ALCOVE GEOMETRY AND A TRANSLATION PRINCIPLE FOR THE BRAUER ALGEBRA

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Abstract. There are similarities between algebraic Lie theory and a geometric description of the blocks of the Brauer algebra. Motivated by this, we study the alcove geometry of a certain reflection group action. We provide analogues of translation functors for a tower of recollement, and use these to construct Morita equivalences between blocks containing weights in the same facet. Moreover, we show that the determination of decomposition numbers for the Brauer algebra can be reduced to a study of the block containing the weight 0. We define parabolic Kazhdan-Lusztig polynomials for the Brauer algebra and show in certain low rank examples that they determine standard module decomposition numbers and filtrations.

1. Introduction and Motivation

The Brauer algebra $B_n(\delta)$ was introduced in 1937 [Bra37], to be in Schur-Weyl duality with the symplectic or orthogonal groups over $\mathbb{C}$ (for suitable integer values of δ). However it may be defined over an arbitrary ring K, for any $n \in \mathbb{N}$ and $\delta \in K$. It has integral representations (in the sense of [Ben91]) that pass to simple modules over suitable splitting fields, constructed by Brown [Bro55]. This raises the problem of determining simple decomposition matrices for these key modules (which we will for now refer to as cell modules), and hence for indecomposable projective modules, over other extensions to a field k. This long-standing problem remains open. In this paper we will first develop some tools to solve this problem, by constructing a formal 'weight space' with a geometry and associated functors on the module categories, and second propose a combinatorial framework (over $\mathbb{C}$) in which the answer might be couched.

When the algebra is semisimple, the decomposition matrices are trivial. Over $\mathbb{C}$ this is true generically [Bro55], and a series of papers by Hanlon and Wales [HW89b, HW89a, HW90, HW94] culminated in the conjecture that this was true for all non-integer values of δ. This was proved by Wenzl [Wen88]. We will consider the non-semisimple cases.

The problem can be addressed in two parts: first working over $\mathbb{C}$, and then over fields of prime characteristic. (The latter can be anticipated to be significantly harder, as the representation theory of $B_n(\delta)$ *contains* the representation theory of the symmetric group Σ_n.) A significant step towards an answer came with the determination of the blocks of the algebra over $\mathbb{C}$ (and later of a geometric linkage principle in any characteristic different from 2) [CDM09a, CDM09b]. These results built on earlier work of Doran, Hanlon, and Wales [DWH99], and were obtained by using functors that allow the algebras for all n to be treated together (as previously used in [MW98, CGM03], and in part following the pioneering work

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of Green [Gre80]). An alternative approach to the characteristic 0 result via characteristic p has recently been developed by Donkin and Tange [DT].

The key observation that underpins the various geometric considerations in this paper is that the cell modules of $B_n(\delta)$ may be indexed by certain orbits of lattice points in the Euclidean space $\mathbb{E}^{\mathbb{N}}$. The orbits are those of a reflection group A, where A is the limit of the usual type-A reflection group action on Euclidean N-space $\mathbb{E}^N$. The A-action is a parabolic in the limit type-D reflection group action on $\mathbb{E}^{\mathbb{N}}$ (as in finite rank), and the orbits of the D-action on coset space $\mathbb{E}^{\mathbb{N}}/A$ describe the blocks of $B_n(\delta)$-mod over $\mathbb{C}$. For this reason we will work over $\mathbb{C}$ in this paper. Using this parabolic/reflection group formulation we are able to:

(1) determine a translation principle (Morita equivalences between certain blocks);
(2) compute appropriate Brauer analogues of the parabolic Kazhdan-Lusztig polynomials that determine decomposition matrices in Lie (quantum group) theory;
(3) use these to encode the structure of the algebra in many special cases (for example in low rank, with the obvious conjecture that this extends to all cases).

Our methodology, and the structure of the paper, can be summarised as follows. The well-established root system/Weyl group analysis of high-weight theory reduces many questions in Lie representation theory (of algebraic and quantum groups) to geometry and combinatorics [Deo87, Jan03, Soe97b], once the Weyl group and affine Weyl group action on weight space has been determined. Of course these Weyl groups are reflection groups, one a parabolic in the other [Hum90], facilitating, for example, an alcove geometric description of blocks.

Note the obvious analogy with the role of reflection groups described above. It was this which motivated our formulation of the results in [CDM09b] (guided by success with a similar approach to other 'diagram' algebras [MW03]). In Lie theory the Euclidean space is finite and the reflection group is infinite by virtue of being affine; here it is by virtue of unbounded rank. Nonetheless, all the geometric and combinatorial machinery goes through unchanged. The development of this analogy in Section 3 lies at the heart of our methodology.

Arguably one of the most beautiful machines that exists for computing decomposition matrices in any setting is the method of (parabolic) Kazhdan-Lusztig polynomials in Lie theory [AJS94, Soe97b]. Not all of the assumptions of this set-up hold for the Brauer algebra, but in Section 8 we show how to bring the two theories close enough together that parabolic Kazhdan-Lusztig (pKL) polynomials suitable for the Brauer algebra may be computed.

In the group (or quantum group) case one has pKL polynomials associated to alcoves in the alcove geometry, determining (at least for the q-group over $\mathbb{C}$, and its Ringel dual Hecke algebra quotient [Erd94]) decomposition matrices in alcove blocks. In general there is more than one block intersecting an alcove, but there is also a translation principle [Jan03], which states that all these blocks are Morita equivalent, and hence do indeed have the same decomposition matrices. In our case the pKL method formally assigns the same decomposition matrix to every alcove block. One is therefore led to seek a form of translation principle.

The Brauer algebras $B_n(\delta)$ as n varies form a tower via an idempotent construction. In [CMPX06] was introduced a general axiom scheme for studying such a tower as a *tower of* *recollement.* The advantage of studying algebras in such a tower is the existence of four functors: induction, restriction, globalisation, and localisation, which relate the representation theories of the different algebras in a compatible manner.

In Section 4 we will show how towers of recollement, when combined with a suitable description of the blocks in the tower, give rise to analogues of translation functors and corresponding Morita equivalences. These functors are defined using induction or restriction functors followed by projection onto a block, and are similar in spirit to α-induction and α-restriction functors for the symmetric group [Rob61].

We apply this translation theory to the Brauer algebras in Section 6, with the aim of proving that two blocks corresponding to weights in the same facet have the same representation theory (Corollary 6.8). However, in order to do this we will need some additional functors, generalisations of induction and restriction, which are introduced in Section 5. We will also see that when $\delta < 0$ there are translation equivalences *between* certain facets, which raises interesting questions as to the true geometric structure underlying the representation theory.

We can also consider an analogue of translation 'onto a wall' in Lie theory for towers of recollement. Using this we show that the decomposition matrix for the Brauer algebra is determined by the decomposition matrix for the block containing the weight 0 (Theorem 6.14).

In Section 7 we consider various graphs associated to each block (or each facet), and show that they are in fact all isomorphic. For alcove graphs we can define associated Kazhdan-Lusztig polynomials; using the graph isomorphisms these polynomials can more generally be associated to any block graph. In the final section we show that when $\delta = 1$ these polynomials correctly predict decomposition numbers and filtrations in the alcove case for standard modules in low rank examples.

2. A review of Brauer algebra representation theory

In this section we will very briefly summarise the basic representation theory of the Brauer algebras that will be needed in what follows. Details can be found in [CDM09a, CDM09b]. In this paper we will restrict our attention to the case where the ground field is $\mathbb{C}$.

The Brauer algebra $B_n(= B_n(\delta))$ is a finite dimensional algebra with parameter $\delta \in \mathbb{C}$. When $\delta \notin \mathbb{Z}$ this algebra is semisimple, so we will henceforth assume that δ is an integer. *We will also assume that $\delta \neq 0$.*

It will be convenient to use the usual graphical presentation of Brauer algebras. An (n, m) *Brauer algebra diagram* will consists of a rectangular frame with n marked points on the northern edge and m on the southern edge called *nodes*. Each of these sets will be numbered from 1 to n (respectively m) from left to right. Each node is joined to precisely one other by a line; lines connecting the northern and southern edge will be called *propagating lines* and the remainder *(northern or southern) arcs*.

Multiplication of two (n, n) diagrams is by concatenation, where any diagram obtained with a closed loop is set equal to δ times the same diagram with the loop removed. Two diagrams are equivalent if they connect the same pairs of nodes. The algebra obtained by taking linear combinations of (n, n) diagrams is a realisation of B_n. Note that $\mathbb{C}\Sigma_n$

is isomorphic to the subalgebra of B_n spanned by diagram with only propagating lines. Moreover, B_n is generated by this subalgebra together with the elements $X_{i,j}$ with $1 \leq i, j \leq n$ consisting of $n-2$ propagating lines and arcs joining i and j on the northern (respectively) southern edges.

The Brauer algebra can also be constructed via 'iterated inflations' of the symmetric group [KX01], and thus is a cellular algebra. If $\delta \neq 0$, then it is even quasihereditary. The standard modules $\Delta_n(\lambda)$ are parametrised by partitions of $n, n-2, \ldots, 1/0$ (where the final term depends on the parity of n), and we will denote the set of such by Λ_n. If $\delta \neq 0$ then the same set parametrises the simple modules.

We have the following explicit construction of standard modules. Consider a Brauer diagram with n northern nodes and $n-2t$ southern nodes, and with no southern arcs. Such a diagram must have exactly t northern arcs. We will denote this diagram by $X_{v,1,\sigma}$, where v denotes the configuration of northern arcs, 1 represents the fixed southern boundary, and $\sigma \in \Sigma_{n-2t}$ is the permutation obtained by setting $\sigma(i) = j$ if the ith propagating northern node from the left is connected to the southern node labelled by j.

The elements v arising as above will be called *partial one-row diagrams*, and the set of such will be denoted by $V_{n,t}$. If a node i in $w \in V_{n,t}$ is not part of a northern arc we say that it is *free*. The vector space spanned by the set of diagrams of the form $X_{w,1,id}$ where $w \in V_{n,t}$ will be denoted I_n^t. Note that Σ_m acts on I_n^t on the right by permuting the southern nodes.

Given λ a partition of $m = n-2t$, let S^λ denote the Specht module (as defined in [JK81]) corresponding to λ for Σ_m. Then the standard module $\Delta_n(\lambda)$ can be realised (see [DWH99, Section 2] or [CDM09a, Section 2]) in the following manner. As a vector space we have

$$\Delta_n(\lambda) = I_n^t \otimes S^\lambda. \tag{1}$$

An element b of B_n acts on $d \in I_n^t$ from the right by diagram multiplication. If the resulting product has fewer than m propagating lines then we define the action of b on $d \otimes S^\lambda$ to be 0. Otherwise the product will result in a diagram with exactly m propagating lines, but these may now be permuted. We transfer this permutation (thought of as an element of Σ_m) through the tensor product to act on S^λ.

For $\delta \neq 0$ (or $n > 2$) there is an idempotent $e_n \in B_n$ such that $e_n B_n e_n \cong B_{n-2}$, and so there are associated *localisation* and *globalisation* functors $F_n : B_n\text{-mod} \to B_{n-2}\text{-mod}$ and $G_n : B_n\text{-mod} \to B_{n+2}\text{-mod}$ given on objects by $F_n(M) = e_n M$ and

$$G_n(M) = B_{n+2}e_{n+2} \otimes_{B_n} M.$$

In this way we can regard B_n-mod as a full subcategory of B_{n+2}-mod, and hence $\Lambda_n \subset \Lambda_{n+2}$. We set $\Lambda = \lim_{n\to\infty}(\Lambda_n \cup \Lambda_{n+1})$; note that $\Lambda = \cup_{n\geq 0}\Lambda_n$, the set of all partitions. We will abuse terminology and say that two labels are in the same block when the associated standard modules are in the same block.

In order to describe the main results in [CDM09a] we will need some additional terminology. Recall that for a partition λ (which we will identify with its Young diagram), the *content* of the box in row i and column j of the diagram is defined to be $j - i$. A pair of partitions $\mu \subset \lambda$ is said to be *δ-balanced* if the following conditions are satisfied:

(1) the boxes in the skew partition can be paired so that the contents of each pair sum to $1 - \delta$;

(2) if the skew partition contains boxes labelled by $1-\frac{\delta}{2}, -\frac{\delta}{2}$ and there is only one such box in the bottom row then the number of pairs of such boxes is even.

We say that two general partitions λ and μ are *δ-balanced* if the pairs $\lambda\cap\mu \subset \lambda$ and $\lambda\cap\mu\subset\mu$ are both δ-balanced. The importance of the δ-balanced condition is clear from the following result [CDM09a, Corollary 6.7]:

Theorem 2.1. *Two partitions λ and μ are in the same block for B_n if and only if they are δ-balanced.*

Denote by $V_\delta(\lambda)$ the set of partitions μ such that μ and λ are δ-balanced. Note that if $\mu \in V_\delta(\lambda)$ then so too are $\lambda\cap\mu$ and $\lambda\cup\mu$. Thus $V_\delta(\lambda)$ forms a lattice under the inclusion relation. We say that μ is a *maximal balanced subpartition of* λ if $\mu\in V_\delta(\lambda)$ and there does not exist $\tau\in V_\delta(\lambda)$ with $\mu\subset\tau\subset\lambda$. One of the main steps in the proof of Theorem 2.1 is [CDM09a, Theorem 6.5], which shows that if μ is a maximal balanced subpartition of λ then

$$\mathrm{Hom}_n(\Delta_n(\lambda), \Delta_n(\mu)) \neq 0.$$

The standard module $\Delta_n(\lambda)$ has simple head $L_n(\lambda)$ and all other composition factors are of the form $L_n(\mu)$ where $\mu\supset\lambda$ lies in the same block as λ [CDM09a, Proposition 4.5]. If λ and μ are such a pair with $|\lambda/\mu| = 2$ then

$$[\Delta_n(\mu) : L_n(\lambda)] = \dim \mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) = 1 \tag{2}$$

by [DWH99, Theorem 3.4 and the remarks after Theorem 3.1] (see [CDM09a, Theorem 4.4]). If $\mu\subset\lambda$ are two weights in the same block and $\lambda/\mu = (a^b)$ for some a and b with a even then we also have [CDM09a, Proposition 5.1] that

$$[\Delta_n(\mu) : L_n(\lambda)] = 1. \tag{3}$$

In general, if $\lambda\vdash n$ and $\mu\vdash m$ with $m\leq n$ the exactness of the localisation functor implies that

$$[\Delta_N(\mu) : L_N(\lambda)] = [\Delta_n(\mu) : L_n(\lambda)]$$

for all $N > n$.

The algebra B_n embeds inside B_{n+1}, and so we may consider the associated induction and restriction functors ind_n and res_{n+1}. If λ and μ are partitions we write $\lambda \rhd \mu$, or $\mu\lhd\lambda$ if the Young diagram for μ is obtained from that for λ by removing one box. Then [DWH99, Theorem 4.1 and Corollary 6.4] we have short exact sequences

$$0 \to \bigoplus_{\mu\lhd\lambda} \Delta_{n+1}(\mu) \to \mathrm{ind}_n\,\Delta_n(\lambda) \to \bigoplus_{\mu\rhd\lambda}\Delta_{n+1}(\mu) \to 0 \tag{4}$$

and

$$0 \to \bigoplus_{\mu\lhd\lambda} \Delta_{n-1}(\mu) \to \mathrm{res}_n\,\Delta_n(\lambda) \to \bigoplus_{\mu\rhd\lambda}\Delta_{n-1}(\mu) \to 0. \tag{5}$$

The restriction rule for simples is not so straightforward. However we do have by [CDM09a, Lemma 7.1] that if μ is a partition obtained from λ by removing one box then

$$[\mathrm{res}_n\, L_n(\lambda) : L_{n-1}(\mu)] \neq 0. \tag{6}$$

The next two results are new, and show how the local data in (4) and (5) can be applied to explicit decomposition number calculations (which illustrates one of the motivations for the tower of recollement formalism in [CMPX06]).

Proposition 2.2. *Suppose that by removing m boxes from $\lambda \vdash n$ it is possible to reach a partition $\mu \vdash n-m$ such that $\Delta_{n-m}(\mu)$ is a projective B_{n-m}-module. Then the simple module $L_n(\lambda)$ does not appear as a composition factor in any $\Delta_n(\nu)$ with ν of degree less than $n-2m$.*

Proof. Suppose that $L_n(\lambda)$ does occur as a composition factor of $\Delta_n(\nu)$. Then by Brauer-Humphreys reciprocity [Don98, Proposition A2.2(iv)] the projective cover $P_n(\lambda)$ of $L_n(\lambda)$ has a standard module filtration with $\Delta_n(\nu)$ as a factor.

As $\Delta_{n-m}(\mu)$ is projective, so is $\mathrm{ind}_{n-1}\cdots\mathrm{ind}_{n-m}\Delta_{n-m}(\mu)$. By repeated application of (4) we see that this contains $L_n(\lambda)$ in its head, and so must have as a summand $P_n(\lambda)$. By [CDM09a, Lemma 2.6 and (2.2)] we have that

$$\mathrm{ind}_{B_{n-m}}^{B_n} = \mathrm{res}_{B_n}^{B_{m+n}} G_{n+m-2}\cdots G_{n-m+2}G_{n-m}$$

and

$$G_r(\Delta_r(\lambda)) \cong \Delta_{r+2}(\lambda).$$

Therefore by repeated application of (5) to $\Delta_{n+m}(\mu)$ we see that $P_n(\lambda)$ cannot have a standard module filtration with $\Delta_n(\nu)$ as a factor. This gives the desired contradiction and so we are done. □

Remark 2.3. Note that any standard module which is alone in its block must be projective. Thus there are many circumstances where Proposition 2.2 will be easy to apply. Indeed, this case will be sufficient for our purposes.

If $\mu \subset \lambda$ are two partitions then their skew λ/μ can be regarded as a series of disjoint partitions; when considering such differences we will list the various partitions in order from top right to bottom left. Thus a skew partition $((22)^2)$ will consist of two disjoint partitions of the form (22).

Proposition 2.4. *If $\mu \subset \lambda$ is a balanced pair with $\lambda/\mu = ((22)^2)$ or $\lambda/\mu = ((1)^4)$ then*

$$\mathrm{Hom}_n(\Delta_n(\lambda), \Delta_n(\mu)) \neq 0.$$

Proof. We may assume that $\lambda \vdash n$ by localisation. If $\lambda/\mu = ((22)^2)$ let λ' be λ less one of the two removable boxes in λ/μ, and μ' be the partition μ together with the addable box from the other component of the skew. If $\lambda/\mu = ((1)^4)$ then let λ' be λ with any one of the boxes in λ/μ removed, and μ' be μ together with the unique box in λ/μ making this a balanced pair.

In each case μ' is a maximal balanced subpartition of λ' and so by [CDM09a, Theorem 6.5] we have that

$$\mathrm{Hom}_{n-1}(\Delta_{n-1}(\lambda'), \Delta_{n-1}(\mu')) \neq 0.$$

By [CDM09a, Corollary 6.7] and (5) the only term in the block labelled by λ' in the standard filtration of $\mathrm{res}_n \Delta_n(\mu)$ is $\Delta_n(\mu')$. Therefore by Frobenius reciprocity we have

$$\mathrm{Hom}(\mathrm{ind}_{n-1}\Delta_{n-1}(\lambda'), \Delta_n(\mu)) \cong \mathrm{Hom}(\Delta_{n-1}(\lambda'), \mathrm{res}_n \Delta_n(\mu)) \neq 0. \tag{7}$$

Now λ is not the only weight in its block in the set of weights labelling term in the standard filtration of $\mathrm{ind}_{n-1}\,\Delta_{n-1}(\lambda')$. However, by [CDM09a, Lemma 4.10] it follows from (7) that

$$\mathrm{Hom}_n(\Delta_n(\lambda), \Delta_n(\mu)) \neq 0$$

as required. □

In [CDM09b] we identified partitions labelling Brauer algebra modules with elements of $\mathbb{Z}^N$ (for suitable N) only after transposition of the original partition to form its conjugate. Henceforth when we regard Λ_n (or Λ) as a subset of $\mathbb{Z}^\infty$ it will always be via this *transpose map* $\lambda \to \lambda^T$.

3. Brauer analogues of Weyl and affine Weyl groups

We wish to identify reflection groups associated to the Brauer algebra which play the role of the Weyl and affine Weyl groups for reductive algebraic groups. First let us recall the properties of Weyl groups which we wish to replicate.

In Lie theory a Weyl group W is a reflection group acting on a Euclidean weight space with the following properties:

(1) There is an integral set of weights on which W acts via a 'dot' action $(W, \cdot)$,
(2) The reflection hyperplanes of W under this action break space up into chambers (and other facets),
(3) A complete set of weights indexing simple (or standard) modules coincides with the weights in a single chamber under the dot action, namely that containing the zero weight. Such weights are said to be *dominant*.

Thus the selection of an indexing set for the dominant weights is taken care of by the Weyl group (and its dot action). In positive characteristic p or at a quantum lth root of unity there is then a second stage, the introduction of an affine extension of W (with action depending on p or l), which has orbits whose intersection with the dominant weights determine the blocks.

This affine extension defines an additional set of reflecting hyperplanes, which break the set of weights up into a series of chambers (now called alcoves) and other facets. We refer to this configuration of facets, together with the action of the affine extension, as the *alcove geometry* associated to the particular Lie theory in question.

The alcove geometry controls much of the representation theory of the corresponding reductive group. In particular, we typically have a *translation principle* which says that there are Morita equivalences between blocks which intersect a given facet, and so much of the representation theory does not depend on the weight itself but only on the facet in which it lies.

We will show how a version of the above programme can be implemented for the Brauer algebra from scratch.

Let $\mathbb{E}^n$ be the $\mathbb{R}$-vector space with basis $e_1, \ldots, e_n$, and $\underline{n} = \{1, \ldots, n\}$. We will define various reflections on $\mathbb{E}^n$ corresponding to the standard action of the type D Weyl group. Let (ij) be the reflection in the hyperplane in $\mathbb{E}^n$ through the origin which takes e_i to e_j and fixes all other unit vectors, and $(ij)_-$ to be the reflection in the hyperplane perpendicular to $e_i + e_j$ which takes e_i to $-e_j$. We define

$$W_a(n) = \langle (i,j), (i,j)_- : i \neq j \in \underline{n} \rangle$$

which is the type D Weyl group. Note that it has a a subgroup

$$W(n) = \langle (i,j) : i \neq j \in \underline{n} \rangle$$

which is just the type A Weyl group (isomorphic to Σ_n).

As explained in the previous section, the Brauer algebras B_n as n varies form a tower. Thus it is natural to consider all such algebras simultaneously. In order to do this we will work with the infinite rank case. Note also that orbits of the finite Weyl group $W(n)$ are not sufficient to define an indexing set for the simple B_n-modules (one needs to consider $W(n+1)$-orbits, but then this group is not a subgroup of $W_a(n)$), unlike the infinite rank case.

Let $\mathbb{E}^\infty$ be the $\mathbb{R}$-vector space consisting of (possibly infinite) linear combinations of the elements $e_1, e_2, \ldots$. We say that $\lambda \in \mathbb{Z}^\infty$ has *finite support* if only finitely many components of λ are non-zero, and write $\mathbb{Z}^f$ for the set of such elements. (We define $\mathbb{E}^f$ similarly.) Thus, with the obvious embedding of $\mathbb{Z}^n$ inside $\mathbb{Z}^{n+1}$, we have that $\mathbb{Z}^f = \lim_{n\to\infty} \mathbb{Z}^n$. We say that an element $\lambda = (\lambda_1, \lambda_2, \ldots)$ of $\mathbb{Z}^f$ is *dominant* if $\lambda_i \geq \lambda_{i+1}$ for all i. (Note that any such element must lie in $\mathbb{N}^\infty$.) Embed $\mathbb{E}^n$ inside $\mathbb{E}^\infty$ in the obvious way, and let W_a, and W be the corresponding limits of $W_a(n)$, and $W(n)$. Clearly the space $\mathbb{Z}^f$ is closed under the action of W_a. We will call elements in $\mathbb{Z}^f$ *weights*. Dominant weights are precisely those which label standard modules for the Brauer algebras, and by analogy with Lie theory we will denote the set of such weights by X^+. General elements of $\mathbb{E}^\infty$ will be called *vectors*. We will use Greek letters for weights and Roman letters for general vectors.

Given a reflection group G (or the corresponding set of hyperplanes $\mathbb{H}$), we say that a vector is *regular in G (or in $\mathbb{H}$)* if it lies in the interior of a chamber, i.e. in some connected component of $\mathbb{E}^\infty \backslash \cup_{X \in \mathbb{H}} X$. Otherwise we say the vector is *singular*. In the case $G = W_a$ we shall call chambers *alcoves* to emphasise the distinction between this and the W case. For $v \in \mathbb{E}^\infty$ we define the *degree of singularity*

$$s(v) = |\{\{i,j\} : v_i = \pm v_j, \quad i < j\}|$$

(which need not be finite in general). Note that a vector v is regular in W_a if and only if $s(v) = 0$. The next lemma is clear.

Lemma 3.1. *(i) There is a chamber A^+ of the action of W on $\mathbb{E}^\infty$ consisting of all strictly decreasing sequences.*
(ii) The boundary of A^+ consists of all non-strictly decreasing sequences.

Recall that in Lie theory we typically consider a shifted reflection group action with respect to some fixed element ρ. It will be convenient to consider a similar adjustment here. Let $-2\omega = (1, 1, \ldots, 1) \in \mathbb{E}^\infty$ and $\rho_0 = (0, -1, -2, \ldots)$. For $\delta \in \mathbb{Z}$ define

$$\rho_\delta = \rho_0 + \delta\omega.$$

For $w \in W_a$ and $v \in \mathbb{E}^\infty$ let

$$w \cdot_\delta v = w(v + \rho_\delta) - \rho_\delta$$

where the right-hand side is given by the usual reflection action of W_a on $\mathbb{E}^\infty$. Note that $\mathbb{Z}^f$ is closed under this action of W_a. We say that a weight λ is *δ-regular* if $\lambda + \rho_\delta$ is regular, and define the *degree of δ-singularity* of λ to be $s(\lambda + \rho_\delta)$.

Proposition 3.2. *Let $\lambda \in \mathbb{Z}^f$.*
(i) For $w \in W$ the weight $w \cdot_\delta \lambda$ does not depend on δ. Moreover, if $w \neq 1$ and $\lambda \in X^+$ then $w \cdot_\delta \lambda \notin X^+$.
(ii) If $\lambda \in X^+$ then $\lambda + \rho_\delta$ can only lie on an $(ij)_-$-hyperplane.
(iii) We have $\lambda \in X^+$ if and only if $\lambda + \rho_\delta \in A^+$.

Proof. (i) Note that

$$(ij)(\lambda + \rho_0 + \delta\omega) - \rho_0 - \delta\omega = (ij)(\lambda + \rho_0) - \rho_0.$$

(ii) and (iii) are clear. □

The description of the blocks of the Brauer algebra in characteristic zero in Theorem 2.1 was given the following geometric reformulation in [CDM09b]:

Theorem 3.3. *Two standard modules $\Delta_n(\lambda)$ and $\Delta_n(\lambda')$ for B_n are in the same block if and only if λ^T and λ'^T are in the same $(W_a(n), \cdot_\delta)$-orbit.*

Remark 3.4. In summary, we have shown that there is a space that plays a role analogous to a weight space (in Lie theory) for the Brauer algebra, together with an action of the type A Coxeter group which plays the role of the Weyl group, while the corresponding type D Coxeter group plays the role of the affine Weyl group.

We will now consider the geometry of facets induced by W_a inside A^+. We will call reflection hyperplanes *walls*, and for any collection of hyperplanes we will call a connected component of the set of points lying on the intersection of these hyperplanes but on no other a *facet*. (Then an alcove is a facet corresponding to the empty collection of hyperplanes.)

It will be convenient to have an explicit description of the set of vectors in a given facet. For vectors in A^+ (which will be the only ones which concern us) these facets are defined by the hyperplanes $v_i = -v_j$ for some $i \neq j$. For $v = (v_1, v_2, v_3, ...)$ in A^+, note that for all $i \in \mathbb{N}$ we have

$$|\{j \,:\, |v_j| = |v_i|\}| \leq 2.$$

We will call v_i a *singleton* if v_i is the only coordinate with modulus $|v_i|$, and the pair v_i, v_j a *doubleton* if $|v_i| = |v_j|$ and $i \neq j$.

For a given facet F with $v \in F$, a vector $v' \in A^+$ lies in F if and only if $|v'_i| = |v'_j|$ whenever $|v_i| = |v_j|$, and $|v'_i| > |v'_j|$ whenever $|v_i| > |v_j|$. Therefore an alcove (where every v_i is a singleton) is determined by a permutation π from $\mathbb{N}$ to $\mathbb{N}$ where $|v_{\pi(n)}|$ is the nth smallest modulus occurring in v. Note that not every permutation corresponds to an alcove in this way. Further, if $i < \pi(1)$ then $v_i > 0$, while if $i > \pi(1)$ then $v_i < 0$.

For more general facets we replace the permutation π by a function $f : \mathbb{N} \to \mathbb{N} \cup (\mathbb{N} \times \mathbb{N})$ such that $f(n)$ is the coordinate, or pair of coordinates, where the nth smallest modulus in v occurs. For example, if

$$v = (6, 4, 2, 1, 0, -2, -3, -5, \dots)$$

then the facet containing v corresponds to a function whose first four values are $f(1) = 5$, $f(2) = 4$, $f(3) = (3, 6)$, and $f(4) = 7$.

We will denote by A_0 the alcove corresponding to the identity permutation. Thus A_0 consists of all $v \in A^+$ such that $|v_1| < |v_2|$ and $v_2 < 0$. It is easy to see that, for any $\delta \geq 0$, the weight 0 is δ-regular, with the vector $0 + \rho_\delta$ in A_0; in this case we will call the $(W_a, \cdot_\delta)$-alcove the *δ-fundamental alcove.*

Lemma 3.5. *For $\delta \geq 0$ the set of weights in the δ-fundamental alcove is*

$$\{\lambda \in X^+ : \lambda_1 + \lambda_2 \leq \delta\}.$$

Proof. By our discussion above, the desired set of weights is precisely the set of dominant λ such that $x = \lambda + \rho_\delta$ and $|x_1| < |x_2|$. But this means that

$$\lambda_1 - \frac{\delta}{2} < \frac{\delta}{2} + 1 - \lambda_2$$

which implies the result. □

Remark 3.6. Although our alcove geometry is reminiscent of that arising in positive characteristic Lie theory, there are also some striking differences. Consider for example the case when $\delta = 1$. The alcove A_0 is non-empty and contains the two weights 0 and (1). The next lowest alcove contains $(2, 1)$ and $(2, 2)$, and the third contains $(3, 2, 1)$ and $(3, 1, 1)$. However, the associated facets with singularity 1 are not necessarily finite (in $\mathbb{E}^\infty$); for example there is a facet consisting of the weights (2) and the n-tuple $(1, \ldots, 1)$ for all $n \geq 2$. In particular, not every weight on a wall is adjacent to a weight in an alcove.

Recall that for $\delta \in \mathbb{N}$ the Brauer algebra $B_r(\delta)$ is in Schur-Weyl duality with $\mathrm{O}_\delta(\mathbb{C})$ acting on the rth tensor product of the natural representation.

Theorem 3.7. *Suppose that $\delta \in \mathbb{N}$. The elements of Λ_∞ corresponding to weights in the δ-fundamental alcove are in bijection with the set of partitions labelling the irreducible representations which arise in a decomposition of tensor powers of the natural representation of $\mathrm{O}_n(\mathbb{C})$.*

Proof. For $\mathrm{O}_n(\mathbb{C})$ tensor space components are labelled by partitions whose first and second columns sum to at most n (see for example [GW98, Theorem 10.2.5]). The result now follows by comparing with Lemma 3.5 via the transpose map on partitions. □

Remark 3.8. The above result shows that the fundamental alcove arises naturally in the representation theory of $\mathrm{O}_n(\mathbb{C})$.

Suppose that $\delta < 0$. Choose $m \in \mathbb{N}$ so that $\delta = -2m$ (if δ is even) or $\delta = -2m + 1$ (if δ is odd). It is easy to see that 0 is δ-singular of degree m. Indeed, any dominant weight λ is δ-singular of degree at least m. Thus there are no regular dominant weights for $\delta < 0$. Instead of the δ-fundamental alcove, we can consider the *δ-fundamental facet* containing 0, for which we have

Lemma 3.9. *For $\delta < 0$ of the form $-2m$ or $-2m+1$ the set of weights in the δ-fundamental facet is $\{0\}$.*

Proof. We consider the case $\delta = -2m$; the odd case is similar. The element $0 + \rho_\delta$ equals

$$(m, m-1, \ldots, 0, -1, \ldots)$$

and hence our facet consists of all vectors of the form

$$(t, t-1, \ldots, -t+1, -t, v_{|\delta|}, v_{|\delta|+1}, \ldots)$$

where the sequence $-t, v_{|\delta|} \ldots$ is decreasing (as any other weight would be non-dominant). But this implies that $Y_\delta = \{0\}$. ☐

This is very different from the case $\delta > 0$. However we do have

Theorem 3.10. *Suppose that $\delta = -2m$. The set of elements of Λ_∞ corresponding to weights which can be obtained from 0 via a sequence of one box additions only involving intermediate weights of singularity m is in bijection with the set of partitions labelling the irreducible representations which arise in a decomposition of tensor powers of the natural representation of $\mathrm{Sp}_{2m}(\mathbb{C})$.*

Proof. First note that we can clearly add boxes in the first m coordinate of 0 without changing the degree of singularity. In order to change the $(m+1)$st coordinate in our path, we will have to pass through some point of the form

$$(a_1, a_2, \ldots, a_m, 1, -1, -2, \ldots).$$

where $a_1 > a_2 > \cdots > a_m > 1$. But this implies that the first $m+1$ coordinates of the vector all pair up with the corresponding negative values later down the vector, and so this is a singular vector of degree $m+1$. The result now follows from the description of tensor space components (see for example [GW98, Theorem 10.2.5]). ☐

We will see in Section 6 that there is a sense in which the set of weights occurring in Theorem 3.10 can be regarded as playing the role of an alcove in the $\delta < 0$ case.

4. A translation principle for towers of recollement

Towers of recollement were introduced in [CMPX06] as an axiom scheme for studying various families of algebras. The Brauer algebra over $\mathbb{C}$ was shown to satisfy these axioms in [CDM09a]. We will prove a general result about Morita equivalences in such towers, and later apply it to the Brauer algebra. In this section we will work over a general field k.

We begin by reviewing what it means for a family of k-algebras $A = \{A_n : n \in \mathbb{N}\}$ to form a tower of recollement. Further details can be found in [CMPX06]. The tower of recollement formalism involves six axioms (A1–6); however only the first five will be needed in what follows, and we will concentrate on these.

Axiom A1: *In each A_n with $n \geq 2$ there exists an idempotent e_n such that $e_n A_n e_n \cong A_{n-2}$.*

This axiom provides a pair of functors: *localisation* F_n from A_n-mod to A_{n-2}-mod, and *globalisation* G_n from A_n-mod to A_{n+2}-mod given on objects by

$$F_n M = e_n M \qquad \text{and} \qquad G_n M = A_{n+2} e_{n+2} \otimes_{A_n} M.$$

The functor F_n is exact, G_n is right exact, and G_n is left adjoint to F_{n+2}. This gives a full embedding of A_n-mod inside A_{n+2}-mod. Let Λ_n be an indexing set for the simple A_n-modules. Globalisation induces an embedding of Λ_n inside Λ_{n+2}, and we take Λ to be the disjoint union of $\lim_n \Lambda_{2n}$ and $\lim_n \Lambda_{2n+1}$, whose elements we call *weights.*

Axiom A2: *The algebras A_n are quasihereditary, with heredity chain induced by the e_{n-2i} with $0 \leq i \leq n/2$ via the isomorphisms in A1.*

This axiom implies that there exists a standard A_n-module $\Delta_n(\lambda)$ for each weight λ in Λ_n, such that the associated simple $L_n(\lambda)$ arises as its head. Further we have

$$G_n(\Delta_n(\lambda)) \cong \Delta_{n+2}(\lambda)$$

and

$$F_n(\Delta_n(\lambda)) \cong \begin{cases} \Delta_{n-2}(\lambda) & \text{if } \lambda \in \Lambda_{n-2} \\ 0 & \text{otherwise} \end{cases}$$

In particular, the full embedding induced by G_n implies that

$$\mathrm{Hom}(\Delta_n(\lambda), \Delta_n(\mu)) \cong \mathrm{Hom}(\Delta_{n+2}(\lambda), \Delta_{n+2}(\mu)) \tag{8}$$

for all $\lambda, \mu \in \Lambda_n$.

Axiom A3: *There exist algebra inclusions $A_n \subset A_{n+1}$ for each $n \geq 0$.*

The usual induction and restriction functors associated with such an algebra include will be denoted ind_n from A_n-mod to A_{n+1}-mod and res_n from A_n-mod to A_{n-1}-mod. Induction is a right exact functor, while restriction is exact.

Axiom A4: *For $n \geq 1$ we have $A_n e_n \cong A_{n-1}$ as an (A_{n-1}, A_{n-2})-bimodule.*

This axiom implies the compatibility of induction and restriction with globalisation and localisation: in particular we have

$$\mathrm{res}_{n+2} G_n(M) \cong \mathrm{ind}_n M$$

for all M in A_n-mod.

For the next axiom we will need some additional notation. If an A_n-module M has a filtration by standard modules then the multiplicities occurring are well-defined, and we denote by $\mathrm{supp}_n(M)$ the multiset of labels for standards which arise. Further we partition the set Λ_n into a disjoint union of sets Λ_n^m, with $m = n - 2t$ for some $t \geq 0$, where the elements of Λ_n^m are precisely the elements in Λ_m (regarded as a subset of Λ_n) not occurring in Λ_{m-2}.

Axiom A5: *For each $\lambda \in \Lambda_n^m$ the module $\mathrm{res}_n \Delta_n(\lambda)$ has a filtration by standard modules, and*

$$\mathrm{supp}_{n-1}(\mathrm{res}_n \Delta_n(\lambda)) \subseteq \Lambda_{n-1}^{m-1} \cup \Lambda_{n-1}^{m+1}.$$

For simplicity we will denote $\mathrm{supp}_{n-1}(\mathrm{res}_n \Delta_n(\lambda))$ by $\mathrm{supp}_{n-1}(\lambda)$. The embedding of Λ_n in Λ_{n+2} induces an embedding of $\mathrm{supp}_n(\lambda)$ inside $\mathrm{supp}_{n+2}(\lambda)$, which becomes an identification if $\lambda \in \Lambda_{n-2}$. We denote by $\mathrm{supp}(\lambda)$ the set $\mathrm{supp}_n(\lambda)$ with $n >> 0$.

Suppose that we have determined the blocks of such a family of algebras (or at least a necessary condition for being in the same block: a linkage principle); we are thinking of the cases where we have an alcove geometry at hand, but will avoid stating the result in that form. Let res_n^λ be the functor $\mathrm{pr}_{n-1}^\lambda \mathrm{res}_n$ and ind_n^λ be the functor $\mathrm{pr}_{n+1}^\lambda \mathrm{ind}_n$ where pr_n^λ is projection onto the block containing λ for A_n. Note that res_n^λ is exact and ind_n^λ is right exact. We will regard these functors as analogues of *translation functors* in Lie theory, and as there will show that under certain conditions they induce Morita equivalences.

Let $\mathcal{B}_n(\lambda)$ denote the set of weights in the block of A_n which contains λ. Our embedding of Λ_n into Λ_{n+2} induces an embedding of $\mathcal{B}_n(\lambda)$ into $\mathcal{B}_{n+2}(\lambda)$, and we denote by $\mathcal{B}(\lambda)$ the corresponding limiting set. We will consider the intersection of $\mathcal{B}(\lambda)$ with various multisets; in such cases we say that an element is the *unique* element in this intersection if it is the only element occurring in both sets and also has multiplicity one in the multiset. With this convention we will say that two elements λ and λ' are *translation equivalent* if for all weights $\mu \in \mathcal{B}(\lambda)$ there is a unique element $\mu' \in \mathcal{B}(\lambda') \cap \mathrm{supp}(\mu)$, and μ is the unique element in $\mathcal{B}(\lambda) \cap \mathrm{supp}(\mu')$.

In an alcove geometry where blocks correspond to orbits under some group of reflections it is a routine exercise to check that this condition can be restated as: (i) The weight λ' is the unique element of $\mathcal{B}(\lambda') \cap \mathrm{supp}(\lambda)$, and (ii) The weight λ is the unique element of $\mathcal{B}(\lambda) \cap \mathrm{supp}(\lambda')$.

When λ and λ' are translation equivalent then we will denote by $\theta : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda')$ the bijection taking μ to μ'. We will see that translation equivalent weights belong to Morita equivalent blocks.

We will put a very crude partial order on weights in $\mathcal{B}(\lambda)$ by saying that $\lambda > \mu$ if there exists n such that $\mu \in \Lambda_n$ and $\lambda \in \Lambda_{n+2t}$ for some $t \in \mathbb{N}$, but $\lambda \notin \Lambda_n$. Note that this is the opposite of the standard order arising from the quasi-hereditary structure; in our main example of the Brauer algebra this will enable us to work with the natural order on the size of partitions. In the following proposition, by a unique element in a multiset we mean one with multiplicity one.

Proposition 4.1. *Let A be a tower of recollement. Suppose that $\lambda \in \Lambda_n$ and $\lambda' \in \Lambda_{n-1}$ are translation equivalent, and that $\mu \in \mathcal{B}_n(\lambda)$ is such that the μ' is in $\mathcal{B}_{n-1}(\lambda')$. Then we have that*

$$\mathrm{res}_n^{\lambda'} L_n(\mu) \cong L_{n-1}(\mu') \qquad \textit{and} \qquad \mathrm{ind}_{n-1}^{\lambda} L_{n-1}(\mu') \cong L_n(\mu) \tag{9}$$

for all $\mu \in \mathcal{B}_n(\lambda)$. Further if $\tau \in \mathcal{B}_n(\lambda)$ is such that τ' is in $\mathcal{B}_{n-1}(\lambda')$ then we have

$$[\Delta_n(\mu) : L_n(\tau)] = [\Delta_{n-1}(\mu') : L_{n-1}(\tau')] \tag{10}$$

and

$$\mathrm{Hom}(\Delta_n(\mu), \Delta_n(\tau)) \cong \mathrm{Hom}(\Delta_{n-1}(\mu'), \Delta_{n-1}(\tau')). \tag{11}$$

Proof. We begin with (9). Consider the exact sequence

$$\Delta_n(\mu) \longrightarrow L_n(\mu) \longrightarrow 0.$$

Applying $\mathrm{res}_n^{\lambda'}$ we obtain by our assumptions the exact sequence

$$\Delta_{n-1}(\mu') \longrightarrow \mathrm{res}_n^{\lambda'} L_n(\mu) \longrightarrow 0$$

and hence $\mathrm{res}_n^{\lambda'} L_n(\mu)$ has simple head $L_{n-1}(\mu')$, and possibly other composition factors $L_{n-1}(\tau')$ with $\tau' > \mu'$.

If $L(\tau')$ is in the socle of $\operatorname{res}_n^{\lambda'} L_n(\mu)$ then we have by our assumptions and Frobenius reciprocity that

$$\begin{aligned}\operatorname{Hom}(\Delta_n(\tau), L_n(\mu)) &= \operatorname{Hom}(\operatorname{ind}_{n-1}^{\lambda} \Delta_{n-1}(\tau'), L_n(\mu)) \cong \operatorname{Hom}(\operatorname{ind}_{n-1} \Delta_{n-1}(\tau'), L_n(\mu)) \\ &\cong \operatorname{Hom}(\Delta_{n-1}(\tau'), \operatorname{res}_n L_n(\mu)) \cong \operatorname{Hom}(\Delta_{n-1}(\tau'), \operatorname{res}_n^{\lambda'} L_n(\mu)) \neq 0.\end{aligned}$$

As $\Delta_n(\tau)$ has simple head $L_n(\tau)$ this implies that $\tau = \mu$ and hence $\tau' = \mu'$. Therefore $\operatorname{res}_n^{\lambda'} L_n(\mu) \cong L_{n-1}(\mu')$ as required.

Next consider the exact sequence

$$\Delta_{n-1}(\mu') \longrightarrow L_{n-1}(\mu') \longrightarrow 0.$$

Applying $\operatorname{ind}_{n-1}^{\lambda}$ we obtain by our assumptions the exact sequence

$$\Delta_n(\mu) \longrightarrow \operatorname{ind}_{n-1}^{\lambda} L_{n-1}(\mu') \longrightarrow 0.$$

and hence $\operatorname{ind}_{n-1}^{\lambda} L_{n-1}(\mu')$ has simple head $L_n(\mu)$, and possibly other composition factors $L_n(\tau)$ with $\tau > \mu$. Now apply $\operatorname{res}_n^{\lambda'}$ to obtain the exact sequence

$$\Delta_{n-1}(\mu') \longrightarrow \operatorname{res}_n^{\lambda'} \operatorname{ind}_{n-1}^{\lambda} L_{n-1}(\mu') \longrightarrow 0.$$

Then $\operatorname{res}_n^{\lambda'} \operatorname{ind}_{n-1}^{\lambda} L_{n-1}(\mu')$ has simple head $L_{n-1}(\mu')$ and possibly other composition factors $L_{n-1}(\tau')$ with $\tau' > \mu'$ corresponding to those in $\operatorname{ind}_{n-1}^{\lambda} L_{n-1}(\mu')$. We have

$$\operatorname{Hom}(L_{n-1}(\mu'), \operatorname{res}_n^{\lambda'} \operatorname{ind}_{n-1}^{\lambda} L_{n-1}(\mu')) \cong \operatorname{Hom}(\operatorname{ind}_{n-1}^{\lambda} L_{n-1}(\mu'), \operatorname{ind}_{n-1}^{\lambda} L_{n-1}(\mu')) \neq 0$$

and hence $L_{n-1}(\mu')$ must appear in the socle of $\operatorname{res}_n^{\lambda'} \operatorname{ind}_{n-1}^{\lambda} L_{n-1}(\mu')$. This forces

$$\operatorname{res}_n^{\lambda'} \operatorname{ind}_{n-1}^{\lambda} L_{n-1}(\mu') \cong L_{n-1}(\mu')$$

and as we already have that $\operatorname{res}_n^{\lambda'}$ takes simples to simples we deduce that

$$\operatorname{ind}_{n-1}^{\lambda} L_{n-1}(\mu') \cong L_n(\mu)$$

which completes our proof of (9). Now (10) follows immediately as $\operatorname{res}_n^{\lambda'} \Delta_n(\mu) \cong \Delta_{n-1}(\mu')$, while (11) follows from

$$\begin{aligned}\operatorname{Hom}(\Delta_n(\mu), \Delta_n(\tau)) &\cong \operatorname{Hom}(\operatorname{ind}_{n-1}^{\lambda} \Delta_{n-1}(\mu'), \Delta_n(\tau)) \\ &\cong \operatorname{Hom}(\Delta_{n-1}(\mu'), \operatorname{res}_n^{\lambda'} \Delta_n(\tau)) \cong \operatorname{Hom}(\Delta_{n-1}(\mu'), \Delta_{n-1}(\tau')).\end{aligned}$$

□

Let $P_n(\lambda)$ denote the projective cover of $L_n(\lambda)$. As our algebras are quasihereditary we have that $P_n(\lambda)$ has a filtration by standard modules with well-defined filtration multiplicities (see for example [Don98, Appendix]); we denote the multiplicity of $\Delta_n(\mu)$ in such a filtration by $(P_n(\lambda) : \Delta_n(\mu))$.

Proposition 4.2. *Suppose that $\lambda \in \Lambda_n$ and $\lambda' \in \Lambda_{n-1}$ are translation equivalent. Then for all $\mu \in \mathcal{B}_n(\lambda)$ with $\mu' \in \mathcal{B}_{n-1}(\lambda')$ we have*

$$\operatorname{ind}_{n-1}^{\lambda} P_{n-1}(\mu') \cong P_n(\mu). \tag{12}$$

If $\mu \in \mathcal{B}_{n-2}(\lambda)$ we have

$$\operatorname{res}_n^{\lambda'} P_n(\mu) \cong P_{n-1}(\mu'). \tag{13}$$

Proof. We begin with (12). The functor ind_{n-1} takes projectives to projectives, and hence so does $\mathrm{ind}_{n-1}^{\lambda}$. We must show that inducing an indecomposable projective gives an indecomposable projective with the right weight.

Suppose we have an exact sequence

$$\mathrm{ind}_{n-1}^{\lambda} P_{n-1}(\mu') \to L_n(\tau) \to 0$$

for some $\tau \in \mathcal{B}_n(\lambda)$. Then we have

$$\begin{aligned} 0 &\neq \mathrm{Hom}_n(\mathrm{ind}_{n-1}^{\lambda} P_{n-1}(\mu'), L_n(\tau)) \\ &\cong \mathrm{Hom}_{n-1}(P_{n-1}(\mu'), \mathrm{res}_n^{\lambda'} L_n(\tau)) \cong \mathrm{Hom}_{n-1}(P_{n-1}(\mu'), L_n(\tau')) \end{aligned}$$

by Proposition 4.1. Therefore we must have $\mu' = \tau'$, and hence $\mu = \tau$ and

$$\mathrm{Hom}_n(\mathrm{ind}_{n-1}^{\lambda} P_{n-1}(\mu'), L_n(\mu)) \cong k.$$

This implies that $\mathrm{ind}_{n-1}^{\lambda} P_{n-1}(\mu')$ has simple head $L_n(\mu)$ and hence is isomorphic to $P_n(\mu)$.

Next we consider (13). As $A_n e_n$ is a direct summand of the left A_n-module A_n, it is a projective A_n-module. Moreover, as $e_n A_n e_n \cong A_{n-2}$ we have that $A_n e_n$ contains precisely those indecomposable projective A_n-modules labelled by weights in Λ_{n-2}. By axiom 4 we have that

$$\mathrm{res}_{n-1} A_n e_n \cong A_{n-1}$$

as a left A_{n-1}-module. This implies that for $\mu \in \Lambda_{n-2}$, the module $\mathrm{res}_n P_n(\mu)$ (and hence $\mathrm{res}_n^{\lambda'} P_n(\mu)$) is projective.

As $\mathrm{res}_n^{\lambda'}$ is an exact functor and $P_n(\mu)$ has simple head $L_n(\mu)$ we know from Proposition 4.1 that

$$\mathrm{res}_n^{\lambda'} P_n(\mu) = P_n(\mu') \oplus Q \tag{14}$$

for some projective A_{n-1}-module Q. However, by Brauer-Humphreys reciprocity for quasi-hereditary algebras [Don98, A.2.2(iv)] and Proposition 4.1 we have

$$(P_n(\mu) : \Delta_n(\tau)) = [\Delta_n(\tau) : L_n(\mu)] = [\Delta_{n-1}(\tau') : L_{n-1}(\mu')] = (P_{n-1}(\mu') : \Delta_{n-1}(\tau')).$$

As $\mathrm{res}_n^{\lambda'}$ is exact and takes $\Delta_n(\tau)$ to $\Delta_{n-1}(\tau')$ this implies that $Q = 0$. □

We would like to argue that two blocks labelled by translation equivalent weights are Morita equivalent. However, the fact that not every projective module restricts to a projective in (13) causes certain complications.

Lemma 4.3. *If* $\lambda \in \Lambda_n$ *then*

$$G_n(P_n(\lambda)) \cong P_{n+2}(\lambda).$$

Proof. By [ASS06, Chapter I, Theorem 6.8] $G_n(P_n(\lambda))$ is an indecomposable projective. We have an exact sequence

$$P_n(\lambda) \to \Delta_n(\lambda) \to 0$$

And hence as G_n is right exact and takes standards to standards we obtain

$$G_n(P_n(\lambda)) \to \Delta_{n+2}(\lambda) \to 0.$$

This implies that $G_n(P_n(\lambda)) \cong P_{n+2}(\lambda)$. □

Lemma 4.4. *If $\mu, \tau \in \Lambda_n$ then*

$$\mathrm{Hom}_n(P_n(\mu), P_n(\tau)) \cong \mathrm{Hom}_{n+2}(P_{n+2}(\mu), P_{n+2}(\tau))$$

and this extends to an algebra isomorphism

$$\mathrm{End}_n(\bigoplus_{\mu\in\Gamma} P_n(\mu)) \cong \mathrm{End}_{n+2}(\bigoplus_{\mu\in\Gamma} P_{n+2}(\mu))$$

for any subset Γ of Λ_n.

Proof. See [ASS06, Chapter I, Theorem 6.8]. □

The algebra A_n decomposes as a direct sum of indecomposable projective modules:

$$A_n = \bigoplus_{\lambda\in\Lambda_n} P_n(\lambda)^{d_{n,\lambda}}$$

for some integers $d_{n,\lambda}$. There is a corresponding decomposition of $1 \in A_n$ as a sum of (not necessarily primitive) orthogonal idempotents $1 = \sum_{\lambda\in\Lambda_n} e_{n,\lambda}$ where $A_n e_{n,\lambda} = P_n(\lambda)^{d_{n,\lambda}}$. As Λ_n decomposes as a union of blocks the algebra A_n decomposes as a direct sum of (block) subalgebras

$$A_n = \bigoplus_{\lambda} A_n(\lambda)$$

where the sum runs over a set of block representatives and

$$A_n(\lambda) = \bigoplus_{\mu\in\mathcal{B}_n(\lambda)} P_n(\mu)^{d_{n,\mu}}.$$

Now let $\Gamma \subset \mathcal{B}_n(\lambda)$ and consider the idempotent $e_{n,\Gamma} = \sum_{\gamma\in\Gamma} e_{n,\gamma}$. We define the algebra $A_{n,\Gamma}(\lambda)$ by

$$A_{n,\Gamma}(\lambda) = e_{n,\Gamma} A_n(\lambda) e_{n,\Gamma}.$$

By Lemma 4.4 we have that $A_{n,\Gamma}(\lambda)$ and $A_{m,\Gamma}(\lambda)$ are Morita equivalent for all m such that $\Gamma \subset \mathcal{B}_m(\lambda)$.

Theorem 4.5. *Suppose that λ and λ' are translation equivalent with $\lambda \in \Lambda_n$, and set*

$$\Gamma = \theta(\mathcal{B}_n(\lambda)) \subset \mathcal{B}_{n+1}(\lambda').$$

Then $A_n(\lambda)$ and $A_{n+1,\Gamma}(\lambda')$ are Morita equivalent. In particular, if there exists an n such that $|\mathcal{B}_n(\lambda)| = |\mathcal{B}_{n+1}(\lambda')|$ then $A_n(\lambda)$ and $A_{n+1}(\lambda')$ are Morita equivalent.

Proof. We will show that the basic algebras corresponding to $A_n(\lambda)$ and $A_{n+1,\Gamma}(\lambda')$ are isomorphic; i.e. that

$$\mathrm{End}_n(\bigoplus_{\mu\in\mathcal{B}_n(\lambda)} P_n(\mu)) \cong \mathrm{End}_{n+1}(\bigoplus_{\nu'\in\Gamma} P_{n+1}(\nu')).$$

By Lemma 4.4 it is enough to show that

$$\mathrm{End}_{n+2}(\bigoplus_{\mu\in\mathcal{B}_n(\lambda)} P_{n+2}(\mu)) \cong \mathrm{End}_{n+1}(\bigoplus_{\nu'\in\Gamma} P_{n+1}(\nu')).$$

Suppose that $\mu, \tau \in \mathcal{B}_n(\lambda)$. Then by Lemma 4.4 and Proposition 4.2 we have

$$\begin{aligned}\mathrm{Hom}_{n+2}(P_{n+2}(\mu), P_{n+2}(\tau)) &\cong \mathrm{Hom}_{n+2}(\mathrm{ind}_{n+1}^{\lambda} P_{n+1}(\mu'), P_{n+2}(\tau)) \\ &\cong \mathrm{Hom}_{n+1}(P_{n+1}(\mu'), \mathrm{res}_{n+2}^{\lambda'} P_{n+2}(\tau)) \\ &\cong \mathrm{Hom}_{n+1}(P_{n+1}(\mu'), P_{n+1}(\tau'))\end{aligned}$$

Next we will show that these isomorphisms are also compatible with the multiplicative structure in each of our algebras.

Let P, Q, and R be indecomposable projectives for A_{n+2} labelled by elements from $\mathcal{B}_n(\lambda)$. Then there exist indecomposable projectives P', Q' and R' labelled by elements in $\mathcal{B}_{n+1}(\lambda')$ such that

$$P = \mathrm{ind}_{n+1}^{\lambda} P' \qquad Q = \mathrm{ind}_{n+1}^{\lambda} Q' \qquad Q' = \mathrm{res}_{n+2}^{\lambda'} Q \qquad R' = \mathrm{res}_{n+2}^{\lambda'} R.$$

An isomorphism α giving a Frobenius reciprocity of the form

$$\mathrm{Hom}_{n+1}(M, \mathrm{res}_{n+2} N) \cong \mathrm{Hom}_{n+2}(\mathrm{ind}_{n+1} M, N)$$

is given by the map taking ϕ to $\alpha(\phi)$ where

$$\alpha(\phi)(a \otimes m) = a\phi(m)$$

for all $a \in A_{n+2}$ and $m \in M$, and extending by linearity. (Recall that ind_{n+1} is just the function $A_{n+2} \otimes_{A_{n+1}} -$.) Given

$$\begin{array}{ccccc}
\mathrm{Hom}_{n+2}(P,Q) & \times & \mathrm{Hom}_{n+2}(Q,R) & \longrightarrow & \mathrm{Hom}_{n+2}(P,R) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\mathrm{Hom}_{n+2}(\mathrm{ind}_{n+1}^{\lambda} P', \mathrm{ind}_{n+1}^{\lambda} Q') & \times & \mathrm{Hom}_{n+2}(\mathrm{ind}_{n+1}^{\lambda} Q', R) & \to & \mathrm{Hom}_{n+2}(\mathrm{ind}_{n+1}^{\lambda} P', R) \\
\alpha\uparrow & & \alpha\uparrow & & \alpha\uparrow \\
\mathrm{Hom}_{n+1}(P', \mathrm{res}_{n+2}^{\lambda'} \mathrm{ind}_{n+1}^{\lambda} Q')) & \times & \mathrm{Hom}_{n+1}(Q' \mathrm{res}_{n+2}^{\lambda'} R) & \longrightarrow & \mathrm{Hom}_{n+1}(P' \mathrm{res}_{n+2}^{\lambda'} R) \\
\cong\uparrow & & \cong\uparrow & & \cong\uparrow \\
\mathrm{Hom}_{n+1}(P', Q') & \times & \mathrm{Hom}_{n+1}(Q', R') & \longrightarrow & \mathrm{Hom}_{n+1}(P', R') \\
\phi & & \psi & & \psi\circ\phi
\end{array} \tag{15}$$

we need to check that $\alpha(\psi \circ \phi) = \alpha(\psi) \circ \alpha(\phi)$. We have

$$\alpha(\phi)(\sum_i a_i \otimes p_i) = \sum_i a_i \phi(p_i)$$

where $a_i \in A_{n+2}$ and $p_i \in P'$. As $\phi(p_i) \in Q' \cong \mathrm{res}_{n+2}^{\lambda'}(\mathrm{ind}_{n+1}^{\lambda} Q')$ we have

$$\phi(p_i) = \sum_j a'_j \otimes q_j$$

where $a'_j \in A_{n+2}$ and $q_j \in Q'$. Now

$$\begin{aligned}(\alpha(\psi) \circ \alpha(\phi))(\sum_i a_i \otimes p_i) &= \alpha(\psi)\Big(\sum_i a_i \big(\sum_j a'_j \otimes q_j\big)\Big) \\ &= \alpha(\psi)\Big(\sum_{i,j} a_i a'_j \otimes q_j\Big) = \sum_{i,j} a_i a'_j \psi(q_j)\end{aligned}$$

where the second equality follows from the action of A_{n+2} on $\mathrm{ind}_{n-1} Q'$. On the other hand

$$\begin{aligned}\alpha(\psi\circ\phi)(\sum_i a_i\otimes p_i) &= \sum_i a_i(\psi\circ\phi)(p_i)\\ &= \sum_i a_i\psi(\sum_j a'_j\otimes q_j)\\ &= \sum_i a_i\sum_j a'_j\psi(q_j)=\sum_{i,j}a_ia'_j\psi(q_j)\end{aligned}$$

and hence $\alpha(\psi\circ\phi)=\alpha(\psi)\circ\alpha(\phi)$ as required. □

Next we will consider how we can relate the cohomology of $A_{n+1,\Gamma}(\lambda')$ to that of $A_{n+1}(\lambda')$ and hence compare the cohomology of $A_n(\lambda)$ with that of $A_{n+1}(\lambda')$. We say that a subset $\Gamma\subset\mathcal{B}_n(\lambda)$ is *saturated* if $\mu\in\Gamma$ and $\nu\in\mathcal{B}_n(\lambda)$ with $\nu>\mu$ implies that $\nu\in\Gamma$. A subset $\Gamma\subset\mathcal{B}_n(\lambda)$ is *cosaturated* if $\mathcal{B}_n(\lambda)\backslash\Gamma$ is saturated.

Lemma 4.6. *The set $\Gamma=\theta(\mathcal{B}_n(\lambda))$ is cosaturated in $\mathcal{B}_{n+1}(\lambda')$.*

Proof. We need to show that if $\mu'\in\mathcal{B}_{n+1}(\lambda')\backslash\Gamma$ and $\nu'\in\mathcal{B}_{n+1}(\lambda')$ with $\nu'>\mu'$ then $\nu'\in\mathcal{B}_{n+1}(\lambda')\backslash\Gamma$. Suppose for a contradiction that $\nu'\in\Gamma$. Then $\nu=\theta^{-1}(\nu')\in\mathcal{B}_n(\lambda)$ and $\mu=\theta^{-1}(\mu')\notin\mathcal{B}_n(\lambda)$. As $\mu\in\mathcal{B}(\lambda)$ we must have $|\mu|\geq n+2$, and as $\mu'\in\mathrm{supp}(\mu)$ we have

$$|\mu'|\geq n+2\pm 1\geq n+1.$$

Now $|\nu|\leq n$ and so

$$|\nu'|\leq n\pm 1\leq n+1$$

but this contradicts the assumption that $\mu'<\nu'$. □

If $\Gamma\subset\mathcal{B}_n(\lambda)$ is cosaturated then by [Don98, A.3.11] the algebra $A_{n,\Gamma}(\lambda)$ is quasihereditary with standard modules given by

$$\{e_{n,\Gamma}\Delta_n(\mu):\mu\in\Gamma\}.$$

Moreover if X is any A_n-module having a Δ-filtration with factors $\Delta_n(\mu)$ for $\mu\in\Gamma$, and Y is any A_n-module, then for all $i\geq 0$ we have [Don98, A.3.13]

$$\mathrm{Ext}^i_n(X,Y)=\mathrm{Ext}^i_{A_n(\lambda)}(X,Y)\cong\mathrm{Ext}^i_{A_{n,\Gamma}(\lambda)}(e_{n,\Gamma}X,e_{n,\Gamma}Y).$$

Combining the above remarks with Theorem 4.5 and Lemma 4.6 we obtain

Corollary 4.7. *If $\lambda\in\Lambda_n$ and $\lambda'\in\Lambda_{n+1}$ are translation equivalent then for all $i\geq 0$ and for all $\mu\in\mathcal{B}_n(\lambda)$ we have*

$$\mathrm{Ext}^i_n(\Delta_n(\lambda),\Delta_n(\mu))\cong\mathrm{Ext}^i_{n+1}(\Delta_{n+1}(\lambda'),\Delta_{n+1}(\mu')).$$

We will say that two weights λ and λ' are in the same *translation class* if they are related by the equivalence relation generated by translation equivalence. Then analogues of (10), (11), Theorem 4.5 and Corollary 4.7 also hold for weights in the same translation class.

In Lie theory one can consider translation between two weights in the same facet (corresponding to the case considered above), or from one facet to another. We will now give an analogue of translation onto a wall for a tower of recollement.

We will say that λ' *separates* λ^- and λ^+ if $\lambda^-\neq\lambda^+$ and

(i) The weight λ' is the unique element of $\mathcal{B}(\lambda') \cap \operatorname{supp}(\lambda^-)$.
(ii) The weight λ' is the unique element of $\mathcal{B}(\lambda') \cap \operatorname{supp}(\lambda^+)$.
(iii) The weights λ^+ and λ^- are the unique pair of elements of $\mathcal{B}(\lambda^-) \cap \operatorname{supp}(\lambda')$.

Whenever we consider a pair of weights λ^- and λ^+ separated by λ' we shall always assume that $\lambda^- < \lambda^+$.

Theorem 4.8. *(i) If $\lambda' \in \Lambda_{n-1}$ separates λ^- and λ^+ then*
$$\operatorname{res}_n^{\lambda'} L_n(\lambda^+) \cong L_{n-1}(\lambda').$$
(ii) If further we have $\operatorname{Hom}(\Delta_n(\lambda^+), \Delta_n(\lambda^-)) \neq 0$ then
$$\operatorname{res}_n^{\lambda'} L_n(\lambda^-) = 0$$
and $\operatorname{ind}_{n-1}^{\lambda^-} \Delta_n(\lambda')$ is a nonsplit extension of $\Delta_n(\lambda^-)$ by $\Delta_n(\lambda^+)$ and has simple head $L_n(\lambda^+)$.

Proof. Arguing as in the proof of Proposition 4.1 we see that $\operatorname{res}_n^{\lambda'} L_n(\lambda^i)$ is either 0 or has simple head $L_{n-1}(\lambda')$ for $i \in \{\pm\}$. Also, any other composition factors $L_{n-1}(\tau')$ of $\operatorname{res}_n^{\lambda'} L_n(\lambda^i)$ must satisfy $\tau' > \lambda'$. Note that by assumption we have a short exact sequence
$$0 \longrightarrow \Delta_n(\lambda^-) \longrightarrow \operatorname{ind}_{n-1}^{\lambda^-} \Delta_{n-1}(\lambda') \longrightarrow \Delta_n(\lambda^+) \longrightarrow 0 \tag{16}$$
and hence
$$\operatorname{Hom}(\Delta_{n-1}(\tau'), \operatorname{res}_n^{\lambda'} L_n(\lambda^i)) \cong \operatorname{Hom}(\operatorname{ind}_{n-1}^{\lambda^-} \Delta_{n-1}(\tau'), L_n(\lambda^i)) \tag{17}$$
is non-zero when $i = +$ and $\tau' = \lambda'$, and is zero when $\tau' > \lambda'$ by our assumptions. This completes the proof of (i).

Now suppose that $\operatorname{Hom}(\Delta_n(\lambda^+), \Delta_n(\lambda^-)) \neq 0$. Then we have that
$$[\Delta_n(\lambda^-) : L_n(\lambda^+)] \neq 0.$$
By exactness and the first part of the Theorem, the unique copy of $L_{n-1}(\lambda')$ in
$$\operatorname{res}_n^{\lambda'} \Delta_n(\lambda^-) \cong \Delta_{n-1}(\lambda')$$
must come from $\operatorname{res}_n^{\lambda'} L_n(\lambda^+)$, and hence $\operatorname{res}_n^{\lambda'} L_n(\lambda^-)$ cannot have simple head $L_{n-1}(\lambda')$. But this implies by the first part of the proof that $\operatorname{res}_n^{\lambda'} L_n(\lambda^-) = 0$. Therefore the Hom-space in (17) must be zero when $\tau' = \lambda'$ and $i = 1$, which implies that (16) is a non-split extension whose central module has simple head $L_n(\lambda^+)$ as required. □

Suppose that λ' and λ^+ are weights with $\lambda' < \lambda^+$ and $\lambda' \in \operatorname{supp}(\lambda^+)$ such that for every weight $\tau' \in \mathcal{B}(\lambda')$ either (i) there is a unique weight $\tau^+ \in \mathcal{B}(\lambda^+) \cap \operatorname{supp}(\tau')$ and τ' is the unique weight in $\mathcal{B}(\lambda') \cap \operatorname{supp}(\tau^+)$, or (ii) there exists $\tau^-, \tau^+ \in \mathcal{B}(\lambda^+)$ such that τ' separates τ^- and τ^+. Then we say that λ' is in the *lower closure* of λ^+. If further
$$\operatorname{Hom}(\Delta_n(\tau^+), \Delta_n(\tau^-)) \neq 0$$
whenever $\tau' \in \mathcal{B}(\lambda')$ separates τ^- and τ^+ in $\mathcal{B}(\lambda^+)$ then we shall say that $\mathcal{B}(\lambda^+)$ has *enough local homomorphisms* with respect to $\mathcal{B}(\lambda')$.

Proposition 4.9. *Suppose that $\lambda' \in \Lambda_{n-1}$ is in the lower closure of $\lambda^+ \in \Lambda_n$, and that $\mathcal{B}(\lambda^+)$ has enough local homomorphisms with respect to $\mathcal{B}(\lambda')$. Then*
$$[\Delta_{n-1}(\lambda') : L_{n-1}(\mu')] = [\Delta_n(\lambda^+) : L_n(\mu^+)].$$

Proof. We have by our assumptions that
$$\mathrm{res}_n^{\lambda'} \Delta_n(\lambda^+) \cong \Delta_{n-1}(\lambda').$$
As $\mathrm{res}_n^{\lambda'}$ is an exact functor, it is enough to determine its effect on simples $L_n(\mu^+)$ in $\Delta_n(\lambda^+)$. If there exists μ' separating μ^+ from μ^- then the result follows from Theorem 4.8, while if μ^+ is the only element in $\mathcal{B}(\lambda^+) \cap \mathrm{supp}(\mu')$ then if follows as in the proof of Proposition 4.1. □

Thus, as long as there are enough local homomorphisms, the decomposition numbers for $\Delta_n(\lambda)$ determine those for all weights in the lower closure of λ.

We can generalise the results of this section up to Corollary 4.7 by replacing res_n^λ and ind_n^λ by any pair of functor families R_n and I_n with the following properties.

Let A be a tower of recollement, with $\lambda, \lambda' \in \Lambda$ having corresponding blocks $\mathcal{B}(\lambda)$ and $\mathcal{B}(\lambda')$, and fix $i \in \mathbb{N}$. Suppose that we have functors
$$R_n : A_n\text{-mod} \to A_{n-i}\text{-mod}$$
for $n \geq i$ and
$$I_n : A_n\text{-mod} \to A_{n+i}\text{-mod}$$
for $n \geq 0$ satisfying

(i) The functor I_n is left adjoint to R_{n+i} for all n.
(ii) The functor R_n is exact and I_n is right exact for all n where they are defined.
(iii) There is a bijection $\theta : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda')$ taking μ to μ' such that for all $n \geq i$, if $\mu \in \mathcal{B}_n(\lambda)$ and $\mu' \in \mathcal{B}_{n-i}(\lambda')$ then
$$R_n\Delta_n(\mu) \cong \Delta_{n-i}(\mu') \qquad \text{and} \qquad I_{n-i}\Delta_{n-i}(\mu') = \Delta_n(\mu)$$
and $R_n\Delta_n(\mu) = 0$ otherwise.
(iv) If $\Gamma_n = \theta(\mathcal{B}_n(\lambda)) \subset \mathcal{B}_m(\lambda')$ for some m then Γ_n is cosaturated in $\mathcal{B}_m(\lambda')$.
(v) There exists $t \in \mathbb{N}$ such that for all n and for all $\mu \in \mathcal{B}_{n-t}(\lambda)$ the module $R_nP_n(\mu)$ is projective.

Then we say that λ and λ' are (R, I)-*translation equivalent*. In this case the proofs of Propositions 4.1 and 4.2 go through essentially unchanged, and we get

Theorem 4.10. *Suppose that λ and λ' are (R, I)-translation equivalent and $n \geq i$. Then for all $\mu \in \mathcal{B}_n(\lambda)$ with $\mu' \in \mathcal{B}_{n-i}(\lambda')$ we have*
$$R_nL_n(\mu) \cong L_{n-i}(\mu'), \qquad I_{n-i}L_{n-i}(\mu') \cong L_n(\mu) \qquad \text{and} \qquad I_{n-i}P_{n-i}(\mu') \cong P_n(\mu)$$
and if $\mu \in \mathcal{B}_{n-t}(\lambda)$ then
$$R_nP_n(\mu) \cong P_{n-i}(\mu').$$
Moreover, if the adjointness isomorphism
$$\alpha : \mathrm{Hom}_{n-i}(M, R_n(N)) \to \mathrm{Hom}_n(I_{n-i}(M), N)$$
is multiplicative (i.e. makes the diagram (15) commute) then there is a Morita equivalence between $A_n(\lambda)$ and $A_{n+i,\Gamma_{n+i}}(\lambda')$ and for all $\mu, \tau \in \mathcal{B}_n(\lambda)$ and $j \geq 0$ we have
$$\mathrm{Ext}_n^j(\Delta_n(\mu), \Delta_n(\tau)) \cong \mathrm{Ext}_{n+i}^j(\Delta_{n+i}(\mu'), \Delta_{n+i}(\tau')).$$

5. A generalised restriction/induction pair

We wish to show (in Section 6) that two weights in the same facet for the Brauer algebra give rise to Morita equivalent blocks (at least when we truncate the blocks to have the same number of simples). However, the usual induction and restriction functors are not sufficient to show this except in the alcove case. To remedy this, in this section we will consider a variation on the usual induction and restriction functors.

First consider B_2 with $\delta \neq 0$. It is easy to see that this is a semisimple algebra, with a decomposition

$$1 = e + e^- + e^+$$

of the identity into primitive orthogonal idempotents given by the elements in Figure 1.

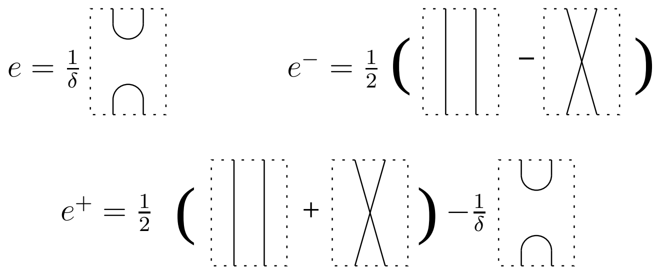

Figure 1. Idempotents in B_2

There are three standard modules for this algebra, which we will denote by

$$\begin{aligned} & \Delta_2(0) &= < e > \\ S^- = \ & \Delta_2(1,1) &= < e^- > \\ S^+ = \ & \Delta_2(2) &= < e^+ > \end{aligned}$$

For $n \geq 2$ consider the subalgebra $B_{n-2} \otimes B_2 \subseteq B_n$ obtained by letting B_{n-2} act on the leftmost $n-2$ lines and B_2 act on the rightmost pair of lines. We will view elements of B_{n-2} and B_2 as elements of B_n via this embedding. Note that under this embedding the two algebras obviously commute with each other.

In particular, for any B_n-module M the vector spaces $e^\pm M$ are B_{n-2}-modules. Thus we have a pair of functors $\mathrm{res}_n^\pm$ from B_n-mod to B_{n-2}-mod given on objects by the map $M \longmapsto e^\pm M$. Note that these functors can also be defined as

$$\mathrm{res}_n^\pm M = e^\pm \,\mathrm{res}^{B_n}_{B_{n-2}\otimes B_2} M.$$

We have

$$\begin{aligned} \mathrm{Hom}_{B_n}(\mathrm{ind}^{B_n}_{B_{n-2}\otimes B_2}(N \boxtimes S^\pm), M) &\cong \mathrm{Hom}_{B_{n-2}\otimes B_2}(N \boxtimes S^\pm, \mathrm{res}^{B_n}_{B_{n-2}\otimes B_2} M) \\ &\cong \mathrm{Hom}_{B_{n-2}}(N, e^\pm \,\mathrm{res}^{B_n}_{B_{n-2}\otimes B_2} M) \end{aligned}$$

and so the functors $\mathrm{ind}^\pm_{n-2}$ from B_{n-2}-mod to B_n-mod given by

$$\mathrm{ind}^\pm_{n-2} N = \mathrm{ind}^{B_n}_{B_{n-2}\otimes B_2}(N \boxtimes S^\pm)$$

are left adjoint to $\mathrm{res}_n^\pm$.

Lemma 5.1. *Let N be a B_{n-2}-module. Then we have*

$$\operatorname{ind}_{n-2}^{\pm} N \cong B_n e^{\pm} \otimes_{B_{n-2}} N$$

as B_n-modules, where the action on the right-hand space is by left multiplication in B_n.

Proof. Define a map

$$\phi : B_n \otimes_{B_{n-2}\otimes B_2} (N \boxtimes S^{\pm}) \to B_n e^{\pm} \otimes_{B_{n-2}} N$$

by

$$b \otimes (n \otimes e^{\pm}) \longmapsto be^{\pm} \otimes n.$$

We first show that this is well-defined. Let $b = b' b_{n-2} b_2$ for some $b_{n-2} \in B_{n-2}$ and $b_2 \in B_2$. Then

$$\begin{aligned} \phi(b \otimes (n \otimes e^{\pm}) - b' \otimes (b_{n-2} n \otimes b_2 e^{\pm})) &= b' b_{n-2} b_2 e^{\pm} \otimes n - b' b_2 e^{\pm} \otimes b_{n-2} n \\ &= b' b_2 e^{\pm} b_{n-2} \otimes n - b' b_2 e^{\pm} \otimes b_{n-2} n = 0 \end{aligned}$$

as required. The map ϕ is clearly a B_n-homomorphism. We also have a map

$$\psi : B_n e^{\pm} \otimes_{B_{n-2}} N \to B_n \otimes_{B_{n-2}\otimes B_2} (N \boxtimes S^{\pm})$$

given by

$$be^{\pm} \otimes n \longmapsto be^{\pm} \otimes (n \otimes e^{\pm}).$$

It is easy to check that ψ is well-defined and that $\psi\phi = \mathrm{id}$ and $\phi\psi = \mathrm{id}$. ☐

Let $e_{n,4}$ be the idempotent in B_n shown in Figure 2.

$\frac{1}{\delta^2}$

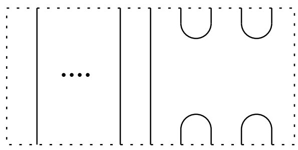

Figure 2. An idempotent in B_n

Lemma 5.2. *As left B_n- and right B_{n-2}-modules we have*

$$e^{\pm} B_{n+2} e_{n+2,4} \cong B_n e^{\pm}.$$

Proof. Consider the map from $e^{\pm} B_{n+2} e_{n+2,4}$ to $B_n e^{\pm}$ given on diagrams as shown in Figure 3. The grey shaded regions show the actions of B_n from above, of B_{n-2} from below, and the dark shaded region the action of the element $e^{\pm}$. All lines in the diagrams except those indicated remain unchanged; the two southern arcs in the left-hand diagram are removed, and the ends of the pair of lines acted on by $e^{\pm}$ are translated clockwise around the boundary from the northern to the southern side. This gives an isomorphism of vector spaces, and clearly preserves the actions of B_n and B_{n-2}. ☐

Corollary 5.3. *The module $\operatorname{res}_n^{\pm}(P_n(\lambda))$ is projective for all $\lambda \in \Lambda_{n-4}$.*

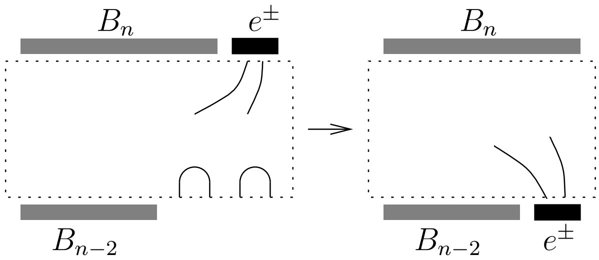

FIGURE 3. Realising the isomorphism between $e^{\pm}B_{n+2}e_{n+2,4}$ and $B_n e^{\pm}$

Proof. First note that $B_n e_{n,4}$ is a projective B_n-module. Moreover, as $e_{n,4}B_n e_{n,4} \cong B_{n-4}$ we have that $B_n e_{n,4}$ contains precisely the indecomposable projectives labelled by elements of Λ_{n-4}. By Lemma 5.2 we have that $e^{\pm}B_n e_{n,4} \cong B_{n-2}e^{\pm}$ as left B_{n-2}-modules, and hence $\mathrm{res}_n^{\pm}(P_n(\lambda))$ is projective for all $\lambda \in \Lambda_{n-4}$. □

Corollary 5.4. *We have an isomorphism of functors*

$$\mathrm{ind}_n^{\pm} \cong \mathrm{res}_{n+4}^{\pm} G_{n+2}G_n.$$

Proof. By the definition of G_n and G_{n+2} we have

$$\begin{aligned} \mathrm{res}_{n+4}^{\pm} G_{n+2}G_n(N) &= \mathrm{res}_{n+4}^{\pm}(B_{n+4}e_{n+4,2} \otimes_{B_n} N) \\ &= e^{\pm}B_{n+4}e_{n+4,4} \otimes_{B_n} N \\ &\cong B_{n+2}e^{\pm} \otimes_{B_n} N \end{aligned}$$

where the final isomorphism follows from Lemma 5.2. But by Lemma 5.1 this final module is isomorphic to $\mathrm{ind}_n^{\pm} N$. □

Corollary 5.4 is an analogue of the relation between induction, restriction and globalisation in [CDM09a, Lemma 2.6(ii)], corresponding to axiom (A4) for a tower of recollement.

Given two partitions λ and μ, we write $\lambda \rhd\rhd^{+} \mu$, or $\mu \lhd\lhd^{+} \lambda$, if μ can be obtained from λ by removing two boxes and λ/μ is not the partition $(1,1)$. Similarly we write $\lambda \rhd\rhd^{-} \mu$, or $\mu \lhd\lhd^{-} \lambda$ if μ can be obtained from λ by removing two boxes and $\lambda/\mu \neq (2)$. We will write $\mu \rhd\lhd \lambda$ if μ is obtained from λ by removing a box and then adding a box. Finally, let $r(\lambda)$ denote the number of removable boxes in λ.

The next theorem describes the structure of $\mathrm{res}_n^{\pm} \Delta_n(\lambda)$, and so is an analogue of the usual induction and restriction rules in [DWH99, Theorem 4.1 and Corollary 6.4] (and use the same strategy for the proof).

Theorem 5.5. *Suppose that λ is a partition of $m = n - 2t$ for some $t \geq 0$.*
(i) There is a filtration of B_{n-2}-modules

$$W_0 \subseteq W_1 \subseteq W_2 = \mathrm{res}_n^{\pm} \Delta_n(\lambda)$$

with

$$W_0 \cong \bigoplus_{\mu \lhd\lhd^{\pm} \lambda} \Delta_{n-2}(\mu) \qquad\qquad W_2/W_1 \cong \bigoplus_{\mu \rhd\rhd^{\pm} \lambda} \Delta_{n-2}(\mu)$$

and

$$W_1/W_0 \cong \Delta_{n-2}(\lambda)^{r(\lambda)} \oplus \bigoplus_{\substack{\mu \rhd \lhd \lambda \\ \mu \neq \lambda}} \Delta_{n-2}(\mu)$$

where any $\Delta_{n-2}(\mu)$ *which does not make sense is taken as* 0.
(ii) There is a filtration of B_{n+2}*-modules*

$$U_0 \subseteq U_1 \subseteq U_2 = \mathrm{ind}_n^{\pm} \Delta_n(\lambda)$$

with

$$U_0 \cong \bigoplus_{\mu \lhd \lhd^{\pm} \lambda} \Delta_{n+2}(\mu) \qquad U_2/U_1 \cong \bigoplus_{\mu \rhd \rhd^{\pm} \lambda} \Delta_{n+2}(\mu)$$

and

$$U_1/U_0 \cong \Delta_{n+2}(\lambda)^{r(\lambda)} \oplus \bigoplus_{\substack{\mu \rhd \lhd \lambda \\ \mu \neq \lambda}} \Delta_{n+2}(\mu)$$

Proof. Part (ii) follows from part (i) by Corollary 5.4. For the rest of the proof we will work with the concrete realisation of standard modules given in Section 2. By definition we have

$$\mathrm{res}_n^{\pm} \Delta_n(\lambda) = e^{\pm} I_n^t \otimes_{\Sigma_m} S^{\lambda}$$

and we will represent an element $e^{\pm} X_{w,1,id} \otimes x$ in this space diagrammatically as shown in Figure 4.

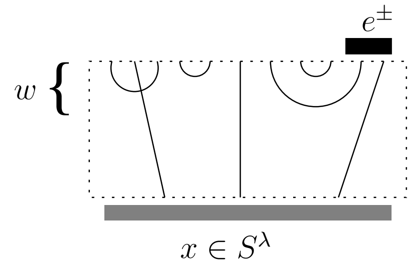

FIGURE 4. Representing the element $e^{\pm} X_{w,1,id} \otimes x$ in $e^{\pm} I_n^t \otimes_{\Sigma_m} S^{\lambda}$

We are now in a position to define the various spaces W_0, W_1, and W_2. Choose a fixed basis $V(\lambda)$ for S^{λ} and set

$$V_{n,t}^0 = \{w \in V_{n,t} : n-1 \text{ and } n \text{ are free in } w\}$$

$$V_{n,t}^1 = \{w \in V_{n,t} : n-1 \text{ is on an arc and } n \text{ is free in } w\}$$

$$V_{n,t}^2 = \{w \in V_{n,t} : n-1 \text{ is linked to } j \text{ and } n \text{ is linked to } i \text{ in } w \text{ with } i < j \leq n-2\}.$$

Then for $0 \leq i \leq 2$ we set

$$W_i = \mathrm{span}\{e^{\pm} X_{w,1,id} \otimes x : w \in V_{n,t}^j \text{ with } j \leq i \text{ and } x \in V(\lambda)\}$$

Note that if $w \in V_{n,t}^1 \cup V_{n,t}^2$ and w' is obtained from w by swapping nodes $n-1$ and n, then

$$e^{\pm} X_{w,1,id} = \pm e^{\pm} X_{w',1,id}.$$

Moreover, if there is an arc linking nodes $n-1$ and n in w then

$$e^{\pm}X_{w,1,id} = 0.$$

Thus we have that

$$W_2 = e^{\pm}\Delta_n(\lambda)$$

and W_0 and W_1 are submodules of $e^{\pm}\Delta_n(\lambda)$.

We first show that

$$W_0 \cong I^t_{n-2} \otimes_{\Sigma_{m-2}} \sigma^{\pm}S^{\lambda} \tag{18}$$

where $\sigma^{\pm}$ represents the symmetriser/antisymmetriser on the last two lines in Σ_m and $\Sigma_{m-2} \subset \Sigma_m$ acts on the first $m-2$ lines. Note that

$$\begin{aligned} \sigma^{\pm}S^{\lambda} &= \sigma^{\pm}\operatorname{res}^{\Sigma_m}_{\Sigma_{m-2}\times\Sigma_2} S^{\lambda} \\ &= \sigma^{\pm}\left(\bigoplus_{\mu\vdash m-2,\ \nu\vdash 2} c^{\lambda}_{\mu,\nu}(S^{\mu}\boxtimes S^{\nu})\right) = \bigoplus_{\mu\vdash m-2} c^{\lambda}_{\mu,*}S^{\mu} \end{aligned}$$

where $*$ equals (2) for σ^+ and $(1,1)$ for σ^-. As

$$c^{\lambda}_{\mu,(2)} = \begin{cases} 1 & \text{if } \mu \lhd \lhd^{+}\lambda \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad c^{\lambda}_{\mu,(1,1)} = \begin{cases} 1 & \text{if } \mu \lhd \lhd^{-}\lambda \\ 0 & \text{otherwise} \end{cases}$$

it will follow from (1) that

$$W_0 \cong \bigoplus_{\mu\lhd\lhd^{\pm}\lambda} \Delta_{n-2}(\mu)$$

as required.

Note that for $w \in V^0_{n,t}$ the lines from $n-1$ and n are propagating in $X_{w,1,id}$, and so we have

$$e^{\pm}X_{w,1,id}\otimes S^{\lambda} = X_{w,1,id}e^{\pm}\otimes S^{\lambda} = X_{w,1,id}\otimes\sigma^{\pm}S^{\lambda}.$$

For $w \in V^0_{n,t}$ define $\overline{w} \in V_{n-2,t}$ by removing nodes $n-1$ and n, and a map

$$\phi_0 : W_0 \longrightarrow I^t_{n-2}\otimes_{\Sigma_{m-2}}\sigma^{\pm}S^{\lambda}$$

by

$$e^{\pm}X_{w,1,id}\otimes x = X_{w,1,id}\otimes\sigma^{\pm}x \longmapsto X_{\overline{w},1,id}\otimes\sigma^{\pm}x.$$

It is clear that ϕ_0 is an isomorphism of vector spaces, and commutes with the action of B_{n-2}. This proves (18).

Next we will show that

$$W_1/W_0 \cong I^{t-1}_{n-2}\otimes_{\Sigma_m}\operatorname{ind}^{\Sigma_m}_{\Sigma_{m-1}}\operatorname{res}^{\Sigma_m}_{\Sigma_{m-1}} S^{\lambda}. \tag{19}$$

Note that

$$\operatorname{ind}^{\Sigma_m}_{\Sigma_{m-1}}\operatorname{res}^{\Sigma_m}_{\Sigma_{m-1}} S^{\lambda} = \operatorname{ind}^{\Sigma_m}_{\Sigma_{m-1}}\big(\bigoplus_{\nu\lhd\lambda} S^{\nu}\big) = \bigoplus_{\nu\lhd\lambda}\big(\operatorname{ind}^{\Sigma_m}_{\Sigma_{m-1}} S^{\nu}\big) = (S^{\lambda})^{r(\lambda)} \oplus \bigoplus_{\substack{\mu\,\rhd\,\lhd\lambda \\ \mu\neq\lambda}} S^{\mu}$$

and so it will follow from (1) that

$$W_1/W_2 \cong \Delta_{n-2}(\lambda)^{r(\lambda)} \bigoplus_{\substack{\mu\,\rhd\,\lhd\lambda \\ \mu\neq\lambda}} \Delta_{n-2}(\mu)$$

as required.

We will need an explicit description of $\mathrm{ind}_{\Sigma_{m-1}}^{\Sigma_m} \mathrm{res}_{\Sigma_{m-1}}^{\Sigma_m} S^\lambda$. The quotient Σ_m/Σ_{m-1} has coset representatives

$$\{\tau_i = (i,m) : 1 \leq i \leq m\}$$

where $(m,m) = 1$. Therefore $\mathrm{ind}_{\Sigma_{m-1}}^{\Sigma_m} \mathrm{res}_{\Sigma_{m-1}}^{\Sigma_m} S^\lambda$ has a basis

$$\{(i,x) : 1 \leq i \leq m, x \in V(\lambda)\}$$

and the action of $\theta \in \Sigma_m$ is given by

$$\theta(i,x) = (j, \theta' x)$$

where $\theta\tau_i = \tau_j\theta'$ for a unique $1 \leq j \leq m$ and $\theta' \in \Sigma_{m-1}$.

For $1 \leq i \leq m$ set

$$\sigma_i = (i, m, m-1, m-2, \ldots, i+1)$$

and for $w \in V_{n,t}^1$ define $\overline{w} \in V_{n-2.t-1}$ by removing the nodes $n-1$ and n and removing the arc from $n-1$ (which will thus introduce a new free node elsewhere in $\overline{w}$). Now we can define a map

$$\phi_1 : W_1/W_0 \to I_{n-2}^{t-1} \otimes_{\Sigma_m} \mathrm{ind}_{\Sigma_{m-1}}^{\Sigma_m} \mathrm{res}_{\Sigma_{m-1}}^{\Sigma_m} S^\lambda$$

by

$$e^{\pm} X_{w,1,id} \otimes x \longmapsto X_{\overline{w},1,id}\sigma_i \otimes (m,x)$$

if node $n-1$ is linked to node i in w. This is illustrated graphically in Figure 5.

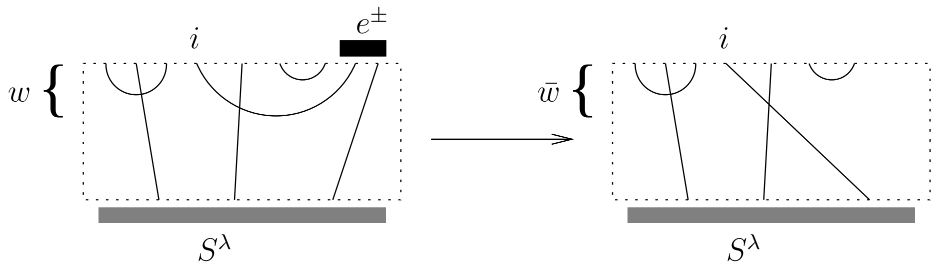

FIGURE 5. An example of the effect of the map ϕ_1

Note that for every $v \in V_{n-2,t-1}$ there are exactly m elements $w \in V_{n,t}^1$ satisfying $\overline{w} = v$, as $n-1$ can be joined to any of the m free vertices in v. Note also that

$$\sigma_i = (i,m)(m-1, m-2, \ldots, i+1, i) = (i,m)\sigma_i'$$

where $\sigma_i' \in \Sigma_{m-1}$, and so $\sigma_i(m,x) = (i, \sigma_i' x)$.

Given $v \in V_{n-2,t-1}$, $1 \leq i \leq m$, and $x \in V(\lambda)$ pick $w \in V_{n,t}^1$ with $\overline{w} = v$ and $n-1$ joined to the ith free node. Then

$$\begin{aligned} \phi_1(e^{\pm} X_{w,1,id} \otimes (\sigma_i')^{-1} x) &= X_{v,1,id}\sigma_i \otimes (m, (\sigma_i')^{-1} x) \\ &= X_{v,1,id} \otimes (i, \sigma_i'(\sigma_i')^{-1} x) = X_{v,1,id} \otimes (i,x) \end{aligned}$$

and so ϕ_1 is surjective. Moreover

$$\dim W_1/W_2 = m|V_{n-2,t-1}| \dim S^\lambda = \dim I_{n-2}^{t-1} \otimes \mathrm{ind}_{\Sigma_{m-1}}^{\Sigma_m} \mathrm{res}_{\Sigma_{m-1}}^{\Sigma_m} S^\lambda$$

and so ϕ_1 is an isomorphism of vector spaces. It remains to show that ϕ_1 commutes with the action of B_{n-2}.

First consider the action of $\tau \in \Sigma_{n-2}$. The actions of ϕ_1 and τ can be seen to commute by the schematic diagram in Figure 6, noting that $\tau(\overline{w}) = \overline{\tau(w)}$.

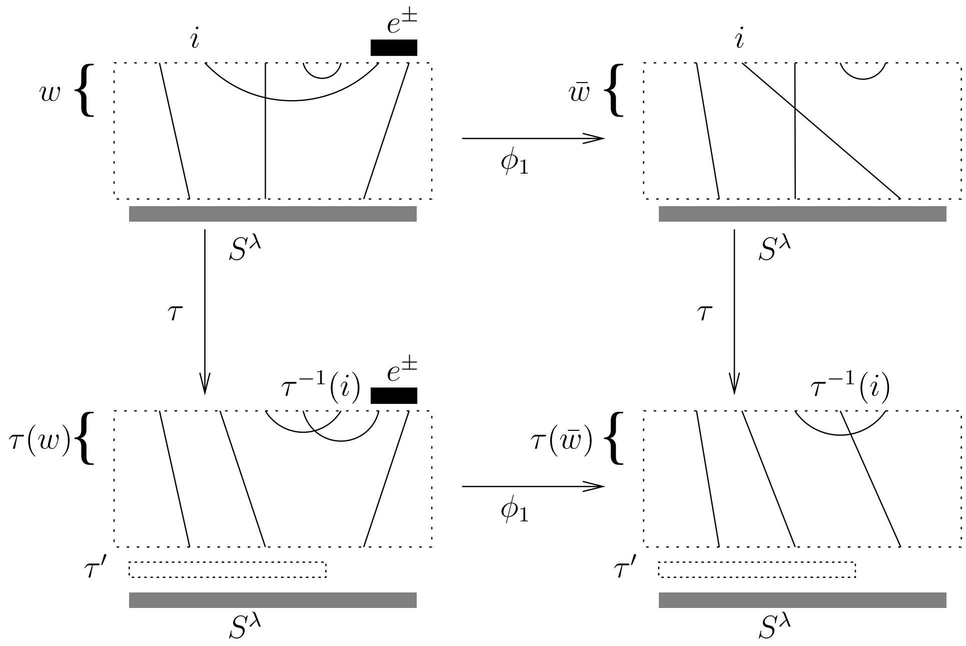

FIGURE 6. A diagrammatic illustration that $\phi_1\tau = \tau\phi_1$

Next consider the action of $X_{jk} \in B_{n-2}$. If $j, k \neq i$ then it is clear that X_{jk} commutes with ϕ_1. Now consider the action of X_{ij}. There are two cases: (i) j is a free node in w, and (ii) j is linked to some node k in w. Case (i) is illustrated schematically in Figure 7. The lower left diagram in Figure 7 represents 0 as it lies in W_0. The lower right diagram represents 0 as there is a decrease in the number of propagating lines. Therefore the dotted arrow is an equality and the diagram commutes.

Case (ii) is illustrated schematically in Figure 8. Again we see that X_{ij} commutes with the action of ϕ_1, and so we have shown that ϕ_1 is a B_{n-2}-isomorphism. This completes the proof of (19).

Finally, we shall show that

$$W_2/W_1 \cong I_{n-2}^{t-2} \otimes_{\Sigma_{m-2}} \mathrm{ind}_{\Sigma_m \times \Sigma_2}^{\Sigma_{m+2}}(S^{\lambda} \boxtimes S^{\pm}). \tag{20}$$

As for restriction we have

$$\mathrm{ind}_{\Sigma_m \times \Sigma_2}^{\Sigma_{m+2}}(S^{\lambda} \boxtimes S^{\pm}) \cong \bigoplus_{\mu \rhd\rhd^{\pm} \lambda} S^{\mu}$$

and so it will follow from (1) that

$$W_2/W_1 \cong \bigoplus_{\mu \rhd\rhd^{\pm} \lambda} \Delta_{n-2}(\mu)$$

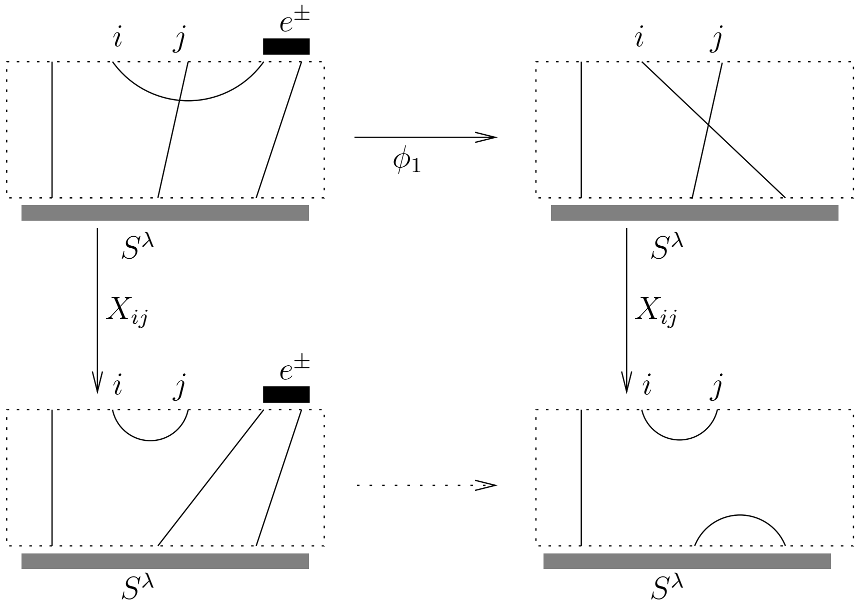

FIGURE 7. The action of X_{ij} and ϕ_1: case (i)

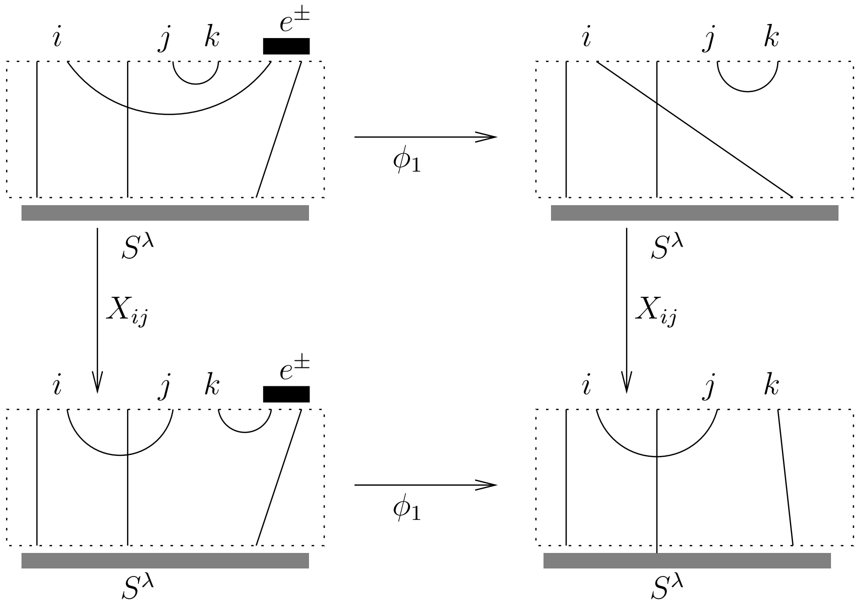

FIGURE 8. The action of X_{ij} and ϕ_1: case (ii)

which will complete the proof.

We will need an explicit description of $\mathrm{ind}_{\Sigma_m \times \Sigma_2}^{\Sigma_{m+2}}(S^\lambda \boxtimes S^\pm)$. The quotient $\Sigma_{m+2}/(\Sigma_m \times \Sigma_2)$ has coset representatives

$$\{\tau_{ij} = (i, m+1)(j, m+2) : 1 \leq i < j \leq m+2\}$$

where $(m+1,m+1)=(m+2,m+2)=1$. Therefore $\mathrm{ind}_{\Sigma_m\times\Sigma_2}^{\Sigma_{m+2}}(S^\lambda\boxtimes S^\pm)$ has a basis

$$\{(i,j;x\otimes\sigma^\pm):1\leq i<j\leq m+2, x\in V(\lambda)\}$$

and the action of $\theta\in\Sigma_{m+2}$ is given by

$$\theta(i,j;x\otimes\sigma^\pm)=(k,l;\theta'(x\otimes\sigma^\pm))$$

where $\theta\tau_{ij}=\tau_{kl}\theta'$ for a unique $1\leq k<l\leq m+2$ and $\theta'\in\Sigma_m\times\Sigma_2$.

For $1\leq r<s\leq m+2$ set

$$\sigma_{r,s}=(r,s,s-1,s-2,\dots,r+1)$$

and for $w\in V_{n,t}^2$ define $\overline{w}\in V_{n-2,t-2}$ by removing the nodes $n-1$ and n and removing the arcs from $n-1$ and n (which will thus introduce two new free nodes elsewhere in $\overline{w}$). Now we can define a map

$$\phi_2:W_2/W_1\longrightarrow I_{n-2}^{t-2}\otimes_{\Sigma_{m+2}}\mathrm{ind}_{\Sigma_m\times\Sigma_2}^{\Sigma_{m+2}}(S^\lambda\boxtimes S^\pm)$$

by

$$e^\pm X_{w,1,id}\otimes x\longmapsto X_{\overline{w},1,id}\sigma_{j,m+2}\sigma_{i,m+1}\otimes(m+1,m+2;,x\otimes\sigma^\pm)$$

if $n-1$ is linked to j and n is linked to i in w. This is illustrated graphically in Figure 9.

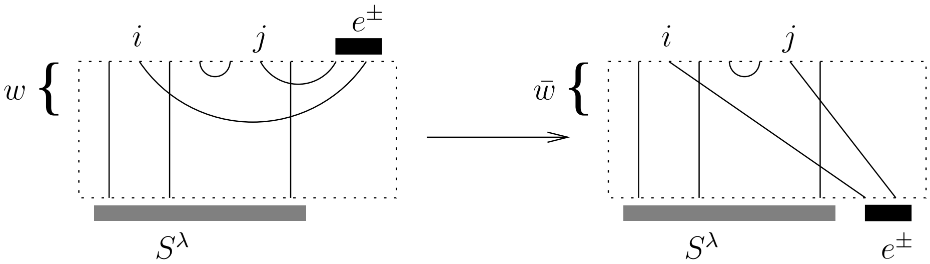

FIGURE 9. An example of the effect of the map ϕ_2

Arguing as for ϕ_1 we can show that ϕ_2 is an isomorphism of vector spaces. Thus we will be done if we can show that ϕ_2 commutes with the action of B_{n-2}.

First consider the action of $\tau\in\Sigma_{n-2}$. The actions of ϕ_2 and τ are illustrated schematically in Figure 10. Again we use the fact that $\tau(\overline{m})=\overline{\tau(w)}$, while in the bottom pair of diagrams we have used the action of $e^\pm$ on each side, which in each case gives a coefficient of ±1. We see that the actions of ϕ_2 does commute with τ as required.

It remains to check that ϕ_2 commutes with the action of $X_{kl}\in B_{n-2}$. If $\{k,l\}$ is disjoint from $\{i,j\}$ then it is clear that $X_{k,l}$ commutes with ϕ_2. If $k=i$ and $l=j$ it is easy to verify that

$$X_{ij}e^\pm X_{w,1,id}=0\quad\text{and}\quad X_{i,j}\phi_2(e^\pm X_{w,1,id}\otimes x)=0.$$

Thus we just have to check what happens when $k=i$ and $l\neq j$. There are two cases: (i) l is a free node in w, and (ii) l is linked to some node h in w.

Case (i) is illustrated schematically in Figure 11. The lower left diagram in Figure 11 represents 0 as it lies in W_1. The lower right diagram represents 0 as there is a decrease in

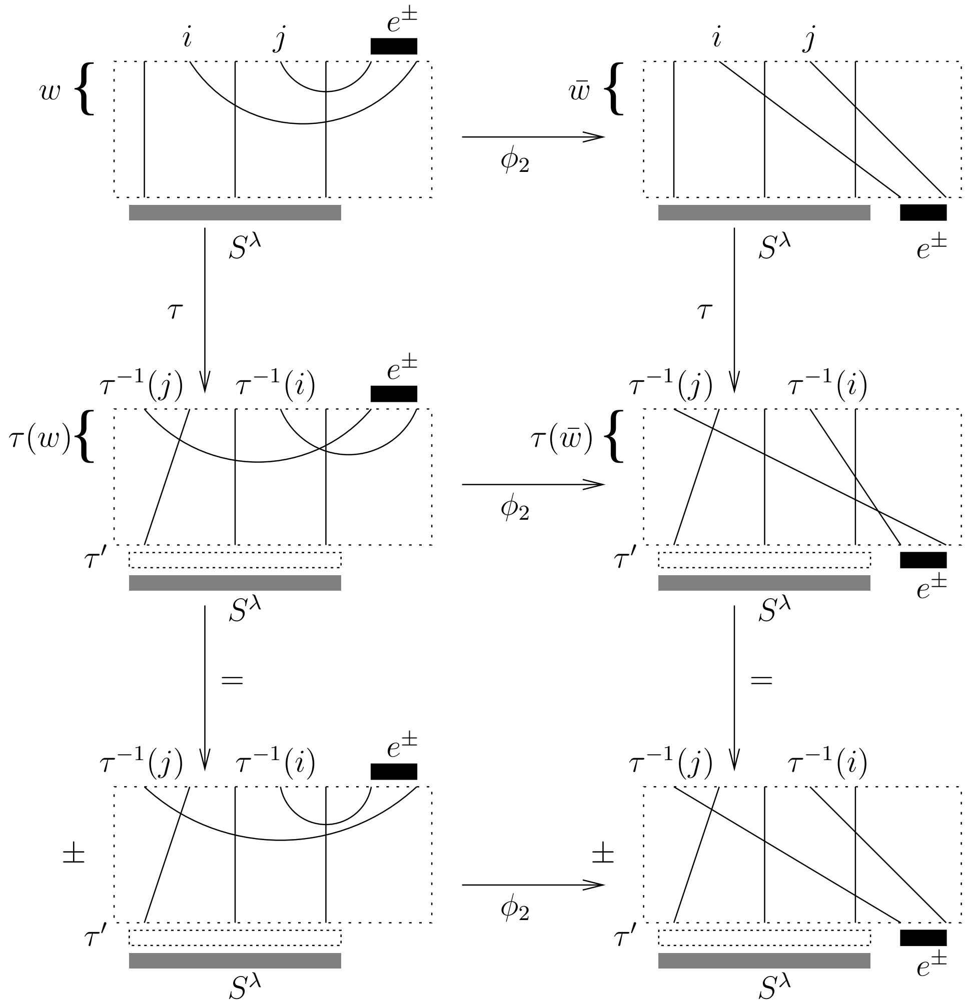

FIGURE 10. A diagrammatic illustration that $\phi_2\tau = \tau\phi_1$

the number of propagating lines. Therefore the dotted arrow is an equality and the diagram commutes.

Case (ii) is illustrated schematically in Figure 12. Again we see that X_{il} commutes with the action of ϕ_2, and so we are done. □

6. Translation equivalence for the Brauer algebra

In Section 4 we saw how translation equivalence of weights for a tower of recollement implies Morita equivalences of the corresponding blocks (when the blocks are truncated to contain the same number of simples). We will now reinterpret this in the language of alcove geometry in the case of the Brauer algebra.

Given a partition λ, we saw in (5) that the set $\mathrm{supp}(\lambda)$ consists of those partitions obtained from λ by the addition or subtraction of a box from λ, all with multiplicity one. We denote by $\lambda \pm \epsilon_i$ the composition obtained by adding/subtracting a box from row i of λ.

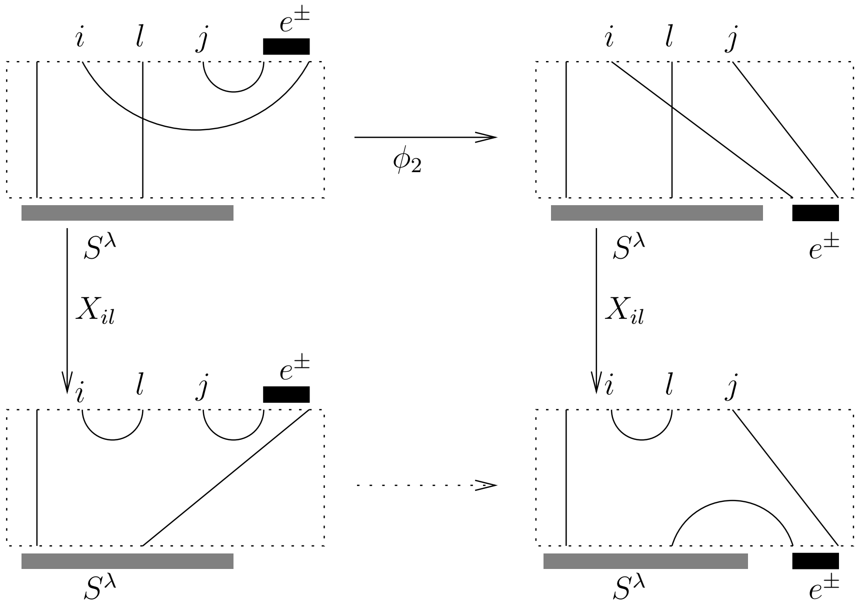

FIGURE 11. The action of X_{il} and ϕ_2: case (i)

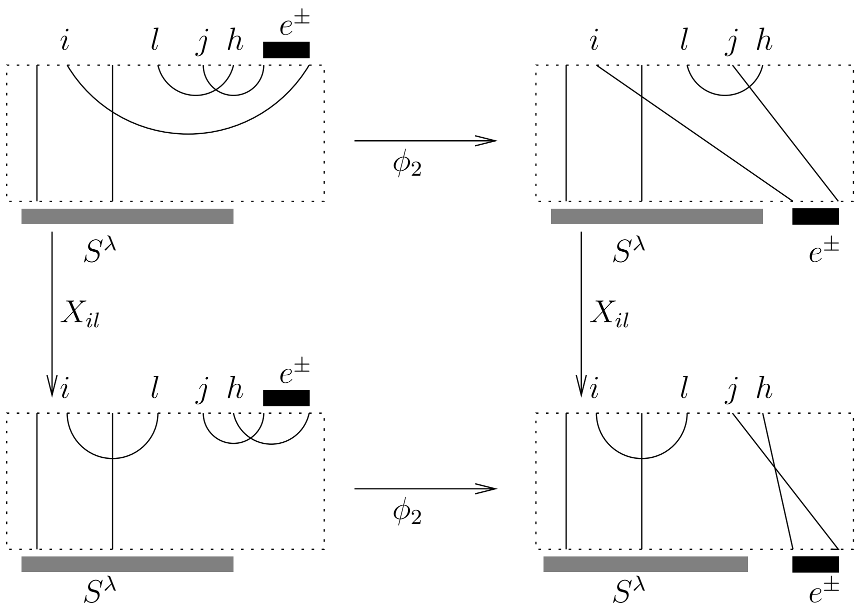

FIGURE 12. The action of X_{il} and ϕ_2: case (ii)

Lemma 6.1. *If $\lambda' = \lambda \pm \epsilon_i$ then there cannot exist a reflection hyperplane separating λ from λ'.*

Proof. Suppose that R is a reflection hyperplane between λ and λ', and denote their respective reflections by $r(\lambda)$ and $r(\lambda')$. Either λ is the reflection of λ' or the line from λ to λ' is not orthogonal to the hyperplane.

The former case is impossible as two weights differing by one box cannot be in the same block. For the latter case, note that the distance between λ and λ' is one. Therefore at least one of the distances from λ to $r(\lambda)$ and λ' to $r(\lambda')$ is less than one. But this is impossible, as $r(\lambda)$ and $r(\lambda')$ are also elements of the lattice of weights. □

Given a facet F, we denote by $\overline{F}$ the closure of F in $\mathbb{E}^\infty$. This will consist of a union of facets.

Lemma 6.2. *If $\lambda' \in \mathrm{supp}(\lambda)$ and $\lambda' \in F$ for some facet F then*

$$|\mathcal{B}(\lambda') \cap \mathrm{supp}(\lambda)| > 1$$

if and only if $\lambda \in \overline{F} \backslash F$.

Proof. We first show that if $|\mathcal{B}(\lambda') \cap \mathrm{supp}(\lambda)| > 1$ then $\lambda \in \overline{F}\backslash F$. By the interpretation of blocks in terms of contents of partitions (Theorem 2.1) there is precisely one other weight λ'' in $\mathrm{supp}(\lambda)$ in the same block as λ'. Also, one of these weights is obtained from λ by adding a box, and one by subtracting a box. But this implies that λ' is the reflection of λ'' about some hyperplane H; this reflection must fix the midpoint on the line from λ' to λ'', which is λ, and so $\lambda \in H$.

By Lemma 6.1 there is no hyperplane separating λ from λ'. However to complete the first part of the proof we still need to show that if $\lambda' \in H'$ for some hyperplane H' then $\lambda \in H'$ too. First note that by Proposition 3.2(ii) H and H' must be $(i,j)_-$ and $(k,l)_-$ hyperplanes respectively for some quadruple i,j,k,l. It is easy to check that either $(i,j)_-$ fixes H' or $(i,j)_- \cdot_\delta H'$ is an (i,l)-hyperplane. But $\lambda'' \in (i,j)_- \cdot_\delta H'$ is dominant and so $(i,j)_-$ must fix H'. Hence $\lambda'' \in H'$ and as λ is the midpoint between λ' and λ'' we must have $\lambda \in H'$.

For the reverse implication, suppose that $\lambda \in \overline{F}\backslash F$. Then for all hyperplanes H' with $\lambda' \in H'$ we have $\lambda \in H'$ and there is (at least) one hyperplane H with $\lambda \in H$ and $\lambda' \notin H$. Suppose that H is an $(i,j)_-$-hyperplane, and consider $\lambda'' = (i,j)_- \cdot_\delta \lambda'$.

If $\lambda'' \in X^+$ then we are done. Otherwise by Lemma 6.1 we have that λ'' must lie on the boundary of the dominant region, and hence in some (k,l)-hyperplane $\tilde{H}$. Now $(i,j)_- \cdot_\delta \lambda'' = \lambda'$ and hence $\lambda' \in (i,j)_- \cdot_\delta \tilde{H} = H' \neq \tilde{H}$ (as $\lambda' \in X^+$). Therefore we must have $\lambda \in H'$. But λ is fixed by $(i,j)_-$ and so $\lambda \in \tilde{H} \cap H'$. This implies that $\lambda \notin X^+$ which is a contradiction. Thus we have shown that $\lambda'' \in X^+$ and so $|\mathcal{B}(\lambda') \cap \mathrm{supp}(\lambda)| > 1$. □

Theorem 6.3. *If λ is in an alcove then μ is in the same translation class as λ if and only if it is in the same alcove.*

Proof. By Lemmas 6.1 and 6.2 it is enough to show that if μ is in the same alcove as λ then μ can be obtained from λ by repeatedly adding or subtracting a box without ever leaving this alcove.

Suppose that λ and μ are in the same alcove, and set $x = \lambda + \rho_\delta$ and $y = \mu + \rho_\delta$, the corresponding vectors in A^+. Recall that there is a permutation π defining the alcove A introduced in Section 3. We may assume that $|x_{\pi(1)}| \leq |y_{\pi(1)}|$. Consider the sequence obtained by repeatedly adding (or subtracting) 1 from $y_{\pi(1)}$ until we obtain $x_{\pi(1)}$. At each stage the vector v obtained is of the form $\tau + \rho_\delta$ for some weight τ, and the sequence of weights thus obtained are such that each consecutive pair are translation equivalent. Now we repeat the process to convert $y_{\pi(2)}$ into $x_{\pi(2)}$ (note that $y_{\pi(2)}$ and $x_{\pi(2)}$ have the same sign, and so the chain of weights constructed will always have $\pi(2)$-coordinate satisfying the defining conditions for the alcove). We continue in this manner until we have converted y into x. This constructs a chain of translation equivalent weights connecting λ and μ and so we are done. □

Remark 6.4. Theorem 6.3 shows that the geometry on the weight space for B_n comes naturally from the induction and restriction functors when the alcoves are non-empty (i.e. for $\delta > 0$).

We would like to extend Theorem 6.3 to the case of two weights in the same facet. However, not all weights in the same facet are in the same translation class. To see this, note that a hyperplane is defined by the equation $x_i = -x_j$ for some fixed pair i and j. Any modification of a weight in such a hyperplane by adding or subtracting a single box cannot alter the value of the ith or jth coordinate without leaving the hyperplane. However, we will see that if we also use the modified translation functors introduced in Section 5 then we do get the desired equivalences within facets.

Let $\mathrm{supp}^2(\lambda) = \mathrm{supp}(\mathrm{supp}(\lambda))$. This set consists of those partitions obtained from λ by adding two boxes, removing two boxes, or adding a box and removing a box.

Lemma 6.5. *Suppose that $\lambda, \tilde{\lambda} \in X^+$ with $\tilde{\lambda} \in \mathrm{supp}^2(\lambda)$. If λ and $\tilde{\lambda}$ are in the same facet then*

$$\mathrm{supp}^2(\lambda) \cap \mathcal{B}(\tilde{\lambda}) = \{\tilde{\lambda}\}.$$

Proof. We take $\lambda' \in \mathrm{supp}^2(\lambda)$ with $\lambda' \neq \tilde{\lambda}$ and show that the above assumptions imply that $\lambda' \notin \mathcal{B}(\lambda)$. There are six possible cases.

(i) Suppose that $\tilde{\lambda} = \lambda - \epsilon_i + \epsilon_j$ and $\lambda' = \lambda - \epsilon_k + \epsilon_l$. For these two weights to be in the same block the boxes ϵ_i and ϵ_l must pair up (and so must ϵ_j and ϵ_k) in the sense of condition (1) for a balanced partition. This implies that there is a simple reflection $(i,l)_-$ taking $\lambda - \epsilon_i$ to $\lambda + \epsilon_l$, which fixes λ. Hence λ is in the $(i,l)_-$-hyperplane. However, $\tilde{\lambda}$ is not in this hyperplane, contradicting our assumption that they are in the same facet.

(ii) Suppose that $\tilde{\lambda} = \lambda + \epsilon_i + \epsilon_j$ and $\lambda' = \lambda + \epsilon_k + \epsilon_l$. For these two weights to be in the same block the elements $\epsilon_i, \epsilon_j, \epsilon_k, \epsilon_l$ must all be distinct and ϵ_i and ϵ_j must pair up (and so must ϵ_k and ϵ_l). Thus there is a reflection taking λ to $\tilde{\lambda}$, which contradicts our assumption.

(iii) Suppose that $\tilde{\lambda} = \lambda - \epsilon_i - \epsilon_j$ and $\lambda' = \lambda - \epsilon_k - \epsilon_l$. This is similar to (ii).

(iv) Suppose that $\tilde{\lambda} = \lambda - \epsilon_i + \epsilon_j$ and $\lambda' = \lambda + \epsilon_k + \epsilon_l$. For these two weights to be in the same block we must have $j = l$ (say). But then $\lambda + \epsilon_k + \epsilon_j$ is the reflection of $\lambda - \epsilon_i + \epsilon_j$ through the $(i,k)_-$-hyperplane, and hence $\lambda + \epsilon_j$ is in the $(i,k)_-$-hyperplane. Therefore λ is also in this hyperplane, but $\tilde{\lambda}$ is not, which gives a contradiction.

(v) Suppose that $\tilde{\lambda} = \lambda - \epsilon_i + \epsilon_j$ and $\lambda' = \lambda - \epsilon_k - \epsilon_l$. This is similar to (iv).

(vi) Suppose that $\tilde{\lambda} = \lambda + \epsilon_i + \epsilon_j$ and $\lambda' = \lambda - \epsilon_k - \epsilon_l$. First note that ϵ_i and ϵ_j cannot pair up (as this would imply that λ and $\tilde{\lambda}$ are not in the same facet). So for these two weights to be in the same block we must have ϵ_i pairing up with ϵ_k (say) and ϵ_j pairing up with ϵ_l.

But then $\lambda - \epsilon_k$ is the reflection of $\lambda + \epsilon_i$ through the $(i,k)_-$-hyperplane, which implies that λ is in this hyperplane but $\tilde{\lambda}$ is not, which gives a contradiction. □

Lemma 6.6. *Suppose that $\lambda, \tilde{\lambda} \in X^+$ with $\tilde{\lambda} = \lambda - \epsilon_i + \epsilon_j \in \mathrm{supp}^2(\lambda)$, and that λ lies on the $(ij)_-$-hyperplane. Then λ and $\tilde{\lambda}$ are in the same facet. Moreover, if $\mu = w \cdot_\delta \lambda \in X^+$ for some $w \in W$ then $\tilde{\mu} = w \cdot_\delta \tilde{\lambda}$ satisfies*

$$\tilde{\mu} = \mu - \epsilon_s + \epsilon_t$$

for some s, t.

Proof. The fact that λ and $\tilde{\lambda}$ are in the same facet is clear. Now suppose that $\mu = w \cdot_\delta \lambda$ and $\tilde{\mu} = w \cdot_\delta \tilde{\lambda}$. Then $\tilde{\mu} = \mu + \beta$ where $\beta = \pm(\epsilon_s + \epsilon_t)$ or $\beta = \epsilon_s - \epsilon_t$ for some s, t. Suppose for a contradiction that $\beta = \pm(\epsilon_s + \epsilon_t)$. Note that μ and $\tilde{\mu}$ are in the same facet, and so for any $(k,l)_-$-hyperplane on which μ and $\tilde{\mu}$ lie we must have that $\beta = \tilde{\mu} - \mu$ lies on the *unshifted* $(k,l)_-$-hyperplane. This implies that $s \neq k, l$ and $t \neq k, l$.

We have a sequence of dominant weights μ, $\mu' = \mu \pm \epsilon_s$ and $\tilde{\mu} = \mu \pm (\epsilon_s + \epsilon_t)$ which are each at distance 1 from their neighbours in the sequence. We have already seen that they all lie on the same set of hyperplanes. Moreover, by Lemma 6.1 there cannot exist a hyperplane separating μ from μ' or μ' from $\tilde{\mu}$. So μ, μ' and $\tilde{\mu}$ all lie in the same facet.

Now consider the image of these three weights under w^{-1}. We get a corresponding sequence λ, λ' and $\tilde{\lambda}$. These weights must also lie in a common facet (and hence λ' is dominant) and are distance 1 from their neighbours. This forces $\lambda' = \lambda - \epsilon_i$ or $\lambda' = \lambda + \epsilon_j$. However λ' cannot be in the same facet as λ as it does not lie on the $(i,j)_-$-hyperplane, which gives the desired contradiction. □

Let $\mathrm{res}_n^{\lambda,\pm} = \mathrm{pr}^\lambda \mathrm{res}_n^\pm$ and $\mathrm{ind}_n^{\lambda,\pm} = \mathrm{pr}^\lambda \mathrm{ind}_n^\pm$. We say that λ and μ are in the same $(\pm)$-*translation class* if there is a chain of dominant weights

$$\lambda = \lambda^0, \lambda^1, \ldots, \lambda^r = \mu$$

such that either $\lambda^{i+1} \in \mathrm{supp}(\lambda^i)$ with λ^i and λ^{i+1} translation equivalent or $\lambda^{i+1} \in \mathrm{supp}^2(\lambda^i)$ with $\lambda^{i+1} = \lambda^i + \epsilon_s - \epsilon_t$ (for some s and t) and λ^i and λ^{i+1} are $(\mathrm{res}^{\lambda^i,\pm}, \mathrm{ind}^{\lambda^{i+1},\pm})$-translation equivalent.

Suppose that $\lambda, \tilde{\lambda} \in \Lambda_n$ with $\tilde{\lambda} = \lambda - \epsilon_i + \epsilon_j$, and that λ lies in the $(i,j)_-$-hyperplane. By Lemmas 6.5 and 6.6 we have a bijection $\theta : \mathcal{B}(\lambda) \to \mathcal{B}(\tilde{\lambda})$ which restricts to a bijection $\theta : \mathcal{B}_n(\lambda) \to \mathcal{B}_n(\tilde{\lambda})$. By Corollary 5.3, Theorem 5.5, Lemmas 6.5 and 6.6, and standard properties of ind and res it is clear that weights λ and $\tilde{\lambda}$ are $(\mathrm{res}^{\lambda,\pm}, \mathrm{ind}^{\tilde{\lambda},\pm})$ translation equivalent. It is also easy to see that the adjointness isomorphism is multiplicative. Thus we can apply Theorem 4.10 and get a Morita equivalence between the two blocks $B_n(\lambda)$ and $B_n(\tilde{\lambda})$.

Theorem 6.7. *If λ and μ are in the same facet then they are in the same $(\pm)$-translation class.*

Proof. By Lemmas 6.2 and 6.5 it is enough to show that if λ and μ are in the same facet then there is a chain of dominant weights

$$\lambda = \lambda^0, \lambda^1, \ldots, \lambda^r = \mu$$

in the same facet such that $\lambda^{i+1} \in \mathrm{supp}(\lambda^i)$ or $\lambda^{i+1} \in \mathrm{supp}^2(\lambda^i)$ for each i.

Let $x = \lambda + \rho_\delta$ and $y = \mu + \rho_\delta$, and recall that in Section 3 we associated a function f to each facet (rather than just a permutation π as for an alcove). The proof now proceeds exactly as for the alcove case (Theorem 6.3) replacing π by f, until we reach some point where $f(i) = (k, l)$. In this case we repeatedly add (or subtract) a box from y_k and subtract (or add) a box to y_l until we reach x_k and x_l. In each of these steps we obtain some $\lambda^{i+1} = \lambda^i \pm (\epsilon_k - \epsilon_l) \in \mathrm{supp}^2(\lambda)$. □

Applying the results on translation and (R, I)-translation equivalence from Section 4 we deduce

Corollary 6.8. *If λ and λ' are in the same facet and μ and μ' are such that $\lambda, \mu \in \Lambda_n$, and $\lambda', \mu' \in \Lambda_m$, and μ' is the unique weight in $\mathcal{B}(\lambda')$ in the same facet as μ, then we have:*
(i)
$$[\Delta_n(\lambda) : L_n(\mu)] = [\Delta_m(\lambda') : L_m(\mu')]$$
(ii)
$$\mathrm{Hom}_n(\Delta_n(\lambda), \Delta_n(\mu)) \cong \mathrm{Hom}_m(\Delta_m(\lambda'), \Delta_m(\mu'))$$
(iii)
$$\mathrm{Ext}^i_n(\Delta_n(\lambda), \Delta_n(\mu)) \cong \mathrm{Ext}^i_m(\Delta_m(\lambda'), \Delta_m(\mu'))$$
for all $i \geq 1$. If further $\mathcal{B}_n(\lambda)$ and $\mathcal{B}_m(\lambda')$ contain the same number of simples then the corresponding blocks are Morita equivalent.

By [DWH99, Theorem 3.4] there are always enough local homomorphisms for the Brauer algebra. Further, by Lemma 6.2 any weight that is not adjacent to a weight in a less singular facet is translation equivalent to a weight of smaller total degree. Thus if $\delta > 0$ then every weight can be reduced to a weight in some alcove by translation equivalence and repeated applications of Proposition 4.9. This implies

Corollary 6.9. *If $\delta > 0$ then the decomposition numbers $[\Delta_n(\lambda) : L_n(\mu)]$ for arbitrary λ and μ are determined by those for λ and μ in an alcove.*

Note that the restriction on δ is necessary, as for $\delta < 0$ there are no weights in an alcove. In fact for $\delta = -2m$ or $\delta = -2m + 1$ any dominant weight is δ-singular of degree at least m. For the rest of this section we will see what more can be said in such cases.

We will denote the set of all partitions λ with at most m non-zero parts by $\Lambda^{\leq m}$, and the set of those with at most $m + 1$ non-zero parts with $\lambda_{m+1} \leq 1$ by $\Lambda^{\leq m,1}$. (Note that $\Lambda^{\leq m}$ is precisely the set of weights considered in Theorem 3.10.) Such weights lie in a union of facets, but we shall see that together they play a role analogous to that played by the fundamental alcove in the $\delta > 0$ case.

We begin by noting

Proposition 6.10. *(i) For $\delta = -2m$, every δ-singular weight of degree m is in the same block as a unique element of $\Lambda^{\leq m}$.*
(ii) For $\delta = -2m + 1$, every δ-singular weight of degree m is in the same block as a unique element of $\Lambda^{\leq m,1}$.

Proof. (i) Suppose that $\delta = -2m$ and let λ be a δ-singular weight of degree m. Then $\lambda + \rho_\delta$ is of the form

$$(\ldots, x_1, \ldots, x_2, \ldots, x_m, \ldots, (0), \ldots, -x_m, \ldots, -x_2, \ldots, -x_1, \ldots, -n, -(n+1), \ldots)$$

where the only elements of equal modulus are those of the form $\pm x_i$, and the bracketed 0 may or may not appear. Note that as λ is a finite weight the tail of $\lambda + \rho_\delta$ will equal the tail of ρ_δ, i.e. has value $-n$ in position $m+1+n$ for all $n >> 0$, and we assume that this holds for the n in the expression above (and similar expressions to follow).

First suppose that $\lambda + \rho_\delta$ contains 0. Then $\lambda + \rho_\delta$ is in the same W_a-orbit as

$$\mu + \rho_\delta = (x_1, x_2, \ldots, x_m, 0, \ldots, -x_m, \ldots, -x_2, \ldots, -x_1, \ldots, -n, -(n+1), \ldots).$$

Thus the $n-1$ coordinates between the entries 0 and $-n$ must be strictly decreasing, which forces

$$\mu + \rho_\delta = (x_1, x_2, \ldots, x_m, 0, -1, -2, -3, \ldots).$$

Hence we deduce that $\mu \in \Lambda^{\leq m}$ as required.

Next suppose that $\lambda + \rho_\delta$ does not contain 0. Then there are two cases depending on the parity of the number of positive entries in $\lambda + \rho_\delta$. The first case is when $\lambda + \rho_\delta$ is in the same W_a-orbit as

$$\mu + \rho_\delta = (x_1, x_2, \ldots, y, \ldots, x_m, \ldots, -x_m, \ldots, -x_2, \ldots, -x_1, \ldots, -n, -(n+1), \ldots)$$

where y is some positive integer and all entries after x_m are negative. Arguing as above we see that

$$\mu + \rho_\delta = (x_1, x_2, \ldots, y, \ldots, x_m, -1, -2, -3, \ldots).$$

But this vector is δ-singular of degree $m+1$, which contradicts our assumptions on λ.

The second case is when $\lambda + \rho_\delta$ is in the same W_a-orbit as

$$\mu + \rho_\delta = (x_1, x_2, \ldots, x_m, \ldots, -x_m, \ldots, -x_2, \ldots, -x_1, \ldots, -n, -(n+1), \ldots)$$

where all entries after x_m are negative. But this implies that $\mu + \rho_\delta$ has n strictly decreasing coordinates between the entries 0 and $-n$, which is impossible.

The argument for (ii) is very similar. We see that $\lambda + \rho_\delta$ is in the same W_a-orbit as either

$$\mu + \rho_\delta = (x_1, x_2, \ldots, x_m, \ldots, -x_m, \ldots, -x_2, \ldots, -x_1, \ldots, -n - \frac{1}{2}, -(n+1) - \frac{1}{2}, \ldots)$$

or

$$\mu + \rho_\delta = (x_1, x_2, \ldots, y, \ldots, x_m, \ldots, -x_m, \ldots, -x_2, \ldots, -x_1, \ldots, -n - \frac{1}{2}, -(n+1) - \frac{1}{2}, \ldots)$$

where in each case all entries after x_m are negative.

In the first case we deduce as above that

$$\mu + \rho_\delta = (x_1, x_2, \ldots, x_m, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots)$$

and so $\mu \in \Lambda^{\leq m} \subset \Lambda^{\leq m,1}$ as required. In the second case, as $\mu + \rho_\delta$ must be δ-singular of degree m, we deduce that

$$\mu + \rho_\delta = (x_1, x_2, \ldots, y, \ldots, x_m, -\frac{1}{2}, -\frac{3}{2}, - \ldots, -y+1, \hat{y}, -y-1 \ldots)$$

where $\hat{y}$ denotes the omission of the entry y. But this element is in the same W_a-orbit as

$$\nu + \rho_\delta = (x_1, x_2, \ldots, x_m, \frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \ldots)$$

(by swapping y and $-\frac{1}{2}$ with a change of signs, and rearranging to get a decreasing sequence). Thus λ is in the same W_a-orbit as ν, and $\nu = (\nu_1, \ldots, \nu_m, 1) \in \Lambda^{\leq m,1}$ as required. □

Although the weights in $\Lambda^{\leq m}$ (respectively in $\Lambda^{\leq m,1}$) lie in several different facets, the next result shows that all these facets have equivalent representation theories.

Proposition 6.11. *Let $\delta = -2m$ (respectively $\delta = -2m+1$) and $\lambda \in \Lambda^{\leq m}$ (respectively $\lambda \in \Lambda^{\leq m,1}$). Then λ and λ' are translation equivalent if and only if $\lambda' \in \Lambda^{\leq m}$ (respectively $\lambda' \in \Lambda^{\leq m,1}$).*

Proof. Note that all weights in $\Lambda^{\leq m}$ (respectively in $\Lambda^{\leq m,1}$) are δ-singular of degree m, and that any pair of such weights can be linked by a chain of weights in the same set differing at each stage only by the addition or subtraction of a single block. By Lemma 6.2 we see that any such pair is translation equivalent.

For the reverse implication, we first consider the $\delta = -2m$ case, with $\lambda \in \Lambda^{\leq m}$, and suppose that $\lambda' \in \mathrm{supp}(\lambda)$ is not an element of $\Lambda^{\leq m}$. Then we must have $\lambda' = (\lambda_1, \lambda_2, \ldots, \lambda_m, 1)$. Now $x = \lambda + \rho_\delta$ and $x' = \lambda' + \rho_\delta$ differ only in the $m+1$st coordinate, which is 0 or 1 respectively. If f is the function associated to the facet containing x and f' is the corresponding function for x' then the only difference is that $f(1) = m+1$ while $f'(1) = (m+1, m+2)$. Thus $\lambda' \in \overline{F}\backslash F$, where F is the facet containing λ, and so by Lemma 6.2 this pair cannot be translation equivalent.

The case $\delta = -2m+1$ is similar. Arguing as above we have that $\lambda' = (\lambda_1, \ldots, \lambda_m, 1, 1)$ or $\lambda' = (\lambda_1, \ldots, \lambda_m, 2)$, and in each case it is easy to show that the pair λ and λ' are not translation equivalent. □

Combining Propositions 6.10 and 6.11 we deduce that for $\delta < 0$ and $\lambda, \lambda' \in \Lambda^{\leq m}$ (respectively $\lambda, \lambda' \in \Lambda^{\leq m,1}$) there is a bijection $\theta : \mathcal{B}(\lambda) \to \mathcal{B}(\lambda')$ which as before we will denote by $\theta(\mu) = \mu'$. Applying the results from Section 4 we obtain

Corollary 6.12. *Let $\delta < 0$ and $\lambda, \lambda' \in \Lambda^{\leq m}$ (respectively $\lambda, \lambda' \in \Lambda^{\leq m,1}$). If $\lambda, \lambda' \in \Lambda_n$ and $\lambda', \mu' \in \Lambda_l$ then (i–iii) of Corollary 6.8 hold. If further $\mathcal{B}_n(\lambda)$ and $\mathcal{B}_l(\lambda')$ contain the same number of elements then the corresponding blocks are Morita equivalent.*

As in Corollary 6.9, we obtain the following application of Proposition 4.9.

Corollary 6.13. *If $\delta < 0$ then the decomposition numbers $[\Delta_n(\lambda) : L_n(\mu)]$ for arbitrary λ and μ are determined by those for λ and μ in a singular facet of degree m.*

Combining Corollaries 6.9 and 6.13 with our earlier remarks we obtain

Theorem 6.14. *For $\delta \in \mathbb{Z}$ non-zero the decomposition numbers $[\Delta_n(\lambda) : L_n(\mu)]$ for arbitrary λ and μ are determined by those for λ and μ in $\mathcal{B}(0)$.*

Thus (at least at the level of decomposition numbers) is it enough to restrict attention to a single block of the Brauer algebra.

Remark 6.15. The decomposition numbers for the module $\Delta_n(0)$ are known by [CDM09a, Proposition 5.1 and Theorem 5.2].

We would also like the representation theory to be independent of $\delta \in \mathbb{Z}$, in the sense that it should depend only on the geometry of facets. For weights in alcoves, this would in large part follow if we could show that decomposition numbers are given by some kind of parabolic Kazhdan-Lusztig polynomials. In the remaining sections we will consider some evidence for this.

7. Block graphs for the Brauer algebra

Recall from Section 2 the definition of a maximal balanced partition. Let $\mathbf{MBS}_\delta(\lambda)$ be the directed graph with vertex set $V_\delta(\lambda)$ and edge $\mu \to \tau$ if μ is a maximal δ-balanced subpartition of τ.

The above graph appears to depend both on λ and δ, while the alcove geometry associated to W_a does not. Let **Alc** be the directed graph with vertex set the set of alcoves for W_a in A^+, and an edge $A \to B$ if the closures of A and B meet in a hyperplane and this hyperplane separates A_0 and B. (Note that the former condition corresponds to $B = (ij)_- A$ for some reflection $(ij)_-$.)

Our goal in this section is to show that all the graphs $\mathbf{MBS}_\delta(\lambda)$ are in fact isomorphic, and are isomorphic to the alcove graph **Alc**. This will be our first evidence that the representation theory depends only on the geometry of facets.

Recall that for $\lambda \in X^+$ we have $\lambda + \rho_\delta \in A^+$ (the set of strictly decreasing sequences in $\mathbb{E}^\infty$) and the δ-dot action of W_a on λ corresponds to the usual action of W_a on $\lambda + \rho_\delta$. For the rest of this section we will work with the usual action of W_a on A^+.

For $v \in A^+$ we define

$$V(v) = W_a v \cap A^+.$$

We define a partial order on A^+ by setting $x \leq y$ if $y - x \in \mathbb{E}^f$ and all entries in $y - x$ are non-negative. For $v \in A^+$ we define a directed graph $\mathbf{G}(v)$ with vertex set $V(v)$ and arrows given as follows. If $x, y \in V(v)$, we set $x \to y$ if and only if $x < y$ and there is no $z \in V(v)$ with $x < z < y$. The reason for introducing this graph is clear from

Proposition 7.1. *For $\lambda \in X^+$ we have $\mathbf{MBS}_\delta(\lambda^T) \cong \mathbf{G}(\lambda + \rho_\delta)$.*

Proof. By Theorem 3.3 we have a bijection between $V_\delta(\lambda^T)$ and $V(\lambda + \rho_\delta)$. Moreover, for $\mu^T, \nu^T, \tau^T \in V_\delta(\lambda^T)$ we have $\mu^T \subset \nu^T \subset \tau^T$ if and only if $\mu + \rho_\delta < \nu + \rho_\delta < \tau + \rho_\delta$. Thus the two graph structures on these vertex sets are preserved under the correspondence. $\square$

Recall the definition of singletons from Section 3. Define v_{reg} to be the subsequence of v consisting only if its singletons. For example, if v begins $(9, 8, 7, 0, -1, -2, -7, -9, -11, \ldots)$ then v_{reg} begins $(8, 0, -1, -2, -11, \ldots)$. Note that if $v \in A^+$ then $v_{reg} \in A^+$ and $|(v_{reg})_i| \neq |(v_{reg})_j|$ for all $i \neq j$. Therefore v_{reg} is a regular element in $\mathbb{E}^\infty$ (as it does not lie on any reflecting hyperplane). We define the *regularisation map* Reg : $A^+ \longrightarrow A^+$ by setting

$$\mathrm{Reg}(v) = v_{reg}.$$

The key result about the regularisation map is

Proposition 7.2. *For all $v \in A^+$ we have*

$$\mathbf{G}(v) \cong \mathbf{G}(v_{reg}).$$

Proof. We first observe that the map Reg gives rise to a bijection between $V(v)$ and $V(v_{reg})$. For the set of doubletons is an invariant of the elements in $V(v)$, and given this set there is a unique way of adding the doubletons into an element of $V(v_{reg})$ keeping the sequence strictly decreasing. Now suppose that $x, y \in A^+$ and $a \in \mathbb{R}$ are such that

$$s = (x_1, ..., x_i, a, x_{i+1}, ...) \in A^+ \quad \text{and} \quad t = (y_1, ..., y_j, a, y_{j+1}, ...) \in A^+.$$

Then it is easy to see that $x < y$ if and only if $s < t$. However, this implies that the set of edges coincide under the map Reg, as required. □

Corollary 7.3. *For all $v, v' \in A^+$ we have*

$$\mathbf{G}(v) \cong \mathbf{Alc}\,.$$

Hence for all $\delta, \delta' \in \mathbb{Z}$ and $\lambda, \lambda' \in X^+$ we have

$$\mathbf{MBS}_\delta(\lambda) \cong \mathbf{MBS}_{\delta'}(\lambda').$$

Proof. Note that any $v \in \mathrm{Reg}(A^+)$ lies inside an alcove. For any vector $v \in \mathrm{Reg}(A^+)$ the maximal weights below v in the same orbit lie in the alcoves below and adjacent to the alcove containing v. Thus for $v \in \mathrm{Reg}(A^+)$ it is clear that we have $\mathbf{G}(v) \cong \mathbf{Alc}$. Now the result follows for general v from Proposition 7.1, and in its **MBS** form from Proposition 7.2. □

It will be convenient to give **Alc** the structure of a graph with coloured edges. An edge in **Alc** corresponds to reflection from an alcove A to an alcove B through the facet separating them. The action of W_a on weights induces a corresponding action on facets, and we shall say that two edges have the same colour if and only if the corresponding facets lie in the same orbit.

We conclude this section with one final graph $\mathbf{Par}_e^+$ isomorphic to **Alc**, whose structure can be described explicitly.

We fix the element $v = (-1, -2, -3, -4, ...) \in A^+$. Using the action of W_a we can see that every $x \in V(v)$ corresponds uniquely to a strictly decreasing partition with an even number of parts, obtained by ignoring all parts of x which are negative. For example, the element $x = (6, 5, 3, 1, -2, -4, -7, -8, ...)$ corresponds to $(6, 5, 3, 1)$ while v corresponds to $\emptyset$. Thus if we write P_e^+ for the set of strictly decreasing partitions with an even number of parts then we have a bijection

$$\phi\,:\, V(v) \longrightarrow P_e^+.$$

Consider the usual partial order $\subseteq$ on P_e^+ given by inclusion of partitions (viewed as Young diagrams). It is clear that the partial order $\leq$ on $V(v)$ corresponds to the partial order $\subseteq$ on P_e^+ under the bijection ϕ. Define a graph $\mathbf{Par}_e^+$ with vertex set P_e^+ and an arrow $\lambda \to \mu$ if and only if $\lambda \subset \mu$ and there is no $\nu \in P_e^+$ with $\lambda \subset \nu \subset \mu$. It is easy to verify that the map ϕ induces a graph isomorphism between $\mathbf{G}(v)$ and $\mathbf{Par}_e^+$.

The graph $\mathbf{Par}_e^+$ can easily be described explicitly as follows. For $\lambda, \mu \in P_e^+$, there is an arrow $\lambda \to \mu$ if and only if either

$$\lambda = (\lambda_1, \dots, \lambda_n) \quad \text{and} \quad \mu = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots, \lambda_n) \tag{21}$$

or

$$\lambda = (\lambda_1, \dots, \lambda_n) \text{ with } \lambda_n \geq 3 \quad \text{and} \quad \mu = (\lambda_1, \dots, \lambda_n, 2, 1). \tag{22}$$

To see this, first observe that in both cases there is no $\nu \in P_e^+$ with $\lambda \subset \nu \subset \mu$. Moreover, if $\lambda, \mu \in P_e^+$ with $\lambda \subset \mu$ then μ can be obtained from λ by applying (21) and (22) repeatedly.

8. Coxeter systems and parabolic Kazhdan-Lusztig polynomials

In this section we will introduce parabolic Kazhdan-Lusztig polynomials associated with the pair (W_a, W). We briefly review the relevant theory; details can be found in [Hum90] and [Soe97b].

Recall that a Coxeter system is a pair (G, S) consisting of a group G and a set S of generators of G such that all relations in G are of the form

$$(ss')^{m(s,s')} = 1$$

where $m(s, s) = 1$ and $m(s, s') = m(s', s) \geq 2$ otherwise (including the possibility that $m(s, s') = \infty$ denoting no relation between s and s'). Note that the group G does not need to be finite (although this is often assumed). Given a Coxeter system, the associated Coxeter graph is the graph with vertices the elements of S, and $m(s, s') - 2$ edges between s and s' (or no edges when $m(s, s') = \infty$). For example the D_∞ Coxeter graph is given by the graph shown in Figure 13.

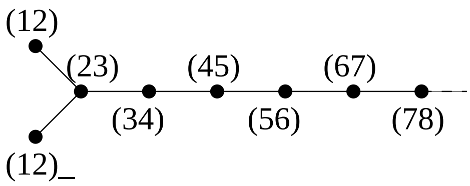

Figure 13. The type D_∞ Coxeter system

It is easy to verify that our group W_a is generated by the elements $\{(12)_-, (i\,i+1) : i \geq 1\}$ and satisfies the relations given by the Coxeter graph in Figure 13. Thus it must be a quotient of the Coxeter group of type D_∞. However, for each choice of n, the subsystem generated by the first n generators is precisely the type D_n system (see [Bou68, Planche IV]), and so there can be no further relations, and our group is the type D_∞ Coxeter group associated to the given generators.

Given a Coxeter system (G, S), any subgroup G' generated by a subset S' of S defines a parabolic subsystem (G', S'). In our case the group W clearly arises in this way from the generators of the form $(i\,i+1)$ and so is a type A_∞ parabolic subgroup of W_a.

When $\delta \geq 0$ there is a bijection from W_a to the set of alcoves, given by $w \longmapsto w.0$. We will henceforth identify elements of W_a with alcoves via this map. Under this bijection the standard length function on our Coxeter system associated to W_a (given in terms of the number of terms occurring in a reduced expression for w) corresponds to the number of reflection hyperplanes between 0 and $w \cdot_\delta 0$.

We define W^a to be the subset of W_a corresponding to the alcoves in X^+. By Proposition 3.2(ii) we then have a bijection

$$W \times W^a \to W_a.$$

We are thus in a position to define D_∞/A_∞ parabolic Kazhdan-Lusztig polynomials following [Deo87] (although we use the notation of [Soe97b, Section 3]). Their precise definition and general properties need not concern us, instead we will give a recursive construction corresponding to stepping away from the root of **Alc**. To do this we will first need to define a partial order on weights (and alcoves).

Two weights λ and μ such that $\mu = w.\lambda$ for some reflection w, lie in different components of the space formed by removing this hyperplane. We say that $\lambda < \mu$ if λ is in the component containing the fundamental alcove. This extends to give a partial order on weights, which in turn induces a partial order on alcoves. This agrees with the path-from-root order on **Alc**. Two alcoves are said to be *adjacent* if there is precisely one reflecting hyperplane between them (i.e. they are adjacent in **Alc**).

Suppose that ν and μ are dominant weights in adjacent alcoves with $\nu = s \cdot_\delta \mu > \mu$. Given a dominant weight $\lambda \in W_a \cdot_\delta \mu$ we define $\kappa_\lambda(\nu, \mu)$ to be the unique weight such that

$$(\kappa_\lambda(\nu, \mu), \lambda) = (w \cdot_\delta \mu, w \cdot_\delta \nu)$$

i.e. $(\kappa_\lambda(\nu, \mu), \lambda)$ is an edge of the same colour as (μ, ν) in **Alc**.

We next define certain polynomials $n_{\nu,\lambda}$ (in an indeterminate v) for regular weights λ and μ in the following recursive manner. Let e_λ as λ runs over the regular weights be a set of formal symbols.

(i) We set $n_{\nu,\lambda} = 0$ if $\lambda \not\leq \nu$ or $\lambda \notin W_a \cdot_\delta \nu$ or either λ or ν is non-dominant.
(ii) We set $n_{0,0} = 1$ and $N(0) = e_0$.
(iii) For each $\nu > 0$ regular dominant, there exists some μ regular dominant below it such that $\mu = s \cdot_\delta \nu$ and (ν, μ) are in adjacent alcoves. Pick any such μ. Then for any dominant λ with $\lambda = w \cdot_\delta \nu$ for some w and $\kappa = \kappa_\lambda(\nu.\mu)$ we set

$$\hat{n}_{\nu,\lambda} = \mathrm{pr}_+(\kappa)\left(n_{\mu,\kappa} + v^{l(\kappa)-l(\lambda)} n_{\mu,\lambda}\right)$$

where $\mathrm{pr}_+(\kappa) = 1$ if $\kappa \in X^+$ and $\mathrm{pr}_+(\kappa) = 0$ otherwise. Note that for $\kappa \in X^+$ we have $l(\kappa) - l(\lambda) = -1$ if $\kappa < \lambda$, respectively $+1$ if $\kappa > \lambda$. Let $\hat{N}(\nu)$ be the sum

$$\hat{N}(\nu) = \sum_\lambda \hat{n}_{\nu,\lambda} e_\lambda.$$

and $R(\nu)$ be the set of $\lambda < \nu$ such that $\hat{n}_{\nu,\lambda}(0) \neq 0$. Then

$$N(\nu) = \hat{N}(\nu) - \sum_{\lambda \in R(\nu)} \hat{n}_{\nu,\lambda}(0) N(\lambda)$$

and $n_{\nu,\lambda}$ is the coefficient of e_λ in $N(\nu)$.

It is a consequence of (Deodhar's generalisation of) Kazhdan-Lusztig theory that this process is well defined (so does not depend on the choice of μ in step (iii)), and that each $n_{\nu,\lambda}$ is a polynomial in v with $n_{\nu,\lambda}(0) \neq 0$ only if $\lambda = \nu$.

9. Some low rank calculations for $\delta = 1$

To illustrate the various constructions so far, we will consider the case when $\delta = 1$, and examine the regular block containing the weight 0. First we calculate the associated parabolic Kazhdan-Lusztig polynomials, and then we compare these with the representation theoretic results.

We will also need to consider the block containing (1). As this is in the same alcove as 0 these two blocks are translation equivalence. However, in this simple case case we do not obtain any simplification to the calculations by applying the results from Section 6; instead the results can be considered as a verification of the general theory in this special case.

In Figure 14 we have listed all dominant weights of degree at most 16 that are in the same block as the weight 0. We will abbreviate weights in the same manner as partitions (and so write for example (1^3) instead of $(1,1,1)$). An edge between two weights indicates that they are in adjacent alcoves, and the label $(ij)_-$ corresponds to the reflection hyperplane between them. (Clearly only weights of the form $(ij)_-$ can arise as such labels.)

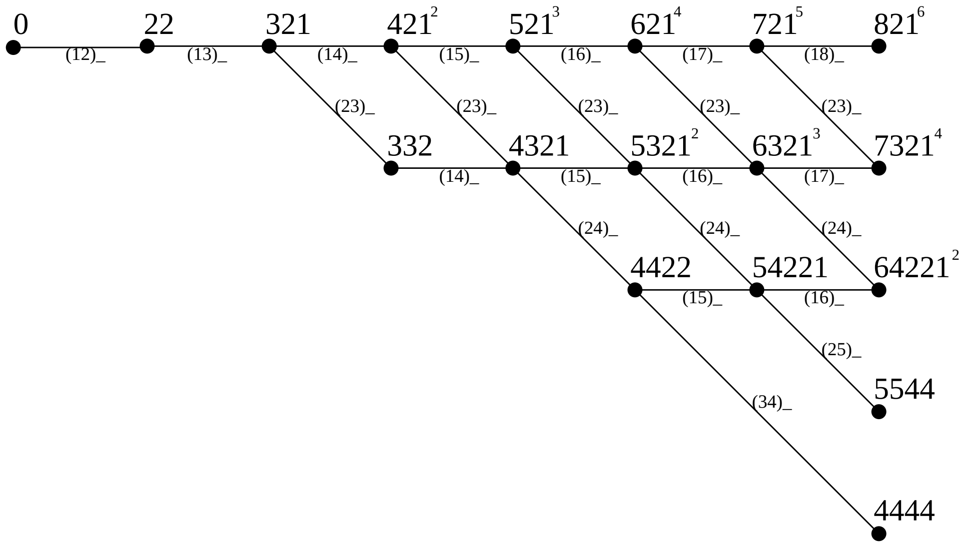

Figure 14. The block of the weight 0 for $\delta = 1$, up to degree 16

Given this data we can now compute the $n_{\lambda,\mu}$. The final results are shown in Figure 15. We start with the weight 0 having $n_{(0),(0)} = 1$. Reflecting through $(12)_-$ we obtain the weight (22), and we see that $n_{(22),(0)} = v$. (Note that the term $n_{\lambda,\lambda}$ is always 1.) Continuing we reflect (22) through $(13)_-$ to obtain (321). As $(12)_-(13)_- \cdot_\delta 0 = (2,-1,3)$ is not dominant we see that the only non-zero term apart from $n_{(321),(321)}$ is $n_{(321),(22)} = v$. Identical arguments give all polynomials $n_{\nu,\lambda}$ where ν is on the top row of Figure 14. For the second row we obtain four terms as ν and μ both give dominant weights under the action of $(1i)_-(23)_-$ for suitable i (as the parallelogram with ν as highest term has identically labelled parallel sides).

For (4422) we must observe that

$$(14)_-(23)_-(24)_- \cdot_\delta (4321) = (22)$$

μ \ λ	0	22	321	4211	332	521^3	4321	621^4	53211	4422	721^5	6321^3	54221	821^6	7321^4	64221^2	552^3	4^4
0	1																	
22	v	1																
321	.	v	1															
4211	.	.	v	1														
332	.	.	v	.	1													
521^3	.	.	.	v	.	1												
4321	.	.	v^2	v	v	.	1											
621^4	.	.	.	.	.	v	.	1										
53211	.	.	.	v^2	.	v	v	.	1									
4422	.	v^2	v	.	.	.	v	.	.	1								
721^5	.	.	.	.	.	.	.	v	.	.	1							
6321^3	.	.	.	.	.	v^2	.	v	v	.	.	1						
54221	.	.	.	.	.	.	v^2	.	v	v	.	.	1					
821^6	.	.	.	.	.	.	.	.	.	.	v	.	.	1				
7321^4	.	.	.	.	.	.	.	v^2	.	.	v	v	.	.	1			
64221^2	.	.	.	.	.	.	.	.	v^2	.	.	v	v	.	.	1		
552^3	.	.	.	.	v^2	.	v	.	.	.	.	.	v	.	.	.	1	
4^4	v^2	v	.	.	.	.	.	.	.	v	.	.	.	.	.	.	.	1

FIGURE 15. The polynomials $n_{\lambda,\mu}$ for $\delta = 1$ and $|\lambda| \leq 16$

and similar results give the remaining cases. In all of these cases we have no constant terms arising at any stage (apart from in $n_{\lambda,\lambda}$), and hence $N(\nu) = \hat{N}(\nu)$ for every weight considered.

Next we will determine the structure of certain low rank standard modules for $B_n(1)$ in the block containing 0. These will then be compared with the Kazhdan-Lusztig polynomials calculated above. We will proceed in stages, and will also need to consider the structure of modules in the block containing (1). The submodule structure of modules will be illustrated diagrammatically, where a simple module X is connected by a line to a simple module Y above it if there is a non-split extension of X by Y. Note that in this section we follow the usual labelling of modules by partitions (as in [CDM09a]) and *not* via the transpose map by weights.

9.1. **The case $n \leq 6$.** When $n = 0$ or $n = 2$ we have

$$\Delta_n(0) = L_n(0)$$

by quasihereditary (and the absence of any other simples in the same block).

When $n = 4$ we have

$$\Delta_4(22) = L_4(22) \qquad \text{and} \qquad \Delta_4(0) = \begin{array}{c} L_4(0) \\ | \\ L_4(22) \end{array}$$

by quasi-hereditary and (2).

When $n = 6$ we have

$$\Delta_6(321) = L_6(321) \qquad \text{and} \qquad \Delta_6(22) = \begin{array}{c} L_6(22) \\ | \\ L_6(321) \end{array}$$

as in the case $n = 4$. For the remaining module $\Delta_6(0)$ we know that $[\Delta_6(0) : L_6(22)] = 1$ by localising to $n = 4$. Applying Proposition 2.2 with $\mu = (32)$ (as this weight is minimal in its block) we see that $L_6(321)$ cannot occur in $\Delta_6(0)$. Hence we have that

$$\Delta_6(0) = \begin{array}{c} L_6(0) \\ | \\ L_6(22). \end{array}$$

The odd n cases are very similar. Arguing as above we see that

$$\Delta_1(1) = L_1(1), \qquad \Delta_3(21) = L_3(21), \qquad \Delta_3(1) = \begin{array}{c} L_3(1) \\ | \\ L_6(21) \end{array}$$

and for $n = 5$ that

$$\Delta_5(311) = L_5(311), \qquad \Delta_5(21) = \begin{array}{c} L_5(21) \\ | \\ L_5(311) \end{array}, \qquad \Delta_5(1) = \begin{array}{c} L_5(1) \\ | \\ L_5(21). \end{array}$$

9.2. **The case $n = 7$.** As above, we deduce from quasi-hereditary and (2) that

$$\Delta_7(41^3) = L_7(41^3), \qquad \text{and} \qquad \Delta_7(311) = \begin{array}{c} L_7(311) \\ | \\ L_5(41^3). \end{array}$$

For the remaining two standard modules, all composition multiplicities are known (by localising to the case $n = 5$) except for those for the 'new' simple $L_7(41^3)$. However, this does not occur in $\Delta_7(21)$ or in $\Delta_7(1)$ by an application of Proposition 2.2 with $\mu = (41^2)$ (as this weight is minimal in its block). Thus we have that

$$\Delta_7(21) = \begin{array}{c} L_7(21) \\ | \\ L_8(311) \end{array} \qquad \text{and} \qquad \Delta_7(1) = \begin{array}{c} L_7(1) \\ | \\ L_7(21). \end{array}$$

9.3. **The case $n = 8$.** This is similar to the preceding case. We have that

$$\Delta_8(421^2) = L_8(421^2), \qquad \Delta_8(332) = L_8(332), \qquad \text{and} \qquad \Delta_8(321) = \begin{array}{ccc} & L_8(321) & \\ \swarrow & & \searrow \\ L_8(421^2) & & L_8(332). \end{array}$$

For the remaining two standard modules, all composition multiplicities are known (by localising to the case $n = 6$) except for those involving $L_8(421^2)$ and $L_8(332)$. However, neither of these occurs in $\Delta_8(22)$ or in $\Delta_8(0)$ by an application of Proposition 2.2 with $\mu = (322)$, respectively $\mu = (3211)$. Thus we have that

$$\Delta_8(22) = \begin{array}{c} L_8(22) \\ | \\ L_8(321) \end{array} \qquad \text{and} \qquad \Delta_8(0) = \begin{array}{c} L_8(0) \\ | \\ L_8(22). \end{array}$$

9.4. **The case** $n = 9$**.** In this case we have 6 standard modules, labelled by (1), (21), (311), (41^3), (51^4) and (3^3). By (2) there is a homomorphism from each of these standards to the preceding one in the list, except in the case of (3^3). For this weight we instead use (3) which tells us that

$$[\Delta_9(311) : L_9(3^3)] = 1.$$

As in earlier cases, we have that

$$\Delta_9(51^4) = L_9(51^4), \qquad \Delta_9(3^3) = L_9(3^3), \qquad \text{and} \qquad \Delta_9(41^3) = \begin{array}{c} L_9(41^3) \\ | \\ L_9(3^3). \end{array}$$

The module $L_9(51^4)$ cannot occur in any other standards, by applying Proposition 2.2 with $\mu = (41^4)$, and similarly $L_9(3^3)$ can only occur in $\Delta_9(311)$, by taking $\mu = (331)$. By the above observations and localisation to $n = 7$ we deduce that

$$\Delta_9(311) = \begin{array}{ccc} & L_9(311) & \\ \diagup & & \diagdown \\ L_9(41^3) & & L_9(3^3) \end{array} \qquad \Delta_9(21) = \begin{array}{c} L_9(21) \\ | \\ L_9(311) \end{array} \qquad \Delta_9(1) = \begin{array}{c} L_9(1) \\ | \\ L_9(21). \end{array}$$

We will need to consider $\operatorname{res}_{10} \Delta_{10}(321)$. For this we need to understand the various standard modules arising in the short exact sequence (5) in this case. First note that (32) and (221) are the unique weights in their respective blocks when $n = 9$. For the weights (331), (322) and (3211) there is exactly one larger weight in the same block in each case, respectively (4311), (4221), and (3321).

It follows from the above remarks, (2), and (5) that $\operatorname{res}_{10} \Delta_{10}(321)$ has a short exact sequence

$$0 \longrightarrow A \longrightarrow \operatorname{res}_{10} \Delta_{10}(321) \longrightarrow B \longrightarrow 0 \tag{23}$$

where

$$A \cong L_9(221) \oplus \begin{array}{c} L_9(311) \\ L_9(41^3)\ \ L_9(3^3) \end{array} \oplus L_9(32) \tag{24}$$

and

$$B \cong \begin{array}{c} L_9(421) \\ L_9(432) \end{array} \oplus \begin{array}{c} L_9(331) \\ L_9(4311) \end{array} \oplus \begin{array}{c} L_9(322) \\ L_9(4221) \end{array} \oplus \begin{array}{c} L_9(3211) \\ L_9(3321). \end{array} \tag{25}$$

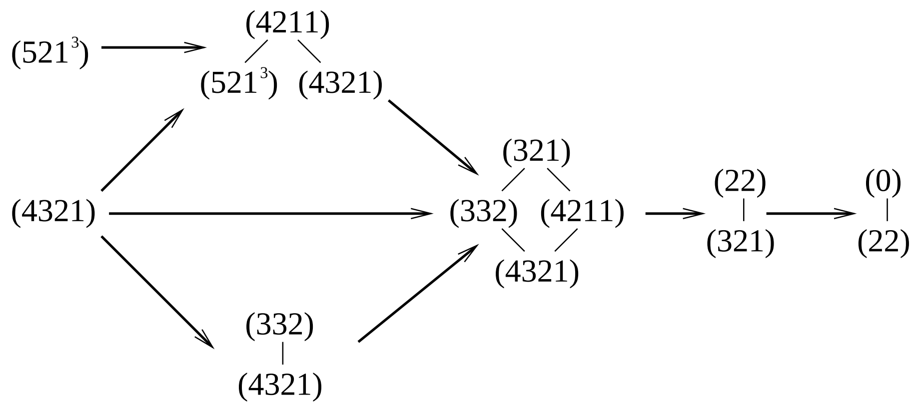

FIGURE 16. The block containing 0 when $n = 10$

9.5. **The case** $n = 10$**.** From now on, we will summarise the results obtained for each value of n in a single diagram, together with an explanation of how they were derived. In each such diagram we shall illustrate the structure of individual modules as above, but label simple factors just by the corresponding partition. We will indicate the existence of a homomorphism between two modules by an arrow. (It will be clear which standard module is which by the label of the simple in the head.)

For $n = 10$ we claim that the structure of the block containing (0) is given by the data in Figure 16. The structure of the modules $\Delta_{10}(521^3)$, $\Delta_{10}(4321)$ and $\Delta_{10}(4211)$, follows exactly as in the preceding cases for partitions of n and $n-2$. For $\Delta_{10}(332)$ we also need to note that $(332) \not\subset (521^3)$, and so $L_{10}(521^3)$ cannot occur.

To see that $L_{10}(521^3)$ cannot occur anywhere else it is enough to note (by Proposition 2.2) that $\Delta_9(421^3)$ is projective. Similarly $L_{10}(4321)$ cannot occur in the standards $\Delta_{10}(22)$ and $\Delta_{10}(0)$ as $\Delta_8(431)$ is projective. The structure of $\Delta_{10}(0)$ and $\Delta_{10}(22)$ then follows by localisation to $n = 8$.

The only remaining module is $\Delta_{10}(321)$. It is clear that this must have at most the four factors shown. The multiplicities of $L_{10}(4211)$ and $L_{10}(332)$ must be 1 by localisation to the case $n = 8$. It remains to show that the final factor has multiplicity 1, that there is a map to the module from $\Delta_{10}(4321)$, and that the module structure is as shown.

Consider $\mathrm{res}_{10} L_{10}(4321) = \mathrm{res}_{10} \Delta_{10}(4321)$. By (5), and the simplicity of standard modules $\Delta_n(\lambda)$ when $\lambda \vdash n$, this has simple factors

$$L_9(432) \qquad L_9(4311) \qquad L_9(4221) \qquad L_9(3321). \tag{26}$$

If we consider $\mathrm{res}_{10} \Delta_{10}(4211)$ and $\mathrm{res}_{10} \Delta_{10}(332)$, using the structure of $\Delta_{10}(4211)$ and $\Delta_{10}(332)$ given above, it is easy to show that neither $\mathrm{res}_{10} L_{10}(4211)$ nor $\mathrm{res}_{10} L_{10}(332)$ contain any of the factors in (26). Comparing with (23), (24), and (25) we see that

$$[\Delta_{10}(321) : L(4321)] \leq 1.$$

Further, either this simple does occur, or the simples in (26) all occur in $\mathrm{res}_{10} L_{10}(321)$.

By (6) we have that $\mathrm{res}_{10} L_{10}(4211)$ contains $L_9(421)$ and $\mathrm{res}_{10} L_{10}(332)$ contains $L_9(331)$. (In fact, all factors can be easily determined.) Comparing with (25) we see that both

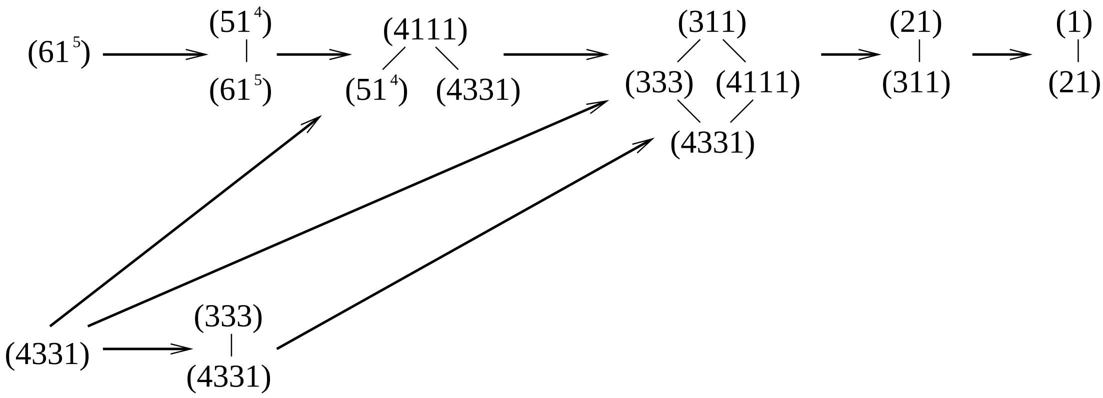

FIGURE 17. The block containing (1) when $n = 11$

$L_{10}(4211)$ and $L_{10}(331)$ must occur above the simple whose restrictions contribute the terms in (26). Thus $L_{10}(4321)$ must occur and forms the socle of $\Delta_{10}(321)$. This completes our verification of the various claims above, and so the block structure is as shown in Figure 16.

9.6. **The case** $n = 11$**.** We claim that the structure of the block containing (1) is given by the data in Figure 17. The structure of the modules $\Delta_{11}(61^5)$, $\Delta_{11}(4331)$, $\Delta_{11}(51^4)$ and $\Delta_{11}(333)$ follows as for $n = 10$ for partitions of n and $n-2$.

The modules $\Delta_{10}(51^5)$ and $\Delta_8(431)$ are projective. Therefore by Proposition 2.2 the simple $L_{11}(61^5)$ cannot occur in any of the remaining standards, and $L_{11}(4331)$ cannot occur in $\Delta_{11}(21)$ or $\Delta_{11}(1)$. The structure of these latter two modules now follows by localisation to the case $n = 9$. To see that the structure of $\Delta_{11}(41^3)$ is as illustrated follows from (3) and localisation.

The only remaining module is $\Delta_{11}(311)$. By localising this contains $L_{11}(31^2)$, $L_{11}(3^3)$ and $L_{11}(41^3)$, all with multiplicity. We have eliminated all other possible factors except $L_{11}(4331)$. We proceed as for the module $\Delta_{10}(321)$ above. Restriction of $\Delta_{11}(311)$ contains $L_{10}(433)$ with multiplicity one, and this can only arise in the restriction of $L_{11}(4311)$. Arguing as in the $n = 10$ case we also see that $L_{11}(4311)$ must coincide with the socle of $\Delta_{11}(311)$, and so we are done.

9.7. **The case** $n = 12$**.** We claim that the structure of the block containing (0) is given by the data in Figure 18. As usual, the structure of the modules labelled by partitions of n and $n-2$ is straightforward. The simple $L_{12}(621^4)$ cannot occur anywhere else as $\Delta_{11}(521^4)$ is projective.

The modules $\Delta_{10}(42211)$ and $\Delta_8(2^4)$ are both projective. Therefore $L_{12}(53211)$ cannot occur in any standard labelled by a partition of 6 or smaller, while $L_{12}(4422)$ cannot occur in $\Delta_{12}(0)$. The structure of $\Delta_{12}(0)$ is then clear by localisation.

Next consider $\Delta_{12}(332)$. This cannot contain $L_{12}(53211)$ as $(332) \not\subset (53211)$, so it is enough by localisation to verify that $L_{12}(4422)$ cannot occur. But this is clear, as the restriction of this simple contains $L_{11}(4421)$, which is not in the same block as any of the standard modules in the restriction of $\Delta_{12}(332)$.

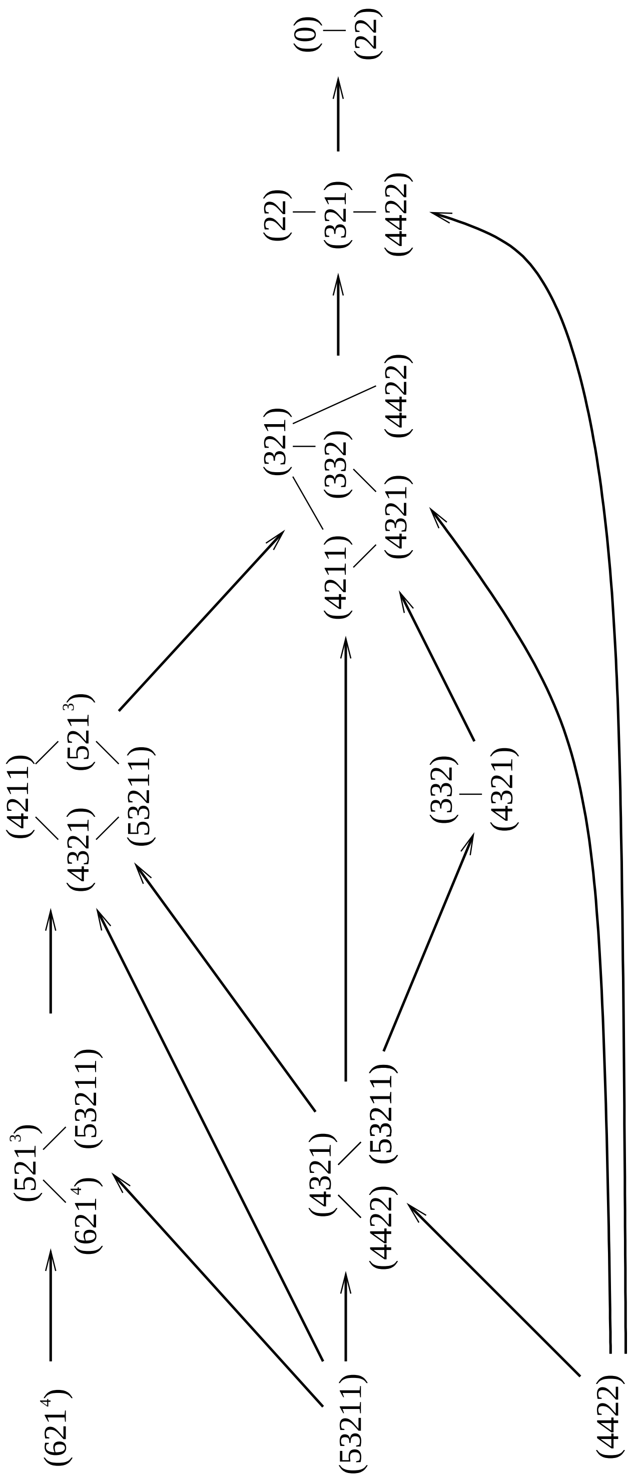

FIGURE 18. The block containing 0 when $n = 12$

It remains to determine the structure of $\Delta_{12}(321)$ and $\Delta_{12}(22)$. In each case we know the multiplicity of all composition factors by localisation and the remarks above, except for $L_{12}(4422)$. Using Proposition 2.4 we see that we have a non-zero homomorphism from $\Delta_{12}(4422)$ into each of the two standards, and so the multiplicity of $L_{12}(4422)$ in each case is at least one. The restriction of each of these standards contains precisely one copy of $L_{11}(4322)$ and $L_{11}(4421)$, and so $L_{12}(4422)$ must occur with multiplicity one in each standard.

To determine the location of $L_{12}(4422)$ in $\Delta_{12}(321)$ note that it cannot occur below any composition factor other than $L_{12}(321)$, as this would contradict the existence of homomorphisms from $\Delta_{12}(4211)$ and $\Delta_{12}(332)$ into $\Delta_{12}(321)$. Therefore the structure of this module must be as shown. For $\Delta_{12}(22)$, we have that the simple $L_{11}(32)$ in the restriction of $L_{12}(321)$ occurs above $L_{12}(4421)$, and hence the structure of $\Delta_{12}(22)$ must be as shown.

9.8. **A comparison with Kazhdan-Lusztig polynomials.** Suppose that W is a Weyl group, with associated affine Weyl group W_p. Soergel has shown [Soe97b, Soe97a] that (provided p is not too small) the value of the parabolic Kazhdan-Lusztig polynomials $n_{\lambda\mu}$ (evaluated at $v = 1$) associated to $W \subset W_p$ determine the multiplicity of the standard module $\Delta_q(\lambda)$ in the indecomposable $T_q(\mu)$ for the quantum group U_q associated to W where q is a pth root of unity.

In the case of the quantum general linear group, Ringel duality [Erd94] translates this into a result about decomposition numbers for the Hecke algebra of type A, where now μ labels a simple module and λ a Specht module. Further, Rouquier has conjectured that the coefficient of v^t occurring in $n_{\lambda\mu}$ should correspond to the multiplicity of the simple D^μ in the tth layer of the Jantzen filtration of the Specht module S^λ.

In this spirit, we can compare our results in this section with the polynomials in Figure 9 for $n \leq 12$. We see that in each case, the value of $n_{\lambda,\mu}(1)$ from Figure 9 is exactly the multiplicity of $L_n(\mu)$ in $\Delta_n(\lambda)$, and that there is a filtration of $\Delta_n(\lambda)$ corresponding to the powers of v occurring in the polynomials for the $L_n(\mu)$s. This, together with the other Lie-like phenomena we have observed leads us to ask

Question 9.1. *(i) For the Brauer algebra with Kazhdan-Lusztig polynomials as defined in Section 9, is it true for weights in an alcove that*

$$[\Delta_n(\lambda) : L_n(\mu)] = n_{\lambda,\mu}(1) \ ?$$

(ii) Is there a (Jantzen?) filtration of $\Delta_n(\lambda)$ such that the multiplicity of a simple $L_n(\mu)$ in the tth layer is given by the coefficient of v^t in $n_{\lambda,\mu}$?

As we have noted, the results in this section answer both parts in the affirmative when $n \leq 12$ and $\delta = 1$.

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