}$. Each of these classes is S-closed. We may generalize these facts in the following way.

<u>Theorem 1·6</u> If $\mathfrak{X} = \underline{S}\,\mathfrak{X}$ and $\mathfrak{X}$ is persistent, then each of the classes $\underline{\mathfrak{X}}$, $\hat{\mathfrak{X}}$, $\acute{\mathfrak{X}}$, $\grave{\mathfrak{X}}$ and $\mathring{\mathfrak{X}}$ is s-closed.

<u>Proof</u> $\underline{\mathfrak{X}}$; Let H be a subgroup of a $\underline{\mathfrak{X}}$-group G and let x be a non-trivial element of H. Then there exist normal subgroups M and N of G with $N \leqslant M$ such that $x \in M - N$ and $(M/N)\ \mathfrak{X}\ (G/N)$. By 1·5 $((H \cap M)/(H \cap N))\ \mathfrak{X}\ (H/(H \cap N))$ and, since $x \in (H \cap M) - (H \cap N)$, we have shown that $H \in \underline{\mathfrak{X}}$.

$\hat{\mathfrak{X}}$; Let H be a subgroup of a $\hat{\mathfrak{X}}$-group G and let $\{H_\sigma, K_\sigma : \sigma \in \Sigma\}$ be a $\mathfrak{X}$-series in G. Now $H - 1 = \bigcup_{\sigma\in\Sigma}((H \cap H_\sigma) - (H \cap K_\sigma))$, $H \cap H_\tau \leqslant H \cap K_\sigma$ if $\tau < \sigma$, $H \cap K_\sigma \triangleleft H \cap H_\sigma \triangleleft H$ and $H \cap K_\sigma \triangleleft H$ for all $\sigma \in \Sigma$. By 1·5 $((H \cap H_\sigma)/(H \cap K_\sigma))\ \mathfrak{X}\ (H/(H \cap K_\sigma))$ and so $\{H \cap H_\sigma, H \cap K_\sigma ; \sigma \in \Sigma\}$ is a $\mathfrak{X}$-series in H. Therefore $H \in \hat{\mathfrak{X}}$.

$\acute{\mathfrak{X}}$; Let H be a subgroup of a $\mathfrak{X}$-group G. There exists an ascending $\mathfrak{X}$-series $\{K_\beta ; \beta < \alpha\}$ in G. By the preceding part of this theorem, $\{H \cap K_\beta ; \beta < \alpha\}$ is a $\mathfrak{X}$-series in H. But this is an ascending series so $H \in \mathfrak{X}$.

$\grave{\mathfrak{X}}$, $\mathring{\mathfrak{X}}$; The proof of the s-closure of these classes is similar to that used for $\acute{\mathfrak{X}}$.

When X is not $\underline{S}$-closed we may be able to use the following alternative to 1·6 :

<u>Theorem 1·7</u> If X satisfies $(*)$ and is persistent, then each of the classes $\underline{X}$, $\hat{X}$, $\acute{X}$, $\grave{X}$ and $\mathring{X}$ is S_n-closed.

<u>Proof</u> $\underline{X}$; Let H be a normal subgroup of a $\underline{X}$-group G and let x be a non-trivial element of H. Then there exist normal subgroups A and B of G, with $B \leq A$, such that $x \in A - B$ and $(A/B)\ X\ (G/B)$. By $(*)$, $((H \cap A)/(H \cap B))\ X\ (G/(H \cap B))$ and, because X is persistent, $((H \cap A)/(H \cap B))\ X\ (H/(H \cap B))$. But $x \in (H \cap A) - (H \cap B)$ and so we have proved that $H \in \underline{X}$.

$\hat{X}$; Let H be a normal subgroup of a $\hat{X}$-group G and let $\{H_\sigma, K_\sigma ; \sigma \in \Sigma\}$ be a X-series in G. Then, as in the proof of 1·6 , $\{H \cap H_\sigma, H \cap K_\sigma ; \sigma \in \Sigma\}$ is a X-series in H. Also $((H \cap H_\sigma)/(H \cap K_\sigma))\ X\ (G/(H \cap K_\sigma))$ and so $((H \cap H_\sigma)/(H \cap K_\sigma))\ X\ (H/(H \cap K_\sigma))$, as required.

$\acute{X}, \grave{X}, \mathring{X}$; These follow in exactly the same way as in 1·6.

We now establish some conditions on subgroup theoretical properties which imply R- and C- closure on the series classes. A preliminary result is required.

<u>Lemma 1·8</u> If X satisfies $(+)$ then $\hat{X}$, $\acute{X}$, $\grave{X}$ and $\mathring{X}$ are all D_0-closed.

<u>Proof</u> Let $G = H \times K$ where H and K are $\hat{X}$-groups. Then $G/H \cong K \in \hat{X}$ and so we have X-series $\{\Lambda'_\sigma , V'_\sigma ; \sigma \in S\}$ and $\{\Lambda''/H_\tau , V''/H_\tau ; \tau \in T\}$ in H and G/H respectively.

We may choose S and T such that $S \cap T = \phi$.

Let $W = S \cup T$ and define a linear ordering in W such that every element of S precedes every element of T and the original orderings in S and T prevail. For each $\omega \in W$ define :

$$\Lambda_\omega = \begin{cases} \Lambda'_\omega & \text{if } \omega \in S \\ \Lambda''_\omega & \text{if } \omega \in T \end{cases}$$

$$V_\omega = \begin{cases} V'_\omega & \text{if } \omega \in S \\ V''_\omega & \text{if } \omega \in T \end{cases}$$

We now prove that $\{\Lambda_\omega, V_\omega \; ; \; \omega \in W\}$ is a X-series in G.

Choose $\omega \in W$. If $\omega \in T$ then Λ''_ω and V''_ω are normal in G and if $\omega \in S$ then Λ'_ω and V'_ω are normal in H, which is a direct factor of G. So, in either case, Λ_ω and V_ω are normal subgroups of G. It is clear that $V_\omega \leq \Lambda_\omega$ for all $\omega \in W$.

Let x be a non-trivial element of G. If $x \in H$ then $x \in \bigcup_{\omega \in S} (\Lambda'_\omega - V'_\omega) \subseteq \bigcup_{\omega \in W} (\Lambda_\omega - V_\omega)$ and if $x \notin H$ then xH is a non-trivial element of G/H so that $xH \in (\Lambda''_\omega/H) - (V''_\omega/H)$ for some $\omega \in T$. Then $x \in \Lambda''_\omega - V''_\omega$ and $x \in \bigcup_{\omega \in T} (\Lambda''_\omega - V''_\omega) \subseteq \bigcup_{\omega \in W} (\Lambda_\omega - V_\omega)$. Therefore $G - 1 = \bigcup_{\omega \in W} (\Lambda_\omega - V_\omega)$.

Now let $\alpha, \beta \in W$ with $\alpha < \beta$. If α and β are both elements of S then $\Lambda_\alpha = \Lambda'_\alpha \leq V'_\beta = V_\beta$ while if α and β are both elements of T then $\Lambda_\alpha = \Lambda''_\alpha \leq V''_\beta = V_\beta$. Finally if $\alpha \in S$ and $\beta \in T$ then $\Lambda_\alpha = \Lambda'_\alpha \leq H \leq V''_\beta = V_\beta$.

Therefore $\Lambda_\alpha \leq V_\beta$ for all $\alpha, \beta \in W$ with $\alpha < \beta$.

Choose $\omega \in W$. If $\omega \in S$ then by 1·3 we have $(\Lambda_\omega / V_\omega)\ \mathfrak{X}\ (G / V_\omega)$ because $V_\omega \leq \Lambda_\omega \leq H \,\delta\, G$. If, on the other hand, $\omega \in T$ then $((\Lambda''_\omega / H)/(V''_\omega / H))\ \mathfrak{X}\,((G/H)/(V''_\omega)H))$ so that, by 1·1, $(\Lambda_\omega / V_\omega)\ \mathfrak{X}\ (G / V_\omega)$. Therefore $\hat{\mathfrak{X}} = D_0\hat{\mathfrak{X}}$.

A similar proof will show that $\acute{\mathfrak{X}} = D_0\acute{\mathfrak{X}}$ because if S and T are well-ordered, the ordering imposed on $S \cup T$ is also a well-ordering. Also if S and T are both finite then $S \cup T$ is finite and so $\mathring{\mathfrak{X}} = D_0 \mathring{\mathfrak{X}}$.

If we let $G = H \times K$ where H and K are $\grave{\mathfrak{X}}$-groups, so that $\{ V'_\sigma \ ; \sigma \in S \}$ and $\{ V''_\tau / H \ ; \tau \in T \}$ are descending $\mathfrak{X}$-series in H and G/H respectively, we can prove G to be a $\grave{\mathfrak{X}}$ group as follows .

Let $W = S \cup T$ but define a reverse well-ordering of W in such a way as to make every element of T precede every element of S and the original orderings in S and T prevail. Define V_ω as before. Again V_ω is normal in G for all $\omega \in W$.

Let x be a non-trivial element of G. If $x \in H$ then there exists an ordinal α such that $x \in V'_{\alpha+1} - V'_\alpha$ so $x \in V_{\alpha+1} - V_\alpha$. If $x \notin H$ then there exists an ordinal number α such that $xH \in (V''_{\alpha+1}/H) - (V''_\alpha/H)$ so $x \in V_{\alpha+1} - V_\alpha$.

Choose $\omega \in W$. If $\omega \in S$ then, by 1·3 , $(V_{\omega+1} / V_\omega)\ \mathfrak{X}\ (G/ V_\omega)$ and if $\omega \in T$ then, by 1·1, $(V_{\omega+1} / V_\omega)\ \mathfrak{X}\ (G/ V_\omega)$, as required.

This result enables us to prove the following theorem, the proof of which is similar to that of a theorem of Scott and Sonneborn [11] .

<u>Theorem 1·9</u> If X satisfies (†) then $\hat{X}$ and $\grave{X}$ are both C-closed and $\acute{X}$ is D-closed.

<u>Proof</u> Let $G = \underset{\alpha\varepsilon A}{Cr} G_\alpha$ where $G_\alpha \varepsilon \hat{X}$ for each $\alpha \varepsilon A$. By 1·8 the theorem is true if A is finite. Thus we may assume that A is infinite. Let A be well-ordered and let its ordinal number be γ . Then $\gamma = \lambda + n$ for some limit ordinal λ and $0 \leq n < \omega$. If $n > 0$ then

$$G = (\underset{\alpha<\lambda}{Cr} G_\alpha) \times (\overset{n}{\underset{r=0}{Dr}} G_{\lambda+r}).$$

Again by 1·8 we may assume that $n = 0$. Therefore we may suppose that $A = \{\alpha ; \alpha < \lambda\}$ for some limit ordinal λ .

If $\beta < \lambda$ let H_β consist of all functions x of A such that $x(\alpha) = 1$ if $\alpha < \beta$. Then clearly $H_\beta \lhd G$ for all $\beta < \lambda$ and $H_{\beta+1} \leq H_\beta$ for all $\beta < \lambda$. Also $G = H_0$.

Let β be a non-zero limit ordinal. Let $h \varepsilon H_\beta$ and $\delta < \beta$. If $\alpha < \delta$ then $\alpha < \beta$ and so $h(\alpha) = 1$. Therefore $h \varepsilon H_\delta$ and so $H_\beta \leq \bigcap_{\delta<\beta} H_\delta$. Let $h \varepsilon \bigcap_{\delta<\beta} H_\delta$. If $\alpha < \beta$ then $\alpha+1 < \beta$ and so $h \varepsilon H_{\alpha+1}$. Therefore $h(\alpha) = 1$ so $h \varepsilon H_\beta$. Then we have proved that $H_\beta = \bigcap_{\delta<\beta} H_\delta$. If $h \varepsilon H_\lambda$ then $h(\alpha) = 1$ for all $\alpha < \lambda$ and so $h = 1$. Therefore $H_\lambda = 1$ and we have a descending normal series

$$G = H_0 \rhd H_1 \rhd H_2 \rhd \ldots H_\lambda = 1$$

Also, for any $\beta < \lambda$, $H_\beta/H_{\beta+1} \cong G_\beta \varepsilon \hat{X}$. Let K be a term of a X -series between H_β and $H_{\beta+1}$. Then $K \triangleleft H_\beta \delta G$ so that $K \triangleleft G$. If A/B is a factor of this X-series then $(A/B) \times (H_\beta/B)$ and, by (†) , we have $(A/B) \times (G/B)$. Thus the union of the X-series between the H_β's is a X-series in G. Therefore $G \varepsilon \hat{X}$.

The result for $\grave{X}$ is proved in the same way.

Now let $G \varepsilon D\acute{X}$. As above we may suppose that $G = \underset{\alpha<\lambda}{\mathrm{Dr}}\ G_\alpha$ for some limit ordinal λ . If $\beta < \lambda$ we now define H_β to consist of all functions $x : A \rightarrow G$ such that $x(\alpha) = 1$ if $\alpha \geq \beta$. Then, for each $\beta < \lambda$, $H_\beta \triangleleft G$ and $H_\beta \leq H_{\beta+1}$, and $H_1 = 1$. Then we have an ascending normal series

$$1 = H_1 \triangleleft H_2 \triangleleft \dots H_\lambda = G$$

Now, for any $\beta < \lambda$, $H_{\beta+1}/H_\beta \cong G_\beta \varepsilon \acute{X}$. As above we may prove that each term of a X-series between H_β and $H_{\beta+1}$ is normal in G and that every factor A/B of such a series satisfies $(A/B) \times (G/B)$. Therefore $G \varepsilon \acute{X}$.

We turn now to the question of residual closure.

<u>Theorem 1·10</u> (i) If $X = \underline{S}X$ and X is persistent and satisfies (+), then $\hat{X}$ and $\grave{X}$ are R-closed.

(ii) $\underline{X} = R\underline{X}$

(iii) $\overset{*}{X} = R\overset{*}{X}$

<u>Proof</u> (i) Since $R \leq \langle S,C \rangle$, this result follows from 1·6 and 1·9.

(ii) Let $G \varepsilon R\underline{X}$ and let x be a non-trivial element of G. Then there exists a normal subgroup N of G such that $x \notin N$ and $G/N \varepsilon \underline{X}$. But xN is a non-trivial element of G/N so there exist normal subgroups K/N and H/N of G/N, with $K \leqslant H$, such that $xN \varepsilon (H/N) - (K/N)$ and $\frac{H/N}{K/N} \; X \; \frac{G/N}{K/N}$. Therefore $x \varepsilon H - K$ and, by 1·1, $(H/K) \; X \; (G/K)$. Therefore $G \varepsilon \underline{X}$.

(iii) Let $G \varepsilon R\overset{*}{X}$ and let x be a non-trivial element of G. Then there exists a normal subgroup N of G such that $x \notin N$ and $G/N \varepsilon \overset{*}{X}$. But, since xN is a non-trivial element of G/N, there exists a normal subgroup M of G with $N \leqslant M$ such that $x \notin M$ and $\frac{(xN)^{G/N}M/N}{M/N} \; X \; \frac{G/N}{M/N}$. Now

$(xN)^{G/N}M/N = (x^G N/N).(M/N) = x^G M/N$. Therefore

$\frac{x^G M/N}{M/N} \; X \; \frac{G/N}{M/N}$ and, by 1·1, $(x^G M/M) \; X \; (G/M)$ and we have proved $G \varepsilon \overset{*}{X}$.

The proof of our next theorem requires the following proposition.

<u>Proposition 1.11</u> Suppose $X = \underline{H}X$. If A,B and C are subgroups of a group G, with $B \lhd G$, $C \lhd G$, $B \leqslant A$ and $(A/B) \; X \; (G/B)$, then $(AC/BC) \; X \; (G/BC)$.

<u>Proof</u> Define the function $\phi : G/B \to G/BC$ by $\phi(gB) = gBC$ for each $g \varepsilon G$. Then ϕ is an epimorphism and $\phi(A/B) = AC/BC$. Therefore $(AC/BC) \; X \; (G/BC)$.

<u>Theorem 1.12</u> If X satisfies $(*)$ and if M is a minimal normal subgroup of a $\overset{*}{X}$-group G then $M \; X \; G$. If, in addition, $X = \underline{H}X$, then $G/M \varepsilon \overset{*}{X}$.

Proof Let x be a non-trivial element of M. Then there exists a normal subgroup N of G such that $x \notin N$ and $(x^G N/N)\ X\ (G/N)$. Since X satisifes $(*)$, we have $((M \cap x^G N)/(M \cap N))\ X\ (G/(M \cap N))$. By the minimality of M, $M \cap N = 1$ or $M \leq N$. But $x \in M$ and $x \notin N$ so the only possibility is that $M \cap N = 1$. Therefore $(M \cap x^G N)\ X\ G$. Again by the minimality of M, $x^G = M$ so that $M \cap x^G N = M \cap MN = M$. Therefore $M\ X\ G$.

Suppose now that $X = \underline{H}X$ and let xM be a non-trivial element of G/M. Then we must prove that there exists a normal subgroup E of G ; with $M \leq E$, such that $x \notin E$ and $(x^G E/E)\ X\ (G/E)$. This is sufficient by 1·1.

Now there exists a normal subgroup K of G such that $x \notin K$ and $(x^G K/K)\ X\ (G/K)$. By the minimality of M, either $M \cap K = 1$ or $M \leq K$. If $M \leq K$ then we may take $E = K$. So we may assume that $M \cap K = 1$.

Since $X = \underline{H}X$, we have $(x^G KM/KM)\ X\ (G/KM)$. Therefore if $x \notin KM$ we may take $E = KM$. Thus we may assume further that $x \in KM$.

Suppose $x = km$, where $k \in K$ and $m \in M$. We know that $k \neq 1$ because $x \notin M$. Therefore there exists a normal subgroup L of G such that $k \notin L$ and $(k^G L/L)\ X\ (G/L)$. Let $P = K \cap L$. If $x \in PM$ then $x = pm'$, say, where $p \in P$ and $m' \in M$. Thus $km = pm'$ and so $p^{-1}k = m'm^{-1} \in M \cap K = 1$. Consequently $p = k$ which contradicts the fact that $k \notin L$. Therefore $x \notin PM$.

Since $(k^G L/L)\ \mathfrak{X}\ (G/L)$ we have by $(*)$ that $(k^G L \cap K/L \cap K)\ \mathfrak{X}\ (G/L \cap K)$. But $k^G L \cap K = k^G(L \cap K)$, so that $(k^G P/P)\ \mathfrak{X}\ (G/P)$. Therefore, since $\mathfrak{X} = \underline{H}\mathfrak{X}$, we have by 1·11 that $(k^G PM/PM)\ \mathfrak{X}\ (G/PM)$. But $x^G PM = k^G PM$. Therefore, letting $E = PM$, we have $(x^G E/E)\ \mathfrak{X}\ (G/E)$, as required.

We observe that the condition on $\mathfrak{X}$ in Theorem 1·12 ensures by 1·4(ii) that the theorem is also true for $\underline{\mathfrak{X}}$-groups.

So far, we have only discussed the classes $\overset{*}{\mathfrak{X}}$, $\underline{\mathfrak{X}}$ and the $\mathfrak{X}$-series classes. The following definitions provide further ways of defining classes of groups in terms of subgroup theoretical properties. In each case $\mathfrak{X}$ is any subgroup theoretical property. Several of these definitions are after Robinson [1] .

Definition The $\mathfrak{X}$-centre of a group G, denoted by $\zeta^{\mathfrak{X}}(G)$, is defined by $\zeta^{\mathfrak{X}}(G) = \langle H \leq G\ ;\ H\ \mathfrak{X}\ G \rangle$.

Notice that it follows from the definition of a subgroup theoretical property that $\zeta^{\mathfrak{X}}(G) \lhd G$ is always true.

Definition The upper $\mathfrak{X}$-central series of G is the series $\{\zeta^{\mathfrak{X}}_{\alpha}(G)\ ;\ \alpha \text{ an ordinal}\}$ defined by $\zeta^{\mathfrak{X}}_{0}(G) = 1$, $\zeta^{\mathfrak{X}}_{1}(G) = \zeta^{\mathfrak{X}}(G)$ and if $\zeta^{\mathfrak{X}}_{\beta}(G)$ has been defined for all $\beta < \alpha$ then : if α is a limit ordinal , $\zeta^{\mathfrak{X}}_{\alpha}(G) = \bigcup_{\beta<\alpha} \zeta^{\mathfrak{X}}_{\beta}(G)$, and otherwise

$$\zeta^{\mathfrak{X}}_{\alpha}(G)/\zeta^{\mathfrak{X}}_{\alpha-1}(G) = \zeta^{\mathfrak{X}}(G/\zeta^{\mathfrak{X}}_{\alpha-1}(G)) .$$

Definition The limit of the upper $\mathfrak{X}$-central series of G, denoted by $\overline{\zeta^X}(G)$, is called the <u>X-hypercentre</u> of G .

<u>Definition</u> If $N \leq G$, $\gamma^X(N,G) = \bigcap\{M \triangleleft G;\ M \leq N, (N/M)\ X\ (G/M)\}$

<u>Definition</u> The <u>lower X-central series</u> of G is the series $\{\gamma_\alpha{}^X(G) :\ \alpha$ an ordinal$\}$ defined by $\gamma_1{}^X(G) = G$ and if $\gamma_\beta^X(G)$ has been defined for all ordinals $\beta < \alpha$ then : if α is a limit ordinal $\gamma_\alpha^X(G) = \bigcap_{\beta<\alpha} \gamma_\beta^X(G)$, otherwise

$$\gamma_\alpha^X(G) = \gamma^X(\gamma_{\alpha-1}^X(G), G).$$

<u>Definition</u> The limit of the lower X-central series of G, denoted by $\overline{\gamma^X}(G)$, is called the <u>X-hypocentre</u> of G.

Some results concerning these concepts can be found in [1] . Notice, though, that the definition in this thesis of $\gamma^X(N,G)$ does not require N to be normal in G, whereas that in [1] does.

Subgroup theoretical properties can also be defined in terms of group theoretical functions, as follows.

<u>Definition</u> A <u>group theoretical function</u> α is a function which assigns to each group G a subgroup $\alpha(G)$ such that $\theta(\alpha(G)) = \alpha(\theta(G))$ for each isomorphism θ out of G.

<u>Definition</u> If α is a group theoretical function, the <u>subgroup theoretical property associated with α</u> , also denoted by α , is defined by saying that $H\ \alpha\ G$ if and only if $H \leq \alpha(G)$.

We also make the following definition of a monotone group theoretical function.

Definition A group theoretical function is monotone if $L \leq G$ always implies that $\alpha(L) \leq \alpha(G)$.

Examples of group theoretical functions and associated subgroup theoretical properties are $\mathfrak{V}$ and $\mathfrak{V}^*$, for a variety $\mathfrak{V}$, and ζ_1. Here $H \mathfrak{V} G$ if H is contained in the $\mathfrak{V}$-verbal subgroup of G, $H \mathfrak{V}^* G$ if H is a $\mathfrak{V}$-marginal subgroup of G and $H \zeta_1 G$ if H is a central subgroup of G. Of these $\mathfrak{V}$ is monotone while $\mathfrak{V}^*$ and ζ_1 are not. Clearly $\zeta_1 = \mathfrak{A}^*$.

The proof of the following lemma is very easy and is omitted.

Lemma 1·13 (i) α is a subgroup theoretical property.

(ii) $\alpha = \underline{S}\alpha$.

(iii) If whenever $K \leq G$ it follows that $\alpha(G) \cap K \leq \alpha(K)$, then α is persistent.

(iv) If α is monotone then $H \alpha K \leq G$ implies $H \alpha G$.

(v) $\zeta^{\alpha}(G) = \alpha(G)$ so that $\zeta^{\alpha}(G) \alpha G$ is always true.

We consider now the generalization of marginal and verbal subgroups of Hulse and Lennox [2]. For a normal subgroup K of a group G, they define $\Phi(K,G)$ to be the subgroup of G generated by all elements of the form

$$(\omega(g_1,\ldots,g_n))^{-1}\,\omega(g_1,\ldots,g_ik,\ldots,g_n)$$

where $g_1,\ldots,g_n \in G$, $k \in K$ and $\omega(x_1,\ldots,x_n) \in w$, where w is the set of words defining a variety $\mathfrak{V}$. They also define $\Phi^*(K,G)$ to be the set of all elements k of K such that

$$\omega(g_1,\ldots,g_n) = \omega(g_1,\ldots,g_ik,\ldots,g_n)$$

whenever $\omega(x_1,\ldots,x_n) \in w$, $g_1,\ldots,g_n \in G$ and $1 \leq i \leq n$.

It may be seen that $\Phi(K,G)$ and $\Phi^*(K,G)$ are normal subgroups of G. Also $\Phi(G,G) = \mathfrak{V}(G)$ and $\Phi^*(G,G) = \mathfrak{V}^*(G)$, the usual verbal and marginal subgroups of G, respectively.

Hulse and Lennox go on to define the lower G-Φ-marginal series of K to be :

$$K = \Phi(K,G) \geq \Phi_1(K,G) \geq \dots \geq \Phi_n(K,G) \geq \dots$$

where, for $n > 0$, $\Phi_n(K,G) = \Phi(\Phi_{n-1}(K,G),G)$. They define the upper G-Φ-marginal series of K to be ;

$$1 = \Phi_0^*(K,G) \leq \Phi_1^*(K,G) \leq \dots \leq \Phi_n^*(K,G) \leq \dots$$

where, for $n > 0$, $\Phi_n^*(K,G)$ is defined by

$$\Phi_n^*(K,G)/\Phi_{n-1}^*(K,G) = \Phi^*(G/\Phi_{n-1}^*(K,G)) \cap K/\Phi_{n-1}^*(K,G).$$

When $K = G$ they obtain the lower and upper Φ-marginal series of G with terms $\Phi_n(G,G)$ and $\Phi_n^*(G,G)$, which they abbreviate to $\Phi_n(G)$ and $\Phi_n^*(G)$, respectively.

Hulse and Lennox define a group to be Φ-nilpotent if there exists a series

$$G = G_0 \geq G_1 \geq \dots \geq G_n = 1$$

where $G_i \triangleleft G$ and $G_{i-1}/G_i \leq \mathfrak{V}^*(G/G_i)$ for $i = 1,\dots,n$. They denote the class of Φ-nilpotent groups by $\mathfrak{N}_\Phi$.

As mentioned earlier, the subgroup theoretical property associated with the group theoretical function $\mathfrak{V}^*$ is just the property "is a $\mathfrak{V}$-marginal subgroup of". Some properties of $\mathfrak{V}^*$ are demonstrated in our next proposition.

Proposition 1·14 (i) $\mathfrak{V}^*$ is a subgroup theoretical property.

(ii) $\mathfrak{V}^* = \underline{s}\mathfrak{V}^*$.

(iii) $\mathfrak{V}^*$ is persistent.

(iv) $\zeta^{\mathfrak{V}^*}(G) = \mathfrak{V}^*(G)$ and so $\zeta^{\mathfrak{V}^*}(G)\,\mathfrak{V}^*\,G$ is always true.

(v) $\mathfrak{V}^*$ satisfies (+).

(vi) $\mathfrak{V}^*$ satisfies (∗).

(vii) $\mathfrak{V}^* = \underline{H}\mathfrak{V}^*$.

(viii) $(N/\gamma^{\mathfrak{V}^*}(N,G))\,\mathfrak{V}^*(G/\gamma^{\mathfrak{V}^*}(N,G))$ whenever $N \triangleleft G$.

Proof Parts (i) to (iv) follow from 1·13.

(v) Suppose $G = K \times L$ and $H\,\mathfrak{V}^*\,K$. Since $\mathfrak{V}^*(G) = \mathfrak{V}^*(K) \times \mathfrak{V}^*(L)$ we have $H \leq \mathfrak{V}^*(K) \leq \mathfrak{V}^*(G)$, as required.

(vi) Suppose X,Y and N are normal subgroups of a group G with $Y \leq X$ and $(X/Y)\,\mathfrak{V}^*(G/Y)$. Let $g_1,\ldots,g_r$ be elements of G and suppose that $n \in N \cap X$ and $\omega(z_1,\ldots,z_r) \in W$. Since $X/Y \leq \mathfrak{V}^*(G/Y)$, we have

$$\omega(g_1Y, \ldots g_rY) = \omega(g_1Y,\ldots,g_i nY, \ldots g_rY)$$

so that

$$(\omega(g_1,\ldots,g_r))^{-1}\,\omega(g_1,\ldots,g_i n,\ldots,g_r) \in Y\,.$$

But

$$(\omega(g_1,\ldots,g_r))^{-1}\,\omega(g_1,\ldots,g_i n,\ldots,g_r) \in \Phi(N,G)$$

and $\Phi(N,G) \leq N$. Therefore

$$\omega(g_1,\ldots,g_r)(Y \cap N) = \omega(g_1,\ldots,g_i n,\ldots,g_r)(Y \cap N)$$

so that $n(Y \cap N) \in \mathfrak{V}^*(G/Y \cap N)$, and we have proved that $\mathfrak{V}^*$

satisfies $(*)$.

(vii) Suppose $N \leq \mathfrak{V}^*(G)$ and let θ be a homomorphism out of G. Let $g_1,\ldots,g_r \in G$ and $n \in N$. Then

$$\begin{aligned}\omega(\theta(g_1),\ldots,\theta(g_i n),\ldots,\theta(g_r)) &= \theta(\omega(g_1,\ldots,g_i n,\ldots,g_r)) \\ &= \theta(\omega(g_1,\ldots,g_r)) \\ &= \omega(\theta(g_1),\ldots,\theta(g_r)).\end{aligned}$$

Therefore $\theta(n) \in \mathfrak{V}^*(\theta(G))$ and so $\theta(N)\mathfrak{V}^*\theta(G)$, as required.

(viii) Let $L = \gamma^{\mathfrak{V}^*}(N,G)$. That is,
$L = \bigcap\{M \triangleleft G \;;\; M \leq N,\; N/M \leq \mathfrak{V}^*(G/M)\}$. For all such M, if $n \in N$, we have

$$\omega(g_1,\ldots,g_r)M = \omega(g_1,\ldots,g_i n,\ldots,g_r)M.$$

Therefore

$$(\omega(g_1,\ldots,g_r))^{-1}\,\omega(g_1,\ldots,g_i n,\ldots,g_r) \in M$$

and so

$$(\omega(g_1,\ldots,g_r))^{-1}\,\omega(g_1,\ldots,g_i n,\ldots,g_r) \in L.$$

Thus

$$\omega(g_1,\ldots,g_r)L = \omega(g_1,\ldots,g_i n,\ldots,g_r)L$$

which implies that $nL \in \mathfrak{V}^*(G/L)$. Therefore $(N/L)\mathfrak{V}^*(G/L)$, as required.

Part (iv) of the last proposition, together with the next lemma, shows that we have a characterization of the upper and lower Φ-marginal series of Hulse and Lennox.

Lemma 1·15 $\Phi(K,G) = \gamma^{\mathfrak{V}^*}(K,G)$.

Proof Let M be a normal subgroup of G with $M \leq K$ and $(K/M)\,\mathfrak{V}^*(G/M)$, and let x be a generator of $\Phi(K,G)$, say

$$x = (\omega(g_1,\ldots,g_r))^{-1}\omega(g_1,\ldots,g_i k,\ldots,g_r)$$

where $g_1,\ldots,g_r \in G$ and $k \in K$. Now $kM \in \mathfrak{V}^*(G/M)$ and so

$$\omega(g_1M, \ldots, g_rM) = \omega(g_1M, \ldots, g_ikM, \ldots, g_rM).$$

Therefore

$$\omega(g_1,\ldots,g_r) \equiv \omega(g_1,\ldots,g_ik,\ldots,g_r) \pmod M.$$

Thus $x \in M$ and $\Phi(K,G) \leq \gamma^{\mathfrak{V}^*}(K,G)$.

But, as observed by Hulse and Lennox,

$$K/\Phi(K,G) \leq \Phi^*(G/\Phi(K,G)) = \mathfrak{V}^*(G/\Phi(K,G)).$$

Therefore $\gamma^{\mathfrak{V}^*}(K,G) \leq \Phi(K,G)$, as required.

Thus we have proved,

Theorem 1·16 The lower (respectively upper) Φ-marginal series of a group G is identical to the lower (respectively upper) $\mathfrak{V}^*$-central series of G and $\mathcal{H}_\Phi = \overset{0}{\mathfrak{V}}^*$.

The following facts about $\mathfrak{V}^*$ are direct consequences of 1·4(ii) and 1·12.

Corollary 1·17 (i) $\underline{\mathfrak{V}^*} = \overset{*}{\mathfrak{V}}^*$

(ii) If M is a minimal normal subgroup of a $\underline{\mathfrak{V}^*}$-group G, then $M \leq \mathfrak{V}^*(G)$ and $G/M \in \underline{\mathfrak{V}^*}$.

We define the operations C_F and C_n on classes of groups as follows :

Definition Let $\mathfrak{X}$ be a class of groups. Then $G \in C_F\mathfrak{X}$

if for each non-trivial element x of G there exist subgroups H and K of G with $H \lhd K \lhd G$, $x \in K - H$ and $K/H \in \mathfrak{X}$. We say G has an $\underline{\mathfrak{X}\text{-factor covering}}$.

Definition Let $\mathfrak{X}$ be a class of groups. $G \in C_n\mathfrak{X}$ if for each non-trivial element x of G there exist normal subgroups H and K of G with $H \leq K$, $x \in K - H$ and $K/H \in \mathfrak{X}$. We say G has a $\underline{\text{normal } \mathfrak{X}\text{-factor covering}}$.

Both these definitions appear in [12] but with different notation.

We have the following result for any variety $\mathfrak{V}$.

Theorem 1·18 $C_n\underline{\mathfrak{V}^*} = C_F\mathfrak{V} = C_n\mathfrak{V} = C_n\hat{\mathfrak{V}}^*$.

Proof Let $G \in C_n\underline{\mathfrak{V}^*}$ and let x be a non-trivial element of G. Then there exist normal subgroups M and N of G such that $N \leq M$, $x \in M - N$ and $M/N \in \underline{\mathfrak{V}^*}$. But xN is a non-trivial element of M/N and so there exist normal subgroups H/N and K/N of M/N such that $K \leq H$, $xN \in (H/N) - (K/N)$ and

$$\frac{H/N}{K/N} \leq \mathfrak{V}^*\left(\frac{M/N}{K/N}\right) = \frac{L/N}{K/N}, \text{ say.}$$

Now $\frac{L/N}{K/N} \lhd \frac{M/N}{K/N} \lhd \frac{G/N}{K/N}$ and so $\frac{L/N}{K/N} \lhd \frac{G/N}{K/N}$ which shows that $L \lhd G$. Also $x \in L$ and $x \notin K$. But $K \lhd L$ and $L/K \cong \frac{L/N}{K/N} \in \mathfrak{V}$.

We have proved that $C_n\underline{\mathfrak{V}^*} \leq C_F\mathfrak{V}$. Since it is obvious that $C_n\mathfrak{V} \leq C_n\hat{\mathfrak{V}}^* \leq C_n\underline{\mathfrak{V}^*}$, it only remains to prove that $C_F\mathfrak{V} \leq C_n\mathfrak{V}$.

Let x be a non-trivial element of a $C_F\mathfrak{V}$-group G. Then there exist subgroups M and N of G such that $M \lhd N \lhd G$, $x \in M - N$ and $M/N \in \mathfrak{V}$. Therefore $\mathfrak{V}(M) \leq N$ and we have $x \in M - \mathfrak{V}(M)$. But $\mathfrak{V}(M) \lhd M \lhd G$ so $\mathfrak{V}(M) \lhd G$ and we have proved that $G \in C_n\mathfrak{V}$.

By taking $\mathfrak{V}$ to be the class $\mathfrak{A}$ of abelian groups, we see that the classes $\underline{\mathfrak{V}^*}$ and $C_n\mathfrak{V}$ are not, in general, Q-closed. For $\mathfrak{A}^*$ is just the subgroup theoretical property "is a central subgroup of" and so $\underline{\mathfrak{A}^*}$ is the class of groups with a central factor covering. Also $C_n\mathfrak{A}$ is the class of groups with a normal abelian covering. It is clear that $C_n\mathfrak{A}$ contains the class of SI-groups and $\underline{\mathfrak{A}^*}$ contains the class of groups with a central series. Thus both classes contain the class of free groups.

Many of the classes considered here are not Q-closed. However, we can make use of the homomorphism properties of Petty [6].

<u>Definition</u> Let A be a non-empty set. A non-empty set $\mathfrak{F}$ of subsets of A is called a <u>filter</u> if ;

(i) $\phi \notin \mathfrak{F}$

(ii) $S_1, S_2 \in \mathfrak{F}$ implies that $S_1 \cap S_2 \in \mathfrak{F}$

(iii) $S \in \mathfrak{F}$ and $S \subseteq T$ implies that $T \in \mathfrak{F}$.

We order filters naturally by saying $\mathfrak{F}_1 \leq \mathfrak{F}_2$ if $S \in \mathfrak{F}_1$ always implies $S \in \mathfrak{F}_2$.

<u>Definition</u> A filter is an <u>ultrafilter</u> if it is maximal in the natural ordering of filters.

It can be seen that any filter can be enlarged to an ultrafilter. The following result is in [13] .

Lemma A filter $\mathfrak{F}$ on A is an ultrafilter if and only if given any subset T of A either $T \in \mathfrak{F}$ or $A - T \in \mathfrak{F}$.

Let $\{R_\alpha ; \alpha \in A\}$ be a set of groups and let $\overline{R} = \underset{\alpha \in A}{\mathrm{Cr}} R_\alpha$. Again we consider $\overline{R}$ as the set of functions

$\{x : A \to \bigcup_{\alpha \in A} R_\alpha ; x(\alpha) \in R_\alpha$ for all $\alpha \in A\}$. Let $\mathfrak{F}$ be a filter on A. For elements f and g of $\overline{R}$ define $f \sim g$ if $\{\alpha \in A ; f(\alpha) = g(\alpha)\} \in \mathfrak{F}$. Then $\sim$ is an equivalence relation. Let $I = \{f \in \overline{R} ; f \sim 1\} \triangleleft \overline{R}$. Let $\overline{R}/\mathfrak{F}$ denote the set of equivalence classes in $\overline{R}$ determined by $\sim$.

Definition $\overline{R}/I$ is a reduced direct product of the R_α's.

Definition If $\mathfrak{F}$ is an ultrafilter then $\overline{R}/I$ is an ultraproduct of the R_α's.

If P is a property of group homomorphisms such that all isomorphisms satisfy P, Petty defines an operation H_p on group theoretical classes by saying that, for any class $\mathfrak{X}$, $G \in H_p\mathfrak{X}$ if there exist an $\mathfrak{X}$-group K and an epimorphism $h : K \to G$ with h satisfying P. For any class $\mathfrak{X}$ such that $\mathfrak{X} = H_p\mathfrak{X}$ he says that P is a weak homomorphic image closure property of $\mathfrak{X}$. In particular, he defines the following operations on group theoretical classes.

Definition $G \in H_R\mathfrak{X}$ if G is isomorphic to a reduced direct product of $\mathfrak{X}$-groups.

Definition $G \in H_u\mathfrak{X}$ if G is isomorphic to an ultraproduct of $\mathfrak{X}$-groups.

Definition $G \varepsilon H_w\mathfrak{X}$ if there exist an $\mathfrak{X}$-group K and an epimorphism $h:K \to G$ such that the kernel of h is contained in the limit of the upper $\mathfrak{V}^*$ - central series of K, where $\mathfrak{V}$ is the variety determined by the set of words w.

Notice that $H_u \leq H_R$. Notice also that p being a weak homomorphic image closure property does not necessarily imply that $H_p \leq Q$. However for classes of groups $\mathfrak{X}$ such that $\mathfrak{X} = C\mathfrak{X}$, $H_R\mathfrak{X}$, $H_u\mathfrak{X}$ and $H_w\mathfrak{X}$ are all contained in $Q\mathfrak{X}$.

Petty proved the following theorem in [6] .

Theorem For a variety $\mathfrak{V}$, $C_n\mathfrak{V}$, $\underline{\mathfrak{V}^*}$, $\hat{p}_n\mathfrak{V}$, $\hat{p}\mathfrak{V}$ and $\hat{\mathfrak{V}}^*$ are all $\langle H_R,H_w \rangle$ -closed classes.

Petty also defines a generalized local property in the following way.

Definition Let $P = \{K_\alpha, H_\alpha \ ; \ \alpha \varepsilon A\}$ be a set of subgroups of a group G. P is a local factor system of G if $K_\alpha \lhd H_\alpha$ for all $\alpha \varepsilon A$ and if F is a finite subset of G - 1 then there exists an α in A such that $F \subseteq H_\alpha - K_\alpha$.

Definition $G \varepsilon L_F\mathfrak{X}$ if G has a local factor system $\{K_\alpha, H_\alpha \ ; \ \alpha \varepsilon A\}$ such that $H_\alpha/K_\alpha \varepsilon \mathfrak{X}$ for all $\alpha \varepsilon A$.

Petty then proves that $L \leq L_F \leq \langle S,H_u \rangle$ which yields :

Corollary $C_n\mathfrak{V}$, $\underline{\mathfrak{V}^*}$, $\hat{p}_n\mathfrak{V}$, $\hat{p}\mathfrak{V}$ and $\hat{\mathfrak{V}}^*$ are all L_F-closed classes, and so are all L-closed classes.

These closure properties will be of recurring interest to us.

Two other subgroup theoretical properties which are of interest are those of being an $\mathfrak{X}$-subgroup and being a normal $\mathfrak{X}$-subgroup, for a class of groups $\mathfrak{X}$. We denote these by $\mu_{\mathfrak{X}}$ and $\nu_{\mathfrak{X}}$, respectively. When it is clear which class $\mathfrak{X}$ is being considered, we omit the suffices.

Obviously $\nu \leqslant \mu$ and $C_n\mathfrak{X} = \underline{\nu}$. We also have the following result.

<u>Proposition 1·19</u> (i) μ and ν are subgroup theoretical properties.

(ii) $\mu = \underline{S}\mu$ if $\mathfrak{X} = S\mathfrak{X}$. $\nu \neq \underline{S}\nu$ in general.

(iii) μ and ν are persistent.

(iv) μ and ν satisfy (*) if $\mathfrak{X} = S_n\mathfrak{X}$.

(v) μ and ν satisfy (†).

(vi) If $\mathfrak{X} = Q\mathfrak{X}$, then $\mu = \underline{H}\mu$ and $\nu = \underline{H}\nu$.

(vii) If $\mathfrak{X} = R\mathfrak{X}$, then $(N/\gamma^{\mu}(N,G))\mu(G/\gamma^{\mu}(N,G))$, and $(N/\gamma^{\nu}(N,G))\mu(G/\gamma^{\nu}(N,G))$ whenever $N \leqslant G$.

(viii) $\zeta^{\mu}(G)\mu G$ is not true in general. $\zeta^{\nu}(G)\nu G$ if $\mathfrak{X} = N\mathfrak{X}$.

<u>Proof</u> The proof of this proposition is easy except for the following :

(iv) : Suppose $\mathfrak{X} = S_n\mathfrak{X}$ and let X,Y and N be normal subgroups of a group G such that $Y \leqslant X$. If $(X/Y)\mu(G/Y)$ then $X \cap N/Y \cap N \cong (X \cap N)Y/Y \lhd X/Y \in \mathfrak{X}$. If $(X/Y)\ \nu\ (G/Y)$ then $(X/Y)\ \mu\ (G/Y)$ so $X \cap N/Y \cap N \in \mathfrak{X}$. But $X \cap N/Y \cap N \lhd G/Y \cap N$, as required.

(viii): Consider the class $\mathfrak{C}$ of cyclic groups. If $G \notin \mathfrak{C}$ then clearly $G\ \mu\ G$ is not true. But any group is generated by its cyclic subgroups so G $= \zeta^{\mu}(G)$.

On the other hand, if $\mathfrak{X} = N\mathfrak{X}$ we have $\zeta^{\nu}(G) \varepsilon N\mathfrak{X} = \mathfrak{X}$ so, since $\zeta^{\nu}(G)$ is clearly a normal subgroup of G, $\zeta^{\nu}(G)\ \nu\ G$.

We observe that $\zeta^{\nu}(G)$ is just the $\mathfrak{X}$-radical of G, and that the upper ν-central series of G is just the upper $\mathfrak{X}$-series of G for any group G.

Finally, in this chapter, we have the following corollary, which is a direct consequence of 1·4 and 1·12.

<u>Corollary 1·22</u> (i) $\underline{\mu} = \overset{*}{\mu}$ and $\underline{\nu} = \overset{*}{\nu}$

(ii) If $\mathfrak{X} = S_n\mathfrak{X}$ and M is a minimal normal subgroup of a $\overset{*}{\mu}$-group G then $M \varepsilon \mathfrak{X}$ and $G/M\ \varepsilon\ \overset{*}{\mu}$.

(iii) If $\mathfrak{X} = S_n\mathfrak{X}$ and M is a minimal normal subgroup of a $\overset{*}{\nu}$-group G then $M \varepsilon \mathfrak{X}$ and $G/M\ \varepsilon\ \overset{*}{\nu}$.

II $\mathfrak{X}$- Centrality and $\mathfrak{X}_D$-Centrality

For any $< S,Q,D_0 >$-closed class of groups $\mathfrak{X}$, Stanley defined, in [3], the $\mathfrak{X}$-centre of a group G to be the set of all elements x of G for which there exists a normal subgroup N of G, depending on x, such that $[x,N] = 1$ and $G/N \in \mathfrak{X}$. He denoted the $\mathfrak{X}$-centre of G by $H_1(G:\mathfrak{X})$ and proved $H_1(G:\mathfrak{X})$ to be a characteristic subgroup of G.

Making a slight generalization of the property ϕ, defined in [5], we define the subgroup theoretical property $\phi_{\mathfrak{X}}$ as follows :

Definition Let G be a group. If $H \leq H_1(G:\mathfrak{X})$ we say that H is an $\mathfrak{X}$-central subgroup of G, and write $H \, \phi_{\mathfrak{X}} \, G$.

Arrell, in [4], defined a descending series $\{K_\sigma ; \sigma \in \Sigma\}$ to be an $\mathfrak{X}_D$-central series if, for each $\sigma \in \Sigma$,

$$\bigcap \{[K_\sigma,N] \; ; \; N \triangleleft G, G/N \in \mathfrak{X}\} \leq K_{\sigma+1} .$$

Again generalizing a property in [5], we define the subgroup theoretical property $\psi_{\mathfrak{X}}$ thus :

Definition Let G be a group. If H is a subgroup of G such that $\bigcap\{[H,N] \; ; \; N \triangleleft G, G/N \in \mathfrak{X}\} = 1$, we say that H is an $\mathfrak{X}_D$-central subgroup of G and write $H \, \psi_{\mathfrak{X}} \, G$.

When it is clear which class $\mathfrak{X}$ is being considered, we omit the suffices from $\phi_{\mathfrak{X}}$ and $\psi_{\mathfrak{X}}$. The next two theorems carry over from [5].

Theorem 2·1 (i) ϕ is a subgroup theoretical property.

(ii) $\phi = \underline{H}\phi$.

(iii) ϕ satisfies $(*)$.

(iv) $(N/\gamma^{\phi}(N,G))\ \phi\ (G/\gamma^{\phi}(N,G))$ is not true in general.

(v) $\zeta^{\phi}(G)\ \phi\ G$ is always valid.

Theorem 2·2 (i) ψ is a subgroup theoretical property.

(ii) $\psi < \underline{H}\psi$ in general.

(iii) ψ satisfies $(*)$.

(iv) $(N/\gamma^{\psi}(N;G))\ \psi\ (G/\gamma^{\psi}(N,G))$ always holds for $N \leq G$.

(v) $\zeta^{\psi}(G)\ \psi\ G$ is not true in general.

We now add :

Theorem 2·3 (i) $\phi = \underline{S}\phi$ and $\psi = \underline{S}\psi$

(ii) ϕ and ψ satisfy (†).

(iii) ϕ is persistent, and ψ is persistent if $\mathfrak{X} = S\mathfrak{X}$.

Proof The first part is very easy to prove. That ϕ satisfies (†) follows from proposition 3(iii) of [3] which proves that if K is a direct factor of G then $H_1(K : \mathfrak{X}) = H_1(G : \mathfrak{X}) \cap K$. Suppose $F\ \psi\ H\ \delta\ G$. Let $x \in \bigcap \{ [F,L] \ ; \ L \triangleleft G, G/L \in \mathfrak{X} \}$ and let N be a normal subgroup of H such that $H/N \in \mathfrak{X}$. There exists a normal subgroup K of G such that $G = H \times K$. Let $M = KN$. Then $M \triangleleft G$ and $G/M \cong H/N \in \mathfrak{X}$. Therefore $x \in [F,M]$. But $[F,M] = [F,N]$ because $[H,K] = 1$.

We have shown that $x \in [F,N]$ for any normal subgroup N of H such that $H/N \in \mathfrak{X}$. But $\bigcap \{ [F,N] \ ; \ N \triangleleft H, H/N \in \mathfrak{X} \} = 1$ Therefore $x = 1$, proving that $F\ \psi\ G$.

To prove part (iii), suppose first that $H \leq K \leq G$ and $H \phi G$. Then $H \leq H_1(G : \mathfrak{X}) \cap K$. But by proposition 3(i) of [3] , $H_1(G : \mathfrak{X}) \cap K \leq H_1(K : \mathfrak{X})$. Therefore $H \phi K$. Suppose now that $H \leq K \leq G$ and $H \psi G$. Then $\bigcap\{[H,N] \; ; N \lhd G, G/N \in \mathfrak{X}\} = 1$. Let $x \in \bigcap \{ [H,M] \; ; \; M \lhd K, K/M \in \mathfrak{X}\}$ and let N be a normal subgroup of G such that $G/N \in \mathfrak{X}$. Now $K \cap N \lhd K$ and $K/K \cap N \cong KN/N \in S\mathfrak{X}$ and we are assuming that $\mathfrak{X} = S\mathfrak{X}$. Therefore $x \in [H,K \cap N] \leq [H,N]$ for all normal subgroups N of G such that $G/N \in \mathfrak{X}$. Thus $x = 1$ and so $H \psi K$.

As a consequence of some earlier results we have

Theorem 2·4 (i) $\underline{\phi} = \underline{\psi} = \overset{*}{\phi} = \overset{*}{\psi}$

(ii) $\hat{\phi} \leq \underline{\phi}$

Proof (i) follows from 1·4 (ii) and 2·3 and from theorem 3 of [5] , which states that $\overset{*}{\phi} = \overset{*}{\psi}$. (ii) follows from 1·4(i).

As an immediate consequence of 1·6, we have

Theorem 2·5 $\underline{\phi}$, $\hat{\phi}$, $\acute{\phi}$, $\grave{\phi}$, $\overset{o}{\phi}$, $\acute{\psi}$, $\grave{\psi}$ and $\overset{o}{\psi}$ are all S-closed if $\mathfrak{X} = S\mathfrak{X}$.

As observed in [5] , the class $\overset{*}{\mathfrak{X}}$ defined in [7] is identical with the class $\overset{*}{\phi}_{\mathfrak{X}}$. In particular, the class of residually central groups is just the class $\overset{*}{\phi}_1$. Also the classes of groups with a central factor covering and groups with a central series, Z, are just the classes $\underline{\phi_1}$ and $\hat{\phi}_1$, respectively. Phillips and Roseblade [10] have discovered a $\overset{*}{\phi}_1$-group which is not a $\hat{\phi}_1$-group. However, 2·4(i) gives us :

Corollary 2·6 $\underline{\phi_1} = \overset{*}{\phi}_1$

It is also observed in [5] that $\overset{o}{\phi}_1 = \overset{o}{\psi}_1 = \mathfrak{N}$, $\acute{\phi}_1 = \acute{\psi}_1 =$ ZA and $\grave{\phi}_1 = \grave{\psi}_1 =$ ZD.

We shall now examine some connections between classes of groups defined in terms of ϕ and ψ for any $\langle S,Q,D_0\rangle$-closed class $\mathfrak{X}$. The following two lemmas will be helpful.

Lemma 2·7 If A and B are normal subgroups of a group G and if $(A/B)\ \phi\ (G/B)$ then there exists an ascending series $\{G_\alpha\}$ from B up to A of normal subgroups of G, such that for each ordinal α,

$$(G_{\alpha+1}/G_\alpha)\ \psi\ (G/G_\alpha)\ .$$

Proof Let $G_0 = B$. If $G_0 < A$ we define a series as follows : If α is a limit ordinal let $G_\alpha = \bigcup_{\beta<\alpha} G_\beta$. If $\alpha-1$ exists and if $G_{\alpha-1} < A$ choose an $x \in A - G_{\alpha-1}$ and let $G_\alpha = x^G G_{\alpha-1}$. It is easy to see that $B \leq G_\alpha \triangleleft G$ for each ordinal α, and that the series reaches A.

Let α be an ordinal number and let $x \in G_{\alpha+1}$. Since $x \in A$ there exists a normal subgroup N of G, with $B \leq N$ and $G/N \in \mathfrak{X}$, such that $[x^G,N] \leq B$. Therefore $[G_{\alpha+1},N] = [x^G G_\alpha,N] \leq G_\alpha$. Thus $(G_{\alpha+1}/G_\alpha)\ \psi\ (G/G_\alpha)$.

Lemma 2·8 If A and B are normal subgroups of a group G and if $(A/B)\ \psi\ (G/B)$ then there exists a descending series $\{G_\alpha\}$ from A down to B of normal subgroups of G, such that for each ordinal α , $(G_\alpha/G_{\alpha+1})\ \phi\ (G/G_{\alpha+1})$.

Proof Let $G_0 = A$. If $G_0 > B$ we define a series as follows : If α is a limit ordinal let $G_\alpha = \bigcap_{\beta<\alpha} G_\beta$. If $\alpha-1$ exists and if $G_{\alpha-1} > B$ choose an $x \in G_{\alpha-1} - B$. Now $x \in A - B$ and so there exists a normal subgroup N of G with $B \leq N$ and $G/N \in \mathfrak{X}$ such that $xB \notin [A/B,N/B] = [A,N]B/B$. Therefore $x \notin B[G_{\alpha-1},N]$. Let $G_\alpha = B[G_{\alpha-1},N]$. It is easy to see that $G_\alpha \triangleleft G$ for all ordinals α and that the series reaches B.

Let α be an ordinal number. Since there exists a normal subgroup N of G with $B \leq N$ and $[G_\alpha,N] \leq G_{\alpha+1}$, we have $(G_\alpha/G_{\alpha+1})\ \phi\ (G/G_{\alpha+1})$, as required.

These two lemmas provide the proof of :

Theorem 2·9 $\acute{\phi} \leq \acute{\psi}$, $\grave{\psi} \leq \grave{\phi}$ and $\hat{\phi} = \hat{\psi}$.

An indication of the size of some of the classes defined in terms of ϕ and ψ has been given by Stanley : in [7] he proves that $\overset{*}{\phi}_1 \leq \overset{*}{\phi}_{\mathfrak{X}} \leq \overset{*}{\phi}_1(R\mathfrak{X})$ and in [3] he proves that $\mathring{\phi}_1 \leq \mathring{\phi}_{\mathfrak{X}} \leq \mathring{\phi}_1(R\mathfrak{X})$ and that $\acute{\phi}_1 \leq \acute{\phi}_{\mathfrak{X}} \leq \acute{\phi}_1(R\mathfrak{X})$.
We now add

Theorem 2·10 $\hat{\phi}_1 \leq \hat{\phi}_{\mathfrak{X}} \leq \hat{\phi}_1(R\mathfrak{X})$, $\grave{\phi}_1 \leq \grave{\phi}_{\mathfrak{X}} \leq \grave{\phi}_1(R\mathfrak{X})$ and

$$\mathring{\psi}_1 \leq \mathring{\psi}_{\mathfrak{X}} \leq \mathring{\psi}_1(R\mathfrak{X}).$$

Proof It is clear that $\hat{\phi}_1 \leq \hat{\phi}_{\mathfrak{X}}$; $\grave{\phi}_1 \leq \grave{\phi}_{\mathfrak{X}}$ and $\mathring{\psi}_1 \leq \mathring{\psi}_{\mathfrak{X}}$.

Let $G \in \hat{\phi}_{\mathfrak{X}}$ and let $\{H_\sigma, K_\sigma \ ; \ \sigma \in \Sigma\}$ be a $\phi_{\mathfrak{X}}$-series in G. Let R be the $\mathfrak{X}$-residual of G and let $M_\sigma = H_\sigma \cap R$ and $N_\sigma = K_\sigma \cap R$ for each $\sigma \in \Sigma$. Then M_σ and N_σ are normal subgroups of R and $N_\sigma \leq M_\sigma$ for all $\sigma \in \Sigma$.
Let x be a non-trivial element of R. Then for some $\sigma \in \Sigma$, $x \in (H_\sigma \cap R) - (K_\sigma \cap R)$ so $R - 1 = \bigcup_{\sigma\in\Sigma} M_\sigma - N_\sigma$.
Suppose $\tau < \sigma$. Then $M_\tau = H_\tau \cap R \leq K_\sigma \cap R = N_\sigma$.
Since $\phi_{\mathfrak{X}}$ satisfies (*), $(M_\sigma/N_\sigma)\phi_{\mathfrak{X}}(G/N_\sigma)$ for all $\sigma \in \Sigma$.

Let $m \in M_\sigma$. Then there exists a normal subgroup P of G, with $N_\sigma \leq P$, such that $[m,P] \leq N_\sigma$ and $G/P \in \mathfrak{X}$.
Therefore $R \leq P$ and so $[m,R] \leq N_\sigma$ for all $m \in M_\sigma$.
Thus $[M_\sigma,R] \leq N_\sigma$ so that $M_\sigma/N_\sigma \leq \zeta_1(R/N_\sigma)$. Therefore $\{M_\sigma,N_\sigma \ ; \ \sigma \in \Sigma\}$ is a central series in R so $R \in \hat{\phi}_1$ and $G \in \hat{\phi}_1(R\mathfrak{X})$.

Now suppose $G \in \grave{\phi}_{\mathfrak{X}}$ and let $\{H_\sigma ; \sigma \leq \alpha\}$ be a descending $\phi_{\mathfrak{X}}$-series in G. Let $M_\sigma = H_\sigma \cap R$ for all $\sigma \leq \alpha$. Then $R \in \grave{\phi}_1$ because $\{M_\sigma ; \sigma \leq \alpha\}$ is a descending central series. Therefore $G \in \grave{\phi}_1(R\mathfrak{X})$.

Suppose now that $G \in \mathring{\psi}_{\mathfrak{X}}$ and let $\{H_i \ ; \ 1 \leq i \leq n\}$ be a finite $\psi_{\mathfrak{X}}$-series in G. Now $(H_{i+1}/H_i)\ \psi_{\mathfrak{X}}\ (G/H_i)$ for each $1 \leq i \leq n$ so, since $[R,H_{i+1}] \leq \bigcap\{[N,H_{i+1}] \ ; N \triangleleft G, G/N \in \mathfrak{X}\} \leq H_i$ we have $[R/M_i, H_{i+1}/M_i] = 1$, where $M_i = H_i \cap R$ $(1 \leq i \leq n)$. Therefore $\{M_i \ ; \ 1 \leq i \leq n\}$ is a finite central series in R and $G \in \mathring{\phi}_1(R\mathfrak{X})$.

The following result follows very easily from 1·8 and 1·9 :

Corollary 2·11 (i) $\hat{\phi}_{\mathfrak{X}}$, $\grave{\phi}_{\mathfrak{X}}$ and $\grave{\psi}_{\mathfrak{X}}$ are all C-closed.

(ii) $\acute{\phi}_{\mathfrak{X}}$ and $\acute{\psi}_{\mathfrak{X}}$ are D-closed.

(iii) $\overset{o}{\phi}_{\mathfrak{X}}$ and $\overset{o}{\psi}_{\mathfrak{X}}$ are D_0-closed.

(iv) $\underline{\phi_{\mathfrak{X}}}$ is R-closed.

Some of the classes under consideration are shown in Fig. 1.

We note that by proposition 9 of [3] , $\overset{o}{\phi}_{\mathfrak{X}} = \langle S,Q,D_0\rangle \overset{o}{\phi}_{\mathfrak{X}}$ and $\acute{\phi}_{\mathfrak{X}} = \langle S,Q,D\rangle \acute{\phi}_{\mathfrak{X}}$, and that by theorem A of [4] , $\overset{o}{\psi}_{\mathfrak{X}} = \langle S,\dot{D}_0\rangle \overset{o}{\psi}_{\mathfrak{X}}$ and $\grave{\psi}_{\mathfrak{X}} = \langle S,D_0\rangle \grave{\psi}_{\mathfrak{X}}$. We have improved the last of these results and proved several of them in a different manner.

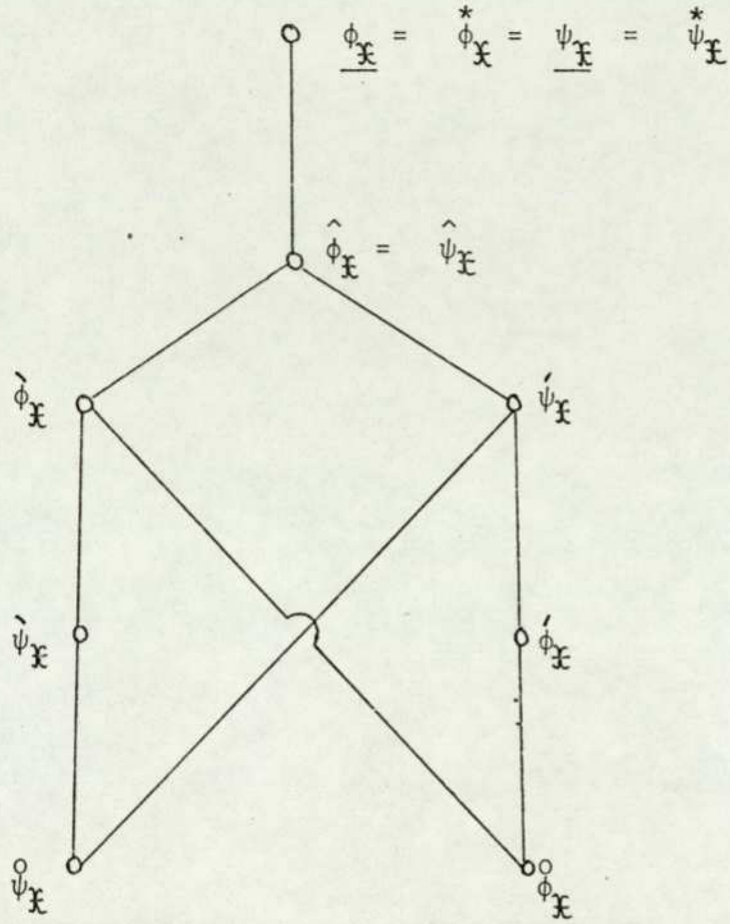
The upper $\mathfrak{X}$-central series defined in [3] by the rules $H_\alpha(G:\mathfrak{X})/H_{\alpha-1}(G:\mathfrak{X}) = H_1(G/H_{\alpha-1}(G:\mathfrak{X}):\mathfrak{X})$ if α-1 exists and $H_\alpha(G:\mathfrak{X}) = \bigcup_{\lambda<\alpha} H_\lambda(G:\mathfrak{X})$ if α is a limit ordinal, is just the upper $\phi_{\mathfrak{X}}$-central series. For any ordinal number α , we define the class of $\mathfrak{X}(\alpha)$-groups as follows :

Definition $G \in \mathfrak{X}(\alpha)$ if $G = H_\alpha(G:\mathfrak{X})$.

We continue to use Stanley's notation of $\overline{H}(G:\mathfrak{X})$ for the $\phi_{\mathfrak{X}}$ -hypercentre of G, thus $G \in \acute{\phi}_{\mathfrak{X}}$ if and only if $G = \overline{H}(G:\mathfrak{X})$.

We now include some results about some subclasses of $\hat{\phi}_{\mathfrak{X}}$. It is assumed that $\mathfrak{X}$ and $\mathfrak{Y}$ are $\langle S,Q,D_0\rangle$-closed classes.

Proposition 2·12 (i) If $\mathfrak{X} \leq \mathfrak{Y}$ then $H_\alpha(G:\mathfrak{X}) \leq H_\alpha(G:\mathfrak{Y})$ for all ordinals α .


$\underline{\phi}_{\mathfrak{X}} = \overset{*}{\phi}_{\mathfrak{X}} = \underline{\psi}_{\mathfrak{X}} = \overset{*}{\psi}_{\mathfrak{X}}$
$\hat{\phi}_{\mathfrak{X}} = \hat{\psi}_{\mathfrak{X}}$
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$\acute{\psi}_{\mathfrak{X}}$
$\grave{\psi}_{\mathfrak{X}}$
$\acute{\phi}_{\mathfrak{X}}$
$\mathring{\psi}_{\mathfrak{X}}$
$\mathring{\phi}_{\mathfrak{X}}$

Fig. 1

(ii) $H_\alpha(G:\mathfrak{X}\cap\mathfrak{Y}) \leq H_\alpha(G:\mathfrak{X})\cap H_\alpha(G:\mathfrak{Y})$ for all ordinals α.

(iii) If $\mathfrak{X}\leq\mathfrak{Y}$ then $\mathfrak{X}(\alpha)\leq\mathfrak{Y}(\alpha)$ for all ordinals α.

(iv) $(\mathfrak{X}\cap\mathfrak{Y})(\alpha)\leq\mathfrak{X}(\alpha)\cap\mathfrak{Y}(\alpha)$.

(v) $H_1(G:\mathfrak{X}\cap\mathfrak{Y}) = H_1(G:\mathfrak{X})\cap H_1(G:\mathfrak{Y})$.

(vi) $(\mathfrak{X}\cap\mathfrak{Y})(1) = \mathfrak{X}(1)\cap\mathfrak{Y}(1)$.

(vii) $H_\alpha(G:\mathfrak{X})\in\mathfrak{X}(\alpha)$ for all ordinals α.

(viii) $\mathfrak{X}(j+k)\leq\mathfrak{X}(j)\,\mathfrak{X}(k)$ for any integers j and k.

(ix) $\hat{\phi}_{\mathfrak{X}}\leq\hat{p}(\mathfrak{X}(1))$, $\acute{\phi}_{\mathfrak{X}}\leq\acute{p}(\mathfrak{X}(1))$, $\grave{\phi}_{\mathfrak{X}}\leq\grave{p}(\mathfrak{X}(1))$ and $\mathring{\phi}_{\mathfrak{X}}\leq p(\mathfrak{X}(1))$.

Proof of 2·12 (vii) Proposition 3 of [3] states that if K is a subgroup of G and α is an ordinal number, then $H_\alpha(G:\mathfrak{X})\cap K\leq H_\alpha(K:\mathfrak{X})$. By putting $K = H_\alpha(G:\mathfrak{X})$ we obtain $H_\alpha(G:\mathfrak{X})\leq H_\alpha(H_\alpha(G:\mathfrak{X}):\mathfrak{X})$, as required.

The proof of the rest of 2·12 is very easy and is omitted, as is the proof of our next proposition.

Proposition 2·13 $\mathfrak{N}\mathfrak{X}\leq\mathfrak{X}(1)\mathfrak{X}\cap\mathfrak{X}(2)$.

Definition Let $\mathfrak{X}$ be a class of groups. We say that $\mathfrak{X}$ is a good class if, for any group G, $G/\zeta_1(G)\in\mathfrak{X}$ always implies $G\in\mathfrak{X}$.

Examples of good classes include the classes of nilpotent and soluble groups and, as is shown by 2·16, the classes of locally nilpotent and locally soluble groups.

Lemma 2·14 (i) If X is a finitely generated subgroup of $H_1(G:\mathfrak{X})$ then there exists a normal subgroup N of G such that $N \cap X \leq \zeta_1(X)$ and $G/N \in \mathfrak{X}$.

(ii) If $G \in \mathfrak{F} \cap \mathfrak{X}(1)$ then there exists a normal subgroup N of G such that $N \leq \zeta_1(G)$ and $G/N \in \mathfrak{X}$.

(iii) If $\mathfrak{X}$ is a good class then $\mathfrak{X}(1) \cap \mathfrak{F} \leq \mathfrak{X}$.

Proof (i) Suppose $X = \langle x_1, \ldots, x_n \rangle$. For each i, $1 \leq i \leq n$, there exists a normal subgroup N_i of G such that $[x_i, N_i] = 1$ and $G/N_i \in \mathfrak{X}$. Let $N = \bigcap_{i=1}^{n} N_i$. Then $[X,N] = 1$ so that $X \cap N \leq \zeta_1(X)$ and $G/N \in s\, D_0\mathfrak{X} = \mathfrak{X}$.

(ii) follows immediately from (1).

(iii) If $N \leq \zeta_1(G)$ and $G/N \in \mathfrak{X}$ then $G/\zeta_1(G) \in Q\mathfrak{X} = \mathfrak{X}$ so that, if $\mathfrak{X}$ is a good class, $G \in \mathfrak{X}$.

Lemma 2·15 Let $\mathfrak{X}$ be a good class such that $\mathfrak{F} \cap \mathfrak{X} \leq \mathfrak{Y}$ for some class $\mathfrak{Y}$. Then $\mathfrak{F} \cap \mathfrak{X}(c) \leq \mathfrak{Y}(c)$ for all integers c.

Proof By 2·14(iii) , $\mathfrak{X}(1) \cap \mathfrak{F} = \mathfrak{X} \cap \mathfrak{F} \leq \mathfrak{Y} \leq \mathfrak{Y}(1)$. Suppose that $\mathfrak{F} \cap \mathfrak{X}(c-1) \leq \mathfrak{Y}(c-1)$ and let $G \in \mathfrak{F} \cap \mathfrak{X}(c)$. Then by corollary 6 of [3] , $G/H_1(G:\mathfrak{X}) \in \mathfrak{F} \cap \mathfrak{X}(c-1) \leq \mathfrak{Y}(c-1)$.

Let $x \in H_1(G:\mathfrak{X})$. Then there exists a normal subgroup N of G such that $[x,N] = 1$ and $G/N \in \mathfrak{X}$. Thus $G/N \in \mathfrak{F} \cap \mathfrak{X} \leq \mathfrak{Y}$ and we have shown that $H_1(G:\mathfrak{X}) \leq H_1(G:\mathfrak{Y})$.

Therefore $G/H_1(G:\mathfrak{Y}) \varepsilon Q(\mathfrak{Y}(c-1)) = \mathfrak{Y}(c-1)$ by proposition 9 of [3] and so, again by corollary 6 of [3], $G \varepsilon \mathfrak{Y}(c)$.

It is well known that if $\mathfrak{X} = \langle S,Q,D_0 \rangle \mathfrak{X}$ then so is $L\mathfrak{X}$. For good classes, the following properties of $L\mathfrak{X}$ also follow.

<u>Proposition 2·16</u> If $\mathfrak{X}$ is a good class then

(i) $L\mathfrak{X}$ is a good class.

(ii) $\mathfrak{X}(1) \leqslant L\mathfrak{X} = (L\mathfrak{X})(1) = L(\mathfrak{X}(1))$

<u>Proof</u> (i) Suppose $G/\zeta_1(G) \varepsilon L\mathfrak{X}$ and let H be a finitely generated subgroup of G. Now
$H\zeta_1(G)/\zeta_1(G) \cong H/H \cap \zeta_1(G)$, which is finitely generated and so $H\zeta_1(G)/\zeta_1(G) \varepsilon \mathfrak{X}$. But $H \cap \zeta_1(G) \leqslant \zeta_1(H)$ and so $H/\zeta_1(H) \varepsilon Q\mathfrak{X} = \mathfrak{X}$. But $\mathfrak{X}$ is a good class so $H \varepsilon \mathfrak{X}$. Therefore $G \varepsilon L\mathfrak{X}$.

(ii) By 2·14, $\mathfrak{X}(1) \leqslant L\mathfrak{X}$. Clearly $L\mathfrak{X} \leqslant (L\mathfrak{X})(1)$. By 2·15, $\mathfrak{G} \cap (L\mathfrak{X})(1) \leqslant \mathfrak{X}(1)$ and so $(L\mathfrak{X})(1) \leqslant L(\mathfrak{X}(1))$ because each of the classes under consideration is S-closed. Since $\mathfrak{X}(1) \leqslant L\mathfrak{X}$, we have $L(\mathfrak{X}(1)) \leqslant L(L\mathfrak{X}) = L\mathfrak{X}$, as required.

For a good class $\mathfrak{X}$, we can obtain another bound on $\mathfrak{X}(1)$ by means of the following lemma.

<u>Lemma 2·17</u> If $\mathfrak{X}$ is a good class and $x \varepsilon H_1(G:\mathfrak{X})$ then $x^G \varepsilon \mathfrak{X}$.

Proof Let $X = x^G$. Then $X \lhd G$, $X \leq H_1(G:\mathfrak{X})$ and $d_G(X) < \infty$ so we have, by lemma 7(ii) of [3], that $G/C_G(X) \in \mathfrak{X}$. Thus, since $X/X \cap C_G(X) \cong XC_G(X)/C_G(X)$, we have $X/X \cap C_G(X) \in S\mathfrak{X} = \mathfrak{X}$. But $X \cap C_G(X) = \zeta_1(X)$ and so we have $X/\zeta_1(X) \in Q\mathfrak{X} = \mathfrak{X}$. Therefore $X \in \mathfrak{X}$.

Corollary 2·18 If $\mathfrak{X}$ is a good class then $\mathfrak{X}(1) \leq N\mathfrak{X}$.

Proof Let $G \in \mathfrak{X}(1)$. Then $G = H_1(G:\mathfrak{X}) = \pi\{x^G : x \in H_1(G:\mathfrak{X})\} \in N\mathfrak{X}$.

We also have the following connection between the upper $\mathfrak{X}$-central series of a group and its upper $\mathfrak{X}$-series, that is, its upper $\nu_{\mathfrak{X}}$-central series. We write $\rho_\alpha(G:\mathfrak{X})$ for $\zeta_\alpha^{\nu_{\mathfrak{X}}}(G)$.

Lemma 2·19 If $\mathfrak{X}$ is a good class and $\mathfrak{X} = N_0\mathfrak{X}$ then $H_\alpha(G:\mathfrak{X}) \leq \rho_\alpha(G:\mathfrak{X})$ for every ordinal number α.

Proof For all ordinals α, let $H_\alpha = H_\alpha(G:\mathfrak{X})$ and $R_\alpha = \rho_\alpha(G:\mathfrak{X})$. Let $x \in H_1$. Then by 2·17 $x^G \in \mathfrak{X}$. Thus $H_1 \leq R_1$ and the result is true for $\alpha = 1$. Suppose the result is true for all groups for all ordinals $\beta < \alpha$. If α is a limit ordinal then $H_\alpha = \bigcup_{\beta<\alpha} H_\beta \leq \bigcup_{\beta<\alpha} R_\beta = R_\alpha$. Otherwise α-1 exists. Let $x \in H_\alpha$ so that $xH_{\alpha-1} \in H_1(G/H_{\alpha-1}:\mathfrak{X})$. Thus $xH_{\alpha-1} \in R_1(G/H_{\alpha-1}:\mathfrak{X})$ by the case $\alpha = 1$. Therefore $xH_{\alpha-1}$ belongs to a product of finitely many normal $\mathfrak{X}$-subgroups of $G/H_{\alpha-1}$. Since $\mathfrak{X} = N_0\mathfrak{X}$, we have that $xH_{\alpha-1}$ is an element of a normal $\mathfrak{X}$-subgroup $N/H_{\alpha-1}$ of $G/H_{\alpha-1}$.

Now $x^G H_{\alpha-1}/H_{\alpha-1} \leq N/H_{\alpha-1} \in \mathfrak{X}$. Therefore

$x^G/(x^G \cap H_{\alpha-1}) \in \mathfrak{X}$. But, by assumption, $H_{\alpha-1} \leq R_{\alpha-1}$ so $x^G \cap H_{\alpha-1} \leq x^G \cap R_{\alpha-1}$ and so $x^G/(x^G \cap R_{\alpha-1}) \in Q\mathfrak{X} = \mathfrak{X}$. Thus $x^G R_{\alpha-1}/R_{\alpha-1} \in \mathfrak{X}$ so that $(xR_{\alpha-1})^{G/R_{\alpha-1}} \in \mathfrak{X}$. Therefore $xR_{\alpha-1} \in R_1(G/R_{\alpha-1} : \mathfrak{X})$ and so $x \in R_\alpha$, as required.

We now define a root class of groups. Our definition follows Robinson [1] . Stanley [3] omits the necessity of S-closure.

<u>Definition</u> A class of groups $\mathfrak{X}$ is a <u>root class</u> if $\mathfrak{X} = S\mathfrak{X}$ and whenever we have subgroups G_2 and G_3 of a group G such that $G_2 \triangleleft G_3 \triangleleft G$, $G/G_3 \in \mathfrak{X}$ and $G_3/G_2 \in \mathfrak{X}$ it always follows that there exists a normal subgroup G_4 of G such that $G_4 \leq G_2$ and $G/G_4 \in \mathfrak{X}$.

We observe that, for $\langle S,Q \rangle$-closed classes of groups, the condition of being a root class is equivalent to p-closure. We also observe that, since $N_0 \leq \langle p,Q \rangle$, if $\mathfrak{X} = \langle S,Q,p \rangle \mathfrak{X}$ then $\mathfrak{X} = N_0\mathfrak{X}$ and so $\mathfrak{X} = D_0\mathfrak{X}$.

Theorem C of [3] proves that if $\mathfrak{X}$ is N_0-closed and a root class, then $\overset{o}{\phi}_{\mathfrak{X}}$ and $\phi'_{\mathfrak{X}}$ are N_0-closed. This may be perhaps expressed more simply by

<u>Theorem 2·20</u> If $\mathfrak{X} = \langle S,Q,p \rangle \mathfrak{X}$ then $\overset{o}{\phi}_{\mathfrak{X}}$ and $\phi'_{\mathfrak{X}}$ are N_0-closed.

<u>Lemma 2·21</u> If $\mathfrak{X}$ is a good $\langle S,Q,p \rangle$-closed class of groups,

then $\overset{*}{\phi}_{\mathfrak{X}} \cap \mathfrak{F} \leq \mathfrak{X}$.

Proof If $\mathfrak{X}$ is a good class then by 2·14(iii), $H_1(G:\mathfrak{X}) \in L\mathfrak{X}$. Therefore if $G \in \mathfrak{X}(1) \cap \mathfrak{F}$, we have $G = H_1(G:\mathfrak{X}) \in L\mathfrak{X} \cap \mathfrak{F} \leq \mathfrak{X}$. Suppose $\mathfrak{X}(c) \cap \mathfrak{F} \leq \mathfrak{X}$ and let $G \in \mathfrak{X}(c+1) \cap \mathfrak{F}$. Then $G/H_1(G:\mathfrak{X}) \in \mathfrak{X}(c) \cap \mathfrak{F}$, by corollary 6 of [3] , and $H_1(G:\mathfrak{X}) \in \mathfrak{X}(1) \cap \mathfrak{F}$ by 2·12(vii). Therefore $G \in p\mathfrak{X} = \mathfrak{X}$. Thus $\overset{o}{\phi}_{\mathfrak{X}} \cap \mathfrak{F} \leq \mathfrak{X}$. But, by corollary 7 of [7] , $\overset{*}{\phi}_{\mathfrak{X}} \cap \mathfrak{F} = ZA\mathfrak{X} \cap \mathfrak{F} = \mathfrak{H}\mathfrak{X} \cap \mathfrak{F}$. By theorem A of [3] , $\mathfrak{H}\mathfrak{X} \leq \overset{o}{\phi}_{\mathfrak{X}}$. Therefore $\overset{*}{\phi}_{\mathfrak{X}} \cap \mathfrak{F} \leq \overset{o}{\phi}_{\mathfrak{X}} \cap \mathfrak{F} \leq \mathfrak{X}$.

Corollary 2·22 If $\mathfrak{X}$ is a good $\langle S,Q,p \rangle$-closed class of groups, then $\overset{*}{\phi}_{\mathfrak{X}}$ is a class of generalized $\mathfrak{X}$-groups.

Lemma 2·23 If $\mathfrak{X} = \langle S,Q,p \rangle\mathfrak{X}$ and N is a normal $\mathfrak{X}$-subgroup of G with $d_G(N) < \infty$ and $N \leq H_1(G:\mathfrak{X})$, then $H_1(G/N:\mathfrak{X}) = H_1(G:\mathfrak{X})/N$.

Proof Let $xN \in H_1(G/N:\mathfrak{X})$. Then there exists a normal subgroup K of G with $N \leq K$ such that $[x,K] \leq N$ and $G/K \in \mathfrak{X}$. Let $C = C_G(N)$. Then $G/C \in \mathfrak{X}$ by lemma 7 of [3] , so that if $L = K \cap C$, we have $G/L \in \mathfrak{X}$.

Define $\phi : L \to N$ by $\phi(\ell) = [\ell,x]$ for all $\ell \in L$. This is a homomorphism because $[\ell_1\ell_2,x]$ is easily seen to be equal to $[\ell_1,x]\,[\ell_1,x,\ell_2]\,[\ell_2,x]$ so that, since $[\ell_1,x] \in [L,x] \leq [K,x] \leq N$, we have $[\ell_1,x,\ell_2] \in [N,L] \leq [N,C] = 1$, yielding
$\phi(\ell_1\ell_2) = [\ell_1\ell_2,x] = [\ell_1,x]\,[\ell_2,x] = \phi(\ell_1)\phi(\ell_2)$.

The kernel of ϕ is $C_G(x) \cap L$. Therefore, since $N \varepsilon \mathfrak{X}$ we have $L/(C_G(x) \cap L) \varepsilon \mathfrak{X}$. Since $\mathfrak{X}$ is a root class there exists a normal subgroup M of G with $M \leqslant C_G(x) \cap L$ and $G/M \varepsilon \mathfrak{X}$. Therefore $[x,M] = 1$ and so $x \varepsilon H_1(G:\mathfrak{X})$. Therefore $xN \varepsilon H_1(G:\mathfrak{X})/N$, as required.

<u>Corollary 2·24</u> If $G \varepsilon \phi'_{\mathfrak{X}}$, where $\mathfrak{X} = \langle S,Q,p\rangle\mathfrak{X}$ and if $H_1(G:\mathfrak{X}) \varepsilon \mathfrak{F} \cap \mathfrak{X}$ then $G \varepsilon \mathfrak{F} \cap \mathfrak{X}$.

<u>Proof</u> Since $H_1(G:\mathfrak{X})$ itself satisfies the conditions of lemma 2·23, we have $H_1(G/H_1(G:\mathfrak{X}):\mathfrak{X}) = H_1(G:\mathfrak{X})/H_1(G:\mathfrak{X})$. Therefore $G = \bar{H}(G:\mathfrak{X}) = H_1(G:\mathfrak{X}) \varepsilon \mathfrak{F} \cap \mathfrak{X}$.

We note that lemma 2·23 does not hold when $\mathfrak{X}$ is taken to be the class of nilpotent groups. This is shown by the infinite dihedral group, $G = \langle x,y \,;\, x^y = x^{-1}, y^2 = 1 \rangle$. For let $X = \langle x \rangle$. Then $X \triangleleft G$ and because X and G/X are abelian, we have $X \leqslant H_1(G:\mathfrak{N})$. Also $d_G(X) < \infty$ because $X \varepsilon \mathfrak{F}$. But $G \varepsilon \mathfrak{F}$ and $G \notin \mathfrak{N}$ and so by 2·16(ii), $G \notin \mathfrak{N}(1)$. That is, $H_1(G:\mathfrak{N}) < G$. Therefore $H_1(G:\mathfrak{N})/X < G/X = H_1(G/X:\mathfrak{N})$.

We now consider $\mathfrak{V}$-centrality and $\mathfrak{V}_D$-centrality for any variety $\mathfrak{V}$. We shall prove some weak homomorphic image closure properties and some generalized local theorems as introduced in section 1. Hereafter, $\mathfrak{V}$ always denotes a variety.

There is a simplification of the general situation in this case, as is shown by :

Theorem 3·1 $\quad \grave{\phi}_{\mathfrak{V}} = \grave{\psi}_{\mathfrak{V}}, \quad \acute{\phi}_{\mathfrak{V}} = \acute{\psi}_{\mathfrak{V}}$ and $\mathring{\phi}_{\mathfrak{V}} = \mathring{\psi}_{\mathfrak{V}}$.

Proof Let $G \in \grave{\phi}_{\mathfrak{V}}$. Then there exists a descending series

$$G = G_0 \geq G_1 \geq \dots G_\gamma = 1$$

in G such that $(G_\alpha/G_{\alpha+1})\phi_{\mathfrak{V}}(G/G_{\alpha+1})$ for each ordinal $\alpha < \gamma$. Let $x \in G_\alpha$. Then there exists a normal subgroup N of G such that $[x,N] \leq G_{\alpha+1}$ and $G/N \in \mathfrak{V}$. Therefore $\mathfrak{V}(G) \leq N$ so $[x,\mathfrak{V}(G)] \leq G_{\alpha+1}$ for every element x of G_α. Therefore $[G_\alpha, \mathfrak{V}(G)] \leq G_{\alpha+1}$ and so $(G_\alpha/G_{\alpha+1})\psi_{\mathfrak{V}}(G/G_{\alpha+1})$. Thus $G \in \grave{\psi}_{\mathfrak{V}}$. The reverse inclusion is by 2·9.

Now let $G \in \acute{\psi}_{\mathfrak{V}}$ so that we have a series

$$1 = G_0 \leq G_1 \leq \dots G_\gamma = G$$

in G such that $(G_{\alpha+1}/G_\alpha)\psi_{\mathfrak{V}}(G/G_\alpha)$ for each ordinal $\alpha < \gamma$. That is, $\bigcap\{[G_{\alpha+1}/G_\alpha, N/G_\alpha] ; (N/G_\alpha) \lhd (G/G_\alpha), G/N \in \mathfrak{V}\} = 1$. Therefore $[G_{\alpha+1}/G_\alpha, \mathfrak{V}(G)G_\alpha/G_\alpha] = 1$ and since $G/\mathfrak{V}(G)G_\alpha \in \mathfrak{V}$, we have $(G_{\alpha+1}/G_\alpha)\phi_{\mathfrak{V}}(G/G_\alpha)$ and so $G \in \acute{\phi}_{\mathfrak{V}}$. The reverse inclusion is again by 2·9.

Let $\mathcal{S}$ be a $\phi_{\mathfrak{V}}$-series of finite length in a group G. Since any series of finite length is obviously both ascending and descending, $\mathcal{S}$ is a descending $\phi_{\mathfrak{V}}$-series, and so a descending $\psi_{\mathfrak{V}}$-series . But $\mathcal{S}$ has finite length and so $G \varepsilon \mathring{\psi}_{\mathfrak{V}}$. We may prove $\mathring{\psi}_{\mathfrak{V}} \leq \mathring{\phi}_{\mathfrak{V}}$ in the same way.

From some results in section 2 and by 3.1, we have the inclusions shown in Fig. 2 for any variety $\mathfrak{V}$.

It can be seen, by considering the variety of trivial groups, that all the inclusions shown are proper. For $\mathring{\phi}_1 = \mathcal{H}$, $\acute{\phi}_1 =$ ZA, $\grave{\phi}_1 =$ ZD and, as observed in section 2, $\hat{\phi}_1 =$ Z and $\overset{*}{\phi}_1$ is the class of residually central groups. As observed in section 2, there exists a $\overset{*}{\phi}_1$-group which is not a Z-group and it is well known that none of the classes $\mathcal{H}$, ZA, ZD and Z coincide.

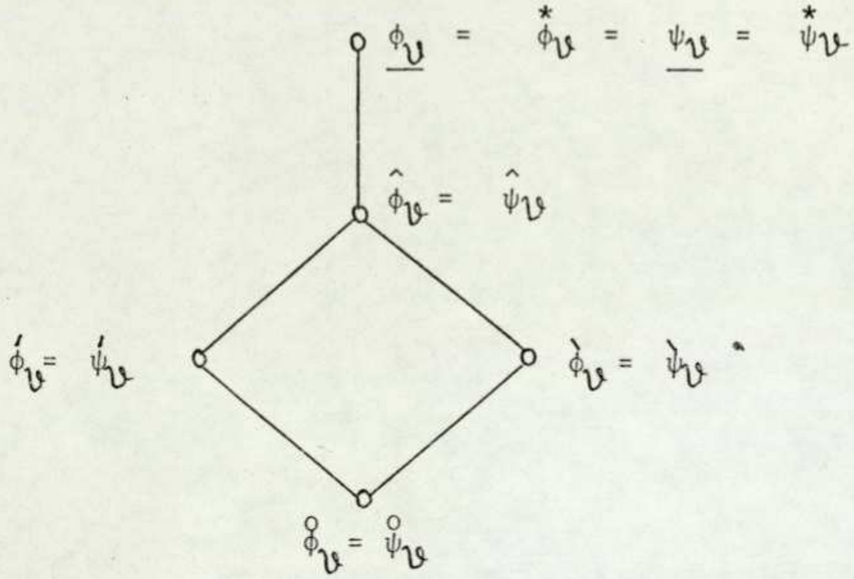
In order to prove our weak homomorphic image closure theorems and generalized local theorems we use the following method, as described in [1] .

Let G be any group and let $\{H_\sigma, K_\sigma ; \sigma \varepsilon \Sigma\}$ be a series in G. This series determines a binary relation $\subset$ on G, defined by $x \subset y$ if either $x = 1$, or $x \neq 1$ and $\sigma(x) \leq \sigma(y)$ where $\sigma(x)$ is the unique element of Σ such that $x \varepsilon H_{\sigma(x)} - K_{\sigma(x)}$.
It may be proved that :

i) $x \subset y$ and $y \subset z$ imply that $x \subset z$.

ii) Either $x \subset y$ or $y \subset x$ or both.

iii) $x \subset 1$ implies that $x = 1$.



$\underline{\phi_{\vartheta}} = \overset{*}{\phi}_{\vartheta} = \underline{\psi_{\vartheta}} = \overset{*}{\psi}_{\vartheta}$
$\hat{\phi}_{\vartheta} = \hat{\psi}_{\vartheta}$
$\acute{\phi}_{\vartheta} = \acute{\psi}_{\vartheta}$
$\grave{\phi}_{\vartheta} = \grave{\psi}_{\vartheta}$
$\mathring{\phi}_{\vartheta} = \mathring{\psi}_{\vartheta}$

Fig. 2

iv) $x \subset y$ and $z \subset y$ imply that $xz^{-1} \subset y$.

v) $y \subset x^y$ implies that $y \subset x$.

If the series is normal, we also have

vi) $x^y \subset x$ for all x and y.

If the series is a $\mathfrak{V}$-series, we have

vii) given $x_i \subset y$ $(i = 1,\ldots,n)$ and w a word in n variables determining $\mathfrak{V}$, it follows that $y \not\subset w(x_1,\ldots,x_n)$.

On the other hand, if $\subset$ is a binary relation, on a group G, satisfying i) to v) then it determines a series in G. For, define an equivalence relation ~ on G by $x \sim y$ if $x \subset y$ and $y \subset x$ and let Σ be the set of all ~-equivalence classes other than $\{1\}$. Define a linear ordering $<$ on Σ by $\sigma < \tau$ if $\sigma \neq \tau$ and there exist an x belonging to σ and a y belonging to τ such that $x \subset y$.

Let $H_\sigma = \{x \in G;\ x \subset y \text{ for some } y \in \sigma\}$ and $K_\sigma = \bigcup_{\tau<\sigma} H_\tau$. Then it is not difficult to prove that $S = \{H_\sigma, K_\sigma\ ;\ \sigma \in \Sigma\}$ is a series in G. Also, if $\subset$ satisfies vi) then S is a normal series and if $\subset$ satisfies vii) then S is a $\mathfrak{V}$-series.

Thus there is a one-to-one correspondence between series in a group G and binary relations on G satisfying i) to v).

We now make the following definition :

<u>Definition</u> Let S be a non-empty set. Then a <u>local system</u> $\mathfrak{L}$ on S is a collection of subsets of S such that each finite subset of S lies within some member of $\mathfrak{L}$.

Notice that with this definition, G is a locally-$\mathfrak{X}$ group if and only if G has a local system consisting of $\mathfrak{X}$-groups.

For any set S, we denote by $S^{[n]}$ the set $\{(S_1,\ldots,S_n)\ ;\ S_i \in S$ for each $1 \leq i \leq n\}$.

The following lemma is proved in [1] :

<u>Lemma</u> Let $\mathfrak{L}$ be a local system on a set S, let F be a finite set and let n be a positive integer. Suppose that for each $H \in \mathfrak{L}$ there is a function α_H: $H^{[n]} \to F$. Then there is a function $\alpha : S^{[n]} \to F$ such that for every subset $\{x_1,\ldots,x_m\}$ of $S^{[n]}$ there is an $H \in \mathfrak{L}$ such that $x_i \in H^{[n]}$ and $\alpha(x_i) = \alpha_H(x_i)$ for each $i = 1,\ldots,m$.

We write $\alpha = \lim\limits_{H\to S} \alpha_H$.

Suppose now that $\mathfrak{L}$ is a local system of subgroups of a group G and that each $H \in \mathfrak{L}$ possesses a series determined by a binary relation $\subset_H$ on H. Define $\alpha_H : H^{[2]} \to \{0,1\}$ by :

$$\alpha_H(x,y) = \begin{cases} 1 \text{ if } x \subset_H y. \\ 0 \text{ otherwise} \end{cases} .$$

The properties i) to vii) may be stated purely as properties of the function α_H. By the lemma, $\alpha = \lim\limits_{H\to S} \alpha_H$ exists. It can easily be seen that the properties i) to vii) are <u>hereditary</u>, that is, they are inherited by α from the α_H's. Therefore we obtain a series in G of the same kind as the series in the subgroups H.

<u>Lemma 3·2</u> A series is a $\mathfrak{V}$-central series if and only if the relation $\subset$ also satisfies $x \subset y \neq 1$ implies that $y \not\subset [x,z]$ for all $x,y \in G$ and all $z \in \mathfrak{V}(G)$.

<u>Proof</u> Suppose that G has a $\mathfrak{V}$-central series and that $x \subset y \neq 1$. If $x = 1$ then $[x,z] = 1$ for all $z \in \mathfrak{V}(G)$ so that $y \not\subset [x,z]$. If $x \neq 1$ then $\sigma(x) \leq \sigma(y)$. Suppose, for a contradiction, that there exist $x,y \in G$ and $z \in \mathfrak{V}(G)$ such that $y \subset [x,z]$. Then $\sigma(y) \leq \sigma([x,z])$.

If either $\sigma(x) < \sigma(y)$ or $\sigma(y) < \sigma([x,z])$ then $\sigma(x) < \sigma([x,z])$ so that $H_{\sigma(x)} \leq K_{\sigma([x,z])}$. But $x \in H_{\sigma(x)}$ so $[x,z] \in H_{\sigma(x)} \leq K_{\sigma([x,z])}$, which is a contradiction. Therefore we may assume that $\sigma(x) = \sigma([x,z])$.

It is easy to see that, if $\{H_\sigma, K_\sigma : \sigma \in \Sigma\}$ is a $\mathfrak{V}$-central series in G, then, for each $\sigma \in \Sigma$, $[H_\sigma, \mathfrak{V}(G)] \leq K_\sigma$. Since $\sigma(x) = \sigma([x,z])$ we have $[x,z] \in [H_{\sigma([x,z])}, \mathfrak{V}(G)] \leq K_{\sigma([x,z])}$, which again is a contradiction. Therefore $y \not\subset [x,z]$ for all $x,y \in G$ and $z \in \mathfrak{V}(G)$.

Conversely, suppose the condition holds. A generator of $[H_\sigma, \mathfrak{V}(G)]$ has the form $[x,z]$ where $x \in H_\sigma$ and $z \in \mathfrak{V}(G)$. Then $x \subset y$ for some $y \in \sigma$. Since $y \not\subset [x,z]$, we have by property ii) that $[x,z] \subset y \in \sigma$. Therefore $\sigma([x,z]) < \sigma$ and so $[x,z] \in H_{\sigma([x,z])} \leq K_\sigma$.

Therefore $[H_\sigma, \mathfrak{V}(G)] \leq K_\sigma$. But $G/\mathfrak{V}(G) \in \mathfrak{V}$ so that $\{H_\sigma, K_\sigma \; ; \sigma \in \Sigma\}$ is a $\mathfrak{V}$-central series.

Let $G = \underset{\alpha \in A}{\mathrm{Cr}} \; G_\alpha$. Then G is the set of all functions $\{x: A \to \bigcup_{\alpha \in A} G_\alpha \; ; \; x(\alpha) \in G_\alpha$ and $(xy)(\alpha) = x(\alpha)y(\alpha)$ for each $\alpha \in A\}$. For a particular $\alpha \in A$, we may associate G_α with the subset of G consisting of all the functions $x(\alpha)$. We may make similar associations with subgroups of G_α defined in terms of group theoretical functions. We make such associations in our next proposition.

<u>Proposition 3·3</u> If $G = \underset{\alpha \in A}{\mathrm{Cr}} \; G_\alpha$ and if $g \in \mathfrak{V}(G)$ then $g(\alpha) \in \mathfrak{V}(G_\alpha)$.

<u>Proof</u> Choose an $\alpha \in A$ and let $V = \mathfrak{V}(G_\alpha)$. Let

$$B = \{x: A \to \bigcup_{\alpha \in A} G_\alpha \; ; \; x(\beta) \in G_\beta \; , \; x(\beta)y(\beta) = (xy)(\beta) \text{ and } x(\alpha) = 1\}.$$

Then $B \triangleleft G$, $G_\alpha \cap B = 1$ and $G = G_\alpha B$. Now

$$\begin{aligned} G/VB &= G_\alpha B/VB \\ &= G_\alpha VB/VB \\ &\cong G_\alpha/(VB \cap G_\alpha) \\ &= G_\alpha/\,V(B \cap G_\alpha) \\ &= G_\alpha/V \end{aligned}$$

Then $G/VB \in \mathfrak{V}$ so that if $g \in \mathfrak{V}(G)$ then $g \in VB$. Say $g = vb$ where $v \in V$ and $b \in B$. Then, since $b(\alpha) = 1$, we have

$$g(\alpha) = (vb)(\alpha) = v(\alpha) \in \mathfrak{V}(G_\alpha)$$

as required.

Recalling the properties of Petty, discussed in section 1, we are now able to prove :

<u>Theorem 3·4</u> $\hat{\phi}_{\mathfrak{V}} = H_u\hat{\phi}_{\mathfrak{V}} = L_F\hat{\phi}_{\mathfrak{V}}$ and so $\hat{\phi}_{\mathfrak{V}} = L\hat{\phi}_{\mathfrak{V}}$.

<u>Proof</u> Let $K \in H_u\,\hat{\phi}_{\mathfrak{V}}$. Then there exists a set $\{G_\alpha ; \alpha \in A\}$ of $\hat{\phi}_{\mathfrak{V}}$-groups and an ultrafilter $\mathcal{U}$ on A such that $K \cong G/N$ where $G = \underset{\alpha \in A}{\mathrm{Cr}}\, G_\alpha$ and $N = \{x \in G ; I(x) \in \mathcal{U}\}$ with

$I(x) = \{\alpha \in A; x(\alpha) = 1_\alpha\}$ for each $x \in G$.

For each $\alpha \in A$, there exists a function $f_\alpha : G_\alpha \times G_\alpha \to \{0,1\}$ defined by

$$f_\alpha(x(\alpha),y(\alpha)) = \begin{cases} 1 \text{ if } x(\alpha) \subset_{G_\alpha} y(\alpha) \\ 0 \text{ otherwise} \end{cases}$$

and f_α defines a $\phi_{\mathfrak{V}}$-series on G_α .

For each $(x,y) \in G \times G$, define

$T(x,y) = \{ \alpha \in A ; f_\alpha(x(\alpha),y(\alpha)) = 1 \}$

and define the function $f : G/N \times G/N \to \{0,1\}$ by

$$f(xN,yN) = \begin{cases} 1 \text{ if } T(x,y) \in \mathcal{U} \\ 0 \text{ otherwise .} \end{cases}$$

(Petty, in [6] , proves that this is a function).

The normal series properties are hereditary. For the $\mathfrak{V}$-centrality, suppose xN and yN are elements of G/N and $zN \in \mathfrak{V}(G/N)$. Now $\mathfrak{V}(G/N) = \mathfrak{V}(G)N/N$ so $z = rn$ for some $r \in \mathfrak{V}(G)$ and $n \in N$.

Suppose $f(xN,yN) = 1$, then $T(x,y) \in \mathcal{U}$.

Let $\alpha \in T(x,y)$ so that $f_\alpha(x(\alpha),y(\alpha)) = 1$. Then $x(\alpha) \subset_{G_\alpha} y(\alpha)$ and by 3·3, $r(\alpha) \in \mathfrak{V}(G_\alpha)$. Therefore, by 3·2, $y(\alpha) \not\subset [x(\alpha),r(\alpha)]$ and so $f_\alpha(y_\alpha, [x(\alpha),r(\alpha)]) = 0$. Thus $\alpha \notin T(y,[x,r])$.

So $T(x,y) \subseteq A - T(y,[x,r])$ so that, since $T(x,y) \in \mathcal{U}$, $A - T(y,[x,r]) \in \mathcal{U}$. Therefore, because $\phi \notin \mathcal{U}$, $T(y,[x,r]) \notin \mathcal{U}$. So we have $f(yN,[x,r]N) = 0$.

Now $zN = rN$ so $[x,z]N = [x,r]N$. Therefore $f(yN,[x,z]N) = 0$.

Thus we have proved that $\hat{\phi}_{\mathfrak{V}} = H_u\hat{\phi}_{\mathfrak{V}}$. But by 2·5, $\hat{\phi}_{\mathfrak{V}} = S\hat{\phi}_{\mathfrak{V}}$ and so by 4·2 of [6], $\hat{\phi}_{\mathfrak{V}} = L_F\hat{\phi}_{\mathfrak{V}}$, and since $L \leq L_F$, we have $\hat{\phi}_{\mathfrak{V}} = L\hat{\phi}_{\mathfrak{V}}$.

We can improve this result to include H_R-closure for certain classes by using the following lemma :

<u>Lemma 3·5</u> If $\mathfrak{X}$ is a class of groups such that $\mathfrak{X} = \langle H_u, R\rangle\mathfrak{X}$ then $\mathfrak{X} = H_R\mathfrak{X}$.

<u>Proof</u> Let $K \in H_R\mathfrak{X}$. Then there exists a set $\{G_\alpha ; \alpha \in A\}$ of $\mathfrak{X}$-groups and a filter $\mathfrak{F}$ on A such that $K \cong G/N$ where $G = \underset{\alpha\in A}{\mathrm{Cr}}\ G_\alpha$ and $N = \{x \in G : I(x) \in \mathfrak{F}\}$. Now $\mathfrak{F}$ is equal to the intersection of all the ultrafilters $\mathcal{U}$ containing $\mathfrak{F}$ and so $N = \{x \in G ; I(x) \in \bigcap_{\mathcal{U}} \mathcal{U}\}$. For each $\mathcal{U}$,

let $M_{\mathcal{U}} = \{ x \in G \; ; I(x) \in \mathcal{U}\}$. Then it is easy to see that $N = \bigcap_{\mathcal{U}} M_{\mathcal{U}}$. Therefore $G/N \in R H_u \mathfrak{X} = \mathfrak{X}$ so that $K \in \mathfrak{X}$.

Corollary 3·6 $\hat{\phi}_{\mathcal{V}} = H_R \hat{\phi}_{\mathcal{V}}$.

As observed in section I, $H_u \leqslant H_R$. Thus in order to prove H_u - and H_R - closure for a class of groups it is sufficient to prove the second. The following method can be useful in certain cases :

Suppose that if G is a group then for each finite subset F of G there is an associated subgroup S(F) , normal in G, satisfying :

i) If $F = \{x_1, \ldots, x_n\}$ then $S(x_1N, \ldots, x_nN) \leqslant S(F)N/N$ for any normal subgroup N of G.

ii) If $G = \mathrm{Cr}_{\alpha \in A} G_\alpha$ and $F = \{x_1, \ldots, x_n\}$ then $S(F) \leqslant \mathrm{Cr}_{\alpha \in A} S(x_1(\alpha), \ldots, x_n(\alpha))$.

We now consider classes of groups $\mathfrak{X}$ which can be characterized by $G \in \mathfrak{X}$ if and only if for each finite subset F of G, not all of whose elements are 1 , we have $F \nsubseteq S(F)$.

Lemma 3·7 Any class $\mathfrak{X}$ defined in the above manner is H_R-closed.

Proof Let $K \in H_R \mathfrak{X}$. Then there exists a set $\{G_\alpha \; ; \alpha \in A\}$ of $\mathfrak{X}$- groups and a filter $\mathcal{F}$ on A such that $K \cong G/N$ where $G = \mathrm{Cr}_{\alpha \in A} G_\alpha$ and $N = \{g \in G \; ; I(g) \in \mathcal{F} \}$, where

$I(g) = \{ \alpha \varepsilon A \; ; \; g(\alpha) = 1 \}$.

Suppose $K \notin \mathfrak{X}$. Then there exists a finite subset $F = \{x_1 \ldots, x_n\}$ of G, not all of whose elements belong to N, such that

$$\{x_1N, \ldots, x_nN\} \subseteq S(x_1N, \ldots, x_nN) \subseteq S(F)N/N .$$

Therefore , for each $i = 1, \ldots, n$, $x_i \varepsilon S(F)N$. Say $x_i = y_i n_i$ where $y_i \varepsilon S(F)$ and $n_i \varepsilon N$. Since $S(F) \leq \underset{\alpha \varepsilon A}{Cr} S(x_1(\alpha), \ldots, x_n(\alpha))$, we have $y_i(\alpha) \varepsilon S(x_1(\alpha), \ldots, x_n(\alpha))$ for each $i = 1, \ldots, n$.

Now for each $i = 1, \ldots, n$ we have $I(n_i) \varepsilon \mathfrak{F}$ and so $\bigcap_{i=1}^{n} I(n_i) \varepsilon \mathfrak{F}$. Suppose $\alpha \varepsilon \bigcap_{i=1}^{n} I(n_i)$. Then $n_i(\alpha) = 1$ for each $i = 1, \ldots, n$ and so

$$x_i(\alpha) = y_i(\alpha) \varepsilon S(x_1(\alpha), \ldots, x_n(\alpha)) ,$$

Therefore

$$\{x_i(\alpha), \ldots, x_n(\alpha)\} \subseteq S(x_1(\alpha), \ldots, x_n(\alpha)) .$$

and so $x_i(\alpha) = 1$ for each $i = 1, \ldots, n$. Therefore $\alpha \varepsilon I(x_i)$, so $\bigcap_{i=1}^{n} I(n_i) \subseteq I(x_i)$ and so we have $I(x_i) \varepsilon \mathfrak{F}$ for each $i = 1, \ldots, n$.

Therefore $x_i \varepsilon N$ for each $i = 1, \ldots, n$, which is a contradiction, and we have proved that $K \varepsilon \mathfrak{X}$.

We may also consider the classes $\mathfrak{X}$ characterized by $G \varepsilon \mathfrak{X}$ if and only if for each subset F of G, consisting of exactly n elements not all of which are 1 , we have $F \nleq S(F)$. Note that 3·7 still holds in this case. In the special case where F consists of exactly one (non-trivial) element, we have the following examples : If $S(x) = \mathfrak{V}(x^G)$ we have $\mathfrak{X} = C_n\mathfrak{V}$, if $S(x) = \Phi(x^G, G)$ then $\mathfrak{X} = \underline{\mathfrak{V}^*}$ and, by a remark in [7] , if $S(x) = [x^G, \mathfrak{V}(G)]$ then $\mathfrak{X} = \overset{*}{\phi}_{\mathfrak{V}}$. Thus we have proved

<u>Theorem 3·8</u> $C_n\mathfrak{V}$, $\underline{\mathfrak{V}^*}$ and $\overset{*}{\phi}_{\mathfrak{V}}$ are all H_R-closed.

By 1.14 and 1.6, $\underline{\mathfrak{V}^*} = S\underline{\mathfrak{V}^*}$ and by 2·3 and 1·6, $\overset{*}{\phi}_{\mathfrak{V}} = S\,\overset{*}{\phi}_{\mathfrak{V}}$. That $C_n\mathfrak{V} = S\,C_n\mathfrak{V}$ is an immediate consequence of the following proposition :

<u>Proposition 3·9</u> If $\mathfrak{X} = S\mathfrak{X}$ then $C_n\mathfrak{X} = S\,C_n\mathfrak{X}$.

<u>Proof</u> Let $H \leq G$ where $G \in C_n\mathfrak{V}$ and let x be a non-trivial element of H. Then there exist normal subgroups M and N of G with $N \leq M$ such that $x \in M - N$ and $M/N \in \mathfrak{X}$.

Now $(H \cap M)N/N \leq M/N \in S\mathfrak{X} = \mathfrak{X}$ and so $(H \cap M)/(H \cap N) \in \mathfrak{X}$. But $H \cap N$ and $H \cap M$ are both normal subgroups of H and $x \in (H \cap M) - (H \cap N)$, as required.

Since, as has already been observed, $L \leq L_F \leq \langle H_u, S \rangle$, we have now proved

<u>Theorem 3·10</u> $C_n\mathfrak{V}$, $\underline{\mathfrak{V}^*}$ and $\overset{*}{\phi}_{\mathfrak{V}}$ are all L_F-closed and so L-closed.

The L_F-closure of $\overset{*}{\phi}_{\mathfrak{V}}$ improves theorem 10 of [7] which proves L-closure. Our next proposition is useful in proving the H_W-closure of $\overset{*}{\phi}_{\mathfrak{V}}$.

Proposition 3·11 Let α and β be subgroup theoretical properties defined by group theoretical functions α and β, respectively, and suppose that $\beta = \underline{H}\beta$. If $\{A_\lambda\}$ is an ascending α-series and $\{B_\lambda\}$ is the upper β-central series of a group G and if $\alpha(G) \leqslant \beta(G)$ for all groups G then $A_\lambda \leqslant B_\lambda$ for all ordinals λ.

Proof Suppose the result is true for all ordinals $\gamma < \lambda$. If λ is a limit ordinal then $A_\lambda = \bigcup_{\gamma<\lambda} A_\gamma \leqslant \bigcup_{\gamma<\lambda} B_\gamma = B_\lambda$. Otherwise $\lambda-1$ exists and $A_\lambda/A_{\lambda-1} \leqslant \alpha(G/A_{\lambda-1}) \leqslant \beta(G/A_{\lambda-1})$. That is, $(A_\lambda/A_{\lambda-1})\ \beta\ (G/A_{\lambda-1})$ and so $(A_\lambda B_{\lambda-1}/B_{\lambda-1})\ \beta\ (G/B_{\lambda-1})$. Therefore $A_\lambda \leqslant A_\lambda B_{\lambda-1} \leqslant B_\lambda$, as required.

Theorem 3·12 $\overset{*}{\phi}_{\mathfrak{V}} = H_W \overset{*}{\phi}_{\mathfrak{V}}$.

Proof In order to prove the theorem, we may choose any $\overset{*}{\phi}_{\mathfrak{V}}$-group G and show that for each normal subgroup N of G with $N \leqslant \overline{\mathfrak{V}^*(G)}$, we have $G/N \in \overset{*}{\phi}_{\mathfrak{V}}$. Since $[\mathfrak{V}^*(G), \mathfrak{V}(G)] = 1$ and $H_1(G:\mathfrak{V}) = C_G(\mathfrak{V}(G))$ hold for all groups G, we have $\mathfrak{V}^*(G) \leqslant H_1(G:\mathfrak{V})$. Therefore, by 3·11, it is sufficient to prove that $G/N \in \overset{*}{\phi}_{\mathfrak{V}}$ for each $N \triangleleft G$ with $N \leqslant \overline{H}(G:\mathfrak{V})$. We prove this in our next proposition.

Proposition 3·13 If $G \in \overset{*}{\phi}_{\mathfrak{V}}$ and N is a normal subgroup of G, contained in $\overline{H}(G:\mathfrak{V})$, then $G/N \in \overset{*}{\phi}_{\mathfrak{V}}$.

Proof For any ordinal α, let $H_\alpha = H_\alpha(G:\mathfrak{V})$. We prove by induction that for each ordinal α, if $G \in \overset{*}{\phi}_{\mathfrak{V}}$, $N \triangleleft G$ and $N \leqslant H_\alpha$ then $G/N \in \overset{*}{\phi}_{\mathfrak{V}}$. The case $\alpha = 1$ is dealt with in lemma 5 of [7]. Suppose the proposition is true for all ordinals $\beta < \alpha$.

If α is a limit ordinal, choose an element x of G, not belonging to N. Then $x \notin H_\beta \cap N$ for every $\beta < \alpha$. But $G/(H_\beta \cap N) \in \overset{*}{\phi}_{\mathfrak{V}}$ by assumption and so $x \notin [x^G, \mathfrak{V}(G)]\,(H_\beta \cap N)$ for each $\beta < \alpha$. Therefore $x \notin [x^G, \mathfrak{V}(G)](H_\alpha \cap N) = [x^G, \mathfrak{V}(G)]N$ so that $G/N \in \overset{*}{\phi}_{\mathfrak{V}}$.

Otherwise $\alpha-1$ exists and $G/N \cong (G/N \cap H_{\alpha-1})/(N/N \cap H_{\alpha-1})$. Let $x(N \cap H_{\alpha-1}) \in N/N \cap H_{\alpha-1}$. Then $x \in H_\alpha$ and so there exists a normal subgroup M of G such that $H_{\alpha-1} \leqslant M$, $G/M \in \mathfrak{V}$ and $[x,M] \leqslant H_{\alpha-1}$. But $x \in N$ so that $[x,M] \leqslant H_{\alpha-1} \cap N$. Therefore $N/N \cap H_{\alpha-1} \leqslant H_1(G/N \cap H_{\alpha-1};\mathfrak{V})$. Also $G/N \cap H_{\alpha-1} \in \overset{*}{\phi}_{\mathfrak{V}}$ by assumption and so $G/N \in \overset{*}{\phi}_{\mathfrak{V}}$.

We observe that $\overset{o}{\phi}_{\mathfrak{V}} < L\overset{o}{\phi}_{\mathfrak{V}} < L\grave{\phi}_{\mathfrak{V}}$ in general because $\overset{o}{\phi}_1 = \mathfrak{N} < L\mathfrak{N} = L\overset{o}{\phi}_1$ and $L\mathfrak{N} < LZD = L\grave{\phi}_1$. We are unable to decide whether $L\overset{o}{\phi}_{\mathfrak{V}} = L\phi'_{\mathfrak{V}}$ in general.

We also record the following lemma.

Lemma 3·14 $\mathfrak{V}^* < \phi_{\mathfrak{V}}$

Proof If $H\ \mathfrak{V}^*\ G$ then $H \leqslant \mathfrak{V}^*(G) \leqslant H_1(G:\mathfrak{V})$ so that $H\ \phi_{\mathfrak{V}}\ G$.

Consider the symmetric group of degree three, S_3 . Now S_3 is metabelian and has trivial centre. The alternating group of degree three, A_3 , is a normal subgroup of S_3 and A_3 and S_3/A_3 are abelian. Therefore $A_3 \leq H_1(S_3:\mathfrak{A})$ but $A_3 \nleq \mathfrak{z}_1(S_3) = \mathfrak{A}^*(S_3)$.

This shows that $N \, \phi_{\mathfrak{A}} \, G$ is true but $N \, \mathfrak{A}^* G$ is not.

We now examine the class $\mathfrak{V}(1)$ for a variety $\mathfrak{V}$. By theorem B of [3] , $\mathfrak{V}(1)$ is a variety. The proof of the following proposition is easy and is omitted.

<u>Proposition 3.15</u> $\mathfrak{V}(1)$ is just the variety of central-by-$\mathfrak{V}$ groups.

It is well known that if $\mathfrak{V}_1$ and $\mathfrak{V}_2$ are varieties then $\mathfrak{V}_2(G) \leq \mathfrak{V}_1(G)$ for all groups G if and only if $\mathfrak{V}_1 \leq \mathfrak{V}_2$. Thus, for any group G, $(\mathfrak{V}(1))(G) \leq \mathfrak{V}(G)$. We can, by means of the following lemma, identify $(\mathfrak{V}(1))(G)$ exactly.

<u>Lemma 3·16</u> Let $\mathfrak{V}$ be the variety determined by a set of words W in variables $z_1, z_2, \ldots$. Let $W' = \{[w(z_1,\ldots,z_n),z_{n+1}]; \quad w \in W\}$.
Then $\mathfrak{V}(1)$ is precisely the variety determined by W'.

<u>Proof</u> Let $G \in \mathfrak{V}(1)$ and choose a word w , belonging to W , and an element x of G. Suppose w is a word in n variables. Then there exists a normal subgroup N of G with $[N,x] = 1$ and $G/N \in \mathfrak{V}$. For any $g_1,\ldots,g_n \in G$ we have $w(g_1,\ldots,g_n) \in N$

and so $[w(g_1,\ldots,g_n),x] = 1$.

On the other hand, let G be a group such that for any word w, belonging to W, in n variables and any elements $g, g_1,\ldots,g_n$ of G, we have $[w(g_1,\ldots,g_n),g] = 1$. Therefore $[\mathfrak{V}(G),g] = 1$. Thus $\mathfrak{V}(G) \leq \mathfrak{Z}_1(G)$ so $G \in \mathfrak{V}(1)$ by 3·15.

Corollary 3·17 $(\mathfrak{V}(1))(G) = [\mathfrak{V}(G),G]$.

We now introduce a condition that holds for many varieties .

Definition A variety $\mathfrak{V}$ satisfies (X) if for all groups G, $\mathfrak{V}(\mathfrak{V}(G)) \leq [\mathfrak{V}(G),G]$.

Examples of varieties satisfying (X) include $\mathfrak{N}_c$ and $\mathfrak{A}^d$ for any positive integers c and d. A variety not satisfying (X) is the trivial class $\mathfrak{1}$, for let G be any non-trivial abelian group. Then $\mathfrak{1}(G) = G$ so $\mathfrak{1}(\mathfrak{1}(G)) = G$ but $[\mathfrak{1}(G),G] = [G,G] = 1$.

Lemma 3·18 If $\mathfrak{V}$ satisfies (X) then $\mathfrak{V}(G)/(\mathfrak{V}(1))(G) \in \mathfrak{V}$.

Proof By 3·17 we have $\mathfrak{V}(G)/(\mathfrak{V}(1))(G) = \mathfrak{V}(G)/[\mathfrak{V}(G),G]$. Also if $\mathfrak{V}$ satisfies (X) then

$$\mathfrak{V}(\mathfrak{V}(G)/[\mathfrak{V}(G),G]) = \mathfrak{V}(\mathfrak{V}(G))\,[\mathfrak{V}(G),G]/[\mathfrak{V}(G),G] = 1 ,$$

as required.

In theorem 1·18 we proved that $C_n\hat{\mathfrak{V}}^* = C_F\mathfrak{V} = C_n\mathfrak{V} = C_n\mathfrak{V}^*$

for any variety $\mathfrak{V}$. We may use 3·18 to improve this result for varieties satisfying (X), as follows :

Theorem 3·19 If $\mathfrak{V}$ satisfies (X) then

$$C_n \hat{\mathfrak{V}}^* = C_F\mathfrak{V} = C_n\mathfrak{V} = C_n \underline{\mathfrak{V}^*} = C_n(\mathfrak{V}(1)) =$$
$$C_n\widehat{\mathfrak{V}(1)}^* = C_F(\mathfrak{V}(1)) = C_n \underline{\mathfrak{V}(1)^*} \; .$$

Proof By 1·18 it is sufficient to prove that $C_n(\mathfrak{V}(1)) = C_n\mathfrak{V}$. Let $G \in C_n(\mathfrak{V}(1))$ and let x be a non-trivial element of G. Then there exist normal subgroups K and L of G, with $K \leq L$, such that $x \in L - K$ and $L/K \in \mathfrak{V}(1)$.

If $x \notin \mathfrak{V}(L)$ then $x \in L - \mathfrak{V}(L)$ and, since $\mathfrak{V}(L) \lhd L \lhd G$, we have no more to prove . Suppose, then, that $x \in \mathfrak{V}(L)$. We know that $L/K \in \mathfrak{V}(1)$ so $(\mathfrak{V}(1))(L) \leq K$ and $x \in \mathfrak{V}(L) - (\mathfrak{V}(1))(L)$. Also 3·18 shows that $\mathfrak{V}(L)/(\mathfrak{V}(1))(L) \in \mathfrak{V}$. But $\mathfrak{V}(L)$ and $(\mathfrak{V}(1))(L)$ are both characteristic in L and so are normal in G, as required.

The following lemma is interesting in that it requires no condition on the variety $\mathfrak{V}$.

Lemma 3·20 For any variety $\mathfrak{V}$, the variety $\mathfrak{V}(1)$ satisfies (X).

Proof Let G be a group. Then $(\mathfrak{V}(1))((\mathfrak{V}(1))(G)) = (\mathfrak{V}(1))[\mathfrak{V}(G),G]$

$$= [\mathfrak{V}([\mathfrak{V}(G),G]), [\mathfrak{V}(G),G]] \leq [[\mathfrak{V}(G),G] , G]$$
$$= [(\mathfrak{V}(1))(G),G] \; .$$

Lemma 3·21 For any variety $\mathfrak{V}$, $\overset{*}{\phi}_{\mathfrak{V}} \leq C_n \mathfrak{V}(1)$.

Proof Let x be a non-trivial element of a $\overset{*}{\phi}_{\mathfrak{V}}$-group G. Now $\overset{*}{\phi}_{\mathfrak{V}} = \underline{\phi}_{\mathfrak{V}}$ by 2·4 (i) and so there exist normal subgroups K and L of G with $L \leq K$, such that $x \in K - L$ and $K/L \leq H_1(G/L:\mathfrak{V})$. But by 2·12 (vii) we have $H_1(G/L:\mathfrak{V}) \in \mathfrak{V}(1)$ so $K/L \in \mathfrak{V}(1)$, as required.

This enables us to prove

Theorem 3·22. (i) For any variety $\mathfrak{V}$, $\overset{*}{\phi}_{\mathfrak{V}(1)} \leq C_n\mathfrak{V}(1)$.

(ii) If $\mathfrak{V}$ satisfies (X), then $\overset{*}{\phi}_{\mathfrak{V}(1)} \leq C_n\mathfrak{V}$.

In particular, if $\mathfrak{V}$ satisfies (X) then $\overset{*}{\phi}_{\mathfrak{V}} \leq C_n\mathfrak{V}$.

Proof By 3·19, 3·20, and 3·21, $\overset{*}{\phi}_{\mathfrak{V}(1)} \leq C_n((\mathfrak{V}(1))(1)) = C_n(\mathfrak{V}(1))$. By 3·19, if $\mathfrak{V}$ satisfies (X) then $C_n(\mathfrak{V}(1)) = C_n\mathfrak{V}$ so $\overset{*}{\phi}_{\mathfrak{V}(1)} \leq C_n\mathfrak{V}$.

We take as a corollary :

Corollary 3·23 For all positive integers C, $C_n \mathfrak{H}_C = C_n\mathfrak{A}$ so that $\overset{*}{\phi}_{\mathfrak{H}_C} \leq C_n\mathfrak{A}$.

Proof By 3·22 (ii) this is true when C = 1. Suppose that $C_n \mathfrak{H}_{C-1} = C_n\mathfrak{A}$. Then $C_n\mathfrak{H}_C \leq C_n(\mathfrak{H}_{C-1}(1)) =$

$C_n\mathfrak{H}_{C-1} = C_n\mathfrak{A}$ by 3·19.

We also record :

Lemma 3·24 $\underline{\mathfrak{V}^*} \leq \overset{*}{\phi}_{\mathfrak{V}} \leq \overset{*}{\phi}_1\mathfrak{V} \leq \underline{\mathfrak{A}^*}\mathfrak{V}$.

Proof $\underline{\mathfrak{V}^*} \leq \underline{\phi_{\mathfrak{V}}} = \overset{*}{\phi}_{\mathfrak{V}}$ by 3·14 and $\overset{*}{\phi}_{\mathfrak{V}} \leq \overset{*}{\phi}_1\mathfrak{V}$ by lemma 1 of [7] . If x is a non-trivial element of a $\overset{*}{\phi}_1$ -group G then there exists a normal subgroup N of G with $x \notin N$ such that $[x,G] \leq N$. Thus $x \in x^G N - N$ and $x^G N/N \leq \zeta_1(G/N) = \mathfrak{A}^*(G/N)$. Therefore $\overset{*}{\phi}_1 \leq \underline{\mathfrak{A}^*}$ so $\overset{*}{\phi}_1\mathfrak{V} \leq \underline{\mathfrak{A}^*}\mathfrak{V}$, as required.

It is an immediate consequence of the definitions that $\hat{P}_n\mathfrak{V} \leq C_n\mathfrak{V}$. However, $\hat{P}\mathfrak{V} \nleq C_n\mathfrak{V} = C_F\mathfrak{V}$ in general. This is demonstrated by the construction of non-abelian simple SN-groups by P. Hall [14] . As is noted in [8] , these are clearly not $C_F\mathfrak{A}$ -groups. We will show, in lemma 6·8 (iv) that $\overset{*}{\phi}_1 \nleq SN^*$. Thus $\overset{*}{\phi}_{\mathfrak{A}} \nleq SN^* = \acute{P}\mathfrak{A}$. Thus we have that $\overset{*}{\phi}_{\mathfrak{V}} \nleq \acute{P}\mathfrak{V}$ in general.

We remark finally that it is easy to prove that $\hat{\phi}_{\mathfrak{V}} \leq \hat{P}_n(\mathfrak{V}(1))$ and that similar results hold for the classes $\acute{\phi}_{\mathfrak{V}}$, $\grave{\phi}_{\mathfrak{V}}$ and $\overset{\circ}{\phi}_{\mathfrak{V}}$.

IV The Class of Groups $[\mathfrak{V}]$

Here we offer a generalization of the class of residually commutable groups introduced in [9] and studied, amongst other places, in [1] .

Definition Let $\mathfrak{V}$ be a non-trivial variety determined by a set of words W. We define the class $[\mathfrak{V}]$ by saying that $G \in [\mathfrak{V}]$ if for any $w(x_1,\dots,x_n) \in W$ and any $g_1,\dots,g_n \in G$, not all trivial, there exists a normal subgroup N of G such that $w(g_1,\dots,g_n) \in N$ but at least one $g_i \notin N$.

It is easy to see that the class of residually commutable groups is just the class $[\mathfrak{A}]$. It is also clear that $G \in [\mathfrak{V}]$ if and only if given $w(x_1,\dots,x_n) \in W$ and $g_1,\dots,g_n \in G$, not all trivial, then $g_i \notin w(g_1,\dots,g_n)^G$ for at least one i, $1 \leq i \leq n$.

We now establish some closure properties of $[\mathfrak{V}]$.

Theorem 4·1 $[\mathfrak{V}] = \langle S,R,L\rangle [\mathfrak{V}]$.

Proof Suppose H is a subgroup of a $[\mathfrak{V}]$-group G and let $w(x_1,\dots,x_n) \in W$ and $h_1,\dots,h_n \in H$ such that at least one h_i is non-trivial. Then for some i, $1 \leq i \leq n$, we have $h_i \notin w(h_1,\dots,h_n)^G$ and, a fortiori, $h_i \notin w(h_1,\dots,h_n)^H$. Therefore $H \in [\mathfrak{V}]$ and so $[\mathfrak{V}] = S[\mathfrak{V}]$.

Let $G \in R[\mathfrak{V}]$, $w(x_1,\dots,x_n) \in W$ and $g_1,\dots,g_n \in G$, not all trivial. Suppose $g_i \neq 1$. Then there exists a normal subgroup N of G such that $g_i \notin N$ and $G/N \in [\mathfrak{V}]$. So g_iN is a non-trivial element of G/N. Therefore there exists a normal subgroup M of G with $N \leq M$, $w(g_1N, \ldots, g_nN) \in M/N$ and $g_jN \notin M/N$ for some j, $1 \leq j \leq n$. But $w(g_1N, \ldots, g_nN) = w(g_1, \ldots, g_n)N$ and so $w(g_1, \ldots, g_n) \in M$. But $g_j \notin M$ and so $G \in [\mathfrak{V}]$. Therefore $[\mathfrak{V}] = R[\mathfrak{V}]$.

To prove the local closure of $[\mathfrak{V}]$ we assume that $G \notin [\mathfrak{V}]$. Then there exists a word $w(x_1, \ldots, x_n)$ in W and elements $g_1, \ldots, g_n$ of G, not all trivial, such that $g_i \in w(g_1, \ldots, g_n)^G$ for all $i = 1, \ldots, n$. Suppose that $g_i = w(g_1, \ldots, g_n)^{k_{i1}} \ldots w(g_1, \ldots g_n)^{k_{im}}$ where $k_{i1}, \ldots, k_{im} \in G$.

Let $K_i = \{k_{i1}, \ldots, k_{im}\}$ and $M = \langle g_1, \ldots, g_n, K_1, \ldots, K_n \rangle$. Then M is finitely generated and $g_i \in w(g_1, \ldots, g_n)^M$ for each i, $1 \leq i \leq n$. Therefore $M \notin [\mathfrak{V}]$ and so $G \notin L[\mathfrak{V}]$.

Thus we have proved that $[\mathfrak{V}] = L[\mathfrak{V}]$.

In general, $[\mathfrak{V}]$ is not Q-closed, as is shown by $[\mathfrak{A}]$ which contains the class of residually nilpotent groups and hence all free groups. However, we can prove that $[\mathfrak{V}] = H_R[\mathfrak{V}]$ when $\mathfrak{V}$ is determined by a single word by use of the following proposition, the proof of which is very easy and is omitted.

<u>Proposition 4·2</u> If $\mathfrak{V}$ is a variety determined by a single word $w(x_1, \ldots, x_n)$ then a group G is a $[\mathfrak{V}]$-group

if and only if given $g_1,\ldots,g_n \in G$, not all trivial,

$\{g_1,\ldots g_n\} \not\subseteq w(g_1,\ldots,g_n)^G$.

Theorem 4·3 If $\mathfrak{V}$ is a variety determined by a single word $w(x_1,\ldots,x_n)$, then $[\mathfrak{V}] = H_R[\mathfrak{V}]$.

Proof It is easy to see that if $N \triangleleft G$ then

$(w(g_1N,\ldots,g_nN))^{G/N} = w(g_1,\ldots,g_n)^G N/N$ and that if

$G = \underset{\alpha\varepsilon A}{Cr}\ G_\alpha$ then $(w(g_1,\ldots,g_n))^G \leqslant \underset{\alpha\varepsilon A}{Cr}\ (w(g_1(\alpha),\ldots g_n(\alpha)))^{G_\alpha}$.

By letting $F = \{g_1,\ldots,g_n\}$ and $S(F) = (w(g_1,\ldots,g_n))^G$ we obtain the result by 3·7.

Corollary 4·4 If $\mathfrak{V}$ is a variety determined by a single word then $[\mathfrak{V}] = L_F[\mathfrak{V}]$.

We now give an indication of the size of $[\mathfrak{V}]$. The following theorem shows that, if $\mathfrak{V}$ satisfies (X), then $[\mathfrak{V}]$ contains all the classes under consideration in the last section. This result, together with a remark in the previous section, also shows that $\hat{p}_n\mathfrak{V} \leqslant [\mathfrak{V}]$, a generalization of the well-known result of Ayoub [9] that SI-groups are residually commutable.

Theorem 4·5 $C_F\mathfrak{V} \leqslant [\mathfrak{V}]$

Proof Let W be the set of words determining $\mathfrak{V}$. Let $G \varepsilon C_F\mathfrak{V}$, $w(x_1,\ldots,x_n) \varepsilon W$ and $g_1,\ldots,g_n \varepsilon G$, not all trivial.

We may assume that $w(g_1,\ldots,g_n) \neq 1$. Then there exist subgroups K and L of G with $K \lhd L \lhd G$ such that $w(g_1,\ldots,g_n) \in L - K$ and $L/K \in \mathfrak{V}$. Suppose that $g_1,\ldots,g_n \in L$. Then $w(g_1,\ldots,g_n) \leq \mathfrak{V}(L) \leq K$, which is a contradiction. Therefore at least one g_i does not belong to L and we have proved $G \in [\mathfrak{V}]$.

Hall, in Theorem A of [14], gives an example of an $\overline{SN}$-group G which is simple and non-abelian. G is not a residually commutable group. To prove this, we take non-trivial elements x and y of G and suppose $G \in [\mathfrak{A}]$. Then there exists a normal subgroup N of G such that $[x,y] \in N$ but at least one of x and y is not an element of N. But, since G is simple, N = 1 and so $[x,y] = 1$. This shows that G is abelian, which is not true. Therefore $G \notin [\mathfrak{A}]$. Thus $\overline{SN} \nleq [\mathfrak{A}]$ and so $SN \nleq [\mathfrak{A}]$. So we have proved that $\hat{p}\mathfrak{V} \nleq [\mathfrak{V}]$ in general.

In fact, the above example proves more. For suppose that $G \in [\mathfrak{A}(1)]$. Then for all elements x,y and z of G, not all trivial, there exists a normal subgroup N of G such that $[x,y,z] \in N$ and at least one of x,y and z does not belong to N. Thus $[x,y,z] = 1$ and so $G \in \mathfrak{N}_2$, which is a contradiction because G is simple and non-abelian. Therefore $G \notin [\mathfrak{A}(1)]$ and we have :

<u>Lemma 4·6</u> $\hat{p}\mathfrak{V} \nleq [\mathfrak{V}(1)]$ in general.

On the other hand, we are able to see the effect of the minimal condition on normal subgroups by means of the following lemma.

Lemma 4·7 If M is a minimal normal subgroup of a $[\mathfrak{V}]$-group G then $M \in \mathfrak{V}$ and $G/M \in [\mathfrak{V}]$.

Proof Suppose M is a minimal normal subgroup of a $[\mathfrak{V}]$-group G and let $w(x_1,\ldots,x_n) \in W$ and $g_1,\ldots,g_n \in M$, not all trivial. Suppose $w(g_1,\ldots,g_n) \neq 1$. Now there exists a normal subgroup N of G such that $w(g_1,\ldots,g_n) \in N$ and $g_i \notin N$ for some i, $1 \leqslant i \leqslant n$. Then $w(g_1,\ldots,g_n) \in M \cap N$ and so, by the minimality of M, $M \leqslant N$. Therefore $g_i \in N$ for each i, $1 \leqslant i \leqslant n$, which is a contradiction. Therefore $w(g_1,\ldots,g_n) = 1$ for all $w(x_1,\ldots,x_n) \in W$ for all $g_1,\ldots,g_n \in M$. That is, $M \in \mathfrak{V}$.

To prove the second part of the lemma we again suppose M is a minimal normal subgroup of $G \in [\mathfrak{V}]$ and let $w(x_1,\ldots,x_n) \in W$. We choose $g_1M,\ldots,g_nM \in G/M$, not all trivial. Then there exists a normal subgroup N of G such that $w(g_1,\ldots,g_n) \in N$ but $g_i \notin N$ for at least one i, $1 \leqslant i \leqslant n$. If $M \leqslant N$ then $w(g_1M,\ldots,g_nM) = w(g_1,\ldots,g_n)M \in N/M$ and $g_iM \notin N/M$ so we have finished.

We may suppose then that $M \cap N = 1$. If $g_i \notin NM$ for some i, $1 \leqslant i \leqslant n$, then since $w(g_1M,\ldots,g_nM) \in NM/M$ again we have finished. Thus we may assume that $g_i \in NM$ for all i, $1 \leqslant i \leqslant n$.

For each i, $1 \leqslant i \leqslant n$, let $g_i = n_im_i$ where $n_i \in N$ and $m_i \in M$. Then at least one n_i is not trivial because $g_i \notin M$ for at least one g_i. Therefore there exists a normal subgroup

L of G such that $w(n_1,\ldots,n_n) \in L$ but $n_i \notin L$ for at least one i, $1 \leq i \leq n$. Let $P = N \cap L$. Choose an i for which $n_i \notin L$ and suppose $g_i \in PM$. Then $g_i = pm$, say, where $p \in P$ and $m \in M$. Then $n_i m_i = pm$ so $p^{-1}n_i = mm_i^{-1} \in N \cap M = 1$. Therefore $n_i = p \in L$, which is a contradiction.

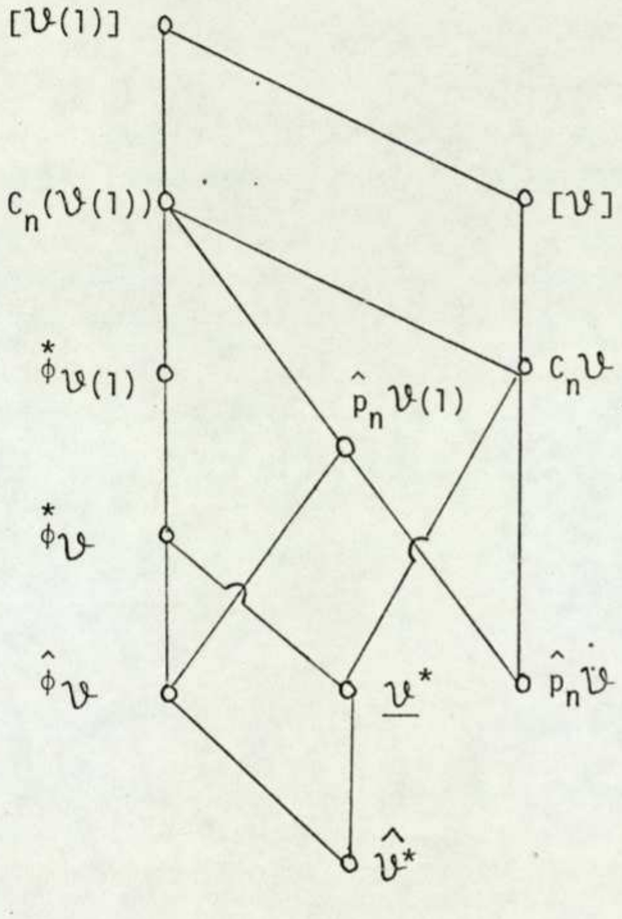
Thus for some i, $1 \leq i \leq n$, we have $g_i \notin PM$. Now $PM/M \triangleleft G/M$ and $w(g_1,\ldots,g_n) = w(n_1,\ldots,n_n)w(m_1,\ldots,m_n)$ because $[M,N] = 1$. Therefore $w(g_1M,\ldots,g_nM) = w(n_1,\ldots,n_n)M \in PM/M$, while one $g_iM \notin PM/M$, proving that $G/M \in [\mathfrak{V}]$.

The following corollary, which is known when $\mathfrak{V} = \mathfrak{A}$, is an immediate consequence of 4·7.

<u>Corollary 4·8</u> $[\mathfrak{V}] \cap \check{\mathfrak{M}}_n \leq p'_n\mathfrak{V}$.

Some of the classes discussed in Chapters 3 and 4 have been proved to have the inclusions shown in Fig. 3 for any variety $\mathfrak{V}$.

When $\mathfrak{V}$ satisfies (X) we have the simpler diagram shown in Fig. 4.



$[\mathcal{V}(1)]$
$C_n(\mathcal{V}(1))$
$[\mathcal{V}]$
$\overset{*}{\phi}\mathcal{V}(1)$
$C_n\mathcal{V}$
$\hat{p}_n\mathcal{V}(1)$
$\overset{*}{\phi}\mathcal{V}$
$\hat{\phi}\mathcal{V}$
$\underline{\mathcal{V}}^*$
$\hat{p}_n\mathcal{V}$
$\hat{\mathcal{V}}^*$

Fig. 3

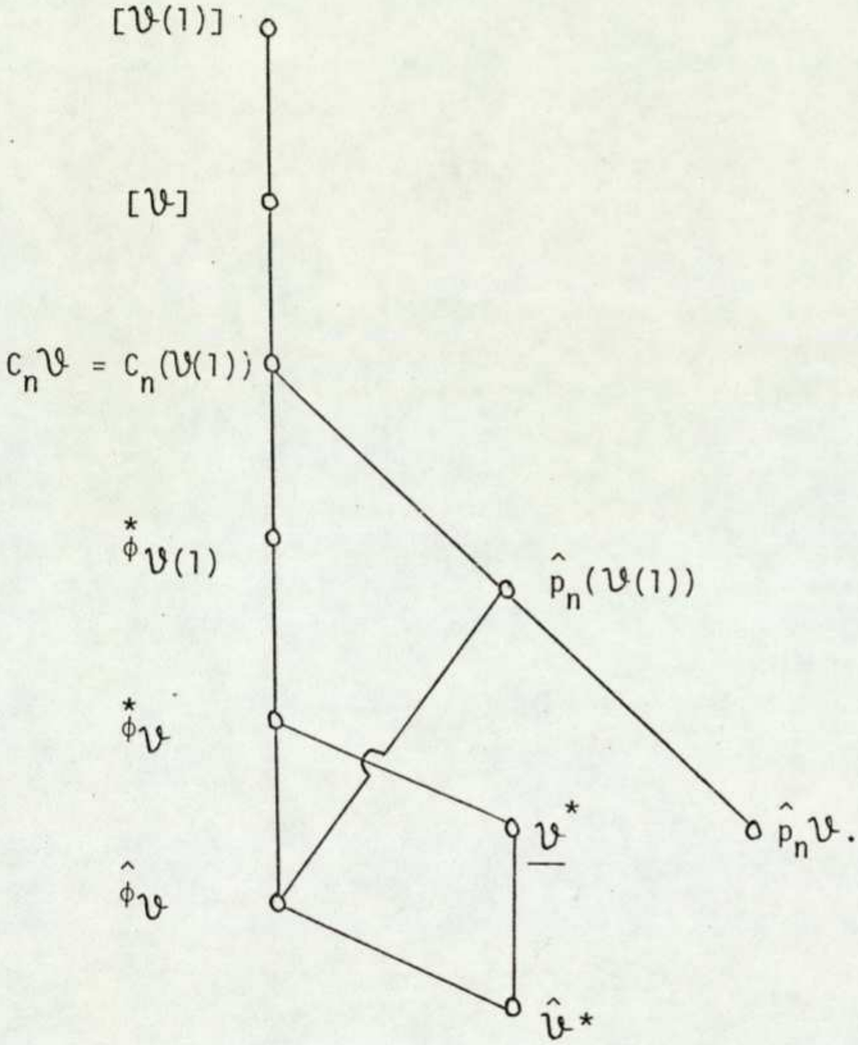

$[\mathfrak{V}(1)]$
$[\mathfrak{V}]$
$C_n\mathfrak{V} = C_n(\mathfrak{V}(1))$
$\overset{*}{\phi}\mathfrak{V}(1)$
$\hat{p}_n(\mathfrak{V}(1))$
$\overset{*}{\phi}\mathfrak{V}$
$\underline{\mathfrak{V}}^*$
$\hat{p}_n\mathfrak{V}.$
$\hat{\phi}\mathfrak{V}$
$\hat{\mathfrak{V}}^*$

Fig. 4

As mentioned in [7] a group G is a $\overset{*}{\phi}_{\mathfrak{V}}$-group if and only if for each non-trivial element x of G, we have $x \notin [x^G, \mathfrak{V}(G)]$. For varieties $\mathfrak{V}_1, \ldots, \mathfrak{V}_n$ we may define a new class $(\mathfrak{V}_1, \ldots, \mathfrak{V}_n)$ in the following way.

<u>Definition</u> A group G is a $(\mathfrak{V}_1, \ldots, \mathfrak{V}_n)$-group if, for each non-trivial element x of G, $x \notin [x^G, \mathfrak{V}_1(G), \ldots, \mathfrak{V}_n(G)]$.

For example $(\mathfrak{V}) = \overset{*}{\phi}_{\mathfrak{V}}$. Another example is given by our next lemma.

<u>Lemma 4·9</u> $\qquad (\overbrace{1,\ldots,1}^{C}) = \underline{\mathfrak{N}^*_C}$

<u>Proof</u> Suppose $G \in \underline{\mathfrak{N}^*_C}$ and let x be a non-trivial element of G. Then there exists a normal subgroup N of G such that $x \notin N$ and $x^G N/N \leq \mathfrak{N}^*_C(G/N) = \zeta_C(G/N)$. Therefore $xN \in \zeta_C(G/N)$.

It may be seen by induction that $x \in \zeta_C(G)$ if and only if $[x,\overbrace{G,\ldots,G}^{C}] = 1$ for any group G. Thus we have $[xN,\overbrace{G/N,\ldots,G/N}^{C}] = 1$ so that $[x,\overbrace{G,\ldots G}^{C}] \leq N$.
Therefore $x \notin [x^G,\overbrace{G,\ldots,G}^{C}]$.

On the other hand, suppose $G \in (\overbrace{1,\ldots,1}^{C})$ and let x be a non-trivial element of G and $N = [x^G,\overbrace{G,\ldots,G}^{C}]$. Then $x \notin N$. Also $[xN,\overbrace{G/N\ldots,G/N}^{C}] = [x,\overbrace{G,\ldots,G}^{C}]N/N = 1$.
Therefore $xN \in \zeta_C(G/N) = \mathfrak{N}^*_C(G/N)$, as required.

The example $(\mathfrak{V}) = \overset{*}{\phi}_{\mathfrak{V}}$ shows that $(\mathfrak{V}_1,\ldots,\mathfrak{V}_n)$ is not Q-closed in general. However, we do have

<u>Lemma 4·10</u> $(\mathfrak{V}_1,\ldots,\mathfrak{V}_n) = \langle S,R\rangle(\mathfrak{V}_1,\ldots,\mathfrak{V}_n)$.

The proof of this is very easy and is omitted.

We may prove that $(\mathfrak{V}_1,\ldots,\mathfrak{V}_n)$ is L-closed as follows. This proof is very similar to that of Theorem 10 of [7] .

<u>Theorem 4·11</u> $(\mathfrak{V}_1,\ldots,\mathfrak{V}_n) = L(\mathfrak{V}_1,\ldots,\mathfrak{V}_n)$ for any varieties $\mathfrak{V}_1,\ldots,\mathfrak{V}_n$.

<u>Proof</u> Let $G \in L(\mathfrak{V}_1,\ldots,\mathfrak{V}_n)$ and let H be a finitely generated subgroup of G. Then if x is a non-trivial element of H we have $x \notin [x^H, \mathfrak{V}_1(H),\ldots,\mathfrak{V}_n(H)]$. Let I_x be the set of all finite subsets of G containing x, and if $S \in I_x$ let J(S) be the set of all finitely generated subgroups of G containing S.

For $S \in I_x$ let $K_x(S) = \bigcap_{H\in J(S)} x^H$ and let $L_x^{(i)}(S) = \bigcap_{H\in J(S)} \mathfrak{V}_i(H)$. Also let $K_x = \bigcup_{S\in I_x} K_x(S)$ and

$L_x^{(i)} = \bigcup_{S\in I_x} L_x^{(i)}(S)$, for i = 1,...,n.

Then we may prove :

(i) $S_1, S_2 \in I_x$ with $S_1 \subseteq S_2$ implies $K_x(S_1) \leq K_x(S_2)$ and

$L_x^{(i)}(S_1) \leq L_x^{(i)}(S_2)$.

(ii) K_x and each $L_x^{(i)}$ are subgroups of G.

(iii) Each $L_x^{(i)}$ is a normal subgroup of G.

(iv) $G/L_x^{(i)} \in \mathfrak{V}_i$.

(v) $x^G \leq K_x$.

Now suppose that $x \in [x^G, \mathfrak{V}_1(G), \ldots, \mathfrak{V}_n(G)]$. By (iv) and (v) we have $x \in [K_x, L_x^{(1)}, \ldots, L_x^{(n)}]$. Then there exist finite subsets A of K_x and B_i of $L_x^{(i)}$ $(i = 1,\ldots,n)$ such that $x \in [A, B_1, \ldots, B_n]$.

We can obtain an $S \in I_x$ such that $A \subseteq K_x(S)$ and $B_i \subseteq L_x^{(i)}(S)$, $i = 1,\ldots,n$ in the same way as in Theorem 10 of [7] . Therefore if $H \in J(S)$ we have $[A, B_1, \ldots, B_n] \subseteq [x^H, \mathfrak{V}_1(H), \ldots, \mathfrak{V}_n(H)]$, which yields a contradiction. Therefore $G \in (\mathfrak{V}_1, \ldots, \mathfrak{V}_n)$.

V The Class of Groups $\mathfrak{H}(1)$

As we have already said, an $\mathfrak{H}(1)$ group is a group G such that $G = H_1(G:\mathfrak{H})$. It follows immediately that a group G is an $\mathfrak{H}(1)$-group if and only if for each element x of G there exists an integer n, depending on x, such that $[x,\gamma_n(G)] = 1$. We see from 2·16 (ii) that $\mathfrak{H}(1) \leq L\mathfrak{H}$. In fact, $\mathfrak{H}(1)$ is a class of hypercentral groups, as is shown by

<u>Theorem 5·1</u> $ZA_\omega \leq \mathfrak{H}(1) \leq ZA_{\omega+1}$

<u>Proof</u> Let $G \in ZA_\omega$ and let $x \in G$. Since $G = \zeta_\omega(G) = \bigcup_{n<\omega} \zeta_n(G)$, there exists an integer n such that $x \in \zeta_n(G)$. But it is well-known that, for all groups G and for all integers n, $[\zeta_n(G), \gamma_n(G)] = 1$. Therefore $[x, \gamma_n(G)] = 1$ and we have proved that $G \in \mathfrak{H}(1)$.

To prove that $\mathfrak{H}(1) \leq ZA_{\omega+1}$, we let $G \in \mathfrak{H}(1)$ and prove that $G' \leq \zeta_\omega(G)$. Let x and y be elements of G. Then there exist integers h and k such that $[x,\gamma_{h+1}(G)] = 1$ and $[y,\gamma_{k+1}(G)] = 1$. If $h = 0$ or $k = 0$ then $x \in \zeta_1(G)$ or $y \in \zeta_1(G)$ respectively. In either case $[x,y] = 1$, and the result is true. Therefore we may assume that $h + k \geq 1$.

We now need lemma 4·26of [1] which states that if G is any group, $H = C_G(\gamma_{h+1}(G))$ and $K = C_G(\gamma_{k+1}(G))$ where h and k are integers with $h + k > 0$, then $[H,K] \leq \zeta_{h+k-1}(G)$. In the notation of this lemma, $x \in H$ and $y \in K$ and so $[x,y] \in \zeta_{h+k-1}(G)$.

Thus we have proved that $G' \leqslant \zeta_{\omega}(G)$. Therefore $G = \zeta_{\omega+1}(G)$, as required.

We will now demonstrate some more inclusions between $\mathfrak{H}(1)$ and some classes of generalized nilpotent groups. Here, N_2 denotes the class of groups in which every subgroup is descendant.

Theorem 5·2 (i) All $\mathfrak{H}(1)$-groups are Fitting groups

(ii) $\mathfrak{H}(1) \leqslant ZD_{\omega+1}$

(iii) $\mathfrak{H}(1) \leqslant N_2$

Proof (i) If $G \in \mathfrak{H}(1)$ then by 2·17 $x^G \in \mathfrak{H}$ for every element x of G. Therefore G is a Fitting group.

(ii) If x is an element of an $\mathfrak{H}(1)$-group G then there exists an integer n such that $[x,\gamma_n(G)] = 1$. Therefore $[x,\gamma_{\omega}(G)] = 1$ for all $x \in G$. Thus $\gamma_{\omega+1}(G) = [G,\gamma_{\omega}(G)] = 1$.

(iii) Let H be a subgroup of an $\mathfrak{H}(1)$-group G. For integers $n = 1,2,\ldots$ let $H_n = \gamma_n(G)H$ and let $I = \bigcap_{n=1}^{\infty} H_n$. Then $G = H_1 \geqslant H_2 \geqslant \ldots \geqslant I$ is a descending series. For, let $x \in \gamma_{n+1}(G)$ and $y \in \gamma_n(G)$ and let h and h_1 be elements of H. Then $[xh,yh_1] = [x,h_1]^h [x,y]^{h_1 h} [h,h_1] [h,y]^{h_1}$ $\in \gamma_{n+1}(G)H = H_{n+1}$. Therefore $[H_{n+1},H_n] \leqslant H_{n+1}$ which shows that $H_{n+1} \triangleleft H_n$ for $n = 1,2,\ldots$

Also $H \triangleleft I$. For, let $h \in H$ and $x \in I$. Now there exists an integer k such that $[h,\gamma_k(G)] = 1$. Since $x \in I$ we have $x \in H_k$ and so $x = g_k h_k$, say, where $g_k \in \gamma_k(G)$ and $h_k \in H$. Therefore $h^x = h^{g_k h_k} = (h[h,g_k])^{h_k} = h^{h_k} \in H$.

Behaviour at the only limit ordinal is correct by the definition of I and therefore there exists a descending series

$$G = H_1 \geq H_2 \geq \ldots \geq I \geq H$$

of length less than or equal to $\omega+1$, from G down to H. Therefore H is descendant in G so $G \in N_2$.

The connections between $\mathcal{H}(1)$ and some other classes of generalized nilpotent groups are shown in Fig. 5.

Various questions seem to remain unanswered about the class N_2 ; for instance, is every N_2-group locally nilpotent ? is every N_2-group a ZD-group ? And is the class of residually nilpotent groups included in N_2 ? However, we do have the following result.

<u>Theorem 5·3</u> (i) There exists a $ZA_{\omega+1}$-group which is not an N_2-group.

(ii) There exists a Fitting group which is not an N_2-group.

<u>Proof</u> (i) Let G be the locally dihedral 2-group. That is, $G = \langle A,t\rangle$ where A is a 2^∞-group generated by a_0 (=1), $a_1, a_2, \ldots$ and $a_i^t = a_i^{-1}$ for all integers i and $t^2 = 1$. It may easily be seen that $\zeta_n(G) = \langle a_1, \ldots, a_n\rangle$. Therefore $\zeta_\omega(G) = A$ and so $G = \zeta_{\omega+1}(G)$.

Now $a_i = tt^{a_{i+1}}$ and so $G = t^G$. But if $\langle t\rangle$ is descendant in G then there exists a proper normal subgroup of G containing $\langle t\rangle$, which is a contradiction. Therefore $G \notin N_2$.

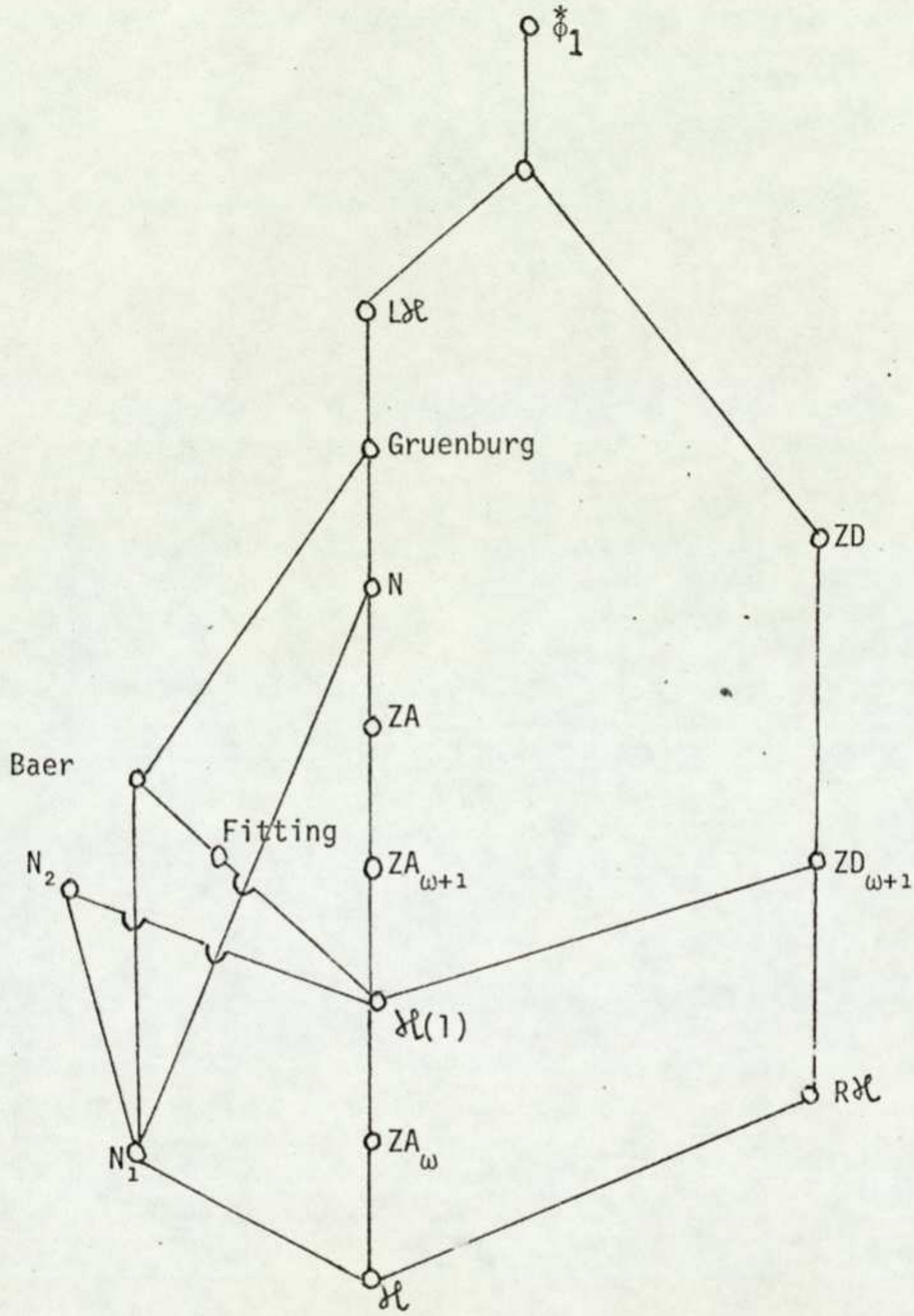

$\overset{*}{\phi}_1$
$L\mathcal{H}$
Gruenburg
ZD
N
ZA
Baer
Fitting
N_2
$ZA_{\omega+1}$
$ZD_{\omega+1}$
$\mathcal{H}(1)$
$R\mathcal{H}$
N_1
ZA_ω
$\mathcal{H}$

Fig. 5

(ii) The group of Zassenhaus which is studied in [15] , beginning on page 86, is a Fitting group (ibid page 114) but has a non-descendant subgroup (ibid page 89).

Although it is well-known that $\mathfrak{H} < ZA_\omega$, so that $\mathfrak{H} < \mathfrak{H}(1)$, we have been unable to decide whether there exists an $\mathfrak{H}(1)$-group which is not a ZA_ω-group.

<u>Theorem 5·4</u> Apart from the case mentioned above, there are no connections between $\mathfrak{H}(1)$ and other classes in the diagram other than those shown.

<u>Proof</u> (i) We construct a group which is residually nilpotent and a Fitting group but is not a $\acute{\phi}_{\mathfrak{H}}$-group and, in particular, is not an $\mathfrak{H}(1)$-group :

Let M be the Mclain group $M(\mathbb{Z}_{>0},F)$ for any field F. Although M is not characteristically simple, it does have trivial centre because $\mathbb{Z}_{>0}$ has no last element. While the proof of Theorem 6·22 of [1] seems to be false, we prove M to be residually nilpotent in the following way.

We have $\gamma_1(M) = M = \langle 1 + a e_{ij} \; ; a \in F, \; j - i \geqslant 1 \rangle$. Suppose $\gamma_r(M)$ is contained in the subgroup of M generated by all finite sums of the form

$$1 + a_1 e_{i_1 j_1} + a_2 e_{i_2 j_2} + \ldots + a_p e_{i_p j_p}$$

where $a_n \in F$ and $j_n - i_n \geqslant r$ for each $n = 1,\ldots,p$. Then

$$\gamma_{r+1}(M) = [\gamma_r(M), M]$$

$$\leqslant [\{1 + a_1 e_{i_1 j_1} + \ldots + a_p e_{i_p j_p} \; ; \; j_n - i_n \geq r\} , \{1 + a e_{ij} ; j - i \geq 1\}]^M.$$

Let $g = 1 + a_1 e_{i_1 j_1} + \ldots + a_p e_{i_p j_p}$. Then

$$[g, 1 + ae_{ij}] = g^{-1}(1 - ae_{ij}) g (1 + ae_{ij})$$

$$= g^{-1}(g - ae_{ij} g)(1 + ae_{ij})$$

$$= g^{-1}(g + ag\, e_{ij} - a\, e_{ij}\, g)$$

because $e_{ij}\, g\, e_{ij} = 0$. Thus

$$[g, 1 + ae_{ij}] = 1 + g^{-1}(ag\, e_{ij} - ae_{ij}\, g)$$
$$= 1 + ag^{-1} f$$

say, where $f = g\, e_{ij} - e_{ij}\, g$. Now

$$f = (1 + \sum_{k=1}^{p} a_k e_{i_k j_k}) e_{ij} - e_{ij}(1 + \sum_{k=1}^{p} a_k e_{i_k j_k})$$

$$= \sum_{k=1}^{p} a_k e_{i_k j_k} e_{ij} - \sum_{k=1}^{p} a_k e_{ij} e_{i_k j_k} .$$

Thus f may be represented as a sum of the form

$$\sum_{\ell=1}^{m} b_\ell e_{i_\ell j_\ell}$$

where $j_\ell - i_\ell \geq r+1$.

But g^{-1} belongs to $\gamma_r(M)$ and so is of the form

$$1 + \sum_{k=1}^{n} d_k e_{s_k t_k} \quad \text{where} \quad t_k - s_k \geq r \quad \text{for each } k = 1,\dots,n.$$

Therefore

$$g^{-1}f = \left(1 + \sum_{k=1}^{n} d_k e_{s_k t_k}\right) \sum_{\ell=1}^{m} b_\ell e_{i_\ell j_\ell}$$

$$= \sum_{\ell=1}^{m} b_\ell e_{i_\ell j_\ell} + \sum_{k=1}^{n} \sum_{\ell=1}^{m} d_k b_\ell e_{s_k t_k} e_{i_\ell j_\ell} .$$

Thus $ag^{-1}f$ may also be represented as a sum of the form

$$\sum_{\ell=1}^{m} b_\ell e_{i_\ell j_\ell}$$

where each $j_\ell - i_\ell \geq r+1$, for some m. Therefore

$$[g, 1 + ae_{ij}] = 1 + \sum_{\ell=1}^{\bar{m}} b_\ell e_{i_\ell j_\ell}$$

for some integer m. It is easy to see that conjugation of the left hand side of this equation will leave the right hand side in the same form .

We have proved that $\gamma_{r+1}(M)$ is contained in the subgroup of M generated by all finite sums of the form

$$1 + a_1 e_{i_1 j_1} + \dots + a_m e_{i_m j_m}$$

where each $a_n \in F$ and $j_n - i_n \geq r+1$. It follows that $\gamma_\omega(M)$ is contained in the subgroup of M generated by all finite sums of this form where $a_n \in F$ and $j_n - i_n \geq \lambda + 1$ for all $\lambda < \omega$. This is just the trivial subgroup and so we have proved that $M \in R\mathfrak{H}$.

Now, $1 + ae_{i,i+1} \in \gamma_1(M)$ for all $a \in F$ for all integers i.

Suppose $1 + a\, e_{i,i+r} \in \gamma_r(M)$ for all $a \in F$ for all integers i.

Then $1 + a\, e_{i,i+r+1} = [1 + e_{i,i+1}, 1 + a\, e_{i+1,i+r+1}] \in \gamma_{r+1}(M)$.

Let $1 + a\, e_{ij} \in M$ with $a \neq 0$. Then for all positive integers n, $1 + a\, e_{j,j+n} \in \gamma_n(M)$. Now $[1 + a\, e_{ij}, 1 + a\, e_{j,j+n}] =$

$1 + a^2 e_{i,j+n} \neq 1$ and so $[1 + ae_{ij}, \gamma_n(M)] \neq 1$ for all positive integers n. Therefore $1 + a\, e_{ij} \notin H_1(G:\mathfrak{H})$ for all $a \in F$ for all integers i and j. Thus $H_1(G:\mathfrak{H}) = 1$ by theorem 6.21 (iii) of [1] . Therefore $G \notin \phi'_{\mathfrak{H}}$.

This, of course, also proves that $\mathfrak{H}(1) < ZD_{W+1}$. Since all Mclain groups are Fitting groups, we have also proved that $\mathfrak{H}(1)$ is strictly contained in the class of Fitting groups.

(ii) There exists a $ZA_{\omega+1}$-group which is not a Baer group, for example the locally dihedral 2-group. This shows that $\mathfrak{H}(1) < ZA_{\omega+1}$.

(iii) There exists an N_1-group which is not hypercentral. For example, Heineken and Mohamed [16] have constructed a group which has all its subgroups subnormal but has trivial centre. This shows that $N_1 \nleq \mathfrak{H}(1)$.

(iv) Glušcov [17] proves that the group $G = \mathop{Dr}\limits_{n=2}^{\infty} M_n(P)$ where $M_n(P)$ is the group of all n × n unitriangular matrices over any field P, is a ZA_ω-group and is residually nilpotent. He proceeds to show that some factor group of G is a ZA_ω-group and a $ZD_{\omega+1}$-group but is not residually nilpotent. Thus $\mathfrak{H}(1) \nleq R\mathfrak{H}$.

(v) We construct a ZA_ω-group which is not an N_1-group as follows :

Let $A = \mathop{Dr}\limits_{n=1}^{\infty} C_n$ where $C_n = \langle a_n \rangle$ with $|a_1| = 2$, $|a_2| = 4$ and $|a_{2n-1}| = |a_{2n}| = 2^{n+1}$, for $n = 2,3,\ldots$. Define $t_i : A \to A$ by $a_i^{t_i} = a_i a_{i+1}^2$ and $a_i^{t_j} = a_i$ $(i \neq j)$ for $i = 1,2,\ldots$. Each t_i is an automorphism of A and $t_i = 1 + 2e_{i,i+1}$. We consider the group $T = \langle t_1, t_2, \ldots \rangle$.

Note that $[1 + \alpha e_{ij}, 1 + \beta e_{jk}] = 1 + \alpha\beta e_{ik}$ and that $[1 + \alpha e_{ij}, 1 + \beta e_{k\ell}] = 1$ if $j \neq k$ and $i \neq \ell$. In particular, $[1 + \alpha e_{rs}, t_s] = 1 + 2\alpha e_{r,s+1}$,

$[1 + \alpha e_{rs}, t_{r-1}] = 1 - 2\alpha e_{r-1,s}$ and $[1 + \alpha e_{rs}, t_n] = 1$

if $n \neq s$ and $n \neq r - 1$. Also $[t_r, t_{r+1}] = 1 + 4e_{r,r+2}$ and $[t_r, t_s] = 1$ if $s \neq r+1$ and $s \neq r-1$. Let $d_{\alpha\beta} = 1 + 2^{\beta-\alpha} e_{\alpha\beta}$ $(\beta > \alpha)$.

<u>Lemma 5·5</u> Suppose $\alpha \geq 1$. Then

(i) $d_{\alpha\beta} = 1$ if $\beta > 2\alpha + 1$

(ii) $d_{\alpha\beta} \neq 1$ if $\alpha < \beta \leq 2\alpha + 1$

Proof (i) Suppose β is even, say $\beta = 2k$. Then $|a_\beta| = 2^{k+1}$. We have $2k > 2\alpha + 1$ so $k - \alpha \geq 1$. Therefore $\beta - \alpha = 2k - \alpha \geq k + 1$. Thus

$$a_\beta^{2^{\beta-\alpha}} = a_\beta^{2^{k+1+r}} \quad \text{for some integer } r. \text{ But}$$

$$a_\beta^{2^{k+1+r}} = (a_\beta^{2^{k+1}})^{2^r} = 1 .$$

Suppose now that β is odd, say $\beta = 2k - 1$. Then again $|a_\beta| = 2^{k+1}$. We have $2k - 1 > 2\alpha + 1$ so $k - \alpha > 1$. Therefore $\beta - \alpha = 2k - 1 - \alpha > k$ and so $\beta - \alpha \geq k + 1$, as before.

(ii) Assuming now that $\alpha < \beta \leq 2\alpha + 1$, suppose β is even, say $\beta = 2k$. Then $|a_\beta| = 2^{k+1}$. We have $2k \leq 2\alpha + 1$ so that $k \leq \alpha$. Thus $k < \alpha + 1$ so that $\beta - k < \alpha + 1$, that is $k + 1 > \beta - \alpha$. Therefore $|a_\beta| > 2^{\beta-\alpha}$ and so $a_\alpha^{d_{\alpha\beta}} \neq 1$.

Suppose that β is odd, say $\beta = 2k + 1$. Then $|a_\beta| = 2k + 2$. We have $2k + 1 \leq 2\alpha + 1$ so $\alpha \geq k$. Thus $\alpha > k - 1$ so that $k + 1 < \alpha + 2$. Therefore $\beta - k < \alpha + 2$ so that $k + 2 > \beta - \alpha$. Therefore $a_\alpha^{d_{\alpha\beta}} \neq a_\alpha$, as required.

Observe now that $[d_{\alpha\beta}, t_\beta] = d_{\alpha,\beta+1}$, $[d_{\alpha\beta}, t_{\alpha-1}] = d^{-1}_{\alpha-1,\beta}$ and $[d_{\alpha\beta}, t_n] = 1$ if $n \neq \beta$ and $n \neq \alpha - 1$. We may now prove :

Lemma 5·6 If $1 \leqslant n \leqslant \alpha$ then $d_{\alpha,2\alpha-n+2} \in \zeta_n(T)$.

Proof Let $n = 1$ so that $d_{\alpha,2\alpha-n+2} = d_{\alpha,2\alpha+1}$. Now

$$[d_{\alpha,2\alpha+1}, t_{2\alpha+1}] = d_{\alpha,2\alpha+2} = 1$$

and

$$[d_{\alpha,2\alpha+1}, t_{\alpha-1}] = d_{\alpha-1,2\alpha-1} = 1,$$

both by 5·5(i), and $[d_{\alpha,2\alpha+1}, t_m] = 1$ for all other values of m. Therefore $d_{\alpha,2\alpha+1} \in \zeta_1(T)$.

Suppose the result is true for all integers 1,2,...,n-1.
Now

$$[d_{\alpha,2\alpha-n+2}, T] = < [d_{\alpha,2\alpha-n+2}, t_i] \; ; \; i = 1,2,... >^T$$

$$= < d_{\alpha-1,2\alpha-n+2}, d_{\alpha,2\alpha-n+3} >^T.$$

But

$$d_{\alpha-1,2\alpha-n+2} = d_{\alpha-1,2(\alpha-1)-(n-2)+2} \in \zeta_{n-2}(T)$$

if $n > 2$ and $d_{\alpha-1,2\alpha-n+2} = 1$ if $n \leqslant 2$ by 5·5 (i).
Also

$$d_{\alpha,2\alpha-n+3} = d_{\alpha,2\alpha-(n-1)+2} \in \zeta_{n-1}(T)$$

if $n > 1$ and $d_{\alpha,2\alpha-n+3} = 1$ if $n = 1$, by 5·5(i). Therefore $[d_{\alpha,2\alpha-n+2}, T] \leqslant \zeta_{n-1}(T)$ and so $d_{\alpha,2\alpha-n+2} \in \zeta_n(T)$.

This yields :

Corollary 5·7 $t_{n-1} \in \zeta_n(T)$ for $n = 2,3,\ldots$.

Proof For the case $n = 2$, we have to prove that $t_1 \in \zeta_2(T)$.

Now

$$[t_1,T] = \langle [t_1,t_i] \; ; \; i = 1,2,\ldots \rangle^T$$

$$= \langle [t_1,t_2] \rangle^T$$

$$= \langle 1 + 4e_{13} \rangle^T = \langle d_{13} \rangle^T .$$

Setting $\alpha = n = 1$, we have

$$d_{13} = d_{\alpha,2\alpha-n+2} \in \zeta_n(T) = \zeta_1(T).$$

Therefore $t_1 \in \zeta_2(T)$.

Now suppose that $n \geqslant 3$. We have

$$[t_{n-1},T] = \langle [t_{n-1}, t_i] \; ; \; i = 1,2,\ldots \rangle^T$$

$$= \langle d_{n-2,n} , d_{n-1,n+1} \rangle^T .$$

Now

$$d_{n-2,n} = d_{n-2,2(n-2)-(n-2)+2} \in \zeta_{n-2}(T)$$

and

$$d_{n-1,n+1} = d_{n-1,2(n-1)-(n-1)+2} \in \zeta_{n-1}(T).$$

Therefore $t_{n-1} \in \zeta_n(T)$.

Thus T is a ZA_ω-group. In order to prove that T is not an N_1-group, let $H = \langle d_{\alpha,\alpha+2} \; ; \; \alpha = 1,2,\ldots \rangle$.

We have :

Lemma 5·8 $d_{\alpha,\alpha+2n+1} \in \gamma TH^n$ for all α .

Proof We consider first the case n = 1. We have

$$[H,T] = \langle [d_{\alpha,\alpha+2}, t_i] \; ; \; i = 1,2,\ldots \rangle^T. \text{ But}$$

$$[d_{\alpha,\alpha+2}, t_i] = \begin{cases} d^{-1}_{\alpha-1,\alpha+2} & \text{if } i = \alpha - 1 \\ d_{\alpha,\alpha+3} & \text{if } i = \alpha + 2 \\ 1 & \text{otherwise} \end{cases} .$$

Therefore $[H,T] = \langle d_{\alpha,\alpha+3} \; ; \; \alpha = 1,2,\ldots \rangle^T$, as required.

Suppose the result is true for n - 1. If $\alpha < 2n$ then $d_{\alpha,\alpha+2n+1} = 1$ by 5·5 (i). Thus we may suppose that $\alpha \geq 2n$. Then $d_{\alpha,\alpha+2n+1} = [d_{\alpha,\alpha+2n-1}, d_{\alpha+2n-1,\alpha+2n+1}]$.

But $d_{\alpha,\alpha+2n-1} \in \gamma TH^{n-1}$ by hypothesis. Therefore

$d_{\alpha,\alpha+2n+1} \in [\gamma TH^{n-1}, H] = \gamma TH^n$.

Corollary 5·9 H is not a subnormal subgroup of T.

Proof Any element of H has the form $1 + k$ where k is a linear combination of e_{ij}'s. These e_{ij}'s are linearly independent and so k is unique. But for each of these e_{ij}'s, j-i is even. Therefore $d_{2n,4n+1} = 1 + 2^{2n+1} e_{2n,4n+1} \notin H$.

Thus $\gamma TH^n \not\leq H$ for all n. But it is well-known that, if $X \leq Y$, X is subnormal in Y if and only if $\gamma YX^i \leq X$ for some integer i.

This completes the proof of theorem 5·4.

We also record the following fact about $\mathfrak{H}(1)$-groups.

Lemma 5·10 $\quad \mathfrak{H}(1) \cap \mathfrak{N}\mathfrak{G} \leq \mathfrak{H}$.

Proof Let G be an $\mathfrak{H}(1)$-group and suppose that N is a normal nilpotent subgroup of G such that G/N is finitely generated. We proceed by induction on the nilpotency class of N.

Suppose N is abelian. Now $G/N \in \mathfrak{G} \cap \mathfrak{H}(1) \leq \mathfrak{H}$ and so there exists an integer k such that $\gamma_k(G) \leq N$. Also G = FN, say, where F has a finite number of generators $f_1, \ldots, f_n$. For each $i (1 \leq i \leq n)$ there exists a normal subgroup M_i of G such that $[f_i, M_i] = 1$ and $G/M_i \in \mathfrak{H}$. Let

$$M = \bigcap_{i=1}^{n} M_i \text{ . Then } [F,M] = 1 \text{ and } G/M \in SD_0\mathfrak{H} = \mathfrak{H} \text{ .}$$

Therefore there exists an integer ℓ such that $\gamma_\ell(G) \leq M$. Thus $[F, \gamma_\ell(G)] = 1$. Let $m = \max\{k, \ell\}$ and let $g \in \gamma_m(G)$. Then g centralizes F and, since N is abelian, g centralizes N. Therefore $g \in \zeta_1(G)$. We have proved that $\gamma_m(G) \leq \zeta_1(G)$ so $G \in \mathfrak{H}$. Therefore $\mathfrak{H}(1) \cap \mathfrak{A}\mathfrak{G} \leq \mathfrak{H}$.

Suppose that $\mathfrak{H}(1) \cap \mathfrak{N}_c\mathfrak{G} \leq \mathfrak{H}$ and let $N \in \mathfrak{N}_{c+1}$. Then $N/\zeta_1(N) \in \mathfrak{N}_c$ so $G/\zeta_1(N) \in \mathfrak{H}$ by hypothesis. Therefore there exists an integer k such that $\gamma_k(G) \leq \zeta_1(N)$. Also G = FN , say , where F is finitely generated. As above, there exists an integer ℓ such that $[F, \gamma_\ell(G)] = 1$. Let $m = \max\{k, \ell\}$ and let $g \in \gamma_m(G)$. Then g centralizes F and, since $\gamma_m(G) \leq \zeta_1(N)$, g centralizes N. Therefore $g \in \zeta_1(G)$. Thus $G/\zeta_1(G) \in \mathfrak{H}$ and so $G \in \mathfrak{H}$, as required.

We observed earlier that $\mathfrak{H} < \mathfrak{H}(1)$. However, the relationship $\mathfrak{X} < \mathfrak{X}(1)$ does not hold for all classes $\mathfrak{X}$ of generalized nilpotent groups, as is shown by the following lemma. The proof of the lemma is easy and so is omitted . We note that the classes of Baer and Gruenberg groups are just $N\mathfrak{A}$ and $\acute{N}\mathfrak{A}$, respectively.

<u>Lemma 5·11</u> Let F denote the class of Fitting groups (temporary notation). Then $F = F(1)$, $N\mathfrak{A} = (N\mathfrak{A})(1)$, $\acute{N}\mathfrak{A} = (\acute{N}\mathfrak{A})(1)$, $ZA = ZA(1)$ and $L\mathfrak{H} = (L\mathfrak{H})(1)$.

We note that $\mathfrak{H}(2)$ is not a class of generalized nilpotent groups, as is shown by the symmetric group of degree 3, S_3. This group is metabelian so, by 2·13, $S_3 \in \mathfrak{A}(2) \leq \mathfrak{H}(2)$. But S_3 is finite and is not nilpotent. We also observe that $\mathfrak{X} < \mathfrak{X}(2)$ for each of the classes mentioned in 5·11, since $S_3 \in \mathfrak{H}(2) \leq \mathfrak{X}(2)$. The group T, constructed in 5·4(v), is not an N_1-group but $T \in ZA_{\omega} \leq \mathfrak{H}(1) \leq N_1(1)$. Therefore $N_1 < N_1(1)$.

Stanley [3] proves that $\mathfrak{H}(1) = \langle S,Q,D \rangle \mathfrak{H}(1)$. We can see, however, that $\mathfrak{H}(1) < p(\mathfrak{H}(1))$, $\mathfrak{H}(1) < R(\mathfrak{H}(1))$ and $\mathfrak{H}(1) < C(\mathfrak{H}(1))$. For $\mathfrak{H}(1) < \mathfrak{H}(2) \leq p(\mathfrak{H}(1))$ by 2·12 (ix) and by 5·4 (i), $\mathfrak{H}(1) < R(\mathfrak{H}(1))$. But $R \leq C$ so $\mathfrak{H}(1) < C(\mathfrak{H}(1))$.

For locally nilpotent groups, and so for $\mathfrak{H}(1)$-groups, the maximal condition on subgroups is equivalent to the condition of being finitely generated. In contrast, we have :

<u>Lemma 5·12</u> $\mathfrak{H}(2) \cap \hat{\mathfrak{M}} < \mathfrak{H}(2) \cap \mathfrak{G}$

Proof Let $G = \langle a,b \; ; \; a^b = a^2 \rangle$. G is well known to be metabelian but not to satisfy max. By 2·13, $G \in \mathfrak{A}(2) \leq \mathfrak{H}(2)$ and G is finitely generated.

The class $\mathfrak{H}(2)$ is not contained in the class of locally nilpotent groups, but we do have the following result .

Lemma 5·13 $\mathfrak{J} \cap \mathfrak{H}(2) = \mathfrak{J} \cap \mathfrak{A}\mathfrak{H}$

Proof By 2·13, $\mathfrak{A}\mathfrak{H} \leq \mathfrak{H}(2)$. Let $G \in \mathfrak{J} \cap \mathfrak{H}(2)$. Then $G/H_1(G:\mathfrak{H}) \in \mathfrak{J} \cap \mathfrak{H}(1) \leq \mathfrak{H}$. Therefore there exists an integer k such that $\gamma_k(G) \leq H_1(G:\mathfrak{H})$. Also, since finitely generated nilpotent groups are finitely presented, we have by lemma 1·43 of [1] that $d_G(H_1(G:\mathfrak{H})) < \infty$. Therefore by lemma 7(ii) of [3] we have $G/C_G(H_1(G:\mathfrak{H})) \in \mathfrak{H}$. Therefore there exists an integer ℓ such that $\gamma_\ell(G) \leq C_G(H_1(G:\mathfrak{H}))$.

Let $m = \max\{k,\ell\}$. Then $\gamma_m(G)' \leq [\gamma_k(G), \gamma_\ell(G)] = 1$. Therefore $G \in \mathfrak{A}\mathfrak{H}$.

VI Generalized Soluble Groups

We are able to use the subgroup theoretical property ϕ_{σ} to construct several new classes of generalized soluble groups. We begin by considering the relationship between the class $\sigma(1)$ and some well-known classes of generalized soluble groups.

Theorem 6·1 (i) $\sigma < \sigma(1)$

(ii) $\sigma(1) < SI^*$

(iii) $\sigma(1) < L\sigma$

(iv) $\sigma(1) < SD_{\omega+1}$

(v) $R\sigma \nleq \sigma(1)$

Proof To prove (i) we prove that the group T, discussed in Chapter 5, is not a soluble group. This is sufficient to prove that $\sigma < \sigma(1)$ because we have proved in 5·7 that $T \in ZA_{\omega}$ and $ZA_{\omega} \leq \mathcal{H}(1) \leq \sigma(1)$. We need the following lemma.

Lemma 6·2 $\gamma_r(T) = \langle d_{\alpha\beta} \;;\; \beta - \alpha \geq r \rangle$

Proof Let $R = \langle d_{\alpha\beta} \;;\; \beta - \alpha \geq r \rangle$. Now if $\beta - \alpha \geq r$ we have

$$d_{\alpha\beta}^{t_i} = d_{\alpha\beta}[d_{\alpha\beta}, t_i] = \begin{cases} d_{\alpha\beta}d_{\alpha,\beta+1} & \text{if } i = \beta \\ d_{\alpha\beta}d_{\alpha-1,\beta} & \text{if } i = \alpha - 1 \\ d_{\alpha\beta} & \text{otherwise.} \end{cases}$$

Also $d_{\alpha,\beta+1} \in R$ and $d_{\alpha-1,\beta} \in R$ so $d_{\alpha\beta}^{t_i} \in R$ for all i for any α, β with $\beta - \alpha \geq r$. Similarly $d_{\alpha\beta}^{t_i^{-1}} \in R$. Therefore $R \lhd T$.

We now prove that $T/R \in \mathfrak{N}_{r-1}$. It is sufficient to prove that $[t_{i_1},\ldots,t_{i_r}] \equiv 1 \pmod R$ for any $i_1,\ldots,i_r$.

But $[t_{i_1},\ldots,t_{i_r}]$ is either 1 or $1 \pm 2^r e_{\alpha\beta}$ where $\beta - \alpha \geq r$.
For, if r = 2,

$$[t_{i_1}, t_{i_2}] = \begin{cases} 1 + 4e_{i_1, i_1+2} \\ 1 - 4e_{i_1-1, i_1+1} \\ 1 \quad \text{otherwise} \end{cases}.$$

Suppose the result is true for r-1. Then $[t_{i_1},\ldots,t_{i_r}] = 1$ or $[t_{i_1},\ldots,t_{i_r}] = [1 \pm 2^{r-1} e_{\alpha\beta}, t_{i_r}]$, where $\beta - \alpha = r-1$.

$$\text{But } [1 \pm 2^{r-1} e_{\alpha\beta}, t_{i_r}] = \begin{cases} 1 \pm 2^r e_{\alpha,\beta+1} & \text{if } i_r = \beta \\ 1 \mp 2^r e_{\alpha-1,\beta} & \text{if } i_r = \alpha - 1 \\ 1 & \text{otherwise} \end{cases}$$

Since $(\beta+1)-\alpha = \beta-(\alpha-1) = r$, this gives the result. Thus $[t_{i_1},\ldots,t_{i_r}] \equiv 1 \pmod R$, as required. Therefore $\gamma_r(T) \leq R$.

Now it is easy to prove by induction that if $\beta - \alpha \geq r > 1$ then

$$\begin{aligned} d_{\alpha\beta} &= 1 + 2^{\beta-\alpha} e_{\alpha\beta} \\ &= [t_\alpha, t_{\alpha+1}, \ldots, t_{\beta-1}] \\ &\in \gamma_{\beta-\alpha}(T) \leq \gamma_r(T) \end{aligned}$$

so that $R \leq \gamma_r(T)$, as required.

To complete the proof of 6·1 (i) we use the following result :

Lemma 6·3 $\quad T^{(n)} = \gamma_{2^n}(T)$.

Proof $\quad$ This is clearly true if n = 1. Suppose it is true for n-1. Take $d_{\alpha\beta}$ where $\beta - \alpha \geq 2^n$. Choose δ such that $\alpha < \delta < \beta$, $\delta - \alpha \geq 2^{n-1}$ and $\beta - \delta \geq 2^{n-1}$.

Then

$$\begin{aligned} d_{\alpha\beta} &= 1 + 2^{\beta-\alpha}e_{\alpha\beta} \\ &= [1 + 2^{\delta-\alpha}e_{\alpha\delta}, 1 + 2^{\beta-\delta}e_{\delta\beta}] \\ &\quad \varepsilon[\gamma_{2^{n-1}}(T), \gamma_{2^{n-1}}(T)] \\ &= [T^{(n-1)}, T^{(n-1)}] = T^{(n)}. \end{aligned}$$

Therefore $\gamma_{2^n}(T) \leq T^{(n)}$. The reverse inclusion is true for any group.

Thus, since T is not nilpotent, we have for all integers n, that $T^{(n)} \neq 1$ so that T is not soluble.

We proceed now with the proof of the other parts of theorem 6·1. We prove inclusions first and prove that these inclusions are strict at the end.

(ii) Let $G \in \mathfrak{S}(1)$. Then by 2·18 we have $G \in N\mathfrak{S}$. Therefore if G is non-trivial, G has a non-trivial soluble normal subgroup, N, say. There exists an integer n such that $N^{(n)}$ is a non-trivial abelian subgroup of N. But

$$N^{(n)} \trianglelefteq N \trianglelefteq G \text{ so } N^{(n)} \trianglelefteq G.$$

Thus any non-trivial $\mathfrak{S}(1)$-group has a non-trivial normal abelian subgroup. By the Q-closure of $\mathfrak{S}(1)$, this proves that $\mathfrak{S}(1) \leq SI^*$.

(iii) $\mathfrak{S}(1) \leq L\mathfrak{S}$ by 2·16 (ii)

(iv) It is easy to see that $G \in \mathfrak{S}(1)$ if and only if for each element x of G there exists an integer n, depending on x, such that $[x, G^{(n)}] = 1$. Therefore, if $G \in \mathfrak{S}(1)$, it follows that $G^{(\omega+1)} = [G^{(\omega)}, G^{(\omega)}] \leq [G, G^{(\omega)}] = 1$. Therefore $G \in SD_{\omega+1}$.

The rest of theorem 6·1 is a consequence of theorem 8·11 of [1]. For Robinson proves the existence of an SI*-group which is not SD and so is not $\mathfrak{S}(1)$, a locally soluble group which is not SD and so is not $\mathfrak{S}(1)$ and a residually soluble (and so $SD_{\omega+1}$) group which is not $\overline{SI}$ and so is not SI* and so not $\mathfrak{S}(1)$.

Theorem 6·1 shows that no class of generalized soluble groups known to the author coincides with the class of $\mathfrak{S}(1)$-groups.

We also have the following information about $\mathfrak{S}(1)$-groups.

Lemma 6·4 (i) $\mathfrak{S}(1) < R(\mathfrak{S}(1)) \leq SD$

(ii) $L\mathfrak{S} = (L\mathfrak{S})(1) = L(\mathfrak{S}(1))$

Proof (i) By 6·1 (v), $\mathfrak{S}(1) < R(\mathfrak{S}(1))$. Also $R(\mathfrak{S}(1)) \leq R(SD) = SD$.

(ii) This follows immediately from 2·16 (ii)

Turning now to groups with ascending σ-central series, we have

<u>Lemma 6·5</u> (i) $\overset{o}{\phi}_{\sigma} \leq SD$

(ii) $L\overset{o}{\phi}_{\sigma} \leq SI$

(iii) $\acute{\phi}_{\sigma} \leq SJ^*$

(iv) $L\acute{\phi}_{\sigma} \leq \overline{\overline{SN}}$

<u>Proof</u> (i) $\overset{o}{\phi}_{\sigma} \leq p(\sigma(1)) \leq p(SD)$ by 6·1 (iv). But $SD = \grave{p}\mathfrak{A}$ so $\overset{o}{\phi}_{\sigma} \leq p\,\grave{p}\mathfrak{A} = \grave{p}\mathfrak{A} = SD$.

(ii) $L\overset{o}{\phi}_{\sigma} \leq L\,SD \leq L\,SI = SI$.

(iii) $\acute{\phi}_{\sigma} \leq \acute{p}_n(\sigma(1)) \leq \acute{p}_n\,SI^*$ by 6·1 (ii). But $SI^* \leq SJ^* = \acute{p}_{sn}\mathfrak{A}$ so $\acute{\phi}_{\sigma} \leq \acute{p}_n\acute{p}_{sn}\mathfrak{A} = \acute{p}_{sn}\mathfrak{A} = SJ^*$.

(iv) $L\acute{\phi}_{\sigma} \leq L\,SJ^* \leq L\,\overline{\overline{SN}} = \overline{\overline{SN}}$.

We prove in our next theorem that the class $\overset{*}{\phi}_{\sigma}$ of residually σ-central groups is a class of generalized soluble groups. We also prove, in contrast, that its subclass $\overset{*}{\phi}_{\mathfrak{H}}$ is not.

<u>Theorem 6·6</u> $\overset{*}{\phi}_{\sigma}$ is a class of generalized soluble groups but $\overset{*}{\phi}_{\mathfrak{H}}$ is not a class of generalized soluble groups .

<u>Proof</u> As observed in [3] and [7] , $\mathfrak{X} \leq \acute{\phi}_{\mathfrak{X}} \leq \overset{*}{\phi}_{\mathfrak{X}}$ for any $\langle S,Q,D_0\rangle$ -closed class of groups $\mathfrak{X}$. By 2·21, $\overset{*}{\phi}_{\sigma} \cap \mathfrak{F} \leq \sigma$ and so $\overset{*}{\phi}_{\sigma}$ is a class of generalized soluble groups.

In order to prove that $\overset{*}{\phi}_{\mathfrak{H}}$ is not a class of generalized soluble groups, we prove that the symmetric group of degree 4, S_4, is not a $\overset{*}{\phi}_{\mathfrak{H}}$-group. Now the only normal subgroups of the soluble group S_4 are $1 = S_4^{(3)}$, $K_4 = S_4^{(2)}$, that is Klein's four group, $A_4 = S_4'$ and S_4 itself.

Now $S_4/S_4'' = S_4/K_4 \cong S_3$ and since $\zeta_1(S_3) = 1$ we have $S_4/S_4'' \notin \mathfrak{H}$. Suppose $S_4 \in \overset{*}{\phi}_{\mathfrak{H}}$ and let $x = (12)(34)$. Then there exist normal subgroups M and N of S_4 such that $N \leq M$, $x \notin N$, $[x,M] \leq N$ and $S_4/M \in \mathfrak{H}$. Since $x \notin N$ we have $N = 1$ because $x \in S_4''$. Also, since $S_4/M \in \mathfrak{H}$, we have $M = S_4'$ or $M = S_4$. Let $y = (124)$. Now $y \in A_4 = S_4'$ and so $[x,y] \in [x,M] = 1$. But $[x,y] = (14)(23) \neq 1$ which is a contradiction. Therefore $S_4 \notin \overset{*}{\phi}_{\mathfrak{H}}$.

This, of course, also proves that $\overset{o}{\phi}_{\mathfrak{H}}$ and $\phi'_{\mathfrak{H}}$ are not classes of generalized soluble groups. However, we will show, in our next lemma, some of the connections between these classes and some other classes of interest.

<u>Lemma 6·7</u> (i) $ZA < \phi'_{\mathfrak{H}} < SI^*$.

(ii) $\phi'_{\mathfrak{H}} < \overset{*}{\phi}_{\mathfrak{H}}$

(iii) $L\mathfrak{H} \nleq \phi'_{\mathfrak{H}}$

(iv) $\phi'_{\mathfrak{H}} < L\phi'_{\mathfrak{H}}$

(v) $L\phi'_{\mathfrak{H}} < \overline{\overline{SI}}$

<u>Proof</u> (i) Since $ZA = \phi'_1$ it is clear that $ZA \leq \phi'_{\mathfrak{H}}$. As observed in Chapter 5, the symmetric group of degree 3, S_3, is a finite $\mathfrak{H}(2)$-group but is not nilpotent. Thus $S_3 \in \phi'_{\mathfrak{H}}$

but $S_3 \nleq ZA$.

Let $G \in \acute{\phi}_{\mathfrak{H}}$. Then $G = H_\alpha(G:\mathfrak{H})$ for some ordinal α . Therefore , by 2·19, $G \leq \rho_\alpha(G:\mathfrak{H})$ so that $G \in SI^*$. The Mclain group $M(\mathbb{Z}_{>0},F)$, discussed in Chapter 5, is a Fitting group and so an SI*-group, but is not a $\acute{\phi}_{\mathfrak{H}}$-group.

(ii),(iii) and (iv) The Mclain group $M(\mathbb{Z}_{>0},F)$ is a Fitting group, and so is in the class $\overset{*}{\phi}_{\mathfrak{H}}$ and is locally nilpotent and so is a $L\acute{\phi}_{\mathfrak{H}}$-group. But it is not a $\acute{\phi}_{\mathfrak{H}}$ -group, by the proof of 5·4(i).

(v) $L\acute{\phi}_{\mathfrak{H}} \leq LSI^* \leq L\overline{\overline{SI}} = \overline{\overline{SI}}$.

We include here some results concerning other relevant classes.

<u>Lemma 6·8</u> (i) $\overset{*}{\phi}_{\mathfrak{A}} < [\mathfrak{A}]$

(ii) $\sigma \nleq \overset{*}{\phi}_{\mathfrak{A}}$

(iii) $\overset{*}{\phi}_{\mathfrak{A}} \nleq SD$

(iv) $\overset{*}{\phi}_{1} \nleq SN^*$

<u>Proof</u> (i) and (ii) By 3.22 and 4·5 , $\overset{*}{\phi}_{\mathfrak{A}} \leq C_n\mathfrak{A} \leq [\mathfrak{A}]$. But the group S_4, discussed in the proof of 6·6, is soluble, and so is an $[\mathfrak{A}]$-group, but is not a $\overset{*}{\phi}_{\mathfrak{H}}$-group and so is not a $\overset{*}{\phi}_{\mathfrak{A}}$-group.

(iii) The Mclain group $M(Q,F)$, for any field F, is locally nilpotent and $L\mathfrak{H} \leq \overset{*}{\phi}_1 \leq \overset{*}{\phi}_{\mathfrak{A}}$. But $M(Q,F)$ is characteristically simple and is not abelian. Therefore $M(Q,F)$ is perfect and so is not an SD-group.

(iv) Robinson [1] gives an example of a group G which is locally nilpotent and has trivial Gruenberg radical.

Therefore $G \in L\mathfrak{N} \leq \overset{*}{\phi}_1$ but $G \notin SN^*$.

<u>Lemma 6·9</u> (i) $\mathfrak{A}(1) \leq \mathfrak{A}^2 \leq \mathfrak{A}(2)$

(ii) $\sigma = p(\mathfrak{A}(1))$

(iii) $\mathfrak{N} \leq \overset{o}{\phi}_{\mathfrak{A}} = \mathfrak{N}\mathfrak{A}$

(iv) $ZA \leq \acute{\phi}_{\mathfrak{A}} = ZA\mathfrak{A}$

<u>Proof</u> (i) Let $G \in \mathfrak{A}(1)$. Then by 3·15 $G' \leq \zeta_1(G)$ so $G \in \mathfrak{A}^2$. We have $\mathfrak{A}^2 \leq \mathfrak{A}(2)$ by 2·13 .

(ii) This follows from (i).

(iii) and (iv) These follow from theorem A of [3] .

Some inclusions are shown in Fig. 6.

We will now show the effects of some finiteness conditions on some of the classes under discussion.

<u>Lemma 6·10</u> (i) A $\overset{o}{\phi}_{L\mathfrak{N}}$-group satisfying the minimal condition on subgroups is a soluble Černikov group.

(ii) $\overset{*}{\phi}_{\sigma} \cap \check{\mathfrak{M}}_n \leq SI^*$

(iii) A $\overset{*}{\phi}_{\sigma}$-group satisfying the minimal condition on subnormal subgroups is a soluble Černikov group.

<u>Proof</u> (i) By 2·16 (ii), $\overset{o}{\phi}_{L\mathfrak{N}} \leq p((L\mathfrak{N})(1)) = pL\mathfrak{N}$.
By the <S,Q> -closure of $\check{\mathfrak{M}}$ we have $pL\mathfrak{N} \cap \check{\mathfrak{M}} \leq p(L\mathfrak{N} \cap \check{\mathfrak{M}})$.
But it is well-known that locally nilpotent groups satisfying

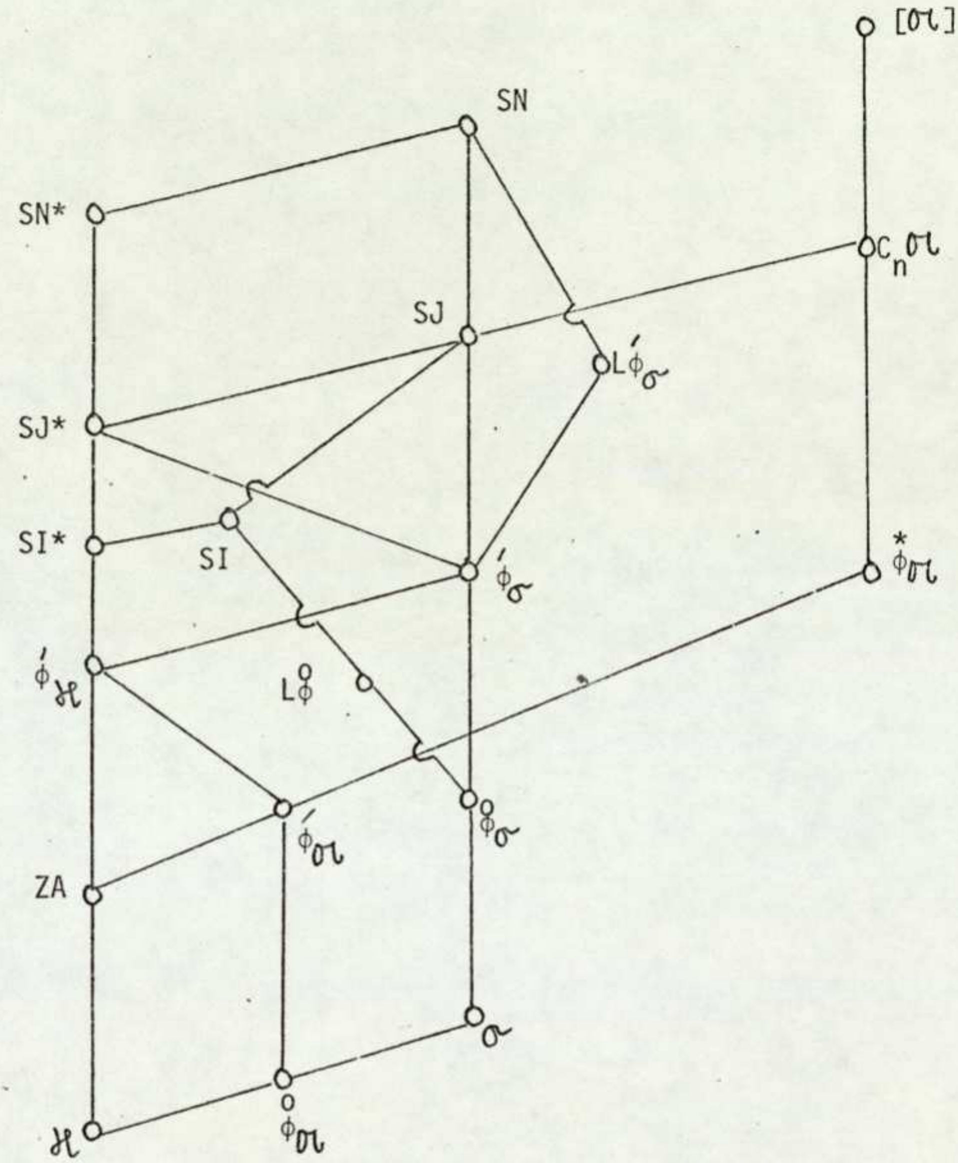

[$\mathfrak{A}$]
SN
SN*
$C_n\mathfrak{A}$
SJ
$L\phi'_\sigma$
SJ*
SI*
SI
ϕ'_σ
$\phi^*_\mathfrak{A}$
$\phi'_\mathcal{H}$
$L\phi^o$
ϕ^o_σ
$\phi'_\mathfrak{A}$
ZA
σ
$\phi^o_\mathfrak{A}$
$\mathcal{H}$

Fig. 6

the minimal condition are Černikov groups (see, for example, theorem 3·16 of [1]). Therefore

$$\overset{\circ}{\phi}_{L\mathfrak{H}} \cap \check{\mathfrak{M}} \leq p(\mathfrak{A}\mathfrak{F} \cap L\mathfrak{H}) \leq \mathfrak{C}$$

and the result follows from the p-closure of the class of Černikov groups.

(ii) By Corollary 7 of [7], $\overset{*}{\phi}_{\mathfrak{C}} \cap \check{\mathfrak{M}}_n \leq ZA\mathfrak{C} \leq SI^*$.

(iii) By Theorem 3·12 of [1], an SI*-group satisfying the minimal condition on subnormal subgroups is a soluble Černikov group.

Lemma 6·11 $\phi'_{\mathfrak{H}} \cap \hat{\mathfrak{M}} \leq p\mathfrak{C}$

Proof By a corollary to Theorem A of [3],

$$\begin{aligned} \phi'_{\mathfrak{H}} \cap \hat{\mathfrak{M}} &\leq ZA\mathfrak{H} \cap \hat{\mathfrak{M}} \\ &\leq (ZA \cap \hat{\mathfrak{M}})(\mathfrak{H} \cap \hat{\mathfrak{M}}) \\ &= (\mathfrak{H} \cap \hat{\mathfrak{M}})^2 \leq p\mathfrak{C} \end{aligned}$$

Lemma 6·12 For locally finite groups, $L\overset{o}{\phi}_{\mathfrak{V}} = L\phi'_{\mathfrak{V}} = \overset{*}{\phi}_{\mathfrak{V}}$ for any variety $\mathfrak{V}$.

Proof By lemma 1 of [7], and Theorem A of [3],

$$\begin{aligned} \overset{*}{\phi}_{\mathfrak{V}} \cap L\mathfrak{F} &\leq L(\overset{*}{\phi}_1\mathfrak{V} \cap \mathfrak{F}) \\ &\leq L(\mathfrak{H}\mathfrak{V}) = L\overset{o}{\phi}_{\mathfrak{V}}. \end{aligned}$$

Also $\overset{*}{\phi}_{\mathfrak{V}} = L\overset{*}{\phi}_{\mathfrak{V}}$, as required.

Corollary 6·13 $\overset{*}{\phi}_{\mathfrak{A}} \cap L\mathfrak{F} \leq Lp\mathfrak{C}$

<u>Proof</u> $\overset{*}{\phi}_{\mathfrak{A}} \cap L\mathfrak{F} \leq L\overset{o}{\phi}_{\mathfrak{A}} \cap L\mathfrak{F}$

$\leq L(\sigma \cap \mathfrak{F}) \leq L\, p\mathfrak{C}$.

The question may be asked whether periodic $\overset{*}{\phi}_{\mathfrak{X}}$-groups are locally finite. The answer is negative as can be seen from an example of Golod [19] which is periodic, finitely generated, infinite and residually nilpotent but is not nilpotent. This is, of course, a $\overset{*}{\phi}_1$-group but it is not locally finite.

This shows that, in general, the equivalence of the conditions "periodic" and "locally finite" for a class of groups $\mathfrak{X}$ does not imply their equivalence for the class $\overset{*}{\phi}_{\mathfrak{X}}$. Since $\overset{*}{\phi}_1 \leq \overset{*}{\phi}_{\mathfrak{X}}$ for all $\langle S,Q,D_0\rangle$-closed classes $\mathfrak{X}$, we have shown that for all $\langle S,Q,D_0\rangle$-closed classes $\mathfrak{X}$, there exists a periodic $\overset{*}{\phi}_{\mathfrak{X}}$-group which is not locally finite.

We do, however, have the following weaker result.

<u>Lemma 6·14</u> If every periodic $\mathfrak{X}$-group is locally finite, then every periodic $\phi'_{\mathfrak{X}}$-group is locally finite.

<u>Proof</u> Suppose the condition is true for a class of groups $\mathfrak{X}$. Let G be a periodic $\mathfrak{X}(1)$-group and let $F = \langle x_1,\ldots,x_n\rangle$ be a finitely generated subgroup of G. Then F is also a periodic $\mathfrak{X}(1)$-group. Therefore for each i, $1 \leq i \leq n$, there exists a normal subgroup N_i of F such that $[x_i,N_i] = 1$ and $G/N_i \in \mathfrak{X}$. Let $N = \bigcap_{i=1}^{n} N_i$. Then F/N can be

embedded in the direct product of a finite number of periodic $\mathfrak{X}$-groups. By the $\langle S, D_0 \rangle$-closure of $\mathfrak{X}$ and the class of periodic groups, we have that F/N is a periodic $\mathfrak{X}$-group and so is locally finite, by assumption. But F is finitely generated and so F/N is finite. This implies that N is finitely generated. But $N \leq \zeta_1(F)$ and $\zeta_1(F)$ is a periodic, abelian group and so is locally finite. Therefore N is locally finite and, since $L\mathfrak{F} = pL\mathfrak{F}$, F is locally finite. But F is finitely generated so F is finite and G is locally finite, as required.

Now $\acute{\phi}_{\mathfrak{X}} < \acute{p}(\mathfrak{X}(1))$ so by the $\langle S, Q \rangle$-closure of the class of periodic groups, the class of periodic $\acute{\phi}_{\mathfrak{X}}$-groups is contained in the class $\acute{p}L\mathfrak{F} = L\mathfrak{F}$.

VII FC-Groups

When we restrict our discussion by considering only FC-groups we are able to prove that many of the classes under discussion are equivalent. First we observe that the class of FC-groups is just the class $\mathfrak{F}(1)$.

We have the following preliminary result.

Proposition 7·1 Let $\mathfrak{X}$ be an $\langle S,Q,D_0\rangle$-closed class of groups. Then $L\mathfrak{X} \cap \mathfrak{F}(1) \leq \mathfrak{X}(1) \cap \mathfrak{F}(1)$, with equality if $\mathfrak{X}$ is a good class.

Proof It is clear that $L\mathfrak{X} \cap \mathfrak{F}(1) \leq (L\mathfrak{X})(1) \cap \mathfrak{F}(1)$, which is equal to $(L\mathfrak{X} \cap \mathfrak{F})(1)$ by 2·12 (vi). Since $L\mathfrak{X} \cap \mathfrak{F} \leq \mathfrak{X}$ we have by 2·12 (iii) that $L\mathfrak{X} \cap \mathfrak{F}(1) \leq \mathfrak{X}(1)$.

On the other hand, if $\mathfrak{X}$ is a good class then $\mathfrak{X}(1) \leq L\mathfrak{X}$ by 2·16 (ii).

Taking $\mathfrak{X} = \mathfrak{F}$ shows that $L\mathfrak{X} \cap \mathfrak{F}(1)$ is not equal to $\mathfrak{X}(1) \cap \mathfrak{F}(1)$ in general.

The proof of our next proposition is very easy and is omitted.

Proposition 7·2 Let $\mathfrak{X}$ be a Q-closed class of generalized nilpotent groups or of generalized soluble groups. Then $\mathfrak{X} \cap \mathfrak{F}(1) \leq \mathfrak{N}(1)$ or $\mathfrak{X} \cap \mathfrak{F}(1) \leq \sigma(1)$, respectively.

Recall that a group is Baer-nilpotent if every finite section of the group is nilpotent. The class of Baer-nilpotent groups is obviously Q-closed and so we have :

Corollary 7·3 Every Baer-nilpotent FC-group is an $\mathfrak{H}(1)$-group.

Theorem 7·4 If $\mathfrak{X}$ is an S_n-closed class of generalized soluble groups, then $\mathfrak{X} \cap \mathfrak{F}(1) \leq \mathfrak{S}(1)$.

Proof Let $G \in \mathfrak{X} \cap \mathfrak{F}(1)$. It is well-known that a group G is an FC-group if and only if $x^G \in \mathfrak{F}\mathfrak{C}$ for every element x of G (see, for example, Corollary 3 to Theorem 4·32 of [1]). Therefore for each element x of G there exists a subgroup N of G, normal in x^G, such that $x^G/N \in \mathfrak{C}$ and $N \in \mathfrak{F} \cap \mathfrak{X} \leq \mathfrak{S}$. Thus $x^G \in \mathfrak{S}$ for all elements x of G.

Let $F = \{x_1, \ldots, x_n\}$ be a finite subset of G. Now for each $i = 1, \ldots, n$, $x_i \in x_i^G$ and $x_i^G \in \mathfrak{S}$. Therefore

$$F \subseteq \prod_{i=1}^{n} x_i^G \in N_0\mathfrak{S} = \mathfrak{S}.$$ Therefore $G \in L\mathfrak{S}$. But

$L\mathfrak{S} \cap \mathfrak{F}(1) = \mathfrak{S}(1)$ by 7·1.

We are unable as yet to answer the question "does every S_n-closed (or even S-closed) class $\mathfrak{X}$ of generalized nilpotent groups satisfy $\mathfrak{X} \cap \mathfrak{F}(1) \leq \mathfrak{H}(1)$?". However, we shall prove that $\overset{*}{\phi}_1$, the class of residually central groups, satisfies $\overset{*}{\phi}_1 \cap \mathfrak{F}(1) \leq ZA_\omega$ (recall that $ZA_\omega \leq \mathfrak{H}(1)$). We shall obtain this result as a consequence of some more general results.

The following definition of a local system is due to Kurosh. We shall refer to such local systems as Kurosh-local systems to avoid confusion with the definition in Chapter 3.

<u>Definition</u> A <u>Kurosh-local system</u> of a group G is a set $\mathfrak{L}$ of subgroups of G such that :

(i) If $g \in G$ then there exists an $L \in \mathfrak{L}$ such that $g \in L$.

(ii) If L_1 and L_2 belong to $\mathfrak{L}$ then there exists an L belonging to $\mathfrak{L}$ such that $L_1 \cup L_2 \subseteq L$.

<u>Definition</u> $G \in L_1\mathfrak{X}$ if G has a Kurosh-local system of $\mathfrak{X}$-subgroups.

<u>Definition</u> $G \in L_2\mathfrak{X}$ if every finitely generated subgroup of G is an $\mathfrak{X}$-group.

With these definitions and the usual definition of the closure operation L, that is the one found in our list of notation, it is easy to see that $L_2\mathfrak{X} \leq L_1\mathfrak{X} \leq L\mathfrak{X}$ for any class of groups $\mathfrak{X}$, with equalities if $\mathfrak{X} = S\mathfrak{X}$.

<u>Lemma 7·5</u> Let $\mathfrak{X}$ and $\mathfrak{Y}$ be classes of groups. Suppose a group G has a Kurosh-local system $\mathfrak{L}$ such that for each $L \in \mathfrak{L}$ there exists a normal subgroup R(L) of L with $R(L) \in \mathfrak{X}$ and $L/R(L) \in \mathfrak{Y}$ and such that whenever $L_1 \leq L_2$ it follows that $R(L_1) \leq R(L_2)$. Then $G \in (L\mathfrak{X})(LQ\mathfrak{Y})$.

Proof Let $H = \bigcup_{L \in \mathfrak{L}} R(L)$. If a and b are elements of H then $a \in R(L_1)$ and $b \in R(L_2)$ for some L_1 and L_2 belonging to $\mathfrak{L}$. Therefore there exists an $L \in \mathfrak{L}$ such that $L_1 \cup L_2 \subseteq L$ and so a and b belong to R(L). Therefore $ab^{-1} \in R(L) \subseteq H$, proving that $H \leq G$. Now let $a \in H$ and $g \in G$. Then $a \in R(L_1)$ and $g \in L_2$ for some L_1 and L_2 belonging to $\mathfrak{L}$. Therefore there exists an L belonging to $\mathfrak{L}$ such that $L_1 \cup L_2 \subseteq L$ so $a \in R(L)$ and $g \in L$. Therefore $a^g \in R(L) \leq H$ and we have proved that $H \triangleleft G$.

Let $\{h_1,\ldots,h_n\}$ be a finite set of elements of H. Then $h_i \in R(L_i)$, say, for $i = 1,\ldots,n$. There exists an L belonging to $\mathfrak{L}$ such that $\bigcup_{i=1}^{n} L_i \subseteq L$ and so $h_i \in R(L)$ for each $i = 1,\ldots,n$. But $R(L) \in \mathfrak{X}$ and so $H \in L\mathfrak{X}$.

Let $\{g_1H,\ldots,g_nH\}$ be a finite subset of G/H. Then $g_i \in L_i$, say, for each $i = 1,\ldots,n$. Again, there exists an L belonging to $\mathfrak{L}$ such that $\bigcup_{i=1}^{n} L_i \subseteq L$ and so $\{g_1,\ldots g_n\} \subseteq L$. Now $R(L) \leq L \cap H$. Thus $LH/H \cong L/(L \cap H) \in Q\mathfrak{Y}$. Therefore, since $\{g_1H,\ldots,g_nH\} \subseteq LH/H$, we have proved that $G/H \in LQ\mathfrak{Y}$, completing the proof of the lemma.

This result yields several corollaries. For example, we may use it to derive the following result of Gardiner, Hartley and Tompkinson [20] about formations :

Definition Let Σ be a class of groups. A class $\mathfrak{F}$ (temporary notation) is a Σ -formation if $\mathfrak{F} \leq \Sigma$,

$\mathfrak{F} = Q\mathfrak{F}$ and $\Sigma \cap R\mathfrak{F} = \mathfrak{F}$.

<u>Corollary 7·6</u> Let $\Sigma = QS\Sigma$ and let $\mathfrak{F}$ be an S-closed Σ-formation. Suppose $\mathfrak{X} = S\mathfrak{X}$ and $\mathfrak{X}\mathfrak{F} \leq \Sigma$. Then $L(\mathfrak{X}\mathfrak{F}) \leq (L\mathfrak{X})(L\mathfrak{F})$.

<u>Proof</u> Let $G \in L(\mathfrak{X}\mathfrak{F})$ and take $\mathfrak{L}$ to be the set of all finitely generated subgroups of G. Then $\mathfrak{L}$ is a Kurosh-local system of G. Since $\mathfrak{X}\mathfrak{F} \leq \Sigma$ and $\mathfrak{X}, \mathfrak{F}$ and Σ are all S-closed we have, for each $L \in \mathfrak{L}$, $L \in \mathfrak{X}\mathfrak{F} \cap \Sigma$.

Take R(L) to be $\rho^*(L:\mathfrak{F})$. Then $R(L) \in S\mathfrak{X} = \mathfrak{X}$. Also $L/R(L) \in \Sigma \cap R\mathfrak{F} = \mathfrak{F}$. The S-closure of $\mathfrak{F}$ implies that if $L_1 \leq L_2$ then $R(L_1) \leq R(L_2)$. For if $N \triangleleft L_2$ with $L_2/N \in \mathfrak{F}$ then $L_1/(L_1 \cap N) \cong L_1N/N \leq L_2/N$ so $R(L_1) \leq L_1 \cap N \leq N$.

Therefore G satisfies the conditions of 7.5. The result follows by the Q-closure of $\mathfrak{F}$.

When we consider Σ to be the class of all groups, it is easy to see that any variety $\mathfrak{V}$ is a Σ -formation. Thus the next corollary follows from 7·6.

<u>Corollary 7·7</u> If $\mathfrak{X} = S\mathfrak{X}$ and $\mathfrak{V}$ is a variety then $L(\mathfrak{X}\mathfrak{V}) \leq (L\mathfrak{X})\mathfrak{V}$.

In general, $L(\mathfrak{X}\mathfrak{V}) < (L\mathfrak{X})\mathfrak{V}$. For example, the corollary to theorem 6·25 of [1] proves that there exists a $(L\mathfrak{H})\mathfrak{C}$ -group G which is not a $L\mathfrak{S}$ -group. Thus $G \in (L\mathfrak{H})\mathfrak{A}$ but $G \notin L(\mathfrak{H}\mathfrak{A})$. We may now prove :

Theorem 7·8 If $\mathfrak{V}$ is a variety then $\overset{*}{\phi}_{\mathfrak{V}} \cap \mathfrak{F}(1) \leq \phi'_{\mathfrak{V}}$.

Proof Let $G \in \overset{*}{\phi}_{\mathfrak{V}} \cap \mathfrak{F}(1)$ and let $H = \overline{H}(G:\mathfrak{V})$, the limit of the upper $\mathfrak{V}$-central series of G. By theorem 6 of [7], $G/H \in \overset{*}{\phi}_{\mathfrak{V}} \cap \mathfrak{F}(1)$. Now $\zeta_1(G/H) \leq H_1(G/H:\mathfrak{V}) = 1$ and so, by theorem 4·32 (ii) of [1], $G/H \in L\mathfrak{F}$.

Therefore, $G/H \in \overset{*}{\phi}_{\mathfrak{V}} \cap L\mathfrak{F} \leq L(\overset{*}{\phi}_{\mathfrak{V}} \cap \mathfrak{F})$. But, by lemma 1 of [7], $\overset{*}{\phi}_{\mathfrak{V}} \leq \overset{*}{\phi}_1\mathfrak{V}$ and so $G/H \in L(\overset{*}{\phi}_1\mathfrak{V} \cap \mathfrak{F})$ $\leq L(\mathfrak{H}\mathfrak{V}) \leq (L\mathfrak{H})\mathfrak{V}$ by 7·7. But $G/H \in \mathfrak{F}(1)$ so by 7·1 we have $G/H \in (L\mathfrak{H} \cap \mathfrak{F}(1))\mathfrak{V} \leq (\mathfrak{H}(1))\mathfrak{V} \leq ZA\mathfrak{V} \leq \phi'_{\mathfrak{V}}$ by theorem A of [3]. Therefore $G = H$ and so $G \in \phi'_{\mathfrak{V}}$.

We take as a corollary our result about residually central groups.

Corollary 7·9 $\overset{*}{\phi}_1 \cap \mathfrak{F}(1) \leq ZA_{\omega}$.

Proof By 7·8, $\overset{*}{\phi}_1 \cap \mathfrak{F}(1) \leq ZA$. But $ZA \leq L\mathfrak{H}$ and by lemma 4 of [21], $L\mathfrak{H} \cap \mathfrak{F}(1) \leq ZA_{\omega}$.

This result, together with 7·3, shows that all classes $\mathfrak{X}$ of generalized nilpotent groups, known to the author, satisfy $\mathfrak{X} \cap \mathfrak{F}(1) \leq ZA_{\omega}$.

In contrast to this, we shall prove that $\mathfrak{H}(1) \cap \mathfrak{F}(1) \nleq N_1$. For the group T, defined in Chapter 5, is an $\mathfrak{H}(1)$-group but is not an N_1 group, as proved in 5·4. We shall now prove that $T \in \mathfrak{F}(1)$. We start with :

<u>Lemma 7·10</u> (i) For all i, $|t_i| < \infty$.

(ii) For all α, β with $\alpha < \beta$, $|d_{\alpha\beta}| < \infty$.

<u>Proof</u> (i) Recall $a_i^{t_j} = a_i$ if $i \neq j$. Suppose i is even, say i = 2n. Now

$$a_{2n}^{t_{2n}^{2^{n+1}}} = a_{2n}(a_{2n+1})^{2.2^{n+1}} = a_{2n}(a_{2(n+1)-1})^{2^{(n+1)+1}} = a_{2n}.$$

Suppose i is odd, say i = 2n-1. Now

$$a_{2n-1}^{t_{2n-1}^{2^n}} = a_{2n-1}(a_{2n})^{2.2^n} = a_{2n-1}(a_{2n})^{2^{n+1}} = a_{2n-1} .$$

(ii) Suppose β is even, say $\beta = 2n$. We may assume, by 5·5(i) that $2n \leq 2\alpha + 1$ which implies that $\alpha - n \geq 0$. Now

$$a_\alpha^{d_{\alpha,2n}^{\alpha-n+2}} = a_\alpha(a_{2n})^{2^{2n-\alpha+\alpha-n+2-1}} = a_\alpha(a_{2n})^{2^{n+1}} = a_\alpha .$$

Suppose β is odd, say $\beta = 2n-1$. Again, we may assume that $2n-1 \leq 2\alpha + 1$ which implies that $\alpha - n \geq -1$. Now

$$a_\alpha^{d_{\alpha,2n-1}^{\alpha-n+3}} = a_\alpha(a_{2n-1})^{2^{2n-1-\alpha+\alpha-n+3-1}}$$

$$= a_\alpha(a_{2n-1})^{2^{n+1}} = a_\alpha .$$

<u>Lemma 7·11</u> For all $d_{\alpha\beta} \in T$, we have $d_{\alpha\beta}^{T} \in \mathfrak{F}$.

<u>Proof</u> By 5·5 (i), if $\beta \geq 2\alpha + 2$ then $d_{\alpha\beta}^{T} = 1$ for all α . Suppose that for all α , $d_{\alpha,\alpha+r}^{T} \in \mathfrak{F}$. We prove that $d_{\alpha,\alpha+r-1}^{T} \in \mathfrak{F}$.

Now $[d_{\alpha,\alpha+r-1},T] = \langle [d_{\alpha,\alpha+r-1},t_i] \; ; \; i = 1,2,\ldots \rangle^{T}$

$= \langle d_{\alpha,\alpha+r}, d_{\alpha-1,\alpha+r-1} \rangle^{T} = d_{\alpha,\alpha+r}^{T} \; d_{\alpha-1,\alpha+r-1}^{T}$.

So, by hypothesis, $[d_{\alpha,\alpha+r-1},T] \in N_0 \mathfrak{F} = \mathfrak{F}$. But

$d_{\alpha,\alpha+r-1}^{T} = \langle d_{\alpha,\alpha+r-1} , [d_{\alpha,\alpha+r-1},T] \rangle$ and

$[d_{\alpha,\alpha+r-1},T] \triangleleft d_{\alpha,\alpha+r-1}^{T}$.

Therefore, since $|d_{\alpha,\alpha+r-1}| < \infty$, we have

$d_{\alpha,\alpha+r-1}^{T} \, / \, [d_{\alpha,\alpha+r-1},T] \in \mathfrak{F}$ and so

$d_{\alpha,\alpha+r-1}^{T} \in \mathfrak{F}$.

<u>Corollary 7·12</u> For each $t_i \in T$, we have $t_i^{T} \in \mathfrak{F}$.

<u>Proof</u> Since $t_i^{T} = \langle t_i, [\, t_i, T \,] \rangle$ and

$[t_i,T] = \langle [t_i,t_j] \; ; \; j = 1,2,\ldots \rangle^{T} = \langle d_{i,i+2}, d_{i-1,i+1} \rangle^{T}$

$= d_{i,i+2}^{T} \; d_{i-1,i+1}^{T} \in \mathfrak{F}$ by 7·11 , we may complete the proof in the same way as in 7·11.

This yields :

<u>Corollary 7·13</u> T is a periodic FC-group.

<u>Proof</u> We have proved that every generator of T is an FC-element of T. But it is well-known that the FC-elements of any group form a subgroup. Therefore $T \in \mathfrak{F}(1)$.

We have also proved, in 7·10 (i), that every generator of T is periodic. But, because T is an FC-group, we have, by a result of Neumann [22] , that T is periodic.

Thus we have proved the existence of a periodic $(\mathfrak{H} \cap \mathfrak{F})(1)$-group which is not an N_1-group. We also have the following result, which contrasts with the fact that $(\mathfrak{H} \cap \mathfrak{F})(1) \leq ZA_\omega$.

<u>Theorem 7·14</u> $(\mathfrak{H} \cap \mathfrak{F})(1) \nleq ZD_\omega$.

<u>Proof</u> The group of Glušcov [17] , $G = \operatorname*{Dr}_{n=2}^{\infty} M_n(P)$, where $M_n(P)$ is the group of all $n \times n$ unitriangular matrices over a field P, was observed, in section 5, to give rise to a factor group which is an $\mathfrak{H}(1)$-group but is not a ZD_ω- group.

Now, if P is the Galois field of p^m elements, then

$|M_n(P)| = p^{\binom{n}{2} m}$ and so G, being a direct product of finite groups, is an FC-group. Thus the factor group is an $(\mathfrak{H} \cap \mathfrak{F})(1)$-group.

Finally, we consider the effect of FC-conditions on some larger classes.

Theorem 7·15 (i) $\overset{*}{\phi}_{\mathfrak{H}} \cap \mathfrak{F}(1) \leq SI^*_{\omega 2}$

(ii) $\overset{*}{\phi}_{\sigma} \cap \mathfrak{F}(1) \leq L\, p\, \mathfrak{C}$.

Proof (i) Let $G \in \overset{*}{\phi}_{\mathfrak{H}} \cap \mathfrak{F}(1)$. By lemma 1 of [7] and 7·9, we have

$$G \in \overset{*}{\phi}_1(R\mathfrak{H}) \cap \mathfrak{F}(1)$$

$$\leq (\overset{*}{\phi}_1 \cap \mathfrak{F}(1))(R\mathfrak{H} \cap \mathfrak{F}(1))$$

$$\leq (\overset{*}{\phi}_1 \cap \mathfrak{F}(1))^2$$

$$\leq ZA^2_{\omega} \leq SI^*_{\omega 2} \quad .$$

(ii) Now $\overset{*}{\phi}_{\sigma}$ is an S_n-closed class of generalized soluble groups by 2·4, 2·5 and 6·6. Therefore by 7·4, $\overset{*}{\phi}_{\sigma} \cap \mathfrak{F}(1) \leq \sigma(1)$.

Let $G \in \mathfrak{J} \cap \overset{*}{\phi}_{\sigma} \cap \mathfrak{F}(1)$. Then

$$G \in \mathfrak{J} \cap \sigma(1) \cap \mathfrak{F}(1)$$

$$\leq \mathfrak{J} \cap L\sigma \cap \mathfrak{F}(1)$$

$$\leq \sigma \cap \mathfrak{J} \cap \mathfrak{F}(1)$$

$$\leq \sigma \cap \hat{\mathfrak{M}} = p\,\mathfrak{C}$$

by 6·1 (iii) and lemma 2·1 of [23] , as required.

<u>Lemma 7·16</u> If $\mathfrak{X} = \langle S,Q,D_0\rangle \mathfrak{X}$ then

$$L\mathfrak{X} \cap \mathring{\Phi}_{\mathfrak{F}} \leq \mathring{\Phi}_{\mathfrak{X}\cap\mathfrak{F}}$$

<u>Proof</u> By 7·1, $L\mathfrak{X} \cap \mathfrak{F}(1) \leq (\mathfrak{X}\cap\mathfrak{F})(1)$. Suppose that $L\mathfrak{X} \cap \mathfrak{F}(c) \leq (\mathfrak{X}\cap\mathfrak{F})(c)$ and let $G \in L\mathfrak{X} \cap \mathfrak{F}(c+1)$. Let $x \in H_1(G:\mathfrak{F})$. Then there exists a normal subgroup N of G such that $[x,N] = 1$ and $G/N \in \mathfrak{F} \cap QL\mathfrak{X} \leq \mathfrak{F}\cap\mathfrak{X}$. Therefore $H_1(G:\mathfrak{X}) = H_1(G:\mathfrak{X}\cap\mathfrak{F})$ and so

$$G/H_1(G:\mathfrak{X}\cap\mathfrak{F}) = G/H_1(G:\mathfrak{X}) \in L\mathfrak{X} \cap \mathfrak{F}(c)$$

by corollary 6 of [3] . Therefore

$$G/H_1(G:\mathfrak{X}\cap\mathfrak{F}) \in L\mathfrak{X} \cap \mathfrak{F}(c) \leq (\mathfrak{X}\cap\mathfrak{F})(c)$$

by hypothesis. Again by corollary 6 of [3] , we have $G \in (\mathfrak{X}\cap\mathfrak{F})(c)$, as required.

<u>Theorem 7·17</u> $L\mathfrak{H} \cap \mathring{\Phi}_{\mathfrak{F}} \leq \mathring{\Phi}_{\mathfrak{H}\cap\mathfrak{F}} \leq L\,p\mathfrak{C} \cap \mathring{\Phi}_{\mathfrak{F}}$

<u>Proof</u> Lemma 7·16 proves that $L\mathfrak{H} \cap \mathring{\Phi}_{\mathfrak{F}} \leq \mathring{\Phi}_{\mathfrak{H}\cap\mathfrak{F}}$. Let $G \in (\mathfrak{H}\cap\mathfrak{F})(c) \cap \mathfrak{Y}$. Then $G \in \mathfrak{F}(c) \cap \mathfrak{Y} \leq \mathfrak{F}(c) \cap \hat{\mathfrak{H}}$ by theorem 1 of [23] . Therefore $G \in (\mathfrak{F}(1))^c \cap \hat{\mathfrak{H}} \leq (p\mathfrak{C})\mathfrak{F}$ by theorem 3 of [23] . Thus

$$\begin{aligned} G \in (p\mathfrak{C})\mathfrak{F} \cap \mathfrak{H}(c) &\leq p\mathfrak{C}(\mathfrak{F}\cap\mathfrak{H}(c)) \\ &\leq p\mathfrak{C}(\mathfrak{F}\cap\mathfrak{H}(1))^c \\ &\leq p\mathfrak{C}(\mathfrak{F}\cap\mathfrak{H})^c = p\mathfrak{C} \end{aligned}$$

as required.

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