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RESEARCH ARTICLE

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On the solvability of the Lie algebra $\mathrm{HH}^1(B)$ for blocks of finite groups

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Abstract

We give some criteria for the Lie algebra $\mathrm{HH}^1(B)$ to be solvable, where B is a p-block of a finite group algebra, in terms of the action of an inertial quotient of B on a defect group of B.

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1 | INTRODUCTION

The Lie algebra structure of the first Hochschild cohomology of a block of a finite group algebra sits at the crossroads of the representation theory of a block as a part of the wider theory of representations of finite-dimensional algebras and the fusion systems and their invariants that can be associated with block algebras. This Lie algebra is therefore one of the ingredients that has the potential to feed into an understanding of the connections between the global and local structure of block algebras. The purpose of this paper is to contribute to investigating this connection.

Let p be a prime number and k a field of characteristic p. A block of a finite group algebra kG is an indecomposable direct factor B of kG as an algebra. A defect group of a block B of kG is a maximal p-subgroup P of G such that kP is isomorphic to a direct summand of B as a kP-kP-bimodule. The results in this paper are a contribution to the broader theme investigating connections between Hochschild cohomology and fusion systems of blocks. More precisely, the main results of this paper relate the Lie algebra structure of $\mathrm{HH}^1(B)$, notably the solvability of this Lie algebra, to the action of an inertial quotient E on a defect group of the block.

For P a finite p-group, we denote by $\Phi(P)$ the Frattini subgroup of P; this is the smallest normal subgroup of P such that $P/\Phi(P)$ is elementary abelian. If E is a finite group acting on P, then this action induces an action of E on $P/\Phi(P)$. In this way, we can regard $P/\Phi(P)$ as an $\mathbb{F}_pE$-module. If in addition E has order prime to p, then $P/\Phi(P)$ is a semisimple $\mathbb{F}_pE$-module. The following results have in common that the property of this module being multiplicity free is the key ingredient for the first Hochschild cohomology to be solvable as a Lie algebra.

Theorem 1.1. *Let G be a finite group, and assume that k is large enough for the subgroups of G. Let B be a block of kG with a non-trivial abelian defect group P and a non-trivial inertial quotient E acting freely on $P \setminus \{1\}$. If the $\mathbb{F}_pE$-module $P/\Phi(P)$ is multiplicity free, then the Lie algebra $\mathrm{HH}^1(B)$ is solvable. The converse holds if p is odd.*

In the course of the proof we will describe more precise results on the Lie algebra structure of $\mathrm{HH}^1(B)$. One key ingredient is a stable equivalence of Morita type between the block B and the semidirect product $k(P \rtimes E)$, due to Puig. Another key ingredient is the next result which investigates the Lie algebra structure of $\mathrm{HH}^1(kP)^E$. We denote by $[P,E]$ the subgroup of P generated by the set of elements of the form $({}^e u)u^{-1}$, where $u \in P$ and $e \in E$ (this is the hyperfocal subgroup in P of $P \rtimes E$).

Theorem 1.2. *Let P be a non-trivial finite abelian p-group and E a finite p'-group acting on P. Suppose $[P,E] = P$.*

(i) *Every E-stable derivation on kP has image contained in the Jacobson radical $J(kP)$.*
(ii) *If the $\mathbb{F}_pE$-module $P/\Phi(P)$ is multiplicity free, then $\mathrm{HH}^1(kP)^E$ is a solvable Lie algebra. The converse holds if p is odd.*

The two theorems above will be proved in Section 6. When the acting p'-group E is abelian as well, we can be more precise. See Section 4 for the notation and basic facts on twisted group algebras. The following result will be proved in Section 7.

Theorem 1.3. *Let P be a non-trivial finite abelian p-group and E an abelian p'-subgroup of $\mathrm{Aut}(P)$. Let $\alpha \in Z^2(E; k^\times)$ inflated to $P \rtimes E$ via the canonical surjection $P \rtimes E \to E$. Suppose $[P,E] = P$.*

(i) *Every class in $\mathrm{HH}^1(k_\alpha(P \rtimes E))$ is represented by a derivation on $k_\alpha(P \rtimes E)$ with image contained in the Jacobson radical $J(k_\alpha(P \rtimes E))$.*
(ii) *If the $\mathbb{F}_pE$-module $P/\Phi(P)$ is multiplicity free, then the Lie algebra $\mathrm{HH}^1(k_\alpha(P \rtimes E))$ is solvable. The converse holds if p is odd.*

If one replaces twisted group algebras by group algebras of corresponding central extensions, then Theorem 1.3 admits an equivalent reformulation, in which the acting group E need not act faithfully and need not be abelian so long as its image in $\mathrm{Aut}(P)$ is abelian; see Theorem 7.4 below. We illustrate the above results in conjunction with the structure theory of normal defect blocks in Theorem 6.2 and Corollary 7.7, and we determine under what circumstances the Lie algebra $\mathrm{HH}^1(B)$ is simple or solvable for blocks B with elementary abelian defect of rank 2 and abelian inertial quotient in Example 8.4.

2 | BACKGROUND MATERIAL

Let k be a field. Let A be an associative unital k-algebra. A *derivation on A* is a k-linear map $f : A \to A$ satisfying the Leibniz rule $f(ab) = f(a)b + af(b)$, for all $a, b \in A$. The Leibniz rule implies that any derivation f on A vanishes at all central idempotents; in particular, $f(1) = 0$. The set $\mathrm{Der}(A)$ of all derivations on A is a Lie subalgebra of $\mathrm{End}_k(A)$ with Lie bracket $[f, g] = f\circ g - g\circ f$, for all $f, g \in \mathrm{End}_k(A)$. If $c \in A$, then the map $[c, -]$ sending $a \in A$ to the additive commutator $[c, a] = ca - ac$ is a derivation. The derivations of this form are called *inner derivations on A*, and the subspace $\mathrm{IDer}(A)$ of inner derivations is an ideal in the Lie algebra $\mathrm{Der}(A)$.

For M an A-A-bimodule, regarded as an $A \otimes_k A^{\mathrm{op}}$-module, the Hochschild cohomology of A with coefficients in M is the graded k-module $\mathrm{HH}^*(A; M) = \mathrm{Ext}^*_{A\otimes_k A^{\mathrm{op}}}(A; M)$. We set $\mathrm{HH}^*(A) = \mathrm{HH}^*(A; A)$. Then $\mathrm{HH}^*(A)$ is a graded-commutative algebra, and $\mathrm{HH}^*(A; M)$ is a graded right $\mathrm{HH}^*(A)$-module. We have canonical identifications $\mathrm{HH}^0(A) \cong \mathrm{Z}(A)$ and $\mathrm{HH}^1(A) \cong \mathrm{Der}(A)/\mathrm{IDer}(A)$; see, for instance, Weibel [26, Lemma 9.2.1]. If f is a derivation on A and α a k-algebra automorphism of A, then $\alpha^{-1}\circ f\circ\alpha$ is a derivation on A, and if f is an inner derivation, then so is $\alpha^{-1}\circ f\circ\alpha$. Thus, if E is a group acting on A by k-algebra automorphisms, then this action induces an action of E on $\mathrm{HH}^1(A)$ by Lie algebra automorphisms, and the subspace $\mathrm{HH}^1(A)^E$ of E-fixed points in $\mathrm{HH}^1(A)$ is a Lie subalgebra of $\mathrm{HH}^1(A)$. We will need the following well-known facts.

Lemma 2.1 (cf. [18, Lemma 3.1]). *Let A be a finite-dimensional associative unital k-algebra. For every derivation f on A, we have $f(Z(A)) \subseteq Z(A)$.*

Lemma 2.2 (cf. [19, Lemma 2.4]). *Let A be a finite-dimensional associative unital k-algebra. Suppose that A has a separable subalgebra C such that $A = C \oplus J(A)$. Every class in $\mathrm{HH}^1(A)$ has a representative $f \in \mathrm{Der}(A)$ satisfying $C \subseteq \ker(f)$.*

We note that in [19, Lemma 2.4, Proposition 2.8] the algebra A is assumed to be split, but the proof there shows that this is not needed so long as in the previous Lemma A is assumed to have a separable subalgebra C satisfying $A = C \oplus J(A)$. By the Malcev–Wedderburn Theorem, this is equivalent to requiring $A/J(A)$ to be separable, in which case we have $C \cong A/J(A)$. If f is a derivation on A which vanishes on C and sends $J(A)$ to $J(A)^m$ for some positive integer m, then in fact $\mathrm{Im}(f) \subseteq J(A)^m$. The following proposition is a slight variation of [19, Proposition 2.8], with essentially unchanged proofs, making repeatedly use of the Leibniz rule. We denote by $\ell\ell(A)$ the *Loewy length of A*; this is the smallest positive integer m such that $J(A)^m = 0$.

Proposition 2.3. *Let A be a finite-dimensional associative unital k-algebra. For $m \geqslant 1$, denote by $\mathrm{Der}_m(A)$ the subspace of $\mathrm{Der}(A)$ consisting of all derivations $f : A \to A$ such that $\mathrm{Im}(f) \subseteq J(A)^m$. The following hold.*

(i) *For any positive integers m, n, we have $[\mathrm{Der}_m(A), \mathrm{Der}_n(A)] \subseteq \mathrm{Der}_{m+n-1}(A)$.*
(ii) *The space $\mathrm{Der}_1(A)$ is a Lie subalgebra of $\mathrm{Der}(A)$, and for any positive integer m, the space $\mathrm{Der}_m(A)$ is a Lie ideal in $\mathrm{Der}_1(A)$.*
(iii) *The space $\mathrm{Der}_2(A)$ is a nilpotent ideal in $\mathrm{Der}_1(A)$. More precisely, if $\ell\ell(A) \leqslant 2$, then $\mathrm{Der}_2(A) = 0$, and if $\ell\ell(A) > 2$, then the nilpotency class of $\mathrm{Der}_2(A)$ is at most $\ell\ell(A) - 2$.*

Corollary 2.4. *With the notation and hypotheses of Proposition 2.3, the following hold.*

(i) *If* $[\mathrm{Der}_1(A), \mathrm{Der}_1(A)] \subseteq \mathrm{Der}_2(A)$, *then* $\mathrm{Der}_1(A)$ *is a solvable Lie algebra.*
(ii) *If* $[\mathrm{Der}_1(A), \mathrm{Der}_1(A)] \subseteq \mathrm{Der}_2(A) + \mathrm{IDer}(A)$, *then the image of* $\mathrm{Der}_1(A)$ *in* $\mathrm{HH}^1(A)$ *is a solvable Lie algebra.*
(iii) *If the canonical map* $\mathrm{Der}_1(A) \to \mathrm{HH}^1(A)$ *is surjective, and if* $[\mathrm{Der}_1(A), \mathrm{Der}_1(A)] \subseteq \mathrm{Der}_2(A) + \mathrm{IDer}(A)$, *then* $\mathrm{HH}^1(A)$ *is a solvable Lie algebra.*
(iv) *If the canonical map* $\mathrm{Der}_1(A) \to \mathrm{HH}^1(A)$ *is surjective, then the image of* $\mathrm{Der}_2(A)$ *is a nilpotent ideal in* $\mathrm{HH}^1(A)$. *Furthermore, suppose* $\mathrm{HH}^1(A) = L + D_2$ *where* L *is a Lie subalgebra and* D_2 *is the image of* $\mathrm{Der}_2(A)$, *then* $\mathrm{HH}^1(A)$ *is solvable if and only if* L *is solvable.*

Proof. The statements (i) and (ii) follow from Proposition 2.3(iii) and the assumptions. Statement (iii) are immediate consequences of (ii). As for statement (iv), suppose $\mathrm{HH}^1(A) = L + D_2$ as in the statement. If L is not solvable, then $\mathrm{HH}^1(A)$ is not solvable. Suppose L is solvable, and $L(n) = 0$ for some positive integer n, where $L(n)$ denotes the nth derived Lie algebra of L. Since D_2 is an ideal, it follows that for any $i \geqslant 1$ we have $\mathrm{HH}^1(A)(i) \subseteq L(i) + D_2$. Thus, $\mathrm{HH}^1(A)(n) \subseteq D_2$. The statement follows since D_2 is nilpotent. □

We denote by $[A, A]$ the subspace spanned by the set of additive commutators $[a, b] = ab - ba$, where a, $b \in A$. Since $[a, b]c = abc - bac = abc - acb + acb - bac = a[b, c] + [ac, b]$ for all a, b, $c \in A$, we have $[A, A]A = A[A, A]$, and this is the smallest ideal such that the corresponding quotient of A is commutative.

Lemma 2.5. *Let* A *be a finite-dimensional associative unital* k*-algebra. Every derivation on* A *preserves the subspace* $[A, A]$ *and the ideal* $[A, A]A$, *and induces a derivation on* $A/[A, A]A$. *Under this correspondence, an inner derivation on* A *is mapped to zero. In particular, this correspondence induces a Lie algebra homomorphism* $\mathrm{HH}^1(A) \to \mathrm{HH}^1(A/[A, A]A)$.

Proof. If f is a derivation on A, then $f([a, b]) = f(a)b + af(b) - f(b)a - bf(a) = [f(a), b] + [a, f(b)]$, and hence f preserves the subspace $[A, A]$ and hence also the ideal $[A, A]A$, using the Leibniz rule. Thus f induces a derivation on $A/[A, A]A$. If f is inner, then the image of f is contained in $[A, A]$, and the rest follows easily. □

Suppose now that k has prime characteristic p. If $A = kG$ for some finite group G, then the largest commutative quotient of kG is kG/G', where G' is the commutator subgroup of G. Thus, $[kG, kG]kG = I(kG')kG$. One can verify this also directly by noting the relation between additive and multiplicative commutators $xyx^{-1}y^{-1} - 1 = [x, y]x^{-1}y^{-1}$ for all x, $y \in G$. Thus, Lemma 2.5 specialises to the following observation.

Lemma 2.6. *Let* G *be a finite group. Denote by* G' *the commutator subgroup. Every derivation on* kG *induces a derivation on* kG/G', *and every inner derivation on* kG *induces the zero map on* kG/G'. *Through this correspondence, the canonical surjection* $G \to G/G'$ *induces a Lie algebra homomorphism* $\mathrm{HH}^1(kG) \to \mathrm{HH}^1(kG/G')$.

Proof. This is a special case of Lemma 2.5, using the equality $[kG, kG]kG = I(kG')kG$ mentioned above. □

We use without further comment the standard fact that for P a finite p-group the augmentation ideal $I(kP)$ in kP is equal to the Jacobson radical $J(kP)$. We denote by $\Phi(P)$ the Frattini subgroup of P; this is the smallest normal subgroup of P such that the quotient $P/\Phi(P)$ is elementary abelian, with the convention $\Phi(P) = 1$ if $P = 1$. The following is well known; we sketch a proof for convenience.

Lemma 2.7. *Let P be a finite p-group and E a subgroup of* $\mathrm{Aut}(P)$*. The map sending $y \in P$ to $y - 1 \in J(\mathbb{F}_p P)$ induces an isomorphism of $\mathbb{F}_p E$-modules*

$$P/\Phi(P) \cong J(\mathbb{F}_p P)/J(\mathbb{F}_p P)^2.$$

Proof. Set $J = J(\mathbb{F}_p P)$. Let $x, y \in P$. Then $(x-1)(y-1) \in J^2$. Since $(x-1)(y-1) = (xy-1) - (x-1) - (y-1)$, it follows that $xy - 1$ and $(x-1)+(y-1)$ have the same image in J/J^2. Thus, the map $x \mapsto x - 1$ induces a surjective group homomorphism $P \to J/J^2$. Since the right side is an abelian group, the kernel of this group homomorphism contains the commutator subgroup of P, and since $\mathbb{F}_p$ has characteristic p, the kernel contains also x^p for all $x \in P$. Thus, the map $x \mapsto x - 1$ yields a surjective group homomorphism $P/\Phi(P) \to J/J^2$. Both sides are easily seen to have the same dimension, equal to the rank of the elementary abelian p-group $P/\Phi(P)$. □

Note that the unit element of $P/\Phi(P)$ is mapped to the zero element in $J(\mathbb{F}_p P)/J(\mathbb{F}_p P)^2$ in the Lemma 2.7. We will further need the following observation regarding the hyperfocal subgroup $[P, E]$ of $P \rtimes E$ in P.

Lemma 2.8. *Let P be a finite p-group and E a finite group of order prime to p which acts on P. The following are equivalent.*

(i) *We have $[P, E] = P$.*
(ii) *We have $[P/\Phi(P), E] = P/\Phi(P)$.*
(iii) *The $\mathbb{F}_p E$-module $J(\mathbb{F}_p P)/J(\mathbb{F}_p P)^2$ has no non-zero trivial direct summand.*
(iv) *The kE-module $J(kP)/J(kP)^2$ has no non-zero trivial direct summand.*

Proof. Clearly $[P/\Phi(P), E]$ is the image of $[P, E]$ under the canonical surjection $P \to P/\Phi(P)$, so (i) implies (ii) trivially. If $[P, E]$ is a proper subgroup of P, then so is its image in $P/\Phi(P)$ since $\Phi(P)$ is the intersection of all maximal subgroups of P. Thus (ii) implies (i). By standard facts on coprime group actions, we have $P/\Phi(P) = [P/\Phi(P), E] \times C_{P/\Phi(P)}(E)$, thus (ii) is equivalent to the statement $C_{P/\Phi(P)}(E) = 1$. Under the isomorphism $P/\Phi(P) \cong J(\mathbb{F}_p P)/J(\mathbb{F}_p P)^2$ from Lemma 2.7 this is equivalent to (iii). Since $J(kP) = k \otimes_{\mathbb{F}_p} J(\mathbb{F}_p P)$ and similarly for $J(kP)^2$, we have $J(kP)/J(kP)^2 \cong k \otimes_{\mathbb{F}_p} J(\mathbb{F}_p P)/J(\mathbb{F}_p P)$ as kE-modules. Setting $U = J(\mathbb{F}_p P)/J(\mathbb{F}_p P)^2$, the equivalence of (iii) and (iv) follows from the canonical isomorphisms $k \otimes_{\mathbb{F}_p} U^E \cong k \otimes_{\mathbb{F}_p} \mathrm{Hom}_{F_pE}(\mathbb{F}_p, U) \cong \mathrm{Hom}_{kE}(k, k \otimes_{\mathbb{F}_p} U) \cong (k \otimes_{\mathbb{F}_p} U)^E$, where for the second isomorphism we make use of the well-known fact [15, Corollary 1.12.11] on scalar extensions of homomorphism spaces. □

For convenience we draw attention to the following obvious fact.

Lemma 2.9. *Let P be a finite p-group and E a finite group of order prime to p which acts on P. Every element in* $\mathrm{HH}^1(kP)^E$ *has a representative in* $\mathrm{Der}(kP)^E$.

Proof. The canonical surjection $\mathrm{Der}(kP) \to \mathrm{HH}^1(kP)$ is E-stable, so remains surjective upon taking E-fixed points as E is a p'-group. □

We will need the following fact from [11].

Lemma 2.10 (cf. [11, Lemma 5.1]). *Let P be a finite p-group and α an automorphism of P with no non-trivial fixed point. Then the kP-module $(kP)_\alpha$, with $u \in P$ acting on $x \in P$ by $ux\alpha(u)^{-1}$, is projective.*

Proof. The hypothesis on α implies that if u runs over all elements of P, then so does $u\alpha(u)^{-1}$. Thus, the given action of P on itself is transitive because the P-orbit of 1 is P. The lemma follows. □

3 | THE KÜNNETH FORMULA AND SOLVABILITY OF HH^1

Let k be a field. For A an associative unital k-algebra and m a positive integer, we denote as before by $\mathrm{Der}_m(A)$ the space of derivations on A with image contained in $J(A)^m$. Given two algebras A, B, the solvability of $\mathrm{HH}^1(A)$ and $\mathrm{HH}^1(B)$ does not necessarily imply the solvability of $\mathrm{HH}^1(A \otimes_k B)$. The following observation implies that the slightly stronger condition from Corollary 2.4(iii) does extend to tensor products.

Proposition 3.1. *Let A, B be two associative unital k-algebras. Suppose that the canonical maps $\mathrm{Der}_1(A) \to \mathrm{HH}^1(A)$ and $\mathrm{Der}_1(B) \to \mathrm{HH}^1(B)$ are surjective. Then the map $\mathrm{Der}_1(A \otimes_k B) \to \mathrm{HH}^1(A \otimes_k B)$ is surjective. Suppose further that $[\mathrm{Der}_1(A), \mathrm{Der}_1(A)] \subseteq \mathrm{Der}_2(A) + \mathrm{IDer}(A)$ and that $[\mathrm{Der}_1(B), \mathrm{Der}_1(B)] \subseteq \mathrm{Der}_2(B) + \mathrm{IDer}(B)$. Then*

$$[\mathrm{Der}_1(A \otimes_k B), \mathrm{Der}_1(A \otimes_k B)] \subseteq \mathrm{Der}_2(A \otimes_k B) + \mathrm{IDer}(A \otimes_k B).$$

In particular, the Lie algebra $\mathrm{HH}^1(A \otimes_k B)$ is solvable.

The proof of this proposition is based on the Künneth formula

3.2.

$$\mathrm{HH}^1(A \otimes_k B) \cong \mathrm{Z}(A) \otimes_k \mathrm{HH}^1(B) \oplus \mathrm{HH}^1(A) \otimes_k \mathrm{Z}(B),$$

where we use the canonical identifications $\mathrm{HH}^0(A) \cong \mathrm{Z}(A)$ and $\mathrm{HH}^0(B) \cong \mathrm{Z}(B)$. This formula extends in the obvious way to tensor products of more than two algebras. The Künneth isomorphism 3.2 is induced by with the map sending $z \otimes g$ to the derivation

$$a \otimes b \mapsto az \otimes g(b)$$

on $A \otimes_k B$, where $a \in A, b \in B, z \in \mathrm{Z}(A)$ and g is a derivation on B, together with the map sending $f \otimes w$ to the derivation

$$a \otimes b \mapsto f(a) \otimes bw$$

on $A \otimes_k B$, where f is a derivation on A and $w \in Z(B)$. A trivial verification shows that if $g = [d, -]$ for some $d \in B$ is an inner derivation on B, then the derivation on $A \otimes_k B$ corresponding to $z \otimes g$ is inner, equal to $[z \otimes d, -]$. Similarly, if $f = [c, -]$ for some $c \in A$ is an inner derivation on A, then the derivation on $A \otimes_k B$ corresponding to $f \otimes w$ is inner, and equal to $[c \otimes w, -]$.

The Lie bracket can be followed through the Künneth isomorphism as follows. Given two derivations g, g' on B and $z, z' \in Z(A)$, the Lie bracket of the derivations corresponding to $z \otimes g$, $z' \otimes g'$ is given by

3.3.

$$[z \otimes g, z' \otimes g'] = zz' \otimes [g, g'],$$

or explicitly, the right side is the map

$$a \otimes b \mapsto azz' \otimes [g, g'](b).$$

Similarly, given two derivations f, f' on A and $w, w' \in Z(B)$, and identifying $f \otimes w$ with the derivation $a \otimes b \mapsto f(a) \otimes bw$, the Lie bracket of the derivations $f \otimes w$, $f' \otimes w'$ is given by

3.4.

$$[f \otimes w, f' \otimes w'] = [f, f'] \otimes ww',$$

or explicitly, the right side is the map

$$a \otimes b \mapsto [f, f'](a) \otimes bww'.$$

The formulas 3.3 and 3.4 show that the two summands in the Künneth decomposition 3.2 are both Lie subalgebras. Applied with $w = w' = 1_B$ and $z = z' = 1_A$, these formulas show that $HH^1(A)$ and $HH^1(B)$ are isomorphic to Lie subalgebras of $HH^1(A \otimes_k B)$, so if one of $HH^1(A)$, $HH^1(B)$ is not solvable, then neither is $HH^1(A \otimes_k B)$. Note though that the solvability of both $HH^1(A)$, $HH^1(B)$ need not imply the solvability of $HH^1(A \otimes_k B)$. By Lemma 2.1 we have $f(z) \in Z(A)$ and $g(w) \in Z(B)$. We denote by $z \cdot f$ (resp. $w \cdot g$) the derivation on A (resp. on B) given by $(z \cdot f)(a) = zf(a)$ (resp. $(w \cdot g)(b) = wg(b)$). The Lie bracket $[f \otimes w, z \otimes g]$ is given by

3.5.

$$[f \otimes w, z \otimes g] = f(z) \otimes w \cdot g - z \cdot f \otimes g(w),$$

or equivalently, the right side is the map

$$a \otimes b \mapsto f(az) \otimes wg(b) - f(a)z \otimes g(bw) = af(z) \otimes wg(b) - zf(a) \otimes bg(w).$$

In particular, we have

$$[f \otimes 1, z \otimes g] = f(z) \otimes g.$$

Indeed, the first formula uses the Leibniz rule applied to the terms $f(az)$ and $g(bw)$, followed by cancelling two terms, and the last formula follows from applying this to $w = 1$ and using $g(1) = 0$. The formula 3.5 shows that the two summands in the Künneth formula do not necessarily commute, so this is not, in general, a direct product of Lie algebras. We note the following consequence of this formula, used in the proof of Proposition 3.1.

Lemma 3.6. *Let A, B be finite-dimensional associative unital k-algebras, and $z \in \mathrm{Z}(A)$, $w \in \mathrm{Z}(B)$, f a derivation on A, and g a derivation on B. Suppose $\mathrm{Im}(f) \subseteq J(A)$ and $\mathrm{Im}(g) \subseteq J(B)$. Then $[f \otimes w, z \otimes g]$, regarded as a derivation on $A \otimes_k B$, has image contained in $J(A \otimes_k B)^2$.*

Proof. The hypotheses and formula 3.5 imply that the image of the derivation $[f \otimes w, z \otimes g]$ on $AA \otimes_k B$ is contained in $J(A) \otimes J(B) \subseteq (J(A) \otimes B)(A \otimes J(B)) \subseteq J(A \otimes_k B)^2$. □

Proof of Proposition 3.1. Given a derivation f on A with image contained in $J(A)$ and an element $w \in \mathrm{Z}(B)$, the derivation on $A \otimes_k B$ corresponding to $f \otimes w$ has image contained in $J(A) \otimes_k B \subseteq J(A \otimes_k B)$. Similarly, given a derivation g on B and $z \in \mathrm{Z}(A)$, the derivation on $A \otimes_k B$ corresponding to $z \otimes g$ has image contained in $J(A \otimes_k B)$. The Künneth formula 3.2 implies the first statement. The second statement follows from combining the formulas 3.4, 3.3 and Lemma 3.6. The solvability of $\mathrm{HH}^1(A \otimes_k B)$ follows from Corollary 2.4. □

Lemma 3.7. *Let A, B be finite-dimensional associative unital k-algebras. Suppose that the canonical map $\mathrm{Der}_1(B) \to \mathrm{HH}^1(B)$ is surjective. Then the space*

$$Z(A) \otimes_k \mathrm{HH}^1(B) \ \oplus \ \mathrm{HH}^1(A) \otimes_k J(Z(B)),$$

identified to its image in $\mathrm{HH}^1(A \otimes_k B)$, is a Lie ideal in $\mathrm{HH}^1(A \otimes_k B)$. In particular, if $\mathrm{HH}^1(A)$ is non-zero, then $\mathrm{HH}^1(A \otimes_k B)$ is not a simple Lie algebra.

Proof. This follows from the formulas 3.4 and 3.3, together with the fact that if g is a derivation on B with image contained in $J(B)$, then, by Lemma 2.1, g sends $Z(B)$ to $Z(B) \cap J(B) = J(Z(B))$. If $\mathrm{HH}^1(A) \neq 0$, then the space displayed in the statement does not contain $\mathrm{HH}^1(A) \otimes 1_B$, so this is a proper ideal. □

If the algebra B is separable, then the Künneth formula yields an isomorphism

3.8.

$$\mathrm{HH}^1(A \otimes_k B) \cong \mathrm{HH}^1(A) \otimes_k \mathrm{Z}(B).$$

We will need one further special case of the Künneth formula for finite group algebras. Given finite groups G, H, a kG-module U and a kH-module V, we have a natural graded k-linear isomorphism

3.9.

$$\mathrm{H}^*(G \times H; U \otimes_k V) \cong \mathrm{H}^*(G; U) \otimes_k \mathrm{H}^*(H; V),$$

where the grading on the right side is the total grading. Explicitly, for any positive integer n, we have

3.10.

$$\mathrm{H}^n(G\times H; U\otimes_k V) \cong \oplus_{(i,j)} \mathrm{H}^i(G;U)\otimes_k \mathrm{H}^j(H;V),$$

where (i,j) runs over all pairs of non-negative integers such that $i+j=n$. See, for instance, [1, Theorem 3.5.6]. For $n=1$, this yields an isomorphism

3.11.

$$\mathrm{H}^1(G\times H; U\otimes_k V)\cong U^G\otimes_k \mathrm{H}^1(H;V) \oplus \mathrm{H}^1(G;U)\otimes_k V^G.$$

Under this isomorphism an element in $U^G\otimes_k \mathrm{H}^1(H;V)$ given by $u\otimes\tau$ for some $u\in U^G$ and some $\tau\in Z^1(H;V)$ corresponds to the element in $\mathrm{H}^1(G\times H;U\otimes_k V)$ given by the 1-cocycle $(x,y)\mapsto u\otimes\tau(y)$. The analogous statement holds for elements in the second summand.

4 | CALCULATIONS IN TWISTED GROUP ALGEBRAS

One of the standard tools for calculating the Hochschild cohomology of a finite group algebra is the centraliser decomposition, which is shown in [25, Lemma 3.5] to carry over to crossed products, and in particular therefore to twisted group algebras. We review very briefly what we will need in this paper; for more background material, see for instance [15, Section 1.2].

Let G be a finite group and let k be a field. Let $\alpha\in Z^2(G;k^\times)$. The twisted group algebra $k_\alpha G$ has a k-basis $\{\hat g \mid g\in G\}$ in bijection with the elements of G. The multiplication in $k_\alpha G$ is given by $\hat g\hat h=\alpha(g,h)\widehat{gh}$, for $g,h\in G$, extended bilinearly to $k_\alpha G$. The identity element in $k_\alpha G$ is $\alpha(1,1)^{-1}\hat 1$, and hence, for $g\in G$, the inverse of $\hat g$ in $k_\alpha G$ is given by

4.1.

$$\hat g^{-1}=\alpha(1,1)^{-1}\alpha(g,g^{-1})^{-1}\widehat{g^{-1}},$$

where g^{-1} is the inverse of g in G. The isomorphism class of $k_\alpha G$ depends only on the class of α in $H^2(G;k^\times)$, and we may therefore assume that α is *normalised*; that is, $\alpha(g,1)=1=\alpha(1,g)$ for all $g\in G$. This is equivalent to requiring that $\hat 1$ remains the identity element in $k_\alpha G$. We note that if α is normalised, then the inverse of $\hat g$ in $k_\alpha G$ is equal to $\hat g^{-1}=\alpha(g,g^{-1})^{-1}\widehat{g^{-1}}$. A short calculation shows that the conjugation action in $k_\alpha G$ is given by

4.2.

$${}^{\hat g}\hat h=\hat g\hat h\hat g^{-1}=\lambda(g,h)\widehat{{}^g h},$$

where $g,h\in G$ and where $\lambda(g,h)\in k^\times$ is given by the formula

$$\lambda(g,h)=\alpha(g,h)\alpha(gh,g^{-1})\alpha(g,g^{-1})^{-1}\alpha(1,1)^{-1}.$$

In particular, we have $\lambda(1,h) = 1 = \lambda(g,1)$. If N is a normal subgroup of G and $\alpha \in Z^2(G/N; k^\times)$ inflated to G via the canonical surjection, still denoted by α, and if we assume in addition that α is normalised, then for $g \in N$ and $h \in G$ the above formula yields $\lambda(g,h) = 1$, hence in that case we have

4.3.

$$^{\hat{g}}\hat{h} = \widehat{^{g}h}.$$

For M a $k_\alpha G$-$k_\alpha G$-bimodule, we have a standard adjunction isomorphism

4.4.

$$\mathrm{HH}^*(k_\alpha G; M) \cong \mathrm{H}^*(G; M),$$

where $g \in G$ acts on $m \in M$ by $^{g}m = \hat{g} m \hat{g}^{-1}$, having checked that this is well defined. Note that while M is considered as a kG-module, the cohomology $\mathrm{H}^*(G; M)$ still depends on α, even though α does not explicitly appear in the notation. In particular, with $M = k_\alpha G$, the group G acts on $k_\alpha G$ with $g \in G$ acting by conjugation with $\hat{g}$, and we have a graded isomorphism $\mathrm{HH}^*(k_\alpha G) \cong \mathrm{H}^*(G; k_\alpha G)$, which is the first step towards the centraliser decomposition of $\mathrm{HH}^*(k_\alpha G)$ in the proof of [25, Lemma 3.5]. We will need the isomorphism 4.4 in degree 1, where this is given explicitly as follows.

Lemma 4.5. *Let G be a finite group, k a field and $\alpha \in Z^2(G; k^\times)$. Let M be a $k_\alpha G$-$k_\alpha G$-bimodule. Let $d : k_\alpha G \to M$ be a k-linear map and $\tau : G \to M$ a map such that $d(\hat{g}) = \tau(g)\hat{g}$ for all $g \in G$. Then d is a derivation if and only if τ is a 1-cocycle. Moreover, the correspondence $\tau \mapsto d$ induces an isomorphism $\mathrm{H}^1(G; M) \cong \mathrm{HH}^1(k_\alpha G; M)$.*

Proof. Let $g, h \in G$. We have

$$d(\hat{g}\hat{h}) = \alpha(g,h) d(\widehat{gh}) = \alpha(g,h)\tau(gh)\widehat{gh} = \tau(gh)\hat{g}\hat{h}$$

and

$$d(\hat{g})\hat{h} + \hat{g} d(\hat{h}) = \tau(g)\hat{g}\hat{h} + \hat{g}\tau(h)\hat{h} = (\tau(g) + {}^{g}\tau(h))\hat{g}\hat{h}.$$

Thus, d is a derivation if and only if τ is a 1-cocycle. We have $\tau(g) = m - {}^{g}m$ for some $m \in M$ if and only if $d(\hat{g}) = \tau(g)\hat{g} = m\hat{g} - {}^{g}m\hat{g} = m\hat{g} - \hat{g}m = [m, \hat{g}]$. Thus, d is an inner derivation if and only if τ is a 1-coboundary. The result follows. □

By standard facts on group cohomology, this lemma implies that if N is a normal subgroup of G of index invertible in k, then, using [3, Proposition III.10.4] and the isomorphism 4.4 with N instead of G, we have an isomorphism

4.6.

$$\mathrm{HH}^*(k_\alpha G; M) \cong \mathrm{HH}^*(k_\alpha N; M)^{G/N} \cong \mathrm{H}^*(N; M)^{G/N},$$

where we use the same letter α for the restriction of α to $N \times N$. The action of G/N on the last two terms is induced by the conjugation action of G on $k_\alpha N$, on N, and on M, where we note that N acts as identity on $HH^*(k_\alpha N; M)$. (This is just the version for twisted group algebra of the arguments in the proof of [20, Theorem 3.2]). If $M = k_\alpha G$ and if α is the inflation to $G \times G$ of a 2-cocycle in $Z^2(G/N; k^\times)$, then $k_\alpha G = kN \oplus k_\alpha(G \setminus N)$, and hence, still assuming that the index of N in G is invertible in k, the first isomorphism in 4.6 specialises to

4.7.

$$HH^*(k_\alpha G) \cong HH^*(kN)^{G/N} \oplus HH^*(kN; k_\alpha(G \setminus N))^{G/N},$$

which shows in particular that $HH^1(kN)^{G/N}$ is a Lie subalgebra of $HH^1(k_\alpha G)$.

In what follows, we will frequently identify the elements in G to their images in $k_\alpha G$. In that case, for two elements $g, h \in G$, we will denote by gh the product in the group and by $g \cdot h$ the product in $k_\alpha G$.

5 | DERIVATIONS ON $k_\alpha(P \rtimes E)$ AND E-STABLE DERIVATIONS ON kP

Let p be a prime and k a field of characteristic p. We will apply the above calculations in twisted group algebras to groups of the form $P \rtimes E$ for some finite p-group P, some p'-subgroup E of $\mathrm{Aut}(P)$ and some $\alpha \in Z^2(E; k^\times)$ inflated to $P \rtimes E$ via the canonical surjection $P \rtimes E \to E$. The resulting 2-cocycle in $Z^2(P \rtimes E; k^\times)$ will abusively again be denoted by the same letter α. That is, for $u, v \in P$ and $x, y \in E$ we have

$$\alpha(ux, vy) = \alpha(x, y).$$

If we assume in addition that α is normalised, then $\alpha(x, y)$ is equal to 1 if one of x, y is trivial. Note that this implies in particular that kP is a subalgebra of $k_\alpha(P \rtimes E)$ and that $k_\alpha(P \rtimes E)$ is isomorphic to $k(P \rtimes E)$ as a kP-kP-bimodule (cf. [15, Corollary 5.3.8]). The conjugation action of E on kP and on $k_\alpha(P \rtimes E)$ induces an action of E on $HH^*(kP; k_\alpha(P \rtimes E))$.

Lemma 5.1. *Let P be a finite p-group and E a p'-subgroup of* $\mathrm{Aut}(P)$. *Let $\alpha \in Z^2(E; k^\times)$ inflated to $P \rtimes E$ via the canonical surjection $P \rtimes E \to E$.*

(i) *We have canonical graded isomorphisms*

$$\begin{aligned} HH^*(k_\alpha(P \rtimes E)) &\cong (HH^*(kP; k_\alpha(P \rtimes E))^E \\ &\cong HH^*(kP)^E \;\oplus\; (\oplus_{e \in E \setminus \{1\}} HH^*(kP; kP \cdot e))^E. \end{aligned}$$

(ii) *If E is abelian, then E stabilises every summand in the last direct sum in (i), and we have canonical graded isomorphisms*

$$HH^*(k_\alpha(P \rtimes E)) \cong \oplus_{e \in E} HH^*(kP; kP \cdot e)^E \;\cong\; \oplus_{e \in E} H^*(P; kP \cdot e)^E.$$

(iii) *If E acts freely on $P \setminus \{1\}$, then for all positive integers n we have*

$$\mathrm{HH}^n(k_\alpha(P \rtimes E)) \cong \mathrm{HH}^n(kP)^E.$$

For $n = 1$, this is a Lie algebra isomorphism.

Proof. Since E is a p'-group, the statements (i) and (ii) follow from the isomorphism 4.6. Statement (iii) is well-known (see e.g. [11, Proposition 5.2] or [20, Theorem 3.2]) and follows from (i) and together with the fact that $H^n(P; kP \cdot e) = 0$ for $e \neq 1$ by Lemma 2.10, for any positive integer n. □

For the sake of completeness, we show that the Lie algebra embedding of $\mathrm{HH}^1(kP)^E$ into $\mathrm{HH}^1(k_\alpha(P \rtimes E))$ is induced by a canonical map at the level of derivations.

Proposition 5.2. *Let P be a finite p-group and E a finite p'-group acting on P. Let $\alpha \in Z^2(E; k^\times)$, inflated to $P \rtimes E$ via the canonical surjection $P \rtimes E \to E$.*

(i) *Every E-stable derivation f on kP extends uniquely to a derivation $\hat{f}$ on $k_\alpha(P \rtimes E)$ with $k_\alpha E \subseteq \ker(\hat{f})$.*
(ii) *The correspondence $f \mapsto \hat{f}$ induces an injective Lie algebra homomorphism $\mathrm{HH}^1(kP)^E \to \mathrm{HH}^1(k_\alpha(P \rtimes E))$.*
(iii) *If E acts freely on $P \setminus \{1\}$, then the Lie algebra homomorphism in (ii) is an isomorphism.*

Proof. We may assume that α is normalised; that is, $\alpha(1, x) = 1 = \alpha(x, 1)$ for $x \in P \rtimes E$. Since α is inflated to $P \rtimes E$ via the canonical surjection it follows that for $u \in P$ and $y \in E$ we have $\alpha(u, y) = 1 = \alpha(y, u)$. Equivalently, the image in $k_\alpha(P \rtimes E)$ of the product uy (resp. yu) in $P \rtimes E$ is equal to the product $u \cdot y$ (resp. $y \cdot u$) in $k_\alpha(P \rtimes E)$.

Let $f \in \mathrm{Der}(kP)^E$. Define a linear map $\hat{f}$ on $k_\alpha(P \rtimes E)$ by setting

$$\hat{f}(uy) = f(u) \cdot y,$$

where uy is the product in $P \rtimes E$ and where the right side is the product taken in $k_\alpha(P \rtimes E)$. This defines $\hat{f}$ uniquely as a linear map on $k_\alpha(P \rtimes E)$ which extends f and vanishes on $k_\alpha E$. The Leibniz rule implies that if there is a derivation on $k_\alpha(P \rtimes E)$ which extends f and which vanishes on $k_\alpha E$, then it must be equal to $\hat{f}$. It remains to check that $\hat{f}$ is indeed a derivation.

Let $u, v \in P$ and $y, z \in E$. Calculating in $k_\alpha(P \rtimes E)$ and using that α is normalised and inflated to $P \rtimes E$, we have

$$(uy) \cdot (vz) = uyvz\alpha(uy, vz) = u({}^y v)yz\alpha(y, z) = u({}^y v) \cdot y \cdot z.$$

Thus,

$$\hat{f}((uv) \cdot (yz)) = \hat{f}(u({}^y v) \cdot y \cdot z) = f(u({}^y v)) \cdot y \cdot z.$$

We need to show that this is equal to $\hat{f}(uy) \cdot vz + uy \cdot \hat{f}(vz)$. Using that f is E-stable as well as a derivation, together with the comments preceding this proposition, we have

$$\hat{f}(uy)\cdot vz + uy\cdot \hat{f}(vz) = f(u)\cdot y\cdot vz + uy\cdot f(v)\cdot z = f(u)({}^y v)\cdot y\cdot z + u({}^y f(v))\cdot y\cdot z =$$

$$f(u)({}^y v)\cdot y\cdot z + uf({}^y v)\cdot y\cdot z = f(u({}^y v))\cdot y\cdot z,$$

which implies that $\hat{f}$ is a derivation on $k_\alpha(P\rtimes E)$. The construction of $\hat{f}$ implies that the assignment $f\mapsto \hat{f}$ is a Lie algebra homomorphism $\mathrm{Der}(kP)^E\to \mathrm{Der}(k_\alpha(P\rtimes E))$. If f is inner, hence equal to $[c,-]$ for some $c\in kP$, then $\hat{f}(uy)=f(u)y=[c,u]y$. The E-stability of f implies $[c,{}^y u]=[{}^y c,{}^y u]$. This holds for all $u\in P$, and hence $[c,-]$ and $[{}^y c,-]$ have the same restriction to kP, which is equal to f. Thus, we may replace c by $\frac{1}{|E|}\mathrm{Tr}_1^E(c)$, and then $\hat{f}=[c,-]$, showing that $\hat{f}$ is an inner derivation. Conversely, if $\hat{f}$ is an inner derivation, then $\hat{f}=[d,-]$ for some $d\in k_\alpha(P\rtimes E)$ which centralises $k_\alpha E$. Writing $d=\sum_{e\in E} c_e e$ for some $c_e\in kP$, one sees that $f=[c_1,-]$, so f is inner. This, together with Lemma 2.9, shows that the assignment $f\mapsto\hat{f}$ induces an injective Lie algebra homomorphism $\mathrm{HH}^1(kP)^E\to \mathrm{HH}^1(k_\alpha(P\rtimes E))$. The last statement follows from Lemma 5.1(iii). □

Remark 5.3. With the notation of Proposition 5.2, if E acts freely on $P\setminus\{1\}$, then $P\rtimes E$ is a Frobenius group. The structural properties of Frobenius groups, as described in [7, Theorem 10.3.1], imply that if k is algebraically closed, then $H^2(E;k^\times)$ is trivial. Thus, α may be chosen to be 1 in that case.

Lemma 5.4. *Let P be a finite p-group and E a finite p'-group acting on P. Set $Q=\Phi(P)$.*

(i) *The canonical surjection $P\to P/Q$ induces a Lie algebra homomorphism $\mathrm{HH}^1(kP)\to \mathrm{HH}^1(kP/Q)$.*
(ii) *If P is abelian, then $J(kQ)\subseteq J(kP)^p$, and the canonical surjection $P\to P/Q$ induces a surjective Lie algebra homomorphism $\mathrm{HH}^1(kP)\to\mathrm{HH}^1(kP/Q)$ with nilpotent kernel.*
(iii) *If P is abelian, then the Lie algebra homomorphism from (ii) induces a surjective Lie algebra homomorphism $\mathrm{HH}^1(kP)^E\to \mathrm{HH}^1(kP/Q)^E$ with nilpotent kernel.*

Proof. Since Q contains the commutator subgroup P' of P, the algebra homomorphism $kP\to kP/Q$ factors through the algebra homomorphism $kP\to kP/P'$. By Lemma 2.6, this homomorphism induces a Lie algebra homomorphism $\mathrm{HH}^1(kP)\to\mathrm{HH}^1(kP/P')$. Thus, we may assume that P is abelian. The kernel of the canonical algebra homomorphism $kP\to kP/Q$ is equal to $J(kQ)kP$. Since P is abelian, the subgroup Q consists of all elements of the for x^p, with $x\in P$. Thus, $J(kQ)$ is spanned by the set of elements of the form $x^p-1=(x-1)^p$. In particular, $J(kQ)\subseteq J(kP)^p$, which is the first statement in (ii). Since P is abelian, any derivation f on kP satisfies $f((x-1)^p)=p(x-1)^{p-1}f(x)=0$. Thus, the kernel of f contains $J(kQ)$. The Leibniz rule implies that f preserves the ideal $J(kQ)kP$. Thus, f induces a derivation on kP/Q. This shows (i).

For the surjectivity statement in (ii), one can either play this back to the case where P is cyclic via the Künneth formula, and then show by direct verification that every derivation on kP/Q is induced by a derivation on kP. Or one can use the formula $\mathrm{HH}^1(kP)\cong kP\otimes_k \mathrm{H}^1(P,k)$ from [8], with the analogous formula for P/Q and the fact that $\mathrm{H}^1(P;k)=\mathrm{H}^1(P/Q;k)$, since $\mathrm{Hom}(P,k)$ can be identified with $\mathrm{Hom}(P/Q,k)$.

It remains to show in (ii) that the kernel of the map $\mathrm{HH}^1(kP)\to\mathrm{HH}^1(kP/Q)$ is nilpotent. Let f be a derivation on kP inducing the zero map on kP/Q. Then the image of f is contained in $J(kQ)kP$, and by the above this is contained in $J(kP)^p$. The result follows from Proposition 2.3 (applied with $C=k\cdot 1$, which is in the kernel of every derivation on kP). This shows (ii).

Since E is a p'-group, statement (iii) is an immediate consequence of (ii). □

Proposition 5.5. *Let P be a finite p-group and E a finite p'-group acting on P. Suppose $[P,E] = P$. Let $e \in Z(E)$. Then kPe is an E-stable kP-kP-bimodule summand of $k(P \rtimes E)$, and for every derivation $f : kP \to kPe$ which is E-stable we have $\mathrm{Im}(f) \subseteq J(kP)e$.*

Proof. Set $J = J(kP)$. Clearly kPe is a kP-kP-bimodule summand of $k(P \rtimes E)$, and it is E-stable because $e \in Z(E)$. Since f is a derivation, we have $f(J^2) \subseteq Je$, which in turn is in the kernel of the augmentation map $kPe \to k$. Since J is E-stable, it follows that the restriction of f to J is an E-stable map, as is the composition with the augmentation map $\epsilon : kPe \to k$. Thus, the map $\epsilon \circ f|_J : J \to k$ sends J^2 to zero, hence factors through the canonical surjection $J \to J/J^2$. We therefore get a commutative diagram of kE-modules

$$\begin{array}{ccc} J & \xrightarrow{f|_J} & kPe \\ \downarrow & & \downarrow \epsilon \\ J/J^2 & \xrightarrow[g]{} & k \end{array}$$

Since $[P,E] = P$, it follows from Lemma 2.8 that J/J^2 has no non-zero trivial summand as a kE-module. Since J/J^2 is also semi-simple as a kE-module this implies that g is zero, hence that f sends kP to Je. □

Corollary 5.6. *Let P be a finite p-group and E a finite p'-group acting on P. Suppose $[P,E] = P$. For every E-stable derivation $f : kP \to kP$ we have $\mathrm{Im}(f) \subseteq J(kP)$.*

Proof. This follows from Proposition 5.5 applied to $e = 1$. □

Statement (ii) in the following corollary is the special case of Theorem 1.3(i) with α the trivial 2-cocycle.

Corollary 5.7. *Let P be a finite p-group and E a finite abelian p'-group acting on P. Suppose $[P,E] = P$.*

- (i) *For every E-stable derivation $f : kP \to k(P \rtimes E)$ we have $\mathrm{Im}(f) \subseteq J(k(P \rtimes E))$.*
- (ii) *Every class in $\mathrm{HH}^1(k(P \rtimes E))$ is represented by a derivation $f : k(P \rtimes E) \to k(P \rtimes E)$ which vanishes on kE, and any such derivation f satisfies $\mathrm{Im}(f) \subseteq J(k(P \rtimes E))$.*

Proof. Since E is a p'-group, the algebra kE is separable, and hence we have $J(k(P \rtimes E)) = J(kP) \cdot k(P \rtimes E))$. Since E is abelian, by Lemma 5.1, f is the sum of E-stable derivations $kP \to kPe$, with $e \in E$, and hence (i) follows from Proposition 5.5. We have $k(P \rtimes E) = kE \oplus J(k(P \rtimes E))$, and thus, by Lemma 2.2, every class in $\mathrm{HH}^1(k(P \rtimes E))$ is represented by a derivation f which vanishes on kE. The Leibniz rule implies that f is then a kE-kE-bimodule endomorphism of $k(P \rtimes E)$. Thus, f is determined by its restriction $f|_{kP} : kP \to k(P \rtimes E)$. It follows from (i) that f sends kP to $J(k(P \rtimes E))$. Since f is in particular a right kE-homomorphism, f sends kPe to $J(k(P \rtimes E))$ for all $e \in E$, whence (ii). □

Lemma 5.8. *Let P be a non-trivial finite abelian p-group, and set $J = J(kP)$. Denote by* $\mathrm{Der}_1(kP)$ *the Lie subalgebra of* $\mathrm{Der}(kP) = \mathrm{HH}^1(kP)$ *of derivations which preserve J. Set $n = \dim_k(J/J^2)$. The canonical map*

$$\mathrm{Der}_1(kP) \to \mathrm{End}_k(J/J^2) \cong \mathfrak{gl}_n(k)$$

is a surjective Lie algebra homomorphism. The kernel of this homomorphism is the nilpotent ideal $\mathrm{Der}_2(kP)$ *of derivations with image contained in J^2.*

Proof. By Lemma 5.4, we may assume that P is elementary abelian. Since n is the rank of P (cf. Lemma 2.7), we may write $P = \prod_{i=1}^n \langle x_i \rangle$. For any two i, j such that $1 \leqslant i, j \leqslant n$ there is a unique derivation $d_{i,j}$ on kP which sends $x_i - 1$ to $x_j - 1$ and $x_{i'} - 1$ to 0, where $i' \neq i$, $1 \leqslant i' \leqslant n$. Since the image of the set $\{x_i - 1\}_{1 \leqslant i \leqslant n}$ in J/J^2 is a k-basis, the first statement follows, and the second statement follows from Proposition 2.3. □

Lemma 5.9. *Let P be a non-trivial finite abelian p-group, and set $J = J(kP)$. Let E be a finite p'-group acting on P such that $[P, E] = P$. Then* $\mathrm{Der}(kP)^E = \mathrm{Der}_1(kP)^E$. *Write*

$$J/J^2 \cong \oplus_{j=1}^r S_j^{\oplus n_j}$$

with pairwise non-isomorphic simple kE-modules S_j and positive integers n_j ($1 \leqslant j \leqslant r$). Then $\mathbb{F}_j = \mathrm{End}_{kE}(S_j)$ *is a finite-dimensional commutative extension field of k, where $1 \leqslant j \leqslant r$, and the canonical map* $\mathrm{Der}_1(kP) \to \mathrm{End}_k(J/J^2)$ *induces a surjective Lie algebra homomorphism*

$$\mathrm{Der}(kP)^E \to \mathrm{End}_{kE}(J/J^2) \cong \prod_{j=1}^r \mathfrak{gl}_{n_j}(\mathbb{F}_j).$$

The kernel of this Lie algebra homomorphism is the nilpotent ideal $\mathrm{Der}_2(kP)^E$ *of E-stable derivations on kP with image in J^2.*

Proof. By Lemma 5.6 we have $\mathrm{Der}(kP)^E = \mathrm{Der}_1(kP)^E$. The surjective map $\mathrm{Der}_1(kP) \to \mathrm{End}_k(J/J^2)$ from Lemma 5.8 remains surjective upon taking E-fixed points since E has order prime to p. Thus this map induces a surjective Lie algebra homomorphism $\mathrm{Der}(kP)^E \to \mathrm{End}_{kE}(J/J^2)$ with a nilpotent kernel as stated. The rest follows from decomposing the semisimple kE-module J/J^2 as a direct sum of its isotypic components. □

Lemma 5.10. *Let P be a non-trivial finite p-group and E a finite p'-group acting on P. The kE-module $J(kP)/J(kP)^2$ is multiplicity free if and only if the $\mathbb{F}_pE$-module $P/\Phi(P)$ is multiplicity free.*

Proof. Since $J(kP)/J(kP)^2 \cong k \otimes_{\mathbb{F}_p} J(\mathbb{F}_pP)/J(\mathbb{F}_pP)^2$, it follows from standard properties of coefficient extensions (e.g. [9, Theorem 9.21.(b)], or [6, Chapter I, Theorem 19.4], or [4, (30.33)]) that the $\mathbb{F}_pE$-module $J(\mathbb{F}_pP)/(J(\mathbb{F}_pP)^2$ is multiplicity free if and only if the kE-module $J(kP)/J(kP)^2$ is multiplicity free. Thus, Lemma 2.7 implies the result. □

6 | PROOF OF THEOREM 1.1 AND THEOREM 1.2

Proof of Theorem 1.2. Let P be a finite abelian p-group and E a finite p'-group acting on P such that $[P, E] = P$. Set $J = J(kP)$. By Lemma 5.10, $P/\Phi(P)$ is multiplicity free as an $\mathbb{F}_pE$-module if and only if J/J^2 is multiplicity free as a kE-module. Since P is abelian, we have $\mathrm{HH}^1(kP) = \mathrm{Der}(kP)$, hence $\mathrm{HH}^1(kP)^E = \mathrm{Der}(kP)^E$. By Lemma 5.6, all E-stable derivations on kP have image in $J(kP)$; that is, $\mathrm{Der}(kP)^E = \mathrm{Der}_1(kP)^E$, where the notation is as in Lemma 5.9. It follows from Lemma 5.9 that $\mathrm{HH}^1(kP)^E$ is solvable if and only if $\mathrm{Der}_1(kP)^E/\mathrm{Der}_2(kP)^E$ is a solvable Lie algebra.

If J/J^2 is multiplicity free as a kE-module, then Lemma 5.9 implies

$$\mathrm{Der}_1(kP)^E/\mathrm{Der}_2(kP)^E \cong \prod_{j=1}^{r} \mathfrak{gl}_1(\mathbb{F}_j)$$

for some commutative extension fields $\mathbb{F}_j$ of k, and hence this Lie algebra is abelian, which implies that $\mathrm{HH}^1(kP)^E$ is solvable.

If J/J^2 is not multiplicity free, then $\mathrm{Der}_1(kP)^E/\mathrm{Der}_2(kP)^E$ has a direct factor isomorphic to $\mathfrak{gl}_n(\mathbb{F})$ for some extension field $\mathbb{F}$ of k and some integer $n \geqslant 2$. Thus, if p is odd, then $\mathfrak{gl}_n(\mathbb{F})$ is not solvable, hence neither is $\mathrm{HH}^1(kP)^E$. This completes the proof of Theorem 1.2. □

In order to complete the proof of Theorem 1.1, we summarise the results in the case of a free p'-action on an abelian p-group.

Theorem 6.1. *Let P be a non-trivial finite abelian p-group and E a p'-subgroup of* $\mathrm{Aut}(P)$ *acting freely on $P \setminus \{1\}$. Then the following hold.*

(i) *We have* $\mathrm{HH}^1(k(P \rtimes E)) \cong \mathrm{HH}^1(kP)^E$ *as Lie algebras.*
(ii) *Suppose E is non-trivial. Every class in* $\mathrm{HH}^1(k(P \rtimes E))$ *is represented by a derivation on $k(P \rtimes E)$ with image contained in the Jacobson radical $J(k(P \rtimes E))$.*
(iii) *Suppose E is non-trivial. If the $\mathbb{F}_pE$-module $P/\Phi(P)$ is multiplicity free, then the Lie algebra* $\mathrm{HH}^1(k(P \rtimes E))$ *is solvable. The converse holds if p is odd.*

Proof. The first statement follows from Lemma 5.1(iii) or Proposition 5.2. The hypothesis $E \neq 1$ in (ii) and (iii) together with the free action of E on $P \setminus \{1\}$ imply $[P, E] = P$. The remaining statements follow from Theorem 1.2. □

Proof of Theorem 1.1. With the notation and hypotheses of Theorem 1.1, by a result of Puig [21, 6.8] (see also [16, Theorem 10.5.1]) there is a stable equivalence of Morita type between B and $k(P \rtimes E)$. By [12, Theorem 10.7], this implies that there is a Lie algebra isomorphism $\mathrm{HH}^1(B) \cong \mathrm{HH}^1(k(P \rtimes E))$. Thus, Theorem 1.1 follows from Theorem 6.1. □

The following observation is a combination of some of the above results in conjunction with the structure theory of blocks with a normal defect group, slightly generalising [11, Proposition 5.2], [20, Theorem 3.2]. We state this here for context and for future reference.

Theorem 6.2. *Let G be a finite group and B a block with a non-trivial normal defect group P and inertial quotient E. Suppose that k is large enough for B. Then* $\mathrm{HH}^1(kP)^E$ *is canonically isomorphic to a Lie subalgebra of* $\mathrm{HH}^1(B)$. *If in addition E acts freely on $P \setminus \{1\}$, then* $\mathrm{HH}^1(kP)^E \cong \mathrm{HH}^1(B)$.

Proof. Let B be a block of a finite group algebra kG with a non-trivial defect group P which is normal in G. Suppose that k is large enough for B. Then, by [14, Theorem A] (see [16, Theorem 6.14.1] for an exposition of related material) the block B is Morita equivalent to a twisted group algebra of the form $k_\alpha(P \rtimes E)$, where E is a p'-subgroup of $\mathrm{Aut}(P)$ and $\alpha \in Z^2(E, k^\times)$, inflated to $P \rtimes E$ via the canonical surjection $P \rtimes E \to E$. Thus $\mathrm{HH}^1(B) \cong \mathrm{HH}^1(k_\alpha(P \rtimes E))$ as Lie algebras. Theorem 6.2 follows from Lemma 5.1 or Proposition 5.2. □

Remark 6.3. Since the Lie algebra $\mathrm{HH}^1(B)$ of a block B of a finite group algebra kG is preserved under stable equivalences of Morita type (cf. [12, Theorem 10.7]), it follows that the conclusion of Theorem 6.2 holds if there is a stable equivalence of Morita type between B and its Brauer correspondent; this includes inertial blocks (these are blocks which are Morita equivalent to their Brauer correspondents via bimodules with endopermutation source; cf. [22, 2.16]).

7 | PROOF OF THEOREM 1.3 AND RELATED RESULTS

Proof of Theorem 1.3. Let P be a non-trivial finite abelian p-group and E an abelian p'-subgroup of $\mathrm{Aut}(P)$ with $[P, E] = P$. Let $\alpha \in Z^2(E; k^\times)$ inflated to $P \rtimes E$ via the canonical surjection $P \rtimes E \to E$. We may assume that α is normalised.

Note that the condition $[P, E] = P$ forces $|P| \geqslant 3$. Since E is an abelian p'-group, by Lemma 5.1, we have a canonical isomorphism

$$\mathrm{HH}^1(k_\alpha(P \rtimes E)) \cong \mathrm{HH}^1(kP)^E \oplus \bigoplus_{e\in E\setminus\{1\}} \mathrm{H}^1(P; kP \cdot e)^E.$$

Again by Lemma 5.1, for each $f \in \mathrm{H}^1(P; kP \cdot e)^E$, the corresponding element in $\mathrm{HH}^1(k_\alpha(P \rtimes E))$ under this isomorphism is represented by $d_f \in \mathrm{Der}(k_\alpha(P \rtimes E))$ such that for each $g \in P$, $d_f(g) = f(g) \cdot g$ and d_f vanishes on E.

Fix a non-identity element $e \in E$. Set $T = C_P(e)$ and $F = [P, \langle e\rangle]$. Since $e \neq 1$, it follows that the group F is non-trivial. Since P is abelian, by [10, Theorem 4.34] or by [7, Theorem 5.2.3], we have $P = T \times F$. Using Formula 4.3 and that P is abelian, we get that conjugation action by $g \in P$ on $kP \cdot e$ is equal to left multiplication with $g({}^e g)^{-1}$. Thus, T acts trivially on $kP \cdot e$ and F acts freely on $P \cdot e$. Using that F is non-trivial, this implies $\mathrm{H}^1(F; kP \cdot e) = 0$. Furthermore, $kP \cdot e$ can be viewed as a $k(T \times F)$-module $k \otimes_k kP \cdot e$ where k is the trivial kT-module, and $kP \cdot e$ is the free kF-module for the conjugation action of F on $kP \cdot e$. By the Künneth formula 3.11, we have

$$\mathrm{H}^1(P; kP \cdot e) \cong \mathrm{H}^1(T \times F; k \otimes_k kP \cdot e) \cong \mathrm{H}^0(T; k) \otimes_k \mathrm{H}^1(F; kP \cdot e) \oplus \mathrm{H}^1(T; k) \otimes_k \mathrm{H}^0(F; kP \cdot e)$$

$$\cong \mathrm{H}^1(T; k) \otimes_k H^0(F; kP \cdot e) \cong \mathrm{H}^1(T; k) \otimes_k (kP \cdot e)^F.$$

Any orbit in $P \cdot e$ under the conjugation action of F is equal to $\{ta \cdot e \,:\, a \in F\}$, where $t \in T$. Thus, $(kP \cdot e)^F$ is spanned by the set $\{t(\sum_{a\in F} a) \cdot e \,:\, t \in T\}$. It follows that the map $t \mapsto t(\sum_{a\in F} a) \cdot e$ induces an isomorphism

$$kT \cong (kP \cdot e)^F,$$

where we note that P acts trivially by conjugation on both sides. Since E is abelian, the direct product decomposition $P = T \times F$ is stable under the action of E. Thus, the condition $[P, E] = P$

implies $[T, E] = T$ and $[F, E] = F$, so neither T nor F has order 2. Since F is non-trivial, so has order at least 3, it follows that the socle element $\sum_{a \in F} a$ of kF is contained in $J(kF)^2$. Thus, $(kP \cdot e)^F \subseteq kT \cdot J(kF)^2 \cdot e \subseteq J(kP)^2 \cdot e$. It follows that every class in $\mathrm{HH}^1(kP; kP \cdot e)^E$ has a representative with image contained in $J(kP)^2 \cdot e$. Let D_2 denote the image of $Der_2(k_\alpha(P \rtimes E))$ in $\mathrm{HH}^1(k_\alpha(P \rtimes E)$. By the above, we have $\bigoplus_{e \in E \setminus \{1\}} \mathrm{H}^1(P; kP \cdot e)^E \subseteq D_2$. Thus,

7.1.

$$\mathrm{HH}^1(k_\alpha(P \rtimes E)) = \mathrm{HH}^1(kP)^E + D_2.$$

Theorem 1.2 implies therefore that every class in $\mathrm{HH}^1(k_\alpha(P \rtimes E))$ is represented by a derivation with image contained in $J(k_\alpha(P \rtimes E))$. Equivalently, the map $\mathrm{Der}_1(k_\alpha(P \rtimes E)) \to \mathrm{HH}^1(k_\alpha(P \rtimes E))$ is surjective. By Corollary 2.4(iv),

7.2. *We have that* $\mathrm{HH}^1(k_\alpha(P \rtimes E))$ *is solvable if and only if* $\mathrm{HH}^1(kP)^E$ *is solvable.*

Note that the action of E on $\mathrm{HH}^1(kP)$ by taking conjugation in the twisted group algebra does not depend on α by Statement 4.3. Thus, this action on $\mathrm{HH}^1(kP)$ coincides with the action of E induced by the usual conjugation of E on P. The rest follows from Theorem 1.2. □

Remark 7.3. The proof above can be refined to give a slightly stronger result. Using an isomorphism from [5, section 3.4], for e a non-trivial element in E, we have $\mathrm{H}^1(T; k) \otimes_k (kP \cdot e)^F \cong \mathrm{H}^1(T; (kP \cdot e)^F)$. By the above proof, this is isomorphic to $\mathrm{H}^1(T; kT) \cong \mathrm{HH}^1(kT)$. By Lemma 5.6, every class in $\mathrm{HH}^1(kT)^E$ has a representative d with image in $J(kT)$, that is, every class in $\mathrm{H}^1(T; kT)^E$ is represented by a 1-cocycle $f \in Z^1(T; kT)$ with image contained in $J(kT)$. Mapping f to the corresponding 1-cocycle f' in $Z^1(T; (kP \cdot e)^F)$, the proof above shows that the image of f' is contained in $J(kT)J(kF)^2 \cdot e \subseteq J(kP)^3 \cdot e$.

As mentioned in the Introduction, Theorem 1.3 admits the following equivalent reformulation.

Theorem 7.4. *Let P be a non-trivial finite abelian p-group and E be a p'-group acting on P. Suppose $[P, E] = P$, $C_E(P) \leqslant \mathrm{Z}(E)$ and that $E/C_E(P)$ is abelian. Every class in $\mathrm{HH}^1(k(P \rtimes E))$ is represented by a derivation on $k(P \rtimes E)$ with image contained in the Jacobson radical $J(k(P \rtimes E))$. If the semisimple $\mathbb{F}_p E$-module $P/\Phi(P)$ is multiplicity free, then the Lie algebra $\mathrm{HH}^1(k(P \rtimes E))$ is solvable. The converse holds if p is odd.*

One way to prove Theorem 7.4 would be to play it back to Theorem 1.3 via the standard correspondence between second cohomology and central group extensions. For convenience, we will present a self-contained proof which directly translates the proof of Theorem 1.3 to the situation of Theorem 7.4.

Proof of Theorem 7.4. Let P be a non-trivial finite abelian p-group and E be a p'-subgroup acting on P. Set $C = C_E(P)$. Suppose $[P, E] = P$, $C \leqslant \mathrm{Z}(E)$ and that E/C is abelian. In particular, since C acts trivially on P, C also lies in the centre of $P \rtimes E$. As before, we will use the canonical isomorphisms $\mathrm{HH}^1(k(P \rtimes E)) \cong \mathrm{H}^1(P \rtimes E; k(P \rtimes E)) \cong \mathrm{H}^1(P; k(P \rtimes E))^E$. As a kP-module with respect to the conjugation action, we have $k(P \rtimes E) \cong \bigoplus_{e \in E} kP \cdot e$. Thus, we get $\mathrm{HH}^1(k(P \rtimes E)) \cong \mathrm{H}^1(P; \bigoplus_{e \in E} kP \cdot e)^E$ Since C is a central p'-subgroup of $P \rtimes E$, and since for every $c \in C$ we have $kP \cdot c \cong kP$ as a kP-module, it follows that $\mathrm{H}^1(P; kP \times C) \cong \mathrm{H}^1(P; kP) \otimes_k kC$. This space is E-invariant, with E acting trivially on kC, and hence

$$\mathrm{H}^1(P; k(P \times C))^E \cong \mathrm{H}^1(P; kP)^E \otimes_k kC \cong \mathrm{HH}^1(kP)^E \otimes_k kC.$$

Using the isomorphism 4.7 (with trivial α) it follows that

$$\mathrm{HH}^1(k(P \rtimes E)) \cong \mathrm{HH}^1(kP)^E \otimes_k kC \ \oplus \ (\oplus_{e \in E \setminus C} \mathrm{HH}^1(kP; kP \cdot e))^E.$$

In particular, $\mathrm{HH}^1(kP)^E \otimes_k kC$ is a Lie subalgebra of $\mathrm{HH}^1(k(P \rtimes E))$. In order to analyse, the second summand let $e \in E \setminus C$. Set $T = C_P(e)$ and $F = [P, \langle e \rangle]$. Since $e \notin C$, it follows that the group F is non-trivial. Since P is abelian, by [10, Theorem 4.34] or by [7, Theorem 5.2.3], we have $P = T \times F$. Notice that for every $g \in P$, the conjugation action of g on $kP \cdot e$ is equal to left multiplication with $g({}^e g)^{-1}$. Thus, T acts trivially on $kP \cdot e$ and F acts freely on $P \cdot e$. Using that F is non-trivial, this implies $\mathrm{H}^1(F; kP \cdot e) = 0$. Furthermore, $kP \cdot e$ can be viewed as a $k(T \times F)$-module $k \otimes_k kP \cdot e$ where k is the trivial kT-module, and $kP \cdot e$ is the free kF-module for the conjugation action of F on $kP \cdot e$. Just as in the proof of Theorem 1.3, by the Künneth formula 3.11, we have

$$\mathrm{H}^1(P; kP \cdot e) \cong \mathrm{H}^1(T; k) \otimes_k (kP \cdot e)^F,$$

and the map $t \mapsto t(\sum_{a \in F} a) \cdot e$ induces an isomorphism

$$kT \cong (kP \cdot e)^F.$$

Since E/C is abelian, or equivalently, since the image of E in $\mathrm{Aut}(P)$ is abelian, it follows that the direct product decomposition $P = T \times F$ is still stable under the action of E. An obvious adaptation of the last part of the proof of Theorem 1.3 yields that $[F, E] = F \neq 1$ has order at least 3, so as before the socle element $\sum_{a \in F} a$ of kF is contained in $J(kF)^2$, whence $(kP \cdot e)^F \subseteq kT \cdot J(kF)^2 \cdot e \subseteq J(kP)^2 \cdot e$. It follows that every class in $\mathrm{H}^1(P; kP \cdot e)$ has a representative with image contained in $J(kP)^2 \cdot e$, so the same holds for $\mathrm{H}^1(P; \oplus_{e \in E \setminus C} kP \cdot e)^E$. Thus, every class in $\mathrm{HH}^1(kP; \oplus_{e \in E \setminus C} kP \cdot e)^E$ has a representative with image contained in $J(kP)^2 \cdot e$. Let D_2 denote the image of $\mathrm{Der}_2(k(P \rtimes E))$ in $\mathrm{HH}^1(k(P \rtimes E))$. By the above, we have $\mathrm{HH}^1(kP; \oplus_{e \in E \setminus C} kP \cdot e)^E \subseteq D_2$, hence

7.5.

$$\mathrm{HH}^1(k(P \rtimes E)) = \mathrm{HH}^1(kP)^E \otimes_k kC + D_2.$$

Theorem 1.2 implies therefore that every class in $\mathrm{HH}^1(k(P \rtimes E))$ is represented by a derivation with image contained in $J(k(P \rtimes E))$. Equivalently, the map $\mathrm{Der}_1(k(P \rtimes E)) \to \mathrm{HH}^1(k(P \rtimes E))$ is surjective. By Corollary 2.4 (iv),

7.6. *We have that the Lie algebra* $\mathrm{HH}^1(k(P \rtimes E))$ *is solvable if and only if* $\mathrm{HH}^1(kP)^E \otimes_k kC$ *is solvable.*

The formula 3.4 implies that this is the case if and only if $\mathrm{HH}^1(kP)^E$ is solvable. The rest follows from Theorem 1.2. □

Corollary 7.7. *Let G be a finite group, B a block with a non-trivial abelian defect group P and an abelian inertial quotient E such that there is a stable equivalence of Morita type between B and its*

Brauer correspondent. Assume that k is large enough. Suppose $[P, E] = P$. *If the* $\mathbb{F}_pE$*-module* $P/\Phi(P)$ *is multiplicity free, then* $\mathrm{HH}^1(B)$ *is a solvable Lie algebra. The converse holds if p is odd.*

Proof. Since the Lie algebra $\mathrm{HH}^1(B)$ of a block B of a finite group algebra kG is preserved under stable equivalences of Morita type (cf. [12, Theorem 10.7]), we may assume that P is normal in G. Arguing as before, by [14, Theorem A] the block B is Morita equivalent to $k_\alpha(P \rtimes E)$ for some $\alpha \in Z^2(E, k^\times)$, inflated to $P \rtimes E$ via the canonical surjection $P \rtimes E \to E$. Thus $\mathrm{HH}^1(B) \cong \mathrm{HH}^1(k_\alpha(P \rtimes E))$ as Lie algebras. The result follows from Theorem 1.3. □

We remark that Broué's abelian defect group conjecture predicts that there should always be a derived equivalence between a block with an abelian defect group and its Brauer correspondent, so in particular there should always be a stable equivalence of Morita type in that situation.

8 | EXAMPLES

Example 8.1. Let k be a field of prime characteristic p. Let P be a non-trivial cyclic p-group and E a p'-subgroup of $\mathrm{Aut}(P)$. Then E is cyclic of order dividing $p-1$ and acts freely on $P \setminus \{1\}$. It follows for instance from [19, Example 5.7] as well as explicit calculations in [20, section 3] that $\mathrm{HH}^1(k(P \rtimes E))$ is solvable if and only if $E \neq 1$ or if $p = 2$, and $\mathrm{HH}^1(k(P \rtimes E))$ is simple if and only if $|P| = p \geqslant 3$ and $E = 1$. It is easy to see that this follows also from combining the results of this paper. If $E \neq 1$, then $\mathrm{HH}^1(k(P \rtimes E))$ is solvable by Theorem 1.3, or also by Theorem 1.1. If $|P| = p \geqslant 3$, then $\mathrm{HH}^1(kP)$ is a simple Witt Lie algebra, if $|P| = p = 2$, then $\mathrm{HH}^1(kP)$ is a solvable non-abelian 2-dimensional Lie algebra, and if P has order p^a for some $a \geqslant 2$, then $\mathrm{HH}^1(kP)$ is not simple, by Lemma 5.4, in conjunction with the fact that $\mathrm{HH}^1(kP)$ has dimension $|P|$. Note that this describes the Lie algebra structure of $\mathrm{HH}^1(B)$ for B a block of a finite group algebra kG with a non-trivial cyclic defect group P and inertial quotient E, with k sufficiently large. Indeed, there is a stable equivalence of Morita type between B and $k(P \rtimes E)$ (see e.g. [16, Theorem 11.1.2]), and hence $\mathrm{HH}^1(B) \cong \mathrm{HH}^1(k(P \rtimes E))$ by [12, Theorem 10.2]. One can show this, of course, also by making use of the fact, due to Rickard [23], that B and $k(P \rtimes E)$ are in fact derived equivalent.

Example 8.2. Let p be an odd prime and P a finite abelian p-group of rank $r \geqslant 1$. Let E be a group of order 2, acting on P, with the non-trivial element of E acting by inversion. This action of E on $P \setminus \{1\}$ is free, so $P \rtimes E$ is a Frobenius group, and hence we have a Lie algebra isomorphism

$$\mathrm{HH}^1(\mathbb{F}_p(P \rtimes E)) \cong \mathrm{Der}(\mathbb{F}_pP)^E.$$

Set $J = J(\mathbb{F}_pP)$. The non-trivial element of E acts as inversion on P, hence also on $P/\Phi(P)$, and therefore as multiplication by -1 on J/J^2 via the isomorphism from Lemma 2.7. Since this action is in the centre of $\mathrm{End}_{\mathbb{F}_p}(J/J^2)$, it follows that $\mathrm{End}_{\mathbb{F}_pE}(J/J^2) = \mathrm{End}_{\mathbb{F}_p}(J/J^2)$. Lemma 5.9 implies that the canonical Lie algebra homomorphism

$$\mathrm{HH}^1(\mathbb{F}_p(P \rtimes E)) \cong \mathrm{Der}(\mathbb{F}_pP)^E \to \mathrm{End}_{\mathbb{F}_pE}(J/J^2) \cong \mathfrak{gl}_r(\mathbb{F}_p)$$

is surjective. In particular, if $r \geqslant 2$, then the Lie algebra $\mathrm{HH}^1(\mathbb{F}_p(P \rtimes E))$ is not solvable, while for $r = 1$ this is a solvable Lie algebra of dimension $\frac{|P|-1}{2}$.

Example 8.3. Let P be a finite elementary abelian p-group of rank $r \geqslant 1$, and let E be a p'-subgroup of $\mathrm{Aut}(P)$. Suppose that P is split semisimple as an $\mathbb{F}_pE$-module. Then P has an E-stable decomposition $P = \prod_{i=1}^{r}\langle x_i\rangle$, where the x_i all have order p; in particular, E is abelian. If the subgroups $C_E(x_i)$, with $1 \leqslant i \leqslant r$, are pairwise distinct proper subgroups of E, then the simple $\mathbb{F}_pE$-modules $\langle x_i\rangle$ are pairwise non-isomorphic non-trivial. In that case, setting $E_i = E/C_E(x_i)$, each E_i is non-trivial, and the canonical surjections $E \to E_i$ induce an isomorphism $E \cong \prod_{i=1}^{r} E_i$. Note that each E_i can be identified with a non-trivial cyclic subgroup of $\mathrm{Aut}(\langle x_i\rangle)$, and hence $P \rtimes E = \prod_{i=1}^{r}\langle x_i\rangle \rtimes E_i$. In particular, P is multiplicity free as an $\mathbb{F}_pE$-module, and $P = [P,E]$. Thus, Theorem 1.2 implies that $\mathrm{HH}^1(kP)^E$ is a solvable Lie algebra. In fact, since $P \rtimes E$ is a direct product of Frobenius groups, one can, using the Künneth formula, even conclude that $\mathrm{HH}^1(k(P \rtimes E))$ is solvable; see Example 8.5 below.

Example 8.4. Let k be an algebraically closed field of characteristic p, let G be a finite group and B a block of kG with an elementary abelian defect group $P \cong C_p \times C_p$ of order p^2 and an abelian inertial quotient E. We identify E with a subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$ when convenient.

The Brauer correspondent block with defect group P is source algebra equivalent to $k_\alpha(P \rtimes E)$ for some $\alpha \in Z^2(E; k^\times)$. By a result of Rouquier [24, 6.3] (see [17, Theorem A.2] for a proof for non-principal blocks), there is a stable equivalence of Morita type between B and $k_\alpha(P \rtimes E)$, and hence, by [12, Theorem 10.2], we have an isomorphism of Lie algebras

$$\mathrm{HH}^1(B) \cong \mathrm{HH}^1(k_\alpha(P \rtimes E)).$$

If $E = 1$, or equivalently, if the block B is nilpotent, then $\mathrm{HH}^1(B) \cong \mathrm{HH}^1(kP)$ is a simple Witt Lie algebra (see [18, Theorem 1.1] and the references in that paper for background).

If $E \neq 1$ and $E \leqslant Z(\mathrm{GL}_2(p))$, then p is odd, we have $[P,E] = P$, and P is a direct sum of two isomorphic simple $\mathbb{F}_pE$-modules. Hence, in that case, $\mathrm{HH}^1(B)$ has a quotient isomorphic to $\mathfrak{gl}_2(\mathbb{F}_p)$, so is non-solvable and non-simple, where we use Lemma 5.9.

If $E \neq 1$ and $[P,E] < P$, then $P \rtimes E \cong Q \times (R \rtimes E)$, where $Q = C_P(E)$ and $R = [P,E]$ both have order p and E is cyclic of order dividing $p-1$ (so p is odd and α is trivial). Since $\mathrm{HH}^1(kQ)$ is a simple Witt Lie algebra and every class in $\mathrm{HH}^1(k(R \rtimes E))$ is represented by a derivation of $k(R \times E)$ with image contained in $J(k(R \times E))$, it follows from Lemma 3.7 that $\mathrm{HH}^1(B)$ is neither simple nor solvable in that case.

If E is a non-trivial abelian but not a central subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$, such that $[P,E] = P$, then P is either simple as an $\mathbb{F}_pE$-module or the sum of two non-isomorphic simple $\mathbb{F}_pE$-modules, and hence $\mathrm{HH}^1(B)$ is solvable in that case, by Theorem 1.3. For instance, if $E = C_2 \times C_2$ and α is non-trivial, then the Brauer correspondent of B is Morita equivalent to the quantum complete intersection described in Example 8.6 below. Another instance arises as follows. By identifying P with the additive group of $\mathbb{F}_{p^2}$, one sees that the cyclic group $\mathbb{F}_{p^2}^\times$ of order $p^2 - 1$ acts regularly on $P \setminus \{1\}$. Let $m \geqslant 3$ be a divisor of $p+1$ coprime to $p-1$, and let C_m a cyclic group of order m acting freely on the non-identity elements of P. Then $[P,E] = P$ and P is a simple $\mathbb{F}_pE$-module, hence $\mathrm{HH}^1(k(P \rtimes E)) \cong \mathrm{HH}^1(kP)^E$ is a solvable Lie algebra by Theorem 6.1.

Example 8.5. Theorem 6.1 can be extended to direct products of Frobenius groups, using the Künneth formula. For $1 \leqslant i \leqslant n$ let P_i be a non-trivial finite abelian p-group and let E_i be a p'-subgroup of $\mathrm{Aut}(P_i)$ acting freely on $P_i \setminus \{1\}$. By Proposition 5.2 we have $\mathrm{HH}^1(k(P_i \rtimes E_i)) \cong \mathrm{HH}^1(kP_i)^{E_i}$, so by Lemma 5.6 the canonical map $\mathrm{Der}_1(k(P_i \rtimes E_i)) \to \mathrm{HH}^1(k(P_i \rtimes E_i))$ are surjective.

Set $P = \prod_{i=1}^n P_i$ and $E = \prod_{i=1}^n E_i$. Proposition 3.1, applied repeatedly, implies that the map $\mathrm{Der}_1(k(P \rtimes E)) \to \mathrm{HH}^1(k(P \rtimes E))$ is surjective. Suppose in addition that $P/\Phi(P)$ is multiplicity free as an $\mathbb{F}_pE$-module, or equivalently, that each $P_i/\Phi(P_i)$ is multiplicity free as an $\mathbb{F}_pE_i$-module, for $1 \leqslant i \leqslant n$. Lemma 5.9 implies

$$[\mathrm{Der}_1(k(P \rtimes E)), \mathrm{Der}_1(k(P \rtimes E))] \subseteq \mathrm{Der}_2(k(P \rtimes E)) + \mathrm{IDer}(k(P \rtimes E)),$$

and in particular, that $\mathrm{HH}^1(k(P \rtimes E))$ is a solvable Lie algebra. In the case where E is abelian, this follows also from Theorem 1.3.

Example 8.6. Suppose that p is an odd prime. Consider the algebra

$$A = k\langle x, y | x^p = y^p = 0,\ xy + yx = 0\rangle.$$

This algebra, which has dimension p^2, arises as basic algebra of the non-principal block of $k(C_p \times C_p) \rtimes Q_8$, with defect group $C_p \times C_p$ and inertial quotient $C_2 \times C_2$. It is shown in [2, Theorem 1.1] amongst other statements on the structure of the Lie algebra $\mathrm{HH}^1(A)$ that $\mathrm{HH}^1(A)$ is a solvable Lie algebra. Theorem 1.3 provides an alternative proof of this fact.

Example 8.7. Let k be an algebraically closed field of prime characteristic p, let G be a finite group, N a normal subgroup such that G/N is a p-group, and let b be a G-stable block of kN. Set $C = kNb$. Then $B = kGb$ is a block of kG. Let P be a defect group of B, and set $Q = P \cap N$. Suppose that P is abelian and that Q has a complement R in P. If p is odd or if R has rank at least 2, then $\mathrm{HH}^1(B)$ is not solvable. Indeed, by a theorem of Koshitani and Külshammer in [13] (see [16, Theorem 10.4.2] for an expository account), we have a k-algebra isomorphism $B \cong kR \otimes_k C$, so by the Künneth formula 3.2, $\mathrm{HH}^1(B)$ has a Lie subalgebra isomorphic to $\mathrm{HH}^1(kR)$, which in turn has a Lie algebra quotient $\mathrm{HH}^1(kR/\Phi(R))$. If p is odd or if R has rank at least 2, then $\mathrm{HH}^1(kR/\Phi(R))$ is a simple Witt Lie algebra.

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