Diagrammatic Kazhdan-Lusztig theory for the (walled) Brauer algebra

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Abstract

We determine the decomposition numbers for the Brauer and walled Brauer algebra in characteristic zero in terms of certain polynomials associated to cap and curl diagrams (recovering a result of Martin in the Brauer case). We consider a second family of polynomials associated to such diagrams, and use these to determine projective resolutions of the standard modules. We then relate these two families of polynomials to Kazhdan-Lusztig theory via the work of Lascoux-Schützenberger and Boe, inspired by work of Brundan and Stroppel in the cap diagram case.

Keywords: Brauer algebra, Kazhdan-Lusztig polynomial

1. Introduction

Classical Schur-Weyl duality relates the representations of the symmetric and general linear groups via their actions on tensor space. The Brauer algebra was introduced in [Bra37] to play the role of the symmetric group in a corresponding duality for the symplectic and orthogonal groups. Over the complex numbers it is generically semisimple [Bro55], indeed it can only be non-semisimple if $\delta \in \mathbb{Z}$ [Wen88]. (A precise criterion for semisimplicity was given by Rui [Rui05].)

Building on work of Doran, Hanlon, and Wales [DWH99] we determined, with Martin, the blocks of the Brauer algebra over $\mathbb{C}$ [CDM09a]. This block structure could be defined in terms of the action of a Weyl group of type D [CDM09b], with a maximal parabolic subgroup of type A determining the dominant weights. The corresponding alcove geometry has associated translation functors which can be used to provide Morita equivalences between weights in the same facet [CDM11]. More recently, Martin [Mar] has shown that the decomposition numbers for the standard modules are given by the corresponding parabolic Kazhdan-Lusztig polynomials.

The walled Brauer algebra was introduced in another generalisation of Schur-Weyl duality, by changing the tensor space on which the symmetric group acts. If instead a

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mixed tensor space (made of copies of the natural module and its dual) is considered, then the walled Brauer algebra plays the role of the symmetric group in the duality. This was introduced independently by a number of authors [Tur89, Koi89, BCH$^+$94]. In [CDDM08] the walled Brauer algebra was analysed in the same spirit as in [CDM09a, CDM09b], and the blocks were again described in terms of the action of a Weyl group — but this time of type A, with a maximal parabolic subgroup of type $A \times A$ determining the dominant weights.

The Kazhdan-Lusztig polynomials associated to (D_n, A_{n-1}) and $(A_n, A_{r-1} \times A_{n-r})$ are two of the infinite families associated with Hermitian symmetric spaces, and have already been considered by a number of authors. Lascoux and Schützenberger [LS81] considered the $(A_n, A_{r-1} \times A_{n-r})$ case and gave an explicit formula for the coefficients in terms of certain special valued graphs. This was extended to the other Hermitian symmetric pairs by Boe [Boe88]. A different combinatorial description was given by Enright and Shelton [ES87] in terms of an associated root system. (A more general situation has also been considered by Brenti [Bre09] who describes the corresponding polynomials in terms of shifted-Dyck partitions.)

The Brauer and walled Brauer algebras are examples of diagram algebras. A quite different diagram algebra was introduced by Khovanov [Kho00, Kho02] in his work on categorifying the Jones polynomial. Brundan and Stroppel have studied generalisations of these algebras, relating them to a parabolic category $\mathcal{O}$ and the general linear supergroup [BSa, BS10, BS11, BSb]. (The case of the principal block was previously considered in [Str09].) Along the way, Kazhdan-Lusztig polynomials of type $(A_n, A_{r-1} \times A_{n-r})$ arise, and Brundan and Stroppel re-express the combinatorial formalism of Lascoux and Schützenberger in terms of certain cap diagrams.

In this paper we will determine the decomposition numbers for the Brauer and walled Brauer algebras by analysing the blocks of these algebras in the (combinatorial) spirit of Brundan and Stroppel. For the Brauer algebra we introduce certain curl diagrams which correspond to the graph formalism in Boe, while the walled Brauer algebra involves only cap diagrams. The decomposition numbers for the Brauer algebra were determined by Martin [Mar]; our methods give a uniform proof that includes the walled Brauer case.

One of the main organisational tools in our earlier work was the notion of a tower of recollement [CMPX06]. We give a slight extension of our earlier theory of translation functors for such towers [CDM11] and use this to reduce the decomposition number problem to a combinatorial exercise. This is then solved using curl diagrams, thus giving a unified proof for the Brauer and walled Brauer cases.

In the Brauer case the combinatorial construction is related to that given in [Mar]. However, using cap and curl diagrams we are able to explicitly calculate certain inverses to the decomposition matrices for both Brauer and walled Brauer. The polynomial entries of these matrices can be used to describe projective resolutions of the standard modules in each case. (Again, this is in the spirit of Brundan and Stroppel.)

We begin in Section 2 with a brief review of the basics of Brauer and walled Brauer representation theory. Section 3 reviews (and slightly extends) the tower of recollement formalism, and the theory of translation functors in this context. Sections 4 and 5 introduce two of the main combinatorial constructions: oriented cap and curl diagrams. These are used in Section 6 to determine the decomposition numbers for our algebras.

After providing a recursive formula for decomposition numbers in Section 7 we define a second family of polynomials using valued cap and curl diagrams in Section 8. These are used to determine projective resolutions of standard modules in Section 9. Finally, the relation between the polynomials associated to valued cap and curl diagrams and the construction of parabolic Kazhdan-Lusztig polynomials by Lascoux-Schützenberger and Boe is outlined in the Appendix.

An alternative graphical construction in the curl case has independently been given by Lejcyk [Lej10]; we are grateful to Catharina Stroppel for this reference (and for other comments on this paper). We would also like to thank Paul Martin for several useful discussions.

2. The Brauer and walled Brauer algebras

We will review some basic results about the representations of the Brauer and the walled Brauer algebra. The two theories are very similar; we will concentrate on the walled Brauer (which is less familiar) and sketch the modifications required for the classical Brauer algebra. Details can be found in [CDDM08] for the walled Brauer algebra, and in [CDM09a] otherwise. We will restrict attention to the case where the ground field is $\mathbb{C}$, and assume that our defining parameter δ is non-zero.

Let $n = r+s$ for some non-negative integers r, s. For $\delta \in \mathbb{C}$, the Brauer algebra $B_n(\delta)$ (which we will often denote just by B_n) can be defined in terms of a basis of diagrams. We will consider certain rectangles with n marked nodes on each of the northern and southern edges. Brauer diagrams are then those rectangles in which all nodes are connected to precisely one other by a line. Lines connecting nodes on the same edge are called *arcs*, while those connecting nodes on opposite edges are called *propagating lines.* The product AB of diagrams A and B is given by concatenating A above B, to form a diagram C which may contain some number (t say) of closed loops. To form a diagram in our basis we set C equal to $\delta^t C'$ where C' is the diagram obtained from C by deleting all closed loops.

Now decorate all Brauer diagrams in B_n with a vertical wall separating the first r nodes on each edge from the final s nodes on each edge. The walled Brauer algebra $B_{r,s}(\delta)$ (or just $B_{r,s}$) is then the subalgebra of B_n generated by those Brauer diagrams in which arcs cross the wall, while propagating lines do not.

For $\delta \neq 0$ let $e_{r,s}$ be δ^{-1} times the diagram with all nodes connected vertically in pairs except for those adjacent to the wall, which are connected across the wall. This is an idempotent, and we have an algebra isomorphism

$$B_{r-1,s-1} \cong e_{r,s} B_{r,s} e_{r,s}.$$

Via this isomorphism we have an exact *localisation* functor

$$F_{r,s} : B_{r,s}\text{-mod} \longrightarrow B_{r-1,s-1}\text{-mod}$$

taking a module M to $e_{r,s}M$, and a right exact *globalisation* globalisation functor $G_{r-1,s-1}$ in the opposite direction taking a module N to $B_{r,s}e_{r,s} \otimes_{e_{r,s}B_{r,s}e_{r,s}} N$. There is a similar idempotent $e_n \in B_n$ and algebra isomorphism $B_{n-2} \cong e_n B_n e_n$, giving rising to corresponding localisation and globalisation functors F_n and G_n. (For the basic properties of localisation and globalisation see [Gre80] or [CMPX06].)

Let Σ_r denote the symmetric group on r symbols, and set $\Sigma_{r,s} = \Sigma_r \times \Sigma_s$. There is an isomorphism

$$B_{r,s}/B_{r,s}e_{r,s}B_{r,s} \cong \mathbb{C}\Sigma_{r,s}$$

and this latter algebra has simple modules labelled by $\Lambda^{r,s}$, the set of pairs of partitions of r and s respectively. By standard properties of localisation it follows that if $r, s > 0$ then the set of simple modules for $B_{r,s}$ is labelled by

$$\Lambda_{r,s} = \Lambda^{r,s} \cup \Lambda_{r-1,s-1}.$$

As $B_{r,0} \cong B_{0,r} \cong \mathbb{C}\Sigma_r$ we deduce that $\Lambda_{r,s}$ consists of all pairs $\lambda = (\lambda^L, \lambda^R)$ such that λ^L is a partition of $r - t$ and λ^R is a partition of $s - t$ for some $t \geq 0$. We say that such a bipartition is of *degree* $\deg(\lambda) = (r - t, s - t)$, and put a partial order on degrees by setting $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$.

Let Λ^n denote the set of partitions of n. Then by similar arguments we see that the labelling set Λ_n for simple B_n-modules is given recursively by $\Lambda_n = \Lambda^n \cup \Lambda_{n-2}$ and so Λ_n consists of all partitions λ of $n - 2t$ for some $t \geq 0$. We say that such a partition is of *degree* $\deg(\lambda) = n - 2t$.

The $e_{r-t,s-t}$ with $0 \leq t \leq \min(r, s)$ induce a heredity chain in $B_{r,s}$, and so we can apply the theory of quasihereditary algebras. In particular for each $\lambda \in \Lambda_{r,s}$ there is an associated standard module $\Delta_{r,s}(\lambda)$ with simple head $L_{r,s}(\lambda)$ and projective cover $P_{r,s}(\lambda)$. The standard modules have an explicit description in terms of walled Brauer diagrams and Specht modules for the various $\Sigma_{r-t,s-t}$, and determining the decomposition numbers for these modules in terms of their simple factors is equivalent to determining the simple modules themselves. In the same way the B_n are quasihereditary, with standard modules $\Delta_n(\lambda)$, with simple modules $L_n(\lambda)$, and projective covers $P_n(\lambda)$.

By general properties of our heredity chain we have

$$G_{r,s}\Delta_{r,s}(\lambda) \cong \Delta_{r+1,s+1}(\lambda)$$

and

$$F_{r,s}\Delta_{r,s}(\lambda) \cong \begin{cases} \Delta_{r-1,s-1}(\lambda) & \text{if } \lambda \in \Lambda_{r-1,s-1} \\ 0 & \text{if } \lambda \in \Lambda^{r,s} \end{cases}$$

We define a partial order on the set of all partitions (or all bipartitions) by setting $\lambda \leq \mu$ if $\deg(\lambda) < \deg(\mu)$ or $\lambda = \mu$. This is the opposite of the partial order induced by the quasihereditary structure on Λ_n or $\Lambda_{r,s}$. Thus the decomposition multiplicity

$$[\Delta_{r,s}(\lambda) : L_{r,s}(\mu)]$$

is zero unless $\lambda \leq \mu$, and is independent of (r, s) provided that $\lambda, \mu \in \Lambda_{r,s}$ (and similarly for the Brauer case).

As our algebra is quasihereditary each projective module $P_{r,s}(\lambda)$ has a filtration by standard modules. The multiplicity of a given standard $\Delta_{r,s}(\mu)$ in such a filtration is well-defined, and we denote it by

$$D_{\lambda\mu} = (P_{r,s}(\lambda) : \Delta_{r,s}(\mu)).$$

By Brauer-Humphreys reciprocity we have

$$D_{\lambda\mu} = [\Delta_{r,s}(\mu) : L_{r,s}\ (\lambda)]$$

(and hence $D_{\lambda\mu}$ is independent of r and s). Again, analogous results hold for the Brauer algebra, and we shall denote the corresponding filtration multiplicities by $D_{\lambda\mu}$ also.

The algebra $B_{r,s}$ can be identified with a subalgebra of $B_{r+1,s}$ (respectively of $B_{r,s+1}$) by inserting an extra propagating line immediately to the left (respectively to the right) of the wall. The corresponding restriction functors will be denoted $\mathrm{res}^L_{r+1,s}$ and $\mathrm{res}^R_{r,s+1}$, with associated induction functors $\mathrm{ind}^L_{r,s}$ and $\mathrm{ind}^R_{r,s}$. Similarly, B_n is a subalgebra of B_{n+1} giving associated functors ind_n and res_{n+1}

We will identify a partition with its associated Young diagram, and let $\mathrm{add}(\lambda)$ (respectively $\mathrm{rem}(\lambda)$) denote the set of boxes which can be added singly to (respectively removed singly from) λ such that the result is still a partition. Given such a box ϵ, we denote the associated partition by $\lambda+\epsilon$ (respectively $\lambda-\epsilon$). If we wish to emphasise that ϵ lies in a given row (i say) then we may denote it by ϵ_i.

By [CDDM08, Theorem 3.3] we have

Proposition 2.1. *Suppose that $\lambda=(\lambda^L,\lambda^R)\in\Lambda^{r-t,s-t}$.*
(a) (i) If $t=0$ then

$$\mathrm{res}^L_{r,s}\,\Delta_{r,s}(\lambda^L,\lambda^R)\cong\bigoplus_{\epsilon\in\mathrm{rem}(\lambda^L)}\Delta_{r-1,s}(\lambda^L-\epsilon,\lambda^R).$$

(ii) If $t>0$ then there is a short exact sequence

$$0\longrightarrow\bigoplus_{\epsilon\in\mathrm{rem}(\lambda^L)}\Delta_{r-1,s}(\lambda^L-\epsilon,\lambda^R)\longrightarrow\mathrm{res}^L_{r,s}\,\Delta_{r,s}(\lambda)\longrightarrow\bigoplus_{\epsilon\in\mathrm{add}(\lambda^R)}\Delta_{r-1,s}(\lambda^L,\lambda^R+\epsilon)\longrightarrow 0.$$

(b) There is also a short exact sequence

$$0\longrightarrow\bigoplus_{\epsilon\in\mathrm{rem}(\lambda^L)}\Delta_{r,s+1}(\lambda^L-\epsilon,\lambda^R)\longrightarrow\mathrm{ind}^R_{r,s}\,\Delta_{r,s}(\lambda)\longrightarrow\bigoplus_{\epsilon\in\mathrm{add}(\lambda^R)}\Delta_{r,s+1}(\lambda^L,\lambda^R+\epsilon)\longrightarrow 0$$

where the first sum equals 0 if $\lambda^L=\emptyset$.

Remark 2.2. There is a similar result for $\mathrm{res}^R_{r,s}$ and $\mathrm{ind}^L_{r,s}$ replacing $\mathrm{rem}(\lambda^L)$ by $\mathrm{rem}(\lambda^R)$ and $\mathrm{add}(\lambda^R)$ by $\mathrm{add}(\lambda^L)$.

There is an entirely analogous result for the Brauer algebra, where the terms in the submodule of the restriction (or induction) of $\Delta_n(\lambda)$ are labelled by all partitions obtained by removing a box from λ, and those in the quotient module by all partitions obtained by adding a box to λ. For example, we have a short exact sequence

$$0\longrightarrow\bigoplus_{\epsilon\in\mathrm{rem}(\lambda)}\Delta_{n+1}(\lambda-\epsilon)\longrightarrow\mathrm{ind}_n\,\Delta_n(\lambda)\longrightarrow\bigoplus_{\epsilon\in\mathrm{add}(\lambda)}\Delta_{n+1}(\lambda+\epsilon)\longrightarrow 0$$

where the first sum equals 0 if $\lambda=\emptyset$.

It will be convenient to consider the Brauer and walled Brauer cases simultaneously. In the walled Brauer case we will set $(a)=(r,s)$, with $(a-1)=(r,s-1)$ and $(a+1)=(r+1,s)$. In the Brauer case we will set $(a)=n$ with $(a-1)=n-1$ and $(a+1)=n+1$. Then $\Lambda_{(a)}$ will denote either $\Lambda_{r,s}$ or Λ_n depending on the algebra being considered, and similarly for $\Delta_{(a)}(\lambda)$ and any other objects or functors with subscripts.

3. Translation functors

In [CDM11] we introduced the notion of translation functors for a tower of recollement, and showed how they could be used to generate Morita equivalence between different blocks. After a brief review of this, we will show how this can be applied to the Brauer and walled Brauer algebras. Details can be found in [CDM11, Section 4].

Let A_n with $n \in \mathbb{N}$ form a tower of recollement, with associated idempotents e_n for $n \geq 2$. Let Λ_n denote the set of labels for the simple A_n-modules, which we call weights. We denote the associated simple, standard, and projective modules by $L_n(\lambda)$, $\Delta_n(\lambda)$ and $P_n(\lambda)$ respectively. The algebra embedding arising from our tower structure gives rise to induction and restriction functors ind_n and res_n. For each standard module $\Delta_n(\lambda)$, the module $\mathrm{res}_n \Delta_n(\lambda)$ has a filtration by standard modules with well-defined multiplicities; we denote by $\mathrm{supp}_n(\lambda)$ the multiset of labels for standard modules occurring in such a filtration. We impose a crude partial order on weights by setting $\lambda < \mu$ if there exists n such that $\lambda \in \Lambda_n$ and $\mu \in \Lambda_{n+2}$ but $\mu \notin \Lambda_n$. This is the opposite of the partial order induced by the quasihereditary structure.

In such a tower we have isomorphisms $e_n A_n e_n \cong A_{n-2}$. Thus we also have associated localisation functors F_n and globalisation functors G_n. Globalisation induces an embedding of Λ_n inside Λ_{n+2}, and an associated embedding of $\mathrm{supp}_n(\lambda)$ inside $\mathrm{supp}_{n+2}(\lambda)$, which becomes an identification if $\lambda \in \Lambda_{n-2}$. We denote by $\mathrm{supp}(\lambda)$ the set $\mathrm{supp}_n(\lambda)$ where $n >> 0$.

Let $\mathcal{B}_n(\lambda)$ denote the set of weights labelling simple modules in the same block for A_n as $L_n(\lambda)$. Again there is an induced embedding of $\mathcal{B}_n(\lambda)$ inside $\mathcal{B}_{n+2}(\lambda)$, and we denote by $\mathcal{B}(\lambda)$ the corresponding limit set. Given a weight λ, we denote by pr_n^λ the functor which projects onto the A_n-block containing $L_n(\lambda)$. We then define *translation functors* $\mathrm{res}_n^\lambda = \mathrm{pr}_{n-1}^\lambda \mathrm{res}_n$ and $\mathrm{ind}_n^\lambda = \mathrm{pr}_{n+1}^\lambda \mathrm{ind}_n$.

We say that λ and λ' are *translation equivalent* if for all $\mu \in \mathcal{B}(\lambda)$ there is a unique element $\mu' \in \mathcal{B}(\lambda') \cap \mathrm{supp}(\mu)$, and

$$\mathcal{B}(\lambda) \cap \mathrm{supp}(\mu') = \{\mu\}$$

and every element of $\mathcal{B}(\lambda')$ arises in this way. When λ and λ' are translation equivalent we denote by $\theta : \mathcal{B}(\lambda) \longrightarrow \mathcal{B}(\lambda')$ the bijection taking μ to μ'.

By [CDM11, Propositions 4.1 and 4.2] we have

Theorem 3.1. *Suppose that $\lambda \in \Lambda_n$ and $\lambda' \in \Lambda_{n-1}$ are translation equivalent, and that $\mu \in \mathcal{B}_n(\lambda)$ is such that $\mu' \in \mathcal{B}_{n-1}(\lambda')$.*
(i) We have

$$\mathrm{res}_n^{\lambda'} L_n(\mu) \cong L_{n-1}(\mu') \qquad \mathrm{ind}_{n-1}^{\lambda} L_{n-1}(\mu') \cong L_n(\mu)$$

and

$$\mathrm{ind}_{n-1}^{\lambda} P_{n-1}(\mu') \cong P_n(\mu).$$

(ii) If $\tau \in \mathcal{B}_n(\lambda)$ is such that $\tau' \in \mathcal{B}_{n-1}(\lambda')$ then

$$[\Delta_n(\mu) : L_n(\tau)] = [\Delta_{n-1}(\mu') : L_{n-1}(\tau')]$$

and

$$\mathrm{Hom}(\Delta_n(\mu), \Delta_n(\tau)) \cong \mathrm{Hom}(\Delta_{n-1}(\mu'), \Delta_{n-1}(\tau')).$$

(iii) If $\mu \in \mathcal{B}_{n-2}(\lambda)$ then

$$\operatorname{res}_n^{\lambda'} P_n(\mu) \cong P_{n-1}(\mu').$$

The above result suggests that translation equivalent weights should be in Morita equivalent blocks, but this is not true in general as there will not be a bijection between the simple modules. However, by a suitable truncation of the algebra we do get Morita equivalences.

The algebra A_n decomposes as

$$A_n = \bigoplus_{\lambda \in \Lambda_n} P_n(\lambda)^{m_{n,\lambda}}$$

for some integers $m_{n,\lambda}$. Let $1 = \sum_{\lambda \in \Lambda_n} e_{n,\lambda}$ be the associated orthogonal idempotent decomposition of the identity in A_n. There is also a decomposition of A_n into its block subalgebras

$$A_n = \bigoplus_{\lambda} A_n(\lambda)$$

where the sum runs over a set of block representatives. Now let $\Gamma \subseteq \mathcal{B}_n(\lambda)$ and consider the idempotent $e_{n,\Gamma} = \sum_{\gamma \in \Gamma} e_{n,\gamma}$. We define the algebra $A_{n,\Gamma}(\lambda)$ by

$$A_{n,\Gamma}(\lambda) = e_{n,\Gamma} A(_n(\lambda) e_{n,\Gamma}.$$

By [CDM11, Theorem 4.5 and Corollary 4.7] we have

Theorem 3.2. *Suppose that λ and λ' are translation equivalent, with $\lambda \in \Lambda_n$, and set*

$$\Gamma = \theta(\mathcal{B}_n(\lambda)) \subseteq \mathcal{B}_{n+1}(\lambda').$$

(i) The algebras $A_n(\lambda)$ and $A_{n+1,\Gamma}(\lambda')$ are Morita equivalent. In particular, if $|\mathcal{B}_n(\lambda)| = |\mathcal{B}_{n+1}(\lambda')|$ then $A_n(\lambda)$ and $A_{n+1}(\lambda')$ are Morita equivalent.
(ii) For all $\mu \in \mathcal{B}_n(\lambda)$ we have

$$\operatorname{Ext}^i(\Delta_n(\lambda), \Delta_n(\mu)) \cong \operatorname{Ext}^i(\Delta_{n+1}(\lambda'), \Delta_{n+1}(\mu')).$$

We will say that blocks $\mathcal{B}(\lambda)$ and $\mathcal{B}(\lambda')$ corresponding to translation equivalent weights are *weakly Morita equivalent.*

The notion of translation equivalent weights is motivated by the translation principle in Lie theory, where translation functors give equivalences for weights inside the same facet. Another common situation in Lie theory involves the relationship between weights in a pair of alcoves separated by a wall. There is also an analogue of this in our setting.

We say that λ' *separates* λ^- and λ^+ if

$$\mathcal{B}(\lambda') \cap \operatorname{supp}(\lambda^-) = \{\lambda'\} = \mathcal{B}(\lambda') \cap \operatorname{supp}(\lambda^+)$$

and

$$\mathcal{B}(\lambda^-) \cap \operatorname{supp}(\lambda') = \{\lambda^+, \lambda^-\}.$$

Whenever we consider a pair of weights λ^- and λ^+ separated by λ' we shall always assume that $\lambda^- < \lambda^+$. By [CDM11, Theorem 4.8] we have

Theorem 3.3. *(i) If $\lambda' \in \Lambda_{n-1}$ separates λ^- and λ^+ then*

$$\mathrm{res}_n^{\lambda'} L_n(\lambda^+) \cong L_{n-1}(\lambda').$$

(ii) If further we have $\mathrm{Hom}(\Delta_n(\lambda^+), \Delta_n(\lambda^-)) \neq 0$ then

$$\mathrm{res}_n^{\lambda'} L_n(\lambda^-) = 0$$

and $\mathrm{ind}_{n-1}^{\lambda^-} \Delta_n(\lambda')$ is a nonsplit extension of $\Delta_n(\lambda^-)$ by $\Delta_n(\lambda^+)$ and has simple head $L_n(\lambda^+)$.

Suppose that λ' and λ^+ are weights with $\lambda' < \lambda^+$ such that for every weight $\tau' \in \mathcal{B}(\lambda')$ either (i) there is a unique weight $\tau^+ \in \mathcal{B}(\lambda^+) \cap \mathrm{supp}(\tau')$ and τ' is the unique weight in $\mathcal{B}(\lambda') \cap \mathrm{supp}(\tau^+)$, or (ii) there exists $\tau^-, \tau^+ \in \mathcal{B}(\lambda^+)$ such that τ' separates τ^- and τ^+. Then we say that λ' is in the *lower closure* of λ^+. If every pair of weights μ^- and μ^+ in $\mathcal{B}(\lambda^+)$ separated by some $\mu' \in \mathcal{B}(\lambda')$ satisfy the condition in Theorem 3.3(ii), i.e. satisfy

$$\mathrm{Hom}(\Delta_n(\mu^+), \Delta_n(\mu^-)) \neq 0$$

then then we say that $\mathcal{B}(\lambda^+)$ has *enough local homomorphisms* with respect to $\mathcal{B}(\lambda')$.

We will need one new general result about translation functors for towers of recollement not included in [CDM11].

Proposition 3.4. *Suppose that $\mathcal{B}(\lambda^+)$ has enough local homomorphisms with respect to $\mathcal{B}(\lambda')$. If λ' is in the lower closure of λ^+ then*

$$\mathrm{ind}_n^{\lambda^+} P_n(\lambda') \cong P_{n+1}(\lambda^+).$$

If further $\lambda' \in \Lambda_{n-2}$ then

$$\mathrm{res}_n^{\lambda^+} P_n(\lambda') \cong P_{n-1}(\lambda^+).$$

Proof. The module $\mathrm{ind}_n^{\lambda^+} P_n(\lambda')$ is clearly projective, as induction (and taking a direct summand) takes projectives to projectives.

Suppose that $\tau \in \mathcal{B}(\lambda^+)$ and that

$$\mathrm{res}_{n+1}^{\lambda'} L_{n+1}(\tau) \neq 0.$$

By our assumptions and Theorems 3.1 and 3.3 this implies that $\tau \in \mathrm{supp}(\mu')$ for some $\mu' \in \mathcal{B}(\lambda')$ and $\tau = \mu^+$. From this we see that if

$$\mathrm{Hom}_{n+1}(\mathrm{ind}_n^{\lambda^+} P_n(\lambda'), L_{n+1}(\tau)) = \mathrm{Hom}_n(P_n(\lambda'), \mathrm{res}_{n+1}^{\lambda'} L_{n+1}(\tau))$$

is non-zero then $\tau = \mu^+$ for some $\mu' \in \mathcal{B}(\lambda')$. But

$$\mathrm{Hom}_n(P_n(\lambda'), \mathrm{res}_{n+1}^{\lambda'} L_{n+1}(\mu^+)) = \mathrm{Hom}_n(P_n(\lambda'), \mathrm{pr}_n^{\lambda'} L(\mu')) = \delta_{\lambda'\mu'}$$

by Theorems 3.1 and 3.3. Thus $\mathrm{ind}_n^{\lambda^+} P_n(\lambda')$ has simple head $L_{n+1}(\lambda^+)$, and hence is equal to $P_{n+1}(\lambda^+)$ as required.

Now suppose that further $\lambda' \in \Lambda_{n-2}$. By [CDM11, Lemma 4.3] we have

$$G_{n-2}P_{n-2}(\lambda') \cong P_n(\lambda').$$

By the tower of recollement axioms we have

$$\mathrm{ind}_{n-2}^{\lambda^+} M \cong \mathrm{res}_n^{\lambda^+} G_{n-2}M$$

for any A_{n-2}-module M and hence

$$\mathrm{res}_n^{\lambda^+} P_n(\lambda') \cong \mathrm{res}_n^{\lambda^+} G_{n-2}P_{n-2}(\lambda') \cong \mathrm{ind}_{n-2}^{\lambda^+} P_{n-2}(\lambda') \cong P_{n-1}(\lambda^+)$$

using the first part of the Proposition.

Theorem 3.5. *The Brauer and walled Brauer algebras form towers of recollement, in the latter case by using alternately the functors* res^L *(and* ind^L*) and* res^R *(and* ind^R*), and have enough local homomorphisms.*

Proof. For the tower of recollement claim see [CDM09a] for the Brauer algebra and [CDDM08, Sections 2-3]) for the walled Brauer algebra. The existence of enough local homomorphisms was shown for the Brauer algebra in [DWH99, Theorem 3.4] and for the walled Brauer algebra in [CDDM08, Theorem 6.2].

Thus we can apply the results of this section to the Brauer and walled Brauer algebras.

Definition 3.6. *When using the notation* $\mathrm{ind}_{r,s}^{\lambda}$ *for the walled Brauer algebra, the choice of* $\mathrm{ind}_{r,s}^{L}$ *or* $\mathrm{ind}_{r,s}^{R}$ *will be such that the weight* λ *makes sense for the resulting algebra (and similarly for* $\mathrm{res}_{r,s}^{\lambda}$*).*

Remark 3.7. There are reflection geometries controlling the block structure of the Brauer [CDM09b] and walled Brauer algebras [CDDM08] which we will review shortly. These define a system of facets, and in [CDM09b] it was shown that two weights in the same facet for the Brauer algebra have weakly Morita equivalent blocks in the sense of Theorem 3.2. This required certain generalised induction and restriction functors for the non-alcove cases. Similar functors can be defined for the walled Brauer algebras: it is a routine but lengthy exercise to verify that the construction in [CDM11, Section 5] can be extended to the walled Brauer case. Thus we also have weak Morita equivalences between weights in the same facet in the walled Brauer case.

4. Oriented cap diagrams

In this section we will describe the construction of oriented cap diagrams associated to certain pairs of weights for the walled Brauer algebra. These diagrams were introduced by Brundan and Stroppel in [BSa] to study a generalisation of Khovanov's diagram algebra. We will see later that they give precisely the combinatoric required to describe decomposition numbers for the walled Brauer algebra.

Let $\{\epsilon_i : i \in \mathbb{Z}, i \neq 0\}$ be a set of formal symbols, and set

$$X = \prod_{i\in\mathbb{Z}\setminus\{0\}} \mathbb{Z}\epsilon_i.$$

For $x \in X$ we write

$$x = (\dots, x_{-3}, x_{-2}, x_{-1}; x_1, x_2, x_3, \dots)$$

where x_i is the coefficient of ϵ_i. We define $A^+ \subset X$ by

$$A^+ = \{x \in X : \dots > x_{-3} > x_{-2} > x_{-1}, x_1 > x_2 > x_3 > \dots\}$$

and for $\delta \in \mathbb{Z}$ we define

$$\rho = \rho_\delta = (\cdots, 3, 2, 1; \delta, \delta - 1, \delta - 2, \cdots) \in A^+.$$

Given a bipartition $\lambda = (\lambda^L, \lambda^R)$ with $\lambda^L = (\lambda^L_1, \dots, \lambda^L_r)$ and $\lambda^R = (\lambda^R_1, \dots, \lambda^R_s)$, we define $\bar{\lambda} \in X$ by

$$\bar{\lambda} = (\dots, 0, 0, -\lambda^L_r, -\lambda^L_{r-1}, \dots, -\lambda^L_1; \lambda^R_1, \dots, \lambda^R_s, 0, 0, \dots).$$

Given such a bipartition λ we define

$$x_\lambda = x_{\lambda,\rho} = \bar{\lambda} + \rho_\delta.$$

Note that $x_\lambda \in A^+$. In this way we can embed the sets $\Lambda_{r,s}$ labelling simple modules for $B_{r,s}(\delta)$ as subsets of A^+.

Consider the group W of all permutations of finitely many elements from the set $\mathbb{Z}\backslash\{0\}$ (so $W = \langle (i,j) : i, j \in \mathbb{Z}\backslash\{0\}\rangle$ where (ij) is the usual notation for transposition of a pair i and j). This group acts on X by place permutations.

The main result (Corollary 10.3) in [CDDM08] describes the blocks of $B_{r,s}(\delta)$ in terms of orbits of certain finite reflection groups inside W. However it is easy to see from the proof that this result can be reformulated equivalently as follows.

Theorem 4.1. *Two simple modules $L_{r,s}(\lambda)$ and $L_{r,s}(\mu)$ are in the same block if and only if $x_\lambda = wx_\mu$ for some $w \in W$.*

We will abuse terminology and say that x_λ and x_μ are in the same block if they satisfy the conditions of this theorem.

To each element $x \in A^+$ we wish to associate a diagram with vertices indexed by $\mathbb{Z}$, each labelled with one of the symbols $\circ$, $\times$, $\wedge$, $\vee$. We do this in the following manner. Given $x \in A^+$ define

$$I_\vee(x) = \{x_i : i < 0\} \quad \text{and} \quad I_\wedge(x) = \{x_i : i > 0\}.$$

Now vertex n in the diagram associated to x is labelled by

$$\begin{cases} \circ & \text{if } n \notin I_\vee(x) \cup I_\wedge(x) \\ \times & \text{if } n \in I_\vee(x) \cap I_\wedge(x) \\ \vee & \text{if } n \in I_\vee(x) \backslash I_\wedge(x) \\ \wedge & \text{if } n \in I_\wedge(x) \backslash I_\vee(x). \end{cases} \tag{1}$$

Example 4.2. To illustrate the above construction, consider the bipartition $\lambda = (\lambda^L, \lambda^R)$ where $\lambda^L = (2, 2, 1)$ and $\lambda^R = (3, 2)$, and take $\delta = 2$. Then

$$\rho_\delta = (\dots, 4, 3, 2, 1; 2, 1, 0, -1, -2, \dots)$$

and

$$\bar{\lambda} = (\dots, 0, -1, -2, -2; 3, 2, 0, 0, 0, \dots)$$

and hence

$$x_\lambda = \bar{\lambda} + \rho_\delta = (\dots, 6, 5, 4, 2, 0, -1; 5, 3, 0, -1, -2, -3\dots).$$

Part of the associated diagram is illustrated in Figure 1.

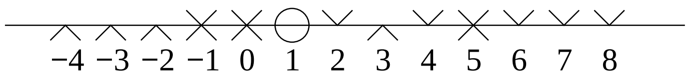

Figure 1: The diagram associated to $((2, 2, 1), (3, 2))$ with $\delta = 2$.

Note that any element in A^+ is uniquely determined by its diagram, and every such diagram corresponds to an element in A^+. For this reason we will use the notation x (or x_λ) for both.

Remark 4.3. It is easy to see that two elements in A^+ are in the same W-orbit if and only if they are obtained from each other by permuting pairwise a finite number of $\wedge$s and $\vee$s.

We define a partial order $\leq$ on A^+ by setting $x < y$ if y is obtained from x by swapping a $\vee$ and a $\wedge$ so that the $\wedge$ moves to the right, and extending by transitivity. Note that if $\lambda, \mu \in \Lambda_{r,s}$ then $x_\lambda \leq x_\mu$ only if λ and μ are in the same block and $\lambda \leq \mu$ (where this is the natural order on bipartitions from Section 2).

Example 4.4. There is only one element in A^+ smaller than the element x_λ in Example 4.2. This corresponds to the diagram in Figure 2.

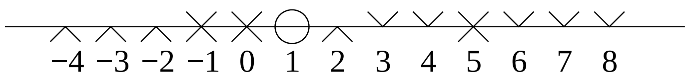

Figure 2: The unique diagram smaller than the diagram in Figure 1.

Remark 4.5. For a bipartition $\lambda = (\lambda^L, \lambda^R)$, the diagram for the element $x_\lambda \in A^+$ is labelled by $\wedge$ for all $n << 0$ and by $\vee$ for all $n >> 0$. Thus there are only finitely many $x < x_\lambda$.

To each bipartition λ (or to each diagram labelled by $\wedge$ for all $n << 0$ and by $\vee$ for all $n >> 0$) we associate a *cap diagram* c_λ in the following (recursive) manner.

In x_λ find a pair of vertices labelled $\vee$ and $\wedge$ in order from left to right that are neighbours in the sense that there are only $\circ$s, $\times$s, or vertices already joined by caps at an earlier stage between them. Join this pair of vertices together with a cap. Repeat this process until there are no more such $\vee$ $\wedge$ pairs. (This will occur after a finite number of steps.) Finally, draw an infinite ray upwards at all remaining $\wedge$s and $\vee$s. Any vertices which are not connected to a ray or a cap are called free vertices.

Example 4.6. In Figures 3 and 4 we give two examples of elements x_λ and their associated cap diagrams.

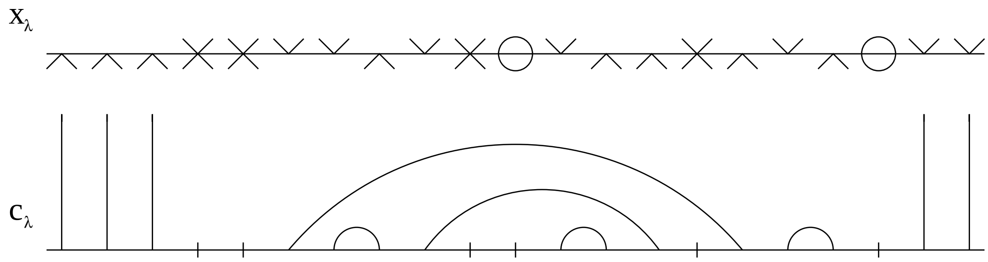

Figure 3: An example of the cap diagram construction.

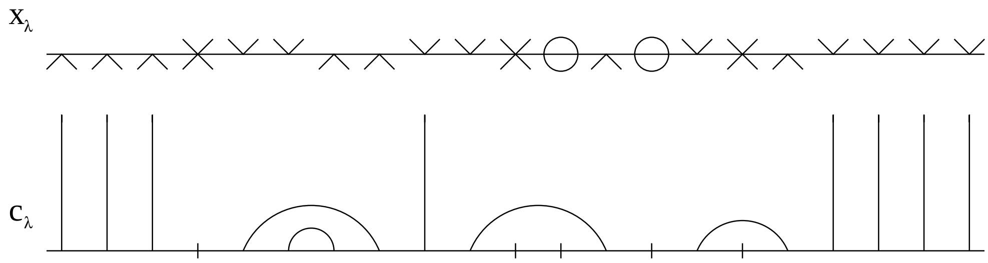

Figure 4: Another example of the cap diagram construction.

To a cap diagram c and an element $x_\lambda \in A^+$ we can associate a *labelled cap diagram* cx_λ by writing each label on a vertex of x_λ underneath the corresponding vertex of c. We call such a diagram an *oriented cap diagram* if the following conditions all hold:

1. each free vertex in c is labelled by a $\circ$ or $\times$ in x_λ;
2. the vertices at the end of each cap in c are labelled by exactly one $\wedge$ and one $\vee$ in x_λ;
3. each vertex at the bottom of a ray in c is labelled by a $\wedge$ or $\vee$ in x_λ;
4. it is impossible to find two rays in c whose vertices are labelled $\vee$ and $\wedge$ in order from left to right in x_λ.

As each cap in an oriented cap diagram is labelled by exactly one $\wedge$ and one $\vee$, these symbols induce an orientation on the cap (as though they were arrows). The *degree* $\deg(cx_\lambda)$ of an oriented cap diagram cx_λ is the total number of clockwise caps that it contains.

Remark 4.7. Given a bipartition λ, the labelled cap diagram $c_\lambda x_\lambda$ is clearly oriented, with all caps having a counterclockwise orientation. Thus the degree of $c_\lambda x_\lambda$ is 0.

For two bipartitions λ and μ we define $d_{\lambda\mu}(q)$ to be $q^{\deg(c_\lambda x_\mu)}$ if (i) λ and μ are in the same W-orbit, and (ii) $c_\lambda x_\mu$ is an oriented cap diagram. We define $d_{\lambda\mu}(q)$ to be 0 otherwise. In other words, $d_{\lambda\mu}(q) \neq 0$ if and only if x_μ is obtained from x_λ by swapping the order of the elements in some of the pairs $\vee$, $\wedge$ which are joined up in c_λ, and in that case $\deg(c_\lambda x_\mu)$ is the number of pairs whose elements have been swapped.

Example 4.8. Let x_λ and c_λ be as in Figure 3. For x_μ as illustrated in Figure 5 we see that $c_\lambda x_\mu$ is an oriented cap diagram with $\deg(c_\lambda x_\mu) = 3$. Hence we have that

$$d_{\lambda\mu}(q) = q^3.$$

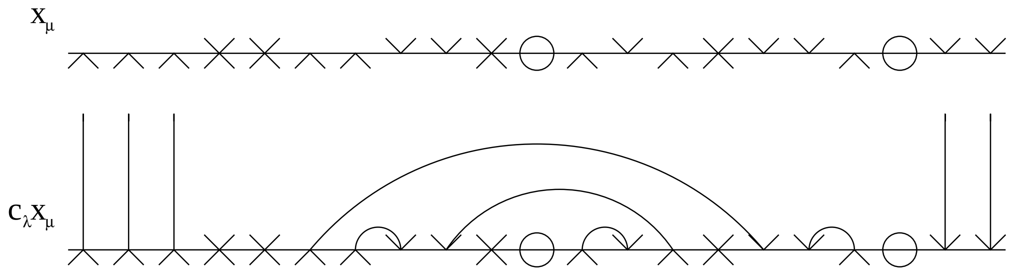

Figure 5: An example of a nontrivial degree calculation.

Remark 4.9. Brundan and Stroppel have shown how to associate weights in a set similar to A^+ to cap diagrams and oriented cap diagrams in order to use this combinatoric to describe the representation theory of parabolic category $\mathcal{O}$ corresponding to type $A \times A$ in type A [BS11], and to the general linear supergroup $\mathrm{GL}(m|n)$ [BSa]. Note the difference between these two sets, and the difference between the assignments of labels in [BSb, (1.6)] and in (1).

We are interested in determining the decomposition numbers for the walled Brauer algebras. As noted in Section 2 this is equivalent to determining the

$$D_{\lambda\mu} = (P_{r,s}(\lambda) : \Delta_{r,s}(\mu)).$$

Our eventual aim is to show

Theorem 4.10. *Given λ and μ in $\Lambda_{r,s}$ we have*

$$D_{\lambda\mu} = d_{\lambda\mu}(1).$$

We will first introduce a corresponding formalism for the Brauer algebra, so that the two cases can be considered simultaneously.

5. Oriented curl diagrams

We will introduce analogues of oriented cap diagrams for use in the ordinary Brauer algebra case. As the two cases will ultimately be very similar, we use the same notation. Which case is being considered later will be clear from context.

Let $\{\epsilon_i : i \in \mathbb{N}\}$ be a set of formal symbols, and set

$$X = \left(\prod_{i\in\mathbb{N}} \mathbb{Z}\epsilon_i\right) \bigcup \left(\prod_{i\in\mathbb{N}} (\mathbb{Z} + \frac{1}{2})\epsilon\right).$$

For $x \in X$ we write

$$x = (x_1, x_2, \ldots)$$

where x_i is the coefficient of ϵ_i. We define $A^+ \subset X$ by

$$A^+ = \{x \in X : x_1 > x_2 > \cdots\}$$

and for $\delta \in \mathbb{Z}$ define

$$\rho = \rho_\delta = (-\frac{\delta}{2}, -\frac{\delta}{2} - 1, -\frac{\delta}{2} - 2, -\frac{\delta}{2} - 3, \ldots) \in A^+.$$

Given a partition λ we define

$$x_\lambda = \lambda + \rho_\lambda \in A^+.$$

Consider the group

$$W = \langle (i,j), (i,j)_- : i \neq j \in \mathbb{N}\rangle$$

where (ij) is the usual notation for transposition of a pair i and j, and $(i,j)_-$ is the element which transposes i and j and also changes their signs. Then W acts naturally on X, with (ij) acting as place permutations, and

$$(ij)_-(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots) = (x_1, x_2, \ldots, -x_j, \ldots, -x_i, \ldots).$$

The blocks of $B_n(\delta)$ are described in [CDM09b, Theorem 4.2] in terms of certain finite reflection groups inside W. Just as in the walled Brauer case, it is easy to see that the following reformulation is equivalent. Here we denote the transpose of a partition λ by λ^T.

Theorem 5.1. *Two simple modules $L_n(\lambda^T)$ and $L_n(\mu^T)$ are in the same block if and only if $x_\lambda = wx_\mu$ for some $w \in W$.*

To each $x \in X$ we wish to associate a diagram. This will have vertices indexed by $\mathbb{N} \cup \{0\}$ if $x \in \prod_{i\in\mathbb{N}} \mathbb{Z}\epsilon_i$ or by $\mathbb{N} - \frac{1}{2}$ if $x \in \prod_{i\in\mathbb{N}}(\mathbb{Z} + \frac{1}{2})\epsilon_i$. Each vertex will be labelled with one of the symbols $\circ$, $\times$, $\vee$, $\wedge$, or $\diamond$. Given $x \in A^+$ define

$$I_\wedge(x) = \{x_i : x_i > 0\} \quad \text{and} \quad I_\vee(x) = \{|x_i| : x_i < 0\}.$$

We also set $I_\diamond(x) = \{x_i : x_i = 0\}$, so $I_\diamond(x)$ can consist of at most one element. Now vertex n in the diagram associated to x is labelled by

$$\begin{cases} \circ & \text{if } n \notin I_\vee(x) \cup I_\wedge(x) \\ \times & \text{if } n \in I_\vee(x) \cap I_\wedge(x) \\ \vee & \text{if } n \in I_\vee(x) \backslash I_\wedge(x) \\ \wedge & \text{if } n \in I_\wedge(x) \backslash I_\vee(x) \\ \diamond & \text{if } n \in I_\diamond(x). \end{cases} \tag{2}$$

Note that every element in A^+ is uniquely determined by its diagram, and every such diagram corresponds to an element in A^+ (provided that 0 is labelled by $\circ$ or $\diamond$). For this reason we will use the notation x (or x_λ) for both.

Example 5.2. Let $\lambda = (4, 3, 2)$ and $\delta = 1$. Then we have

$$\rho_\delta = (-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots,)$$

and

$$x_\lambda = \lambda + \rho_\delta = (\frac{7}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{7}{2}, -\frac{9}{2}, \ldots,).$$

The corresponding diagram is shown in Figure 6.

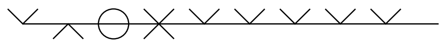

Figure 6: The diagram associated to $\lambda = (4, 3, 2)$ when $\delta = 1$.

Remark 5.3. Consider the case where no $\diamond$ occurs then two elements in A^+ are in the same W-orbit if and only if they are obtained from each other by repeatedly swapping a $\vee$ and a $\wedge$ or replacing two $\vee$s by two $\wedge$s. If we fix a λ where x_λ contains $\diamond$ then we can arbitrarily choose to replace this $\diamond$ by either $\vee$ or $\wedge$, and use the same swapping rules as in the previous case to define a unique choice of $\vee$ or $\wedge$ for $\diamond$ for every other element of the same block. *Thus in what follows we will always assume that a fixed choice of $\vee$ or $\wedge$ has been made for the symbol $\diamond$ for some weight in each block. Our combinatorial constructions will not be affected by this choice (provided we are consistent in a given block).*

We define a partial order $\leq$ on A^+ by setting $x < y$ if y is obtained from x by swapping a $\vee$ and a $\wedge$ so that the $\wedge$ moves to the right, or if y contains a pair of $\wedge$s instead of a corresponding pair of $\vee$s in x, and extending by transitivity. Note that for partitions $\lambda, \mu \in \Lambda_n$ we have $x_\lambda \leq x_\mu$ only if λ and μ are in the same block and $\lambda \leq \mu$ (where this is the natural order on partitions from Section 2).

Remark 5.4. For a fixed partition λ the diagram for x_λ is labelled by $\vee$ for all $n >> 0$. Thus there are only finitely many $x < x_\lambda$.

To each $x_\lambda \in A^+$ we now associate a *curl diagram* c_λ in the following (recursive) fashion.

In x_λ find a pair of vertices labelled $\vee$ and $\wedge$ in order from left to right that are neighbours in the sense that there are only $\circ$s, $\times$s, or vertices already joined by caps at an earlier stage between them. Join this pair of vertices together with a cap. Repeat this process until there are no more such $\vee\ \wedge$ pairs. (This will occur after a finite number of steps.)

Ignoring all $\circ$s, $\times$s and vertices on a cap, we are left with a sequence of a finite number of $\wedge$s followed by an infinite number of $\vee$s. Starting from the leftmost $\wedge$, join each $\wedge$

to the next from the left which has not yet been used, via a clockwise arc around all vertices to the left of the starting vertex and without crossing any other arcs or caps. If there is a free $\wedge$ remaining at the end of this procedure, draw an infinite ray up from this vertex, and draw infinite rays from each of the remaining $\vee$s. We will refer to the arcs connecting $\wedge$s as *curls*.

Example 5.5. An example of this construction is given in Figure 7.

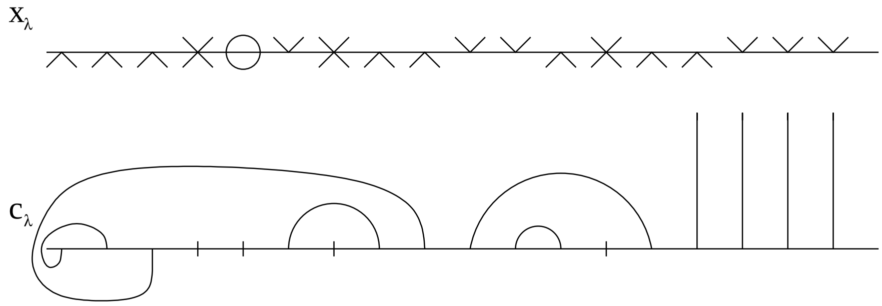

Figure 7: An example of the curl diagram construction.

To a curl diagram c and an element $x_\lambda \in A^+$ we can associate a *labelled curl diagram* cx_λ by writing each label on a vertex of x_λ underneath the corresponding vertex of c. We call such a diagram an *oriented curl diagram* if the following conditions all hold:

1. each free vertex in c is labelled by a $\circ$ or $\times$ in x_λ;
2. the vertices at the end of each cap in c are labelled by exactly one $\wedge$ and one $\vee$ in x_λ;
3. the vertices at the end of each curl in c are labelled by two $\wedge$s or two $\vee$s in x_λ;
4. each vertex at the bottom of a ray in c is labelled by a $\wedge$ or $\vee$ in x_λ;
5. it is impossible to find two rays in c whose vertices are labelled $\vee$ and $\wedge$, or $\wedge$ and $\wedge$, in order from left to right in x_λ.

Each cap or curl in an oriented curl diagram has an orientation induced by the terminal symbols (as though they were arrows). The *degree* $\deg(cx_\lambda)$ of an oriented curl diagram cx_λ is the number of clockwise caps and curls that it contains.

Remark 5.6. Given a partition λ, all caps and curls in the labelled curl diagram $c_\lambda x_\lambda$ are clearly oriented anticlockwise. Thus the degree of $c_\lambda x_\lambda$ is 0.

For two partitions λ and μ we define $d_{\lambda\mu}(q)$ to be $q^{\deg(c_\lambda x_\mu)}$ if (i) λ and μ are in the same W-orbit, and (ii) $c_\lambda x_\mu$ is an oriented curl diagram. We define $d_{\lambda\mu}(q)$ to be 0 otherwise.

Example 5.7. Let x_λ and c_λ be as in Figure 7. For x_μ as illustrated in Figure 8 we see that $c_\lambda x_\mu$ is an oriented curl diagram with $\deg(c_\lambda x_\mu) = 2$. Hence we have that

$$d_{\lambda\mu}(q) = q^2.$$

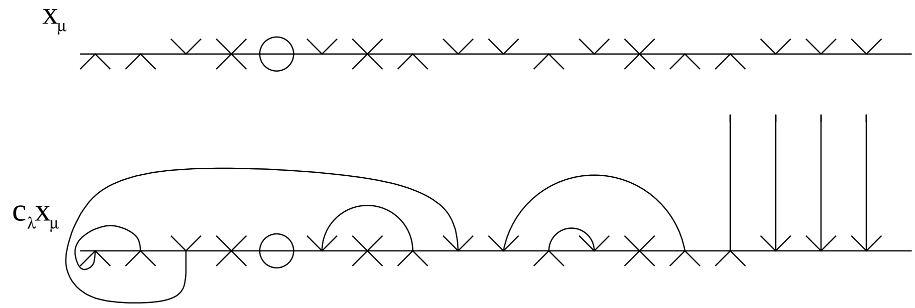

Figure 8: An example of the calculation of the degree of a curl diagram.

We are interested in determining the decomposition numbers for the Brauer algebras (and hence recovering the result of Martin [Mar]). As noted in Section 2 this is equivalent to determining the

$$D_{\lambda\mu} = (P_n(\lambda) : \Delta_n(\mu)).$$

Our eventual aim is to show

Theorem 5.8. *Given λ and μ in Λ_n we have*

$$D_{\lambda\mu} = d_{\lambda\mu}(1).$$

6. Decomposition numbers from oriented cap and curl diagrams

The aim of this section is to prove Theorems 4.10 and 5.8. To do this we will apply the translation functor formalism from Section 3. We will consider the two cases simultaneously as they are very similar. The proof is similar in spirit to the proof of [BS11, Theorem 4.10] and [BSb, Theorem 2.14], but with some complications (as the functors we use are not exact).

Fix $\lambda \in \Lambda_{r,s}$ or Λ_n. We will proceed by induction on the partial order $\leq$ introduced in Section 4 or 5. If x_λ is minimal in its block with respect to the order $\leq$ then we have

$$D_{\lambda\mu} = \delta_{\lambda\mu} = d_{\lambda\mu}(1)$$

for all μ and so we are done.

Suppose that x_λ is not minimal in its block. We proceed by induction on $|\lambda|$. (Note that if $\lambda^L = \emptyset$ or $\lambda^R = \emptyset$ then $x_\lambda = \rho$ is minimal.) Then $x_\lambda c_\lambda$ contains at least one cap or curl.

First consider the cap case: we may choose the cap so that it does not contain any smaller caps (and hence all vertices inside the cap are labelled by $\times$ or $\circ$ only). We call such a cap a *small* cap. There are three cases, which are illustrated in Figure 9. Note that we will henceforth abuse notation and write λ instead of x_λ.

Case (i): The vertex at the point marked with a $\wedge$ is of the form x_i for some $i \in \mathbb{N}$ (Brauer) or some $i \in \mathbb{Z}\backslash\{0\}$ (walled Brauer). In the latter case by the definition of $\wedge$ we must have $i > 0$. Now consider $\lambda' = (\lambda^L, \lambda^R - \epsilon_i)$ or $\lambda' = \lambda - \epsilon_i$. Note that $x_i - 1$ is not

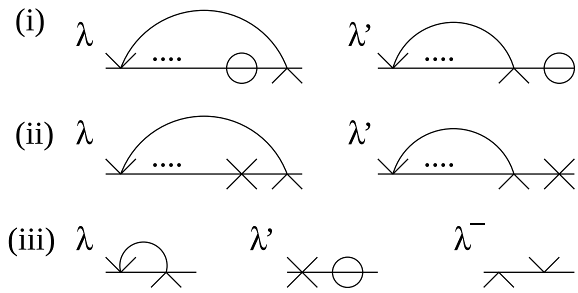

Figure 9: The three possible small cap configurations

an entry in x_λ and hence λ' is a (bi)partition. The diagram associated to λ' is illustrated on the right-hand side of Figure 9(i).

We claim that λ and λ' are translation equivalent; that is for every $\mu \in \mathcal{B}(\lambda)$ there exists a unique $\mu' \in \mathcal{B}(\lambda') \cap \mathrm{supp}(\mu)$ and for every $\mu' \in \mathcal{B}(\lambda')$ there is a unique $\mu \in \mathcal{B}(\lambda) \cap \mathrm{supp}(\mu')$. Indeed, it is easy to see that the only places where x_μ and $x_{\mu'}$ can differ are at the vertices labelled x_i and $x_i - 1$, and the possible cases are illustrated in Figure 10.

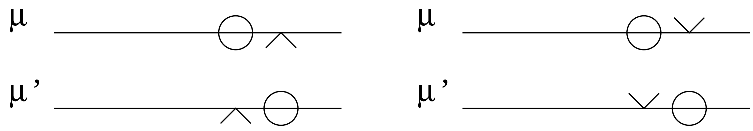

Figure 10: The possible diagrams for x_μ and $x_{\mu'}$ in case (i)

Recall our labelling convention involving (a) from Section 2. By Theorem 3.1 and the inductive hypothesis we have that

$$\begin{aligned} D_{\lambda\mu} &= [\Delta_{(a)}(\mu) : L_{(a)}(\lambda)] \\ &= [\Delta_{(a-1)}(\mu') : L_{(a-1)}(\lambda')] = D_{\lambda'\mu'} = d_{\lambda'\mu'}(1). \end{aligned} \tag{3}$$

But if we ignore the $\times$s and $\circ$s (which play no role other than as place markers in the definition of $d_{\lambda\mu}$) then the cap or curl diagrams c_λ and $c_{\lambda'}$ are identical, and hence

$$d_{\lambda'\mu'}(1) = d_{\lambda\mu}(1). \tag{4}$$

Combining (3) and (4) we see that $D_{\lambda\mu} = d_{\lambda\mu}(1)$ as required.

Case (ii): This is very similar to case (i). The vertex at the point marked with a $\times$ is in the walled Brauer case of the form x_{-j} for some $j \in \mathbb{Z}\backslash\{0\}$, and by the definition of $\times$ we can take $j > 0$. In the Brauer case this vertex is of the form $x_i > 0$ and x_i and $-x_i$ both appear in x_λ, and we choose j so that $x_{\ j} = -x_i$.

Now consider $\lambda' = (\lambda^L - \epsilon_j, \lambda^R)$ or $\lambda' = \lambda - \epsilon_j$. (As before it is easy to verify that λ' is a (bi)partition.) The diagram associated to λ' is illustrated on the right-hand side of Figure 9(ii). As in case (i) the weights λ and λ' are translation equivalent, where the various possibilities for x_μ and $x_{\mu'}$ as before are shown in Figure 11.

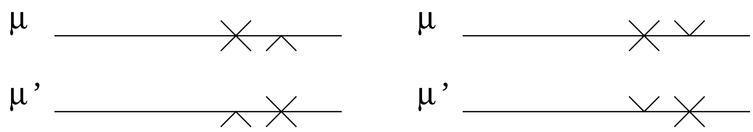

Figure 11: The possible diagrams for x_μ and $x_{\mu'}$ in case (ii)

The rest of the argument proceeds exactly as in case (i).

Case (iii): The vertex at the point marked with a $\wedge$ is of the form x_i for some $i \in \mathbb{N}$ (Brauer) or some $i \in \mathbb{Z}\backslash\{0\}$ (walled Brauer). In the latter case by the definition of $\wedge$ we must have $i > 0$. Now consider $\lambda' = (\lambda^L, \lambda^R - \epsilon_i)$ or $\lambda' = \lambda - \epsilon_i$ (which as before is a (bi)partition), and set $\lambda^+ = \lambda$. Note that there is another element $\lambda^- \in \mathcal{B}(\lambda^+) \cap \mathrm{supp}(\lambda')$; the three diagrams associated to λ^+, λ' and λ^- are illustrated in Figure 9(iii).

Moreover, for each $\mu' \in \mathcal{B}(\lambda')$ there are exactly two elements μ^+ and μ^- in $\mathcal{B}(\lambda) \cap \mathrm{supp}(\mu')$ (which correspond to the same three configurations as for λ^+, λ^-, and λ' at the two points x_i and $x_i - 1$). Also, μ' is the unique element in $\mathcal{B}(\lambda') \cap \mathrm{supp}(\mu^\pm)$. Thus λ' is in the lower closure of λ^+.

For $\mu \in \mathcal{B}(\lambda)$ we have

$$\begin{aligned} D_{\lambda\mu} &= [\Delta_{(a)}(\mu) : L_{(a)}(\lambda)] \\ &= \dim \mathrm{Hom}(P_{(a)}(\lambda^+), \Delta_{(a)}(\mu)) \\ &= \dim \mathrm{Hom}(\mathrm{ind}_{(a-1)}^{\lambda} P_{(a-1)}(\lambda'), \Delta_{(a)}(\mu)) \\ &= \dim \mathrm{Hom}(P_{(a-1)}(\lambda'), \mathrm{res}_{(a)}^{\lambda'} \Delta_{(a)}(\mu)) \end{aligned}$$

where the third equality follows from Proposition 3.4. Now $\mathrm{res}_{(a)}^{\lambda'} \Delta_{(a)}(\mu) \neq 0$ implies that $\mu = \mu^\pm$ with $\mu' \in \mathcal{B}(\lambda') \cap \mathrm{supp}(\mu^\pm)$ and so $D_{\lambda\mu} = 0$ unless $\mu = \mu^\pm \in \mathcal{B}(\lambda) \cap \mathrm{supp}(\mu')$. Note that for any μ not of this form in $\mathcal{B}(\lambda)$ the two vertices labelled x_i and $x_i - 1$ must be either both $\wedge$s or both $\vee$s, which implies that $d_{\lambda\mu} = 0$.

If $\mu = \mu^\pm$ as above then

$$D_{\lambda\mu^\pm} = \dim \mathrm{Hom}(P_{(a-1)}(\lambda'), \Delta_{(a-1)}(\mu')) = D_{\lambda'\mu'} = d_{\lambda'\mu'}(1)$$

by the induction hypothesis. Finally, note that $c_{\lambda'} x_{\mu'}$ is an oriented cap diagram if and only if $c_\lambda x_{\mu^\pm}$ is an oriented cap diagram, and so

$$D_{\lambda\mu^\pm} = d_{\lambda\mu^\pm}(1)$$

as required.

This completes the proof for the walled Brauer algebra. However, for the Brauer algebra the diagram c_λ may contain only curls. We pick the one involving the left-most $\wedge$, and there are five cases, which are illustrated in Figure 12.

Cases (iv-vii): These are very similar to cases (i) and (ii) above. In each case λ' is obtained from λ by swapping the leftmost or right-most end of the curl with the symbol immediately to its left (either $\circ$ or $\times$). Arguing exactly as in cases (i) and (ii) we see that λ and λ' are translation equivalent, and satisfy

$$d_{\lambda\mu}(q) = d_{\lambda'\mu'}(q).$$

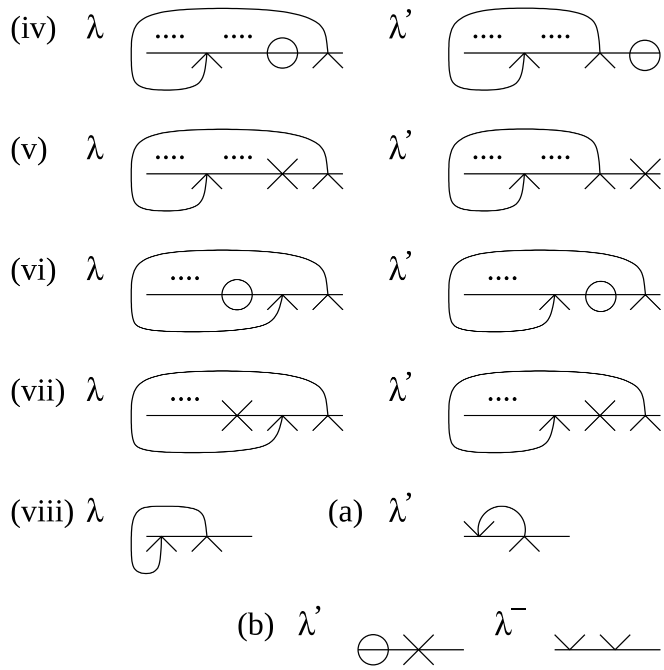

Figure 12: The five possible small curl configurations

Thus the result follows by induction.
Case (viii): We are left with the case where the curl is labelled with (a) $\frac{1}{2}$ and $\frac{3}{2}$, or (b) 0 and 1.

First consider configuration (a). Then the label $\frac{1}{2}$ must occur in the ith entry of x_λ for some i. As $-\frac{1}{2}$ is not in x_λ we have that $\lambda' = \lambda - \epsilon_i$ is a partition. The corresponding diagrams are illustrated in Figure 12(viii)(a). These two elements are translation equivalent, and the result follows as in case (i).

Finally consider configuration (b), and suppose that 0 is in the ith entry of x_λ for some i. As -1 is not in x_λ, we have that $\lambda' = \lambda - \epsilon_i$ is a partition. Setting $\lambda^+ = \lambda$ we see by arguing as in case (iii) that λ' is in the lower closure of λ^+ (with λ^- as illustrated in Figure 12(viii)(b)). The result for this case follows just as in case (iii).

Remark 6.1. We have shown that

$$D_{\lambda\mu} = d_{\lambda\mu}(1)$$

for both the Brauer and walled Brauer algebras. In the Brauer case Martin [Mar] has introduced a similar diagram calculus, but omitting the labels marked with $\times$ or $\circ$ and using caps instead of curls. This allowed him to define versions of the $d_{\lambda\mu}(q)$ and determine the decomposition numbers.

However, the $d_{\lambda\mu}(q)$ encode more than just their values at $q = 1$, and we would like to have a representation-theoretic interpretation of these as polynomials in q. Instead we shall define some closely related polynomials $p_{\lambda\mu}(q)$ and show how these can be related to projective resolutions for our algebras. The definition of this second family of polynomials crucially depends on the distinction between caps and curls in our construction of curl diagrams.

Before defining our second family of polynomials, we consider the relation of the $d_{\lambda\mu}(q)$ to certain Kazhdan-Lusztig polynomials.

7. A recursive formula for decomposition numbers

We will show how the polynomials $d_{\lambda\mu}(q)$ can be calculated recursively. The Brauer and walled Brauer cases will be considered simultaneously. We will then relate this to the conjectured recursive formula for the Brauer algebra given in [CDM11] (and proved in [Mar]).

Proposition 7.1. *(i) Let $\lambda' \in \mathrm{supp}(\lambda)$ be as in one of the cases in Figure 9 or 12, with λ and λ' translation equivalent. Then*

$$d_{\lambda'\mu'}(q) = d_{\lambda\mu}(q).$$

(ii) Suppose that λ contains a small cap as in Figure 9(iii), or a small curl as in Figure 12(viii) with 0 *in x_λ. Denote λ by λ^+ and let λ' and λ^- be as indicated in the corresponding Figure. Then*

$$d_{\lambda^+\mu^+}(q) = d_{\lambda'\mu'}(q)$$

and

$$d_{\lambda^+\mu^-}(q) = q d_{\lambda'\mu'}(q).$$

Also we have

$$d_{\lambda^+\mu^+}(q) = q^{-1} d_{\lambda^-\mu^+}(q) + d_{\lambda^-\mu^-}(q) \tag{5}$$

and

$$d_{\lambda^+\mu^-}(q) = q d_{\lambda^-\mu^-}(q) + d_{\lambda^-\mu^+}(q). \tag{6}$$

Proof. Everything is obvious by construction except for (5) and (6). There are seven cases, which are illustrated in the cap case in Figure 13 and in the curl case in Figure 14.

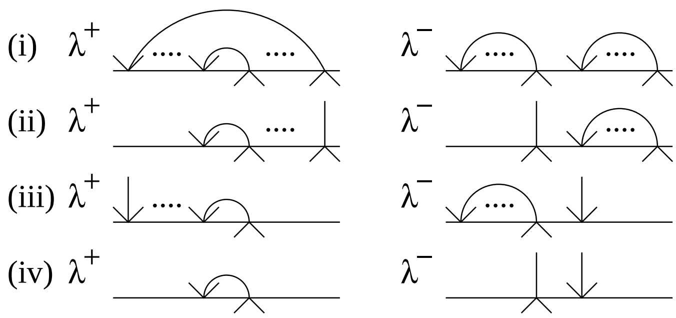

Figure 13: Four small cap configurations

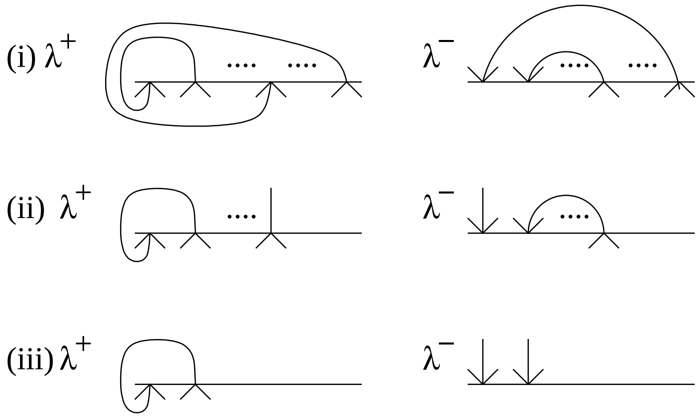

Figure 14: Three small curl configurations

All of the cases are very similar, so we will consider just the case in Figure 13(i). The weights λ^+ and λ^- are illustrated in Figure 15 together with the two possible configurations (a) and (b) for μ^+ and μ^- in the same block as λ^+ and λ^- at the four marked vertices. (The elements μ and μ' must agree at all of the vertices not indicated in the diagram.)

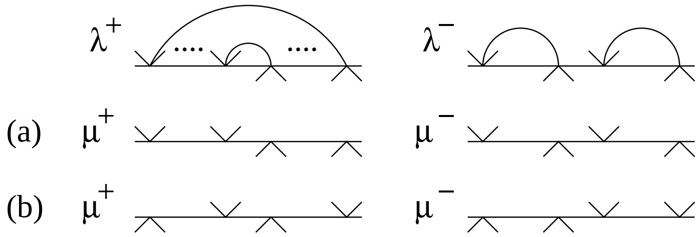

Figure 15: The first cap case

If μ^+ is as in configuration (a) then we have

$$\begin{aligned} d_{\lambda^+\mu^-}(q) &= q d_{\lambda^+\mu^+}(q) \\ d_{\lambda^-\mu^-}(q) &= d_{\lambda^+\mu^+}(q) \\ d_{\lambda^-\mu^+}(q) &= 0 \end{aligned}$$

which implies (5) and (6) as required.

If μ^+ is as in configuration (b) then we have

$$\begin{aligned} d_{\lambda^+\mu^-}(q) &= q d_{\lambda^+\mu^+}(q) \\ d_{\lambda^-\mu^-}(q) &= 0 \\ d_{\lambda^-\mu^+}(q) &= q d_{\lambda^+\mu^+}(q) \end{aligned}$$

which implies (5) and (6) as required. Similar arguments hold in the remaining cases.

In [CDM11] we conjectured that certain parabolic Kazhdan-Lusztig polynomials of type (D, A) gave the decomposition numbers for the Brauer algebra. This conjecture was proved by Martin in [Mar]. Exactly the same conjecture can be made for the walled Brauer case, involving parabolic Kazhdan-Lusztig polynomials of type $(A, A \times A)$.

Corollary 7.2. *The $d_{\lambda\mu}(q)$ for the Brauer and walled Brauer algebras are parabolic Kazhdan-Lusztig polynomials.*

Proof. It follows from (5) and (6) that the recursive formula corresponding to translating a parabolic Kazhdan-Lusztig polynomial holds for the $d_{\lambda\mu}$ as in [Soe97, Section 3]. By definition, the $d_{\lambda\mu}$ are monomials in q with strictly positive degree if $\lambda \neq \mu$ and $d_{\lambda\mu}(q) \neq 0$. This implies that there is no subtraction of lower order terms in the calculation of parabolic Kazhdan-Lusztig polynomials, and hence the $d_{\lambda\mu}(q)$ are indeed parabolic Kazhdan-Lusztig polynomials.

Remark 7.3. There are a number of related (parabolic) Kazhdan-Lusztig polynomials (see [Soe97, Section 3] for the relationship between them). The $d_{\lambda\mu}(q)$ correspond to those labelled by n in [Soe97]. In [LS81] and [Boe88] closed forms are given for certain other Kazhdan-Lusztig polynomials (labelled by m in [Soe97]) arising from types (D, A) and $(A, A \times A)$ (among others). In Section 8 we will recover these from our diagrams by defining new polynomials $p_{\lambda\mu}(q)$. The relation between the $p_{\lambda\mu}(q)$ and the $d_{\lambda\mu}(q)$ will be given in Corollary 9.2.

8. Valued cap and curl diagrams

In this section we will return to the combinatorics of cap and curl diagrams, and define a new family of polynomials associated to pairs of (bi)partitions λ and μ. These are given by a diagrammatic version of the combinatorial formulas for Kazhdan-Lusztig polynomials given in [LS81] and [Boe88]; a discussion of the relation between the two approaches can be found in Appendix A (and in [BS10, Section 5] in the cap case).

Fix $\lambda \in \Lambda_{r,s}$ or Λ_n and $\mu \in \mathcal{B} = \mathcal{B}(\lambda)$. We set $I(\mathcal{B})$ to be the infinite set of non-zero integers indexing the vertices of x_λ labelled by $\vee$ or $\wedge$, but *excluding* the leftmost one. Set $I(\lambda, \mu)$ to be the finite subset of $I(\mathcal{B})$ indexing vertices that are labelled differently in x_λ and in x_μ. For $i \in I(\mathcal{B})$ define

$$\begin{aligned} l_i(\lambda, \mu) \quad = \quad & \#\{j \in I(\lambda, \mu) : j \geq i \text{ and vertex } j \text{ of } x_\lambda \text{ is labelled by } \wedge\} \\ & -\#\{j \in I(\lambda, \mu) : j \geq i \text{ and vertex } j \text{ of } x_\mu \text{ is labelled by } \wedge\}. \end{aligned}$$

Note that $\lambda \geq \mu$ if and only if $l_i(\lambda, \mu) \geq 0$ for all $i \in I(\mathcal{B})$. We set

$$l(\lambda, \mu) = \sum_{i \in I(\mathcal{B})} l_i(\lambda, \mu).$$

Any cap or curl diagram cuts the upper half plane into various open connected regions, which we will call *chambers*. Recall that we say that a cap or curl in c is small if it does not contain any cap or curl inside it. Given a pair of chambers separated by a cap or curl, we say that they are *adjacent* and refer to the one lying below as the *inside* chamber, and the other as the *outside* chamber. The vertices labelled with $\vee$ or $\wedge$ will be called the *non-trivial* vertices.

In the curl diagram case we may have a chamber A (possibly unbounded) inside which there are a series of maximal chambers (i.e. chambers adjacent to A) $A_1, \ldots, A_t$ from left to right not separated by the end of a curl. If A_1 is formed either by a curl or by a cap involving the leftmost non-trivial vertex then we say that $A_1, \ldots, A_t$ forms a *chain*.

A *valued cap diagram* c is a cap diagram whose chambers have been assigned values from the integers such that

(1) all external (unbounded) chambers have value 0;
(2) given two adjacent chambers, the value of the inside chamber is at least as large as the value of the outer chamber.

A *valued curl diagram* c is a curl diagram whose chambers have been assigned values from the integers such that (1) and (2) above hold and also

(3) the value of the chamber defined by a cap or curl connected to or containing inside itself the leftmost non-trivial vertex must be even;
(4) if $A_1, \ldots A_t$ is a chain and the value of A_i is less than or equal to that of A_j for all $1 \leq j < i$ then the value of A_i must be even.

Given a valued cap/curl diagram c, we write $|c|$ for the sum of the values of c.

We are now able to define a new polynomial $p_{\lambda\mu}(q)$ associated to our pair λ and μ in $\mathcal{B}$. If $x_\lambda \not\supseteq x_\mu$ then set $p_{\lambda\mu}(q) = 0$. Otherwise, let $D(\lambda, \mu)$ be the set of all valued cap/curl diagrams obtained by assigning values to the chambers of c_μ in such a way that the value of every small cap or curl is at most $l_i(\lambda, \mu)$, where i indexes the right-most vertex of the cap or curl. Now set

$$p_{\lambda\mu}(q) = q^{l(\lambda,\mu)} \sum_{c \in D(\lambda,\mu)} q^{-2|c|}$$

and write $p_{\lambda\mu}^{(m)}$ for the coefficient of q^m in $p_{\lambda\mu}(q)$. That $p_{\lambda\mu}(q)$ is indeed a polynomial will follow from Proposition 8.2, and hence

$$p_{\lambda\mu}(q) = \sum_{m \geq 0} p_{\lambda\mu}^{(m)} q^m.$$

Example 8.1. In Figure 16 we have illustrated a pair of diagrams x_λ and x_μ together with the curl diagram c_μ and the value of $l_i(\lambda, \mu)$ for each vertex i in our diagram. Thus in this case

$$l(\lambda, \mu) = 2 + 3 + 2 + 2 + 1 + 1 = 11.$$

The various allowable values for the chambers in the curl diagram are indicated in the Figure, where only the chambers marked a and b can be non-zero. We must have $a \in \{0, 2\}$ and $b \in \{0, 1, 2\}$.

Now the valued cap diagram is in $D(\lambda, \mu)$ if and only if

$$(a, b) \in \{(0, 0), (0, 1), (0, 2), (2, 0), (2, 2)\}.$$

For example, note that we cannot have $(a, b) = (2, 1)$ as this configuration would not satisfy condition (4). Thus we see that

$$p_{\lambda\mu}(q) = q^{11}(1 + q^{-2} + 2q^{-4} + q^{-8}) = q^{11} + q^9 + 2q^7 + q^3.$$

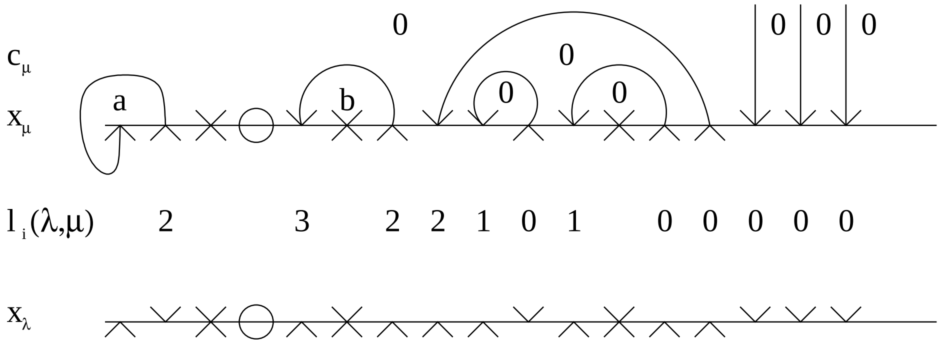

Figure 16: An example of the calculation of $p_{\lambda\mu}(q)$.

Pick a small cap or curl in λ. The possible configurations of caps in λ are given in Figure 9(i-iii) and of curls in Figure 12(iv-viii). Associated weights λ' are shown in each case, with two subcases appearing in Figure 12(viii), together with weights λ^- in Figure 9(iii) and Figure 12(viii)(b). We will show how the values of $p_{\lambda\mu}(q)$ can be calculated from the polynomials $p_{\lambda'\mu'}$ and $p_{\lambda^-\tau}$ for suitable choices of τ, which will give a recursive formula for the $p_{\lambda\mu}$.

Consider the configurations shown in Figure 9(iii) and Figure 12(viii)(b). In both of these cases we will denote λ by λ^+, and then the weights λ^+ and λ^- are separated by λ' and λ' is in the lower closure of λ^+. We will say that an element is *of the form* μ^+ if it is in the same block as λ^+ and has the same configuration of $\wedge$s and $\vee$s as λ^+ at the vertices on the small cap or curl under consideration.

Proposition 8.2. *(i) Let λ and λ' be one of the configurations in Figure 9(i-ii) or Figure 12(iv-vii), or as in Figure 12(viii)(a) where the vertices on the small curl are labelled $\frac{1}{2}$ and $\frac{3}{2}$. Then*

$$p_{\lambda\mu}(q) = p_{\lambda'\mu'}(q)$$

for all $\mu \in \mathcal{B}(\lambda)$.
(ii) Let λ and λ' be configured as in Figure 9(iii) or as in Figure 12(viii)(b) where the vertices on the small curl are labelled 0 *and* 1*. Then setting $\lambda^+ = \lambda$ we have*

$$p_{\lambda^+\mu^+}(q) = p_{\lambda'\mu'}(q) + qp_{\lambda^-\mu^+}(q) \tag{7}$$

and

$$p_{\lambda^+\mu}(q) = qp_{\lambda^-\mu}(q) \tag{8}$$

for all μ not of the form μ^+.

Proof. (Compare with [Boe88, (3.14) Proposition].) In the cases in Figure 9(i-ii) and Figure 12(iv-vii) the weights λ and λ' are translation equivalent. By construction we have in all of these cases that

$$p_{\lambda\mu}(q) = p_{\lambda'\mu'}(q)$$

for all $\mu \in \mathcal{B}(\lambda)$. The case in Figure 12(viii)(a) occurs when the vertices on the small curl are labelled $\frac{1}{2}$ and $\frac{3}{2}$, and again the weights λ and λ' are translation equivalent. The translation equivalence is given by changing the $\pm\frac{1}{2}$ entry in x_μ to $\mp\frac{1}{2}$ in $x_{\mu'}$. Therefore $l_i(\lambda,\mu) = l_i(\lambda',\mu')$ for all $i \in I(\mathcal{B})$ and all other caps and curls are preserved. Thus in this case we also have that

$$p_{\lambda\mu}(q) = p_{\lambda'\mu'}(q)$$

for all $\mu \in \mathcal{B}(\lambda)$.

The two remaining cases are those shown in Figure 9(iii) and Figure 12(viii)(b). In both of these cases the weights λ^+ and λ^- are separated by λ' and λ' is in the lower closure of λ^+. We first consider (7). We claim there is a one-to-one correspondence between $D(\lambda^+,\mu^+)$ and $D(\lambda',\mu') \sqcup D(\lambda^-,\mu^+)$. Let i be the rightmost vertex of the small cap or curl under consideration in x_λ. It is easy to see that

$$l_i(\lambda^+,\mu^+) = l_i(\lambda^-,\mu^+) + 1$$

and that if $i-1$ is the left-most non-trivial vertex then $l_i(\lambda^+,\mu^+)$ is even.

The valued cap/curl diagrams in $D(\lambda^+,\mu^+)$ split into two subsets, those where the value of the small cap/curl under consideration is less than $l_i(\lambda^+,\mu^+)$ and those where the value is equal to $l_i(\lambda^+,\mu^+)$. The first set are exactly the valued cup/curl diagrams in $D(\lambda^-,\mu^+)$.

We will show that the second set is obtained from the set of valued cap/curl diagrams $D(\lambda',\mu')$ by adding to each element a cap/curl joining vertices $i-1$ and i with value $l_i(\lambda^+,\mu^+)$. For $c \in D(\lambda',\mu')$ denote by c^+ the corresponding valued cap/curl diagram with this extra cap/curl. We need to show that c^+ is indeed in $D(\lambda^+,\mu^+)$ to give the desired bijection.

We check that inserting this extra cap/curl with the given value satisfies the condition (1–4) in the definition of a valued cap/curl diagram. (1) is obvious.

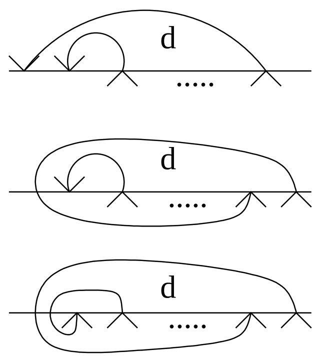

Figure 17: The possible nested cases

For (2), suppose that our small cap/curl is nested inside a larger one d. We may assume that they are adjacent. There are three possible cases, illustrated in Figure 17. Suppose that there is a small cap in the dotted region in Figure 17; if we pick the leftmost such cap and j denotes its right-hand vertex then it is easy to see that

$$l_j(\lambda^+,\mu^+) \leq l_i(\lambda^+,\mu^+).$$

So the value of this small cap is at most $l_i(\lambda^+, \mu^+)$ and hence the value of d is at most $l_i(\lambda^+, \mu^+)$.

If the dotted region in Figure 17 is empty then let j be the vertex at the right-hand end of the cap/curl defining d. If this is a cap then we have

$$l_j(\lambda^+, \mu^+) \leq l_i(\lambda^+, \mu^+)$$

and so the value of d is at most $l_i(\lambda^+ . \mu^+)$. If we have a small cap or curl nested in a curl then

$$l_j(\lambda^+, \mu^+) \leq l_i(\lambda^+, \mu^+) + 1.$$

But d has to be even and $l_i(\lambda^+, \mu^+)$ is even, and so the value of d is at most $l_i(\lambda^+, \mu^+)$.

For (3), as noted above if $i - 1$ is the leftmost non-trivial vertex then $l_i(\lambda^+, \mu^+)$ is even.

Finally for (4), suppose we have a chain of chambers. If our small cap/curl is the leftmost in the chain then denote the vertices of the next chamber along in the chain as shown in Figure 18. By the same argument as in (2) we see that d has value at most $l_i(\lambda^+, \mu^+)$, and as k was the leftmost non-trivial vertex we have that d is even.

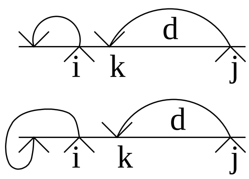

Figure 18: The leftmost chain cases

If there is a chamber to each side of our small cap in the chain then we are in the configuration shown in Figure 19. As before the value of e is at most $l_i(\lambda^+, \mu^+)$. If e has value at most that of d and all other predecessors then removing the small cap at i we have a chain in $D(\lambda', \mu')$ and so d is even as required.

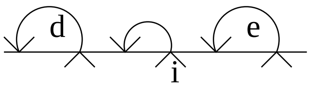

Figure 19: The mid-chain cases

If our small cap is the rightmost in the chain then a similar argument shows that the preceding chamber d in the chain has value at most $l_j(\lambda^+, \mu^+) \leq l_i(\lambda^+, \mu^+)$. If $l_i(\lambda^+, \mu^+)$ is no greater than all preceding values in the chain then $l_i(\lambda^+, \mu^+)$ is at most the value of d, and hence by the preceding inequality the value of d equals $l_i(\lambda^+, \mu^+)$. Removing our small cap gives a chain in $D(\lambda', \mu')$ and hence $l_i(\lambda^+, \mu^+)$ must be even. Thus conditions (1-4) are satisfied and hence $c^+ \in D(\lambda^+, \mu^+)$ as required.

It is also clear that

$$l(\lambda^+, \mu^+) = l(\lambda^-, \mu^+) + 1$$

and

$$l(\lambda^+, \mu^+) = l(\lambda', \mu') + 2l_i(\lambda^+, \mu^+).$$

Hence

$$\begin{aligned}
p_{\lambda^+\mu^+}(q) &= q^{l(\lambda^+,\mu^+)} \sum_{c \in D(\lambda^+,\mu^+)} q^{-2|c|} \\
&= q^{l(\lambda^+,\mu^+)} \sum_{c \in D(\lambda^-,\mu^+)} q^{-2|c|} + q^{l(\lambda^+,\mu^+)} \sum_{c \in D(\lambda',\mu')} q^{-2|c^+|} \\
&= q.q^{l(\lambda^-,\mu^+)} \sum_{c \in D(\lambda^-,\mu^+)} q^{-2|c|} + q^{l(\lambda',\mu')+2l_i(\lambda^+,\mu^+)} \sum_{c \in D(\lambda',\mu')} q^{-2|c|-2l_i(\lambda^+,\mu^+)} \\
&= qp_{\lambda^-\mu^+}(q) + p_{\lambda'\mu'}(q).
\end{aligned}$$

It remains to show that (8) holds. If μ is not of the form μ^+ then it must have a different configuration of $\vee$s and $\wedge$s on the pair of vertices defined by our small cap or curl. Thus the possible configurations are as indicated in Figure 20, where the top row (a-c) corresponds to the small cap case in Figure 9(iii) and the bottom row (d-f) corresponds to the small curl case in Figure 12(viii)(b).

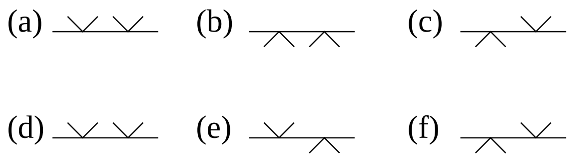

Figure 20: The possible configurations of μ not of the form μ^+

In all six cases we have

$$l(\lambda^+, \mu) = l(\lambda^-, \mu) + 1.$$

Let i be the rightmost of the vertices on the small cap/curl in λ. Note that for all $j \neq i$ we have that

$$l_j(\lambda^+, \mu) = l_j(\lambda^-, \mu) \quad \text{and} \quad l_i(\lambda^+, \mu) = l_i(\lambda^-, \mu) + 1.$$

Now for μ as in Figure 20(a), (c), (d), or (f) there is no cap/curl in c_μ with rightmost vertex i, and so in these cases we have that

$$D(\lambda^+, \mu) = D(\lambda^-, \mu).$$

For μ as in Figure 20(b) or (e) there might be a cap/curl with rightmost vertex i.

If i is the second non-trivial vertex in μ (or λ^+, λ^-), then $l_i(\lambda^-, \mu)$ is even and so $l_i(\lambda^+, \mu)$ is odd. Also the cap/curl in μ involved the first non-trivial vertex in μ and so its value must be even. Hence we again have that

$$D(\lambda^+, \mu) = D(\lambda^-, \mu).$$

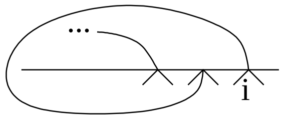

Figure 21: The final configuration of μ

If i is not the second non-trivial vertex then we must have a configuration of the form in Figure 21. Note that

$$l_{i-2}(\lambda^+,\mu) \leq l_i(\lambda^+,\mu) - 1 = l_i(\lambda^-,\mu)$$

and as the values are non-increasing in nested chambers we again have that

$$D(\lambda^+,\mu) = D(\lambda^-,\mu).$$

Thus in all cases we have

$$\begin{aligned} p_{\lambda^+\mu}(q) &= q^{l(\lambda^+,\mu)} \sum_{c\in D(\lambda^+,\mu)} q^{-2|c|} \\ &= qq^{l(\lambda^-,\mu)} \sum_{c\in D(\lambda^-,\mu)} q^{-2|c|} \\ &= qp_{\lambda^-\mu}(q). \end{aligned}$$

9. Projective resolutions of standard modules

We now have the combinatorial framework needed to describe projective resolutions of standard modules for the walled Brauer algebra. This is inspired by the corresponding result for the quasi-hereditary cover of the generalised Khovanov diagram algebra in [BS10, Theorem 5.3] (which itself repeats an argument from [Bru03, Lemma 4.49]).

Theorem 9.1. *For each $\lambda \in \Lambda_{r,s}$ there is an exact sequence*

$$\cdots \longrightarrow P^m_{(a)}(\lambda) \longrightarrow \cdots \longrightarrow P^1_{(a)}(\lambda) \longrightarrow P^0_{(a)}(\lambda) \longrightarrow \Delta_{(a)}(\lambda) \longrightarrow 0$$

where

$$P^i_{(a)}(\lambda) = \bigoplus_{\mu\in\Lambda_{(a)}} p^{(i)}_{\lambda\mu} P_{(a)}(\mu).$$

Proof. Let $\lambda \in \Lambda_{(a)}$. If x_λ is minimal then

$$\Delta_{(a)}(\lambda) = P_{(a)}(\lambda) = P^0_{(a)}(\lambda)$$

and $P^m_{(a)}(\lambda) = 0$ for all $m \geq 0$ and for all (a) with $\lambda \in \Lambda_{(a)}$. Thus we may assume that x_λ is not minimal.

As in Section 6 we choose a cap or a curl in x_λ not containing any smaller caps or curls. We have eight cases to consider as shown in Figures 9 and 12. We proceed by induction on $\deg(\lambda)$. Note that in all cases we have $\deg(\lambda') < \deg(\lambda)$ and in cases (iii) and (viii)(b) we also have $\deg(\lambda^-) < \deg(\lambda)$. So we can assume that the result holds for λ' and λ^-.

In cases (i), (ii), (iv-vii) and (viii)(a) we have by induction a projective resolution of $\Delta_{(a+1)}(\lambda')$ of the form

$$\cdots \longrightarrow P^m_{(a+1)}(\lambda') \longrightarrow \cdots \longrightarrow P^1_{(a+1)}(\lambda') \longrightarrow P^0_{(a+1)}(\lambda') \longrightarrow \Delta_{(a+1)}(\lambda') \longrightarrow 0.$$

In these cases we saw that λ and λ' are translation equivalent. Applying the exact functor $\mathrm{res}^{\lambda}_{(a+1)}$ to this resolution and using Theorem 3.1(iii) and Proposition 8.2(i) and (ii) we get a projective resolution

$$\cdots \longrightarrow P^m_{(a)}(\lambda) \longrightarrow \cdots \longrightarrow P^1_{(a)}(\lambda) \longrightarrow P^0_{(a)}(\lambda) \longrightarrow \Delta_{(a)}(\lambda) \longrightarrow 0$$

as required.

For the cases (iii) and (viii)(b) we set $\lambda^+ = \lambda$. By induction we have projective resolutions of $\Delta_{(a+1)}(\lambda')$ and $\Delta_{(a)}(\lambda^-)$ of the form

$$\cdots \longrightarrow P^m_{(a+1)}(\lambda') \longrightarrow \cdots \longrightarrow P^1_{(a+1)}(\lambda') \longrightarrow P^0_{(a+1)}(\lambda') \longrightarrow \Delta_{(a+1)}(\lambda') \longrightarrow 0 \tag{9}$$

and

$$\cdots \longrightarrow P^m_{(a)}(\lambda^-) \longrightarrow \cdots \longrightarrow P^1_{(a)}(\lambda^-) \longrightarrow P^0_{(a)}(\lambda^-) \longrightarrow \Delta_{(a)}(\lambda^-) \longrightarrow 0. \tag{10}$$

We also have an exact sequence

$$0 \longrightarrow \Delta_{(a)}(\lambda^-) \xrightarrow{f} \mathrm{res}^{\lambda}_{(a+1)} \Delta_{(a+1)}(\lambda') \longrightarrow \Delta_{(a)}(\lambda^+) \longrightarrow 0.$$

Applying $\mathrm{res}^{\lambda}_{(a+1)}$ to (9) and extending f to a chain map using (10) we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\longrightarrow & P^m_{(a)}(\lambda^-) & \longrightarrow \cdots \longrightarrow & P^0_{(a)}(\lambda^-) & \longrightarrow & \Delta_{(a)}(\lambda^-) & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow f & \\
\longrightarrow & \mathrm{res}^{\lambda}_{(a+1)} P^m_{(a+1)}(\lambda') & \longrightarrow \cdots \longrightarrow & \mathrm{res}^{\lambda}_{(a+1)} P^0_{(a+1)}(\lambda') & \longrightarrow & \mathrm{res}^{\lambda}_{(a+1)} \Delta_{(a+1)}(\lambda') & \longrightarrow 0
\end{array}$$

which we extend into a double complex by adding 0s in all remaining rows.

Taking the total complex of this double complex gives an exact sequence

$$\begin{aligned}\cdots \longrightarrow P^m_{(a)}(\lambda^-) \oplus \mathrm{res}^{\lambda}_{(a+1)} P^{m+1}_{(a+1)}(\lambda') \longrightarrow \cdots \\ \cdots \longrightarrow \Delta_{(a)}(\lambda^-) \oplus \mathrm{res}^{\lambda}_{(a+1)} P^0_{(a+1)}(\lambda') \longrightarrow \mathrm{res}^{\lambda}_{(a+1)} \Delta_{(a+1)}(\lambda') \longrightarrow 0.\end{aligned} \tag{11}$$

By Proposition 2.1 there is an obvious injective chain map from

$$\cdots \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \Delta_{(a)}(\lambda^-) \longrightarrow \Delta_{(a)}(\lambda^-) \longrightarrow 0$$

to the complex in (11), and the quotient gives an exact sequence

$$\cdots \longrightarrow P^m_{(a)}(\lambda^-) \oplus \mathrm{res}^{\lambda}_{(a+1)} P^{m+1}_{(a+1)}(\lambda') \longrightarrow \cdots \longrightarrow \mathrm{res}^{\lambda}_{(a+1)} P^0_{(a+1)}(\lambda') \longrightarrow \Delta_{(a)}(\lambda^+) \longrightarrow 0. \tag{12}$$

By Propositions 3.4 and 8.2(iii) we have

$$\operatorname{res}^{\lambda}_{(a+1)} P^{0}_{(a+1)}(\lambda') = \operatorname{res}^{\lambda}_{(a+1)} P_{(a+1)}(\lambda') = P_{(a)}(\lambda^{+}) = P^{0}_{(a)}(\lambda^{+}).$$

For $m > 0$ we have by Proposition 3.4 and Proposition 8.2 that

$$\begin{aligned}
P^{m}_{(a)}(\lambda^{-}) \oplus \operatorname{res}^{\lambda}_{(a+1)} P^{m+1}_{(a+1)}(\lambda') &= \bigoplus_{\mu\in\mathcal{B}(\lambda)} p^{(m)}_{\lambda^{-}\mu} P_{(a)}(\mu) \oplus \bigoplus_{\mu'\in\mathcal{B}(\lambda')} p^{(m+1)}_{\lambda'\mu'} \operatorname{res}^{\lambda}_{(a+1)} P_{(a+1)}(\mu') \\
&= \bigoplus_{\mu\in\mathcal{B}(\lambda)} p^{(m)}_{\lambda^{-}\mu} P_{(a)}(\mu) \oplus \bigoplus_{\mu'\in\mathcal{B}(\lambda')} p^{(m+1)}_{\lambda'\mu'} P_{(a)}(\mu^{+}) \\
&= \bigoplus_{\mu^{+}\in\mathcal{B}(\lambda)} \left(p^{(m)}_{\lambda^{-}\mu^{+}} + p^{(m+1)}_{\lambda'\mu'}\right) P_{(a)}(\mu^{+}) \oplus \bigoplus_{\mu\in\mathcal{B}(\lambda),\mu\neq\mu^{+}} p^{(m)}_{\lambda^{-}\mu} P_{(a)}(\mu) \\
&= \bigoplus_{\mu^{+}\in\mathcal{B}(\lambda)} p^{(m+1)}_{\lambda^{+}\mu^{+}} P_{(a)}(\mu^{+}) \oplus \bigoplus_{\mu\in\mathcal{B}(\lambda),\mu\neq\mu^{+}} p^{(m+1)}_{\lambda^{+}\mu} P_{(a)}(\mu) \\
&= \bigoplus_{\mu\in\mathcal{B}(\lambda)} p^{(m+1)}_{\lambda^{+}\mu} P_{(a)}(\mu) = P^{(m+1)}_{(a)}(\lambda^{+}) = P^{(m+1)}_{(a)}(\lambda).
\end{aligned}$$

Substituting into (12) we obtain the desired projective resolution of $\Delta_{(a)}(\lambda)$.

For fixed (a) we can consider the matrices formed by the $p_{\lambda\mu}(q)$ and the $d_{\lambda\mu}(q)$ with rows and columns indexed respectively by λ and μ in $\Lambda_{(a)}$. The next pair of Corollaries follow from the last Proposition in exactly the same way as in [BS10, Corollaries 5.4 and 5.5].

Corollary 9.2. *The matrix $(p_{\lambda\mu}(-q))$ is the inverse of the matrix $(d_{\lambda\mu}(q))$.*

Corollary 9.3. *We have*

$$p_{\lambda\mu}(q) = \sum_{i\geq 0} q^{i} \dim \operatorname{Ext}^{i}(\Delta(\lambda), L(\mu)).$$

Remark 9.4. We have seen that the walled Brauer algebras have the same combinatoric for decomposition numbers and for projective resolutions of standard modules as the generalised Khovanov diagram algebras studied by Brundan and Stroppel [BSa, BS10, BS11, BSb]. They have shown that a certain infinite dimensional limit of these Khovanov algebras are Morita equivalent to blocks of the general linear supergroup, and that their quasihereditary covers in the finite dimensional setting are Morita equivalent to certain parabolic category $\mathcal{O}$s. It would be very interesting (if true) to determine an analogous relationship between these algebras and the walled Brauer algebra, and to find analogous correspondences for the Brauer algebra.

Appendix A. Kazhdan-Lusztig polynomials

In this section we shall review the constructions of Kazhdan-Lusztig polynomials corresponding to $A_r \times A_s$ inside A_{r+s+1} and A_{n-1} inside D_n given respectively by Lascoux and Schützenberger [LS81, Section 6] and by Boe [Boe88, Section 4 (and Section 3)], and how these can be identified (up to a power of q) with the polynomials associated to valued cap diagrams and valued curl diagrams. In the former case this was already observed in [BS10, Remark 5.1]. An alternative graphical description in the curl case has been given in [Lej10, Section 5]. We will concentrate on the case A_{n-1} inside D_n, as this includes the combinatorics for the other case (as will be noted at the end).

We begin by outlining the construction of Boe [Boe88]. Fixing W of type D_n and a fixed subCoxeter system of type A_{n-1} defines a dominant set of elements in W. These can be identified with words of the form

$$w = w_n \dots w_1$$

where each $w_i \in \{\alpha, \beta\}$, such that the number of αs is even. Because of this parity condition the final element w_1 is redundant and is omitted.

Given a partition λ we will identify the weight x_λ with a word w of the above form in the following manner. Fix $m >> 0$ so that m is the rightmost vertex in x_λ lying on a cap or curl in c_λ, and let n be the number of vertices labelled $\vee$ or $\wedge$ between 0 and m inclusive, and we associate λ to the word w obtained by setting $w_i = \alpha$ (respectively β) if the $(n-i)$th such vertex from the left is $\vee$ (respectively $\wedge$). We will refer to these vertices as the *non-trivial* vertices in x_λ.

Note that the identification letters in w read from left to right correspond to vertices in x_λ read from right to left.

Lascoux-Schützenberger introduced the cyclic monoid Z in the letters α and β [LS81, Section 4]. Rather than repeating their definition, we note that if $w = w'zw''$ with $z \in Z$ then z corresponds to a line segment in x_λ where the non-trivial vertices form a sequence of (possibly nested) caps. If $w = w'\alpha z\beta w''$ then Boe calls α and β a *linked $\alpha\beta$ pair*; this corresponds to a cap in our terminology. If

$$w = w'\alpha z_{2r}\alpha z_{2r-1}\alpha \dots \alpha z_1 \alpha z_0$$

with $z_i \in Z$ then Boe calls the rightmost α *terminal* and each pair of αs separated by some z_{2i} a *linked $\alpha\alpha$ pair.* Under our correspondence linked $\alpha\alpha$ pairs correspond to curls. As Boe omits w_1 but x_λ retains the corresponding point, a terminal α corresponds to either a cap or a curl involving the leftmost non-trivial vertex.

Boe next defines a rooted directed tree associated to the word w. It is routine to verify that this corresponds to the tree with vertices labelled by the chambers for x_λ, where an edge connects chamber A to chamber B if chamber A is adjacent to and surrounds chamber B, and the unbounded chambers (separated by infinite rays) are regarded as a single unbounded chamber via the point at infinity.

Thus the root of the tree corresponds to the unique unbounded chamber, while the terminal nodes correspond to the small chambers. Certain edges in the tree are marked with a plus sign; these correspond to edges which cross either a curl or a cap involving the left-most non-trivial vertex.

Certain pairs of edges in the tree are related by a dotted arrow. We will describe the diagram version; the equivalence of the two is a straightforward exercise. Suppose we have a chamber A (possibly unbounded) inside which there are a series of maximal chambers $A_1, \dots, A_t$ from left to right (possibly containing other chambers inside them) not separated by the end of a curl. If the leftmost chamber A_1 is formed either by a curl or by a cap involving the leftmost non-trivial vertex, then there is a dotted arrow from the edge defined by A_i in A to the edge defined by A_{i+1} in A for $1 \leq i \leq t-1$.

In fact the dotted arrows are redundant in the diagram case: the leftmost chamber in a curl must always be formed either by a curl or by a cap involving the left-most non-trivial vertex, and the same is true in any unbounded chamber with no ray to its left. Chambers formed by caps or with a ray to their left cannot contain curls or the left-most non-trivial vertex. Thus we can omit the dotted arrows in our diagrams without any ambiguity.

Instead of labelling edges with plus signs, we will label chambers by moving any labels to the vertices at the bottom of their respective edges.

Example A 1. An example of the correspondence between curl diagrams and labelled graphs is given in Figure A.22. Here we have included the dotted arrows to emphasise where they occur. Note that the graph must be reflected in the vertical axis under the correspondence with the construction for Boe in terms of words in α and β.

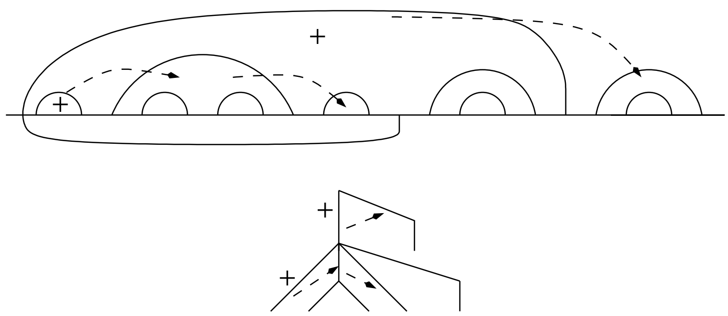

Figure A.22: The diagram graph correspondence

Remark A 2. Our construction appears to depend on the choice of m defined by the rightmost vertex on a cap or curl. However, Boe's construction (in our diagrammatic form) is not affected by the addition of arbitrarily many rays to the right. Thus we can carry out all calculations involving our diagrams in the unbounded setting.

Boe next associates to pairs of words (w, y) a labelling of the tree for w. Under our identifications this corresponds to a valued curl diagram. The polynomial $Q_{y,w}(q)$ defined by Boe by summing over possible labellings corresponds almost exactly to our $p_{\lambda\mu}(q)$. More precisely, if we denote by $w(\lambda)$ and $w(\mu)$ the words in α and β corresponding to λ and μ (as described at the beginning of this section), then we have that

$$p_{\lambda\mu}(q) = q^{l(\lambda,\mu)} Q_{w(\lambda),w(\mu)}(q^{-2}).$$

We have considered the relation between Boe's rooted tree construction and curl diagrams. There is an entirely analogous relation between the rooted tree construction of Lascoux-Schützenberger and cap diagrams. In that case there are no linked $\alpha\alpha$ pairs or terminal αs marked with a plus sign, and thus no chambers contain chains. The remainder of the construction goes through unchanged.

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