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WEIGHT CONJECTURES FOR FUSION SYSTEMS ON AN EXTRASPECIAL GROUP

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Abstract

In a previous paper, we stated and motivated counting conjectures for fusion systems that are purely local analogues of several local-to-global conjectures in the modular representation theory of finite groups. Here, we verify some of these conjectures for fusion systems on an extraspecial group of order p^3, which contain among them the Ruiz–Viruel exotic fusion systems at the prime 7. As a byproduct, we verify Robinson's ordinary weight conjecture for principal p-blocks of almost simple groups G realizing such (nonconstrained) fusion systems.

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1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$, let G be a finite group and let B be a block of kG. One way to encode the local structure of B is via a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F} = \mathcal{F}_S(B)$ is the fusion system of the block on the defect group S of B and $\alpha = (\alpha_Q)_{Q\in\mathcal{F}^c}$ is a certain compatible family of 2-cohomology classes of automorphism groups of centric subgroups of the fusion system. In this context, an Alperin weight is a projective simple module for the twisted group algebra $k_\alpha \operatorname{Out}_{\mathcal{F}}(Q)$, and so the number $\mathbf{w}(B) = \mathbf{w}(\mathcal{F}, \alpha)$ of B-weights can be counted from the information in the pair.

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If $\mathcal{F}$ is a saturated fusion system on a finite p-group, there is nothing preventing one from fixing a compatible family α and imagining $(\mathcal{F}, \alpha)$ arose from a block in the above fashion, even if this is not actually the case. In [KLLS19], we took this point of view. There, we defined two numbers $\mathbf{k}(\mathcal{F}, \alpha)$ and $\mathbf{m}(\mathcal{F}, \alpha)$ that are both stand-ins for the number of 'ordinary irreducible characters' of $(\mathcal{F}, \alpha)$. If $(\mathcal{F}, \alpha)$ comes from a block B, then $\mathbf{m}(\mathcal{F}, \alpha)$ is exactly the local side of the equality appearing in a certain 'summed-up' version of Robinson's ordinary weight conjecture (SOWC), which is supposed to count the number of ordinary irreducible characters in B (see the paragraph preceding [KLLS19, Conjecture 2.3]). However, $\mathbf{k}(\mathcal{F}, \alpha)$ is the local side of an equality that is an immediate consequence of Alperin's weight conjecture (AWC) and which is supposed to count the same number. It was shown in [KLLS19] that $\mathbf{k}(\mathcal{F}, \alpha) = \mathbf{m}(\mathcal{F}, \alpha)$ for all pairs $(\mathcal{F}, \alpha)$, possibly with $\mathcal{F}$ an exotic fusion system, conditional on the validity of AWC for all blocks of finite group algebras. Indeed, our proof of this is a fusion-theoretic version of Robinson's theorem that AWC holds for all blocks if and only if SOWC does.

With Alperin's and Robinson's weight conjectures as a bridge from local to global, one can formulate purely local analogues of several of the local-to-global counting conjectures in block theory and there seems to be no reason why these should not hold also for exotic pairs. It was shown in [LS23] that the exotic Benson–Solomon fusion systems at the prime 2 have only the trivial compatible family, and some of the conjectures were verified for these systems in [Sem23]. The recent paper [KSST24] verifies six of the conjectures for fusion systems on a Sylow p-subgroup of $G_2(p)$, a class which contains 27 exotic systems.

In this paper, we show that a nonconstrained saturated fusion system $\mathcal{F}$ on an extraspecial p-group S of order p^3 and exponent p supports just the trivial compatible family (Theorem 4.4) and that many of the purely local counting conjectures hold for $\mathcal{F}$ (for the list, see Conjecture 2.1). Along with the other fusion systems on extraspecial groups, like those coming from sporadic groups, we give counts of the above representation theoretic quantities (as well as others that take into account defects of characters) for the exotic Ruiz–Viruel fusion systems [RV04]. For example, we show that $\mathbf{m}(\mathrm{RV}_1, 0) = 41$ and $\mathbf{m}(\mathrm{RV}_2, 0) = 33$, that is, that a 7-block with a simple Ruiz–Viruel exotic fusion system at the prime 7 would have 41 or 33 ordinary irreducible characters, respectively, were such a block to exist (it does not [KS08]). Furthemore, in Proposition 5.8, we combine our results with those of [NU09] to observe that Robinson's ordinary weight conjecture (OWC) holds for the principal p-blocks of all almost simple groups G that realise a nonconstrained fusion system $\mathcal{F}$ on an extraspecial group of order p^3 and exponent p. In [Eat04], Eaton shows that OWC is equivalent to the Dade projective conjecture in the sense that a minimal counterexample to one is a minimal counterexample to the other. This latter conjecture has previously been verified for principal p-blocks of some of the groups considered above (see, for example, [An98, AOW03, Nar07]) and in fact for all cases at the prime 3 (see [Usa01]).

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2. Some weight conjectures for fusion systems

We recall in this section just enough detail to state seven conjectures that we verify for nonconstrained fusion systems on extraspecial p-groups, and refer to [KLLS19] and [Lin19a, Sections 8.13–8.15] for more details and additional motivation. For notation involving fusion systems, we follow [AKO11]. As before, we fix an algebraically closed field k of positive characteristic p. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and $\mathcal{F}^c$ the set of $\mathcal{F}$-centric subgroups of S. Assume given a compatible family $\alpha = (\alpha_Q)_{Q\in\mathcal{F}^c}$ of 2-cohomology classes, $\alpha_Q \in H^2(\mathrm{Out}_{\mathcal{F}}(Q), k^\times)$ with coefficients in the multiplicative group of k; see [KLLS19, Definition 4.1] or [Lin19a, Theorem 8.14.5]. If $\mathcal{F}$ is the fusion system of a block, there is a canonical compatible family of *Külshammer–Puig* classes of this form.

Denote by $k_{\alpha_Q}\,\mathrm{Out}_{\mathcal{F}}(Q)$ the twisted group algebra and define

$$\mathbf{w}(\mathcal{F},\alpha) = \sum_{Q\in\mathcal{F}^c/\mathcal{F}} z(k_{\alpha_Q}\,\mathrm{Out}_{\mathcal{F}}(Q)),$$

where $z(-)$ denotes the number of isomorphism classes of projective simple modules for an algebra and the sum runs over a set of representatives for the $\mathcal{F}$-conjugacy classes of centric subgroups Q. Thus, $\mathbf{w}(\mathcal{F},\alpha)$ is the number of Alperin B-weights if $(\mathcal{F},\alpha)$ comes from a block B [Lin19a, Theorem 8.14.4], so B satisfies AWC just when $\ell(B) = \mathbf{w}(\mathcal{F},\alpha)$.

Set

$$\mathbf{k}(\mathcal{F},\alpha) = \sum_{x\in S/\mathcal{F}} \mathbf{w}(C_{\mathcal{F}}(x),\alpha(x)),$$

where $C_{\mathcal{F}}(x)$ is the centralizer subsystem, $\alpha(x)$ is the restriction of α along the inclusion functor $C_{\mathcal{F}}(x)^c \to \mathcal{F}^c$ and the sum runs over fully centralized representatives for $\mathcal{F}$-conjugacy classes of elements. If $(\mathcal{F},\alpha)$ comes from a block B, then using a lemma of Brauer about decomposing conjugacy classes by p-sections, the block satisfies AWC just when the number $\mathbf{k}(B)$ of ordinary irreducible characters in B is $\mathbf{k}(\mathcal{F},\alpha)$.

Finally, we define quantities motivated by OWC [Rob96]. For a nonnegative integer d and a subgroup Q of S, write $\mathrm{Irr}_d(Q)$ for the ordinary characters μ of defect d, that is, such that the p-part of $|Q|/\mu(1)$ is p^d. Let $\mathcal{N}_Q$ be the set of normal chains

$$\sigma = (1 = X_0 < X_1 < \cdots < X_n)$$

of p-subgroups in $\mathrm{Out}_{\mathcal{F}}(Q)$, those such that X_i is normal in X_n for each i and which begin with the trivial subgroup. The length of σ as displayed is n. The sets $\mathcal{N}_Q$ and $\mathrm{Irr}_K^d(Q)$ are $\mathrm{Out}_{\mathcal{F}}(Q)$-invariant, and we let $I(\sigma)$ and $I(\sigma,\mu)$ be the stabilizers in $\mathrm{Out}_{\mathcal{F}}(Q)$ of σ and of the pair (σ,μ), respectively.

For a pair $(\mathcal{F},\alpha)$ and $d \geqslant 0$, set

$$w_Q(\mathcal{F},\alpha,d) = \sum_{\sigma\in\mathcal{N}_Q/\,\mathrm{Out}_{\mathcal{F}}(Q)} (-1)^{|\sigma|} \sum_{\mu\in\mathrm{Irr}_K^d(Q)/I(\sigma)} z(k_\alpha I(\sigma,\mu))$$

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and

$$\mathbf{m}(\mathcal{F}, \alpha, d) = \sum_{Q \in \mathcal{F}^c/\mathcal{F}} w_Q(\mathcal{F}, \alpha, d).$$

If $(\mathcal{F}, \alpha)$ arises from a block B, then OWC is the statement that the number $\mathbf{k}_d(B)$ of ordinary characters of defect d is equal to $\mathbf{m}(\mathcal{F}, \alpha, d)$.

We can now state the seven conjectures from [KLLS19] that we verify for nonconstrained fusion systems on an extraspecial group.

CONJECTURE 2.1. *Let $\mathcal{F}$ be a saturated fusion system on finite p-group S of order p^e and let α be a compatible family. Then:*

(1) $\mathbf{k}(\mathcal{F}, \alpha) = \mathbf{m}(\mathcal{F}, \alpha)$;
(2) $\mathbf{k}(\mathcal{F}, \alpha) \leqslant |S|$;
(3) $\mathbf{w}(\mathcal{F}, \alpha) \leqslant p^s$, *where s is the sectional rank of S;*
(4) *for each positive integer d, we have* $\mathbf{m}(\mathcal{F}, \alpha, d) \geq 0$;
(5) *if S is nonabelian, then* $\mathbf{m}(\mathcal{F}, \alpha, d) \neq 0$ *for some* $d \neq e$;
(6) *if S is nonabelian and $r > 0$ is the smallest positive integer such that S has a character of degree p^r, then r is the smallest positive integer such that* $\mathbf{m}(\mathcal{F}, \alpha, d - r) \neq 0$;
(7) (a) $\mathbf{k}(\mathcal{F}, \alpha)/\,\mathbf{m}(\mathcal{F}, \alpha, e)$ *is at most the number of conjugacy classes of* $[S, S]$.
(b) $\mathbf{k}(\mathcal{F}, \alpha)/\,\mathbf{w}(\mathcal{F}, \alpha)$ *is at most the number of conjugacy classes of S.*

If $(\mathcal{F}, \alpha)$ comes from a block B then Statement (1), respectively Statement (4), holds if AWC, respectively OWC, holds for B as well as for all of its Brauer correspondents. Moreover, each of the other statements in Conjecture 2.1 hold if and only if a known conjecture holds for the block B. Those conjectures are Statement (2) (the $\mathbf{k}(B)$-conjecture), Statement (3) (the Malle–Robinson conjecture [MR17]), Statement (5) (Brauer's height zero conjecture), Statement (6) (the Eaton–Moreto conjecture [EM14]) and Statement (7) (the Malle–Navarro conjecture [MN06]).

We conclude this section by showing that Statement (2) would follow as a consequence of Statement (7)(a).

PROPOSITION 2.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S of order p^e and let α be a compatible family. Then,*

$$\mathbf{m}(\mathcal{F}, \alpha, e) \leq |S : [S, S]|.$$

In particular, Conjecture 2.1(7)(a) implies Conjecture 2.1(2).

PROOF. Since $\mathrm{Out}_{\mathcal{F}}(S)$ is a p'-group, $|\mathcal{N}_S| = 1$ and we have

$$\mathbf{m}(\mathcal{F}, \alpha, e) = \sum_{\mu \in \mathrm{Irr}^e(S)/\,\mathrm{Out}_{\mathcal{F}}(S)} z(k_{\alpha_S} C_{\mathrm{Out}_{\mathcal{F}}(S)}(\mu)).$$

Let $\mathcal{F}_0 = N_{\mathcal{F}}(S)$ and let E be a complement to $\mathrm{Inn}(S)$ in $\mathrm{Aut}_{\mathcal{F}}(S)$. Then, $S \rtimes E$ is a model for $\mathcal{F}_0$. Moreover, there exists a central extension, say G, of $S \rtimes E$ by a p'-group

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and a block B of kG with fusion system $\mathcal{F}_0$ and Külshammer–Puig cocycle at S equal to α_S (see [AKO11, Proposition IV.5.35] and its proof). By [Rob04, Theorem 2.5], the ordinary weight conjecture holds for B, that is, the number of maximal defect characters in B is

$$\sum_{\mu\in \mathrm{Irr}^e(S)/\,\mathrm{Out}_{\mathcal{F}_0}(S)} z(k_{\alpha_S} C_{\mathrm{Out}_{\mathcal{F}_0}(S)}(\mu)),$$

which is $\mathbf{m}(\mathcal{F},\alpha,e)$ since $\mathrm{Out}_{\mathcal{F}_0}(S) = \mathrm{Out}_{\mathcal{F}}(S)$. Thus, by [Kül87, Theorem 1], the inequality $\mathbf{m}(\mathcal{F},\alpha,e) \le |S : [S,S]|$ is a consequence of Brauer's $\mathbf{k}(B)$-conjecture for p-solvable groups, or equivalently the $k(GV)$-conjecture (see [Nag62]). Since this latter statement is known by [GMRS04], the result holds. Finally, if Conjecture 2.1(7)(a) holds, then $\mathbf{k}(\mathcal{F},\alpha) \le |[S,S]^{cl}|\,\mathbf{m}(\mathcal{F},\alpha,e) \le |[S,S]|\,\mathbf{m}(\mathcal{F},\alpha,e) \le |S|$ and so Conjecture 2.1(2) also holds. □

3. Fusion systems on an extraspecial group of order p^3

In this section, we recall the Ruiz–Viruel classification [RV04] of the saturated fusion systems over an extraspecial group S of order p^3 and exponent p while setting up notation. There are three exotic fusion systems over S when $p = 7$.

For the remainder of this section, $S \cong p_+^{1+2}$ denotes an extraspecial group of order p^3 and exponent p. Fix the presentation

$$\langle \mathbf{a},\mathbf{b},\mathbf{c} \mid \mathbf{a}^p = \mathbf{b}^p = \mathbf{c}^p = [\mathbf{a},\mathbf{c}] = [\mathbf{b},\mathbf{c}] = [\mathbf{a},\mathbf{b}]\mathbf{c}^{-1} = 1\rangle$$

for S. Each element of S can then be written in the form $\mathbf{a}^r\mathbf{b}^s\mathbf{c}^t$ for unique nonnegative integers $0 \leqslant r,s,t \leqslant p-1$. Let ε be a fixed primitive pth root of 1 in $\mathbb{C}$. We first list some basic information about S, along with general information about the associated saturated fusion systems on S.

Lemma 3.1. *Let S be an extraspecial p-group of order p^3 and of exponent p, as given above.*

(1) *The map that sends $X := \left(\begin{smallmatrix} x & y \\ z & w\end{smallmatrix}\right)$ to the class $[\varphi] \in \mathrm{Out}(S)$ of the automorphism φ given by*

$$\boldsymbol{a} \longmapsto \boldsymbol{a}^x\boldsymbol{b}^z, \quad \boldsymbol{b} \longmapsto \boldsymbol{a}^y\boldsymbol{b}^w, \quad \boldsymbol{c} \longmapsto \boldsymbol{c}^{\det(X)}$$

is an isomorphism from $\mathrm{GL}_2(p)$ to $\mathrm{Out}(S)$.

(2) *A complete set of S-conjugacy class representatives of elements of S is given by*

$$\{\boldsymbol{a}^i\boldsymbol{b}^j \mid 0 \le i,j \le p-1\} \cup \{\boldsymbol{c}^k \mid 1 \le k \le p-1\}.$$

(3) *$\mathrm{Irr}(S)$ consists of p^2 linear characters $\chi_{u,v}$, $0 \le u,v \le p-1$, and $p-1$ faithful characters φ_u, $1 \le u \le p-1$, of degree p. The characters are given explicitly by*

$$\chi_{u,v}(\boldsymbol{a}^r\boldsymbol{b}^s\boldsymbol{c}^t) := \varepsilon^{ru+sv} \quad \text{and} \quad \varphi_u(\boldsymbol{a}^r\boldsymbol{b}^s\boldsymbol{c}^t) := \begin{cases} p\varepsilon^{ut} & \text{if } r = s = 0,\\ 0 & \text{otherwise.}\end{cases}$$

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PROOF. See [RV04], for example, for parts (1) and (2). Part (3) is contained in [Gor80, Theorem 5.5.4]. □

The elementary abelian subgroups of S of order p^2 are in one-to-one correspondence with the points of the projective line over $\mathbb{F}_p$; set

$$Q_i := \langle \mathbf{c}, \mathbf{ab}^i \rangle \quad (0 \le i \le p-1) \qquad \text{and} \qquad Q_p := \langle \mathbf{c}, \mathbf{b} \rangle.$$

From [AKO11, Section III.6.2], we have the following description of the fusion systems on S.

THEOREM 3.2. *Let $\mathcal{F}$ be a saturated fusion system on S. Then,*

$$\mathcal{F}^{cr} \subseteq \{Q_i \mid 0 \le i \le p\} \cup \{S\}$$

and the following hold:

(1) $\mathrm{Out}_{\mathcal{F}}(S)$ *is a p'-group;*
(2) *for each i and $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\alpha(Q_i) = Q_i, \alpha|_{Q_i} \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$;*
(3) *if $Q_i \in \mathcal{F}^r$, then $\mathrm{SL}_2(p) \le \mathrm{Aut}_{\mathcal{F}}(Q_i) \le \mathrm{GL}_2(p)$. If $Q_i \notin \mathcal{F}^r$, then $\mathrm{Aut}_{\mathcal{F}}(Q_i) = \{\varphi|_{Q_i} \mid \varphi \in N_{\mathrm{Aut}_{\mathcal{F}}(S)}(Q_i)\}$;*
(4) *if $Q_i \in \mathcal{F}^r$ and $\beta \in N_{\mathrm{Aut}_{\mathcal{F}}(Q_i)}(Z(S))$, then β extends to an element of $\mathrm{Aut}_{\mathcal{F}}(S)$.*

Conversely, any fusion system $\mathcal{F}$ over S for which Conditions (1)–(4) hold and for which each morphism is a composition of restrictions of $\mathcal{F}$-automorphism of S and the Q_i is saturated. Moreover, in this case, $\mathcal{F}$ is constrained if and only if at most 1 of the Q_i is centric and radical.

When $\mathcal{F}$ is nonconstrained, the possibilities for $\mathrm{Out}_{\mathcal{F}}(S)$ and $\mathrm{Out}_{\mathcal{F}}(Q_i)$ (and hence, $\mathcal{F}$) are given in [RV04, Tables 1.1 and 1.2]. Apart from the p-fusion systems of $\mathrm{PSL}_3(p)$, $p \ge 3$ and their almost simple extensions, there are thirteen exceptional fusion systems for $3 \le p \le 13$, three of which are exotic.

4. Compatible families

The goal of this section is to show that there are no nonzero compatible families for the nonconstrained fusion systems over S. Recall that the Schur multiplier $M(G)$ of a finite group G is a finite abelian group, which may be defined as the second cohomology group $H^2(G, \mathbb{C}^\times)$. In computing the Schur multiplier of various groups, we make use of its connection with stem extensions. A *stem extension* of a group G is a central extension $1 \to Z \to \widehat{G} \xrightarrow{\pi} G \to 1$ such that $\ker(\pi) \leqslant Z(\widehat{G}) \cap [\widehat{G}, \widehat{G}]$. We often identify $\ker(\pi)$ with Z. If $Z \cong M(G)$ in this situation, then the extension (or π, or $\widehat{G}$) is said to be a *Schur covering* of G. Given a central extension as above, there is an associated inflation-restriction exact sequence

$$1 \to H^1(G, \mathbb{C}^\times) \xrightarrow{\text{inf}} H^1(\widehat{G}, \mathbb{C}^\times) \xrightarrow{\text{res}} \mathrm{Hom}(Z, \mathbb{C}^\times) \xrightarrow{\text{tra}} H^2(G, \mathbb{C}^\times) \xrightarrow{\text{inf}} H^2(\widehat{G}, \mathbb{C}^\times), \quad (4\text{-}1)$$

in which three of the maps are given by inflation or restriction, and the fourth is the transgression map. This is defined by first choosing a cocycle α representing

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the class $[\alpha] \in H^2(G, Z)$ of the extension. For any homomorphism $\varphi \in \mathrm{Hom}(Z, \mathbb{C}^\times)$, post-composition with φ yields a 2-cocycle with values in $\mathbb{C}^\times$, and then $\mathrm{tra}(\varphi)$ is defined as the class $[\varphi \circ \alpha] \in H^2(G, \mathbb{C}^\times)$.

The next lemma collects a number of general results regarding the Schur multiplier, which are used later in special cases. The first four parts are due to Schur. In parts (4) and (5), results of Schur and of Blackburn [Bla72] are quoted, and require the following additional notation. Denote the abelianization of a group G by G^{ab}, and write $G^{\mathrm{ab}} \wedge G^{\mathrm{ab}}$ for the quotient of $G^{\mathrm{ab}} \otimes_{\mathbb{Z}} G^{\mathrm{ab}}$ by the subgroup generated by $a \otimes b + b \otimes a$ as a and b range over G^{ab}.

LEMMA 4.1. *The following hold for a finite group G.*

(1) *If* $1 \to Z \to \widehat{G} \to G \to 1$ *is a stem extension, then the transgression map* tra *in (4-1) is injective and hence,* $Z \cong \mathrm{Hom}(Z, \mathbb{C}^\times)$ *is isomorphic to a subgroup of* $M(G)$.

(2) *There exists a Schur covering* $1 \to Z \to \widehat{G} \xrightarrow{\pi} G \to 1$ *and, for any such covering, the associated transgression map is an isomorphism.*

(3) *If there exists a Schur covering as in Condition (2) such that* $\pi\colon \pi^{-1}(H) \to H$ *is a stem extension of some subgroup H of G, then the restriction map* $M(G) \to M(H)$ *is injective.*

(4) *If* $G = G_1 \times G_2$ *is a direct product, then one has*

$$M(G) \cong M(G_1) \times M(G_2) \times (G_1^{\mathrm{ab}} \otimes_{\mathbb{Z}} G_2^{\mathrm{ab}}).$$

(5) *(Blackburn) Assume* $G = K \wr H$ *is a wreath product and write m for the number of involutions in H. Then,* $M(G)$ *is isomorphic to the direct product of* $M(H)$, $M(K)$, $\frac{1}{2}(|H| - m - 1)$ *copies of* $K^{\mathrm{ab}} \otimes K^{\mathrm{ab}}$ *and m copies of* $K^{\mathrm{ab}} \wedge K^{\mathrm{ab}}$.

PROOF. We refer to [CR90, Lemma 11.42] for part (1) and to [CR90, Theorem 11.43] for part (2). Assume the hypotheses of part (3) and set $\widehat{H} = \pi^{-1}(H)$. Choose a 2-cocycle α representing the class of the central extension $1 \to Z \to \widehat{G} \to G \to 1$. Then, $\alpha|_{H \times H}$ represents the class of the central extension $1 \to Z \to \widehat{H} \to H \to 1$, and the square

$$\begin{array}{ccc} \mathrm{Hom}(Z, \mathbb{C}^\times) & \xrightarrow{\mathrm{tra}_G} & H^2(G, \mathbb{C}^\times) \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{res}} \\ \mathrm{Hom}(Z, \mathbb{C}^\times) & \xrightarrow[\mathrm{tra}_H]{} & H^2(H, \mathbb{C}^\times) \end{array}$$

commutes. As tra_H is injective by part (1) and tra_G an isomorphism by part (2), part (3) follows. Finally, a proof of part (4) may be found in [Wie71, Corollary 3] and part (5) is [Bla72, Theorem 1]; see also [Kar87, Theorems 2.2.10 and 6.3.3]. □

Whenever p is a prime, we write $M(G)_p$ for the p-primary part of $M(G)$, and $M(G)_{p'}$ for the p'-primary part of $M(G)$. The following lemma collects some basic information about the various primary parts of the Schur multiplier.

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LEMMA 4.2. *Let G be a finite group, let p be a prime and let k be an algebraically closed field of characteristic p.*

(1) *$H^2(G, k^\times)$ is isomorphic to the p'-part $M(G)_{p'}$ of the Schur multiplier.*
(2) *If $H \leqslant G$ contains a Sylow p-subgroup of G, then the restriction map $M(G)_p \to M(H)_p$ is injective.*
(3) *If G has cyclic Sylow p-subgroups, then $M(G)_p = 1$.*

PROOF. See [Kar87, Proposition 2.1.14] for part (1). Part (2) is derived from the fact that restriction to H followed by transfer to G is multiplication by the index of H in G, which by assumption is prime to p. Then, part (3) follows from part (2) and the fact that the second cohomology group of a finite cyclic group with coefficients in a divisible group with trivial action is trivial, but see also [CR90, Proposition 11.46]. □

We now specialize to the following computations of Schur multipliers of specific finite groups that appear as subgroups of automorphism groups of centric radicals in certain nonconstrained saturated systems over S.

LEMMA 4.3. *The following hold.*

(1) *If p is any prime and G is a subgroup of $\mathrm{GL}_2(p)$ containing $\mathrm{SL}_2(p)$, then $M(G) = 1$.*
(2) *Let G be a 2-group of maximal class. Then, $M(G) \cong C_2$ if G is dihedral, while $M(G) = 1$ if G is semidihedral or quaternion. If G is a dihedral 2-group and V is any 4-subgroup of G, then the restriction $M(G) \to M(V)$ is injective.*
(3) *Let $G = C_n \wr C_2$ with $n \geqslant 2$. Then, $M(G) \cong C_2$ if n is even and $M(G) = 1$ if n is odd. If $J \leqslant G$ is the homocyclic subgroup of rank 2 and exponent n, then the restriction $M(G) \to M(J)$ is injective.*
(4) *If $G = \mathrm{Out}_{\mathcal{F}}(S)$ for some fusion system $\mathcal{F}$ over S appearing in [RV04, Table 1.2], then $M(G)_{2'} = 1$. Moreover, either $M(G)_2 = 1$, or $M(G)_2 \cong C_2$ and G is D_8, $S_3 \times C_6$, $C_6 \wr C_2$, $D_8 \times C_3$, or $D_{16} \times C_3$.*

PROOF. The fact that $\mathrm{SL}_2(p)$ has trivial multiplier for all primes p is standard: note that all Sylow r-subgroups of $\mathrm{SL}_2(p)$ are cyclic, except when $p \geqslant 3$ and $r = 2$, in which case a Sylow r-subgroup is generalized quaternion. It follows that $M(G)_r = 1$ for all primes r by part (2) above and Lemma 4.2(2). Also, $\mathrm{GL}_2(3)$ has trivial multiplier by part (2) and Lemma 4.2(2) since a Sylow 2-subgroup of $\mathrm{GL}_2(3)$ is semidihedral. So, to finish the proof of part (1), we may assume that $p \geqslant 5$. Let G be a subgroup of $\mathrm{GL}_2(p)$ containing $N = \mathrm{SL}_2(p)$. Since $p \geqslant 5$, N is perfect. Fix a group $\widehat{G}$ having a central subgroup Z such that $\widehat{G}/Z \cong G$. Identify $\widehat{G}/Z$ with G and let $\pi\colon \widehat{G} \to G$ be the canonical projection. We show that there is a complement to Z in $\widehat{G}$. Let $\widehat{N}$ be the preimage of N under π. Then, $\widehat{N}$ contains Z and splits over it, as $M(N) = 1$. Fix any complement N_0 of Z in $\widehat{N}$, so that $\widehat{N} = N_0 \times Z$. Then, $N_0 \cong \mathrm{SL}_2(p)$. We claim that N_0 is normal in $\widehat{G}$; it is clear that N_0 is normal in $\widehat{N}$. In general, conjugation by $g \in \widehat{G}$ sends an element $h \in N_0$ to $h'\zeta_g(h)$, where $h' \in N_0$ and $\zeta_g(h) \in Z$ are uniquely determined. Also, since Z is central in $\widehat{G}$, the assignment $h \mapsto \zeta_g(h)$ is a group homomorphism

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from N_0 to Z. Since N_0 is perfect, it follows that $\zeta_g = 1$ for each $g \in \widehat{G}$, that is, N_0 is normal in $\widehat{G}$. Write quotients by N_0 with pluses. Now, $\widehat{G}^+$ is a central extension of Z by $\widehat{G}/\widehat{N} \cong G/N$, which is a cyclic p'-group. However, we have $M(G/N) = 1$ by Lemma 4.2(3), applied with each prime divisor r of $|G/N|$ in the role of 'p' there. Thus, we may fix a complement K^+ of Z^+ in $\widehat{G}^+$ and let K be the preimage of K^+ in $\widehat{G}$. Then, K is a complement to Z in $\widehat{G}$.

Part (2) is implied by Lemma 4.1 as follows. Let G be a 2-group of maximal class, so that G is dihedral, semidihedral or quaternion. Then, $M(G)$ is a 2-group. Let $\pi\colon \widehat{G} \to G$ be any Schur covering of G with $Z = \ker(\pi)$. Then, since $Z \leqslant [\widehat{G}, \widehat{G}]$, it follows that $\widehat{G}^{\mathrm{ab}} \cong G^{\mathrm{ab}}$ is of order 4. By [Gor80, Theorem 5.4.5], $\widehat{G}$ is of maximal class, so $Z(\widehat{G})$ is of order 2. Then, either $M(G) = 1$ or $Z(\widehat{G}) = Z$ and $M(G) = C_2$. In the latter case, since $\widehat{G}$ is of maximal class, we have $\widehat{G}/Z \cong G$ is dihedral. Conversely, the dihedral group $\widehat{G} = D_{2^{k+1}}$ provides a Schur covering $\pi\colon \widehat{G} \to G$ of $G = D_{2^k}$. Fix a 4-subgroup V of G. Then, as $\pi^{-1}(V)$ is dihedral of order 8, we have that the restriction map $M(G) \to M(V)$ is injective by Lemma 4.1(3). This completes the proof of part (2).

To prove part (3), apply Lemma 4.1(5) with $K = C_n$ and $H = C_2$. Hence, $m = 1$ there. By Lemma 4.2(3) and that result, $M(G) = K^{\mathrm{ab}} \wedge K^{\mathrm{ab}}$. The multiplication map $C_n \otimes C_n \to C_n$ is an isomorphism, where C_n is viewed as an additive group, and under that map, $a \otimes b + b \otimes a$ is sent to $2ab$. Hence, $M(G) \cong C_n/2C_n \cong C_2$ if n is even, and 1 if n is odd. To prove the claim about the restriction to J, we may by Lemma 4.2(2) assume that $n = 2^l$ for some l, and then it suffices by Lemma 4.1(3) to produce a double covering of G which restricts to a stem extension of J. To this end, the group $G = C_{2^l} \wr C_2$ has a presentation with generators x, y and t, and defining relations $x^{2^l} = y^{2^l} = t^2 = xyx^{-1}y^{-1} = 1$ and $txt^{-1} = y$. Consider the group $\widehat{G}$ with generators $\mathbf{x}, \mathbf{y}, \mathbf{t}$ and defining relations $\mathbf{x}^{2^l} = \mathbf{y}^{2^l} = \mathbf{t}^2 = 1$, $\mathbf{txt}^{-1} = \mathbf{y}$ and $\mathbf{z} = [\mathbf{x}, \mathbf{y}]$ is of order 2 and central. Thus, $\widehat{G}/\langle \mathbf{x}^2, \mathbf{y}^2\rangle \cong D_{16}$ and the obvious map $\pi\colon \widehat{G} \to G$ is the pullback of the Schur covering $D_{16} \to D_8$. Let $\mathbf{J}$ be the preimage of J in $\widehat{G}$ and set $Z = \ker(\pi) = \langle \mathbf{z}\rangle$. Then, by construction, $Z \leqslant [\mathbf{J}, \mathbf{J}] \cap Z(\mathbf{J})$, so $\pi\colon \mathbf{J} \to J$ is a stem extension of J. As noted above, this completes the proof of part (3).

We now prove part (4). When G is D_8, SD_{16}, $C_6 \wr C_2$, $S_3 \times C_3$, $S_3 \times C_6$, $D_8 \times C_3$, $D_{16} \times C_3$ or $SD_{32} \times C_3$, the claim follows from parts (1), (2), (3) and Lemma 4.1(4). It remains to consider the groups '$4S_4$' and '$C_3 \times 4S_4$' in [RV04, Table 1.2]. Note that $4S_4$ as appears in [RV04] is the normalizer G in $\mathrm{GL}_2(5)$ of a Sylow 2-subgroup $Q \cong Q_8$ of $\mathrm{SL}_2(5)$. The normalizer N in $\mathrm{SL}_2(5)$ of Q is the commutator subgroup of G, isomorphic to $\mathrm{SL}_2(3)$, and the quotient G/N is cyclic of order 4. Thus, G^{ab} is cyclic of order 4. Therefore, it suffices to show that $M(G) = 1$, for then by Lemma 4.1(4), we have $M(C_3 \times G) = 1$. Since G has Sylow 3-subgroups of order 3, it follows from Lemma 4.2(3) that $M(G) = M_2(G)$. Since $\mathrm{SL}_2(3) \cong Q \rtimes C_3$ is 2-perfect and G/N is cyclic, the exact same argument as given in part (1) applies with Z a 2-group to show that $M(G)_2 = 1$. This completes the proof of part (4) and the lemma. □

Let now k be a fixed algebraically closed field of characteristic p and $\mathcal{F}$ a saturated fusion system on a finite p-group. Recall from [KLLS19, Section 4] that by a

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compatible family for $\mathcal{F}$ (of 2-cohomology classes with coefficients in $k^\times$), we mean an element of $\lim_{[S(\mathcal{F}^c)]} \mathcal{A}^2_{\mathcal{F}}$, where $\mathcal{A}^2(\mathcal{F})\colon [S(\mathcal{F}^c)] \to \mathrm{Ab}$ is the functor sending the $\mathcal{F}$-isomorphism class of a chain $[\sigma] = [Q_0 < \cdots < Q_m]$ of $\mathcal{F}$-centric subgroups to the p'-part $M(\mathrm{Out}_{\mathcal{F}}(\sigma))_{p'} = H^2(\mathrm{Out}_{\mathcal{F}}(\sigma), k^\times)$ of the Schur multiplier of $\mathrm{Out}_{\mathcal{F}}(\sigma)$.

THEOREM 4.4. *Let p be an odd prime, and let $\mathcal{F}$ be a nonconstrained saturated fusion system on an extraspecial p-group S of order p^3 and of exponent p. Then,* $\lim \mathcal{A}^2_{\mathcal{F}} = 0$.

PROOF. Consider the cochain complex $(C^*(\mathcal{A}^2_{\mathcal{F}}), \delta)$ computing the limits of $\mathcal{A}^2_{\mathcal{F}}$ as in [Lin09]. By Lemma 4.3(1), $M(\mathrm{Out}_{\mathcal{F}}(Q))_{p'} = 1$ for all elementary abelian subgroups $Q \in \mathcal{F}^{cr}$. Thus, the zeroth cochain group is $C^0(\mathcal{A}^2_{\mathcal{F}}) = M(\mathrm{Out}_{\mathcal{F}}(S))_{p'} = M(\mathrm{Out}_{\mathcal{F}}(S))$ and the coboundary map

$$\delta^0\colon M(\mathrm{Out}_{\mathcal{F}}(S)) \longrightarrow \bigoplus_{Q\in\mathcal{F}^{cr}, |Q|=p^2} M(\mathrm{Out}_{\mathcal{F}}([Q < S]))$$

is the sum of the restriction maps $M(\mathrm{Out}_{\mathcal{F}}(S)) \to M(\mathrm{Out}_{\mathcal{F}}([Q < S]))$. Thus, to complete the proof, it suffices to show that at least one of these restriction maps is injective.

We regard $\mathrm{Out}_{\mathcal{F}}(S) \leqslant \mathrm{GL}_2(p)$ as acting on $\{Q_i \mid 0 \leqslant i \leqslant p\}$ as it does on the projective line, and then $\mathrm{Out}_{\mathcal{F}}([Q < S])$ is the stabilizer of the point Q. We go through the possibilities for $\mathcal{F}$ appearing in [RV04, Tables 1.1 and 1.2]. Consider first a fusion system $\mathcal{F}$ over S occurring in Table 1.2. Then, $C^0(\mathcal{A}^2_{\mathcal{F}}) = 1$ unless $G := \mathrm{Out}_{\mathcal{F}}(S)$ is listed in Lemma 4.3(4). By inspection of Table 1.2 for those cases, there is some $Q \in \mathcal{F}^{cr}$ with $|Q| = p^2$ that is stabilized by a 4-subgroup V of G. Let D be a Sylow 2-subgroup of G containing V. Then, D is dihedral. As $M(G) = M(G)_2$ by Lemma 4.3(4), the composite $M(G) \to M(D) \to M(V)$ is injective by Lemma 4.2(2) and the last sentence of Lemma 4.3(2), and hence, $M(G) \to M(\mathrm{Out}_{\mathcal{F}}([Q < S]))$ is also injective.

Now, consider a nonconstrained fusion system appearing in Table 1.1. Then, $\mathcal{F}^{cr} = \{Q_0, Q_p\}$ and $G := \mathrm{Out}_{\mathcal{F}}(S)$ may be taken in the normalizer in $\mathrm{GL}_2(p)$ of the subgroup T of diagonal matrices, which stabilizes Q_0 and Q_p. Write $G_0 = \mathrm{Out}_{\mathcal{F}}([Q_0 < S])$, for short. Then, $G_0 = G \cap T$. Assume first that $r \neq p$ is an odd prime. If r divides $p + 1$, then a Sylow r-subgroup of $\mathrm{GL}_2(p)$ is cyclic and hence, $M(G)_r = 1$ by Lemma 4.2(3). Suppose r divides $p - 1$. Then, a Sylow r-subgroup of G is contained in T and so G_0 contains a Sylow r-subgroup of G. Thus, the restriction $M(G)_r \to M(G_0)_r$ is injective by Lemma 4.2(2). It remains to consider $r = 2$. Inspection of [RV04, Table 1.1] shows that either a Sylow 2-subgroup of G stabilizes Q_0, in which case $M(G)_2 \to M(G_0)_2$ is injective by Lemma 4.2(2), or a Sylow 2-subgroup R of G is isomorphic to $C_{2^l} \wr C_2$ for some $l \geqslant 1$ and $R \cap G_0$ is the homocyclic subgroup of R of index 2. The restriction map $M(G)_2 \to M(G_0)_2$ is injective in this latter case by Lemma 4.3(3). □

5. Verification of the conjectures

We now verify a number of the conjectures in Section 1 for a nonconstrained saturated fusion system $\mathcal{F}$ over S. In the 13 exceptional cases, complete proofs by

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hand could be written down, but since we have no reasonably general argument for the specific numerical computations, the conjectures are ultimately verified using computer calculations in Magma [BCP97].

First, we need the following well-known lemma.

LEMMA 5.1. *Let G be a finite group with normal subgroup N. Suppose that V is an inertial projective simple kN-module and that G/N is a cyclic p'-group. Then, G has exactly $|G : N|$ projective simple modules lying over V. In particular, if q is a power of p and $N = \mathrm{SL}_n(q) \le G \le \mathrm{GL}_n(q)$, then $z(kG) = |G : \mathrm{SL}_n(q)|$.*

PROOF. Since G/N is a cyclic group, we have that $H^2(G/N, k^\times)$ is trivial. Thus, the hypotheses of [Lin19b, Corollary 5.3.13] hold. By that result and its proof, we may fix a simple kG-module U with $\mathrm{Res}^G_N(U) \cong V$, and then the collection of isomorphism classes of one-dimensional kG/N-modules is in one-to-one correspondence with the collection of simple kG-modules restricting to V via the map $W \mapsto U \otimes_k W$, where we regard W as a module for G by inflation. Also, the $U \otimes_k W$ are precisely the summands of the induced module $\mathrm{Ind}^G_N V$ and hence, are all projective. This completes the proof of the first statement.

In the special case of the last statement, by results of Steinberg (see [Hum06, Theorems 3.7 and 8.3]), $N = \mathrm{SL}_n(p)$ has exactly one projective simple module, the Steinberg module, which is therefore inertial. Since G/N is a cyclic p'-group in this case and since any projective kG-module is projective also as a kN-module, the last statement follows. □

In a saturated fusion system $\mathcal{F}$ on S, Lemma 3.1(a) and Lemma 3.2(1) allow one to identify $\mathrm{Out}_{\mathcal{F}}(S)$ with a subgroup of $\mathrm{GL}_2(p)$ of order prime to p. Relative to this setup, we identify elements of $\mathrm{Out}_{\mathcal{F}}(S)$ with matrices with respect to the basis $\{\mathbf{a}, \mathbf{b}\}$ (or rather, the basis $\{\mathbf{a}Z(S), \mathbf{b}Z(S)\}$ of S^{ab}). Write $\mathrm{Out}^*_{\mathcal{F}}(S)$ for those elements of $\mathrm{Out}_{\mathcal{F}}(S)$ that have determinant 1. The next lemma gives a general calculation of the quantity $\mathbf{m}(\mathcal{F}, 0, d)$.

LEMMA 5.2. *Let $\mathcal{F}$ be a saturated fusion system on S. Then:*

(1) $\mathbf{m}(\mathcal{F}, 0, 0) = \mathbf{m}(\mathcal{F}, 0, 1) = 0$*;*
(2) $\mathbf{m}(\mathcal{F}, 0, 2) = (p-1)/l \cdot |\mathrm{Out}^*_{\mathcal{F}}(S)^{\mathrm{cl}}|$*, where l denotes the index of* $\mathrm{Out}^*_{\mathcal{F}}(S)$ *in* $\mathrm{Out}_{\mathcal{F}}(S)$*; and*
(3) $\mathbf{m}(\mathcal{F}, 0, 3) = \sum_\mu z(kC_{\mathrm{Out}_{\mathcal{F}}(S)}(\mu))$*, where μ runs over a set of representatives for the* $\mathrm{Out}_{\mathcal{F}}(S)$*-orbits of linear characters of S and where $C_{\mathrm{Out}_{\mathcal{F}}(S)}(\mu)$ denotes the stabilizer of μ in* $\mathrm{Out}_{\mathcal{F}}(S)$.

Thus,

$$\mathbf{m}(\mathcal{F}, 0) = \frac{p-1}{l} \cdot |\mathrm{Out}^*_{\mathcal{F}}(S)^{\mathrm{cl}}| + \sum_{\mu \in \mathrm{Irr}^3(S)/\,\mathrm{Out}_{\mathcal{F}}(S)} z(kC_{\mathrm{Out}_{\mathcal{F}}(S)}(\mu)).$$

PROOF. Each character of $Q_i \cong C_p \times C_p$ is linear, so has defect 2, while each character of S has defect 2 or 3 by Lemma 3.1(3). This shows part (1).

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The quantity $\mathbf{m}(\mathcal{F},0,2)$ is a sum over $\mathcal{F}$-conjugacy classes of centric radicals Q of the quantities $\mathbf{w}_Q(\mathcal{F},0,2)$. Fix first a centric radical Q of order p^2. We claim $\mathbf{w}_Q(\mathcal{F},0,2)=0$. By Theorem 3.2, $\mathrm{Out}_{\mathcal{F}}(Q)$ is a subgroup of $\mathrm{GL}_2(p)$ containing $\mathrm{SL}_2(p)$ with index a, say. First, consider the trivial chain $\sigma=(\overline{Q})\in\mathcal{N}_Q$. Then, $\mathrm{Out}_{\mathcal{F}}(Q)$ stabilizes σ. There are two orbits on characters, one nontrivial and one trivial. Given a nontrivial character μ, the stabilizer of μ in $\mathrm{Out}_{\mathcal{F}}(Q)$ is a Frobenius group with normal subgroup of order p, and so has 0 projective simple modules by [KLLS19, Lemma 4.11]. Also, the trivial character has stabilizer $\mathrm{Out}_{\mathcal{F}}(Q)$, which by Lemma 5.1 has a projective simple modules. Thus, the contribution to $\mathbf{w}_Q(\mathcal{F},0,2)$ from the trivial chain is a. Next, let $\sigma=(\overline{Q}<\overline{S})\in\mathcal{N}_Q$ be the nontrivial chain. The stabilizer $I(\sigma)$ in $\mathrm{Out}_{\mathcal{F}}(Q)$ of σ has a normal subgroup $\overline{S}$ and the quotient by $\overline{S}$ is abelian of order $(p-1)a$. Then, $I(\sigma)$ has one orbit of size 1 containing the trivial character, and one of size $p-1$ consisting of those characters with kernel $Z(S)$. The respective stabilizers have $\overline{S}$ as a normal subgroup, so the contribution from these orbits is 0, by [KLLS19, Lemma 4.11]. The group $\overline{S}$ acts regularly on the remaining points of $\mathbb{P}^1(Q)$, and a subgroup in $I(\sigma)\cap SL(Q)$ of order $p-1$ acts regularly on the characters with kernel a given point. Thus, there is one further orbit with stabilizer of order a and this contributes $-a$ to $\mathbf{w}_Q(\mathcal{F},0,2)$. Hence, $\mathbf{w}_Q(\mathcal{F},0,2)=a-a=0$, as claimed.

Lastly, consider $\mathbf{w}_S(\mathcal{F},0,2)$. Here, there is only the trivial chain with stabilizer $\mathrm{Out}_{\mathcal{F}}(S)$. Note that $\mathrm{Irr}^2(S)$ consists of the $p-1$ faithful characters denoted φ_u in Lemma 3.1. Let l be the index of $\mathrm{Out}^*_{\mathcal{F}}(S)$ in $\mathrm{Out}_{\mathcal{F}}(S)$. The action of $\mathrm{Out}_{\mathcal{F}}(S)$ on $\mathrm{Irr}^2(S)$ is the same as the action on $Z(S)^\#$ (nonidentity elements) and there, $\mathrm{Out}_{\mathcal{F}}(S)$ acts semiregularly via the determinant map. So, there are $(p-1)/l$ orbits, each of size l, and a representative for each orbit has stabilizer $\mathrm{Out}^*_{\mathcal{F}}(S)$. As $\mathrm{Out}^*_{\mathcal{F}}(S)$ is a p'-group, we have $z(k\,\mathrm{Out}^*_{\mathcal{F}}(S))$ is the number of simple $k\,\mathrm{Out}^*_{\mathcal{F}}(S)$-modules, which is the number of conjugacy classes. This completes the proof of part (2).

In part (3), since $\mathrm{Irr}^3(Q_i)$ is empty, there is only one pair (Q,σ) for which the innermost sum in $\mathbf{m}(\mathcal{F},0,3)$ is nonzero, namely the pair $(S,1)$. We thus have $I(\sigma)=\mathrm{Out}_{\mathcal{F}}(S)$ for this pair. Thus,

$$\mathbf{m}(\mathcal{F},0,3)=\sum_{\mu\in\mathrm{Irr}^3(S)/\,\mathrm{Out}_{\mathcal{F}}(S)} z(kC_{\mathrm{Out}_{\mathcal{F}}(S)}(\mu)),$$

which completes the proof of part (3). The last statement then follows from parts (1)–(3), since clearly $\mathbf{m}(\mathcal{F},0,d)=0$ for $d>3$. □

We now calculate $\mathbf{k}(\mathcal{F},0)$.

LEMMA 5.3. *Let $\mathcal{F}$ be a saturated fusion system on S. Then,*

$$\mathbf{k}(\mathcal{F},0)=\mathbf{w}(\mathcal{F},0)+\frac{p-1}{l}\cdot|\mathrm{Out}^*_{\mathcal{F}}(S)^{\mathrm{cl}}|+\sum_{([x]\in S^{\mathrm{cl}}\setminus\{1,c\})/\mathcal{F}} z(kC_{\mathrm{Out}_{\mathcal{F}}(S)}([x])).$$

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PROOF. By definition, we have

$$\mathbf{k}(\mathcal{F},0) = \mathbf{w}(\mathcal{F},0) + \mathbf{w}(C_{\mathcal{F}}(\mathbf{c}),0) + \sum_{[x]\in(S^{\mathrm{cl}}\setminus\{1,\mathbf{c}\})/\mathcal{F}} \mathbf{w}(C_{\mathcal{F}}(x),0),$$

where in the latter sum, $[x]$ runs over S-classes for which $\langle x\rangle$ is fully $\mathcal{F}$-centralized. Let $1 \neq x \in S$ be such that x is not $\mathcal{F}$-conjugate to $\mathbf{c}$ and $\langle x\rangle$ is fully $\mathcal{F}$-centralized. If $Q \leq C_S(x)$ is a $C_{\mathcal{F}}(x)$-centric radical, then $|Q| = p^2$ and $x \in C_S(Q) = Q$. Therefore, $Q = \langle x, \mathbf{c}\rangle$ and $\mathrm{Out}_{C_{\mathcal{F}}(x)}(Q)$ does not contain a copy of $\mathrm{SL}_2(p)$, which is a contradiction. We conclude that $C_S(x) \trianglelefteq C_{\mathcal{F}}(x)$ and

$$\mathbf{w}(C_{\mathcal{F}}(x),0) = z(k\,\mathrm{Out}_{C_{\mathcal{F}}(x)}(C_S(x))) = z(kC_{\mathrm{Out}_{\mathcal{F}}(C_S(x))}(x)),$$

by [KLLS19, Lemma 3.3]. We claim that $C_{\mathrm{Out}_{\mathcal{F}}(C_S(x))}([x]) \cong C_{\mathrm{Out}_{\mathcal{F}}(S)}([x])$. Since $\mathcal{F}$ is saturated, the restriction map

$$\mathrm{res} : C_{\mathrm{Aut}_{\mathcal{F}}(S)}(x) \to C_{\mathrm{Aut}_{\mathcal{F}}(C_S(x))}(x)$$

is surjective with kernel containing $C_{\mathrm{Inn}(S)}(x)$ as a Sylow p-subgroup. If A is a p'-subgroup of $\ker(\mathrm{res})$, then A commutes with both $C_{\mathrm{Inn}(S)}(x) \cong \langle x\rangle$ and $C_S(C_{\mathrm{Inn}(S)}(x))$. Hence, Thompson's $A \times B$ lemma [Gor80, Theorem 5.3.4] implies that $A = 1$, and $C_{\mathrm{Out}_{\mathcal{F}}(S)}([x]) \cong C_{\mathrm{Aut}_{\mathcal{F}}(S)}(x)/C_{\mathrm{Inn}(S)}(x) \cong C_{\mathrm{Out}_{\mathcal{F}}(C_S(x))}(x)$ as claimed.

Clearly, $S \trianglelefteq C_{\mathcal{F}}(\mathbf{c})$ and so

$$\mathbf{w}(C_{\mathcal{F}}(\mathbf{c}),0) = z(k\,\mathrm{Out}_{C_{\mathcal{F}}(\mathbf{c})}(S)) = z(kC_{\mathrm{Out}_{\mathcal{F}}(S)}([\mathbf{c}])) = \frac{p-1}{l} \cdot |\mathrm{Out}^*_{\mathcal{F}}(S)^{\mathrm{cl}}|,$$

where l is as given in Lemma 5.2. This completes the proof. □

REMARK 5.4. From the classification in [RV04], we see that for any nonconstrained fusion system $\mathcal{F}$ over S, one has $l = p-1$ in Lemmas 5.2 and 5.3. Indeed, if $\mathcal{F}$ is nonconstrained, then $Z(S) = ZJ(S)$ is not weakly $\mathcal{F}$-closed and then a result of Glauberman then suggests that $\mathrm{Out}_{\mathcal{F}}(Z(S)) \cong C_{p-1}$ for any nonconstrained fusion system over S; see [Gla71, Theorem 14.14].

LEMMA 5.5. *Let ω be a generator of the multiplicative group $\mathbb{F}_p^\times$.*

(1) *The wreath product $\langle (\begin{smallmatrix}\omega & 0\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}1 & 0\\ 0 & \omega\end{smallmatrix}), (\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix})\rangle \cong C_{p-1} \wr C_2$ has $(p-1)(p+2)/2$ conjugacy classes.*
(2) *$\langle (\begin{smallmatrix}\omega & 0\\ 0 & \omega^{-1}\end{smallmatrix}), (\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix})\rangle \cong C_{p-1}.C_2$ has $(p+5)/2$ conjugacy classes.*
(3) *If $3 \mid p-1$, then $\langle (\begin{smallmatrix}\omega^3 & 0\\ 0 & 1\end{smallmatrix}), (\begin{smallmatrix}\omega & 0\\ 0 & \omega\end{smallmatrix}), (\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix})\rangle$ has $(p-1)(p+8)/6$ conjugacy classes.*
(4) *If $3 \mid p-1$, then $\langle (\begin{smallmatrix}\omega^3 & 0\\ 0 & \omega^{-3}\end{smallmatrix}), (\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix})\rangle$ has $(p+17)/2$ conjugacy classes.*

PROOF. If B denotes the base of the wreath product $G := C_n \wr C_2$, then $Z(G)$ is a cyclic subgroup of order n in B so $B\setminus Z(G)$ is the union of $(n^2-n)/2$ classes. There are n classes of elements in G outside B, which yields $n(n+3)/2$ classes altogether and part (1) holds. Next, if G denotes the group in part (2) and H is the cyclic subgroup of order $p-1$, we see that, apart from $Z(G)$ (of order 2), there are $(p-3)/2$ classes of elements in H. There are two classes of elements outside of H, which yields $(p+5)/2$

classes altogether. A similar argument proves part (4). Finally, we prove part (3). Let G denote the group in question and set $B := \langle (\begin{smallmatrix} \omega^3 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} \omega & 0 \\ 0 & \omega \end{smallmatrix}) \rangle$. We see that $Z(G)$ has order $p-1$ and so $B \backslash Z(G)$ is the union of $(p-1)(p-4)/6$ classes. There are $p-1$ classes of elements in G outside B and this yields $(p-1)(p+8)/6$ classes altogether, as needed. □

We now describe the computations of the quantities $\mathbf{m}(\mathcal{F}, 0)$, $\mathbf{w}(\mathcal{F}, 0)$ and $\mathbf{k}(\mathcal{F}, 0)$ that were carried out in Magma [BCP97] and listed in Tables 2 and 3. The list of nonconstrained saturated fusion systems on S is given in Table 2, based on the list in [RV04, Tables 1.1 and 1.2]. Generators for $\mathrm{Out}_{\mathcal{F}}(S)$ are listed in the third column of Table 2 for the convenience of the reader: in each case, there is exactly one $\mathrm{Out}(S)$-conjugacy class of subgroups isomorphic with $\mathrm{Out}_{\mathcal{F}}(S)$ and a representative is chosen to contain as many diagonal matrices as possible. Then, $\mathrm{Out}_{\mathcal{F}}(S)$-orbit representatives and stabilizers for the actions on linear characters of S and S-conjugacy classes were computed and listed in columns four through seven, using the notation of Lemma 3.1. In each case, $\mathbf{m}(\mathcal{F}, 0, 3)$ is computed using Lemma 5.2(3), by summing up the number of projective simple modules of the stabilizers listed in the fifth column of Table 2. Then, $\mathbf{m}(\mathcal{F}, 0, 3)$ is listed in Table 3. The quantity $\mathbf{m}(\mathcal{F}, 0, 2)$ is computed using Lemmas 5.2(2) and 5.5, which computes the number of conjugacy classes of the various $\mathrm{Out}^*_{\mathcal{F}}(S)$ listed in Table 3. This completes the description of the computation of $\mathbf{m}(\mathcal{F}, 0)$ as the sum of $\mathbf{m}(\mathcal{F}, 0, 2)$ and $\mathbf{m}(\mathcal{F}, 0, 3)$. Then, $\mathbf{w}(\mathcal{F}, 0)$ is calculated using the list of outer automorphism groups of centric radicals in [RV04, Tables 1.1 and 1.2]. For example, we have denoted the three exotic fusion systems at the prime 7 by RV_1, RV_2 and $\mathrm{RV}_2 : 2$ in the tables, where $\mathrm{Out}_{\mathrm{RV}_1}(S) \cong C_6 \wr C_2$, $\mathrm{Out}_{\mathrm{RV}_2}(S) \cong D_{16} \times C_3$ and $\mathrm{Out}_{\mathrm{RV}_2:2}(S) \cong SD_{32} \times C_3$. Note that $\mathrm{RV}_2 : 2$ contains RV_2 as a normal subsystem of index 2. These systems have the following invariants:

- $\mathbf{m}(\mathrm{RV}_1, 0) = 41$ and $\mathbf{w}(\mathrm{RV}_1, 0) = 35$;
- $\mathbf{m}(\mathrm{RV}_2, 0) = 33$ and $\mathbf{w}(\mathrm{RV}_2, 0) = 25$; and
- $\mathbf{m}(\mathrm{RV}_2 : 2, 0) = 42$ and $\mathbf{w}(\mathrm{RV}_2 : 2, 0) = 35$.

Finally, $\mathbf{k}(\mathcal{F}, 0)$ is calculated using Lemma 5.3 by adding $\mathbf{w}(\mathcal{F}, 0)$ and $(p-1)/l|\mathrm{Out}^*_{\mathcal{F}}(S)^{\mathrm{cl}}|$ to the sum of the number of projective simple modules of the stabilizers listed in the seventh column of Table 2.

PROPOSITION 5.6. *Let $\mathcal{F}$ be a nonconstrained saturated fusion system on S. Then, Conjectures 2.1(1)–(7) all hold for $\mathcal{F}$.*

PROOF. This can be easily verified at this point using the tables. □

Our final result, Proposition 5.8, is a verification of OWC for principal p-blocks of all almost simple groups that realise $\mathcal{F}$. We require the following result concerning character degrees of extensions of $\mathrm{PSL}_3(p)$.

LEMMA 5.7. *For $p \geq 3$, the character degrees of* $\mathrm{PSL}_3(p)$, $\mathrm{PGL}_3(p)$, $\mathrm{PSL}_3(p) : 2$ *and* $\mathrm{PGL}_3(p) : 2$ *are listed in Table 4.*

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TABLE 1. Real characters of $\mathrm{PGL}_3(p)$.

$\chi \in \mathrm{GL}_3(p)$	Notes	$\chi(1)$	Real?	No. Real	Descend to $\mathrm{PGL}_3(p)$?	No. Descend
φ_α^1	$\alpha \in \widehat{\mathbb{F}_p^\times}$	1	$\alpha = 1, \iota$	2	$\alpha = 1$	1
$\varphi_\alpha^{p(p+1)}$	$\alpha \in \widehat{\mathbb{F}_p^\times}$	$p(p+1)$	$\alpha = 1, \iota$	2	$\alpha = 1$	1
$\varphi_\alpha^{p^3}$	$\alpha \in \widehat{\mathbb{F}_p^\times}$	p^3	$\alpha = 1, \iota$	2	$\alpha = 1$	1
$\psi_\alpha \boxtimes \beta$	$\alpha, \beta \in \widehat{\mathbb{F}_p^\times}, \alpha \neq \beta$	$p^2 + p + 1$	$\alpha, \beta = 1, \iota$	2	$\alpha = \iota, \beta = 1$	1
$\mathrm{st}_\alpha \boxtimes \beta$	$\alpha, \beta \in \widehat{\mathbb{F}_p^\times}, \alpha \neq \beta$	$p(p^2 + p + 1)$	$\alpha, \beta = 1, \iota$	2	$\alpha = \iota, \beta = 1$	1
$\chi_{\alpha,\beta} \boxtimes \gamma$	$\alpha, \beta, \gamma \in \widehat{\mathbb{F}_p^\times}$, distinct	$(p+1)(p^2 + p + 1)$	$\alpha = \overline{\beta}, \gamma = 1, \iota$	$p - 3$	$\gamma = 1$	$\frac{p-3}{2}$
$\chi_\varphi \boxtimes \alpha$	$\varphi^p \neq \varphi \in \widehat{\mathbb{F}_{p^2}^\times}, \alpha \in \widehat{\mathbb{F}_p^\times}$	$(p-1)(p^2 + p + 1)$	$\varphi\vert_{\mathbb{F}_p^\times} = 1, \alpha = 1, \iota$	$p - 1$	$\alpha = 1$	$\frac{p-1}{2}$

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TABLE 2. $\mathrm{Out}_{\mathcal{F}}(S)$-orbits of $\mathrm{Irr}^3(S)$, $\mathcal{F}$-classes of S^{cl} and their $\mathrm{Out}_{\mathcal{F}}(S)$-stabilisers.

p	$\mathcal{F}$	$\mathrm{Out}_{\mathcal{F}}(S)$	$\mathrm{Irr}^3(S)/\,\mathrm{Out}_{\mathcal{F}}(S)$	Stabilisers	$(S^{\mathrm{cl}}\backslash\{1,\mathbf{c}\})/\mathcal{F}$	Stabilisers
$3 \nmid (p-1)$	$\mathrm{PSL}_3(p)$	$\langle \left(\begin{smallmatrix} \omega & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & \omega \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,0}, \chi_{1,1}$	$C_{p-1}^2, C_{p-1}, C_{p-1}, 1$	$[\mathbf{ab}]$	1
$3 \nmid (p-1)$	$\mathrm{PSL}_3(p):2$	$\langle \left(\begin{smallmatrix} \omega & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & \omega \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,1}$	$C_{p-1} \wr C_2, C_{p-1}, C_2$	$[\mathbf{ab}]$	C_2
$3 \mid (p-1)$	$\mathrm{PSL}_3(p)$	$\langle \left(\begin{smallmatrix} \omega^3 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,0}$, $\chi_{1,1}, \chi_{\omega,1}, \chi_{1,\omega}$	$C_{p-1} \times C_{(p-1)/3}, C_{(p-1)/3}$, $C_{(p-1)/3}, 1, 1, 1$	$[\mathbf{ab}]$, $[\mathbf{ab}^{\omega}]$, $[\mathbf{a}^{\omega}\mathbf{b}]$	$1, 1, 1$
$3 \mid (p-1)$	$\mathrm{PSL}_3(p):2$	$\langle \left(\begin{smallmatrix} \omega^3 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}$, $\chi_{1,1}, \chi_{\omega,1}$	$(C_{p-1} \times C_{(p-1)/3}) : C_2$, $C_{(p-1)/3}, C_2, 1$	$[\mathbf{ab}]$, $[\mathbf{ab}^{\omega}]$	$C_2, 1$
$3 \mid (p-1)$	$\mathrm{PSL}_3(p):3$	$\langle \left(\begin{smallmatrix} \omega & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & \omega \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,0}, \chi_{1,1}$	$C_{p-1}^2, C_{p-1}, C_{p-1}, 1$	$[\mathbf{ab}]$	1
$3 \mid (p-1)$	$\mathrm{PSL}_3(p):S_3$	$\langle \left(\begin{smallmatrix} \omega & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & \omega \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,1}$	$C_{p-1} \wr C_2, C_{p-1}, C_2$	$[\mathbf{ab}]$	C_2
3	${}^2\mathrm{F}_4(2)'$	$\langle \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,1}$	D_8, C_2, C_2	–	–
3	J_4	$\langle \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 2 & 2 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}$	SD_{16}, C_2	–	–
5	Th	$\langle \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 3 \\ -1 & 1 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}$	$4.S_4, C_4$	–	–
7	He	$\langle \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,0}$, $\chi_{1,1}, \chi_{3,1}, \chi_{1,3}$	$S_3 \times C_3, C_3$, $C_3, 1, C_2, C_2$	$[\mathbf{a}]$, $[\mathbf{b}]$, $[\mathbf{ab}^3]$, $[\mathbf{a}^3\mathbf{b}]$	C_3, C_3, C_2, C_2
7	$\mathrm{He}:2$	$\langle \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,1}, \chi_{3,1}$	$S_3 \times C_6, C_3, C_2, C_2$	$[\mathbf{a}]$, $[\mathbf{ab}^3]$	C_3, C_2
7	Fi_{24}'	$\langle \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,1}, \chi_{3,1}$	$S_3 \times C_6, C_3, C_2, C_2$	$[\mathbf{b}]$	C_3
7	Fi_{24}	$\langle \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,1}$	$C_6 \wr C_2, C_6, C_2$	$[\mathbf{b}]$	C_6
7	RV_1	$\langle \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,1}$	$C_6 \wr C_2, C_6, C_2$	–	–
7	O'N	$\langle \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 2 \\ 3 & 0 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,1}, \chi_{1,3}$	$D_8 \times C_3, C_2, 1, C_2$	$[\mathbf{ab}]$	1
7	$\mathrm{O'N}:2$	$\langle \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 4 \\ -1 & 2 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,1}$	$D_{16} \times C_3, C_2, C_2$	$[\mathbf{ab}]$	C_2
7	RV_2	$\langle \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 4 \\ -1 & 2 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,1}$	$D_{16} \times C_3, C_2, C_2$	–	–
7	$\mathrm{RV}_2:2$	$\langle \left(\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 1 \\ 5 & 2 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}$	$SD_{32} \times C_3, C_2$	–	–
13	M	$\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 8 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 10 & 9 \\ 5 & 2 \end{smallmatrix}\right) \rangle$	$\chi_{0,0}, \chi_{0,1}, \chi_{1,1}$	$C_3 \times 4.S_4, C_4, C_3$	$[\mathbf{ab}]$	C_3

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TABLE 3. $\mathbf{w}(\mathcal{F},0)$ and $\mathbf{m}(\mathcal{F},0,d)$ for $d=2,3$.

p	$\mathcal{F}$	$\mathrm{Out}^*_{\mathcal{F}}(S)$	$\mathbf{m}(\mathcal{F},0,2)$	$\mathbf{m}(\mathcal{F},0,3)$	$\mathbf{w}(\mathcal{F},0)$
$3\nmid(p-1)$	$\mathrm{PSL}_3(p)$	C_{p-1}	$p-1$	p^2	p^2-1
$3\nmid(p-1)$	$\mathrm{PSL}_3(p):2$	$C_{p-1}.C_2$	$(p+5)/2$	$p(p+3)/2$	$(p-1)(p+4)/2$
$3\mid(p-1)$	$\mathrm{PSL}_3(p)$	$C_{(p-1)/3}$	$(p-1)/3$	$(p^2+8)/3$	$(p^2-1)/3$
$3\mid(p-1)$	$\mathrm{PSL}_3(p):2$	$C_{(p-1)/3}.C_2$	$(p+17)/6$	$(p+1)(p+8)/6$	$(p-1)(p+10)/6$
$3\mid(p-1)$	$\mathrm{PSL}_3(p):3$	C_{p-1}	$p-1$	p^2	p^2-1
$3\mid(p-1)$	$\mathrm{PSL}_3(p):S_3$	$C_{p-1}.C_2$	$(p+5)/2$	$p(p+3)/2$	$(p-1)(p+4)/2$
3	${}^2\mathrm{F}_4(2)'$	C_4	4	9	9
3	J_4	Q_8	5	9	9
5	Th	$\mathrm{SL}_2(3)$	7	20	20
7	He	C_3	3	20	10
7	He : 2	C_6	6	25	20
7	Fi_{24}'	C_6	6	25	22
7	Fi_{24}	D_{12}	6	35	29
7	RV_1	D_{12}	6	35	35
7	O'N	C_4	4	20	19
7	O'N : 2	C_8	8	25	23
7	RV_2	C_8	8	25	25
7	$\mathrm{RV}_2:2$	Q_{16}	7	35	35
13	M	$\mathrm{SL}_2(3)$	7	55	52

PROOF. The character degrees for $\mathrm{PSL}_3(p)$ are obtained using [SF73, Table 2] and those for $\mathrm{PGL}_3(p)$ are given in [Ste51, Table VIII]. This accounts for the degrees in columns $2,4,6$ of Table 4. The entries in columns 3 and 7 are respectively computed from those of columns 2 and 6 in the following way. The group in columns 2 and 6 is $\mathrm{PGL}_3(p)$ and the group in columns 3 and 7 is $\mathrm{Aut}(\mathrm{PSL}_3(p)) = \mathrm{PGL}_3(p)\langle\tau\rangle$ with τ acting as the transpose inverse automorphism. Since every $\mathrm{GL}_3(p)$-conjugacy class is invariant under taking transposes, an irreducible character of $\mathrm{PGL}_3(p)$ is τ-stable if and only if it is real-valued. Thus, by Clifford theory, if an irreducible character χ of $\mathrm{PGL}_3(p)$ is real, then it is covered by two characters of $\mathrm{Aut}(\mathrm{PSL}_3(p))$ of the same degree as χ and if χ is not real, it is covered by a unique character of $\mathrm{Aut}(\mathrm{PSL}_3(p))$ of degree twice that of χ.

It thus suffices to determine the real characters of $\mathrm{PGL}_3(p)$. By [FH91, Section 5.2], the characters of $\mathrm{GL}_2(p)$ comprise:

- $p-1$ linear characters ψ_α for $\alpha \in \widehat{\mathbb{F}_p^\times} = \mathrm{Hom}(\mathbb{F}_p^\times, \mathbb{C}^\times)$;
- $p-1$ characters st_α of degree p for $\alpha \in \widehat{\mathbb{F}_p^\times}$;
- $\binom{p-1}{2}$ characters $\chi_{\alpha,\beta}$ of degree $p+1$ for distinct $\alpha,\beta \in \widehat{\mathbb{F}_p^\times}$;
- $\binom{p}{2}$ characters χ_φ of degree $p-1$ for $\varphi \in \widehat{\mathbb{F}_{p^2}^\times}$ with $\varphi \neq \varphi^p$.

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TABLE 4. Counts of character degrees of G for $\mathrm{PSL}_3(p) \le G \le \mathrm{Aut}(\mathrm{PSL}_3(p))$.

p $\chi(1)$	$3 \nmid p-1$ $\mathrm{PSL}_3(p)$	$3 \nmid p-1$ $\mathrm{PSL}_3(p):2$	$3 \mid p-1$ $\mathrm{PSL}_3(p)$	$3 \mid p-1$ $\mathrm{PSL}_3(p):2$	$3 \mid p-1$ $\mathrm{PSL}_3(p):3$	$3 \mid p-1$ $\mathrm{PSL}_3(p):S_3$
1	1	2	1	2	3	2
2	–	–	–	–	–	1
$p(p+1)$	1	2	1	2	3	2
$2p(p+1)$	–	–	–	–	–	1
p^3	1	2	1	2	3	2
$2p^3$	–	–	–	–	–	1
p^2+p+1	$p-2$	2	$\frac{p-4}{3}$	2	$p-4$	2
$2(p^2+p+1)$	–	$\frac{p-3}{2}$	–	$\frac{p-7}{6}$	–	$\frac{p-5}{2}$
$p(p^2+p+1)$	$p-2$	2	$\frac{p-4}{3}$	2	$p-4$	2
$2p(p^2+p+1)$	–	$\frac{p-3}{2}$	–	$\frac{p-7}{6}$	–	$\frac{p-5}{2}$
$(p-1)(p^2+p+1)$	$\frac{p(p-1)}{2}$	$p-1$	$\frac{p(p-1)}{6}$	$p-1$	$\frac{p(p-1)}{2}$	$p-1$
$2(p-1)(p^2+p+1)$	–	$\frac{(p-1)^2}{4}$	–	$\frac{(p-1)(p-3)}{12}$	–	$\frac{(p-1)^2}{4}$
$(p+1)(p^2+p+1)$	$\frac{(p-2)(p-3)}{6}$	$p-3$	$\frac{(p-1)(p-4)}{18}$	$p-5$	$\frac{p^2-5p+10}{6}$	$p-3$
$2(p+1)(p^2+p+1)$	–	$\frac{(p-3)(p-5)}{12}$	–	$\frac{(p-7)^2}{36}$	–	$\frac{p^2-8p+19}{12}$
$\frac{(p+1)(p^2+p+1)}{3}$	–	–	3	2	–	–
$2\frac{(p+1)(p^2+p+1)}{3}$	–	–	–	1	–	–
$(p+1)(p-1)^2$	$\frac{p(p+1)}{3}$	–	$\frac{(p+2)(p-1)}{9}$	–	$\frac{(p-1)(p+2)}{3}$	–
$2(p+1)(p-1)^2$	–	$\frac{p(p+1)}{6}$	–	$\frac{(p-1)(p+2)}{18}$	–	$\frac{(p-1)(p+2)}{6}$

For $\chi \in \mathrm{Irr}(\mathrm{GL}_2(p))$ and $\psi \in \widehat{\mathbb{F}_p^\times}$, let $\chi \boxtimes \psi$ denote the character of $\mathrm{GL}_3(p)$ obtained via parabolic induction. Let φ^1, $\varphi^{p(p+1)}$ and φ^{p^3} denote the unipotent characters of $\mathrm{GL}_3(p)$, and set $\varphi_\alpha^{(-)} := \varphi^{(-)} \otimes \alpha$ for $\alpha \in \widehat{\mathbb{F}_p^\times}$. Further, let $\iota \in \widehat{\mathbb{F}_p^\times}$ be of order 2. In terms of these descriptions, column 4 of Table 1 lists $2p+6$ distinct real characters of $\mathrm{GL}_3(p)$ and so these must account for all real characters by [GS11]. Those which descend to $\mathrm{PGL}_3(p)$ are described in column 6. This explains the degrees listed in columns 3 and 7 of Table 4.

The entries in column 5 can be determined in the following fashion. Let $N = \mathrm{PSL}_3(p)$, G be as in column 5 and $G_1 = \mathrm{PGL}_3(p)$. Let $\chi \in \mathrm{Irr}(N)$. By Clifford theory, we have the following.

- If χ is G_1-stable and covered by a real character of G_1, then χ is G-stable (and hence extends to two characters of G).
- If χ is G_1-stable but not covered by any real character of G_1, then χ is not G-stable.
- If χ is not G_1-stable but is covered by a real character of G_1, then one character in the G_1-orbit of χ is G-stable and the other two are permuted by G.

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- If χ is not G_1-stable and is not covered by a real character of G_1, then no character in the G_1-orbit of χ is G-stable.

Which of the four cases applies can be determined from the information in [SF73, Ste51] and from Table 1. □

PROPOSITION 5.8. *Let G be a finite almost simple group with extraspecial Sylow p-subgroups of order p^3 and nonconstrained fusion system at the odd prime p. The principal p-block of G satisfies Robinson's ordinary weight conjecture.*

PROOF. Let N be a finite simple group and $N \leqslant G \leqslant \mathrm{Aut}(N)$. Fix a Sylow p-subgroup $S \cong p_+^{1+2}$ of G with p odd and set $\mathcal{F} = \mathcal{F}_S(G)$. If $|G/N|$ is divisible by p, then there can be at most one radical elementary abelian subgroup of order p^2 and so $\mathcal{F}$ is constrained. We may thus assume that $S \leqslant N$. If $p = 3$, then the result follows by comparing Table 3 with the table after [NU09, Proposition 63]. Now, suppose $p \geqslant 5$. If N is sporadic, then by [RV04, Remark 1.4], G is one of Th, Ru($p = 5$), He, He .2, Fi_{24}', Fi_{24}, O′ N, O′ N .2($p = 7$), M($p = 13$). Character degrees for these groups and their partition into blocks are recovered from GAP [GAP24] using the `PrimeBlocks` command and the result follows by comparison with Table 3. So, we may assume that N is not sporadic. By [NU09, Theorem 31], N is one of $\mathrm{PSL}_3(p)$ or $\mathrm{PSU}_3(p)$. The normalizer of a Sylow p-subgroup in $\mathrm{PSU}_3(p)$ is strongly p-embedded by [GLS98, Theorem 7.6.2(a)] and so S is normal in $\mathcal{F}_S(N)$ in this case. Thus, N is $\mathrm{PSL}_3(p)$ (and this has a nonconstrained p-fusion system). By Lemma 5.7, the character degrees of G are listed in Table 4, and those for its principal p-block are obtained by excluding rows 6 and 7 from this list. Again, the result follows by comparing with Table 3. □

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