HYDRODYNAMIC STABILITY OF SLOWLY VARYING FLOWS

by

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Declaration

Abstract

First the classical Taylor problem of fluid rotating between two concentric cylinders is considered. A method of approximating to the curve of neutral stability is developed. Values of the torque for Taylor-vortex flows are calculated for the radius ratio of 0.95.

The classical Taylor problem of fluid rotating between two concentric cylinders is then altered by making only the outer boundary a small and slowly varying function of the vertical co-ordinate. Modified amplitude equations, both linear and non-linear are found, and modified critical Taylor numbers are calculated. For the type of solution found it appears that the axial wavenumber is uniquely determined and now depends on both the axial and vertical co-ordinates.

List of abbreviations

(r,θ,z)	cylindrical polar co-ordinates.
t	time variable.
Z	slowly varying axial co-ordinate.
x	non-dimensional radial co-ordinate.
ζ	non-dimensional axial co-ordinate.
τ	non-dimensional time variable.
z^*	non-dimensional slowly varying vertical co-ordinate.
η	ratio of inner vertical wall to outer vertical wall.
η_L	'local' ratio of inner vertical wall to outer surface.
$\eta_{L,\infty}$	η_L evaluated at $z^* = \infty$.
T	Taylor number.
T_c	parallel wall critical T.
T_o	parallel wall T on neutral curve other than T_c.
T_L	non-parallel wall 'local' T.
T_{Lc}	parallel wall 'local' critical T.
$T_{Lc,\infty}$	T_{Lc} evaluated at $z^* = \infty$.
T_{crit}	non-parallel wall critical T.
T_{Lcrit}	non-parallel wall 'local' critical T.
$T_{Lcrit,\infty}$	T_{Lcrit} evaluated at $z^* = \infty$.
$T_{Lcrit,N}$	non-linear, non-parallel wall 'local' critical T.
λ	wavenumber.
λ_c	parallel wall critical λ .
λ_o	parallel wall λ on neutral curve other than λ_c.
λ_{Lc}	parallel wall 'local' critical λ .
$\lambda_{Lc,\infty}$	λ_{Lc} evaluated at $z^* = \infty$.
$N_\zeta(\ )$	non-parallel wall observed or physical λ in ζ space.

λ_{Lcrit}	non-parallel wall 'local' critical λ (used in comparing with λ from the parallel wall problem).
$\lambda_{Lcrit,\infty}$	λ_{Lcrit} evaluated at $z^* = \infty$.
$N_{\zeta,N}$	non-linear, non parallel wall $N_\zeta(\ )$.
$\underline{U}$	extended velocity vector.

1. General Introduction

1.1 Hydrodynamic stability

The idea of the mathematical theory of hydrodynamic stability and instability is to give an understanding of how and why laminar (usually steady) flows become unstable and to give criteria for this. Once this has been achieved the next step is to try to understand how turbulence may arise from laminar instability. In most cases the connection between the different stages of fluid flow will be quite complicated. In some cases the theoretical results agree, to a certain extent, with the experimental results, whilst in other cases there are great disagreements between the laminar flow instability theory and experimental evidence.

The subject now has a vast literature and this is not the place to attempt a historical or critical survey. Rather we attempt the more modest task of putting the present work in the context of known work on Taylor–vortex flows and stability of slowly varying flows.

1.2 The Taylor-vortex instability

One of the most famous phenomena associated with instability is that of the Taylor–vortex problem, which was described with a great degree of accuracy both mathematically and experimentally by G. I. Taylor [1]. The problem involved the instability of a fluid between two concentric rotating circular cylinders.

The criterion for this case was first thought of by

Rayleigh [2] earlier who stated, "For the fluid to become unstable the
square of the angular momentum must decrease in an outward radial direction". This statement indicates that the fluid layers closer to the inner cylinder become unstable first and possibly propagate outwards. The inner layer disturbances grow exponentially with time and ~~perhaps~~ eventually dominate the whole fluid for a time. ~~but as the fluid layers near the outer cylinder are potentially stable, the disturbances are soon damped out and the fluid returns to the basic state of Couette flow~~. This stabilization force is due to the fluid viscosity and the destabilization force is the centrifugal force.

If a certain parameter called the Taylor number T (named after G. I. Taylor), which is the ratio of the stabilization force to the destabilizing force and defined by

$$T = \frac{2\Omega_1^2 R_1^2 (R_2 - R_1)^3}{\nu^2 (R_1 + R_2)} , \qquad (1.2.1)$$

is gradually increased theory indicates that for a given fixed $\eta = R_1/R_2$, the ratio of the radii of the inner cylinder to the outer cylinder, the flow changes at a certain critical Taylor number T_c from Couette flow to the Taylor—vortex flow. Here Ω_1 is the angular velocity of the inner cylinder the outer cylinder being at rest.

The results show that if T is greater than T_c then the destabilizing force overcomes the stabilizing force and instability sets in, whilst if T is less than T_c then the stabilizing force overcomes the destabilizing force and the fluid is stable and returns to laminar Couette flow.

The flow resulting from the instability consists of the Couette flow with superimposed toroidal regularly spaced vortices along the axis of the cylinders, with neighbours having an opposite sense of rotation.

In experiments, however, the change from Couette flow to a Taylor–vortex flow does not occur at precisely T_c as given by the theory but over a small finite range of T.

Some of the important work on the theory is mentioned below though we shall not attempt an exhaustive discussion.

As mentioned G. I. Taylor [1] performed initial experimental and theoretical work. He considered the case when two cylinders rotate in opposite directions to each other ($\mu < 0, \mu$ is the ratio of the angular velocities of the outer cylinder to the inner cylinder), and the case with just the outer cylinder at rest ($\mu = 0$). An axisymmetric perturbation (independent of the cylindrical polar co-ordinate θ) was superimposed on the Couette flow and substituted in the equations of motion which were then linearized. The perturbation velocities etc., were taken to be of the form

$$u = e^{\sigma t} u_1(r)\cos \lambda z \ , \ v = e^{\sigma t} v_1(r)\cos \lambda z, \ w = e^{\sigma t} w_1(r)\sin\lambda z \tag{1.2.2}$$

and $u_1(r)$ etc., were expanded as Bessel-Fourier series. This resulted in an eigenvalue problem for the neutral stability problem with $\partial/\partial t \equiv 0$ ($\sigma = 0$) and η fixed with dimensionless parameters T, λ and μ. The resultant equations were solved subject to the boundary conditions and solutions found for the linear problem. To illustrate his results the streamlines were plotted for specific cases of η and μ.

Most of the earlier work on the Taylor problem assumed axisymmetric disturbances and the equations of motion were simplified using the small gap approximation. This assumed that $R_2 - R_1$ is small compared to R_1 so terms of $O(^d/R_1)$ can be neglected. The linear stability problem on simplification yields the eigenvalue problem mentioned. The critical value of T and the corresponding value of λ, λ_c are determined by the condition that $T = T_c$ is a minimum with $\sigma = 0$. Later work by Roberts (3) and DiPrima & Eagles (4) studied this problem more fully with the full linear equations. The latter authors calculated the neutral curve, see FIG I, for $\eta = 0.95$ and $\eta = 0.5$.

Solutions of the non-linear problem were attempted when the Taylor number T is larger than T_c. The linear theory predicts the Taylor-vortex disturbance will grow exponentially with time. Stuart's (5) investigation of the axisymmetric non-linear problem used an energy balance integral method to solve the problem with $\mu = 0$ and so obtain the amplitude of the equilibrium steady Taylor-vortex flow. Work by Davey (6) and Davey, DiPrima & Stuart (7) considered the non-linear cases when the cylinders rotate in the same direction ($\mu > 0$) and the case $\mu = 0$. Davey's (6) initial work concerned the small gap problem with axisymmetric disturbances for $\eta = 0.5$ and $\eta = 0.95$. He considered a small amplitude function of time, A(t), that satisfied

$$\frac{dA}{dt} = \sigma A + a_1 A^3 \qquad (1.2.3)$$

where σ and a_1 are constants. Davey's [6] torque results were related to the past experimental results of G. I. Taylor [8] and Donnelly [9] and compared with the theoretical results of Stuart [5].

Further work on the non-linear problem was done by Davey, DiPrima & Stuart [7]. They considered the more general case of non-axisymmetric disturbances therefore a θ dependence was brought in. The small gap problem was used , but special care must be taken when non-dimensionalizing the θ derivatives, and the instability of the Taylor-vortex flow was studied. The critical Taylor number for instability of the Taylor-vortex flow (not the Couette flow) was found at about 8% above the critical value for the occurrence of these vortices. This type of wavy-vortex flow is mentioned briefly in § 1.3.

Eagles [10] used a fifth order amplitude method for studying axisymmetric and non-axisymmetric disturbances in relation to the stability of Taylor vortex flow for $\eta = 0.95$ and $\mu = 0$. The main difference over the rest of the previous work was the treatment of the full equations of motion using a matrix representation for the latter. This work was continued by Eagles [11] to involve torque calculations with non-axisymmetric disturbances in relation to wavy-vortex flow.

Other work involving rotating cylinders was by Krueger, Gross & DiPrima [12] who showed that for $\mu \lesssim -0.8$ non-axisymmetric disturbances become unstable at Taylor numbers slightly lower than the critical for $\mu = 0$. Krueger & DiPrima [13] studied the stability of a viscous fluid with an axial flow to rotational symmetrical disturbances. This was carried further by Hughes & Reid [14] who investigated the stability of spiral flow between rotating cylinders for $\mu > 0$.

The study of the stability of the basic flow between eccentric cylinders to infinitesimal disturbances of the Taylor-vortex type was done by DiPrima & Stuart [15]~~, [16]~~. The equations of motion were transformed using a modified bipolar co-ordinate system. The eccentricity and the clearance ratio of the cylinders were assumed small. This problem is analogous to that of a journal bearing. The Taylor-vortex flow that occurs at supercritical speeds was studied with respect to the effect on the torque and load carrying capacity of a journal bearing in both papers. It was found that the first vortices occur at the site of maximum clearance when the local Taylor number exceeds a critical value near that calculated in the case of co-axial cylinders.

Carr-Hill [17] considered the stability of a fluid flow between a rotating inner wavy cylinder and an outer stationary concentric cylinder. The inner wavy cylinder was given by $r = R_1(1 + \varepsilon \cos \lambda z)$ instead of the normal case with $r = R_1$. Some of the methods of solution of his problem are similar to those used in this work.

1.3 Problems involved in relation between theory and experiment

In the theoretical work results and conclusions are based on the linear analysis of disturbances to Couette flow, although weakly non-linear effects are included in later work. This theory is only valid near the critical Taylor number and the cylinder length is usually assumed infinite. The linear theory predicts the wavelengths of the cells. The wavelength of these vortices does not vary with the Taylor number as long as the flow is singly periodic (axisymmetric disturbances only) for infinite length cylinders. It is generally agreed that the non-dimensional wavenumber λ is a property of Taylor vortices and is not dependent upon boundary layer effects at the ends.

We will only deal with singly periodic flow because unsymmetrical disturbances are important only at Taylor numbers larger than those we shall consider. It is noted that at these larger Taylor numbers the vortices become modified by a waviness in the azimuthal direction (at typically about a Taylor number 10% higher than T_c when $\eta = 0.95$) and are known as doubly periodic waves, see Nakaya [18], ~~and~~ Eagles [11], F. Schultz-Grunow and H. Hein [34] and D. Coles [35].

There are various discrepancies between the theoretical and experimental work. This is to be expected since the cylinders are of finite length and we must have at least one rotating or non-rotating end plate to stop the fluid falling out !
Possibly due to end effects etc., there are slight discrepancies between theory and experiment in the wavenumber along the

cylindrical tube and the critical Taylor number.

In the present work we allow the radius of the outer surface to vary in such a way that the vortices die away towards the end ; thus end effects in experiments may be less important.

1.4 <u>Experimental work</u>

Burkhalter & Koschmieder's [19] experiments involving rotating solid end plates for the fluid column noted that the end vortex associated with a rotating end boundary appeared at Taylor numbers substantially below the critical Taylor number for $\eta = 0.727$ and $\eta = 0.525$. This coincided with Koschmieder's [20] experiments with non-rotating solid end plates with the Taylor vortices being formed first at the resting end plates. Bulkhalter & Koschmieder's [21] later work mentioned the best experimental approximation to the infinite cylinder case was with a finite fluid column with non-rotating end-plates.

Synder [22] mentions that the end effects are important in determining the flow near the middle of the fluid column (of length L) only if $L \leq 10(R_2 - R_1)$, with approximately 5 cells at each end of the cylinder affected by end effects. In earlier work Synder [23] states that with a field of 22 vortices (2 end cells and 20 Taylor cells) it is possible to find the wavelengths of the Taylor cells have been compressed by 10% as the Taylor number is increased owing to the expansion of these end cells. This is probably due to the low stability of the end cells for $T > T_c$. Burkhalter & Koschmieder [21] do state that variations in the end cells have increasingly less bearing on the other cells if $L(R_2 - R_1)$ is large.

Burkhalter & Koschmieder [21] emphasize in their introduction that the wavelength which one measures in a finite cylinder differs from the wavelength found theoretically for infinitely long cylinders. Later work [19] showed this by varying the initial conditions of the flow. Stable axisymmetric vortices with wavelengths less than the critical can be produced by sudden starts of the inner cylinder, and longer wavelengths with the vortices remaining permanent can be produced if the annulus of the cylinders is filled over its entire length while the cylinders are rotating. Experimental work by Synder [23] found the experimentally measured critical Taylor number and wavelength was 2150 $\pm$ 4% and 2.04 respectively. The theoretical results obtained by Roberts [3] were 2084 and 2.003 respectively for $\eta = 0.727$. Donnelly & Schwarz [24] also found an increase in wavenumber from about 3.2 to 3.35 as T increases from T_c to about 1950 in an apparatus with $\eta = 0.95$. Kogelman & DiPrima [25] however thought that for a given supercritical Taylor number greater than T_c vortices of different sizes or wavelengths can occur.

It seems important to learn if the wavelength of steady axisymmetric supercritical Taylor vortices substantially longer or smaller than those observed so far can be explained. Also can we explain why the end vortex cells form at Taylor numbers below the critical ?

Finally, DiPrima & Eagles [4] remarked, " We know of no mechanism that would allow the wavenumber of a developed Taylor-vortex flow between cylinders of infinite length to vary as T is increased. Shifts in the wavenumber of a developed Taylor-vortex flow would seem to require a bifurcation from that flow ".

The joy of any mathematics must be to try to develop any possible mechanism to explain how experimental results may agree with theoretical results. As mentioned the wavenumber and Taylor number which one measures in a finite cylinder do differ from the wavenumber and Taylor number found theoretically for infinite cylinders in some cases. The question of how the wavenumber etc of a Taylor-vortex flow may vary along the axis has apparently not been considered.

The present work allows the vortices and wavenumber to vary along the axis and may be amenable to experimental verification of the theory in a more precise way than the infinite cylinder theory because although our domain is still infinite the theoretically obtained vortices become very weak at a finite distance.

1.5 Slowly varying flows

The subject of slowly varying flows was briefly introduced by Benney & Rosenblat [26] in which they assumed that x and z variations are 'slow' compared with y variations and so set

$$X = \mu x, \quad Z = \mu z, \quad T = \mu t, \qquad \mu << 1. \qquad (1.5.1)$$

They ask what sort of modification of the original problem would this introduce because of these slow varitions. They went on to recommend the method of multiple scales to find the stability of slowly varying flows, without doing any detailed calculations.

Most of the past work on slowly varying flows has concerned a slow variation with time.

Seminara & Hall [27] introduced the linear stability problem of slowly varying unsteady flows in a curved channel due to slowly varying time pressure gradients. In their paper the asymptotic behaviour of small perturbation waves is determined and yet allowing the amplitude, transverse structure and amplification rate to be slowly varying with time. A solution is sought in the form of an asymptotic expansion in terms of a small parameter, σ , which characterizes the slow variation of the base flow. From their equations they get solutions of the type

$$u(\xi,z,t) = \int_{-\infty}^{\infty} \frac{1}{2} u_a(\xi,\tau) \; [\exp i(az - \theta(t) + c.c.]da \tag{1.5.2}$$

where τ is the 'slow' time variable and is related to the 'fast' time variable t by

$$\tau = \sigma t, \tag{1.5.3}$$

and $d\theta/dt$ is expected to be a function of the slow time variable τ such that

$$\frac{d\theta}{dt} = \Lambda(\tau) . \tag{1.5.4}$$

They then attempt to find asymptotic solutions with $\sigma \to 0$ such that

$$u_\alpha(\xi,\tau) = \sum_{n=0}^{\infty} \sigma^n u_n(\xi,\tau) \tag{1.5.5}$$

to (1.5.2). Upon equating powers of σ and solving certain consistency conditions on higher order equations this leads to the required amplitude equation

$$\frac{dA}{d\tau} + H(\tau)\,A(\tau) = 0. \tag{1.5.6}$$

Furthermore the order σ correction term is also slowly varying with time.

A similar method of solutions was used by Eagles (28) on the stability of slowly varying flow between concentric cylinders. He examines the effect of a slow variation with time of the Taylor number through values around T_c to see if this has an appreciable effect on the instantaneous Taylor number at which the growth of vortices is first observed. The Taylor number T was taken as a function of the slow variable $t^* = \epsilon t$ and found that where the growth rate of a disturbance is small second order effects are important and lead to variations of up to 20% in the value of T at which growth can appear. As with Seminara & Hall (27) he assumed the amplitude and amplification rate to be slowly varying with time and ended up with a similar amplitude equation as (1.5.6).

Drazin (29) however took the slowly varying flows to be a function of length. In his model of a slightly divergent flow in a channel he introduced a 'slow' length variable by

$$X = \epsilon x \tag{1.5.7}$$

and tried to solve his resultant equation for the perturbation u with the following boundary conditions

$$u = 0 \ , \ y = 0 \quad \text{and} \quad y = \pi h(X) \tag{1.5.8}$$

$$u = 0 \ , \ X = \pm D,$$

where $\pi h(X)$ represents the width of the channel and is itself slowly varying. He assumed an asymptotic solution of the form

$$u \sim U_0(y,X)g_0(x,\varepsilon)\exp(st) \quad \text{as} \quad \varepsilon \to 0 \ , \tag{1.5.9}$$

and illustrated his theory with two cases of $h(X)$. See also Gaster[36] and Bouthier.[37]

The stability problem of slowly varying flow in a divergent channel was studied by Eagles & Weissman [30]. The properties of the disturbance were assumed to be slowly varying functions of space. They started with a low order solution of the form

$$\Psi(\eta,X)\exp[\ i(\theta(\xi) - \beta t)\ ] \tag{1.5.10}$$

where X is a slow streamwise co-ordinate. The complex phase function θ, yet to be determined, describes the fast variation but its derivative, the wavenumber $K(X)$, is assumed to be slowly varying

$$\frac{d\theta}{d\xi} = K(X) \ , \tag{1.5.11}$$

(compare this with (1.5.4)). The analysis is carried through so certain boundary conditions and consistency conditions are satisfied. From these they were able to obtain a differential equation for Ψ and $K(X)$ to solve at second order. Again a similar amplitude equation is obtained as in (1.5.6).

Some of the ideas mentioned here and in preceding sections will be used in this thesis.

1.6 The general problem

In the problem we shall tackle the full non-dimensional Navier-Stokes equations with the matrix notation introduced by Eagles[10].. This allows more uniformity of treatment of the ordinary differential equations and makes the computing easier.

The first problem to tackle is to find the critical value T_c for T and the dimensionless wavenumber λ_c in the axial direction for fixed η . This involves linearizing the equations of motion for small amplitudes of disturbance and finding the condition for neutral stability. This method of finding the value of λ_c, T_c or any λ_0, T_0 that lie on the neutral curve is discussed in § 2.1.

We will only be concerned with equilibrium flows where all the velocity functions will be independent of time, that is

$$\frac{\partial}{\partial \tau} \equiv 0 . \qquad (1.6.1)$$

The velocities will not be expanded in powers of an amplitude function X(t) of time as Stuart[5] et al.

In the main case with our example of Chapter 3 and Chapter 4 we shall choose the outer surface to be both a slowly varying function and also to have a variation of $O(\varepsilon^2)$, the equation of this outer surface being of the form

$$x = \frac{1}{2} [1 + \varepsilon^2 f(z^*)] \tag{1.6.2}$$

in non-dimensional co-ordinates defined in (2.3.7), and where

$$z^* = \varepsilon\zeta \quad . \tag{1.6.3}$$

We shall start our expansion procedure using a slowly varying real amplitude function $\psi(z^*)$ etc., in the same manner as in (1.5.10).

We hope that as the Taylor number T is increased there will be a certain $T = T_{crit}$ at which the flow will become unstable. We shall choose our parameters such that the local parallel wall flow is liable to be more unstable at the centre $z^* = 0$ than at the ends $z^* = \pm \infty$. We will therefore perturb about the most unstable parallel wall case and our analysis is such that to the first approximation the wavenumber is λ_c and

$$T = T_c + \varepsilon T_1^* + \varepsilon^2 T_2^* + \ldots \quad . \tag{1.6.4}$$

We are really fixing the wavenumber to first approximation by requiring $T \to T_c$ as $\varepsilon \to 0$ and therefore we expect the new T_{crit} to be somewhere near T_c. We hope that any correction to the wavenumber will be forced at the next order, if a correction term exists.

For any position along the z-axis there is a different local Taylor number T_L and by fixing T and $f(z^*)$ in such a way that we have, near $z^* = 0$

$$T_L > T_{Lc} \, , \tag{1.6.5}$$

where T_{LC} is the local critical Taylor number from the parallel wall problem with local values of η and T, and

$$T_L < T_{Lc} \qquad (1.6.6)$$

as $z^* \to \pm\ \infty$ at the ends of the cylinders, thus we would expect the Taylor-vortex type solutions fading away to zero as $|z^*|$ becomes large.

A possible critical mode of disturbance would occur with

$$\underline{U} \to 0 \quad \text{as} \quad z^* \to \pm\ \infty \qquad (1.6.7)$$

and at the ends we would have purely circumferential Couette flow. It will be shown that any other boundary condition on $\underline{U}$ would not give the lowest critical value of T.

The function $f(z^*)$ must be chosen such that we do not get 'too far' above T_L as we might then expect wavy vortices. This is particularly true for the case as $\eta \to 1$. At the moment we cannot be too precise about this because the situation is rather different from the straight walled case.

We assume a solution for $\underline{U}$ of the type

$$\underline{U} = e^{i\lambda_i \zeta}\ \psi(z^*)\ \underline{u}_{11}(x) + O(\varepsilon) + c.c. \qquad (1.6.8)$$

where $\underline{u}_{11}(x)$ is determined by a local eigenvalue problem. The boundary condition on $\underline{U}$ given by (1.6.7) now becomes

$$\psi(z^*) \to 0 \quad \text{as} \quad z^* \to \pm\ \infty \qquad (1.6.9)$$

and $\psi(z^*)$ is taken as a real function of the slow variable z^*.

This is an asymptotic representation for the fluid flow which at this lowest order tends to the parallel wall result as $\varepsilon \to 0$.

It was found that by choosing $\psi(z^*)$ to be real we forced the $O(\varepsilon)$ correction term together with terms which arise from the slow variation with space of the overall critical disturbance structure to be completely imaginary with respect to the first order term. This will exhibit higher order corrections to the wavenumber of the form

$$K(z^*) = \lambda_c + \varepsilon^2 fn(x, z^*) \qquad (1.6.10)$$

and each component of $\underline{U}$ under consideration will have a different wavenumber which depends on the z* axial co-ordinate and the radial co-ordinate x.

This correction of the solutions (1.6.8) to (1.6.10) will be retained in the full non-linear equation for $\underline{U}$ and so enable a comparison to be made between the theory of finite amplitude Taylor-vortex type flow of our theory with experiment.

2. Parallel Wall Case

2.1 The neutral curve

The equations of motion are linearized for small amplitudes of disturbance and the condition for neutral stability can be found, where the growth rate, a_o, is zero. For given ratios of radii and angular speeds the condition takes the form of an eigenvalue relationship between a velocity parameter T, the Taylor number, and the wave number λ of the periodic disturbance in the ζ direction. For a given wave number the disturbance is amplified if the Taylor number lies above its critical value for that wave number (i.e. $a_o > 0$) and damped if the Taylor number lies below that value (i.e. $a_o < 0$).

Calculating this relationship gives the neutral curve for this linear stability problem. The procedure is to fix λ at a particular value with a_o at zero and vary T until we obtain the lowest value of T that makes the disturbance velocities satisfy the given boundary conditions. This process is repeated for a number of different λ's and calculations show that this curve has a minimum point denoted by λ_c, T_c which are known as the corresponding critical values. These values are important since experiments indicate that Taylor vortices appear when $\lambda \doteq \lambda_c$. See FIG I for a sketch of the neutral curve.

It thus seems of interest to try to calculate an approximation to the neutral curve which involves just one numerical solution of the eigenvalue problem. To this end, we suppose λ_o, T_o is a

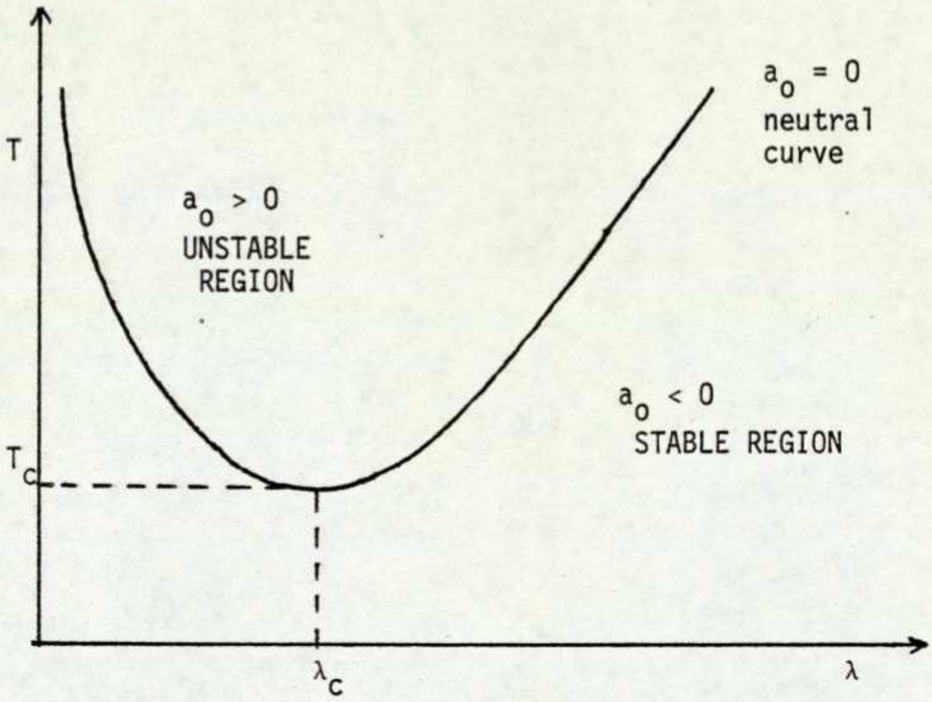

FIG I Variation of the Taylor number T with the wavenumber λ.

point calculated such that it is known to lie on the neutral curve given by $a_o = 0$. We then set

$$\lambda = \lambda_o + \varepsilon \tag{2.1.1}$$

and

$$T = T_o + \varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3 + \dots \tag{2.1.2}$$

with the object of finding T_1, T_2 and T_3 such that (2.1.1) and (2.1.2) comprise a parametric equation of the neutral curve valid for small ε. It should be noted that (2.1.1) is exact, and can be thought of as defining the small parameter ε, and that values for T_1, T_2, ... etc will be given by certain consistency conditions defined later. Some results are obtained by the point on the neutral curve given by $\lambda_o = 3.4$ and $T_o = 1772.97$ which are displayed in TABLE 1.

With this theory a better approximation is found for λ_c, T_c from using the fact that if $\lambda_o = \lambda_c$ and $T_o = T_c$ the value of T_1 should be zero. The effect of different normalizations on the functions $\underline{g}_{31}(x)$, $\underline{g}_{21}(x)$, ... etc, as defined in (2.6.13), (2.6.7) with respect to the neutral curve and the fluid field is also examined.

2.2 Analysis of the base velocities

Let (r,θ,z) denote cylindrical polar co-ordinates such that the z-axis is chosen to lie along the common axis of two concentric cylinders. The inner cylinder of radius R_1 is rotated about its axis with a constant angular veloctity Ω_1 inside the fixed outer cylinder of radius R_2. The gap between the cylinders is filled with a fluid of constant density ρ and kinematic viscosity ν.

If (u, v, w) denote the velocity components and p the pressure of the basic flow, then the Navier-Stokes and continuity equations for viscous incompressible axisymmetric flow are given by :

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u - \frac{u}{r^2}\right), \tag{2.2.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = \nu\left(\nabla^2 v - \frac{v}{r^2}\right), \tag{2.2.2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w, \tag{2.2.3}$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 . \tag{2.2.4}$$

Here $$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} . \tag{2.2.5}$$

There is a basic steady Couette flow solution of the form

$$u = 0,\ v = V_o(r) = Ar + B/r\ ,\ w = 0 \tag{2.2.6}$$

where
$$A = -\frac{R_1^2\,\Omega_1}{R_2^2-R_1^2}\ ,\quad B = \frac{R_1^2\,R_2^2\,\Omega_1}{R_2^2 - R_1^2} \tag{2.2.6a}$$

and the pressure, p, is just a known function of r given by

$$p = p_0(r) = \rho[A^2 r^2/2 + 2AB \log r - \tfrac{1}{2} B^2/r^2 + c] \ . \quad (2.2.7)$$

Here c is a constant.

2.3 The disturbance equations

We now set

$$u = u', \quad v = V_0 + v' , \quad w = w' , \quad p = p_0 + p'$$

in the Navier-Stokes and continuity equations given by (2.2.1) to (2.2.4), where the primed variables are functions of r,z,t and V_0, p_0 are the solutions given by (2.2.6) and (2.2.7) , to obtain the disturbance equations satisfied by u', v', w', p' :

$$\frac{\partial u'}{\partial t} + u' \frac{\partial u'}{\partial r} + w' \frac{\partial u'}{\partial z} - \frac{(v')^2 + 2v'V_0}{r} = -\frac{1}{\rho}\frac{\partial p'}{\partial r} + \nu(\nabla^2 u' - \frac{u'}{r^2}), \quad (2.3.1)$$

$$\frac{\partial v'}{\partial t} + u' \frac{\partial v'}{\partial r} + 2Au' + w' \frac{\partial v'}{\partial z} + \frac{u'v'}{r} = \nu(\nabla^2 v' - \frac{v'}{r^2}), \quad (2.3.2)$$

$$\frac{\partial w'}{\partial t} + u' \frac{\partial w'}{\partial r} + w' \frac{\partial w'}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + \nu \nabla^2 w', \quad (2.3.3)$$

$$\frac{\partial u'}{\partial r} + \frac{u'}{r} + \frac{\partial w'}{\partial z} = 0 \quad (2.3.4)$$

Using equation (2.3.4) and differentiating with respect to r, we can re-arrange (2.3.1) to give

$$\frac{\partial u'}{\partial t} + \nu \frac{\partial^2 w'}{\partial r \partial z} - \nu \frac{\partial^2 u'}{\partial z^2} + \frac{1}{\rho}\frac{\partial p'}{\partial r} - \frac{2v'V_0}{r} = \frac{(v')^2}{r} + \frac{(u')^2}{r} - w' \frac{\partial u'}{\partial z} + u' \frac{\partial w'}{\partial z} \ . \quad (2.3.5)$$

We now define the following constants :

$$d = R_2 - R_1 \ , \ R_o = (R_1+R_2)/2 \ , \ \delta = d/R_o, \ \alpha = 8R_1^2/(R_1+R_2)^2 . \tag{2.3.6}$$

We also introduce the dimensionless variables defined by

$$u' = -\frac{\nu u}{\alpha d} \ , \quad w' = -\frac{\nu w}{\alpha d} \ , \quad v' = \frac{\Omega_1 R_o v}{2}, \ p' = -\frac{\nu^2 \rho}{\alpha d^2} \ p \ ,$$

$$r = R_o + dx \ , \quad z = d\zeta, \quad t = \frac{d^2 \tau}{\nu} \ . \tag{2.3.7}$$

in the usual way (see for example Davey, Di-Prima & Stuart [7] and Eagles [10]).

We also introduce the parameter T, denoting the Taylor number, given by

$$T = \frac{\Omega_1^2 R_1^2 d^3}{\nu^2 R_o} \tag{2.3.8}$$

and η ,

$$\eta = \frac{R_1}{R_2} . \tag{2.3.9}$$

We note that the disturbance equations contain the following dimensionless functions

$$G(x) = 1/(1+\delta x), \ \Omega_o(x) = -\frac{2\eta^2}{1-\eta^2} + \frac{8\eta^2}{(1+\eta)^2(1-\eta^2)} G^2(x). \tag{2.3.10}$$

The dimensionless problem now depends only on the parameter T and η defined above. Thus we are able to write the disturbance equations in matrix form as

$$\frac{\partial \underline{U}}{\partial x} - \underline{A}\,\underline{U} - \underline{B}\frac{\partial \underline{U}}{\partial \tau} = \underline{L}(\underline{U})\underline{U} \ , \tag{2.3.11}$$

where $\underline{A}\,\underline{U}$ denotes the matrix product $\sum_{j=1}^{6} A_{ij}U_j$ etc.

~~Here~~ Following Eagles[10], here

$$\underline{U} = [p, v_o, w_o, u, v, w]^t , \qquad (2.3.12)$$

with t denoting the transpose and v_o, w_o are related to the fluid velocities v and w by

$$v_o = \frac{\partial v}{\partial x} \quad , \quad w_o = \frac{\partial w}{\partial x} \quad ; \qquad (2.3.13)$$

and

$$\underline{A} = \begin{bmatrix} 0 & 0 & -\partial/\partial\zeta & \partial^2/\partial\zeta^2 & -T\Omega_o(x) & 0 \\ 0 & -\delta G & 0 & 1 & -\partial^2/\partial\zeta^2 + \delta^2 G^2 & 0 \\ \partial/\partial\zeta & 0 & -\delta G & 0 & 0 & -\partial^2/\partial\zeta^2 \\ 0 & 0 & 0 & -\delta G & 0 & -\partial/\partial\zeta \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\underline{L}(\underline{U}) = -\frac{1}{\alpha} \begin{bmatrix} \underset{\sim}{0} & \begin{matrix} \delta G\, u - w\partial/\partial\zeta & \alpha T G v/2 & u\partial/\partial\zeta \\ v_o + \delta G v & w\partial/\partial\zeta & 0 \\ w_o & 0 & w\partial/\partial\zeta \end{matrix} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix},$$

and

$$\underset{\sim}{B} = \begin{bmatrix} & -1 & 0 & 0 \\ \underset{\sim}{0} & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix}. \qquad (2.3.14)$$

2.4 The expansion procedure I

The equation given by (2.3.11) has to be solved subject to the physical condition that requires zero disturbance velocities at the two cylinder walls. It is written as follows for a vector with six components.

β_2 : the last three components to be zero at $x = \pm 1/2$. (2.4.1)

Let us consider the steady state linear problem,

$$\frac{\partial \underline{U}}{\partial x} - \underline{A}\,\underline{U} \;\; ; \;\; \beta_2. \qquad (2.4.2)$$

A real solution of (2.4.2) can be written as

$$\underline{U} = e^{i\lambda\zeta}\, \underline{u}(x) + e^{-i\lambda\zeta}\tilde{\underline{u}}(x), \qquad (2.4.3)$$

where λ is a real constant, a tilde denotes the complex conjugate and zero is our eigenvalue with $\underline{u}(x)$ the corresponding eigenfunction of the eigenvalue problem for σ (see Eagles [10]).

We see that the problem for $\underline{u}(x)$ is

$$\frac{du}{dx} - \underline{A}^{(1)}\underline{u} = 0 \;\; ; \;\; \beta_2. \qquad (2.4.4)$$

Here we define

$\{ \underline{A}^{(p)}$ as $\underline{A}$ with $\partial/\partial\zeta$ replaced by $ip\lambda \}$. (2.4.4a)

Fixing λ at λ_o and solving for T such that the last three components of $\underline{u}$ are zero at $x = \pm 1/2$, gives us a point (λ_o, T_o) on the neutral curve for a fixed η . This may be done numerically (see A. Davey[6]) .

Now replacing λ by $\lambda_o + \varepsilon$ and T by $T_o + \varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3 +$ ... etc in (2.4.4), we ask what are the conditions on T_1, T_2, T_3, ... that this perturbation represents the parametric equations of the neutral curve in the region of (λ_o, T_o). Thus equation (2.4.5) becomes

$$\frac{d\underline{u}}{dx} - \underline{A}_o^{(1)}\underline{u} - \varepsilon(\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{u} - \varepsilon^2(\underline{A}_{o2}^{(1)} + T_2\underline{A}_2)\underline{u} -$$

$$\varepsilon^3 T_3\underline{A}_2\underline{u} + O(\varepsilon^4) = 0, \tag{2.4.5}$$

where

$\underline{A}_o^{(1)}$ is $\underline{A}^{(1)}$ except that λ is replaced by λ_o and T by T_o. (2.4.6)

The expressions for the matrices are

$$\underline{A}_{o1}^{(p)} = p\begin{bmatrix} 0 & 0 & -i & -2\lambda_o p & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_o p & 0 \\ i & 0 & 0 & 0 & 0 & 2\lambda_o p \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\underline{A}_{o2}^{(p)} = p^2 \begin{bmatrix} & -1 & 0 & 0 \\ \underset{\sim}{0} & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix} \quad \text{and} \quad \underline{A}_2 = \begin{bmatrix} & 0 & -\Omega_o(x) & 0 \\ \underset{\sim}{0} & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix} . \tag{2.4.7}$$

Now we expand $\underline{u}$ in powers of ε and write $\underline{u} = \underline{u}_1(x) + \varepsilon\underline{u}_2(x) + \varepsilon^2\underline{u}_3(x) + \ldots$, and when we equate powers of ε^n we obtain a set of ordinary differential equations :

$$\mathcal{L}^{(1)}(\underline{u}_1) = 0, \tag{2.4.8}$$

$$\mathcal{L}^{(1)}(\underline{u}_2) = (\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{u}_1 , \tag{2.4.9}$$

$$\mathcal{L}^{(1)}(\underline{u}_3) = (\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{u}_2 + (\underline{A}_{o2}^{(1)} + T_2\underline{A}_2)\underline{u}_1 , \tag{2.4.10}$$

$$\mathcal{L}^{(1)}(\underline{u}_4) = (\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{u}_3 + (\underline{A}_{o2} + T_2\underline{A}_2)\underline{u}_2 + T_3\underline{A}_2\underline{u}_1 . \tag{2.4.11}$$

The present analysis can be continued to higher order in a straightforward manner even though the algebra and computation becomes involved. The expansion has been carried out to fourth order.

Here

$$\mathcal{L}^{(p)} \text{ is the operator } \frac{d}{dx} - \underline{A}_o^{(p)} , \tag{2.4.12}$$

and each $\underline{u}_i(x)$ must satisfy the boundary condition β_2.

The equations (2.4.8) to (2.4.11) may be solved in sequence to give the flow field and an approximation to the neutral curve provided the constants T_1, T_2, T_3, ... etc are chosen correctly.

We set

$$\underline{u}_1(x) = \gamma_1 \underline{f}_{11}(x) \qquad (2.4.13)$$

where $\mathcal{L}^{(1)}(\underline{f}_{11}(x)) = 0$ and $\underline{f}_{11}$ is normalized such that the second component of $\underline{f}_{11}$ evaluated at $x = -1/2$ is equal to 1. Here γ_1 is an unknown real constant at this stage but will be found later by a consistency condition. The idea of using (2.4.13) is if $\underline{f}_{11}$ is a solution of (2.4.8) then so is $\gamma_1\underline{f}_{11}$ since we are dealing with a linear equation. Thus by fixing $\underline{f}_{11}$ in some way, in this case by using the said normalization, we can use a consistency condition later to find γ_1 and we will then have the complete solution for $\underline{u}_1(x)$. It should be noted that upon using this and later normalizations in this chapter the functions given by $\underline{f}_{11}$, $\underline{g}_{21}$, ... etc are found to have real first, second, fourth and fifth components and purely imaginary third and sixth components.

2.5 The adjoint eigenfunction

To solve equations (2.4.9) to (2.4.11) we need to bring in the idea of the adjoint eigenfunction, which for this case is the solution of

$$\frac{d\underset{\sim}{f}}{dx} + [\underline{A}_o^{(1)}]^t \underset{\sim}{f} = 0 \quad ; \quad \beta_1 \tag{2.5.1}$$

where the label β_1 describes the boundary conditions that the first three components of $\underset{\sim}{f}$ are zero at $x = \pm 1/2$ and t denotes the transpose.

Let $\underset{\sim}{f}^a$ be the solution of (2.5.1) then

$$\mathcal{L}^{(1)}(\underset{\sim}{y}) = \underline{R} \quad ; \quad \beta_2 \tag{2.5.2}$$

has a solution if and only if

$$\int_{\text{INTERVAL}} \underset{\sim}{f}^{a,t} \, \underline{R} \, dx = 0 \; . \tag{2.5.3}$$

The proof of (2.5.3) in conjunction with (2.5.1) and (2.5.2) is now given.

Let

$$\mathcal{L}^{(1)}(\underline{u}) = 0 \quad ; \quad \beta_2 \tag{2.5.4}$$

and

$$L^*(\underline{v}) = 0 \quad ; \quad \beta_1 \tag{2.5.5}$$

where $\underline{u} \not\equiv \underset{\sim}{0}$, $\underline{v} \not\equiv 0$ and $L^* \equiv \frac{d}{dx} + [\underline{A}_o^{(1)}]^t$.

Then we are required to solve $\mathcal{L}^{(1)}(\underset{\sim}{y}) = \underline{R}$. We start with the inner product defined by

$$< \underline{v}^t, \mathcal{L}^{(1)}(\underset{\sim}{y}) > \; = \; < \underline{v}^t , \underline{R} > \; = \int_{-\frac{1}{2}}^{\frac{1}{2}} \underline{v}^t \underline{R} \, dx \; . \tag{2.5.6}$$

Note that (2.5.6) can also be written as

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underline{v}^t \left[\frac{d\underset{\sim}{y}}{dx} - \underline{A}_o^{(1)} \underset{\sim}{y}\right] dx . \tag{2.5.7}$$

When we integrate (2.5.7) by parts we obtain

$$[\underline{v}^t \underset{\sim}{y}]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{dv^t}{dx} \underset{\sim}{y} + v^t \underline{A}_o^{(1)} \underset{\sim}{y}\right] dx. \tag{2.5.8}$$

Since the integral in (2.5.8) is a constant and not a vector, it will have the same value as its transpose which is

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{dv^t}{dx} \underset{\sim}{y} + \underline{v}^t \underline{A}_o^{(1)} \underset{\sim}{y}\right]^t dx. \tag{2.5.9}$$

Upon using the properties that the transpose operation on matrices satisfy in (2.5.9) then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\frac{dv^t}{dx} \underset{\sim}{y} + \underline{v}^t \underline{A}_o^{(1)} \underset{\sim}{y}\right]^t dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{y}^t \left[\frac{dv}{dx} + (\underline{A}_o^{(1)})^t \underline{v}\right] dx$$

$$= < \underset{\sim}{y}^t , L^*(\underline{v}) > . \tag{2.5.10}$$

But from (2.5.5), $L^*(\underline{v}) = 0$, therefore

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underline{v}^t \underline{R}\, dx = [\underline{v}^t \underset{\sim}{y}]_{-\frac{1}{2}}^{\frac{1}{2}} \tag{2.5.11}$$

where $[\underline{v}^t \underset{\sim}{y}]_{-\frac{1}{2}}^{\frac{1}{2}}$ denotes the matrix product $[\sum_{j=1}^{6} v_j y_j]_{-\frac{1}{2}}^{\frac{1}{2}}$ and the values of the v_j's and y_j's are given by the boundary conditions β_1, β_2 respectively. That is

$$v_m , y_{m+3} \text{ evaluated at } x = \pm 1/2 \text{ are zero} \tag{2.5.12}$$

for m = 1, 2, 3. When we use (2.5.12) we have the result given by (2.5.3).

If instead the values of y_{m+3} evaluated at x = 1/2 are not zero, and v_{m+3} the corresponding values of $\underline{v}$. Then

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underline{v}^t \, \underline{R} \, dx = [y_4 v_4 + y_5 v_5 + y_6 v_6]_{\text{evaluated at } x=1/2.} \qquad (2.5.13)$$

This last condition will be used later in Chapters 3 and 4 .

It is easy to show that the relation

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \, \underline{A}_{o2} \, \underline{f}_{11} \, dx = 1 \qquad (2.5.14)$$

is satisfied after a suitable scaling of $\underset{\sim}{f}^a$.

2.6 <u>The expansion procedure II</u>

Using (2.4.13), equation (2.4.9) becomes

$$\mathcal{L}^{(1)}(\underline{u}_2) = \gamma_1(\underline{A}_{o1}^{(1)} \, \underline{f}_{11} + T_1 \underline{A}_2 \, f_{11}). \qquad (2.6.1)$$

In fact we can let

$$\underline{u}_i = \gamma_1 \, \underline{f}_{i1} \quad , \; i = 1,2,\ldots . \qquad (2.6.2)$$

Then we obtain

$$\mathcal{L}^{(1)}(\underline{f}_{21}) = \underline{A}_{o1}^{(1)} \, \underline{f}_{11} + T_1 \underline{A}_2 \, \underline{f}_{11} \, . \qquad (2.6.3)$$

For (2.6.3) to have a solution we use the adjoint condition (2.5.3):

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \, (\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{f}_{11} \, dx = 0 \, . \tag{2.6.4}$$

Hence we can find the first perturbation coefficient as

$$T_1 = \frac{-\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \, \underline{A}_{o1}^{(1)} \, \underline{f}_{11} \, dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \, \underline{A}_2 \underline{f}_{11} \, dx} \quad , \tag{2.6.5}$$

and it is now a known quantity.

We set

$$\underline{f}_{21} = \underline{g}_{21}(x) + \beta \underline{f}_{11} \tag{2.6.6}$$

where $\underline{g}_{21}$ is a particular solution satisfying

$$\mathcal{L}^{(1)}(\underline{g}_{21}) = (\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{f}_{11} \quad ; \quad \beta_2 \tag{2.6.7}$$

and is normalized such that the second component of $\underline{g}_{21}$ evaluated at $x = -1/2$ is equal to 0. It can be seen later that different normalizations do not affect the parametric equations of the neutral curve or the final solution for $\underline{U}$. The solution of (2.6.3) has an arbitrary additive multiple of the eigenfunction $\underline{f}_{11}$ and β is a real unknown constant which will be determined by a consistency condition later on.

Using (2.6.6), equation (2.4.10) becomes (with $\underline{u}_3 = \gamma_1 \underline{f}_{31}$),

$$\mathcal{L}^{(1)}(\underline{f}_{31}) = (\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{g}_{21} + \beta(\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{f}_{11} + (\underline{A}_{o2}^{(1)} + T_1\underline{A}_2)f_{11} . \tag{2.6.8}$$

For $\underline{f}_{31}$ to exist, we use the adjoint condition (2.5.3) to obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}(\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\,\underline{g}_{21}\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\underline{A}_{o2}\underline{f}_{11}\,dx + T_2\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\underline{A}_2\underline{f}_{11}\,dx = 0, \tag{2.6.9}$$

since the value of $\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}(\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\,\underline{f}_{11}\,dx$ is zero from

(2.6.4).

Hence we can find the second perturbation coefficient as

$$T_2 = \frac{-\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\,[\underline{A}_{o1}^{(1)}\,\underline{g}_{21} + T_1\underline{A}_2\underline{g}_{21} + \underline{A}_{o2}^{(1)}\,\underline{f}_{11}]\,dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\,\underline{A}_2\underline{f}_{11}\,dx}. \tag{2.6.10}$$

Re-forming equation (2.6.8) and using (2.6.10) we have

$$\begin{aligned}\mathcal{L}^{(1)}(\underline{f}_{31}) = {} & [\,\underline{A}_{o1}^{(1)}\,\underline{g}_{21} + T_1\underline{A}_2\underline{g}_{21} + \underline{A}_{o2}^{(1)}\,\underline{f}_{11} + T_2\underline{A}_2\underline{f}_{11}\,] \\ & + \beta[\underline{A}_{o1}^{(1)}\,\underline{f}_{11} + T_1\underline{A}_2\underline{f}_{11}\,],\end{aligned} \tag{2.6.11}$$

where T_2 is now a known constant, and by examining the right hand side of (2.6.11) we see $\underline{f}_{31}$ may be expressed as

$$\underline{f}_{31} = \underline{g}_{31} + \beta\underline{g}_{21} + \alpha_1\underline{f}_{11}, \tag{2.6.12}$$

with $\underline{g}_{31}$ satisfying

$$\mathcal{L}^{(1)}(\underline{g}_{31}) = \underline{A}_{o1}^{(1)}\,\underline{g}_{21} + T_1\underline{A}_2\underline{g}_{21} + \underline{A}_{o2}^{(1)}\,\underline{f}_{11} + T_2\underline{A}_2\underline{f}_{11}\,. \tag{2.6.13}$$

The function $\underline{g}_{31}$ is normalized in the same way as $\underline{g}_{21}$. The solution of (2.6.11) can yet again have an arbitrary additive multiple of the eigenfunction $\underline{f}_{11}$ with α_1 an unknown real constant determined later on. The term $\beta\underline{g}_{21}$ has been forced by the multiple $\beta\underline{f}_{11}$ of the eigenfunction which occurred in $\underline{f}_{21}$.

The equation for $\underline{f}_{41}$, given by (2.4.11), can be placed in the form

$$\mathcal{L}^{(1)}(\underline{f}_{41}) = [(\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{g}_{31} + (\underline{A}_{o2}^{(1)} + T_2\underline{A}_2)\underline{g}_{21} + T_3\underline{A}_2\underline{f}_{11}] +$$

$$\beta[(\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{g}_{21} + (\underline{A}_{o2}^{(1)} + T_2\underline{A}_2)\underline{f}_{11}] + \alpha_1[(\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{f}_{11}]\,. \tag{2.6.14}$$

The condition for (2.6.14) to have a solution gives

$$T_3 = -\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \underline{f}^{a,t}[(\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\underline{g}_{31} + (\underline{A}_{o2}^{(1)} + T_2\underline{A}_2)\,\underline{g}_{21}]\,dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\,\underline{A}_2\underline{f}_{11}\,dx}\,. \tag{2.6.15}$$

2.7 <u>The forms of solution for $\underline{U}$ with different normalizations</u>

The complete solution for $\underline{U}$ to $O(\varepsilon^3)$ can be shown to be

$$\underline{U} = \gamma_1 e^{i\lambda_0\zeta}[\underline{f}_{11} + \varepsilon\underline{f}_{21} + \varepsilon^2\underline{f}_{31} + \ldots] +$$

$$\gamma_1 e^{-i\lambda_0\zeta}[\underline{f}_{1,-1} + \varepsilon\underline{f}_{2,-1} + \varepsilon^2\underline{f}_{3,-1} + \ldots] \quad (2.7.1)$$

where $$\underline{f}_{i,-j} = \tilde{\underline{f}}_{ij} . \quad (2.7.2)$$

Referring to the form of expressing $\underline{f}_{21}$, $\underline{f}_{31}$ defined in (2.6.6) and (2.6.12) respectively, then substituting for $\underline{f}_{21}$, $\underline{f}_{31}$ in (2.7.1) we have

$$\underline{U} = \gamma_1 e^{i\lambda_0\zeta}[\underline{f}_{11} + \varepsilon(\underline{g}_{21} + \beta\underline{f}_{11}) + \varepsilon^2(\underline{g}_{31} + \beta\underline{g}_{21} + \alpha_1\underline{f}_{11}) + \ldots] + \text{c.c}, \quad (2.7.3)$$

where c.c. stands for the complex conjugate of the preceding expressions. The functions $\underline{g}_{21}$, $\underline{g}_{31}$ satisfy the differential equations (2.6.7) and (2.6.13).

Since the problem is linear we may use an arbitrary normalization on $\underline{U}$ given by $U_2(-1/2,0)$ is equal to 1., that is the second component of $\underline{U}$ evaluated at $x = -1/2$ and at $\zeta = 0$. We obtain from (2.7.3) that

$$1 = 2\gamma_1[1 + \varepsilon\beta + \varepsilon^2\alpha_1 + \ldots], \quad (2.7.4)$$

since the second components of $\underline{f}_{11}$, $\underline{g}_{21}$, $\underline{g}_{31}$ are 1,0,0 respectively at $x = -1/2$.

We equate powers of ε in (2.7.4) and the coefficients γ_1, α_1, β are given by

$$\gamma_1 = 1/2, \quad \alpha_1 = 0, \quad \beta = 0. \quad (2.7.5)$$

Then upon replacing these values in (2.7.3) the solution for $\underline{U}$ is

$$\underline{U} = 1/2e^{i\lambda_0\zeta}[\underline{f}_{11} + \varepsilon\underline{g}_{21} + \varepsilon^2\underline{g}_{31} + \ldots] + \text{c.c.} \quad (2.7.6)$$

Since $\underline{f}_{11}$, $\underline{g}_{21}$, $\underline{g}_{31}$ are complex we can separate them into real and imaginary parts, that is

$$\underline{f}_{11} = \underline{f}_{11}^{(r)} + i\,\underline{f}_{11}^{(i)} \quad (2.7.7)$$

where the index (r) and (i) denote the real and imaginary parts of $\underline{f}_{11}$ respectively. We replace $e^{i\lambda_0\zeta}$ by $\cos\lambda_0\zeta + i\sin\lambda_0\zeta$ and ε by $(\lambda - \lambda_0)$, and $\underline{U}$ can be written as

$$\underline{U} = \underline{f}_{11}^{(r)}\cos\lambda_0\zeta - \underline{f}_{11}^{(i)}\sin\lambda_0\zeta + (\lambda-\lambda_0)\,[\underline{g}_{21}^{(r)}\cos\lambda_0\zeta - \underline{g}_{21}^{(i)}\sin\lambda_0\zeta] +$$

$$(\lambda-\lambda_0)^2\,[\underline{g}_{31}^{(r)}\cos\lambda_0\zeta - \underline{g}_{31}^{(i)}\sin\lambda_0\zeta] + O(\lambda-\lambda_0)^3. \quad (2.7.8)$$

The idea of writing $\underline{U}$ as (2.7.8) will be used in later chapters and to show the form of the real solution for each component of $\underline{U}$.

If $\underline{g}_{21}$, $\underline{g}_{31}$ had a different normalization, we ask would this effect the form of $\underline{U}$ or the value of the integrals used to calculate the constants T_i. We must remember that this is subject to an overall normalization on $\underline{U}$ given by $U_2(-1/2,0) = 1$. For simplicity we consider just one other normalization for $\underline{g}_{21}$ and $\underline{g}_{31}$.

Suppose that $\hat{\underline{g}}_{21}$, $\hat{\underline{g}}_{31}$ are the new solutions of the differential equations (2.6.7) and (2.6.13), that is

$$\mathcal{L}^{(1)}(\hat{\underline{g}}_{21}) = (\underline{A}_{o1}^{(1)} + T_1\underline{A}_2)\,\underline{f}_{11} \quad ; \ \beta_2 \tag{2.7.9}$$

and

$$\mathcal{L}^{(1)}(\hat{\underline{g}}_{31}) = (\underline{A}_{o1}^{(1)} + T_1\,\underline{A}_2)\,\underline{g}_{21} + (\underline{A}_{o2}^{(1)} + T_2\underline{A}_2)\,\underline{f}_{11} \ ; \ \beta_2 \tag{2.7.10}$$

with their second components evaluated at $x = -1/2$ and $\zeta = 0$ being equal to 2. Then comparing the equation for $\underline{g}_{21}$ and $\hat{\underline{g}}_{21}$, we see this forces

$$\hat{\underline{g}}_{21} = \underline{g}_{21} + K\,\underline{f}_{11} \tag{2.7.11}$$

since we can also have an arbitrary multiple of the eigenfunction $\underline{f}_{11}$, but we must also have

$$\hat{g}_{21,2} = g_{21,2} + K\,f_{11,2} \tag{2.7.12}$$

when we use the normalizing condition. This will give

$$K = 2. \tag{2.7.13}$$

We re-arrange (2.7.10) by replacing $\hat{\underline{g}}_{21}$ by (2.7.11), and see

$$\mathcal{L}^{(1)}(\hat{\underline{g}}_{31}) = (\underline{A}_{o1}^{(1)} + T_1A_2)\underline{g}_{21} + (\underline{A}_{o2}^{(1)} + T_2\underline{A}_2)\,\underline{f}_{11} + 2[(\underline{A}_{o1}^{(1)} + T_1A_2)f_{11}]. \tag{2.7.14}$$

When we compare the right hand side of (2.7.14) with the equation for $\underline{g}_{31}$, this forces

$$\hat{\underline{g}}_{31} = \underline{g}_{31} + 2\underline{g}_{21} + K_1\underline{f}_{11} \tag{2.7.15}$$

and the normalizing condition will give

$$K_1 = 2. \tag{2.7.16}$$

The corresponding solution for $\underline{U}$ with this second normalization is

$$\underline{U} = \gamma_1 e^{i\lambda_o\zeta}[\underline{f}_{11} + \varepsilon(\hat{\underline{g}}_{21} + \hat{\beta}\,\underline{f}_{11}) + \varepsilon^2(\hat{\underline{g}}_{31} + \hat{\beta}\hat{\underline{g}}_{21} + \hat{\alpha}\,\underline{f}_{11}) + \ldots] + c.c., \tag{2.7.17}$$

where $\hat{\alpha}_1$, $\hat{\beta}$ are the modified constants α_1, β due to the different normalization used. When the same initial normalization on $\underline{U}$ is used we obtain

$$1 = 2\gamma_1[1 + \varepsilon(2 + \hat{\beta}) + \varepsilon^2(2 + 2\hat{\beta} + \hat{\alpha}) + \ldots] \tag{2.7.18}$$

since the second components of $\underline{f}_{11}$, $\hat{\underline{g}}_{21}$, $\hat{\underline{g}}_{31}$ are 1,2,2 respectively. at $x = -1/2$, $\zeta = 0$.

When we equate powers of ε in (2.7.18) the coefficients γ_1, $\hat{\alpha}$, $\hat{\beta}$ are

$$\gamma_1 = 1/2\ , \quad \hat{\beta} = -2\ , \quad \hat{\alpha} = 2. \tag{2.7.19}$$

Thus noting these values in conjunction with (2.7.11) and (2.7.15) will give the same form for $\underline{U}$ as in (2.7.6).

If $\underline{g}_{21}$ and $\underline{g}_{31}$ were replaced by (2.7.11) and (2.7.15) in the integrals defining T_2 and T_3 we see that their values would remain invariant under the two different normalizations. The reason for using two different normalizations was for a numerical check for both the functions $\underline{g}_{21}$, $\underline{g}_{31}$ and the value of the integrals needed to calculate T_2, T_3.

It is noticed that if $\varepsilon = 0$, we would be left with the linear solution corresponding to the linear problem defined in Eagles [10] for $\lambda = \lambda_o$, $T = T_o$ which is

$$\underline{U} = 1/2[e^{i\lambda_o\zeta}\,\underline{f}_{11} + e^{-i\lambda_o\zeta}\,\underline{f}_{1,-1}] \qquad (2.7.20)$$

or

$$\underline{U} = \underline{f}_{11}^{(r)}\,\cos\lambda_o\zeta - \underline{f}_{11}^{(i)}\,\sin\lambda_o\zeta \quad . \qquad (2.7.21)$$

At this point it was decided to calculate the streamfunction Φ, for $\lambda = \lambda_c$, in order to plot the ~~actual~~ projected streamlines.

In the non-dimensional form the streamfunction Φ is given by

$$\frac{1}{1+\delta x}\,\frac{\partial\Phi}{\partial\zeta} = f_{11,4}(x)\cos\lambda_c\zeta \qquad (2.7.22)$$

and

$$\frac{1}{1+\delta x}\,\frac{\partial\Phi}{\partial\zeta} = f_{11,6}(x)\sin\lambda_c\zeta \quad , \qquad (2.7.23)$$

where $f_{11,4}$ and $f_{11,6}$ are the fourth and sixth component of $\underline{f}_{11}$ respectively.

From (2.7.22) we obtain

$$\Phi = \frac{(1+\delta x)}{\lambda_c} f_{11,4}(x)\sin\lambda_c\zeta + H(x) \qquad (2.7.24)$$

where $H(x)$ is obtained from integrating (2.7.22) with respect to ζ only. If we now differentiate (2.7.24) with respect to x, we have

$$\frac{1}{1+\delta x}\,\frac{\partial\Phi}{\partial x} = \frac{\sin\lambda_c\zeta}{\lambda_c}\left[\frac{df_{11,4}}{dx} + \frac{\delta}{1+\delta x} f_{11,4}\right] + \frac{H'(x)}{1+\delta x} \qquad (2.7.25)$$

$$= f_{11,6}(x)\sin\lambda_c\zeta.$$

From the continuity equation we note

$$\frac{df_{11,4}}{dx} + \delta G(x) f_{11,4} = \lambda_c f_{11,6} \,, \qquad (2.7.26)$$

thus when we use (3.7.26) in (3.7.25) we see

$$H'(x) = 0 \qquad (2.7.27)$$

or

$$H(x) = \text{constant}. \qquad (2.7.28)$$

The value of this constant does not affect any possible streamlines. Thus the streamfunction is

$$\Phi = \frac{(1 + \delta x)}{\lambda_c} f_{11,4}(x) \sin \lambda_c \zeta \, . \qquad (2.7.29)$$

The actual streamlines Φ = constant are given by

$$\frac{(1 + \delta x)}{\lambda_c} f_{11,4}(x) \sin \lambda_c \zeta = c \, . \qquad (2.7.30)$$

To find the value of ζ for which (3.7.30) is true, we first fix the x co-ordinate and solve

$$\zeta = \frac{1}{\lambda_c} \sin^{-1} \left[\frac{c \, . \, \lambda_c}{(1 + \delta x) f_{11,4}} \right] \qquad (2.7.31)$$

where solutions for ζ exist for the various values of c. The streamlines are plotted for various values of c and are shown in FIG II .

The size ~~length~~ of consecutive vortices is subject to

$$\sin \lambda_c \zeta = 0 \qquad (2.7.32)$$

that is

$$\zeta = \frac{\pi}{\lambda_c} \, . \qquad (2.7.33)$$

This implies every $\frac{\pi}{\lambda_c}$ in the non-dimensional ζ co-ordinate contains one vortex. In the dimensional z co-ordinate this becomes

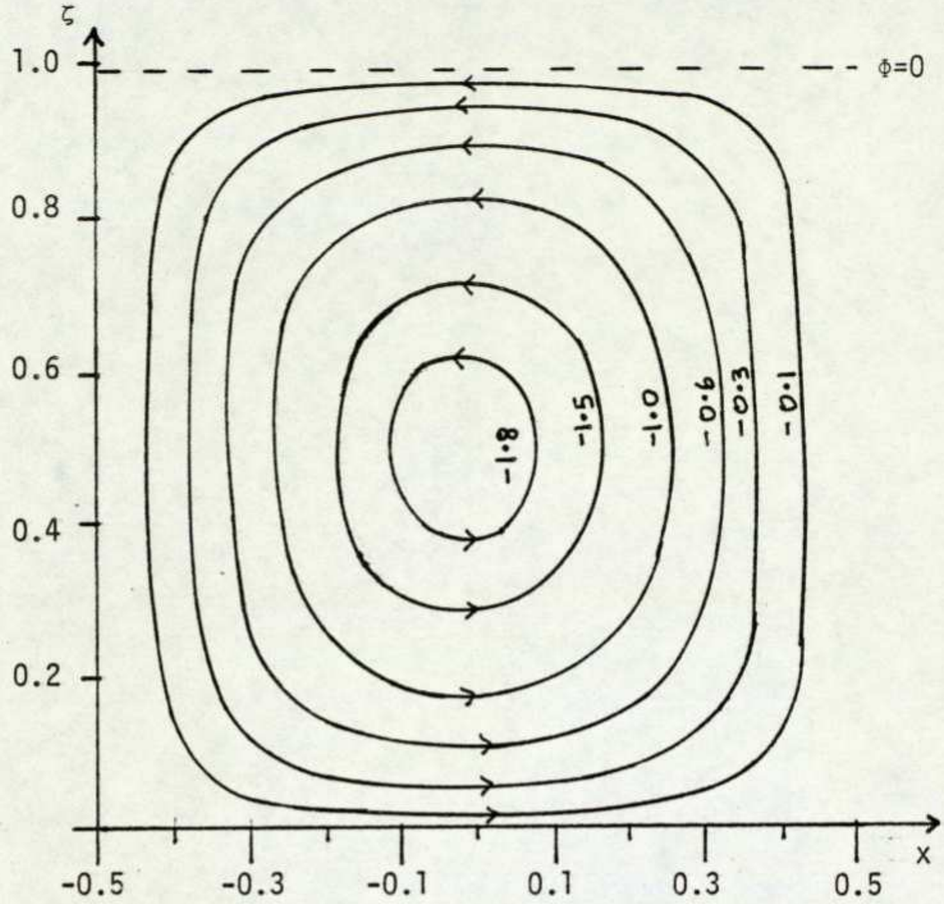

FIG IIa Streamlines for $\eta = 0.5$, $T = T_c$, $\lambda = \lambda_c$ and $c = -0.1, -0.3, -0.6, -1.0, -1.5$ and -1.8.

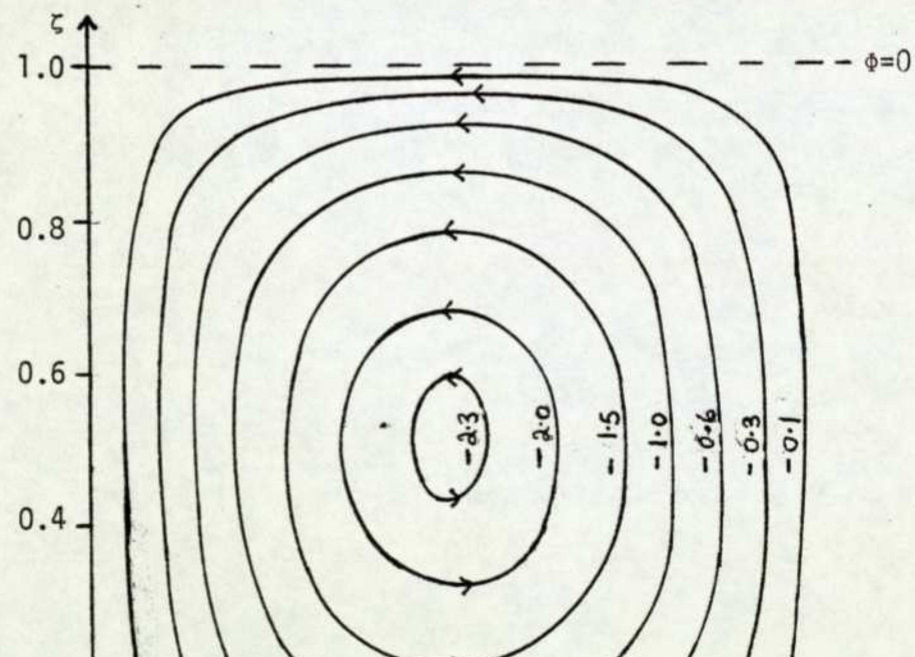

FIG IIb Streamlines for $\eta = 0.95$, $T = T_c$, $\lambda = \lambda_c$ and $c = -0.1, -0.3, -0.6, -1.0, -1.5, -2.0$ and -2.3.

1 vortex for every $z = \frac{d\pi}{\lambda_c}$ units, (2.7.34)

and the number of vortices can be used to check with any experimental results.

2.8 The approximation to the neutral curve

We hope that the analysis we have just carried out in the preceding sections of § 2 will give us a close approximation to the neutral curve over a suitable range of λ . We shall compare the results obtained from using (a) the quadratic, and (b) the cubic approximation to the neutral curve with the directly calculated values obtained from solving the eigenvalue problem a number of times for different λ's.

The approximation we have found for the neutral curve is

$$T = T_0 + (\lambda-\lambda_0)T_1 + (\lambda-\lambda_0)^2 T_2 + (\lambda-\lambda_0)^3 T_3 \qquad (2.8.1)$$

where T_1, T_2 and T_3 are defined in (2.6.5), (2.6.10) and (2.6.15) respectively.

The results obtained for T_1, T_2 and T_3 with,

$$\eta = 0.95, \quad T_0 = 1772.97 \text{ and } \lambda_0 = 3.4 \qquad (2.8.2)$$

were found to be

$$T_1 = 129.0403 \ , \ T_2 = 220.113 \ , \ T_3 = -33.0772. \qquad (2.8.3)$$

Further details of the numerical methods are given in the appendix. We then proceed to form a table as shown in TABLE I, to compare the different approximations to the neutral curve with the corresponding directly calculated values for T given λ.

λ	T	$T^{(2)}$	$e^{(2)}$	$T^{(3)}$	$e^{(3)}$
2.6803	1812.81	1794.11	18.70	1806.44	6.37
2.9	1769.00	1763.48	5.52	1767.61	1.39
3.1	1755.16	1754.07	1.09	1754.96	0.20
3.127	1754.96	1754.15	0.81	1754.82	0.14
3.15	1755.09	1754.47	0.62	1754.98	0.11
3.2	1756.28	1755.97	0.31	1756.23	0.05
3.25	1758.70	1758.57	0.13	1758.68	0.02
3.3	1762.31	1762.27	0.04	1762.30	0.01
3.4695	1782.98	1783.00	-0.02	1782.99	-0.01
3.6259	1812.99	1813.35	-0.36	1812.97	0.02
3.7	1830.69	1831.49	-0.80	1830.60	0.09

TABLE I Variation of T with λ for $\eta = 0.95$ and the cases mentioned below.

Here T are the directly calculated values obtained by solving the eigenvalue problem and are in good agreement with those obtained by DiPrima & Eagles [(4)].

$T^{(2)}$ are the Taylor numbers obtained from the quadratic approximation to the neutral curve.

$T^{(3)}$ are the Taylor numbers obtained from the cubic approximation to the neutral curve.

$e^{(2)}$ and $e^{(3)}$ are the corresponding errors as compared with T.

We next use the approximation for T, given in (2.8.1), to calculate an approximation for λ_c, T_c on the neutral curve. From (2.8.1)

$$\frac{dT}{d\lambda} = T_1 + 2T_2(\lambda-\lambda_o) + 3T_3(\lambda-\lambda_o)^2 + \ldots , \qquad (2.8.4)$$

hence using the fact that at $T = T_c$ and $\lambda = \lambda_c$ the curve has a minimum we obtain

$$3T_3(\lambda_c - \lambda_o)^2 + 2T_2(\lambda_c-\lambda_o) + T_1 = 0 \qquad (2.8.5)$$

to order $(\lambda_c - \lambda_o)^2$.

For the cubic approximation to the neutral curve there are two values given for λ_c. But only the zero closest to that given by the quadratic approximation for λ_c and T_c, that is

$$\lambda_c = \lambda_o - T_1/2T_2 \qquad (2.8.6)$$

$$T_c = T_o - T_1^2 /4T_2 , \qquad (2.8.7)$$

is useful. The zero of (2.8.5) which is relevant is

$$\lambda_c = \lambda_o + \frac{-T_2 + \sqrt{T_2^2 - 3T_1T_3}}{3T_3} . \qquad (2.8.8)$$

If we take the lim of (2.8.8) as $T_3 \to 0$ it can be easily shown we obtain the result of (2.8.6).
When we use (2.8.8) in (2.8.1) it is easy to obtain the corresponding critical Taylor number.

The values of λ_c and T_c obtained from using the quadratic approximation given by (2.8.6) and (2.8.7) with

$$\eta = 0.95 \text{ is } \lambda_c = 3.1069 \text{ and } T_c = 1754.06 \qquad (2.8.9)$$

and for the cubic approximation given by (3.8.8) with

$$\eta = 0.95 \text{ is } \lambda_c = 3.124 \text{ and } T_c = 1754.82 \quad . \qquad (2.8.10)$$

We use TABLE 1 and the above points to compare the accuracy of the approximations to see which is the best approximation to the neutral curve. In fact it can be seen from TABLE 1 an error of less than ·1 in absolute magnitude the range of λ is (3.27, 3.5) for the quadratic approximation, and (3.16, 3.7) for the cubic approximation. From the two ranges it is seen that the addition of the $\varepsilon^3 T_3$ term does make a sizeable difference to the value of the Taylor number, and also increases the accuracy for other values of the Taylor number for a given λ . This is especially the case for the approximation to the critical wavenumber λ_c and the critical Taylor number T_c. The values of which are obtained first by solving the eigenvalue problem a number of times directly, and these points are used to interpolate until a minimum value for T is obtained.

The values obtained for $\eta \doteq 0.95$ using this method are

$$\lambda_c = 3.127 \quad \text{and} \quad T_c = 1754.96, \qquad (2.8.11)$$

this compares quite favourably with the cubic approximation.

The values for λ_c and T_c are quite vital to later work and we intend to obtain a better approximation for these. We use (2.8.11) as my new λ_o and T_o and proceed to calculate T_1, T_2 and T_3 and thus obtain a new approximation for λ_c and T_c. This procedure is repeated until the value of T_1 is approximately zero to a certain number of specified decimal places. This is because the condition for

$$T = T_c + T_1(\lambda-\lambda_c) + T_2(\lambda-\lambda_c)^2 + T_3(\lambda-\lambda_c)^3 + \ldots, \qquad (2.8.12)$$

to represent the neutral curve with a minimum point at $\lambda = \lambda_c$ is that T_1 is equal to zero. After the values for λ_c and T_c are obtained they will be used in all later calculations and T_1 will be taken to be exactly zero. The reasons why will manifest themselves in later chapters.

The values obtained for T_1, T_2, T_3 and the corresponding approximations to λ_c and T_c are :

For $\eta = 0.95$ and $\lambda_o = 3.127$, $T_o = 1754.96$ (2.8.13)

$$T_1 = -0.1806 \; , \; T_2 = 256.567 \; , \; T_3 = -57.9265 \; , \qquad (2.8.14)$$

and when we use (2.8.8) with the above values the new approximation for λ_c and T_c are

3.12735 and 1754.96 respectively. (2.8.15)

These values were considered to be the new λ_o, T_o and a closer approximation was then found with the following values for T_1, T_2 and T_3 .

$$T_1 = -0.0011 \; , \; T_2 = 256.5062 \; , \; T_3 = -57.8862. \qquad (2.8.16)$$

This resulted in the same values for λ_c and T_c as (2.8.15), which will be used for future work in the case $\eta = R_1/R_2 = 0.95$.

The following results were also obtained with $\eta = 0.5$ with the same method :

For $\eta = 0.5$ and $\lambda_o = 3.163$, $T_o = 3099.78$ (2.8.17)

$$T_1 = 0.5140\ , \quad T_2 = 440.108\ , \quad T_3 = -99.8321. \qquad (2.8.18)$$

The first approximation to λ_c and T_c using (2.8.8) and the above values are

3.16242 and 3099.78 respectively. (2.8.19)

These values of λ and T were considered then to be the new λ_0 and T_0 for $\eta = 0.5$. A second closer approximation to λ_c and T_c were found using the new calculated values of T_1, T_2 and T_3,

$$T_1 = 0.0033\ , \quad T_2 = 440.2819\ , \quad T_3 = -99.9442. \qquad (2.8.20)$$

This resulted in the same values for λ_c and T_c as (2.8.19) and for future work will be used as the critical values for the case $\eta = 0.5$.

We also used this method in obtaining values of $\lambda_{L,c}$ and $T_{L,c}$ for fixed η_L in the results of Chapter 4.

In later work we shall use the constants T_2 and T_3 which are to be interpreted as the final constants obtained above in equations (2.8.16) and (2.8.20).

2.9 Introduction to the non-linear problem with respect to torque calculations

This chapter is involved with numerical calculations of the additional torque supplied to the inner cylinder to sustain an equilibrium Taylor-vortex flow of small amplitude between the two cylinders. This is the extra torque required to keep the inner cylinder in motion at a given speed for this particular flow.

We first fix λ at a particular value and calculate the lowest Taylor number that will lie on the neutral curve. We will have a point λ_0, T_0 which lies on the neutral curve. The general idea is to keep λ fixed at λ_0 and increase the Taylor number above T_0, so the point in the λ,T plane is above the neutral curve. We should then be able to obtain an equilibrium Taylor-vortex flow.

In the method adopted, we set

$$\lambda = \lambda_0 \tag{2.9.1}$$

and

$$T = T_0 [1 + \gamma^2] \tag{2.9.2}$$

where γ^2 is assumed small.

The formula for γ^2 can be written as

$$\gamma^2 = \frac{T-T_0}{T_0} , \tag{2.9.3}$$

therefore upon specifying the value of T will fix the value of γ^2. This will enable us to calculate the torque along any line λ equal to a constant. See Davey[6].

An attempt was made to numerically calculate the wavenumber that would give a maximum torque for a given T. The expression for the torque G will be seen to be

$$G = \gamma^2 K_0 + \gamma^4 K_1 + \dots . \tag{2.9.4}$$

The general idea is to use γ^2 as a small parameter so that we can ignore terms of $O(\gamma^6)$ etc. The expression obtained for G seems correct as the results obtained for G when

$$T \to T_o \text{ , with } \lambda \text{ fixed , } G \to 0 \tag{2.9.5}$$

is in accordance with what you expect.

The numerical results obtained from using our expansion procedure for the torque and maximum torque are used to compare with the results of DiPrima & Eagles.[4]

2.10 Expansion of the non-linear equation for small amplitudes

From the theory for the equilibrium Taylor-vortex flow we have if

$$\lambda = \lambda_o \tag{2.10.1}$$

then

$$T = T_o \tag{2.10.2}$$

will represent a point on the neutral curve.

We shall set

$$T = T_o [1 + \gamma^2] \tag{2.10.3}$$

in the steady state, non-linear equation given by (2.3.11), we obtain

$$\frac{\partial \underline{U}}{\partial x} - \underline{A}_o \underline{U} - \gamma^2 T_o \underline{A}_2 \underline{U} = \underline{L}_o (\underline{U}) \underline{U} + \gamma^2 T_o \underline{L}_2 (\underline{U}) \underline{U} \quad ; \quad \beta_2 \tag{2.10.4}$$

subject to the boundary condition (2.4.1).

Here

$$\underline{L}_2(\underline{U}) = \begin{bmatrix} & 0 & -\frac{Gv}{2} & 0 \\ \underset{\sim}{0} & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix} \tag{2.10.5}$$

and $\underline{A}_2$, $\underline{A}_o$ are defined in (2.4.7) and (2.3.14) respectively.

It is noted that both $\underline{A}_o$ and $\underline{L}_o$ contain T_o, and γ^2 will represent a small amplitude of the disturbance for a given T.

We next expand $\underline{U}$ in powers of γ and write

$$\underline{U} = \gamma\underline{u}_1 + \gamma^2\underline{u}_2 + \gamma^3\underline{u}_3 + \ldots \quad , \tag{2.10.6}$$

and when we equate powers of γ^n we obtain a set of partial differential equations given by

$$(\partial/\partial x - \underline{A}_o)\underline{u}_1 = 0 \ , \tag{2.10.7}$$

$$(\partial/\partial x - \underline{A}_o)\underline{u}_2 = \underline{L}_o(\underline{u}_1)\underline{u}_1 \ , \tag{2.10.8}$$

$$(\partial/\partial x - \underline{A}_o)\underline{u}_3 = \underline{L}_o(\underline{u}_1)\underline{u}_2 + \underline{L}_o(\underline{u}_2)\underline{u}_1 + T_o\underline{A}_2\underline{u}_1 \ , \tag{2.10.9}$$

$$\begin{aligned}(\partial/\partial x - \underline{A}_o)\underline{u}_4 = {} & \underline{L}_o(\underline{u}_1)\underline{u}_3 + \underline{L}_o(\underline{u}_2)\underline{u}_2 + \underline{L}_o(\underline{u}_3)\underline{u}_1 \\ & + T_o\underline{L}_2(\underline{u}_1)\underline{u}_1 + T_o\underline{A}_2\underline{u}_2 \ ,\end{aligned} \tag{2.10.10}$$

$$\begin{aligned}(\partial/\partial x - \underline{A}_o)\underline{u}_5 = {} & \underline{L}_o(\underline{u}_1)\underline{u}_4 + \underline{L}_o(\underline{u}_2)\underline{u}_3 + \underline{L}_o(\underline{u}_3)\underline{u}_2 \\ & + \underline{L}_o(\underline{u}_4)\underline{u}_1 + T_o\underline{L}_2(\underline{u}_1)\underline{u}_2 + T_o\underline{L}_2(\underline{u}_2)\underline{u}_1 \\ & + T_o\underline{A}_2\underline{u}_3 \ .\end{aligned} \tag{2.10.11}$$

The boundary condition β_2 applies to equations (2.10.7) to (2.10.11), which is the physical condition that requires zero disturbance velocities at the two cylinder walls.

A real solution of (2.10.7) can be written as

$$\underline{u}_1 = e^{i\lambda_o\zeta}\,\underline{u}_{11}(x) + e^{-i\lambda_o\zeta}\,\underline{u}_{1,-1}(x) \tag{2.10.12}$$

where $\underline{u}_{11}(x)$ satisfies (2.4.4) in conjunction with (2.4.4a).

When we use (2.10.12) in the right hand side of (2.10.8) we see $\underline{u}_2$ can be written as

$$\underline{u}_2 = e^{2i\lambda_o\zeta}\,\underline{u}_{22} + \underline{u}_{20} + e^{-2i\lambda_o\zeta}\,\underline{u}_{2,-2}\ . \tag{2.10.13}$$

The equations and boundary conditions satisfied by $\underline{u}_{22}$, $\underline{u}_{20}$ are

$$\left(\frac{d}{dx} - \underline{A}_o^{(2)}\right)\underline{u}_{22} = \underline{L}_o^{(1)}(\underline{u}_{11})\underline{u}_{11}\ ;\ \beta_2 \tag{2.10.14}$$

$$\left(\frac{d}{dx} - \underline{A}_o^{(0)}\right)\underline{u}_{20} = \underline{L}_o^{(-1)}(\underline{u}_{11})\underline{u}_{1,-1} + \underline{L}_o^{(1)}(\underline{u}_{1,-1})\underline{u}_{11}\cdot\beta_2 \tag{2.10.15}$$

The matrix $\underline{L}_o^{(p)}(\underline{u}_{ij})$ is defined by

$$\underline{L}_o^{(p)}(\underline{u}_{ij}) = -\frac{1}{\alpha}\begin{bmatrix} & \delta G u_{ij,4} - i\lambda_o p u_{ij,6} & \alpha T_o G u_{ij,5}/2 & ip\lambda_o u_{ij,4} \\ \underset{\sim}{0} & u_{ij,2} + \delta G u_{ij,5} & ip\lambda_o u_{ij,6} & 0 \\ & u_{ij,3} & 0 & ip\lambda_o u_{ij,6} \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix} \tag{2.10.16}$$,

$\underline{A}_o^{(p)}$ is defined in (2.4. 6) and $u_{ij,k}$ for $k = 1,6$ represents the six components of the vector $\underline{u}_{ij}$.

To simplify equations (2.10.14), (2.10.15) and later equations we shall use the following notation

$$\underline{R}_{os}(\underline{f}_{ij},\underline{g}_{k\ell}) = \underline{L}_{os}^{(\ell)}(\underline{f}_{ij})\underline{g}_{k\ell} + \underline{L}_{os}^{(j)}(\underline{g}_{k\ell})\underline{f}_{ij}, s=0,2 . \tag{2.10.17}$$

We can now rewrite (2.10.14) and (2.10.15) as

$$\mathcal{L}^{(2)}(\underline{u}_{22}) = \tfrac{1}{2}\underline{R}_{oo}(\underline{u}_{11},\underline{u}_{11}) \quad ; \quad \beta_2 \tag{2.10.18}$$

$$\mathcal{L}^{(0)}(\underline{u}_{20}) = \underline{R}_{oo}(\underline{u}_{11},\underline{u}_{1,-1}) \quad ; \quad \beta_2 \quad . \tag{2.10.19}$$

The operator $\mathcal{L}^{(p)}$ is defined in (2.4.12).

When we use (2.10.12) and (2.10.13) in the right hand side of (2.10.9), we note that $\underline{u}_3$ can be written as

$$\underline{u}_3 = e^{3i\lambda_o\zeta}\underline{u}_{33} + e^{i\lambda_o\zeta}\underline{u}_{31} + e^{-i\lambda_o\zeta}\underline{u}_{3,-1} + e^{-3i\lambda_o\zeta}\underline{u}_{3,-3} . \tag{2.10.20}$$

The equations and boundary conditions satisfied by $\underline{u}_{33}, \underline{u}_{31}$ are

$$\mathcal{L}^{(3)}(\underline{u}_{33}) = \underline{R}_{oo}(\underline{u}_{11},\underline{u}_{22}) \quad ; \quad \beta_2 \tag{2.10.21}$$

$$\mathcal{L}^{(1)}(\underline{u}_{31}) = \underline{R}_{oo}(\underline{u}_{11},\underline{u}_{20}) + \underline{R}_{oo}(\underline{u}_{1,-1},\underline{u}_{22}) + T_o\underline{A}_2\underline{u}_{11} \; ; \; \beta_2 \tag{2.10.22}$$

We find that on repeating this procedure we can write $\underline{u}_4$ and $\underline{u}_5$ as

$$\underline{u}_4 = e^{4i\lambda_o\zeta}\underline{u}_{44} + e^{2i\lambda_o\zeta}\underline{u}_{42} + \underline{u}_{40} + e^{-2i\lambda_o\zeta}\underline{u}_{4,-2} + e^{-4i\lambda_o\zeta}\underline{u}_{4,-4} , \tag{2.10.23}$$

$$\underline{u}_5 = e^{5i\lambda_o\zeta}\underline{u}_{55} + e^{3i\lambda_o\zeta}\underline{u}_{53} + e^{i\lambda_o\zeta}\underline{u}_{51} + e^{-i\lambda_o\zeta}\underline{u}_{5,-1} + e^{-3i\lambda_o\zeta}\underline{u}_{5,-3} + e^{-5i\lambda_o\zeta}\underline{u}_{5,-5} . \tag{2.10.24}$$

We only require the equations for $\underline{u}_{42}$, $\underline{u}_{40}$ from (2.10.23) and the equation for $\underline{u}_{51}$ from (2.10.24) in order to use certain consistency conditions for $\underline{u}_{51}$ to have a solution.

The equations and boundary conditions satisfied by $\underline{u}_{42}$, $\underline{u}_{40}$ and $\underline{u}_{51}$ are

$$\mathcal{L}^{(2)}(\underline{u}_{42}) = \underline{R}_{oo}(\underline{u}_{11},\underline{u}_{31}) + \underline{R}_{oo}(\underline{u}_{1,-1},\underline{u}_{33}) + \underline{R}_{oo}(\underline{u}_{22},\underline{u}_{20}) + \frac{T_o}{2}\underline{R}_{o2}(\underline{u}_{11},\underline{u}_{11}) + T_o\,\underline{A}_2\underline{u}_{22} \quad ; \quad \beta_2 \qquad (2.10.25)$$

$$\mathcal{L}^{(0)}(\underline{u}_{40}) = \underline{R}_{oo}(\underline{u}_{11},\underline{u}_{3,-1}) + \underline{R}_{oo}(\underline{u}_{1,-1},\underline{u}_{31}) + \underline{R}_{oo}(\underline{u}_{22},\underline{u}_{2,-2}) + \underline{R}_{o2}(\underline{u}_{11},\underline{u}_{1,-1}) + \frac{1}{2}\underline{R}_{oo}(\underline{u}_{20},\underline{u}_{20}) + T_o\underline{A}_2\underline{u}_{20} \quad ; \quad \beta_2 \qquad (2.10.26)$$

$$\mathcal{L}^{(1)}(\underline{u}_{51}) = \underline{R}_{oo}(\underline{u}_{11},\underline{u}_{40}) + \underline{R}_{oo}(\underline{u}_{1,-1},\underline{u}_{42}) + \underline{R}_{oo}(\underline{u}_{22},\underline{u}_{3,-1}) + \underline{R}_{oo}(\underline{u}_{20},\underline{u}_{31}) + \underline{R}_{oo}(\underline{u}_{2,-2},\underline{u}_{33}) + T_o\underline{R}_{o2}(\underline{u}_{11},\underline{u}_{20}) + T_o\underline{R}_{o2}(\underline{u}_{1,-1},\underline{u}_{22}) + T_o\underline{A}_2\underline{u}_{31} \quad ; \quad \beta_2 \qquad (2.10.27)$$

2.11 Consistency conditions and formula for Taylor-vortex torque

From the linear theory, a solution for $\underline{u}_{11}(x)$ can be written as

$$\underline{u}_{11}(x) = \mu_o \underline{f}_{11}(x) \qquad (2.11.1)$$

where $\underline{f}_{11}(x)$ satisfies (2.4.13) and the given normalization which follows (2.4.13). Here μ_o is an unknown real constant which will be determined by a consistency condition on a higher order differential equation.

When we use (2.11.1), and let

$$\underline{u}_{2j} = \mu_o^2 \underline{f}_{2j} , \tag{2.11.2}$$

in (2.10.18) and (2.10.19) we then have

$$\mathfrak{L}^{(2)}(\underline{f}_{22}) = \frac{1}{2} \underline{R}_{oo}(\underline{f}_{11}, \underline{f}_{11}) \quad ; \quad \beta_2 \tag{2.11.3}$$

$$\mathfrak{L}^{(0)}(\underline{f}_{20}) = \underline{R}_{oo}(\underline{f}_{11}, \underline{f}_{1,-1}) \quad ; \quad \beta_2 . \tag{2.11.4}$$

Similarly, if we let

$$\underline{u}_{3j} = \mu_o^3 \underline{f}_{3j} \tag{2.11.5}$$

in (2.10.21) and (2.10.22) then we obtain

$$\mathfrak{L}^{(3)}(\underline{f}_{33}) = \underline{R}_{oo}(\underline{f}_{11}, f_{22}) \quad ; \quad \beta_2 \tag{2.11.6}$$

$$\mathfrak{L}^{(1)}(\underline{f}_{31}) = \underline{R}_{oo}(\underline{f}_{11}, \underline{f}_{20}) + \underline{R}_{oo}(\underline{f}_{1,-1}, \underline{f}_{22}) + \frac{T_o \underline{A}_2 \underline{f}_{11}}{\mu_o^2} \quad ; \quad \beta_2 \tag{2.11.7}$$

To solve equation (2.11.7), a consistency condition has to be imposed so that a solution for $\underline{f}_{31}$ will exist. This involves the use of the adjoint function mentioned in § 2.5.

For $\underline{f}_{31}$ to exist, we use the adjoint condition (2.5.3) and obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} [\underline{R}_{oo}(\underline{f}_{11}, \underline{f}_{20}) + \underline{R}_{oo}(\underline{f}_{22}, \underline{f}_{1,-1})] dx + \frac{T_o}{\mu_o^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{A}_2 \underline{f}_{11} dx = 0. \tag{2.11.8}$$

Hence we can find μ_o^2 as

$$\mu_o^2 = \frac{-T_o \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{A}_2 \underline{f}_{11}\, dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} [\underline{R}_{oo}(\underline{f}_{11}, \underline{f}_{20}) + \underline{R}_{oo}(\underline{f}_{22}, \underline{f}_{1,-1})]\, dx} \tag{2.11.9}$$

where μ_o will be taken as the positive root of (2.11.9) and is now a known value.

We set

$$\underline{f}_{31} = \underline{e}_{31} + \omega \underline{f}_{11}(x) \tag{2.11.10}$$

where $\underline{e}_{31}$ is normalized such that the second component of $\underline{e}_{31}$ evaluated at $x = -\frac{1}{2}$ is equal to zero, and ω is a real constant unknown at this stage.

The function $\underline{e}_{31}$ satisfies the differential equation,

$$\mathcal{L}^{(1)}(\underline{e}_{31}) = \underline{R}_{oo}(\underline{f}_{11}, \underline{f}_{20}) + \underline{R}_{oo}(\underline{f}_{22}, \underline{f}_{1,-1}) + T_o \underline{A}_2 \underline{f}_{11} / \mu_o^2 \; ;$$

$$\beta_2$$

and μ_o^2 is now known. (2.11.11)

When we use (2.11.10), (2.11.5), (2.11.2) and (2.11.1) in the right-hand sides of equations (2.10.25) and (2.10.26), and set

$$\underline{u}_{4j} = \mu_o^4 \underline{f}_{4j} \tag{2.11.12}$$

and then

$$\underline{f}_{4j} = \underline{e}_{4j} + \omega \underline{m}_{4j} \tag{2.11.13}$$

in the left-hand sides of (2.10.25) and (2.10.26), we find the functions $\underline{e}_{4j}$ and $\underline{m}_{4j}$ satisfy a set of ordinary differential equations subject to boundary condition β_2 :

$$\mathfrak{L}^{(2)}(\underline{e}_{42}) = \underline{R}_{oo}(\underline{f}_{11}, \underline{e}_{31}) + \underline{R}_{oo}(\underline{f}_{1,-1}, \underline{f}_{33}) + \underline{R}_{oo}(\underline{f}_{22}, \underline{f}_{20}) +$$

$$\frac{T_o}{2}\underline{R}_{o2}(\underline{f}_{11}, \underline{f}_{11})/\mu_o^2 + T_o\,\underline{A}_2\underline{f}_{22}/\mu_o^2 \quad ; \quad \beta_2 \qquad (2.11.14)$$

$$\mathfrak{L}^{(2)}(\underline{m}_{42}) = \underline{R}_{oo}(\underline{f}_{11}, \underline{f}_{11}) \quad ; \quad \beta_2 \qquad (2.11.15)$$

$$\begin{aligned}\mathfrak{L}^{(0)}(\underline{e}_{40}) = {} & \underline{R}_{oo}(\underline{f}_{11}, \underline{e}_{3,-1}) + \underline{R}_{oo}(\underline{f}_{1,-1}, \underline{e}_{31}) \\ & + \underline{R}_{oo}(\underline{f}_{22}, \underline{f}_{2,-2}) + \tfrac{1}{2}\underline{R}_{oo}(\underline{f}_{20}, \underline{f}_{20}) + \\ & T_o\underline{R}_{o2}(\underline{f}_{11}, \underline{f}_{1,-1})/\mu_o^2 + T_o\,\underline{A}_2\underline{f}_{20}/\mu_o^2 \\ & ; \ \beta_2 \qquad (2.11.16)\end{aligned}$$

$$\mathfrak{L}^{(0)}(\underline{m}_{40}) = 2\,\underline{R}_{oo}(\underline{f}_{11}, f_{1,-1}) \quad ; \quad \beta_2 \,. \qquad (2.11.17)$$

From the form of the equations for $\underline{m}_{42}$ and $\underline{m}_{40}$ we can write

$$\underline{m}_{42} = 2\underline{f}_{22} \qquad (2.11.18)$$

and

$$\underline{m}_{40} = 2\underline{f}_{20} \;. \qquad (2.11.19)$$

To solve equation (2.10.27), a consistency condition has to be imposed so that a solution for $\underline{f}_{51}$ will exist. This again involves the use of the adjoint function mentioned in §2.5.

For $\underline{f}_{51}$ to exist we use the adjoint condition (2.5.3) and obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\,\underline{Q}\,dx + \frac{T_o}{\mu_o^2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}[\underline{R}_{o2}(\underline{f}_{11}, \underline{f}_{20}) + \underline{R}_{o2}(\underline{f}_{1,-1}, \underline{f}_{22})$$

$$+ \underline{A}_2\underline{e}_{31}]dx + \omega\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}[\underline{R}_{oo}(\underline{f}_{11}, \underline{m}_{40}) + \underline{R}_{oo}(\underline{f}_{1,-1}, \underline{m}_{42})$$

$$+ \ \underline{R}_{oo}(\underline{f}_{22},\underline{f}_{1,-1}) + \underline{R}_{oo}(\underline{f}_{20},\underline{f}_{11})\,]dx$$

$$+ \omega T_o \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\,\underline{A}_2 f_{11}\,dx/\mu_o^{\,2} \ = \ 0 \qquad (2.11.20)$$

where

$$\underline{Q} = \underline{R}_{oo}(\underline{f}_{11},\underline{e}_{40}) + \underline{R}_{oo}(\underline{f}_{1,-1},\underline{e}_{42}) + \underline{R}_{oo}(\underline{f}_{22},\underline{e}_{3,-1}) +$$

$$\underline{R}_{oo}(\underline{f}_{20},\underline{e}_{31}) + \underline{R}_{oo}(\underline{f}_{2,-2},\underline{f}_{33}) \,. \qquad (2.11.21)$$

The equation (2.11.20) can be simplified when we use (2.11.18), (2.11.19) and (2.11.9) to give ω as

$$\omega = \frac{\mu_o^{\,2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\underline{Q}\,dx + T_o\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}[\underline{R}_{o2}(\underline{f}_{11},\underline{f}_{20}) + \underline{R}_{o2}(\underline{f}_{1,-1},\underline{f}_{22}) + \underline{A}_2\underline{e}_{31}]dx}{2T_o\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\,\underline{A}_2\,\underline{f}_{11}\,dx} \,. \qquad (2.11.22)$$

Now that the coefficients μ_o and ω are known we can calculate the Taylor-vortex torque.

In our approximation, we can calculate the additional torque due to the vortex motion from the velocity disturbance in terms of γ^2. This is given by

$$G = \gamma^2 K_o + \gamma^4 K_1 + \ldots + O(\gamma^6) \qquad (2.11.23)$$

where

$$\gamma^2 = \frac{T-T_o}{T_o} \,, \qquad (2.11.24)$$

and

$$K_o = \mu_o^{\,2} f_{20,2}(-\tfrac{1}{2}) \qquad (2.11.25)$$

$$K_1 = \mu_o^4 e_{40,2}(-\tfrac{1}{2}) + 2\omega\mu_o^4 f_{20,2}(-\tfrac{1}{2}) \, . \tag{2.11.26}$$

Here $f_{ij,2}(-\frac{1}{2})$ denotes the second component of $\underline{f}_{ij}$ evaluated at $x = -\frac{1}{2}$, that is the inner boundary. It is mentioned that the values of the constants K_o and K_1 will remain invariant under whatever sign we had chosen for μ_o .

From (2.11.23) we can numerically calculate the zero's of $(\partial G/\partial\lambda)_{T \text{ fixed}}$ and obtain the maximum torque, λ_t. To do this we first find a point λ_o, T_o on the neutral curve and keep these values fixed, and so compute the coefficients K_o and K_1. The Taylor number T was then fixed at a specific value greater than T_o and the torque calculated from (2.11.23), etc. This procedure was repeated a number of times for various values of λ_o, T_o, which all lie on the neutral curve, but with the Taylor number T kept fixed at its initial value. The values of $\lambda = \lambda_o$ and the resultant values for the torque G are then used to find numerically the zeros of $(\partial G/\partial\lambda)_T$ fixed for a given λ. This value of λ is called the wavenumber for maximum torque λ_t.

2.12 Results of torque calculations

It was decided to do just the one case, using the full equations for $\eta = 0.95$.

The results for the Taylor-vortex torque G obtained from (2.11.23) are given in TABLE II. The calculated wavenumber for maximum torque for T = 1840 and 1920 are given in TABLE III.

These results are compared with those of DiPrima & Eagles [4] for λ = 3.1, 3.2 and 3.3. The results were found to agree very well for T = 1840 ; for T = 1920 there is a much poorer agreement. The difference in results appear to be of $O(\gamma^6)$.

The results for the wavenumber of maximum torque were found to be in poor agreement with that of DiPrima & Eagles [4]. This was probably due to the torque results being too close for the three values of λ chosen, (λ = 3.1, 3.12725, 3.2), for the approximate quadratic curve. The wavenumber for maximum torque was calculated from this curve.

T	$\frac{T-T_o}{T_o}$	λ	$-G$	$-G^a$
1840	0.0401	2.9	0.1094	0.1105
	0.0483	3.1	0.1307	0.1317
	0.0477	3.2	0.1294	0.1300
	0.0441	3.3	0.1206	0.1212
	0.0378	3.4	0.1045	
	0.0485	3.12735	0.1311	
1920	0.0854	2.9	0.2126	0.2215
	0.0939	3.1	0.2359	0.2436
	0.0932	3.2	0.2363	0.2430
	0.0895	3.3	0.2296	0.2353
	0.0829	3.4	0.2159	
	0.0940	3.12735	0.2367	

[a] Taylor-vortex torque results of DiPrima & Eagles[4].

TABLE II Taylor-vortex torque for $\eta = 0.95$

T	λ_t	λ_t^a
1840	3.133	3.164
1920	3.156	3.197

TABLE III Wavenumber for maximum torque (λ_t).

3. Non-parallel Wall Case

3.1 Introduction to the non-parallel wall linear case

In this chapter we are interested in the hydrodynamic stability problem associated with the outer wall being of the form

$$r = R_2 + \varepsilon^2 (R_2 - R_1) F(Z)/2 \qquad (3.1.1)$$

in cylindrical polar co-ordinates (r,θ,z). We call this problem the non-parallel wall problem in order to distinguish it from the parallel wall problem, where the outer wall is given by $r = R_2$. In both cases the inner boundary is $r = R_1$.

The parameter ε defines the slowly varying variable Z, where

$$Z = \varepsilon z , \qquad (3.1.2)$$

and also a small variation in the outer wall.

The base velocities and pressure are expanded in powers of ε and are functions of r and Z only. We obtain solutions of the form

$$u_s = O(\varepsilon^4),\ v_s = V_0(r) + \varepsilon^2 v_2(r,Z) ,\ W_s = \varepsilon^3 w_3(r,Z) \qquad (3.1.3)$$

and

$$p_s = p_0(r) + \varepsilon^2 p_2(r,Z) . \qquad (3.1.4)$$

An additional condition for all the above solutions to be true was

$$u_s , w_s \to 0 \quad \text{as} \quad Z \to \pm\infty , \qquad (3.1.5)$$

because we shall assume a purely (modified) circumferential Couette flow at $Z = \pm\infty$.

The reader should note the assumption made later that F(Z) is even may be dropped up to §3.10 in this chapter. The expansions are valid for all F(z) provided (3.1.5) holds. The boundary condition (3.1.5) will be seen to force F(Z) to have the following property

$$\frac{dF}{dZ} \to 0 \quad \text{as} \quad Z \to \pm\infty . \tag{3.1.5a}$$

We introduce the constants and dimensionless variables defined in (2.3.6) and (2.3.7) along with the slowly varying dimensionless variable

$$z^* = Z/d \tag{3.1.6}$$

into the base velocities and the disturbance equations in the usual manner. See Eagles [10].

The non-dimensionalizing is done in terms of R_1 and R_2. The Taylor number T is

$$T = \frac{\Omega_1^2 R_1^2 d^3}{\nu^2 R_0} . \tag{3.1.7}$$

We are then able to write the disturbance equations in the matrix form shown in (3.5.22).

The boundary conditions on $\underline{U}$ are the physical condition of zero disturbance velocities at the wall of the inner cylinder and at the outer surface. An additional condition imposed on $\underline{U}$ is

$$\underline{U} \to 0 \quad \text{as} \quad z^* \to \pm\infty . \tag{3.1.8}$$

With a suitable choice of F(Z) such that the base flow is locally more stable at $Z = \pm\infty$ than near $Z = 0$ we are able to find a critical overall Taylor number T_{crit} in the form

$$T_{crit} = T_c + \varepsilon^2 T_2^* + \dots , \tag{3.1.9}$$

which appears to be the lowest Taylor number for which neutral solutions exist to the disturbance equations. We are also able to find higher order corrections to the axial wavenumber, which is no longer uniquely defined, but varies with the physical quantity considered and also with x and z^*. In the non-linear case there is also a variation with T.

3.2 Assumptions made about the outer stationary surface

Let (r,θ,z) denote cylindrical polar co-ordinates such that the z-axis is chosen to lie along the axis of a cylinder radius $r = R_1$. This cylinder is rotated about its axis with a constant angular velocity Ω_1 inside a fixed concentric surface radius

$$r = R_2 + \varepsilon^2(R_2-R_1)F(Z)/2 . \tag{3.2.1}$$

The gap between the cylinder and the outer surface is filled with a liquid of Kinematic viscosity ν and constant density ρ .

The variable Z is given by

$$Z = \varepsilon z. \tag{3.2.2}$$

The parameter ε does not only define the slow variable Z but also a small variation of the outer surface. The constant coefficient of F(Z), namely $(R_2-R_1)/2$, was chosen such that the equation of the outer surface in the non-dimensional co-ordinate x, defined in (2.3.7), is given by

$$x = \frac{1}{2}\left[1 + \varepsilon^2 f(z^*)\right]. \tag{3.2.3}$$

To obtain the function $f(z^*)$ we non-dimensionalize Z by

$$Z = dz^* \tag{3.2.4}$$

and place

$$F(dz^*) = f(z^*). \qquad (3.2.5)$$

It can be seen that z^* is a dimensionless slow variable when compared with the dimensionless axial co-ordinate ζ.

To define the problem more the function $f(z^*)$ will be given the following properties :

(i) $f(z^*) = f(-z^*)$. (3.2.6)

This ensures that the outer surface is symmetrical about $z^* = 0$ and enables us to solve the problem for just positive z^*.

(ii) $f(0) = 0$. (3.2.7)

This will give that at $z^* = 0$, the outer surface is given by $x = \frac{1}{2}$.

(iii) $\lim_{z^* \to \pm\infty} f(z^*) = f_\infty$, (3.2.8)

where f_∞ is a negative constant.

(iv) $\lim_{z^* \to \pm\infty} \left(\frac{df}{dz^*}\right) = 0$. (3.2.9)

3.3 Analysis of the base flow

We denote the velocity components of the basic flow by (u_s, v_s, w_s) and the pressure component by p_s. The Navier-Stokes and continuity equations for viscous, incompressible, axisymmetric steady flow are given by (2.2.1) to (2.2.4) with the partial derivatives of u_s, v_s, w_s with respect to time equated to zero.

The boundary conditions on u_s, v_s, w_s are

$$u_s = v_s = w_s = 0 \quad \text{on} \quad r = R_2 + \varepsilon^2 (R_2 - R_1) F(Z)/2, \qquad (3.3.1)$$

$$u_s = w_s = 0 \quad \text{and} \quad v_s = \Omega_1 R_1 \text{ on } r = R_1. \qquad (3.3.2)$$

For steady laminar Couette flow, which is given by $\varepsilon = 0$ in the above, there is a solution of the form

$$u_s = 0., \quad v_s = V_o(r) , \quad w_s = 0. \qquad (3.3.3)$$

where

$$V_o(r) = \frac{-R_1^2 \, \Omega_1}{(R_2^{\,2} - R_1^2)} \cdot \left[r - \frac{R_2^2}{r} \right] \qquad (3.3.4)$$

with the corresponding pressure $p_o(r)$ given by (2.2.7)

For ε small but different from zero we shall expand the base velocities and pressure in powers of ε and write

$$\left.\begin{aligned} u_s &= \varepsilon u_1 + \varepsilon^2 u_2 + \dots , \\ v_s &= V_o(r) + \varepsilon v_1 + \varepsilon^2 v_2 + \dots , \\ w_s &= \varepsilon w_1 + \varepsilon^2 w_2 + \dots , \end{aligned}\right\} \qquad (3.3.5)$$

$$p_s = \varepsilon p_o(r) + \varepsilon p_1 + \varepsilon^2 p_2 + \dots ,$$

where the ellipsis dots stand for all terms with powers of ε^n for which $n \geq 3$. The functions u_i, w_i, v_i, p_i for $i = 1,2,3,\dots$ will be considered to contain just two variables r and Z.

When we use (3.3.5) and the boundary conditions on the inner wall (3.3.2), we can see the base velocities must satisfy

$$V_o = \Omega_1 R_1 \quad \text{on} \quad r = R_1 , \qquad (3.3.6)$$

$$u_i = v_i = w_i = 0 \quad \text{on} \quad r = R_1 \quad \text{for } i = 1,2,3, \dots \qquad (3.3.7)$$

The boundary conditions on the outer wall, $r = R_2 + \varepsilon^2 (R_2 - R_1)F(Z)/2$ are obtained by means of a Taylor expansion of the velocities at $r = R_2$. The base velocity u_s must satisfy

$$u_s = 0 \quad \text{on} \quad r = R_2 + \varepsilon^2 (R_2 - R_1)F(Z)/2 , \qquad (3.3.8)$$

which from (3.3.5), is the same as

$$\varepsilon u_1(R_2 + \varepsilon^2(R_2-R_1)F(Z)/2,Z)+\varepsilon^2 u_2(R_2+ \varepsilon^2(R_2-R_1)F(Z)/2,Z)+\ldots$$
$$= 0 \; . \qquad (3.3.9)$$

When we use the Taylor expansion about $r = R_2$ for the velocity u_s, equate powers of ε^n we obtain the following set of boundary conditions for each u_i :

$$u_1(R_2,Z)=0 \quad , \; (R_2-R_1)F(Z)u_{1r}(R_2,Z)/2 + u_3(R_2,Z) = 0, \qquad (3.3.10)$$
$$u_2(R_2,Z)=0 \quad , \; (R_2-R_1)F(Z)u_{2r}(R_2,Z)/2 + u_4(R_2,Z) = 0 \; ,$$

where

$$u_{ir} = \frac{\partial u_i}{\partial r} \; . \qquad (3.3.11)$$

The boundary conditions for the base velocity w_s are identical to those quoted above and the boundary conditions for v_s are

$$V_o(R_2) = 0 \quad , \; (R_2-R_1)F(Z)V_{or}(R_2)/2+v_2(R_2,Z) = 0,$$
$$(3.3.12)$$
$$v_1(R_2,Z) = 0, \; (R_2-R_1)F(Z)v_{1r}(R_2,Z)/2+v_3(R_2,Z) = 0 \; .$$

From (3.2.2) we notice that any $\partial/\partial z$ in the continuity or Navier-Stokes equations becomes $\varepsilon\partial/\partial Z$. Thus when we use the form of expansion given in (3.3.5) for u_s and w_s, the continuity equation can be rewritten as

$$\varepsilon\left(\frac{\partial u_1}{\partial r} + \frac{u_1}{r}\right) + \varepsilon^2\left(\frac{\partial u_2}{\partial r} + \frac{u_2}{r} + \frac{\partial w_1}{\partial z}\right) + \cdots = 0. \qquad (3.3.13)$$

If we equate powers of ε in (3.3.13) we obtain a set of partial differential equations :

$$\frac{\partial u_1}{\partial r} + \frac{u_1}{r} = 0, \qquad (3.3.14)$$
$$\frac{\partial u_2}{\partial r} + \frac{u_2}{r} + \frac{\partial w_1}{\partial z} = 0, \qquad (3.3.15)$$

$$\frac{\partial u_3}{\partial r} + \frac{u_3}{r} + \frac{\partial w_2}{\partial z} = 0. \qquad (3.3.16)$$

From the boundary conditions on $u_1(r,Z)$, given in (3.3.7) and (3.3.10) , and the differential equation for $u_1(r,Z)$, (3.3.14), the only solution for $u_1(r,Z)$ is zero. That is

$$u_1(r,Z) = 0. \qquad (3.3.17)$$

Now substituting the form of expansion (3.3.5) with $u_1(r,Z)$ equated to zero, in the Navier-Stokes equations and replacing $\partial/\partial z$ by $\varepsilon\,\partial/\partial Z$ we obtain a set of partial differential equations upon equating powers of ε .

From the first Navier-Stokes equation (2.2.1) we find that

$$\frac{V_o^2}{r} = \frac{1}{\rho}\frac{\partial p_o}{\partial r}, \qquad (3.3.18)$$

$$\frac{2v_1V_o}{r} = \frac{1}{\rho}\frac{\partial p_1}{\partial r}, \qquad (3.3.19)$$

$$\frac{2v_2V_o+v_1^2}{r} + \nu\left[\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} - \frac{u_2}{r^2}\right] = \frac{1}{\rho}\frac{\partial p_2}{\partial r}, \qquad (3.3.20)$$

$$\frac{2(V_ov_3+v_2v_1)}{r} + \nu\left[\frac{\partial^2 u_3}{\partial r^2} + \frac{1}{r}\frac{\partial u_3}{\partial r} - \frac{u_3}{r^2}\right] = \frac{1}{\rho}\frac{\partial p_3}{\partial r}. \qquad (3.3.21)$$

From the second Navier-Stokes equation (2.2.2) we have

$$\frac{d^2V_o}{dr^2} + \frac{1}{r}\frac{dV_o}{dr} - \frac{V_o}{r^2} = 0, \qquad (3.3.22)$$

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\frac{\partial v_1}{\partial r} - \frac{v_1}{r^2} = 0, \qquad (3.3.23)$$

$$\frac{u_2}{r}\frac{d}{dr}(rV_o) = \nu\left(\frac{\partial^2 v_2}{\partial r^2} + \frac{1}{r}\frac{\partial v_2}{\partial r} - \frac{v_2}{r^2}\right), \qquad (3.3.24)$$

$$\frac{u_2}{r}\frac{\partial}{\partial r}(rv_1) + \frac{u_3}{r}\frac{d}{dr}(rV_o) + w_1\frac{\partial v_1}{\partial Z} = \nu\left(\frac{\partial^2 v_3}{\partial r^2} + \frac{1}{r}\frac{\partial v_3}{\partial r} - \frac{v_3}{r^2} + \frac{\partial^2 v_1}{\partial Z^2}\right). \tag{3.3.25}$$

And finally from the third Navier-Stokes equation (2.2.3),

$$\frac{1}{\rho}\frac{\partial p_o}{\partial Z} = \nu\left(\frac{\partial^2 w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r}\right), \tag{3.3.26}$$

$$\frac{1}{\rho}\frac{\partial p_1}{\partial Z} = \nu\left(\frac{\partial^2 w_2}{\partial r^2} + \frac{1}{r}\frac{\partial w_2}{\partial r}\right), \tag{3.3.27}$$

$$\frac{1}{\rho}\frac{\partial p_2}{\partial Z} = \nu\left(\frac{\partial^2 w_3}{\partial r^2} + \frac{1}{r}\frac{\partial w_3}{\partial r} + \frac{\partial^2 w_1}{\partial Z^2} - u_2\frac{\partial w_1}{\partial r} - w_1\frac{\partial w_1}{\partial Z}\right). \tag{3.3.28}$$

3.4 Solutions of the equations with the given boundary conditions

Solving (3.3.22) and (3.3.18) gives the solution for V_o and p_o defined in (2.2.6) and (2.2.7), and since these only contain functions of r then

$$\frac{1}{\rho}\frac{\partial p_o}{\partial Z} = 0,$$

which implies, from (3.3.26), that

$$\frac{\partial w_1}{\partial r^2} + \frac{1}{r}\frac{\partial w_1}{\partial r} = 0 . \tag{3.4.1}$$

The solution of (3.4.1) is

$$w_1(r,Z) = C_1(Z)\log r + D_1(Z) \tag{3.4.2}$$

where $C_1(Z)$ and $D_1(Z)$ are determined by the boundary conditions $w_1 = 0$ on $r = R_1$ and $r = R_2$, from which

$$w_1(r,Z) = 0. \tag{3.4.3}$$

Upon replacing this in (3.3.15) and we use the boundary conditions on $u_2(r,Z)$ then

$$u_2(r,Z) = 0. \tag{3.4.4}$$

From equation (3.3.23) we find

$$v_1(r,Z) = C(Z)r + D(Z)/r . \tag{3.4.5}$$

When we solve for C(Z) and D(Z) subject to the boundary conditions given in (3.3.7) and (3.3.12) then

$$v_1(r,Z) = 0. \tag{3.4.6}$$

From (3.3.19) and (3.4.6), we see $p_1(r,Z)$ is only a function of Z. When we use this fact in (3.3.27) and solving for w_2 with the boundary conditions $w_2(R_2,Z) = w_2(R_1,Z) = 0$, the solution is

$$w_2(r,Z) = \frac{1}{4\nu}\frac{dp_1}{dZ}\left[r^2 + \frac{R_1^2 \log(r/R_2)-R_2^2 \log(r/R_1)}{\log(R_2/R_1)}\right]. \tag{3.4.7}$$

To find the unknown function $p_1(Z)$ we use the continuity equation given by (3.3.16) and substitute the above form for w_2 in it. Then

$$\frac{1}{r}\frac{\partial}{\partial r}(-ru_3) = \frac{1}{4\nu}\frac{d^2p_1}{dZ^2}\left[r^2 + \frac{R_1^2\log(r/R_2)-R_2^2\log(r/R_1)}{\log(R_2/R_1)}\right]. \tag{3.4.8}$$

The boundary conditions on u_3 state that $u_3(R_1,Z) = 0$ and $u_3(R_2,Z) = 0$. From this and (3.4.8) , we solve for u_3 as

$$-ru_3(r,Z) = \frac{1}{4\nu}\frac{d^2p_1}{dZ^2}\int_{R_1}^{r} r\left[r^2 + \frac{R_1^2 \log(r/R_2)-R_2^2 \log(r/R_1)}{\log(R_2/R_1)}\right]dr . \tag{3.4.9}$$

The condition that u_3 is zero at $r = R_2$ implies either

$$\frac{d^2 p_1}{dZ^2} = 0 \quad \text{or} \quad \int_{R_1}^{R_2} r\left[r^2 + \frac{R_1^2 \log(r/R_2) - R_2^2 \log(r/R_1)}{\log(R_2/R_1)}\right] dr = 0 \,. \tag{3.4.10}$$

Since it can be shown that the integral in (3.4.10) is non-zero, then

$$\frac{d^2 p_1}{dZ^2} = 0 \quad \text{for all } Z \tag{3.4.11}$$

and therefore the solution for $u_3(r,Z)$ is

$$u_3(r,Z) = 0. \tag{3.4.12}$$

From (3.4.11) the expression for w_2 can be written as

$$w_2 = \frac{b_1}{4}\left[r^2 + \frac{R_1^2 \log(r/R_2) - R_2^2 \log(r/R_1)}{\log(R_2/R_1)}\right] \tag{3.4.13}$$

where b_1 is a constant.

It has been assumed that the velocity components u_s, v_s, w_s are just functions of the two variables r and Z. We assume that at $Z = \pm\infty$ we have purely circumferential Couette flow and therefore the u_s and w_s velocity components disappear, that is

$$w_s(r,\infty) = u_s(r,\infty) = 0. \tag{3.4.14}$$

The velocity w_s at ∞ can be written as

$$w_s(r,\infty) = \varepsilon^2 w_2(r,\infty) + \varepsilon^3 w_3(r,\infty) + \dots = 0. \tag{3.4.15}$$

Comparing coefficients of ε we see

$$w_2(r,\infty) = 0 \quad \text{and} \quad w_3(r,\infty) = 0. \tag{3.4.16}$$

From (3.4.16) the solution for $w_2(r,Z)$, given in (3.4.13) is

$$w_2(r,Z) = 0, \tag{3.4.17}$$

for all r and Z. From which now

$$p_1(Z) = \text{constant}. \tag{3.4.18}$$

Since we know $u_2(r,Z)$ is zero, we solve for v_2 in (3.3.24) to give

$$v_2(r,Z) = C(Z)r + D(Z)/r \tag{3.4.19}$$

with boundary conditions

$$v_2(R_1,Z) = 0 \quad \text{and} \quad v_2(R_2,Z) + (R_2-R_1)F(Z)V_{or}(R_2)/2 = 0, \tag{3.4.19a}$$

with which we can determine $C(Z)$ and $D(Z)$. The solution for $v_2(r,Z)$ is

$$v_2(r,Z) = \frac{\Omega_1 R_1^2 R_2 F(Z)}{(R_2^2 - R_1^2)(R_2 + R_1)} \left[r - \frac{R_1^2}{r} \right]. \tag{3.4.20}$$

Similarly, as for $v_1(r,Z)$, we see the equation for $v_3(r,Z)$ is

$$\frac{\partial^2 v_3}{\partial r^2} + \frac{1}{r}\frac{\partial v_3}{\partial r} - \frac{v_3}{r^2} = 0, \tag{3.4.21}$$

with boundary conditions $v_3(R_1,Z) = v_3(R_2,Z) = 0$ and the only solution with these conditions is given by

$$v_3(r,Z) = 0. \tag{3.4.22}$$

Using the solutions for $V_o(r)$ and $v_2(r,Z)$ in (3.3.20) we integrate the left-hand side with respect to r, then

$$p_2(r,Z) = \frac{\rho\Omega_1{}^2 R_1{}^4 R_2 F(Z)}{(R_2^2 - R_1^2)^2 (R_2 + R_1)} \left[-r^2 + 2(R_1^2 + R_2^2)\log r + \frac{R_1^2 R_2^2}{r^2} \right] + C(Z). \tag{3.4.23}$$

We can find a consistent solution by setting

$$C(Z) = \frac{\rho\Omega_1^2 R_1^4 R_2 F(Z)}{(R_2^2-R_1^2)^2(R_2+R_1)} \left[2(R_1^2+R_2^2)b_o - 2(R_1^2+R_2^2)\log R_2\right]$$

(3.4.24)

where b_o is an unknown constant at this stage. Therefore from (3.3.28), we see

$$\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_3}{\partial r}\right) = \frac{dF}{dZ}\frac{\Omega_1^2 R_1^4 R_2}{(R_2^2-R_1^2)^2(R_2+R_1)}\left[-r^2+2(R_1^2+R_2^2)\{\log r - \log R_2 + b_o\} + \frac{R_1^2R_2^2}{r^2}\right]. \quad (3.4.25)$$

Then b_o is a dimensionless constant depending only on $\eta = R_1/R_2$.

We shall assume a solution of (3.4.25) of the form

$$w_3(r,Z) = \frac{\Omega_1^2 R_1^4 R_2}{\nu(R_2^2-R_1^2)^2(R_2+R_1)}\frac{dF}{dZ} \overset{(3)}{w}(r), \quad (3.4.26)$$

to obtain the following equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw^{(3)}}{dr}\right) = -r^2 + 2(R_1^2+R_2^2)\{\log r - \log R_2 + b_o\} + \frac{R_1^2R_2^2}{r^2},$$

(3.4.27)

subject to the boundary conditions

$$w^{(3)}(R_1) = w^{(3)}(R_2) = 0. \quad (3.4.28)$$

The solution for $w^{(3)}(r)$ using (3.4.27) is

$$w^{(3)}(r) = \frac{-r^4}{16} + (R_1^2+R_2^2)\left[\frac{r^2}{2}\log r - \frac{r^2}{2}(1+\log R_2 - b_o)\right] + \frac{R_1^2R_2^2\log^2 r}{2} + b_1\log r + b_2 \quad (3.4.29)$$

where b_o, b_1, b_2 are constants unknown at this stage, but upon using the boundary conditions (3.4.28) we note $w^{(3)}(r)$ can be rewritten as

$$w^{(3)}(r) = -\frac{1}{16}\left[r^4 + \frac{R_1^4\log(r/R_2)-R_2^4\log(r/R_1)}{\log(R_2/R_1)}\right] + \frac{R_1^2R_2^2}{2}\log(r/R_1)\log(r/R_2)$$

$$+\frac{(R_1^2+R_2^2)}{2}\left[(r^2-R_1^2)\log(r/R_2)+(b_o-1)\left(r^2+\frac{R_1^2\log(r/R_2)-R_2^2\log(r/R_1)}{\log(R_2/R_1)}\right)\right].$$

(3.4.30)

To find the constant b_o, we shall use the continuity equation

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_4) + \frac{\partial w_3}{\partial Z} = 0, \tag{3.4.31}$$

with the boundary conditions

$$u_4(R_2,Z) = u_4(R_1,Z) = 0. \tag{3.4.32}$$

When we use the type of solution for w_3 given in (3.4.26), and simplify $u_4(r,Z)$ by putting

$$u_4(r,Z) = \frac{\Omega_1^2R_1^2R_2^4}{\nu(R_2^2-R_1^2)^2(R_2+R_1)}\frac{d^2F}{dZ^2}u^{(4)}(r), \tag{3.4.33}$$

the differential equation (3.4.31) can be rewritten as

$$\frac{1}{r}\frac{d}{dr}(ru^{(4)}) + w^{(3)}(r) = 0. \tag{3.4.34}$$

The solution of this for $u^{(4)}(r)$ can be written as

$$ru^{(4)}(r) = -\int_{R_1}^{r} r\,w^{(3)}(r)dr. \tag{3.4.35}$$

In our analysis we shall be neglecting powers of ε^4, but we need

(3.4.35) to determine the constant b_o numerically by solving

$$\int_{R_1}^{R_2} r\, w^{(3)}(r)dr = 0 \,, \tag{3.4.36}$$

obtained from the end boundary condition that $u^{(4)}(R_2) = 0$.

The constant b_o can be shown to depend only on $\eta = R_1/R_2$ by transforming the integral relationship (3.4.36) using the non-dimensional co-ordinate x given in (2.3.7). The values obtained for b_o corresponding to the two cases I have chosen are

$$b_o = 0.3358 \quad \text{for} \quad \eta = 0.5 \tag{3.4.37}$$

and

$$b_o = 0.0256 \quad \text{for} \quad \eta = 0.95 \,. \tag{3.4.38}$$

To satisfy the boundary condition at $\pm\infty$ given by (3.4.16) for $w_3(r,Z)$, we see that (3.1.5a) must hold.

Therefore upon neglecting powers of ε^n for $n \geq 4$, the base velocities are represented by

$$u_s = 0,\; v_s = V_o(r) + \varepsilon^2 v_2(r,Z),\; w_s = \varepsilon^3 w_3(r,Z), \tag{3.4.39}$$

where the subscript s indicates that the base velocities are steady and that $V_o(r)$, $v_2(r,Z)$ and $w_3(r,Z)$ are given by equations (3.3.4), (3.4.20) and (3.4.26). The pressure distribution is given by

$$p_s = p_o(r) + \varepsilon^2 p_2(r,Z) \tag{3.4.40}$$

where $p_o(r)$, $p_2(r,Z)$ are defined in (2.2.7) and (3.4.23).

3.5 The disturbance equations and non-dimensionalizing the base velocities

Since we are ignoring powers of ϵ^n for $n \geq 4$, with u' of $O(\epsilon)$ we suppose that the basic flow is disturbed such that the velocity field is of the form

$$u = u' \ , \ v = v_s + v' \ , \ w = w_s + w' \ , \ p = p_s + p' \qquad (3.5.1)$$

where the primed variables are functions of r, z and t, and v_s, w_s, p_s are the solutions given by (3.4.39) and (3.4.40). To obtain the disturbance equations satisfied by u', w', v' and p', we place (3.5.1) in the Navier-Stokes and continuity equations given in (2.2.1) to (2.2.4) and obtain

$$\frac{\partial u'}{\partial t} + u' \frac{\partial u'}{\partial r} + w_s \frac{\partial u'}{\partial z} + w' \frac{\partial u'}{z} - \frac{v'(v'+2v_s)}{r} = -\frac{1}{\rho}\frac{\partial p'}{\partial r} + \nu(\nabla^2 u' - \frac{u'}{r^2}), \qquad (3.5.2)$$

$$\frac{\partial v'}{\partial t} + u' \frac{\partial v'}{\partial r} + u' \frac{\partial v_s}{\partial r} + w_s \frac{\partial v'}{\partial z} + w' \frac{\partial v'}{\partial z} + w' \frac{\partial v_s}{\partial z} + \frac{u'v_s + u'v'}{r} = \nu(\nabla^2 v' - \frac{v'}{r^2}), \qquad (3.5.3)$$

$$\frac{\partial w'}{\partial t} + u' \frac{\partial w'}{\partial r} + u' \frac{\partial w_s}{\partial r} + w_s \frac{\partial w'}{\partial z} + w' \frac{\partial w'}{\partial z} + w' \frac{\partial w_s}{\partial z} = -\frac{1}{\rho}\frac{\partial p'}{\partial z} + \nu\nabla^2 w', \qquad (3.5.4)$$

$$\frac{\partial u'}{\partial r} + \frac{u'}{r} + \frac{\partial w'}{\partial z} = 0. \qquad (3.5.5)$$

Here $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

Using (3.5.5) and differentiating it with respect to r, we can eliminate $\frac{\partial^2 u'}{\partial r^2}$ from the first momentum equation to form

$$\frac{\partial u'}{\partial t} + w_s \frac{\partial u'}{\partial z} - \frac{2v'v_s}{r} + \frac{1}{\rho}\frac{\partial p'}{\partial r} + \nu \frac{\partial^2 w'}{\partial r \partial z} - \nu \frac{\partial^2 w}{\partial z^2} = \frac{(v')^2}{r}$$

$$- w' \frac{\partial u'}{\partial z} + \frac{(u')^2}{r} + u' \frac{\partial w'}{\partial z} . \tag{3.5.6}$$

We now introduce the constants and dimensionless variables as defined in (2.3.6) , (2.3.7) along with the series representation of the base velocities into equations (3.5.3) to (3.5.6) and obtain

$$\frac{\partial p}{\partial x} - \left[\frac{\partial^2 u}{\partial \zeta^2} - \frac{\partial w_o}{\partial \zeta} - T(\Omega_{o,s}(x,\eta) + \varepsilon^2 f(z^*)\Omega_{2,s}(x,\eta))v - \right.$$

$$\left. \varepsilon^3 T \frac{df}{dz^*} W_{3,s}(x,\eta) \frac{\partial u}{\partial \zeta} + \cdots \right] + \frac{\partial u}{\partial \tau} = -\frac{1}{\alpha}\left[- w \frac{\partial u}{\partial \zeta} + \delta G u^2 + \right.$$

$$\left. \alpha \frac{TGv^2}{2} + u \frac{\partial w}{\partial \zeta} \right] , \tag{3.5.7}$$

$$\frac{\partial v_o}{\partial x} - \left[\delta^2 G^2 v - \delta G v_o + (1 - \varepsilon^2 \frac{1}{(1+\eta)} f(z^*))u - \varepsilon^3 \frac{df}{dz^*} V_{2,s}(x,\eta) w + \right.$$

$$\left. \varepsilon^3 T \frac{df}{dz^*} W_{3,s} \frac{\partial v}{\partial \zeta} + \ldots \right] - \frac{\partial v}{\partial \tau} = -\frac{1}{\alpha}\left[u v_o + w \frac{\partial v}{\partial \zeta} + \delta G u v\right], \tag{3.5.8}$$

$$\frac{\partial w_o}{\partial x} - \left[\frac{\partial p}{\partial \zeta} - \delta G w_o - \frac{\partial^2 w}{\partial \zeta^2} + \varepsilon^3 T \frac{df}{dz^*} W_{3,s}(x,\eta) \frac{\partial w}{\partial \zeta} + \varepsilon^3 T \frac{df}{dz^*} \right.$$

$$\left. \frac{dW_{3,s}}{dx} .u + \cdots \right] - \frac{\partial w}{\partial \tau} = -\frac{1}{\alpha}\left[u w_o + w \frac{\partial w}{\partial \zeta}\right], \tag{3.5.9}$$

$$\frac{\partial u}{\partial x} + \delta G u + \frac{\partial w}{\partial \zeta} = 0. \tag{3.5.10}$$

It should be remembered that v_o,w_o are related to the fluid velocities v and w by

$$v_o = \frac{\partial v}{\partial x} , \quad w_o = \frac{\partial w}{\partial x} . \tag{3.5.11}$$

The parameter T is called the Taylor number and is given by

$$T = \frac{\Omega_1^2 R_1^2 d^3}{\nu^2 R_o} , \qquad (3.5.12)$$

whilst the dimensionless function G(x) is defined in (2.3.10).

The relationships between the dimensional base velocities and the dimensionless functions mentioned in (3.5.7) to (3.5.10) are

$$\Omega_{o,s}(x,\eta) + \varepsilon^2 f(z^*)\Omega_{2,s}(x,\eta) + \ldots = \frac{2v_s}{\Omega_1 r} , \qquad (3.5.13)$$

$$\varepsilon^3 \frac{df}{dz^*} V_{2,s}(x,\eta) + \ldots = \frac{2d}{\Omega_1 R_o \alpha} \frac{\partial v_s}{\partial z} , \qquad (3.5.14)$$

$$1 - \varepsilon^2 \frac{f(z^*)}{1+\eta} + \ldots = \frac{2d}{\Omega_1 R_o \alpha} \left[\frac{1}{r}\frac{\partial}{\partial r}(rv_s)\right] \qquad (3.5.15)$$

and

$$\varepsilon^3 T \frac{df}{dz^*} W_{3,s}(x,\eta) + \ldots = \frac{d}{\nu} w_s . \qquad (3.5.16)$$

Here

$$\Omega_{o,s}(x,\eta) = \frac{-2\eta^2}{(1-\eta^2)} + \frac{8\eta^2}{(1+\eta)^2(1-\eta^2)} G^2(x), \qquad (3.5.17)$$

$$\Omega_{2,s}(x,\eta) = \frac{2\eta^2}{(1-\eta^2)(1+\eta)} - \frac{8\eta^4}{(1+\eta)^3(1-\eta^2)} G^2(x), \qquad (3.5.18)$$

and

$$V_{2,s}(x,\eta) = \frac{1}{4(1-\eta)G(x)} - \frac{\eta^2 G(x)}{(1-\eta^2)(1+\eta)} . \qquad (3.5.19)$$

The function $W_{3,s}(x,\eta)$ contains the dimensionless function

$$\sigma(x) = \frac{1+\eta}{2} + (1-\eta)x \qquad (3.5.20)$$

and $W_{3,s}(x,\eta)$ can now be written as

$$\frac{\eta^2}{2(1-\eta)^3(1-\eta^2)^2}\left[\frac{(1+\eta^2)}{2\log\eta}\left\{(\sigma^2-\eta^2)\log\sigma\log\eta+(b_0-1)((\sigma^2-1)\log\eta\right.\right.$$

$$\left.+(1-\eta^2)\log\sigma)\right\}+\frac{\eta^2}{2}\log\sigma\,(\log\sigma-\log\eta)-$$

$$\left.\frac{1}{16\log\eta}\left\{(\sigma^4-1)\log\eta+(1-\eta^4)\log\sigma\right\}\right] . \qquad (3.5.21)$$

The function $\sigma(x)$ is merely the expression of r/R_2.

We are now able to write the disturbance equations in the following matrix form,

$$\frac{\partial \underline{U}}{\partial x}-\underline{A}\,\underline{U}+\varepsilon^2 f(z^*)\underline{C}\,\underline{U}+\varepsilon^3\frac{df}{dz^*}\,\underline{D}\,\underline{U}-\underline{B}\frac{\partial \underline{U}}{\partial \tau}+\ldots$$

$$=\underline{L}(\underline{U})\underline{U}\ , \qquad (3.5.22)$$

where $\underline{U}$, $\underline{A}$, $\underline{B}$, $\underline{L}(\underline{U})$ are defined in (2.3.12) and (2.3.14).

Here

$$\underline{C} = \begin{bmatrix} & 0 & T\Omega_{2,s} & 0 \\ \underset{\sim}{0} & \frac{1}{1+\eta} & 0 & 0 \\ & 0 & 0 & 0 \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix} \quad \text{and} \qquad (3.5.23)$$

$$\underline{D} = \begin{bmatrix} & TW_{3,s}(x,\eta)\frac{\partial}{\partial\zeta} & 0 & 0 \\ & 0 & -TW_{3,s}(x,\eta)\frac{\partial}{\partial\zeta} & V_{2,s}(x,\eta) \\ \underset{\sim}{0} & -T\frac{dW_{3,s}}{dx} & 0 & -TW_{3,s}(x,\eta)\frac{\partial}{\partial\zeta} \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix} . \qquad (3.5.24)$$

The equation (3.5.22) has to be solved subject to the physical condition that requires zero disturbance velocities at the cylinder wall and the outer wall. It is written as follows for a vector with six components,

β_3 : the last three components of $\underline{U}$ to be zero at $x = -1/2$ and

$$x = [1+\varepsilon^2 f(z^*)]/2. \qquad (3.5.25)$$

An additional boundary condition is imposed,

$$\underline{U} \to 0 \quad \text{as} \quad z^* \to \pm\infty \ . \qquad (3.5.26)$$

The solution of (3.5.22) with the boundary conditions given above depends on the parameters T, η and ε and the function $f(z^*)$.

3.6 Explanation of the stability problem and expansion procedure

In the stability problem for the parallel wall case we obtain a neutral curve for fixed η as explained in §2.1 . We remind the reader that in FIG I the disturbance velocities are amplified with respect to time in the unstable area, and Taylor vortices develop. In the stable area of the graph the disturbance velocities are damped with respect to time and Taylor vortices do not form. The curve separating these two cases is called the neutral curve and the disturbance velocities for values of T and λ on this curve are neither amplified or damped. The stability problem as it stands for the parallel wall case does not depend on the axial co-ordinate, ζ, and we hope to bring into play a dependence on ζ in our non-parallel wall case.

We shall impose the boundary condition $\underline{U} \to 0$ as $z^* \to \pm\infty$ in our non-parallel wall neutral stability problem.

This condition makes the disturbance velocities die away from the centre ($z^* = 0$) and tend to zero at the ends ($z^* = \pm\infty$). The flow at the ends will be a modified Couette flow with the base velocities $u_s = w_s = 0$ and we shall be left with only the v_s component.

We envisage that as the Taylor number T, which is defined in terms of the radii at $z^* = 0$, is increased with η and $f(z^*)$ fixed there will be a certain $T = T_{crit}$ for which $\underline{U} \to 0$ as $z^* \to \pm\infty$. We define a local Taylor number as

$$T_L = \frac{\Omega_1^2 R_1^2 d_L^3}{\nu^2 R_{OL}} , \qquad (3.6.1)$$

where

$$d_L = d[1 + \varepsilon^2 f(z^*)/2] \quad \text{and} \quad R_{OL} = R_o[1 + \varepsilon^2 \delta f(z^*)/4]. \qquad (3.6.2)$$

Since Ω_1^2 and ν^2 remain the same irrespective of the position along the z*-axis we divide (3.6.1) by (3.5.12) and obtain

$$T_L = \frac{[1 + \varepsilon^2 f(z^*)/2]^3}{[1 + \varepsilon^2 \delta f(z^*)/4]} T . \qquad (3.6.3)$$

For the value of $T = T_{crit}$ the value of T_L given by replacing T by T_{crit} in (3.6.3) will be defined as

$$T_{Lcrit} = \frac{[1 + \varepsilon^2 f(z^*)/2]^3}{[1 + \varepsilon^2 \delta f(z^*)/4]} T_{crit} . \qquad (3.6.4)$$

For values of $T > T_{crit}$ the flow will presumably become unstable and the linearized disturbance velocities will be amplified with respect to time. At the same time as you proceed along the z* axis from the centre to the ends the disturbance velocities are damped with respect to the z* variable. So the vortices are formed more strongly at the centre but die away as $|z^*|$ increases. For values of $T < T_{crit}$ the flow will presumably be stable.

We choose our parameters such that the local parallel wall flow, that is the local parallel wall problem with local values of η and T, is liable to be more unstable at the centre than at the ends.

For each value of η_L there exists different critical Taylor numbers for the parallel wall problem and these will be denoted by

$$T_{LC} \quad . \tag{3.6.5}$$

We shall say the flow in our non-parallel wall case is locally stable or unstable if

$$T_L < T_{LC} \tag{3.6.6}$$

for local stability and

$$T_L > T_{LC} \tag{3.6.7}$$

for local instability.

A possible sketch of $T_{LC}, T_{L\,crit}$ against η_L, with the boundary condition $\underline{U} \to 0$ as $z^* \to \pm\infty$ is shown in FIG III.

Other possibilities are that the critical disturbance appears with the less restrictive conditions that $\underline{U}$ tends to a constant (other than zero) or is bounded at $\pm\infty$. However, our work will show that the first type of disturbance ($\underline{U} \to 0$ at $z^* \to \pm\infty$) is the most critical, that is appears at the lowest value of T, and we shall consider this in the main.

The advantage of the present problem over the cylinder problem is that our solutions will show Taylor like vortices appearing

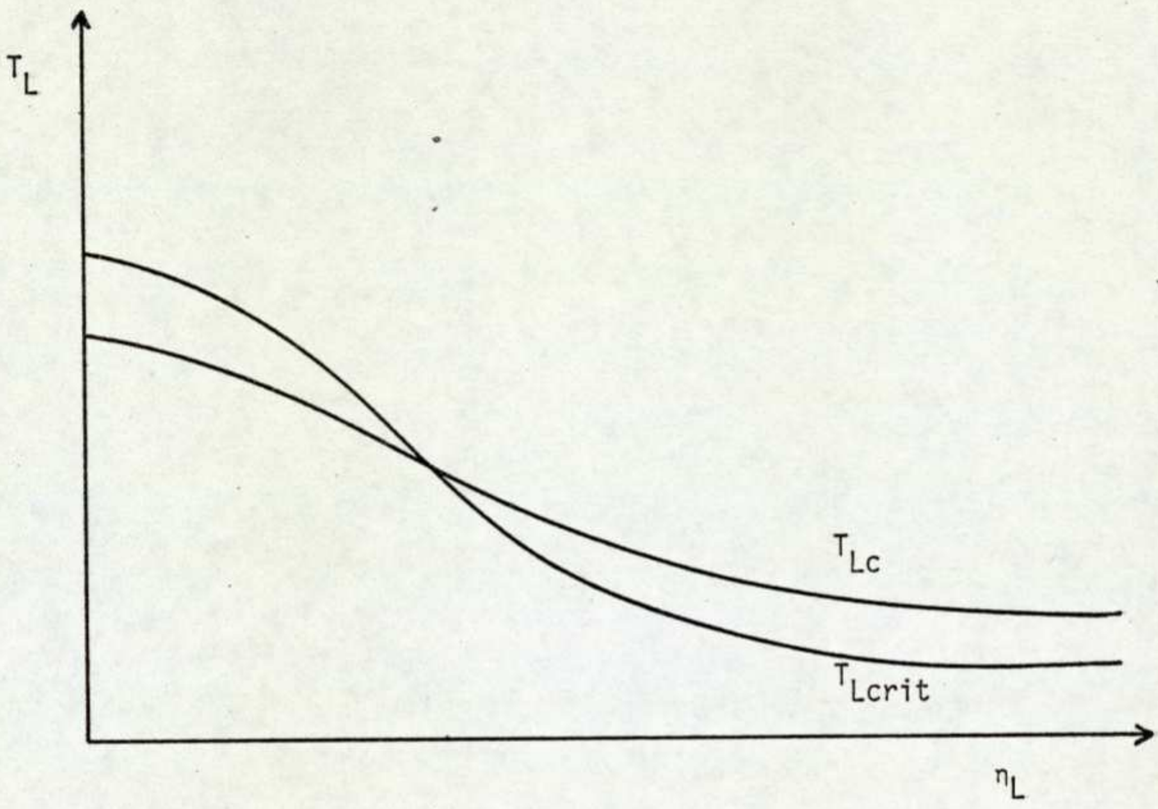

FIG III Sketch of T_{Lc}, T_{Lcrit} against η_L.

at the centre but not at the ends, where the flow will remain a purely circumferential Couette flow. This would take care of possible end effects with respect to experimental evidence. Thus it would be easier to perform experiments to check the theory. Former results and conclusions drawn from linear analysis of the parallel wall case is only valid at or near the critical Taylor number and predicts only the wavelength of the vortex cells which are not close to the end boundaries. With our boundary condition on $\underline{U}$ we hope to find the wavenumber of a Taylor-vortex like flow between surfaces of infinite length to vary as T is increased, thus more agreement with future experimental work to coincide with the theory.

This character of the solution will be retained in the non-linear results of section 3.14, enabling a comparison to be made between the theory of finite amplitude Taylor-vortex like flow of our theory with experiments.

We shall first consider the linear stability problem where the disturbance is assumed to be small enough for linearization to be a valid approximation. Only the neutral stability case with $\partial\underline{U}/\partial\tau$ equal to zero will be considered. Now if $\varepsilon = 0$ or $f(z^*) = 0$ we recover the Taylor problem for the parallel wall case so we expect the disturbance to be a perturbation of the parallel wall case.

We assume that η and $f(z^*)$ are fixed and expand the disturbance velocity, $\underline{U}$, and look for solutions of the form

$$\underline{U} = e^{i\lambda_c\zeta}\underline{u}(x,z^*,\varepsilon) + e^{-i\lambda_c\zeta}\underline{\tilde{u}}(x,z^*,\varepsilon) \qquad (3.6.8)$$

where λ_c is a real constant and $\underline{u}$ can be expanded in powers of ε later, and a tilde denotes the complex conjugate. We now search for the eigenvalue T such that $\underline{U}$ satisfies the boundary conditions given in (3.5.25) and (3.5.26).

We are led to expand T as

$$T = T_c + \varepsilon T_1^* + \varepsilon^2 T_2^* + \varepsilon^3 T_3^* + \dots \qquad (3.6.9)$$

and to use (3.6.8) and (3.6.9) in (3.5.22) with $\partial \underline{U}/\partial\tau \equiv 0$. We shall find that T_1^* and T_2^* are then determined by eigenvalue relationships and that the overall critical Taylor number for disturbances for our problem is given by

$$T_{crit} = T_c + \varepsilon^2 T_2^* + \dots \quad . \qquad (3.6.10)$$

We remind the reader that λ_c and T_c are the critical wavenumber and Taylor number for the cylinder problem with parameters η and T. In Chapter 6, we mention the case where we assume the disturbance can be expanded as

$$\underline{U} = e^{i\lambda_0\zeta}\,\underline{u}(x,z^*,\varepsilon) + \text{c.c.} \qquad (3.6.11)$$

and

$$T = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \dots \qquad (3.6.12)$$

where (λ_0, T_0) are on the same neutral curve as (λ_c, T_c).

Of course, it is not certain that such a disturbance as (3.6.8) will exist. For example, other rates of change with respect to ζ might appear as mentioned in (3.6.11) and (3.6.12). But in fact we have chosen the form of the outer wall such that the type of expansion given in (3.6.8) and (3.6.9) does appear to work. If we had chosen the outer wall to be $r = R_2 + \varepsilon d\, f(z^*)/2$ such an expansion would not be correct.

When we substitute the trial solution given by (3.6.8) and (3.6.9) in (3.5.22), then the equation for $\underline{U}$ becomes

$$\frac{\partial \underline{u}}{\partial x} - \underline{A}_c^{(1)}\underline{u} - \varepsilon \underline{B}_{c1}^{(1)} \frac{\partial \underline{u}}{\partial z^*} - \varepsilon T_1^* \underline{A}_2 \underline{u} - \varepsilon^2 \underline{B}_{c2} \frac{\partial^2 \underline{u}}{\partial z^{*2}} - \varepsilon^2 T_2^* \underline{A}_2 \underline{u}$$

$$+ \varepsilon^2 f(z^*) \underline{C}_c \underline{u} + \varepsilon^3 \frac{df}{dz^*} \underline{D}_c^{(1)} \underline{u} - \varepsilon^3 T_3^* \underline{A}_2 \underline{u} + \varepsilon^3 f(z^*) T_1^*$$

$$\underline{C}_{c1} \underline{u} + \dots = 0. \qquad (3.6.13)$$

Where $\underline{A}_c^{(1)}$ is defined in (2.4.6) with (λ_o, T_o) replaced by (λ_c, T_c), and the expressions for the matrices above are

$$\underline{B}_{c1}^{(m)} = \begin{bmatrix} 0 & 0 & -1 & 2im\lambda_c & 0 & 0 \\ 0 & 0 & 0 & 0 & -2im\lambda_c & 0 \\ 1 & 0 & 0 & 0 & 0 & -2im\lambda_c \\ & & & 0 & 0 & -1 \\ & \underset{\sim}{0} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix},$$

$$\underline{B}_{c2} = \begin{bmatrix} & 1 & 0 & 0 \\ \underset{\sim}{0} & 0 & -1 & 0 \\ & 0 & 0 & -1 \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix}, \quad \underline{C}_{c1} = \begin{bmatrix} & 0 & \Omega_2(x,\eta) & 0 \\ \underset{\sim}{0} & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix},$$

$$(3.6.14)$$

$$\underline{D}_c^{(m)} = \begin{bmatrix} & im\lambda_c T_c W_{3,s} & 0 & 0 \\ \underset{\sim}{0} & 0 & -im\lambda_c T_c W_{3,s} & V_{2,s} \\ & -T_c \frac{dW_{3,s}}{dx} & 0 & -im\lambda_c T_c W_{3,s} \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix}$$

and $\underline{C}_c$ is $\underline{C}$ except that T is replaced by T_c and the matrix $\underline{A}_2$

is given in (2.4.7).

From the above expressions and comparing some of these matrices with those defined in the perturbation about (λ_c, T_c) in the parallel wall case, we see that

$$\underline{B}_{c1}^{(1)} = -i\underline{A}_{c1}^{(1)} \quad \text{and} \quad \underline{B}_{c2} = -\underline{A}_{c2}^{(1)} \quad . \qquad (3.6.15).$$

The equation for $\underline{u}$ has to be solved subject to

$$u_j = 0 \quad \text{on } x = -\frac{1}{2} \quad \text{and} \quad x = [1 + \varepsilon^2 f(z^*)]/2$$

for j = 4,5,6. And (3.6.16)

$$\underline{u} \to 0 \quad \text{as} \quad z^* \to \pm\infty ,$$

where u_j denotes the j^{th} component of $\underline{u}$.

Now we expand $\underline{u}$ in powers of ε in (3.6.13) and write

$$\underline{u} = \underline{u}_1(x,z^*) + \varepsilon\underline{u}_2(x,z^*) + \varepsilon^2\underline{u}_3(x,z^*) + \dots , \qquad (3.6.17)$$

and when we equate powers of ε^n we obtain a set of partial differential equations :

$$\frac{\partial \underline{u}_1}{\partial x} - \underline{A}_c^{(1)} \underline{u}_1 = 0, \qquad (3.6.18)$$

$$\frac{\partial \underline{u}_2}{\partial x} - \underline{A}_c^{(1)}\underline{u}_2 = \underline{B}_{c1}^{(1)} \frac{\partial \underline{u}_1}{\partial z^*} + T_1^* \underline{A}_2\underline{u}_1 , \qquad (3.6.19)$$

$$\frac{\partial \underline{u}_3}{\partial x} - \underline{A}_c^{(1)} \underline{u}_3 = \underline{B}_{c1}^{(1)} \frac{\partial \underline{u}_2}{\partial z^*} + T_1^* \underline{A}_2\underline{u}_2 + \underline{B}_{c2}\frac{\partial^2 \underline{u}_1}{\partial z^{*2}} + T_2^*\underline{A}_2\underline{u}_1$$
$$- f(z^*)\underline{C}_c\underline{u}_1 , \qquad (3.6.20)$$

$$\frac{\partial \underline{u}_4}{\partial x} - \underline{A}_c^{(1)}\underline{u}_4 = \underline{B}_{c1}^{(1)} \frac{\partial \underline{u}_3}{\partial z^*} + T_1^* \underline{A}_2\underline{u}_3 + \underline{B}_{c2} \frac{\partial^2 u_2}{\partial z^{*2}} + T_2^* \underline{A}_2\underline{u}_2 -$$
$$f(z^*)\underline{C}_c\underline{u}_2 + T_3^* \underline{A}_2\underline{u}_1 - f(z^*)T_1^* \underline{C}_{c1}\underline{u}_1 - \frac{df}{dz^*} \underline{D}_c^{(1)}\underline{u}_1 .$$
$$(3.6.21)$$

The boundary conditions on the outer wall $x = [1 + \varepsilon^2 f(z^*)]/2$, are obtained by means of a Taylor expansion of the disturbance velocities about $x = 1/2$. The disturbance velocity $\underline{u}$ has its last three components zero at the two walls. From (3.6.17), we have for the outer wall that

$$u_{1j}([1 + \varepsilon^2 f(z^*)]/2, z^*) + \varepsilon u_{2j}([1 + \varepsilon^2 f(z^*)]/2, z^*) + \dots$$
$$= 0 \quad \text{for} \quad j = 4,5,6. \qquad (3.6.22)$$

When we use the Taylor expansion for the disturbance velocities about $x = 1/2$ and equate powers of ε^n we obtain the following set of boundary conditions for each $\underline{u}_i$ and for $j = 4,5,6$:

$$u_{1j}(1/2,z^*) = 0, \quad u_{3j}(1/2,z^*) + f(z^*)u_{1x,j}(1/2,z^*)/2 = 0 \text{ and}$$
$$u_{2j}(1/2,z^*) = 0, \quad u_{4j}(1/2,z^*) + f(z^*)u_{2x,j}(1/2,z^*)/2 = 0, \qquad (3.6.23)$$

where

$$u_{ix,j} = \frac{\partial u_{i,j}}{\partial x} \quad , \quad u_{ixx,j} = \frac{\partial^2 u_{i,j}}{\partial x^2} \quad , \dots \text{ etc.} \qquad (3.6.24)$$

The boundary condition on the inner wall $x = -1/2$ requires that

$$u_{1j}(-1/2,z^*) = 0, \quad u_{2j}(-1/2,z^*) = 0, \quad u_{3j}(-1/2,z^*) = 0, \dots \qquad (3.6.25)$$

3.7 Solutions of the disturbance equations

A solution of (3.6.18) can be written as

$$\underline{u}_1(x,z^*) = \psi(z^*)\underline{u}_{11}(x), \qquad (3.7.1)$$

to give

$$\mathcal{L}^{(1)}(\underline{u}_{11}(x)) = 0 \quad ; \ \beta_2 . \qquad (3.7.2)$$

Here $\mathcal{L}^{(p)}$ is the operator $\frac{d}{dx} - \underline{A}_c^{(p)}$, and $\underline{u}_{11}(x)$ must satisfy the boundary condition β_2 given in (2.4.1) with the same normalization being used as in the parallel wall case.

The first term given in (3.7.1) contains a real amplitude function, $\psi(z^*)$, which will be defined by integrals of the eigenfunction $\underline{u}_{11}(x)$ and its adjoint. This function will be determined by an amplitude equation which arises at a higher order from a solvability condition imposed at this order.

The boundary condition on $\psi(z^*)$ is that

$$\psi(z^*) \to 0 \quad \text{as} \quad z^* \to \pm\infty \quad . \tag{3.7.3}$$

Using (3.7.1), equation (3.6.19) becomes

$$\frac{\partial \underline{u}_2}{\partial x} - \underline{A}_c^{(1)} \underline{u}_2 = \frac{d\psi}{dz^*} \underline{B}_{c1}^{(1)} \underline{u}_{11} + \psi T_1^* \underline{A}_2 \underline{u}_{11} ; \quad \beta_2 \tag{3.7.4}$$

For $\underline{u}_2(x,z^*)$ to have a solution, we use the corresponding adjoint condition given in (2.5.3) :

$$\frac{d\psi}{dz^*} \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{B}_{c1}^{(1)} \underline{u}_{11}\, dx + \psi T_1^* \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{A}_2 \underline{u}_{11}\, dx = 0. \tag{3.7.5}$$

It is emphasized that from the parallel wall case that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{B}_{c1}^{(1)} \underline{u}_{11}\, dx = 0. \tag{3.7.6}$$

this being the value of T_1 of (2.6.5) when using λ_c, T_c as the base point, and since $\psi(z^*)$ and $\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{A}_2 \underline{u}_{11}$ are known not to be equal to zero, then

$$T_1^* = 0. \tag{3.7.7}$$

Therefore using (3.7.7) in (3.7.4), the latter equation can be replaced by

$$\frac{\partial \underline{u}_2}{\partial x} - \underline{A}_c^{(1)} \underline{u}_2 = \frac{d\psi}{dz^*} \underline{B}_{c1}^{(1)} \underline{u}_{11} \quad ; \quad \beta_2 . \tag{3.7.8}$$

From the boundary conditions on $\underline{u}_2$ and the right-hand side of (3.7.8), $\underline{u}_2$ may be expressed as

$$\underline{u}_2(x,z^*) = \frac{d\psi}{dz^*} \underline{g}_{21}(x) + S(z^*)\underline{u}_{11}(x), \tag{3.7.9}$$

where $\underline{g}_{21}$ satisfies

$$\mathcal{L}^{(1)}(\underline{g}_{21}) = \underline{B}_{c1}^{(1)} \underline{u}_{11} \quad ; \quad \beta_2 \tag{3.7.10}$$

and is normalized such that the second component evaluated at $x = -1/2$ is equal to zero. The function $S(z^*)$ appears since we can have an arbitrary additive multiple of the eigenfunction $\underline{u}_{11}$ and will be for the moment assumed complex. Then the complex conjugate $\tilde{\underline{u}}_2$ is given by

$$\tilde{\underline{u}}_2(x,z^*) = \frac{d\psi}{dz^*} \underline{g}_{2,-1}(x) + \tilde{S}(z^*)\underline{u}_{1,-1}(x) \tag{3.7.11}$$

with

$$\tilde{\underline{u}}_{ij} = \underline{u}_{i,-j} \quad \text{and} \quad \tilde{\underline{g}}_{ij} = \underline{g}_{i,-j} .$$

Two different normalizations were used for $g_{21}(x)$ in order to check the results of later integrals involving $\underline{g}_{21}(x)$. The above normalization was used since this allowed us to make use of the program from the linear parallel wall case. It can be seen by comparing the equations and boundary conditions for $\underline{g}_{21}(x)$ that

$$\underline{g}_{21}(x) = -i\underline{g}_{21}^{(p)}(x) \tag{3.7.12}$$

where the suffix p denotes the corresponding disturbance velocity from the linear parallel wall case.

Using (3.7.7) and (3.7.9), equation (3.6.20) becomes

$$\frac{\partial \underline{u}_3}{\partial x} - \underline{A}_c^{(1)} \underline{u}_3 = \frac{d^2\psi}{dz^{*2}}\left[\underline{B}_{c1}^{(1)}\ \underline{g}_{21} + \underline{B}_{c2}\underline{u}_{11}\right] + \frac{dS}{dz^*}\underline{B}_{c1}^{(1)}\underline{u}_{11}$$

$$+ \psi T_2^* \underline{A}_2 \underline{u}_{11} - f(z^*)\ \psi \underline{C}_c \underline{u}_{11} \qquad (3.7.13)$$

subject to the boundary conditions (for j = 4,5,6),

$$u_{3j}(-1/2,z^*) = 0,\ u_{3j}(1/2,z^*) = -f(z^*)\psi\ u_{11x,j}(1/2)/2. \quad (3.7.14)$$

For $\underline{u}_3(x,z^*)$ to exist, we use the adjoint condition (2.5.11) to obtain

$$\frac{d^2\psi}{dz^{*2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\left[\underline{B}_{c1}^{(1)}\underline{g}_{21} + \underline{B}_{c2}\underline{u}_{11}\right]dx + \psi T_2^* \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\underline{A}_2\underline{u}_{11}\,dx$$

$$- f(z^*)\,\psi\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\underline{C}_c\underline{u}_{11}\,dx = \left[\underset{\sim}{f}^{a,t}\ \underline{u}_3\right]_{-\frac{1}{2}}^{\frac{1}{2}}, \qquad (3.7.15)$$

since the value of $\frac{dS}{dz^*}\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\underline{B}_{c1}^{(1)}\ \underline{u}_{11}\,dx$ is zero from (3.7.6).

When we use (3.7.14) in (3.7.15), we find that

$$\frac{d^2\psi}{dz^{*2}}\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\ [\underline{B}_{c1}^{(1)}\ \underline{g}_{21} + \underline{B}_{c2}\underline{u}_{11}]dx + \psi T_2^* \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\underline{A}_2\underline{u}_{11}\,dx +$$

$$\psi f(z^*)\left[(\underset{\sim}{f}^{a,t}\underline{u}_{11x})/2\Big|_{x=\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\underline{C}_c\underline{u}_{11}\,dx\right] = 0, \qquad (3.7.16)$$

subject to the boundary conditions that

$$\psi(z^*) \to 0 \quad \text{as} \quad z^* \to \pm\infty\ . \qquad (3.7.17)$$

The amplitude equation (3.7.16) can be simplified by using some parallel wall results, namely the integral coefficient of $d^2\psi/dz^{*2}$ can be seen to be equal to

$$- \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \left| \underline{A}_{c1}^{(1)} \underline{g}_{21}^{(p)} + \underline{A}_{c2}^{(1)} \underline{f}_{11} \right| dx.$$ Thus when we use (2.6.10)

we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \left[\underline{B}_{c1}^{(1)} \underline{g}_{21} + \underline{B}_{c2} \underline{u}_{11}\right] dx = T_2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{A}_2 \underline{u}_{11}\, dx. \qquad (3.7.18)$$

Hence (3.7.16) can be rewritten as

$$\frac{d^2\psi}{dz^{*2}} + \psi\left[\frac{T_2^*}{T_2} + a\, f(z^*)\right] = 0, \qquad (3.7.19)$$

subject to (3.7.17) and where

$$a = \frac{1}{T_2}\left[\frac{\frac{1}{2}\, \underset{\sim}{f}^{a,t}(\frac{1}{2}) \underline{u}_{11x}(\frac{1}{2}) - \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{C}_c \underline{u}_{11}\, dx}{\int_{\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{A}_2 \underline{u}_{11}\, dx}\right]. \qquad (3.7.20)$$

Here T_2 is the constant in the parallel wall neutral curve equation $T - T_c = T_2(\lambda - \lambda_c)^2 + \ldots$.

The problem (3.7.19) with (3.7.17) is an eigenvalue problem for T_2^* , since a and T_2 are known constants.

The value of $\left[\underset{\sim}{f}^{a,t} \underline{u}_{11x}\right]_{x=\frac{1}{2}}$ can be found by using the eigenfunction $\underline{u}_{11}(x)$ and its adjoint since

$$\underline{u}_{11x}(\tfrac{1}{2}) = \left[\frac{d\underline{u}_{11}}{dx}\right]_{x=\frac{1}{2}} = \left[\frac{dp_o}{dx}, \frac{dv_o}{dx}, \frac{dw_o}{dx}, \frac{du}{dx}, v_o, w_o\right]^t$$

evaluated at $x = \frac{1}{2}$,

$$\underset{\sim}{f}^{a,t}(\tfrac{1}{2}) = [0, 0, 0, f_4, f_5, f_6]$$ and

$$\frac{du}{dx} = -\delta Gu - i\lambda_c w \text{ (from the continuity equation).}$$

Therefore

$$\underset{\sim}{f}^{a,t}(\tfrac{1}{2})\underline{u}_{11x}(\tfrac{1}{2}) = [f_5 v_o + f_6 w_o]_{x=\frac{1}{2}} \qquad (3.7.21)$$

since $u(\frac{1}{2}) = w(\frac{1}{2}) = 0$ and thus $\left[\frac{du}{dx}\right]_{x=\frac{1}{2}} = 0.$

3.8 The amplitude equation for $\psi(z^*)$

The amplitude equation given by (3.7.19) is identical to the one dimensional Schrödinger's equation,

$$\frac{d^2\Phi}{dx^2} + \frac{2\mu}{h^2}\left[E - U(x)\right]\Phi = 0, \qquad (3.8.1)$$

where U(x) denotes the potential energy of a field and E the energy levels. The similarity between the two equations (3.8.1) and (3.7.19) form a useful comparison, as we can use some results obtained by Landau & Lifshitz (31) and modify them to fit our equation. These are

(i) That there exists a discrete spectrum and also a continuous spectrum of eigenvalues depending on the boundary conditions of $\psi(z^*)$ (3.8.2)

(ii) That none of the eigenvalues of T_2^* of a discrete spectrum are degenerate (3.8.3)

(iii) That if $f(z^*)$ is symmetrical about $z^* = 0$, then the amplitude function must be either even or odd. (3.8.4)

(iv) That if ψ_1 and ψ_2 are two solutions to equation (3.7.19) for different eigenvalues of T_2^* then

$$\int_{-\infty}^{\infty} \psi_1 \psi_2 dz^* = 0, \qquad (3.8.5)$$

that is ψ_1 and ψ_2 are orthogonal.

The normal orthogonality relationship

$$\int_{-\infty}^{\infty} \psi^2 dz^* = 1 \qquad (3.8.6)$$

does not hold, since U is subject to some overall normalization on its second component evaluated at $z^* = 0$, $x = -1/2$ and from this the arbitrary constant that is multiplied by $\psi(z^*)$ is determined.

For the proofs of (3.8.2) to (3.8.5), see Landau & Lifshitz[31].

We next find some integral relationships involving $\psi(z^*)$. When we multiply (3.7.19) by $d\psi/dz^*$ and integrate over $-\infty < z^* < \infty$, then we have

$$\int_{-\infty}^{\infty} \psi'' \psi dz^* = - \int_{-\infty}^{\infty} \left[\frac{T_2^*}{T_2} + a f(z^*)\right] \psi\psi' dz^*. \qquad (3.8.7)$$

Upon integrating by parts we have

$$\frac{1}{2}\left[(\psi')^2\right]_{-\infty}^{\infty} = -\left\{\left[\frac{T_2^*}{T_2} + a f(z^*)\right]\frac{\psi^2}{2}\right\}_{-\infty}^{\infty} + \frac{a}{2}\int_{-\infty}^{\infty}\frac{df}{dz^*}\psi^2 dz^* \qquad (3.8.8)$$

because of the exponential behaviour of ψ at $z^* = \pm\infty$, the resulting integral converges and if

$\psi(z^*) \to 0$ as $z^* \to \pm\infty$ then

$$\frac{d\psi}{dz^*} \to 0 \quad \text{as} \quad z^* \to \pm\infty \quad . \qquad (3.8.9)$$

Therefore from equation (3.8.8)

$$\int_{-\infty}^{\infty} \frac{df}{dz^*} \psi^2 dz^* = 0 \qquad (3.8.10)$$

irrespective of the forms of ψ or $f(z^*)$.

The asymptotic behaviour of ψ as $z^* \to \pm\infty$ is simple, since $f(z^*)$ tends monotonically to a negative constant f_∞. Thus at z^* equal to $\pm\infty$, (3.7.19) becomes

$$\frac{d^2\psi}{dz^{*2}} + \psi\left[\frac{T_2^*}{T_2} + a\,f_\infty\right] = 0. \qquad (3.8.11)$$

The coefficient of ψ is a constant and we define

$$\ell_1 = +\sqrt{\frac{T_2^*}{T_2} + a\,f_\infty} \quad . \qquad (3.8.12)$$

Since $\psi \to 0$ as $z^* \to -\infty$ we have

$$\psi \sim \text{constant} \times e^{\ell_1 z^*}, \qquad (3.8.13)$$

similarly as $z^* \to \infty$ the solution is given asymptotically by

$$\psi \sim \text{constant} \times e^{-\ell_1 z^*} \quad . \qquad (3.8.14)$$

From numerical results it has been calculated that a and T_2 are positive. Therefore for $\psi \to 0$ as $z^* \to \pm\infty$ we have an upper bound for T_2^* given by

$$T_2^* < aT_2(-f_\infty). \qquad (3.8.15)$$

This boundary condition will later (see § 4.2) give the discrete spectrum of eigenvalues for T_2^*.
If instead the asymptotic behaviour of ψ is that $\psi \to$ constant as $z^* \to \pm\infty$ we have

$$T_2^* = aT_2(-f_\infty) \qquad (3.8.16)$$

and T_2^* is positive.

If on the other hand ψ is bounded as $z^* \to \pm\infty$, then

$$\psi \sim A \cos \ell_1 z^* + B \sin \ell_1 z^* \tag{3.8.17}$$

and we have a lower bound for T_2^* given by

$$T_2^* > aT_2(-f_\infty) \tag{3.8.18}$$

and T_2^* is positive. This boundary condition will later (see § 4.2) give the continuous spectrum of eigenvalues for T_2^*. The value of T_{crit} is therefore given by

$$T_{crit} = T_c + \varepsilon^2 T_2^* \tag{3.8.19}$$

to $O(\varepsilon^3)$. The value of T_{Lcrit} defined in (3.6.4) to the same order is

$$T_{Lcrit} = T_c + \varepsilon^2 [T_2^* + T_c(\tfrac{3}{2} - \tfrac{\delta}{4}) f(z^*)] , \tag{3.8.20}$$

upon expanding for small ε. Similarly the local eta η_L is defined as the ratio of the inner wall to the outer and is given by

$$\eta_L = \frac{R_1}{R_2 + \varepsilon^2 d\, f(z^*)/2} , \tag{3.8.21}$$

and for small ε this can be written as

$$\eta_L = \eta - \varepsilon^2 \eta(1 - \eta) f(z^*)/2 + \ldots . \tag{3.8.22}$$

From (3.8.20) we see that if $f(z^*)$ is negative then T_{Lcrit} decreases as z^* increases by an amount dictated by the choice of ε and $f(z^*)$.

3.9 Solution of the disturbance equations II

When (3.7.19) is solved subject to the given boundary conditions we substitute for $d^2\psi/dz^{*2}$ in the right hand side of (3.7.13), whence

$$\frac{\partial \underline{u}_3}{\partial x} - \underline{A}_c^{(1)} \underline{u}_3 = \frac{dS}{dz^*} \underline{B}_{c1}^{(1)} \underline{u}_{11} - \psi f(z^*) [a(\underline{B}_{c1}^{(1)} \underline{g}_{21} + \underline{B}_{c2} \underline{u}_{11}) + \underline{C}_c \underline{u}_{11}]$$

$$- \psi \frac{T_2^*}{T_2} \left[\underline{B}_{c1}^{(1)} \underline{g}_{21} + \underline{B}_{c2} \underline{u}_{11} - T_2 \underline{A}_2 \underline{u}_{11} \right]. \qquad (3.9.1)$$

By examining the right hand side of (3.9.1) we see $\underline{u}_3$ may be expressed as

$$\underline{u}_3(x,z^*) = \psi f(z^*) \underline{h}_{31}(x) + \psi \underline{g}_{31}(x) + \frac{dS}{dz^*} \underline{g}_{21}(x) + P(z^*) \underline{u}_{11}(x) \qquad (3.9.2)$$

and upon equating the functions of z* on either side we have

$$\mathfrak{L}^{(1)}(\underline{h}_{31}) = - \left[a(\underline{B}_{c1}^{(1)} \underline{g}_{21} + \underline{B}_{c2} \underline{u}_{11}) + \underline{C}_c \underline{u}_{11} \right] , \qquad (3.9.3)$$

$$\mathfrak{L}^{(1)}(\underline{g}_{31}) = - \frac{T_2^*}{T_2} \left[\underline{B}_{c1}^{(1)} \underline{g}_{21} + \underline{B}_{c2} \underline{u}_{11} - T_2 \underline{A}_2 \underline{u}_{11} \right] \qquad (3.9.4)$$

and $\underline{g}_{31}$, $\underline{h}_{31}$ are normalized in the same way as $\underline{g}_{21}$. The function $P(z^*)$ appears since again we can have an arbitrary multiple of the eigenfunction $\underline{u}_{11}$ and is assumed to be complex.

The term $\frac{dS}{dz^*} \underline{g}_{21}$ has been forced by the multiple $S(z^*) \underline{u}_{11}$ of the eigenfunction which occured in $\underline{u}_2(x,z^*)$. The boundary conditions on (3.9.3), (3.9.4) are

$$h_{31,j}(-\tfrac{1}{2}) = g_{31,j}(-\tfrac{1}{2}) = 0 \qquad (3.9.5)$$

for j = 4,5,6 and

$$h_{31,j}(\tfrac{1}{2}) = -u_{11x,j}(\tfrac{1}{2})/2 \quad , \quad g_{31,j}(\tfrac{1}{2}) = 0. \qquad (3.9.6)$$

Looking at the equation for $\underline{g}_{31}(x)$ from the linear parallel wall case and comparing the equations and boundary conditions we see that

$$\underline{g}_{31}(x) = \frac{T_2^*}{T_2} \underline{g}_{31}^{(p)}(x) \quad . \qquad (3.9.7)$$

The above relationship will not hold if $\underline{g}_{31}^{(p)}(x)$ was normalized such that its second component is equal to 2 at $z^* = 0$ and $x = -\frac{1}{2}$.

We reform equation (3.6.21) by using (3.7.9) (3.9.2) and replace the term involving $d^3\psi/dz^{*3}$ by differentiating (3.7.19) with respect to z* to give

$$\frac{d^3\psi}{dz^{*3}} = -\frac{T_2^*}{T_2}\frac{d\psi}{dz^*} - a\,f(z^*)\frac{d\psi}{dz^*} - a\,\psi\frac{df}{dz^*}\,, \qquad (3.9.8)$$

and thus the equation for $\underline{u}_4(x,z^*)$ becomes

$$\frac{\partial\underline{u}_4}{\partial x} - \underline{A}_c^{(1)}\underline{u}_4 = \frac{d^2S}{dz^{*2}}[\underline{B}_{c1}^{(1)}\underline{g}_{21} + \underline{B}_{c2}\underline{u}_{11}] + S(z^*)T_2^*\underline{A}_2\underline{u}_{11}$$

$$- S(z^*)f(z^*)\underline{C}_c\underline{u}_{11} + \frac{dP}{dz^*}\underline{B}_{c1}^{(1)}\underline{u}_{11} + \psi T_3^*\underline{A}_2\underline{u}_{11} +$$

$$\frac{d\psi}{dz^*}[\underline{B}_{c1}^{(1)}\underline{g}_{31} + T_2^*\underline{A}_2\underline{g}_{21} - \frac{T_2^*}{T_2}\underline{B}_{c2}\underline{g}_{21}] + \psi\frac{df}{dz^*}[$$

$$\underline{B}_{c1}^{(1)}\underline{h}_{31} - \underline{D}_c^{(1)}\underline{u}_{11} - a\,\underline{B}_{c2}\underline{g}_{21}] + f(z^*)\frac{d\psi}{dz^*}[\underline{B}_{c1}^{(1)}\underline{h}_{31} - C_c\underline{g}_{21} -$$

$$a\,\underline{B}_{c2}\underline{g}_{21}]\,, \qquad (3.9.9)$$

subject to the boundary conditions that

$$u_{4j}(-\tfrac{1}{2},z^*) = 0 \quad\text{and}\quad u_{4j}(\tfrac{1}{2},z^*) = -\;[f(z^*)\frac{d\psi}{dz^*}g_{21x,j}(\tfrac{1}{2})$$

$$+\; f(z^*)S(z^*)u_{11x,j}(\tfrac{1}{2})]/2 \qquad (3.9.10)$$

for j = 4,5,6.

For $\underline{u}_4(x,z^*)$ to exist, we use the adjoint condition (2.5.13) to obtain

$$\frac{d^2S}{dz^{*2}} + S(z^*)\left[\frac{T_2^*}{T_2} + a\,f(z^*)\right] = r_1 f\frac{d\psi}{dz^*} + r_2\psi\frac{df}{dz^*} + r_3\frac{d\psi}{dz^*}$$

$$- \psi\frac{T_3^*}{T_2}\,. \qquad (3.9.11)$$

The boundary conditions on $S(z^*)$ are the same as those on $\psi(z^*)$, that is

$$S(z^*) \to 0 \quad \text{as} \quad z^* \to \pm\infty \quad . \tag{3.9.12}$$

Here

$$r_1 = -\left\{ \frac{[\underset{\sim}{f}^{a,t}\underline{g}_{21,x}]^{x=\frac{1}{2}}/2 + \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}[\underline{B}_{c1}^{(1)}\underline{h}_{31} - \underline{C}_c\underline{g}_{21} - a\,\underline{B}_{c2}\,g_{21}]\,dx}{T_2\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\,\underline{A}_2\underline{u}_{11}\,dx} \right\}, \tag{3.9.13}$$

$$r_2 = -\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\,[\underline{B}_{c1}^{(1)}\,\underline{h}_{31} - \underline{D}_c^{(1)}\,\underline{u}_{11} - a\,\underline{B}_{c2}\underline{g}_{21}]\,dx}{T_2\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\,\underline{A}_2\underline{u}_{11}\,dx} \tag{3.9.14}$$

and

$$r_3 = -\frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\,[\underline{B}_{c1}^{(1)}\underline{g}_{31} + T_2^*\,\underline{A}_2\underline{g}_{21} - \frac{T_2^*}{T_2}\underline{B}_{c2}\underline{g}_{21}]\,dx}{T_2\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\,\underline{A}_2\underline{u}_{11}\,dx} . \tag{3.9.15}$$

For (3.9.11) to have a solution we use the adjoint condition and multiply the right hand side by $\psi(z^*)$, since all second order differential equations are (or can be put in a) self adjoint form, and integrate from -c to c :

$$\lim_{c\to\infty}\left[-\frac{T_3^*}{T_2}\int_{-c}^{c}\psi^2 dz^* + r_1\int_{-c}^{c}\psi\frac{d\psi}{dz^*}f(z^*)dz^* + r_2\int_{-c}^{c}\psi^2\frac{df}{dz^*}dz^* + \right.$$

$$r_3 \int_{-c}^{c} \psi \frac{d\psi}{dz^*} \quad dz^* \Bigg] = 0 \; . \tag{3.9.16}$$

From (3.8.4) we know that $\psi(z^*)$ must be either an odd or an even function. If $\psi(z^*)$ is an even function, $\frac{d\psi}{dz^*}$ is an odd function and therefore

$$\int_{-c}^{c} \psi \frac{d\psi}{dz^*} \, dz^* = \int_{-c}^{c} \psi \frac{d\psi}{dz^*} f(z^*) dz^* = \int_{-c}^{c} \psi^2 \frac{df}{dz^*} \, dz^* = 0 \tag{3.9.17}$$

thus

$$T_3^* \lim_{c\to\infty} \left[\int_{-c}^{c} \psi^2 dz^* \right] = 0. \tag{3.9.18}$$

Hence we can find the perturbation coefficient T_3^*, as

$$T_3^* = 0, \tag{3.9.19}$$

because the integral

$$\int_{0}^{\infty} \psi^2 dz^* \neq 0. \tag{3.9.20}$$

If, on the other hand, $\psi(z^*)$ is an odd function, $\frac{d\psi}{dz^*}$ would be an even function and repeating the same procedure, we obtain equation (3.9.19) again.

For different values of the eigenvalue T_2^* the value of the integrals given by r_1 and r_2 remain invariant and only the value of r_3 changes. To save needlessly computing the values of r_3 for different T_2^* we use the identities (3.6.15),(3.7.12) and (3.9.7) in (3.9.15) and write

$$r_3 = \frac{iT_2^*}{T_2^{\,2}} \left\{ \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} [\underline{A}_{c1}^{(1)} \, \underline{g}_{31}^{(p)} + T_2 \underline{A}_2 \underline{g}_{21}^{(p)} + \underline{A}_{c2} \underline{g}_{21}^{(p)}] dx}{\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \, \underline{A}_2 \underline{u}_{11} \, dx} \right\} . \tag{3.9.21}$$

When we look at the equation of the third perturbation coefficient, about λ_c, T_c, in the parallel wall case (2.6.15), r_3 can be rewritten as

$$r_3 = \frac{-i\ T_3 T_2^*}{T_2^{\ 2}} \ . \tag{3.9.22}$$

It was found that on calculating r_1 and r_2 these two also were purely imaginary and the equation for $S(z^*)$ can now be rewritten as

$$\frac{d^2 S}{dz^{*2}} + S\left[\frac{T_2^*}{T_2} + a\ f(z^*)\right] = i\left[r_{1i} f(z^*)\frac{d\psi}{dz^*} + r_{2i}\psi\frac{df}{dz^*} + r_{3i}\frac{d\psi}{dz^*}\right] \tag{3.9.23}$$

where r_{1i}, r_{2i} and r_{3i} are real quantities. This equation for $S(z^*)$ is to be solved subject to (3.9.12).

3.10 The amplitude equation for $S(z^*)$ and effects of normalization

The differential equation for $S(z^*)$ can be separated into real and imaginary parts, that is

$$S(z^*) = S_r(z^*) + i\ S_i(z^*)\ . \tag{3.10.1}$$

We replace $S(z^*)$ in (3.9.23) by (3.10.1) and equate real and imaginary parts to obtain the following differential equations

$$\frac{d^2 S_r}{dz^{*2}} + S_r\left[\frac{T_2^*}{T_2} + a\ f(z^*)\right] = 0, \tag{3.10.2}$$

and

$$\frac{d^2 S_i}{dz^{*2}} + S_i\left[\frac{T_2^*}{T_2} + a\ f(z^*)\right] = r_{1i}\ f(z^*)\frac{d\psi}{dz^*} + r_{2i}\psi\frac{df}{dz^*} + r_{3i}\frac{d\psi}{dz^*}, \tag{3.10.3}$$

the both of which are subject to the following boundary conditions,

$$S_r(z^*) \to 0,\ S_i(z^*) \to 0 \quad \text{as} \quad z^* \to \pm\infty \ . \tag{3.10.4}$$

When we calculated the values of r_{1i}, r_{2i} and r_{3i} for a fixed T_2^* it was found that r_1 and r_3 remain invariant under different normalizations of $\underline{h}_{31}$, $\underline{g}_{31}$ and $\underline{g}_{21}$ but the value of r_2 varied. The reason why this was so is explained now. :

Suppose that $\hat{\underline{g}}_{21}$, $\hat{\underline{g}}_{31}$, $\hat{\underline{h}}_{31}$ are the new solutions of the differential equations (3.7.10), (3.9.4), (3.9.3) respectively, that is

$$\mathcal{L}^{(1)}(\hat{\underline{g}}_{21}) = \underline{B}_{c1}^{(1)}\ \underline{u}_{11} \qquad ; \quad \beta_2 \tag{3.10.5}$$

$$\mathcal{L}^{(1)}(\hat{\underline{g}}_{31}) = \frac{-T_2^*}{T_2}\left[\underline{B}_{c1}^{(1)}\hat{\underline{g}}_{21} + \underline{B}_{c2}\underline{u}_{11} - T_2\underline{A}_2\underline{u}_{11}\right] \ ; \ \beta_2 \tag{3.10.6}$$

and

$$\mathcal{L}^{(1)}(\hat{\underline{h}}_{31}) = -\left[a(\underline{B}_{c1}^{(1)}\hat{\underline{g}}_{21} + \underline{B}_{c2}\underline{u}_{11}) + \underline{C}_c\underline{u}_{11}\right] , \tag{3.10.7}$$

subject to the boundary conditions given in (3.9.5) and (3.9.6). The functions above are normalized such that their second components evaluated at $x = -1/2$ and $\zeta = 0$ are equal to $-2i$, 2 and 2 respectively. Then comparing the equation for $\underline{g}_{21}$ and $\hat{\underline{g}}_{21}$ along with the normalizing conditions we have

$$\hat{\underline{g}}_{21} = \underline{g}_{21} - 2i\underline{u}_{11} . \tag{3.10.8}$$

If $\hat{S}(z^*)$ is the new multiple of the eigenfunction $\underline{u}_{11}$ in $\underline{u}_2(x,z^*)$ then

$$\hat{S}(z^*) = S(z^*) + 2i\,\frac{d\psi}{dz^*} \ . \tag{3.10.9}$$

We re-arrange (3.10.6) and (3.10.7) by replacing $\hat{\underline{g}}_{21}$ by (3.10.8) and

$$\mathcal{L}^{(1)}(\hat{\underline{g}}_{31}) = \frac{-T_2^*}{T_2}\,[\underline{B}_{c1}^{(1)}\underline{g}_{21} + \underline{B}_{c2}\underline{u}_{11} - T_2\underline{A}_2\underline{u}_{11}] + \frac{2iT_2^*}{T_2}\,\underline{B}_{c1}^{(1)}\underline{u}_{11} \tag{3.10.10}$$

and

$$\mathcal{L}^{(1)}(\hat{\underline{h}}_{31}) = -\left[a(\underline{B}_{c1}^{(1)}\ \underline{g}_{21} + \underline{B}_{c2}\ \underline{u}_{11}) + \underline{C}_{c}\underline{u}_{11}\right] + 2i\,a\,\underline{B}_{c1}^{(1)}\underline{u}_{11}\,. \tag{3.10.11}$$

When we compare the right hand side of (3.10.10) and (3.10.11) with the equations for $\underline{g}_{31}$ and $\underline{h}_{31}$, this forces

$$\hat{\underline{h}}_{31} = \underline{h}_{31} + 2i\,a\,\underline{g}_{21} + \beta\underline{u}_{11} \tag{3.10.12}$$

and

$$\hat{\underline{g}}_{31} = \underline{g}_{31} + \frac{2iT_2^*}{T_2}\,\underline{g}_{21} + \kappa\underline{u}_{11}\,. \tag{3.10.13}$$

The addition of $\beta\underline{u}_{11}$ and $\kappa\underline{u}_{11}$ are due to the fact that $\underline{g}_{31}$, $\underline{h}_{31}$ can have arbitrary multiples of the eigenfunction $\underline{u}_{11}$ and the normalizing conditions will give

$$\kappa = \beta = 2. \tag{3.10.14}$$

The differential equation satisfied by $\hat{S}(z^*)$ is

$$\frac{d^2\hat{S}}{dz^{*2}} + \hat{S}\left[\frac{T^*}{T_2} + a\,f(z^*)\right] = \hat{r}_1\ f(z^*)\,\frac{d\psi}{dz^*} + \hat{r}_2\,\psi\frac{df}{dz^*} + \hat{r}_3\,\frac{d\psi}{dz^*}$$

with boundary conditions $\hat{S}(z^*) \to 0$ as $z^* \to \pm\infty$. (3.10.15)

Where $\hat{r}_1$, $\hat{r}_2$ and $\hat{r}_3$ are similar to r_1, r_2, r_3 except that the functions $\underline{g}_{21}$, $\underline{h}_{31}$, $\underline{g}_{31}$ are replaced by $\hat{\underline{g}}_{21}$, $\hat{\underline{h}}_{31}$, $\hat{\underline{g}}_{31}$. It can be shown using the identities (3.10.12) , (3.10.13) that

$$\hat{r}_1 = r_1 \quad \text{and} \quad \hat{r}_3 = r_3\,. \tag{3.10.16}$$

But the equation for $\hat{r}_2$, upon substituting for $\hat{\underline{g}}_{21}$ etc., becomes

$$\hat{r}_2 = \frac{-\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\left[\underline{B}_{c1}^{(1)}\ \underline{h}_{31} - \underline{D}_{c}^{(1)}\underline{u}_{11} - a\ \underline{B}_{c2}\underline{g}_{21}\right]dx}{T_2\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\ \underline{A}_2\underline{u}_{11}\,dx} -$$

$$\frac{2ia\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}[\underline{B}_{c1}^{(1)}\ \underline{g}_{21} + \underline{B}_{c2}\underline{u}_{11}]dx + 2\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\underline{B}_{c1}^{(1)}\ \underline{u}_{11}\,dx}{T_2\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\ \underline{A}_2\underline{u}_{11}\,dx} . \qquad (3.10.17)$$

When we use (3.7.6) and (3.7.18) in the above, the expression for $\hat{r}_2$ is

$$\hat{r}_2 = r_2 - 2ia. \qquad (3.10.18)$$

After we replace $\hat{r}_1, \hat{r}_2, \hat{r}_3$ by their corresponding values r_1, r_2 and $r_2 - 2ia$, equation (3.10.15) becomes

$$\frac{d^2\hat{S}}{dz^{*2}} + \hat{S}\left[\frac{T_2^*}{T_2} + a\,f(z^*)\right] = r_1 f(z^*)\frac{d\psi}{dz^*} + r_3\frac{d\psi}{dz^*} + r_2\psi\frac{df}{dz^*}$$

$$- 2ia\psi\frac{df}{dz^*} . \qquad (3.10.19)$$

In equation (3.10.19) , we would expect the right hand side to be exactly the same as the right hand side of (3.9.11) but due to the relationship in (3.10.9) this is not the case. When we use (3.10.9) in (3.10.19) we have

$$\frac{d^2S}{dz^{*2}} + S\left[\frac{T_2^*}{T_2} + a\,f(z^*)\right] + 2i\left[\frac{d^3\psi}{dz^{*3}} + \frac{d\psi}{dz^*}\left(\frac{T_2^*}{T_2} + a\,f(z^*)\right)\right]$$

$$= r_1 f(z^*)\frac{d\psi}{dz^*} + r_3\frac{d\psi}{dz^*} + r_2\psi\frac{df}{dz^*} - 2ia\psi\frac{df}{dz^*} . \qquad (3.10.20)$$

Using equation (3.7.19) and differentiating with respect to z^* then

$$\frac{d^3\psi}{dz^{*3}} + \frac{d\psi}{dz^*}\left(\frac{T_2^*}{T_2} + a\,f(z^*)\right) = -a\psi\frac{df}{dz^*} . \qquad (3.10.21)$$

Therefore from (3.10.21) we see (3.10.20) will give the same equation for $S(z^*)$ given in (3.9.11), with $T_3^* = 0$.

It can be seen that the solution for $S(z^*)$ does depend on the normalization chosen but the overall solution for $\underline{U}$ does not. All the relationships between $\underline{h}_{31}$, $\underline{\hat{h}}_{31}$, etc., and the value of the integrals concerned are used as a numerical check on our computer programs.

A solution of (3.10.2) is

$$S_r(z^*) = \text{constant} \cdot \psi(z^*) \qquad (3.10.22)$$

and of (3.10.3) is

$$S_i(z^*) = \text{constant} \cdot \psi(z^*) \quad + \text{ a particular solution,} \qquad (3.10.23)$$

where all the constants are real.

A method of solving (3.10.3) for a particular solution is to set

$$S_i(z^*) = \psi(z^*)R(z^*) \qquad (3.10.24)$$

where ψ satisfies (3.7.19), and the boundary condition (3.9.12) becomes

$$\psi R \to 0 \quad \text{as} \quad z^* \to \pm\infty \; . \qquad (3.10.25)$$

The equation for $R(z^*)$ on substituting (3.10.24) in (3.10.3) can be placed in the form

$$\frac{d}{dz^*}\left[\psi^2\frac{dR}{dz^*}\right] = \psi(z^*)\left[r_{1i}\, f(z^*)\,\frac{d\psi}{dz^*} + r_{2i}\psi\,\frac{df}{dz^*} + r_{3i}\frac{d\psi}{dz^*}\right] . \qquad (3.10.26)$$

We integrate both sides of (3.10.26) with respect to z^* until we can solve for $R(z^*)$. Thus

$$\psi^2\,\frac{dR}{dz^*} = r_{1i}\int_{-\infty}^{z^*}\psi f\,\frac{d\psi}{dz^*}\,dz^* + r_{2i}\int_{-\infty}^{z^*}\psi^2\frac{df}{dz^*}\,dz^* + r_{3i}\int_{-\infty}^{z^*}\psi\frac{d\psi}{dz^*}\,dz^*$$

$$= r_{3i}\,\frac{\psi^2}{2} + r_{1i}\,\frac{\psi^2 f(z^*)}{2} + \frac{(2r_{2i}-r_{1i})}{2}\int_{-\infty}^{z^*}\psi^2\frac{df}{dz^*}\,dz^* \; . \qquad (3.10.27)$$

Provided $\psi^2(z^*)$ does not equal zero for any finite z^* we may divide (3.10.27) by ψ^2 and obtain

$$\frac{dR}{dz^*} = \frac{1}{2}\left[r_{3i} + r_{1i}f(z^*) + \frac{(2r_{2i}-r_{1i})}{\psi^2}\int_{-\infty}^{z^*}\psi^2\frac{df}{dz^*}\,dz^*\right]. \tag{3.10.28}$$

In equation (3.10.26) if $\psi(z^*)$ is assumed to be even, then the left hand side of (3.10.26) is unchanged when the sign of the co-ordinate is reversed, but the sign of the right hand side changes. Therefore

$$R(z^*) \text{ is odd when } \quad \psi(z^*) \text{ is even} \tag{3.10.29}$$

and similarly

$$R(z^*) \text{ is even when } \quad \psi(z^*) \text{ is odd.} \tag{3.10.30}$$

We can still have an arbitrary multiple of the eigenfunction $\psi(z^*)$ in our solution for $S_i(z^*)$, which is <u>not</u> contained in $R(z^*)$.

3.11 The Solutions for $\underline{U}$ and the Stokes Streamfunction Φ

The solution for $\underline{U}$ to $O(\varepsilon)$ is given by

$$\underline{U} = \left[e^{i\lambda_c\zeta}\underline{u}_{11} + e^{-i\lambda_c\zeta}\underline{u}_{1,-1}\right]\psi(z^*) + \varepsilon\left[e^{i\lambda_c\zeta}\underline{g}_{21} + e^{-i\lambda_c\zeta}\underline{g}_{2,-1}\right]\frac{d\psi}{dz^*}$$

$$+\ \varepsilon\left[e^{i\lambda_c\zeta}S(z^*)\underline{u}_{11} + e^{-i\lambda_c\zeta}\tilde{S}(z^*)\underline{u}_{1,-1}\right] + O(\varepsilon^2), \tag{3.11.1}$$

where $\underline{u}_{11}$, $\underline{g}_{21}$ are normalized such that their second component evaluated at $x = -1/2$ are 1 and 0 respectively. The functions given by $\underline{u}_{11}$ and $\underline{g}_{21}$ are found to be of the form

$$\underline{u}_{11} = \begin{bmatrix} R \\ R \\ I \\ R \\ R \\ I \end{bmatrix} \qquad \text{and} \qquad \underline{g}_{21} = \begin{bmatrix} I \\ I \\ R \\ I \\ I \\ R \end{bmatrix}, \tag{3.11.2}$$

where R refers to a real number and I a purely imaginary number.

When we refer to the expression for $S(z^*)$ given in (3.10.1) and substitute it in (3.11.1) we obtain

$$\underline{U} = [e^{i\lambda_c\zeta}\underline{u}_{11} + e^{-i\lambda_c\zeta}\underline{u}_{1,-1}]\psi(z^*) + [e^{i\lambda_c\zeta}\underline{g}_{21} + e^{-i\lambda_c\zeta}\underline{g}_{2,-1}]\frac{d\psi}{dz^*} +$$

$$\varepsilon[e^{i\lambda_c\zeta}\underline{u}_{11} + e^{-i\lambda_c\zeta}\underline{u}_{1,-1}]S_r(z^*) + i\varepsilon[e^{i\lambda_c\zeta}\underline{u}_{11} - e^{-i\lambda_c\zeta}\underline{u}_{1,-1}]S_i(z^*)$$
$$+ 0(\varepsilon^2). \qquad (3.11.3)$$

Since the problem is linear we may use an arbitrary initial normalization on $\underline{U}$ given by $U_2(-1/2,0)$ is equal to 1., that is the second component of $\underline{U}$ evaluated at $x = -1/2$ and $\zeta = 0$. We obtain from (3.11.3)

$$1 = 2\psi(0) + \varepsilon[0]\left(\frac{d\psi}{dz^*}\right)_{z^*=0} + 2\varepsilon S_r(0) + i\varepsilon[0]\, S_i(0) \qquad (3.11.4)$$

because the second component of $\underline{u}_{11}$, $\underline{g}_{21}$ evaluated at $x = -1/2$ is equal to 1,0 respectively. We equate powers of ε in (3.11.4) then

$$\psi(0) = 1/2 \quad \text{and} \quad S_r(0) = 0. \qquad (3.11.5)$$

The only solution for $S_r(z^*)$ with $S_r(0)$ as a boundary condition is

$$S_r(z^*) = 0. \qquad (3.11.6)$$

When we use (3.11.6) in (3.11.3) and separate $\underline{u}_{11}$, $\underline{g}_{12}$ and $e^{i\lambda_c\zeta}$ into real and imaginary parts, the solution for $\underline{U}$ becomes

$$\underline{U} = 2\psi(z^*)\,[\underline{u}_{11}^{(r)}\cos\lambda_c\zeta - \underline{u}_{11}^{(i)}\sin\lambda_c\zeta] + 2\varepsilon\frac{d\psi}{dz^*}[\underline{g}_{21}^{(r)}\cos\lambda_c\zeta - \underline{g}_{21}^{(i)}\sin\lambda_c\zeta]$$

$$- 2\varepsilon\, S_i(z^*)[\underline{u}_{11}^{(r)}\sin\lambda_c\zeta + \underline{u}_{11}^{(i)}\cos\lambda_c\zeta] + 0(\varepsilon^2), \qquad (3.11.7)$$

where the notation $\underline{u}_{11}^{(r)}$ etc., is explained in (2.7.7).

In (3.11.7) we still have an arbitrary multiple of $\psi(z^*)$ in $S_i(z^*)$. If we use the symmetry property of our problem and replace

ε by $-\varepsilon$ in (3.11.7), the solution for $\underline{U}$ should remain the same for fixed values of ζ. Therefore from (3.11.7) we see that

$$S_i(z^*) = -S_i(-z^*) \tag{3.11.8}$$

and there is now no arbitrary multiple of $\psi(z^*)$ in $S_i(z^*)$. From the expression of $\underline{u}_{11}$ and $\underline{g}_{21}$ we know that $\underline{u}_{11}^{(r)}$ and $\underline{g}_{21}^{(i)}$ have the form

$$[R, R, 0, R, R, 0] \quad , \tag{3.11.9}$$

and $\underline{u}_{11}^{(i)}$ and $\underline{g}_{21}^{(r)}$ have the following form

$$[0, 0, R, 0, 0, R]. \tag{3.11.10}$$

If, however, the second normalization is used on $\underline{g}_{21}$, then the corresponding solution will be

$$\begin{aligned}\hat{\underline{U}} = {} & [e^{i\lambda_c\zeta}\underline{u}_{11} + e^{-i\lambda_c\zeta}\underline{u}_{1,-1}]\,\psi(z^*) + \varepsilon[e^{i\lambda_c\zeta}\hat{\underline{g}}_{21} + e^{-i\lambda_c\zeta}\hat{\underline{g}}_{2,-1}] \\ & \frac{d\psi}{dz^*} + \varepsilon[e^{i\lambda_c\zeta}\underline{u}_{11} + e^{-i\lambda_c\zeta}\underline{u}_{1,-1}]\,\hat{S}_r(z^*) + i\varepsilon[e^{i\lambda_c\zeta}\underline{u}_{11} - \\ & e^{-i\lambda_c\zeta}\underline{u}_{1,-1}]\,\hat{S}_i(z^*) + O(\varepsilon^2) \quad .\end{aligned} \tag{3.11.11}$$

When the same initial normalization is used for $\hat{\underline{U}}$ and noting that the second components of $\underline{u}_{11}$ and $\hat{\underline{g}}_{21}$ are 1 and -2i respectively at $x = -1/2$ and $\zeta = 0$, then equating powers of ε on either side we obtain

$$\psi(0) = 1/2 \quad \text{and} \quad \hat{S}_r(0) = 0. \tag{3.11.12}$$

The solution for $\hat{S}_r(z^*)$ is therefore given by (3.11.6) and the solution $\hat{\underline{U}}$ is identical to that for $\underline{U}$ when we note the relationships (3.10.8) and (3.10.9). Again the arbitrary multiple of $\psi(z^*)$ in $S_i(z^*)$ is taken to be zero upon using the symmetry property. Thus we shall use the normalizations for $\underline{u}_{11}$, $\underline{g}_{21}$ pertaining to the first case and the solution for $\underline{U}$ is taken to be (3.11.7).

We shall use the solution (3.11.7) to plot U_1, U_4, U_5 and U_6 along the z* axis for particular values of x, ε and compare these solutions with the parallel wall case. The wavelength of the vortex cells formed will be established by solving

$$\Phi = 0 \tag{3.11.13}$$

where Φ represents the non-dimensional Stokes Streamfunction defined in (3.11.22). The vortices can be plotted by using various values of Φ = constant.

The solutions for U_1, U_4 and U_5 are written as

$$U_k = e^{i\lambda_c\zeta} [\psi u_{11,k}^{(r)} + \varepsilon i \frac{d\psi}{dz^*} g_{21,k}^{(i)} + \varepsilon i S_i(z^*)u_{11,k}^{(r)}] + \text{c.c.} \tag{3.11.14}$$

or

$$U_k = 2 [\psi u_{11,k}^{(r)} \cos \lambda_c\zeta - \varepsilon\sin\lambda_c\zeta (S_i(z^*)u_{11,k}^{(r)} + \frac{d\psi}{dz^*} g_{21,k}^{(i)})] \tag{3.11.15}$$

for k = 1,4,5.

We shall for the sake of neatness and comparison with the linear parallel wall problem write

$$\pm i = e^{\pm i\pi/2}. \tag{3.11.16}$$

The solution for U_6 can now be written as

$$U_6 = e^{i(\lambda_c\zeta+\pi/2)}[\psi u_{11,6}^{(i)} - i\varepsilon g_{21,6}^{(r)} \frac{d\psi}{dz^*} + i\varepsilon S_i(z^*)u_{11,6}^{(i)}] + \text{c.c.} \tag{3.11.17}$$

or

$$U_6 = 2[-\psi u_{11,6}^{(i)} \sin\lambda_c\zeta + \varepsilon\cos\lambda_c\zeta (g_{21,6}^{(r)} \frac{d\psi}{dz^*} - S_i(z^*)u_{11,6}^{(i)})]. \tag{3.11.18}$$

The non-dimensional Stokes Streamfunction is given by

$$\frac{1}{1+\delta x} \frac{\partial\Phi}{\partial\zeta} = U_4 = \text{fourth component of } \underline{U} \tag{3.11.19}$$

and

$$\frac{1}{1+\delta x} \frac{\partial\Phi}{\partial x} = -U_6 = \text{sixth component of } \underline{U}\ . \tag{3.11.20}$$

When we use (3.11.14) we have

$$\frac{1}{1+\delta x}\frac{\partial\Phi}{\partial\zeta} = e^{i\lambda_c\zeta}\left[\psi u^{(r)}_{11,4} + \varepsilon i\frac{d\psi}{dz^*}g^{(i)}_{21,4} + \varepsilon i S_i(z^*)u^{(r)}_{11,4}\right] + \text{c.c.} \tag{3.11.21}$$

thus when we integrate both sides with respect to ζ and use (3.11.16) the solution for Φ is

$$\Phi = \frac{(1+\delta x)}{\lambda_c}e^{i(\lambda_c\zeta-\pi/2)}\left[\psi u^{(r)}_{11,4} + \varepsilon i\frac{d\psi}{dz^*}g^{(i)}_{21,4} + \varepsilon i S_i(z^*)u^{(r)}_{11,4} + \frac{\varepsilon i}{\lambda_c}\frac{d\psi}{dz^*}u^{(r)}_{11,4}\right] + \text{c.c.} \tag{3.11.22}$$

or

$$\Phi = \frac{2(1+\delta x)}{\lambda_c}\left[u^{(r)}_{11,4}\psi(z^*)\sin\lambda_c\zeta + \varepsilon\cos\lambda_c\zeta\left(u^{(r)}_{11,4}S_i(z^*) + g^{(i)}_{21,4}\frac{d\psi}{dz^*} + \frac{u^{(r)}_{11,4}}{\lambda_c}\frac{d\psi}{dz^*}\right)\right] . \tag{3.11.23}$$

In solving (3.11.21), we note the arbitrary function of x given by integrating this equation with respect to ζ is just a constant. This is obtained when we used (3.11.20) and (3.11.22) and compare the resulting expression with (3.11.17), the expression for U_6 does not contain a function of x separate from z*.

The results (3.11.14), (3.11.17) and (3.11.22) are comparable with the parallel wall case where

$$U_k = \gamma e^{i\lambda_c\zeta}u^{(r)}_{11,k} + \text{c.c.}, \tag{3.11.24}$$

$$U_6 = \gamma e^{i(\lambda_c\zeta+\pi/2)}u^{(i)}_{11,6} + \text{c.c.} \tag{3.11.25}$$

and

$$\Phi = \frac{(1+\delta x)}{\lambda_c}\gamma e^{i(\lambda_c\zeta-\pi/2)}u^{(r)}_{11,4} + \text{c.c.} \tag{3.11.26}$$

When we solve (3.11.23) subject to (3.11.13) we see that the solution of this depends on the position in the annulus. We first fix x specify ε and then vary z^* from 0 to a certain value. This will give a string of values for Φ and we interpolate between these to obtain the numerical value of z^* that makes $\Phi = 0$. The obvious values where (3.11.23) are zero are $x = \pm 1/2$, $z^* = 0$ and $z^* = \pm\infty$. For fixed values of ε we shall plot the streamlines given by Φ = constant and compare the vortex cells formed with the vortex cells formed in the parallel wall problem for the same values of Φ = constant. (See FIGS. II, X and XIV).

3.12 The observed or physical wavenumbers

The solutions (3.11.14), (3.11.17) and (3.11.22) show higher order corrections to the wavenumber. Let us write the total phase of U_k given by (3.11.14) as

$$ph(U_k) = \lambda_c\zeta + \arg[\,\psi u_{11,k}^{(r)} + \varepsilon i\frac{d\psi}{dz^*}g_{21,k}^{(i)} + \varepsilon i S_i(z^*)u_{11,k}^{(r)}], \quad (3.12.1)$$

where ph refers to the total phase of U_k and arg as the argument of the expression in brackets.

From (3.12.1) the expression containing arg is equal to

$$\tan^{-1}\left[\frac{\varepsilon\frac{d\psi}{dz^*}g_{21,k}^{(i)} + \varepsilon S_i(z^*)u_{11,k}^{(r)}}{\psi(z^*)u_{11,k}^{(r)}}\right]. \quad (3.12.2)$$

The observed or physical wavenumber for the pressure, U_1, and the two velocity components, U_4 and U_5, are then the derivatives of the total phase. In ζ space they are

$$N_\zeta(U_k) = \frac{\partial}{\partial\zeta}(phU_k) = \lambda_c + \frac{\varepsilon^2\frac{d}{dz^*}\left[\left(\frac{d\psi}{dz^*}g_{21,k}^{(i)} + S_i(z^*)u_{11,k}^{(r)}\right)\Big/\psi u_{11,k}^{(r)}\right]}{1 + \frac{\varepsilon^2}{\psi^2[u_{11,k}^{(r)}]^2}\left[\frac{d\psi}{dz^*}g_{21,k}^{(i)} + S_i(z^*)u_{11,k}^{(r)}\right]^2} \quad (3.12.3)$$

which can be shown to be equal to

$$N_\zeta(U_k) = \lambda_c + \varepsilon^2\left[\frac{dR}{dz^*} + \frac{g^{(i)}_{21,k}}{u^{(r)}_{11,k}}\ \frac{d}{dz^*}\left(\frac{d\psi}{dz^*}\Big/\psi\right)\right] + O(\varepsilon^3) \qquad (3.12.4)$$

after expanding for small ε.

The total phase of the Stokes Streamfunction Φ is

$$\mathrm{ph}(\Phi) = \lambda_c\zeta - \frac{\pi}{2} + \arg\left[\psi u^{(r)}_{11,4} + \varepsilon i\frac{d\psi}{dz^*}g^{(i)}_{21,4} + \varepsilon i S_i(z^*)u^{(r)}_{11,4} + \varepsilon i\frac{d\psi}{dz^*}\frac{u^{(r)}_{11,4}}{\lambda_c}\right] \qquad (3.12.5)$$

and the term containing arg is equivalent to a similar term given in (3.12.2). Thus the observed or physical wavenumber for Φ in ζ is

$$N_\zeta(\Phi) = \lambda_c + \frac{\varepsilon^2\dfrac{d}{dz^*}\left[\left(\dfrac{d\psi}{dz^*}g^{(r)}_{21,4} + S_i(z^*)u^{(r)}_{11,4} + \dfrac{d\psi}{dz^*}\dfrac{u^{(r)}_{11,4}}{\lambda_c}\right)\Big/\psi\, u^{(r)}_{11,4}\right]}{1 + \dfrac{\varepsilon^2}{\psi^2[u^{(r)}_{11,4}]^2}\left[\dfrac{d\psi}{dz^*}g^{(i)}_{21,4} + S_i(z^*)u^{(r)}_{11,4} + \dfrac{d\psi}{dz^*}\dfrac{u^{(r)}_{11,4}}{\lambda_c}\right]^2} \qquad (3.12.6)$$

and can be simplified on expanding for small ε to

$$N_\zeta(\Phi) = N_\zeta(U_4) + \frac{\varepsilon^2}{\lambda_c}\frac{d}{dz^*}\left[\frac{d\psi}{dz^*}\Big/\psi\right] . \qquad (3.12.7)$$

The total phase for the velocity component U_6 is

$$\mathrm{ph}(U_6) = \lambda_c\zeta + \frac{\pi}{2} + \arg\left[\psi u^{(i)}_{11,6} - i\varepsilon\frac{d\psi}{dz^*}g^{(r)}_{21,6} + i\varepsilon S_i(z^*)u^{(i)}_{11,6}\right] \qquad (3.12.8)$$

and the physical wavenumber for U_6 in ζ space is

$$N_\zeta(U_6) = \lambda_c + \frac{\varepsilon^2\dfrac{d}{dz^*}\left[\left(S_i(z^*)u^{(i)}_{11,6} - \dfrac{d\psi}{dz^*}g^{(r)}_{21,6}\right)\Big/\psi\, u^{(i)}_{11,6}\right]}{1 + \dfrac{\varepsilon^2}{\psi^2[u^{(i)}_{11,6}]^2}\left[-\dfrac{d\psi}{dz^*}g^{(r)}_{21,6} + S_i(z^*)u^{(i)}_{11,6}\right]^2} . \qquad (3.12.9)$$

The expression for $N_\zeta(U_6)$ is kept in the above form for the time being, see (3.13.16).

The values of $g_{21,m}$ and $u_{11,m}$ are zero at $x = \pm 1/2$ for $m = 4, 5, 6$ and so we are required to calculate

$$\lim_{x\to\pm 1/2}\left[\frac{g_{21,m}}{u_{11,m}}\right] .$$

We use L'Hospital's rule and note that

$$\lim_{x\to\pm 1/2}\left(\frac{g^{(i)}_{21,5}}{u^{(r)}_{11,5}}\right) = \frac{g^{(i)}_{21,2}}{u^{(r)}_{11,2}} \quad\text{and}\quad \lim_{x\to\pm 1/2}\left(\frac{g^{(r)}_{21,6}}{u^{(i)}_{11,6}}\right) = \frac{g^{(r)}_{21,3}}{u^{(i)}_{11,3}} , \tag{3.12.10}$$

where $u^{(i)}_{11,3}$, $g^{(i)}_{21,5}$ etc., are evaluated at $x = \pm 1/2$ and can be found from the computer program used in solving for $\underline{u}_{11}$ and $\underline{g}_{21}$.

For m = 4, we know from the expansion procedure and the continuity equation that

$$\frac{du^{(r)}_{11,4}}{dx} + \delta G u^{(r)}_{11,4} - \lambda_c\, u^{(i)}_{11,6} = 0 \tag{3.12.11}$$

and

$$\frac{dg^{(i)}_{21,4}}{dx} + \delta G g^{(i)}_{21,4} + \lambda_c g^{(r)}_{21,6} + u^{(i)}_{11,6} = 0. \tag{3.12.12}$$

Therefore we see that $\dfrac{du^{(r)}_{11,4}}{dx}$ and $\dfrac{dg^{(i)}_{21,4}}{dx}$ are zero at $x = \pm 1/2$. We have to use L'Hospital's rule twice to obtain

$$\lim_{x\to\pm 1/2}\left(\frac{g^{(i)}_{21,4}}{u^{(r)}_{11,4}}\right) = \lim_{x\to\pm 1/2}\left(\frac{\frac{d^2}{dx^2}\; g^{(i)}_{21,4}}{\frac{d^2}{dx^2}\; u^{(r)}_{11,4}}\right) . \tag{3.12.13}$$

When we differentiate (3.12.11) and (3.12.12) again we find that

(3.12.13) becomes

$$\lim_{x \to \pm 1/2} \left(\frac{g^{(i)}_{21,4}}{u^{(r)}_{11,4}} \right) = - \left[\frac{1}{\lambda_c} + \frac{g^{(r)}_{21,3}}{u^{(i)}_{11,3}} \right] . \tag{3.12.14}$$

From the equations for N_ζ we see that in all cases the physical wavenumbers depend on the position in the annulus for both the x and z* co-ordinate. In previous theory there was no such variation in the wavenumber.

We want to compare the wavenumbers obtained in (3.12.4), (3.12.7) and (3.12.9) with wavenumbers obtained for the locally scaled parallel wall problem. To find the relationship between the two, we define a new ζ, ζ', to be

$$\zeta' = \frac{z - z_o}{d_L} \tag{3.12.15}$$

where d_L is the local gap width evaluated at the point $z = z_o$. At this point z_o we have local values of η_L and T_L, and T_L is defined in terms of local parameters evaluated at z_o. We proceed as if we were calculating the parallel wall problem, we fix η_L and T_L and solve for λ_L in the normal way with $\partial/\partial\tau \equiv 0$. The disturbance velocities u,v,w are zero at the inner and outer walls, that is in terms of our local variable x_L at $x_L = \pm 1/2$. When we have solved for λ_L the disturbance equation can be written as

$$\underline{U} = e^{i\zeta'\lambda_L} \underline{f}_{11}(x_L) + \text{c.c.} \tag{3.12.16}$$

and when we replace ζ' in (3.12.16) by (3.12.15) we have

$$\underline{U} = e^{\frac{i\lambda_L}{d_L}(z - z_o)} \underline{f}_{11}(x_L) + \text{c.c.} \tag{3.12.17}$$

This becomes in terms of ζ space evaluated at the origin where

$z = d\zeta$,

$$\underline{U} = e^{i\lambda_L \frac{d}{d_L}(\zeta - \frac{z_o}{d})} \underline{f}_{11}(x_L) + \text{c.c.} \qquad (3.12.18)$$

The phase of this disturbance is given by

$$\lambda_L \frac{d}{d_L}(\zeta - \frac{z_o}{d}) \qquad (3.12.19)$$

and therefore the physical or observed wavenumber in ζ space is equal to

$$\lambda_L \frac{d}{d_L} . \qquad (3.12.20)$$

This wavenumber is to be compared with N_ζ's in our non-parallel wall problem. That is the local wavenumber in the parallel wall case corresponding to our non-parallel wall wavenumber is given by

$$N_\zeta = \frac{d}{d_L}\lambda_L \qquad (3.12.21a)$$

or

$$\lambda_L = \frac{d_L}{d} N_\zeta , \qquad (3.12.21b)$$

the quantity $\frac{d_L}{d} N_\zeta$ will be known as the locally scaled wavenumber.

The physical wavelengths is defined to be equal to

$$\frac{2\pi \times \text{local channel width}}{\text{local scaled wave number}} \qquad (3.12.22)$$

$$= \frac{2\pi d_L d}{d_L N_\zeta} = \frac{2\pi d}{N_\zeta} . \qquad (3.12.23)$$

It is noted that the wavenumbers (3.12.4), (3.12.6) and (3.12.9) are all slowly varying.

Therefore, from (3.12.21b) we see that to compare critical wavenumbers with the parallel wall theory, we have

$$\lambda_{Lcrit} = (1 + \varepsilon^2 f(z^*)/2) N_\zeta \qquad (3.12.24)$$

and we shall keep just the ε^2 terms and ignore higher order terms.

This λ_{Lcrit} should not be confused with

$$\lambda_{LC} \tag{3.12.25}$$

which is defined to be the critical wavenumber obtained from solving the parallel wall problem for fixed η .

3.13 The renormalization technique for the solutions for $\underline{U}$, Φ and the observed or physical Wavenumbers

For the solutions for U_k and Φ mentioned in (3.11.14) and (3.11.22), we look at the coefficient of $e^{i\lambda_c\zeta}$ and use the renormalization technique originally developed by Rayleigh [32], who used it to generalize his first scattering from a thin slab to scattering from many slabs. He obtained an expression

$$u = e^{ik_o x}(1 + i\varepsilon\mu x) \tag{3.13.1}$$

for first scattering from one slab. To obtain a solution valid for many slabs he recast (3.13.1) into an exponential of the form

$$u = e^{i(k_o + \varepsilon\mu)x} . \tag{3.13.2}$$

This process of summing expansions is called renormalization.

Thus we use this process in (3.11.14) etc., and we have

$$U_k = \psi u^{(r)}_{11,k} \exp i\left(\lambda_c\zeta + \varepsilon S_i/\psi + \varepsilon \frac{g^{(i)}_{21,k}}{u^{(r)}_{11,k}} \frac{d\psi}{dz^*}/\psi\right) + \text{c.c.} \tag{3.13.3}$$

or

$$U_k = 2\psi u^{(r)}_{11,k} \cos\left(\lambda_c\zeta + \varepsilon S_i/\psi + \varepsilon \frac{g^{(i)}_{21,k}}{u^{(r)}_{11,k}} \frac{d\psi}{dz^*}/\psi\right) \tag{3.13.4}$$

for k = 1, 4 and 5, and for U_6 and Φ we rewrite as

$$U_6 = \psi u^{(i)}_{11,6} \exp i\left(\lambda_c\zeta + \frac{\pi}{2} + \varepsilon S_i/\psi - \varepsilon \frac{g^{(r)}_{21,6}}{u^{(i)}_{11,6}} \frac{d\psi}{dz^*}/\psi\right) + \text{c.c.} \tag{3.13.5}$$

$$\Phi = \frac{(1+\delta x)\psi}{\lambda_c} u_{11,4}^{(r)} \exp i\left(\lambda_c\zeta - \frac{\pi}{2} + \varepsilon S_i/\psi + \varepsilon\left(\frac{g_{21,4}^{(i)}}{u_{11,4}^{(r)}} + \frac{1}{\lambda_c}\right)\frac{d\psi}{dz^*}\Big/\psi\right)$$

$$+ \text{ c.c.} \qquad (3.13.6)$$

or

$$U_6 = -2\psi u_{11,6}^{(i)}\sin\left(\lambda_c\zeta+\varepsilon S_i/\psi - \varepsilon\frac{g_{21,6}^{(r)}}{u_{11,6}^{(i)}}\ \frac{d\psi}{dz^*}\Big/\psi\right), \qquad (3.13.7)$$

and

$$\Phi = \frac{2(1+\delta x)\psi}{\lambda_c} u_{11,4}^{(r)} \sin\left[\lambda_c\zeta + \varepsilon S_i/\psi + \varepsilon\left(\frac{1}{\lambda_c} + \frac{g_{21,4}^{(i)}}{u_{11,4}^{(r)}}\right)\frac{d\psi}{dz^*}\Big/\psi\ \right]. \qquad (3.13.8)$$

From (3.13.3), (3.13.5) and (3.13.7), we see quite clearly there is a different complex phase function for each of the velocity components and the Stoke's stream function. The complex phase functions are given by

$$ph(U_k) = \lambda_c\zeta + \varepsilon S_i/\psi + \varepsilon\frac{g_{21,k}^{(i)}}{u_{11,k}^{(r)}}\frac{d\psi}{dz^*}\Big/\psi \qquad (3.13.9)$$

for k = 1, 4 and 5 and for U_6 and Φ

$$ph(U_6) = \lambda_c\zeta + \frac{\pi}{2} + \varepsilon S_i/\psi - \varepsilon\frac{g_{21,6}^{(r)}}{u_{11,6}^{(i)}}\ \frac{d\psi}{dz^*}\Big/\psi\ , \qquad (3.13.10)$$

$$ph(\Phi) = \lambda_c\zeta - \frac{\pi}{2} + \varepsilon S_i/\psi + \varepsilon\left(\frac{g_{21,4}^{(i)}}{u_{11,4}^{(r)}} + \frac{1}{\lambda_c}\right)\frac{d\psi}{dz^*}\Big/\psi\ . \qquad (3.13.11)$$

The derivatives of these complex phase functions with respect to ζ are seen to be slowly varying with a variation in the x co-ordinate , and for U_k and Φ are in agreement with (3.12.4) and (3.12.7) respectively. The wavenumber for the U_6 component is equal to

$$N_\zeta(U_6) = \lambda_c + \varepsilon^2\frac{dR}{dz^*} + \varepsilon^2\frac{g_{21,6}^{(i)}}{u_{11,6}^{(r)}}\ \frac{d}{dz^*}\left[\frac{d\psi}{dz^*}\Big/\psi\right], \qquad (3.13.12)$$

and will be in agreement with (3.12.9) if the latter was expanded for small ε . However difficulties occur with the U_6 component which are explained later in this section.

With the forms given by (3.13.4), (3.13.7) and (3.13.8) we can more easily find the zeros of the pressure, velocity components and the Stoke's stream function. For example, the next n zeros, after $z^* = 0$, for the Stoke's stream function Φ are given by

$$\lambda_c \zeta + \varepsilon S_i/\psi + \varepsilon\left(\frac{g^{(i)}_{21,4}}{u^{(r)}_{11,4}} + \frac{1}{\lambda_c}\right) \frac{d\psi}{dz^*}\bigg/\psi = n\pi . \tag{3.13.13}$$

The solutions obtained with this method and the method of the previous section will not be compared.

In our expansion procedure in the previous section the functions of x and z^* are bounded and in fact tend to zero as $z^* \to \pm \infty$. Our expansion procedure therefore appears to be uniformly valid.

In this section however the expansion appears to be uniformly valid except for the case of the wavenumber and solution for U_6 due to a certain value of the x co-ordinate, x_o, say. For $x = x_o$, $u^{(i)}_{11,6} = 0$ while $g^{(r)}_{21,6} \neq 0$ for this value of x and it seems from (3.13.7) and (3.13.10) that U_6 is bounded at this value and $N_\zeta(U_6) \to \infty$ as $x \to x_o$. But at this value of x_o, the solution for U_o is given by

$$U_6 = e^{i\lambda_c \zeta} \varepsilon \frac{d\psi}{dz^*} g^{(r)}_{21,6}(x_o) \tag{3.13.14}$$

and the wavenumber of U_6 at this value of x_o is

$$N_\zeta(U_6)_{x=x_o} = \lambda_c. \tag{3.13.15}$$

When we insert $x = x_o$ in the equation for $N_\zeta(U_6)$ given by (3.12.9) we see the answer obtained is in agreement with (3.13.15).

It is pointed out that it is the rule rather than the exception that these expansions break down in regions called regions of nonuniformity. An estimate of the size of this region of nonuniformity can sometimes be obtained by assuming two successive terms to be of the same order. This difficulty is only mentioned for the sake of completeness and no calculations regarding $N_\zeta(U_6)$ will be undertaken.

To calculate the ε^2 correction term in (3.13.15) for values of x near x_o, we need to calculate all of the ε^2 terms. In fact it can be shown that as $x \to x_o$,

$$N_\zeta(U_6) \to \lambda_c + \varepsilon^2 \frac{d}{dz^*} \left[\frac{\psi f h^{(i)}_{31,6} + \psi g^{(i)}_{31,6} + \frac{dS}{dz^*} g^{(r)}_{21,6}}{\frac{d\psi}{dz^*} g^{(r)}_{21,6}} \right] . \quad (3.13.16)$$

The solution of the full linear problem, is according to these perturbation techniques, represented by the first few terms of a perturbation expansion and usually only the first two terms. Although these expansions may be divergent, they can be more useful for a qualitative as well as a quantitative representation of the solution than expansions that are uniformly and absolutely convergent. Nayfeh [33] discusses in great detail perturbation techniques .

3.14 Introduction to the non-linear, non-parallel wall problem

This section is involved with the solution of the steady state non-linear equation for $\underline{U}$ given by

$$\frac{\partial \underline{U}}{\partial x} - \underline{A}\,\underline{U} + \varepsilon^2 f(z^*)\underline{C}\,\underline{U} + \varepsilon^3 \frac{df}{dz^*}\,\underline{D}\,\underline{U} + O(\varepsilon^4) = \underline{L}(\underline{U})\underline{U}. \quad (3.14.1)$$

The boundary conditions on $\underline{U}$ are given in (3.5.25) and (3.5.26).

We first fix λ at a particular value and increase the Taylor number above T_{crit} so the point in the λ,T plane is in the unstable region defined by (3.15.4). We should be able to obtain an equilibrium Taylor-vortex flow for our non-parallel wall case.

In the method adopted, we can only set

$$\lambda = \lambda_c \quad (3.14.2)$$

and

$$T = T_c + \varepsilon^2 T_2', \quad (3.14.3)$$

where ε^2 is assumed small.

We start with an initial solution for $\underline{U}$ of the form

$$\underline{U} = \varepsilon \psi_N(z^*) e^{i\lambda_c \zeta} \underline{f}_{11}(x) + \dots, \quad (3.14.4)$$

where $\psi_N(z^*)$ is the non-linear real amplitude function and is in general different from the linear amplitude function $\psi(z^*)$.

When the analysis is carried through we obtain a non-linear differential equation for $\psi_N(z^*)$,

$$\frac{d^2\psi_N}{dz^{*2}} + \psi_N \left[\frac{T_2'}{T_2} + af(z^*)\right] + b\psi_N^3 = 0 \quad (3.14.5)$$

subject to the boundary condition

$$\psi_N \to 0 \quad \text{as} \quad z^* \to \pm\infty. \quad (3.14.6)$$

The solution for ψ_N can be obtained for different values of T_2' by varying the value of ψ_N at $z^* = 0$.

In a similar manner a second order linear differential equation is obtained for the complex amplitude function $S_N(z^*)$, given by

$$\frac{d^2S_N}{dz^{*2}} + S_N\left[\frac{T_2'}{T_2} + af(z^*)\right] + (2S_N + \tilde{S}_N)b\psi_N^2 =$$

$$r_1 f(z^*)\frac{d\psi_N}{dz^*} + r_2\psi_N\frac{df}{dz^*} + r_3\frac{d\psi_N}{dz^*} + r_4\psi_N^2\frac{d\psi_N}{dz^*} , \qquad (3.14.7)$$

and subject to the boundary condition

$$S_N \to 0 \quad \text{as} \quad z^* \to \pm\infty . \qquad (3.14.8)$$

It was found that the solution for $S_N(z^*)$ was completely imaginary. This was in line with the solution for $S(z^*)$ in the linear case.

From Rayleigh's [32] renormalization technique we found the observed wavenumber for each component varied from the linear results for the wavenumbers. Though no actual calculations were attempted, the form of these 'non-linear' wavenumbers in ζ space will be

$$N_{\zeta,N} = \lambda_c + \varepsilon fn(x,z^*,\zeta) + \varepsilon^2 fn(x,z^*,\zeta). \qquad (3.14.9)$$

3.15 Solution of the non-linear disturbance equations

From the linear theory of the equilibrium Taylor-vortex flow for the non-parallel wall problem we have if

$$\lambda = \lambda_c \qquad (3.15.1)$$

then

$$T = T_{crit} = T_c + \varepsilon^2 T_2^* + O(\varepsilon^4), \qquad (3.15.2)$$

where λ_c, T_c are the critical wavenumber and Taylor number for the parallel wall case with the parameters η and T being evaluated at

$z^* = 0$. The value of T_2^* was an eigenvalue dependent on the boundary conditions at $z^* = \pm\infty$ of the linear amplitude function $\psi(z^*)$ and the form of the outer surface.

The reader is reminded that in the linear case we said the flow would be stable if

$$T < T_{crit} \tag{3.15.3}$$

and unstable if

$$T > T_{crit} \,. \tag{3.15.4}$$

In the following non-linear analysis we shall increase the Taylor number T above T_{crit} so that we would be in the unstable region defined by (3.15.4). We shall ignore terms of $O(\varepsilon^4)$ and therefore set

$$T = T_c + \varepsilon^2 T_2' \,, \tag{3.15.5}$$

where

$$T_2' > T_2^* \,, \tag{3.15.6}$$

in the steady state non-linear equation for $\underline{U}$ given by (3.5.22).

The equation for $\underline{U}$ now becomes,

$$\frac{\partial \underline{U}}{\partial x} - \underline{A}_c \underline{U} - \varepsilon^2 T_2' \underline{A}_2 \underline{U} + \varepsilon^2 f(z^*) \underline{C}_c \underline{U} + \varepsilon^3 \frac{df}{dz^*} \underline{D}_c \underline{U}$$

$$+ O(\varepsilon^4) = \underline{L}_c(\underline{U})\underline{U} + \varepsilon^2 T_2' \underline{L}_{c2}(\underline{U})\underline{U} \,. \tag{3.15.7}$$

The matrices $\underline{A}_c$, $\underline{C}_c$ and $\underline{D}_c$ are given by(2.4.6),(3.5.23) and (3.5.24) with T replaced by T_c. The matrices $\underline{A}_2$ and $\underline{L}_2(\underline{U})$ are given by (2.4.7) and (2.10.5).

The equation for $\underline{U}$ has to be solved subject to the boundary conditions given by (3.5.25) and (3.5.26).

We next expand $\underline{U}$ in powers of ε and $e^{i\lambda_c\zeta}$, and write

$$\underline{U} = \varepsilon e^{i\lambda_c\zeta}\underline{u}_{11}(x,z^*) + \varepsilon^2[e^{i2\lambda_c\zeta}\underline{u}_{22}(x,z^*)+e^{i\lambda_c\zeta}\underline{u}_{21}(x,z^*) + \underline{u}_{20}(x,z^*)] + \varepsilon^3[e^{3i\lambda_c\zeta}\underline{u}_{33}(x,z^*) + e^{2i\lambda_c\zeta}\underline{u}_{32}(x,z^*) + e^{i\lambda_c\zeta}\underline{u}_{31}(x,z^*) + \underline{u}_{30}(x,z^*)] + \ldots + \text{negative powers of } e^{im\lambda_c\zeta}, \quad (3.15.8)$$

in the steady state non-linear equation defined by (3.15.7) . When we equate powers of $\varepsilon^n e^{im\lambda_c\zeta}$ we obtain a set of partial differential equations,

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(1)}\right)\underline{u}_{11} = 0, \quad (3.15.9)$$

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(2)}\right)\underline{u}_{22} = \underline{L}_{c0}^{(1)}(\underline{u}_{11})\underline{u}_{11}, \quad (3.15.10)$$

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(1)}\right)\underline{u}_{21} = \underline{B}_{c1}^{(1)}\frac{\partial \underline{u}_{11}}{\partial z^*}, \quad (3.15.11)$$

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(0)}\right)\underline{u}_{20} = \underline{L}_{c0}^{(-1)}(\underline{u}_{11})\underline{u}_{1,-1} + \underline{L}_{c0}^{(1)}(\underline{u}_{1,-1})\underline{u}_{11}, \quad (3.15.12)$$

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(3)}\right)\underline{u}_{33} = \underline{L}_{c0}^{(2)}(\underline{u}_{11})\underline{u}_{22} + \underline{L}_{c0}^{(1)}(\underline{u}_{22})\underline{u}_{11}, \quad (3.15.13)$$

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(2)}\right)\underline{u}_{32} = \underline{L}_{c0}^{(1)}(\underline{u}_{11})\underline{u}_{21} + \underline{L}_{c0}^{(1)}(\underline{u}_{21})\underline{u}_{11} + \underline{B}_{c1}^{(2)}\frac{\partial \underline{u}_{22}}{\partial z^*} + \underline{L}_{c1}(\underline{u}_{11})\frac{\partial \underline{u}_{11}}{\partial z^*}, \quad (3.15.14)$$

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(1)}\right)\underline{u}_{31} = \underline{L}_{c0}^{(1)}(\underline{u}_{20})\underline{u}_{11} + \underline{L}_{c0}^{(0)}(\underline{u}_{11})\underline{u}_{20} + \underline{L}_{c0}^{(2)}(\underline{u}_{1,-1})\underline{u}_{22} + \underline{L}_{c0}^{(-1)}(\underline{u}_{22})\underline{u}_{1,-1} + \underline{B}_{c2}\frac{\partial^2 \underline{u}_{11}}{\partial z^{*2}} +$$

$$\underline{B}_{c1}^{(1)} \frac{\partial \underline{u}_{21}}{\partial z^*} + T_2' \underline{A}_2 \underline{u}_{11} + \underline{L}_{c1}(\underline{u}_{11}) \frac{\partial \underline{u}_{20}}{\partial z^*} +$$

$$\underline{L}_{c1}(\underline{u}_{20}) \frac{\partial \underline{u}_{11}}{\partial z^*} + \underline{L}_{c1}(\underline{u}_{1,-1}) \frac{\partial \underline{u}_{22}}{\partial z^*} + \underline{L}_{c1}(\underline{u}_{22}) \frac{\partial \underline{u}_{1,-1}}{\partial z^*}$$

$$- f(z^*) \underline{C}_c \underline{u}_{11} , \qquad (3.15.15)$$

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(0)} \right) \underline{u}_{30} = \underline{L}_{c0}^{(-1)}(\underline{u}_{11}) \underline{u}_{2,-1} + \underline{L}_{c0}^{(1)}(\underline{u}_{2,-1}) \underline{u}_{11} +$$

$$\underline{L}_{c0}^{(-1)}(\underline{u}_{21}) \underline{u}_{1,-1} + \underline{L}_{c0}^{(1)}(\underline{u}_{1,-1}) \underline{u}_{21} + \underline{L}_{c1}(\underline{u}_{1,-1}) \frac{\partial \underline{u}_{11}}{\partial z^*}$$

$$+ \underline{B}_{c1}^{(0)} \frac{\partial \underline{u}_{20}}{\partial z^*} . \qquad (3.15.16)$$

The matrices $\underline{A}_c^{(p)}$, $\underline{L}_{c0}^{(p)}(\underline{u}_{ij})$ and $\underline{A}_2$ are defined in (2.4.6) , (2.10.16) and (2.4.7) respectively. The matrices $\underline{C}_c$, $\underline{B}_{c1}^{(p)}$, $\underline{B}_{c2}$ and $\underline{D}_c^{(1)}$ are defined in (3.5.23) and (3.6.14).

Here

$$\underline{L}_{c1}(\underline{u}_{ij}) = -\frac{1}{\alpha} \begin{bmatrix} & -u_{ij,6} & 0 & u_{ij,4} \\ \underset{\sim}{0} & 0 & u_{ij,6} & 0 \\ & 0 & 0 & u_{ij,6} \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix} \qquad (3.15.17)$$

and

$$\underline{u}_{i,-j} = \tilde{\underline{u}}_{ij} , \qquad (3.15.18)$$

while $u_{ij,k}$, k = 1,6, represents the corresponding six components of the vector $\underline{u}_{ij}$.

To find a differential equation for the second amplitude function $S_N(z^*)$, which is introduced later, it is necessary to have the $\varepsilon^4 e^{i\lambda_c \zeta} \underline{u}_{41}(x,z^*)$ term. The partial differential equation for $\underline{u}_{41}(x,z^*)$ is

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(1)} \right)\underline{u}_{41} = \underline{L}_{c0}^{(-1)}(\underline{u}_{22})\underline{u}_{2,-1} + \underline{L}_{c0}^{(1)}(\underline{u}_{20})\underline{u}_{21} + \underline{L}_{c0}^{(0)}(\underline{u}_{21})\underline{u}_{20}$$

$$+ \underline{L}_{c0}^{(2)}(\underline{u}_{2,-1})\underline{u}_{22} + \underline{L}_{c0}^{(1)}(\underline{u}_{30})\underline{u}_{11} + \underline{L}_{c0}^{(-1)}(\underline{u}_{32})\underline{u}_{1,-1} +$$

$$\underline{L}_{c0}^{(0)}(\underline{u}_{11})\underline{u}_{30} + \underline{L}_{c0}^{(2)}(\underline{u}_{1,-1})\underline{u}_{32} + \underline{L}_{c1}(\underline{u}_{11}) \frac{\partial \underline{u}_{20}}{\partial z^*} +$$

$$\underline{L}_{c1}(\underline{u}_{20}) \frac{\partial \underline{u}_{11}}{\partial z^*} + \underline{L}_{c1}(\underline{u}_{1,-1}) \frac{\partial \underline{u}_{22}}{\partial z^*} + \underline{L}_{c1}(\underline{u}_{22}) \frac{\partial \underline{u}_{1,-1}}{\partial z^*} +$$

$$\underline{B}_{c1}^{(1)} \frac{\partial \underline{u}_{31}}{\partial z^*} + T_2' \underline{A}_2 \underline{u}_{21} + \underline{B}_{c2} \frac{\partial^2 \underline{u}_{21}}{\partial z^{*2}} - \frac{df}{dz^*} \underline{D}_c^{(1)} \underline{u}_{11} -$$

$$f(z^*)\underline{C}_c \underline{u}_{21} \ . \qquad (3.15.19)$$

The equations (3.15.10) to (3.15.16) and (3.15.19) will be simplified when the following notation is used,

$$\underline{R}_{c0}(\underline{f}_{ij}, \underline{g}_{k\ell}) = \underline{L}_{c0}^{(\ell)}(\underline{f}_{ij})\underline{g}_{k\ell} + \underline{L}_{c0}^{(j)}(\underline{g}_{k\ell})\underline{f}_{ij} \qquad (3.15.20)$$

and

$$\underline{R}_{c1}(f_{ij}, \underline{g}_{k\ell}) = k\underline{L}_{c1}(\underline{f}_{ij})\underline{g}_{k\ell} + i\underline{L}_{c1}(\underline{g}_{k\ell})f_{ij} \ . \qquad (3.15.21)$$

The non-linear equation for $\underline{U}$ has to be solved subject to the physical condition that requires zero disturbance velocities at the walls given by (3.5.25). Once again the boundary condition on the outer surface, $x = \frac{1}{2}[1 + \varepsilon^2 f(z^*)]$, is obtained by means of a Taylor expansion of the disturbance velocities $\underline{u}_{ij}$'s about $x = \frac{1}{2}$. When we use the Taylor expansion about $x = \frac{1}{2}$ and equate powers of $\varepsilon^n e^{im\lambda_c \zeta}$, we obtain the following set of boundary conditions for the last three components of each $\underline{u}_{ij}$:

$$u_{11,m}(\tfrac{1}{2},z^*) = u_{22,m}(\tfrac{1}{2},z^*) = u_{21,m}(\tfrac{1}{2},z^*) = u_{20,m}(\tfrac{1}{2},z^*) = 0,$$

$$u_{33,m}(\tfrac{1}{2},z^*) = u_{32,m}(\tfrac{1}{2},z^*) = u_{30,m}(\tfrac{1}{2},z^*) = 0, \qquad (3.15.22)$$

$$u_{31,m}(\tfrac{1}{2},z^*) + f(z^*)u_{11\ x,m}(\tfrac{1}{2},z^*)/2=0, u_{41,m}(\tfrac{1}{2},z^*)+f(z^*)u_{21\ x,m}$$

$$(\tfrac{1}{2},z^*)/2 \quad = \quad 0, \quad \text{for } m = 4,5,6.$$

The boundary condition on the inner wall, $x = -\frac{1}{2}$, for each $\underline{u}_{ij}$ are

$$u_{ij,m}(-\tfrac{1}{2},z^*)=0, \text{ for all i and j.} \qquad (3.15.23)$$

A further boundary condition on $\underline{U}$ was that it tends to zero as $z^* \to \pm\ \infty$, this implies that each $\underline{u}_{ij}$ must satisfy

$$\underline{u}_{ij}(x,z^*) \to 0 \quad \text{as} \quad z^* \to \pm\ \infty \ . \qquad (3.15.24)$$

3.16 Solution of the disturbance equations and the differential equation for the non-linear amplitude function $\psi_N(z^*)$

From the linear theory a solution for $\underline{u}_{11}(x,z^*)$ can be written as

$$\underline{u}_{11}(x,z^*) \quad = \quad \psi_N(z^*)\underline{f}_{11}(x) \qquad (3.16.1)$$

to give

$$\mathcal{L}^{(1)}(\underline{f}_{11}) = 0 \qquad ; \qquad \beta_2 \ . \qquad (3.16.2)$$

The operator $\mathcal{L}^{(p)}$ is $\frac{d}{dx} - \underline{A}_c^{(p)}$, and $\underline{f}_{11}(x)$ must satisfy the boundary condition β_2, given in (2.4.1) along with the same normalization as used in the parallel and non-parallel wall linear case. The first term of (3.16.1) contains a real amplitude function $\psi_N(z^*)$ which will be determined by a non-linear differential equation that arises at a higher order from a solvability condition imposed on that higher order system. It should be noted that ψ_N mentioned here will, in general, be different than that of the amplitude function ψ of the linear theory.

The boundary condition on ψ_N is seen to be

$$\psi_N(z^*) \to 0 \quad \text{as} \quad z^* \to \pm\infty \ . \tag{3.16.3}$$

When we use (3.16.1) in (3.15.10) and (3.15.12) the solutions for $\underline{u}_{22}$ and $\underline{u}_{20}$ are of the form

$$\underline{u}_{22}(x,z^*) = \psi_N^2 \, \underline{f}_{22}(x) \tag{3.16.4}$$

and

$$\underline{u}_{20}(x,z^*) = \psi_N^2 \, \underline{f}_{20}(x), \tag{3.16.5}$$

where $\underline{f}_{22}(x)$ and $\underline{f}_{20}(x)$ satisfy

$$\mathcal{L}^{(2)}(\underline{f}_{22}) = \frac{1}{2} \underline{R}_{c0}(\underline{f}_{11}, \underline{f}_{11}) \quad ; \quad \beta_2 \tag{3.16.6}$$

and

$$\mathcal{L}^{(0)}(\underline{f}_{20}) = \underline{R}_{c0}(\underline{f}_{11}, \underline{f}_{1,-1}) \quad ; \quad \beta_2 \ . \tag{3.16.7}$$

It can be seen that by direct comparison with the non-linear parallel wall equations (2.11.3) and (2.11.4) the functions $\underline{f}_{22}$, $\underline{f}_{20}$ are the same in both cases.

Similarly, when we use (3.16.1) in (3.15.11) the equation becomes

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(1)}\right)\underline{u}_{21} = \frac{d\psi_N}{dz^*} \, \underline{B}_{c1}^{(1)} \, \underline{f}_{11} \quad ; \quad \beta_2 \ . \tag{3.16.8}$$

For $\underline{u}_{21}(x,z^*)$ to have a solution we use the adjoint condition

(2.5.3) to obtain

$$\frac{d\psi_N}{dz^*} \int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}^{a,t} \, \underline{B}_{c1}^{(1)} \, \underline{f}_{11} \, dx = 0, \tag{3.16.9}$$

and from the linear theory it was noted that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \tilde{f}^{a,t} \underline{B}_{c1}^{(1)} \, \underline{f}_{11} \, dx = 0. \tag{3.16.10}$$

From the boundary conditions on $\underline{u}_{21}$ and the right-hand side of (3.16.8), $\underline{u}_{21}$ may be expressed as

$$\underline{u}_{21}(x,z^*) = \frac{d\psi_N}{dz^*} \underline{g}_{21}(x) + S_N(z^*)\underline{f}_{11}(x), \tag{3.16.11}$$

where $\underline{g}_{21}(x)$ satisfies (3.7.10) and is identical to the $\underline{g}_{21}(x)$ mentioned in the linear case. The complex function $S_N(z^*)$ will in general be different from $S(z^*)$ of the linear problem. The boundary conditions on $S_N(z^*)$ are

$$S_N(z^*) \to 0 \quad \text{as} \quad z^* \to \pm\infty. \tag{3.16.12}$$

The complex conjugate $\tilde{\underline{u}}_{21}$ is

$$\tilde{\underline{u}}_{21}(x,z^*) = \frac{d\psi_N}{dz^*} \underline{g}_{2,-1}(x) + \tilde{S}_N(z^*)\underline{f}_{1,-1}(x). \tag{3.16.13}$$

When the form of expressions for $\underline{u}_{11}$, $\underline{u}_{21}$, $\underline{u}_{22}$ etc are substituted in (3.15.14) and (3.15.16), the solutions for $\underline{u}_{32}$ and $\underline{u}_{30}$ take the form

$$\underline{u}_{32}(x,z^*) = \psi_N \frac{d\psi_N}{dz^*} \underline{f}_{32}(x) + \psi_N S_N \underline{m}_{32}(x) \tag{3.16.14}$$

and

$$\underline{u}_{30}(x,z^*) = \psi_N \frac{d\psi_N}{dz^*} \underline{f}_{30}(x) + \psi_N(S_N + \tilde{S}_N)\underline{m}_{30}(x). \tag{3.16.15}$$

The equations satisfied by $\underline{f}_{32}$, $\underline{m}_{32}$, $\underline{f}_{30}$ and $\underline{m}_{30}$ are

$$\mathcal{L}^{(2)}(\underline{f}_{32}) = \underline{R}_{c0}(\underline{f}_{11},\underline{g}_{21}) + 2\underline{B}_{c1}^{(2)}\underline{f}_{22} + \underline{R}_{c1}(\underline{f}_{11},\underline{f}_{11})/2, \tag{3.16.16}$$

$$\mathcal{L}^{(2)}(\underline{m}_{32}) = \underline{R}_{c0}(\underline{f}_{11}, \underline{f}_{11}), \tag{3.16.17}$$

$$\mathcal{L}^{(0)}(\underline{f}_{30}) = \underline{R}_{c0}(\underline{f}_{11}, \underline{g}_{2,-1}) + \underline{R}_{c0}(\underline{f}_{1,-1},\underline{g}_{21}) + \underline{R}_{c1}(\underline{f}_{11},\underline{f}_{1,-1}) + 2\underline{B}_{c1}^{(0)}\underline{f}_{20}, \tag{3.16.18}$$

and

$$\mathcal{L}^{(0)}(\underline{m}_{30}) = \underline{R}_{c0}(\underline{f}_{11},\underline{f}_{1,-1}), \tag{3.16.19}$$

all subject to the boundary condition β_2.

When we compare equations (3.16.17) and (3.16.19) with (3.16.6) and (3.16.7) we have

$$\underline{m}_{32}(x) = 2\underline{f}_{22}(x) \tag{3.16.20}$$

and

$$\underline{m}_{30}(x) = \underline{f}_{20}(x). \tag{3.16.21}$$

When (3.16.1) and (3.16.4) etc., are used in (3.15.15) we obtain

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(1)}\right)\underline{u}_{31} = \psi_N^3\,[\underline{R}_{c0}(\underline{f}_{11},\underline{f}_{20}) + \underline{R}_{c0}(\underline{f}_{1,-1},\underline{f}_{22})] + \frac{dS_N}{dz^*}\underline{B}_{c1}^{(1)}\underline{f}_{11} + \frac{d^2\psi_N}{dz^{*2}}[\underline{B}_{c2}\underline{f}_{11} + \underline{B}_{c1}^{(1)}\underline{g}_{21}] + \psi_N T_2'\,\underline{A}_2\underline{f}_{11} - \psi_N f(z^*)\underline{C}_c\underline{f}_{11} \tag{3.16.22}$$

subject to the boundary conditions (for m = 4, 5, 6)

$$u_{31,m}(-\tfrac{1}{2},z^*) = 0, \quad u_{31,m}(\tfrac{1}{2},z^*) = -f(z^*)\psi_N f_{11x,m}(\tfrac{1}{2})/2. \tag{3.16.23}$$

For $\underline{u}_{31}(x,z^*)$ to exist, we use the adjoint condition (2.5.11) to obtain

$$\frac{d^2\psi_N}{dz^{*2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} [\underline{B}_{c1}^{(1)} \underline{g}_{21} + \underline{B}_{c2}\underline{f}_{11}]\, dx + \psi_N f(z^*)[(\underset{\sim}{f}^{a,t}\underline{f}_{11,x})_{x=\frac{1}{2}}/2 -$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} C_c \underline{f}_{11}\, dx] + \psi_N T_2' \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{A}_2 \underline{f}_{11}\, dx +$$

$$\psi_N^3 \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} [\underline{R}_{c0}(\underline{f}_{20}, \underline{f}_{11}) + \underline{R}_{c0}(\underline{f}_{1,-1}, \underline{f}_{22})]\, dx = 0, \quad (3.16.24)$$

since the integral coefficient of $\dfrac{dS_N}{dz^*}$ is zero from (3.7.6).

The non-linear amplitude equation (3.16.24) can be simplified by using some linear results , e.g. (3.7.18), to give

$$\frac{d^2\psi_N}{dz^{*2}} + \psi_N \left[\frac{T_2'}{T_2} + a f(z^*)\right] + b\psi_N^3 = 0 \qquad (3.16.25)$$

subject to (3.16.3).

The constants a and T_2 are defined in (3.7.20), (2.6.10) respectively and

$$b = \frac{\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} [\underline{R}_{c0}(\underline{f}_{20}, \underline{f}_{11}) + \underline{R}_{c0}(\underline{f}_{22}, \underline{f}_{1,-1})]\, dx}{T_2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{A}_2 \underline{f}_{11}\, dx} . \quad (3.16.26)$$

From the parallel wall non-linear theory this constant b is related to the given initial amplitude μ_o. When we use (2.11.9) we find that b is negative and could be rewritten as

$$b = -\frac{T_c}{\mu_o^2 T_2} \qquad (3.16.27)$$

and T_c, T_2 and μ_o^2 are all positive.

3.17 Solution of the disturbance equations (cont) and the differential equation for the amplitude function $S_N(z^*)$

We assume (3.16.25) can be solved subject to the given boundary conditions and we substitute for $d^2\psi_N/dz^{*2}$ in the right-hand side of (3.16.22) , whence

$$\left(\frac{\partial}{\partial x} - \underline{A}_c^{(1)}\right)\underline{u}_{31} = \psi_N^3 [\underline{R}_{c0}(\underline{f}_{20},\underline{f}_{11}) + \underline{R}_{c0}(\underline{f}_{22},\underline{f}_{1,-1}) - b(\underline{B}_{c1}^{(1)}\underline{g}_{21} + \underline{B}_{c2}\underline{f}_{11})] - \psi_N \frac{T_2'}{T_2}[\underline{B}_{c1}^{(1)}\underline{g}_{21} + \underline{B}_{c2}\underline{f}_{11} - T_2\underline{A}_2\underline{f}_{11}] + \frac{dS_N}{dz^*}\underline{B}_{c1}^{(1)}\underline{f}_{11} - \psi_N f(z^*)[a(\underline{B}_{c1}^{(1)}\underline{g}_{21} + \underline{B}_{c2}\underline{f}_{11}) + \underline{C}_c\underline{f}_{11}]. \tag{3.17.1}$$

By examining the right-hand side of (3.17.1) we see $\underline{u}_{31}$ may be expressed as

$$\underline{u}_{31}(x,z^*) = \psi_N^3\underline{f}_{31}(x) + \psi_N\underline{g}_{31}(x) + \psi_N f(z^*)\underline{h}_{31}(x) + \frac{dS_N}{dz^*}\underline{g}_{21}(x) + P_N(z^*)\underline{f}_{11}(x) \tag{3.17.2}$$

and upon equating functions of z* on either side of (3.17.1) we have, $\underline{h}_{31}(x)$ satisfies equation (3.9.3) and its boundary conditions and

$$\mathcal{L}^{(1)}(\underline{f}_{31}) = \underline{R}_{c0}(\underline{f}_{22},\underline{f}_{1,-1}) + \underline{R}_{c0}(\underline{f}_{20},\underline{f}_{11}) - b(\underline{B}_{c1}^{(1)}\underline{g}_{21} + \underline{B}_{c2}\underline{f}_{11}), \tag{3.17.3}$$

$$\mathcal{L}^{(1)}(\underline{g}_{31}) = -\frac{T_2'}{T_2}[\underline{B}_{c1}^{(1)}\underline{g}_{21} + \underline{B}_{c2}\underline{f}_{11} - T_2\underline{A}_2\underline{f}_{11}] \tag{3.17.4}$$

and $\underline{f}_{31}$, $\underline{h}_{31}$ are normalized such that their second components evaluated at $x = -\frac{1}{2}$ are equal to 0. The function $P_N(z^*)$ appears because we can have an arbitrary multiple of the eigenfunction $\underline{f}_{11}$, and is assumed to be complex and different from the $P(z^*)$ mentioned in the linear case. The boundary conditions on (3.17.3), (3.17.4) are

$$f_{31,m}(-\tfrac{1}{2}) = g_{31,m}(-\tfrac{1}{2}) = f_{31,m}(\tfrac{1}{2}) = g_{31,m}(\tfrac{1}{2}) = 0, \tag{3.17.5}$$

for m = 4, 5, 6.

If we look at the same equation for $\underline{g}_{31}(x)$ for the linear parallel wall case and compare the equations and boundary conditions for $\underline{g}_{31}$ we see

$$\underline{g}_{31}(x) = \frac{T_2'}{T_2}\,\underline{g}_{31}^{(p)} \quad . \tag{3.17.6}$$

We reform equation (3.15.19) by using (3.16.1), (3.16.4), (3.16.2) etc., and replace the term involving $d^3\psi_N/dz^{*3}$ by differentiating (3.16.25) with respect to z* to give

$$\frac{d^3\psi_N}{dz^{*3}} = -[3b\psi_N^2\frac{d\psi_N}{dz^*} + \frac{T_2'}{T_2}\frac{d\psi_N}{dz^*} + a\psi_N\frac{df}{dz^*} + af(z^*)\frac{d\psi_N}{dz^*}] \tag{3.17.7}$$

and thus the equation for $\underline{u}_{41}(x,z^*)$ will become

$$(\frac{\partial}{\partial x} - \underline{A}_c^{(1)})\underline{u}_{41} = \psi_N^2\frac{d\psi_N}{dz^*}[\underline{R}_{c0}(\underline{f}_{11},\underline{f}_{30}) + \underline{R}_{c0}(\underline{f}_{32},\underline{f}_{1,-1})$$

$$+ \underline{R}_{c0}(\underline{f}_{22},\underline{g}_{2,-1}) + \underline{R}_{c0}(\underline{f}_{20},\underline{g}_{21}) + 3\underline{B}_{c1}^{(1)}\underline{f}_{31} -$$

$$3b\underline{B}_{c2}\underline{g}_{21} + \underline{R}_{c1}(\underline{f}_{20},\underline{f}_{11}) + \underline{R}_{c1}(\underline{f}_{22},\underline{f}_{1,-1})]$$

$$+ \frac{d\psi_N}{dz^*}\left[T_2' \underline{A}_2\underline{g}_{21} + \underline{B}_{c1}^{(1)}\underline{g}_{31} - \frac{T_2'}{T_2}\underline{B}_{c2}\underline{g}_{21}\right] + \frac{dP_N}{dz^*}\underline{B}_{c1}^{(1)}\underline{f}_{11}$$

$$+ \psi_N \frac{df}{dz^*}\left[\underline{B}_{c1}^{(1)}\underline{h}_{31} - \underline{D}_c^{(1)}\underline{f}_{11} - a\,\underline{B}_{c1}\underline{g}_{21}\right] + \frac{d\psi_N}{dz^*} f(z^*)\left[\underline{B}_{c1}^{(1)}\underline{h}_{31} -\right.$$

$$\left.\underline{C}_c\underline{g}_{21} - a\,\underline{B}_{c1}\underline{g}_{21}\right] + \psi_N^2 (2S_N + \tilde{S}_N)\left[\underline{R}_{c0}(\underline{f}_{11}, f_{20}) + \underline{R}_{c0}\right.$$

$$\left.(\underline{f}_{22}, \underline{f}_{1,-1})\right] + T_2' S_N \underline{A}_2\underline{f}_{11} - S_N f(z^*)\underline{C}_c\underline{f}_{11} + \frac{d^2 S_N}{dz^{*2}}$$

$$\left[\underline{B}_{c1}^{(1)}\underline{g}_{21} + \underline{B}_{c2}\underline{f}_{11}\right] , \tag{3.17.8}$$

subject to the boundary conditions

$$u_{41,m}(-\tfrac{1}{2}, z^*) = 0 \quad \text{and} \quad u_{41,m}(\tfrac{1}{2}, z^*) = -\left[f(z^*)\frac{d\psi_N}{dz^*} g_{21x,m}(\tfrac{1}{2}) + \right.$$

$$\left. f(z^*) S_N(z^*) u_{11x,m}(\tfrac{1}{2})\right] /2 \tag{3.17.9}$$

for m = 4, 5, 6.

For $\underline{u}_{41}(x,z^*)$ to exist, we use the adjoint condition (2.5.11) to obtain

$$\frac{d^2 S_N}{dz^{*2}} + S_N\left[\frac{T_2'}{T_2} + a\,f(z^*)\right] + (2S_N + \tilde{S}_N) b\psi_N^2 =$$

$$r_1 f(z^*)\frac{d\psi_N}{dz^*} + r_2\psi_N\frac{df}{dz^*} + r_{3,N}\frac{d\psi_N}{dz^*} + r_4\psi_N^2\frac{d\psi_N}{dz^*} . \tag{3.17.10}$$

The boundary conditions on $S_N(z^*)$ are the same as those on $\psi_N(z^*)$ that is

$$S_N(z^*) \to 0 \quad \text{as} \quad z^* \to \pm\infty . \tag{3.17.11}$$

Here r_1, r_2 are defined in (3.9.13), (3.9.14) and

$$r_{3,N} = \frac{-\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\left[\underline{B}_{c1}^{(1)}\ \underline{g}_{31} + T_2'\ \underline{A}_2 \underline{g}_{21} - \frac{T_2'}{T_2}\ \underline{B}_{c2}\ \underline{g}_{21}\right]dx}{T_2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\ \underline{A}_2 \underline{f}_{11}\,dx} \tag{3.17.12}$$

$$r_4 = \frac{-\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\ \underline{Q}_2\ dx}{T_2 \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\ \underline{A}_2 \underline{f}_{11}\,dx} \quad . \tag{3.17.13}$$

where

$$\begin{aligned} \underline{Q}_2 &= \underline{R}_{c0}(\underline{f}_{11}, \underline{f}_{30}) + \underline{R}_{c0}(\underline{f}_{32}, \underline{f}_{1,-1}) + \underline{R}_{c0}(\underline{f}_{22}, \underline{g}_{2,-1}) \\ &+ \underline{R}_{c0}(\underline{f}_{20}, \underline{g}_{21}) + \underline{R}_{c1}(\underline{f}_{20}, \underline{f}_{11}) + \underline{R}_{c1}(\underline{f}_{22}, \underline{f}_{1,-1}) \\ &+ 3\underline{B}_{c1}^{(1)} \underline{f}_{31} - 3b\underline{B}_{c2} \underline{g}_{21} \, . \end{aligned} \tag{3.17.14}$$

When we compare the equation for $r_{3,N}$ with (3.9.22) we note that

$$r_{3,N} = -i\, T_3\, T_2' \,/\, T_2^2 \quad . \tag{3.17.15}$$

It was found also on calculating r_1, r_2 and r_4 that these constants were also imaginary. The differential equation for $S_N(z^*)$ can be separated into real and imaginary parts,

$$S_N(z^*) = S_{N,r}(z^*) + i\, S_{N,i}(z^*) \, . \tag{3.17.16}$$

We replace $S_N(z^*)$ by (3.17.16) in (3.17.10) and equate real and imaginary parts, we obtain the following differential equations

$$\frac{d^2 S_{N,r}}{dz^{*2}} + S_{N,r}\left[\frac{T_2'}{T_2} + a\, f(z^*)\right] + 3b\psi_N^2\ S_{N,r} = 0 \tag{3.17.17}$$

and

$$\frac{d^2 S_{N,i}}{dz^{*2}} + S_{N,i}\left[\frac{T_2'}{T_2} + a\,f(z^*)\right] + b\psi_N^2\, S_{N,i} = r_{1i} f(z^*)\,\frac{d\psi_N}{dz^*} +$$

$$r_{2i}\psi_N \frac{df}{dz^*} + r_{3i,N}\frac{d\psi_N}{dz^*} + r_{4i}\psi_N^2 \frac{d\psi_N}{dz^*} \;, \qquad (3.17.18)$$

subject to the following boundary conditions

$$S_{N,r}(z^*)\;,\; S_{N,i}(z^*) \to 0 \quad \text{as} \quad z^* \to \pm\,\infty\,. \qquad (3.17.19)$$

The constant $r_{3i,N}$ is just the imaginary part of (3.17.15).

We can solve for $\psi_N(z^*)$, $S_{N,i}(z^*)$ and $S_{N,r}(z^*)$ by a small parameter perturbation method treating

$$T_2' = T_2^*\,[1 + \delta^2] \qquad (3.17.20)$$

where δ^2 is a small real parameter not related to ε . The solution obtained for $S_{N,r}(z^*)$ with the choice of $f(z^*)$ given in (4.2.1) was identically equal to zero. With this choice of $f(z^*)$ we could only find solutions for $\psi_N(z^*)$ by direct computation, a time factor did not allow us to solve numerically the more complex case of $S_{N,r}(z^*)$ and $S_{N,i}(z^*)$.

For the case chosen in Chapter 4 the perturbation and computation solutions for $\psi_N(z^*)$ were in good agreement for small δ^2 . It was found in both methods the solutions for $\psi_N(z^*)$ depended on the values of T_2' and $\psi_N(0)$.

The inclusion of the non-linear terms involving $\underline{U}$, namely $\underline{L}(\underline{U})\underline{U}$, served mainly to fix the magnitude of $\psi_N(z^*)$ (which was arbitrary fixed in the linear case). This is similar to the non-linear parallel wall case where the amplitude μ_0^2 is fixed by the integral relationship (2.11.9).

We obtain the non-linear solutions for the velocity components U_4, U_5, U_6 and the Stokes stream function Φ in the same way as in §3.11. These non-linear solutions include terms identical to those of the linear case except that all the linear solutions are pre-multiplied by ε, $\psi(z^*)$ and $S(z^*)$ are replaced by $\psi_N(z^*)$ and $S_{N,j}(z^*)$ respectively and plus some extra terms not included in §3.11.

If we let, for k = 4,5,6, $U_{k\chi}$ and Φ_χ represent this 'modified' solution typified by (3.11.14) and $U_{k\Delta}$ and Φ_Δ represent the solutions typified by (3.11.15), then the non-linear solutions for U_k and Φ are

$$U_4 = \varepsilon U_{4\chi} + \varepsilon^2 e^{2i\lambda_c\zeta} \psi_N^2 \ f_{22,4}^{(r)} + \text{c.c.} \tag{3.17.21}$$

$$U_5 = \varepsilon U_{5\chi} + \varepsilon^2 e^{2i\lambda_c\zeta} \psi_N^2 \ f_{22,5}^{(r)} + \text{c.c.} + \varepsilon^2 \psi_N^2 \ f_{20,5}^{(r)} \tag{3.17.22}$$

$$U_6 = \varepsilon U_{6\chi} + \varepsilon^2 e^{2i\lambda_c\zeta + i\frac{\pi}{2}} \psi_N^2 \ f_{22,6}^{(i)} + \text{c.c.} \tag{3.17.23}$$

and

$$\Phi = \varepsilon\Phi_\chi + \frac{(1 + \delta x)}{2\lambda_c} \varepsilon^2 e^{2i\lambda_c\zeta - i\frac{\pi}{2}} \psi_N^2 f_{22,4}^{(r)} + \text{c.c.} \tag{3.17.24}$$

or

$$U_4 = \varepsilon U_{4\Delta} + 2\varepsilon^2 \psi_N^2 \ f_{22,4}^{(r)} \cos 2\lambda_c\zeta \tag{3.17.25}$$

$$U_5 = \varepsilon U_{5\Delta} + 2\varepsilon^2 \psi_N^2 \ f_{22,5}^{(r)} \cos 2\lambda_c\zeta + \varepsilon^2 \psi_N f_{20,5}^{(r)} \tag{3.17.26}$$

$$U_6 = \varepsilon U_6 - 2\varepsilon^2 \psi_N^2 \ f_{22,6}^{(i)} \sin 2\lambda_c\zeta \tag{3.17.27}$$

and

$$\Phi = \Phi_\Delta + \frac{2(1 + \delta x)\varepsilon^2}{\lambda_c} \psi_N^2 \ f_{22,4} \sin 2\lambda_c\zeta \ . \tag{3.17.28}$$

It was found from our computation of the function $\underline{f}_{20}(x)$ that

$$f_{20,4}(x) = f_{20,6}(x) = 0 \text{ for all } x. \tag{3.17.29}$$

It was decided just to plot the non-linear solutions for the U_4 velocity component for various values of ε and T_2' in Chapter 4.

The non-linear wavenumbers $\lambda_{crit,N}$ were not plotted in Chapter 4 due to the incomplete and rather complex nature of the non-linear solutions for U_k and Φ. This is easily shown in the case of U_4. The solution for U_4 given by (3.17.21) can be rewritten as

$$U_4 = \varepsilon e^{i\lambda_c\zeta}\psi_N f_{11,4}^{(r)}\left[1 + \varepsilon e^{i\lambda_c\zeta}\psi_N \frac{f_{22,4}^{(r)}}{f_{11,4}^{(r)}} + \varepsilon\frac{i}{\psi_N}\left\{\frac{d\psi_N}{dz^*}\frac{g_{21,4}^{(i)}}{f_{11,4}^{(r)}} + S_{N,i}(z^*)\right\} + O(\varepsilon^2)\right]. \tag{3.17.30}$$

If we use the renormalization technique of §3.13 the solution (3.17.30) can be replaced by

$$U_4 = \varepsilon e^{\varepsilon\cos\lambda_c\zeta}\psi_N f_{11,4}^{(r)} \cdot e^{i\Lambda(x,\zeta,z^*)} \tag{3.17.31}$$

where

$$\Lambda = \lambda_c\zeta + \varepsilon\left[\psi_N \sin\lambda_c\zeta\frac{f_{22,4}^{(r)}}{f_{11,4}^{(r)}} + \frac{S_N}{\psi_N} + \frac{g_{21,4}^{(i)}}{u_{11,4}^{(r)}}\frac{d\psi_N}{dz^*}\Big/\psi_N\right] + O(\varepsilon^2) \tag{3.17.32}$$

and is a real function.

The total phase for the velocity component U_4 is now given by Λ and the physical wavenumber $N_{\zeta,N}$ is given by $\partial\Lambda/\partial\zeta$. It can easily be shown that in order to find all the $O(\varepsilon^2)$ terms in $\partial\Lambda/\partial\zeta$ we need to include terms of $O(\varepsilon^2)$ in Λ , these terms take the form of $\sin 2\lambda_c\zeta$ multiplied by functions of x and z* etc. Therefore no non-linear wavenumbers were calculated.

4. Solutions For a Specific Case

4.1 Introduction and brief discussion of the results

In this chapter we follow Chapter 3 and calculate numerical results for some special cases. We chose to consider the cases $\eta = 0.5$ ($\varepsilon = 0.1$ and 0.5) and $\eta = 0.95$ ($\varepsilon = 0.1$). Our choice of $f(z^*)$ is given in §4.2.

The linear values of $u_{11,k}$ and $g_{21,k}$ ($k = 4,5$ and 6) for $\eta = 0.5$ and $\eta = 0.95$ are given in TABLE VII and TABLE XVI ; along with the value of $g_{21,k}/u_{11,k}$ at $x = \pm\frac{1}{2}$ for each η. With the use of these tables and formulae mentioned in this chapter the reader can find his own linear solutions for various ε, j and n.

The eigenvalues T_2^* and the eigensolutions $\psi(z^*)$ are found in the linear case with $j = 1$ (see §4.3). It turns out that $\psi(z^*)$ decays quite rapidly with $|z^*|$ in both cases of η but fractionally less for $\eta = 0.5$ than for $\eta = 0.95$, see FIG V. This eigensolution $\psi(z^*)$ is identical for all ε and fixed η , but in ζ co-ordinates the solutions will be different. In fact the larger the value of ε chosen the quicker $\psi \to 0$ and the vortices decay away.

It was shown that the Taylor number T_{Lcrit} depends on the eigenvalue chosen, ε and its position along the vertical axis. The graphs of T_{Lcrit} and T_{Lc} against η_L for both cases of η shows for a particular z^* value, $z^* = |z_c^*|$ say, $T_{Lcrit} \geq T_{Lc}$ for $z^* \leq |z_c^*|$ and $T_{Lcrit} < T_{Lc}$ for $z^* > |z^*|$. This value of z^* increases the smaller the value of ε . For the case with

$\varepsilon = 0.1$, T_{Lcrit} is 0.53% and 0.74% higher than the respective T_{Lc} at $z^* = 0$ for $\eta = 0.5$ and $\eta = 0.95$ respectively, and 0.54% and 0.73% lower than T_{Lc} at $z^* = \pm\infty$. If the value of T_{Lc} at infinity was unknown it was found by interpolation using TABLE IV. For $\varepsilon = 0.5$ with $\eta = 0.5$, T_{Lcrit} is 3.55% higher than T_{Lc} at $z^* = 0$ and 24.82% lower than T_{Lc} at $z^* = \pm\infty$. These larger figures were to be expected due to the high value of ε. The graphs of T_{Lcrit} etc., are given in FIG VI, XI and XV.

Our analysis of the non-parallel wall flow showed that the various disturbance flow quantities do not possess the same wavenumber (except at $z^* = \pm\infty$), there is a variation in x and z^* of $O(\varepsilon^2)$. The variations in the radial direction x have not been given before. See FIG VII, VIII, XII and XVI for typical results.

We choose to demonstrate some results using $\lambda_{Lcrit}(\Phi)$. For $\varepsilon = 0.1$ we found $\lambda_{Lcrit}(\Phi)$ is between 0.42% - 0.58% higher than λ_{Lc} at $z^* = 0$ for $\eta = 0.5$ and between 0.24% - 0.41% higher for $\eta = 0.95$. This is typical of all the wavenumbers concerned. It can be said however that $\lambda_{Lcrit} < \lambda_{Lc}$ at $z^* = \pm\infty$. For $\varepsilon = 0.5$, $\eta = 0.5$ $\lambda_{Lcrit}(\Phi)$ is between 10.53% - 14.43% higher than λ_{Lc} at $z^* = 0$ and 3.37% lower than λ_{Lc} at $z^* = \pm\infty$. For each ε and η the wavenumbers all tend to the same value at $z^* = \pm\infty$, see FIG VII, XII and XVI.

All the solutions of U_i, i = 1,6 and Φ decay away as $z^* \to \pm\infty$. For $\eta = 0.5$, the way in which this occurs is the same for $\varepsilon = 0.1$ and $\varepsilon = 0.5$ because the linear amplitude function $\psi(z^*)$ is identical for both values of ε (in z^* co-ordinates).

The number of times the solutions oscillate depends strongly on the value of ε , the larger the value of ε the smaller the number of times they oscillate along the z^* axis before becoming negligible. The reason why is our solutions involve terms like $\sin\lambda_c\zeta$ and $\cos\lambda_c\zeta$; and $\zeta = z^*/\varepsilon$. Therefore solutions involving $\varepsilon = 0.1$ tend to oscillate approximately five times more often than those of $\varepsilon = 0.5$. A comparison is made between the non-parallel wall linear solutions and the corresponding parallel wall linear solutions (where ζ is replaced by z^*/ε). In all the non-parallel wall cases considered the initial zero's of U_4, U_5 and Φ all appeared before the corresponding zero's of the parallel wall case. This is shown quite well in the case with $\eta = 0.5$ and $\varepsilon = 0.5$, see FIG IX and XIII.

Following G.I. Taylor's[(1)] original work it became of interest to show the streamlines, Φ, of $\eta = 0.5$ with $\varepsilon = 0.1$ and $\varepsilon = 0.5$ to see how they differ from those of the parallel wall case, see FIG II. It was decided a useful criterion for vortices to be observed visually might be if $|c| \geq 0.1$, where c is a constant given by $\Phi = c$. We will only be dealing with the top half of the cylinders since our problem, by choice, is symmetrical. For $\varepsilon = 0.1$, FIG Xa to Xc show the individual vortex cells along the z^* axis. The cell nearest to the centre is very similar to the vortex cell found in the parallel wall case. However, the further we get away from $z^* = 0$ the more the strength of the vortex cells decay . Our results show by the time $z^* > 1.88$ ($\varepsilon = 0.1$) there are no more cells visible where $|c| \geq 0.1$, in all there would be 38 'visible' vortex cells which weaken as z^* increases. A similar result occurs for $\eta = 0.5$ and $\varepsilon = 0.5$ where no more cells with $|c| \geqslant 0.1$ from $z^* = 2.33$, in all there would only be 10 'visible' vortex cells , see FIG XIVa and XIVb.

A note should be made of the boundaries between each vortex cell, the boundaries start off horizontal but gradually curve upwards slightly before becoming horizontal again at $z^* = \pm\ \infty$. A variation like this was expected since $\lambda_{Lcrit}(\Phi)$ does depend on x and z^*.

We also solved the rather more difficult non-linear case. The solutions of the non-linear amplitude function $\psi_N(z^*)$ were calculated numerically and by the perturbation method of §4.5 for Taylor numbers slightly greater than T_{Lcrit} of the linear theory. These Taylor numbers were called $T_{Lcrit,N}$ and involved positive additive corrections of $O(\varepsilon^2)$ to T_{Lcrit}, these corrections were given the letter D and defined in (4.6.3).

The boundary conditions on $\psi_N(z^*)$ were identical to those for the linear amplitude function $\psi(z^*)$. It was found the rate at which $\psi_N \to 0$ as $z^* \to \pm\infty$ depends on D and its relationship with $\psi_N(0)$, see FIG XVII . For large values of D there were difficulties in calculating the numerical solution for ψ_N, the perturbation method not being valid. In the main the non-linear numerical and perturbation solutions for $\psi_N(z^*)$ agree quite well for $\eta = 0.5$ and $\eta = 0.95$ and for $D < 800$.

For small additions of T above T_{Lcrit} a perturbation method was used to solve the non-linear function $S_{N,i}(z^*)$. With this value of $S_{N,i}(z^*)$, and ε and D fixed at specified values, the non-linear solutions of U_k for k = 4,5,6 and Φ were calculated for fixed x values.

These solutions still decay as $z^* \to \pm\infty$ but each different solution has a fixed amplitude depending on the value of $\psi_N(0)$ and D. The non-linear solutions are tabulated in TABLES XXVI, XXVII and XXIX, and drawn for the case $\eta = 0.5$, $\varepsilon = 0.5$, D = 200 and D = 600 in FIG XIX.

The wavelength of these non-linear solutions are slightly shorter than the linear case though there is no obvious change in the number of vortex cells. This implies that the cells near the centre have a shorter wavelength than at the ends. The larger the value of D the bigger the initial amplitude of $\psi_N(z^*)$ and the shorter the initial wavelength of the vortex cells near the centre $z^* = 0$. See FIG XIX.

4.2 The amplitude equation in a special case

We illustrate the analysis of Chapter 3 by means of an example :

$$f(z^*) = -\tanh^2 \omega z^* = \operatorname{sech}^2 \omega z^* - 1, \qquad (4.2.1)$$

where ω is a constant.

The function $f(z^*)$ satisfies (3.2.6) to (3.2.9) and from (3.2.8) and (4.2.1) we see

$$f_\infty = -1 , \qquad (4.2.2)$$

and also from (3.2.3) we have the outer wall at $z^* = \pm\infty$ is

$$x = [1 - \varepsilon^2]/2. \qquad (4.2.3)$$

The amplitude equation (3.7.19) for $\psi(z^*)$ can now be written as

$$\frac{d^2\psi}{dz^{*2}} + \psi\left[\frac{T_2^*}{T_2} - a + a\,\mathrm{sech}^2\omega z^*\right] = 0 \qquad (4.2.4)$$

with the boundary conditions

$$\psi \to 0 \quad \text{as} \quad z^* \to \pm\infty \quad \text{and } \psi(0) = 1/2. \qquad (4.2.5)$$

The last condition of (4.2.5) is obtained from (3.11.5).

Let

$$P_2 = \frac{T_2^*}{T_2} - a \qquad (4.2.6)$$

and P_2 is assumed to be negative. This is because we expect $\psi(z^*)$ to behave like $e^{-\ell_1 z^*}$ at $z^* = \infty$ and $e^{\ell_1 z^*}$ at $z^* = -\infty$, see (3.8.13) to (3.8.15).

We now make the following substitution in (4.2.4),

$$y = \tanh \omega z^* \qquad (4.2.7)$$

and the differential equation for ψ in terms of y is

$$\frac{d}{dy}\left[(1-y^2)\frac{d\psi}{dy}\right] + \left[\frac{a}{\omega^2} + \frac{P_2}{\omega^2(1-y^2)}\right]\psi = 0, \qquad (4.2.8)$$

with the new boundary conditions of

$$\psi \to 0 \quad \text{as } y \to \pm 1 \quad \text{and} \quad \psi(0) = 1/2. \qquad (4.2.9)$$

We introduce the constants

$$\mu = +\sqrt{-P_2/\omega^2} \qquad \text{and } s(s+1) = a/\omega^2 \qquad (4.2.10)$$

in (4.2.8) and obtain

$$\frac{d}{dy}\left[(1-y^2)\frac{d\psi}{dy}\right] + \left[s(s+1) - \frac{\mu^2}{1-y^2}\right]\psi = 0. \qquad (4.2.11)$$

This is the equation for the generalized Legendre polynomials.

We now make a further substitution

$$\psi = (1-y^2)^{\mu/2} W(y) \tag{4.2.12}$$

and therefore obtain

$$\frac{d}{dy}\left[(1-y^2)\frac{d\psi}{dy}\right] = (1-y^2)^{1+\mu/2}\frac{d^2W}{dy^2} - 2(1+\mu)y(1-y^2)^{\mu/2}\frac{dW}{dy} -$$

$$\mu W[(1-y^2)^{\mu/2} - \mu y^2(1-y^2)^{\mu/2-1}]. \tag{4.2.13}$$

Equation (4.2.11) is now of the form

$$(1-y^2)\frac{d^2W}{dy^2} - 2(1+\mu)y\frac{dW}{dy} - [s(s+1)-\mu(\mu+1)]W = 0. \tag{4.2.14}$$

We can transform the above to the hypergeometric differential equation by substituting

$$(1-y)/2 = q \tag{4.2.15}$$

and so finally obtain

$$q(1-q)\frac{d^2W}{dq^2} + (\mu+1)(1-2q)\frac{dW}{dq} - (\mu-s)(\mu+s+1)W = 0. \tag{4.2.16}$$

This equation is now subject to the boundary conditions that

$$W \text{ is bounded as } q \to 0 \text{ and } q \to 1,\ W(0) = 1/2. \tag{4.2.17}$$

The parameters $\hat{a}$,b,c, which occur in the general form of the hypergeometric equation

$$x(1-x)y'' + [c-(\hat{a} + b + 1)x]y' - \hat{a}by = 0 \tag{4.2.18}$$

have in our case the following values :

$$\hat{a} = \mu-s \quad , \quad b = \mu+s+1 \quad , \quad c = \mu+1 \ . \tag{4.2.19}$$

The two solutions of equation (4.2.16) which lead respectively to the even and odd amplitude functions ψ are of the form

$$W_1 = A\,{}_2F_1\,(\mu-s,\ \mu+s+1\ ;\ \mu+1\ ;\ q)\ , \tag{4.2.20}$$

$$W_2 = Bq^{-\mu}\,{}_2F_1(-s,\ s+1\ ;\ 1-s\ ;\ q). \tag{4.2.21}$$

The solution for $\psi(z^*)$ is given by :

$$\psi = A \operatorname{sech}^{\mu}\omega z^* W_1, \tag{4.2.22}$$

where A is a constant.

In order that (4.2.22) should reduce to zero as $y \to -1$, the hypergeometric function in equation (4.2.20) should reduce to a polynomial. This condition means for W_1 that the parameters μ-s or μ+s+1 must be a negative or zero integer. The second case can be discarded since in that case the amplitude function ψ increases exponentially as $z^* \to \pm\infty$. We find, therefore, ${}_2F_1$ is a polynomial of degree n if

$$s-\mu = n \quad \text{for } n = 0,1,2, \ldots . \tag{4.2.23}$$

The eigenvalues for T* are determined from (4.2.23) and are

$$T_2^* = T_2\left[a - \frac{\omega^2}{4} [-(2n+1) + \sqrt{1+4a/\omega^2}]^2\right] . \tag{4.2.24}$$

Furthermore, there are only a finite number of eigenvalues since

$$\mu > 0 \quad , \tag{4.2.25}$$

that is

$$n < s \tag{4.2.26}$$

and hence

$$n < \frac{1}{2} [-1 + \sqrt{1 + 4a/\omega^2}] \text{ for } n = 0,1,2, \ldots . \tag{4.2.27}$$

4.3 Eigenvalues, eigenfunctions and the linear solutions

The corresponding eigenfunction, for each n, is

$$\psi_n(z^*) = A \operatorname{sech}^{s-n} \omega z^* \, {}_2F_1\left(-n, 2s-n+1 ; s-n+1; \frac{1-\tanh\omega z^*}{2}\right) \tag{4.3.1}$$

It is convenient to have the power of sech ωz^* in (4.3.1) to be an integer. Therefore the parameter s must be an integer, this is achieved by choosing the parameter ω^2 by setting

$$s = \frac{1}{2}\left[-1 + \sqrt{1 + 4a/\omega^2}\right] = j \tag{4.3.2}$$

where the integer j is greater than zero.

We obtain from (4.3.2) that

$$\omega^2 = a/[\, j(j+1)\,] \tag{4.3.3}$$

for all n and j = 1,2, ...

When we use (4.3.3) in (4.2.24) and (4.2.26), then

$$T_2^* = aT_2\left[1 - \frac{(j-n)^2}{j(j+1)}\right] \tag{4.3.4}$$

and

$$n < j \, , \tag{4.3.5}$$

respectively.

The eigenfunctions for each n and j can now be written as

$$\psi_{nj}(z^*) = A_{nj}\operatorname{sech}^{j-n}\omega z^* \; {}_2F_1\left(-n, 2j-n+1; j-n+1; \frac{1-\tanh\omega z^*}{2}\right) \tag{4.3.6}$$

where A_{nj} are constants to be determined later.

The number of eigenvalues and the corresponding eigenfunctions depend on the relationship given in (4.3.5). It can be seen that for j = 1 there is only one eigenvalue and eigenfunction given by n = 0, whilst for j = 2 there are two eigenvalues and eigenfunctions given by n = 0 and n = 1 and so on for each j.

The eigenvalues and eigenfunctions for n = 0 are

$$T_2^* = aT_2/(j+1) \quad \text{and} \quad \psi_{0j}(z^*) = A_{0j}\operatorname{sech}^j \omega z^* \tag{4.3.7}$$

for $j = 1,2,3, \ldots$;

$n = 1$ are

$$T_2^* = aT_2(3j-1)/j(j+1) \text{ and } \psi_{1j}(z^*) = A_{1j}\operatorname{sech}^{j-1}\omega z^* \tanh\omega z^* \tag{4.3.8}$$

for $j = 2,3, \ldots$;

$n = 2$ are

$$T_2^* = aT_2(5j-1)/j(j+1) \text{ and } \psi_{2j}(z^*) = A_{2j}\operatorname{sech}^{j-2}\omega z^* \left[\frac{2j-2}{2j-1} - \operatorname{sech}^2\omega z^*\right]. \tag{4.3.9}$$

for $j = 3,4, \ldots$; and so on.

As j increases then ω decreases and the number of discrete eigenvalues increases. As $j \to \infty$ then $\omega \to 0$ and $T_2^* \to 0$. We then have $T_{crit} = T_c$ (to order ε^2) and have recovered the parallel wall case.

In the limit as $n \to j-1$ with j fixed the value of $T_2^* \to aT_2$ and the graphical solutions for ψ_{nj} are given in FIG IV for n = 0, 1 and 2.

The number of times ψ_{nj} cuts the z^*-axis depends on the value of n.

We shall only be concerned with the case given by n = 0 because this gives the lowest value for T at which neutral stability occurs for a fixed ω . Therefore the higher order even functions given in (4.3.9) will be ignored. It can be seen that these higher order even functions will give infinite wavenumbers whenever, for example,

$$\frac{2j-2}{2j-1} - \operatorname{sech}^2\omega z^* = 0 \tag{4.3.10}$$

in the case with n = 2.

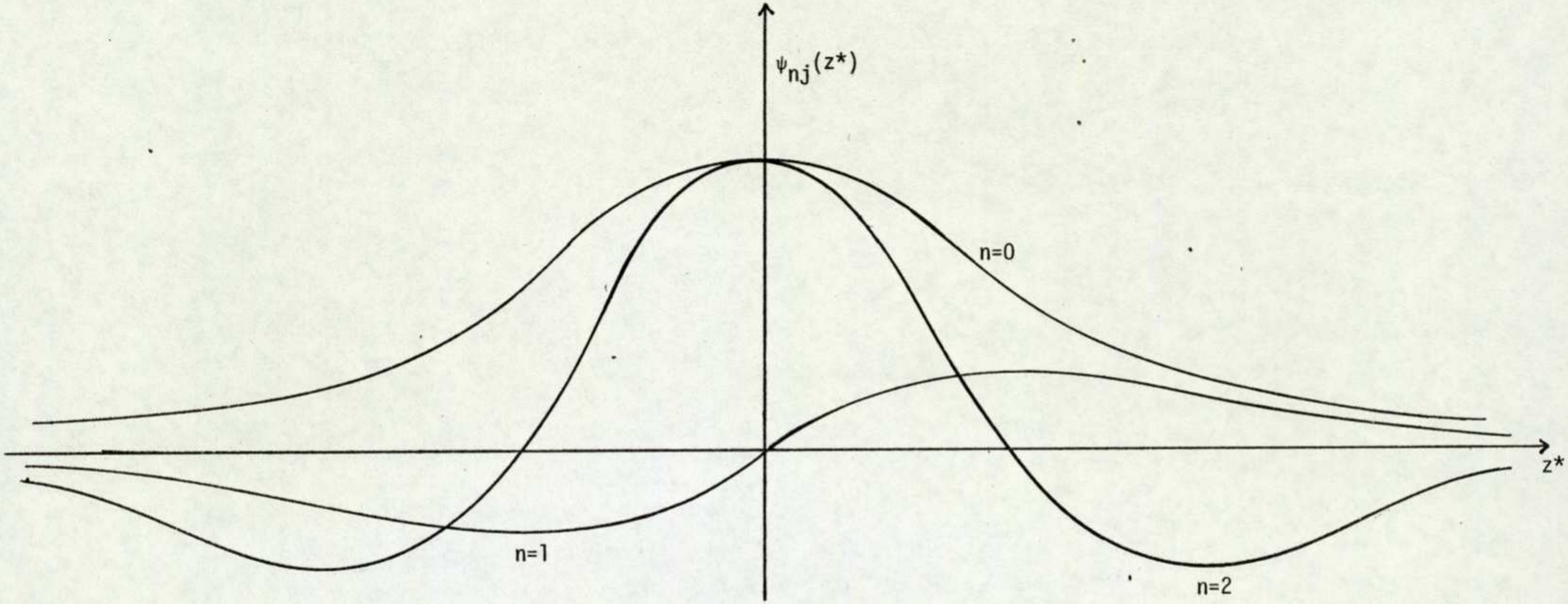

FIG IV Solutions of $\psi_{nj}(z^*)$ for fixed j and n = 0,1,2.

The values of the constants A_{0j} are given by

$$A_{0j} = 1/2 . \tag{4.3.11}$$

We are now only concerned with the eigenvalues and eigenfunctions given by n = 0 but the solution of the amplitude equation for $S_i(z^*)$ will be done for the general case. When we place the form of solution for ψ_{0j}, given in (4.3.7), in (3.10.28) we have for all j that

$$R(z^*) = \frac{1}{2}\left[(r_{3i}-r_{1i})z^* + \frac{(2r_{2i}+jr_{1i})}{(j+1)\omega} \tanh \omega z^*\right]. \tag{4.3.12}$$

Hence we can see that the solutions for U_k and Φ are now given by

$$U_k = \mathrm{sech}^j \omega z^* \left[u_{11,k}^{(r)} \cos\lambda_c\zeta - \varepsilon\sin\lambda_c\zeta \left(R(z^*)u_{11,k}^{(r)} - \omega j \tanh \omega z^*\right.\right.$$

$$\left.\left. g_{21,k}^{(i)} + O(\varepsilon^2)\right]\right. \tag{4.3.13}$$

for k = 1,4,5 whilst the solutions for U_6 and Φ are given by similar expressions [from (3.11.18) and (3.11.23)].

The solution for N_ζ has been given in (3.12.4) and (3.12.7) and can be seen to be

$$N_\zeta(U_k) = \lambda_c + \varepsilon^2\left[\frac{dR}{dz^*} - \omega^2 j\ \mathrm{sech}^2 \omega z^* \frac{g_{21,k}^{(i)}(x)}{u_{11,k}^{(r)}(x)}\right] \tag{4.3.14}$$

and

$$N_\zeta(\Phi) = N_\zeta(U_4) - \varepsilon^2\omega^2 j\ \mathrm{sech}^2 \omega z^*/\lambda_c . \tag{4.3.15}$$

The local critical wavenumbers, λ_{Lcrit}, to be compared with wavenumbers from the parallel wall case are given by

$$\lambda_{Lcrit} = N_\zeta(\ ) - \varepsilon^2\lambda_c \tanh^2 \omega z^*/2 \tag{4.3.16}$$

where the extra ε^2 term is due to (3.12.24).

The local critical Taylor number and the ratio of the inner and outer walls are given by

$$T_{Lcrit} = T_c + \varepsilon^2[T_2^* - T_c(\tfrac{3}{2} - \tfrac{\delta}{4})\tanh^2 \omega z^*] \qquad (4.3.17)$$

and

$$\eta_L = \eta + \varepsilon^2 \eta(1-\eta)\tanh^2 \omega z^*/2. \qquad (4.3.18)$$

4.4 Numerical results for the linear Case

The known values of T_{Lc} and λ_{Lc} calculated from the respective parallel wall case for fixed η_L are given in TABLE IV.

4.4a The linear case with $\eta = 0.5$

The critical values of the wavenumber and Taylor number used in this case were

$$\lambda_c = 3.16242 \quad \text{and} \quad T_c = 3099.78. \qquad (4.4.1)$$

From our computing methods the following constants were found to be given by

$$T_2 = 440.2819 \ , \ a = 7.4918 \ , \ r_{1i} = -1.4179 \ , \ r_{2i} = 1.3302 \qquad (4.4.2)$$

and for the case with $j = 1$ and $n = 0$ the value of r_3 was

$$r_{3i} = 0.8503 \ , \qquad (4.4.3)$$

this value does depend on the eigenvalue T_2^* .

The values of (4.4.2) are used to calculate the eigenvalues and the corresponding eigensolutions for each integer value of j and n described in §4.3. The results are given in TABLE V.
The linear values of $\underline{u}_{11}(x)$ and $\underline{g}_{21}(x)$ for $\eta = 0.5$ and $x = -0.5$ to 0.5 are given in TABLE VII. These, in conjunction with the

constants defined in (4.4.1), (4.4.2) and (4.4.3), the equations (4.3.12) to (4.3.18) will enable the reader to plot other solutions of U_4, U_5, U_6, Φ, λ_{Lcrit}'s etc., for different values of ε. In order to obtain the wavenumbers λ_{Lcrit} we need to find the values of $\underline{g}_{21}(x) / \underline{u}_{11}(x)$ evaluated at $x = \pm \frac{1}{2}$. We use equation (3.12.10) and (3.12.14), and these values are given in TABLE VIII.

If different cases of j and n (other than j = 1, n = 0) need to be considered the solutions can be found in a similar manner. The eigenvalues T_2^* and eigensolutions $\psi(z^*)$ can be found using (4.3.4) and (4.3.6). The new value of r_{3i}(r_{2i} and r_{1i} remain invariant) can be calculated using (3.9.22). The value of T_2^* will be known and the constants T_2 and T_3 are defined in (2.8.20). The solutions for U_4, U_5, U_6, Φ, λ_{Lcrit}'s etc., can now be found.

The graph of $\psi_{01}(z^*)$, which is the envelope of the solutions for the Stokes Streamfunction Φ and $\underline{U}$, is plotted in FIG V. A table of the values of $\psi_{01}(z^*)$ is given in TABLE IX.

The values of ε considered for the case $\eta = 0.5$ are

$$\varepsilon = 0.1 \text{ and } \varepsilon = 0.5 . \qquad (4.4.4)$$

The changing values of η_L, T_{Lcrit} and λ_{Lcrit} for these values of ε, are given in TABLE X and TABLE XI. A graph of η_L against T_{Lc} and T_{Lcrit} is plotted for each value of ε in FIG VI and FIG XI. These graphs show the difference in T_{Lc}, obtained from parallel wall theory, and T_{Lcrit} which is obtained from the non-parallel wall theory.

A graph of η against λ_{Lc} and λ_{Lcrit} is also plotted in FIG VII and FIG XII. These graphs are concerned with the λ_{Lcrit} of the components U_4, U_5 and the Streamfunction Φ . This graph was found to depend on the x co-ordinate, so only the maximum and minimum value of λ_{Lcrit} for each component was plotted. These graphs show that λ_{Lcrit} depends on the x co-ordinate and also the difference between the parallel wall value of λ_{Lc}.

This variation with x of λ_{Lcrit} for each component is given in TABLE XII and TABLE XIII for two fixed z* values. The graph of this relationship of x against λ_{Lcrit} for U_4, U_5 and Φ are shown in FIG VIII.

The terminating values of T_{Lcrit}, η_L and λ_{Lcrit} for all the components mentioned are given at $z^* = \pm\infty$. These values for $\varepsilon = 0.1$ are

$$T_{Lcrit} = 3074.94 \quad , \quad \eta_L = 0.50125, \ \lambda_{Lcrit} = 3.15795 \tag{4.4.5}$$

and for $\varepsilon = 0.5$ are

$$T_{Lcrit} = 2176.59 \quad , \quad \eta_L = 0.5\dot{3} \ , \ \lambda_{Lcrit} = 3.05064. \tag{4.4.6}$$

The solutions for U_4, U_5, U_6 and Φ are summarised for various z* values in TABLE XIV for $\varepsilon = 0.1$ and TABLE XV for $\varepsilon = 0.5$. These solutions are compared with those from the parallel wall case $U_4^{(p)}$, $U_5^{(p)}$, $U_6^{(p)}$ and $\Phi^{(p)}$, so we see how quickly these solutions decay as $z^* \to \pm\infty$. Since all these solutions depend also on the x co-ordinate we shall only plot and tabulate for x = 0.

It can be seen from (4.3.13) that at z* = 0 we have

$$U_k = U_k^{(p)} \tag{4.4.7}$$

and

$$\Phi = \Phi^{(p)} . \tag{4.4.8}$$

The graphs of U_4 and $U_4^{(p)}$ against z^* are drawn in FIG IX and FIG XIII. The graphs of Φ etc are very similar to those of U_4 so they were not plotted.

To show the individual vortices formed it was decided to plot the streamlines given by Φ = const and $\varepsilon = 0.5$ in FIG XIV for various values of this const. The streamlines given by $\eta = 0.5$ and $\varepsilon = 0.1$ were also plotted for a range of z^* and the same values of the constant and are given in FIG X . Compare these streamlines with those of the linear parallel wall case given in FIG II.

These figures show the individual cell shapes of the vortices decaying away quite rapidly for $\varepsilon = 0.5$ as $z^* \to \pm\infty$.

4.4b <u>The linear case with $\eta = 0.95$</u>

Here the critical values of the wavenumber and Taylor number were

$$\lambda_c = 3.12735 \quad \text{and} \quad T_c = 1754.96 . \tag{4.4.9}$$

The following constants were found to be

$$T_2 = 256.5062, \; a = 10.0575, \; r_{1i} = -0.8552, \; r_{2i} = 0.0180 \tag{4.4.10}$$

and for the case with $j = 1$ and $n = 0$ the value of r_3 was

$$r_{3i} = 1.1348 . \tag{4.4.11}$$

The values of (4.4.10) are used to calculate the eigenvalues and the corresponding eigensolutions. The results are given in

TABLE VI. The eigensolutions $\psi_{nj}(z^*)$ are the same as in TABLE V but with a different value for ω .

The linear values of $\underline{u}_{11}$ and $\underline{g}_{21}$ for $\eta = 0.95$ and $x = -0.5$ to 0.5 are given in TABLE XVI, and the values of $\underline{g}_{21} / \underline{u}_{11}$ evaluated at $x = \pm \frac{1}{2}$ are given in TABLE XVII. The same procedure can be followed as in $\eta = 0.5$ so that different solutions for various values of ε, j and n can be found.

The graph of $\psi_{01}(z^*)$ for $\eta = 0.95$ is plotted in FIG V. A table of values of $\psi_{01}(z^*)$ is given in TABLE XVIII.

The value of ε considered for $\eta = 0.95$ is

$$\varepsilon = 0.1 . \tag{4.4.12}$$

Similar results to the earlier case are given in TABLES XIX to XXI and FIGURES XV and XVI .

The terminating values of T_{Lcrit}, η_L and λ_{Lcrit} for all components mentioned are

$$T_{Lcrit} = 1741.76 \; , \; \eta_L = 0.9502 \; , \; \lambda_{Lcrit} = 3.12166. \tag{4.4.13}$$

4.5 Perturbation method for solving $\psi_N(z^*)$ and $S_N(z^*)$

The equation satisfied by $\psi_N(z^*)$ is now

$$\frac{d^2 \psi_N}{dz^{*2}} + \psi_N \left[\frac{T_2'}{T_2} - a + a \operatorname{sech}^2 \omega z^* \right] + b\psi_N^3 = 0 \tag{4.5.1}$$

with boundary conditions

$$\psi_N \to 0 \quad \text{as} \quad z^* \to \pm \infty . \tag{4.5.2}$$

We first obtain a perturbation series for ψ_N assuming that

$$T_2' = T_2^* [1 + \delta^2] \tag{4.5.3}$$

and

$$\psi_N(z^*) = \delta\psi_1(z^*) + \delta^3\psi_3(z^*) + \delta^5\psi_5(z^*) + \ldots, \tag{4.5.4}$$

where δ is a small parameter unrelated to ε and T_2^* is the eigenvalue of problem (3.7.19) with $j = 1$.

The procedure is standard, existence conditions at order δ^{2n+1} determines unknown constants in the solution at order δ^{2n-1} so we omit the details. The solution is

$$\psi_1(z^*) = A \operatorname{sech} \omega z^* \tag{4.5.5}$$

and

$$\psi_3(z^*) = \frac{A^3 b}{\omega^2} \left[\frac{1}{4} - \frac{1}{3} \log_e 2 + \frac{1}{3} \log_e(\operatorname{sech} \omega z^*) \right] . \operatorname{sech}\omega z^*, \tag{4.5.6}$$

where

$$A^2 = -3T_2^* / 2bT_2 . \tag{4.5.7}$$

We next expand

$$S_N(z^*) = \delta S_N^{(1)}(z^*) + \delta^3 S_N^{(3)}(z^*) + \ldots \tag{4.5.8}$$

and use the relationship

$$r_{3i,N} = (1 + \delta^2) r_{3i} \tag{4.5.9}$$

in equation (3.17.18) and again by standard methods (though in this case the work is more complicated) we find

$$S_{N,r}^{(1)} = S_{N,r}^{(3)} = 0. \tag{4.5.10}$$

and

$$S_{N,i} = \delta\psi_1(z^*)R(z^*) + \delta^3(R_2(z^*)\psi_1(z^*) + R(z^*)\psi_3(z^*)) + \ldots \tag{4.5.11}$$

where $S_N(z^*) = S_{N,r} + i\, S_{N,i}$ and $R(z^*)$ is given in (4.3.12).

The value of $R_2(z^*)$ is

$$R_2(z^*) = \frac{1}{2}\,[r_{3i}z^* + \frac{A^2}{2\omega}\;(r_{4i} - \frac{b}{3\omega^2}\,(r_{2i} - \frac{r_{1i}}{2}\,))\tanh\omega z^*\,]. \qquad (4.5.12)$$

The solutions for U_4, U_5, U_6 , Φ can now be given in the form of

$$\varepsilon\delta + \varepsilon\delta^3 + O(\varepsilon\delta^5) + \varepsilon^2\delta + \varepsilon^2\delta^2 + \varepsilon^2\delta^3 + O(\varepsilon^2\delta^4) + O(\varepsilon^3\delta) \text{ etc.} \qquad (4.5.13)$$

The non-linear wavenumbers for the velocity components were not plotted due to the complex nature of the solutions. The non-linear critical Taylor number, however, is given by

$$T_{Lcrit,N} = T_{Lcrit} + \varepsilon^2\delta^2 T_2^* \;, \qquad (4.5.14)$$

where T_{Lcrit} is defined in (4.3.17).

The reader is reminded the analysis of § 4.5 is only valid for the case with j = 1. A similar procedure can be carried out for other values of j but the expansion process becomes rather more complex. It was decided not to proceed any further for other values of j.

The solutions of the full non-linear equations for $\psi_N(z^*)$ were also obtained numerically for the same values of T_2' and for higher values of T_2' where the perturbation technique is not valid. Due to the time factor and computational difficulties it was decided not to calculate the numerical solution for $S_{Nj}(z^*)$ for any values of D.

4.6a The non-linear case with $n = 0.5$ and $j = 1$

Here we use the values for λ_c , T_c, T_2^* etc., given in § 4.4a .

From our computing methods the following additional constants, defined in (3.16.26) and by the imaginary part of (3.17.13), were

$$b = -13.7061 \quad \text{and} \quad r_{4i} = 172.4719. \tag{4.6.1}$$

The values of $f_{20,k}(x)$ and $f_{22,k}(x)$ $(k = 4,5,6)$ for $\eta = 0.5$ and $x = -0.5$ to 0.5 are given in TABLE XXII. The reader is reminded that

$$f_{20,4}(x) = f_{20,6}(x) = 0 \text{ for all } x. \tag{4.6.2}$$

These in conjunction with the values of $u_{11,k}(x)$ and $g_{21,k}(x)$ given in TABLE VII, the constants given in (4.6.1) and elsewhere in §4.4a and the functions $\psi_1(z^*)$, $\psi_3(z^*)$, $R(z^*)$ and $R_2(z^*)$ defined in (4.5.5), (4.5.6), (4.3.12) and (4.5.12) respectively, will enable the reader to find other non-linear solutions of U_4, U_5, U_6 and Φ for various values of ε and δ (with j fixed at 1). These non-linear solutions are given in (3.17.21) to (3.17.28).

From these constants etc., we calculated solutions for $\psi_N(z^*)$ both numerically using the full equation for ψ_N and by the perturbation method for D = 200, D = 600 and D = 1500, where D is given by

$$D = T_2' - T_2^* . \tag{4.6.3}$$

These results for $\psi_N(z^*)$ are given in TABLE XXIII and TABLE XXIV for the perturbation method and the numerical solution respectively. In TABLE XXIV there are also additional solutions of $\psi_N(z^*)$ for larger values of D. A set of values of the initial condition $\psi_N(0)$ against D is also given in TABLE XXV for the numerical solution only.

The graphs of the numerical solutions for $\psi_N(z^*)$, $\psi_N(0)$

against D are given in FIG XVII and FIG XVIII respectively.

The values of ε considered for $\eta = 0.5$ are

$$\varepsilon = 0.1 \text{ and } \varepsilon = 0.5 \ . \tag{4.6.4}$$

At this stage we note that $T_{Lcrit,N}$ is related to T_{Lcrit} by

$$T_{Lcrit,N} = T_{Lcrit} + \varepsilon^2 D \tag{4.6.5}$$

for the numerical and perturbation case. In the perturbation case D defines δ^2 ;

$$\delta^2 = \frac{D}{T_2^*} \ . \tag{4.6.6}$$

Using wholly the perturbation method we are able to obtain results for $T_{Lcrit,N}$ and the non-linear solutions for U_4, U_5, U_6, Φ with D = 200 and D = 600 for both values of ε considered. These non-linear solutions are summarized for various z* values in TABLE XXVI for $\varepsilon = 0.1$ and TABLE XXVII for $\varepsilon = 0.5$. Since all these solutions depend also on the x co-ordinate we shall plot and tabulate for x = 0 only.

The graph of the non-linear solutions for U_4 is drawn for $\eta = 0.5$ and $\varepsilon = 0.5$ for two values of D. These are given in FIG XIX. The graphs of U_4 are not drawn for $\varepsilon = 0.1$ due to the more rapid oscillations of the solutions.

The terminating values of $T_{Lcrit,N}$, $\varepsilon = 0.1$, D = 200 and D = 600 are

$$T_{Lcrit,N} = 3076.94 \text{ and } T_{Lcrit,N} = 3080.94 \tag{4.6.7}$$

respectively. The terminating values for $\varepsilon = 0.5$, D = 200 and D = 600 are

$$T_{Lcrit,N} = 2226.59 \text{ and } T_{Lcrit,N} = 2326.59 \tag{4.6.8}$$

respectively.

4.6b The non-linear case with $\eta = 0.95$ and $j = 1$

The values for λ_c, T_c, T_2^* etc are given in §4.4b for $\eta = 0.95$. From our computing methods we find

$$b = -2.9013 \quad \text{and} \quad r_{4i} = 14.7236 \quad , \tag{4.6.9}$$

with these values we can calculate the solutions for $\psi_N(z^*)$ both numerically and by using our perturbation expansion.

To save time we can compute solutions for $\psi_N(z^*)$ using our previous solutions for $\psi_N(z^*)$ and $\eta = 0.5$. This is done in the following manner.

If $\psi_{N,0.5}(z^*)$ and $\psi_{N,0.95}(y)$ represent the solutions for $\eta = 0.5$ and $\eta = 0.95$ they are given by the following differential equations

$$\frac{d^2\psi_{N,0.5}}{dz^{*2}} + \psi_{N,0.5}\left[\frac{T_2'}{T_2} - a + a\,\mathrm{sech}^2\omega z^*\right] + b\psi_{N,0.5}^3 = 0 \tag{4.6.10}$$

$$\frac{d^2\psi_{N,0.95}}{d^2y} + \psi_{N,0.95}\left[\frac{P_2'}{P_2} - a_1 + a_1\,\mathrm{sech}^2\omega_1 y\right] + b_1\psi_{N,0.95}^3 = 0 \tag{4.6.11}$$

where

$$T_2' = T_2^* + D \quad \text{and} \quad P_2' = P_2^* + D_1 \ , \tag{4.6.12}$$

respectively.

When we substitute

$$z^* = \omega_1 y/\omega \quad \text{and} \quad \psi_{N,0.5} = E\psi_{N,0.95} \tag{4.6.13}$$

in (4.6.10) and compare the two equations we have for $\psi_{N,0.95}$ we have

$$\left(\frac{\omega_1}{\omega}\right)^2 a = a_1 , \quad \left(\frac{\omega_1}{\omega}\right)^2 \frac{T_2'}{T_2} = \frac{P_2'}{P_2} , \quad \left(\frac{\omega_1}{\omega}\right)^2 bE^2 = b_1 . \tag{4.6.14}$$

Therefore given a D and $\psi_{N,0.5}$ we can find D_1 and $\psi_{N,0.95}$ by using (4.6.14) , (4.6.13) and (4.6.12). A table of $\psi_{N,0.95}$ and D_1 are given in TABLE XXV using this method. The relationship between $\psi_N(0)$ and D is given in FIG XVIII. No graphs or tables of $\psi_{N,0.95}$ are given due to this relationship.

The reader is reminded the value of ε considered here is

$$\varepsilon = 0.1 . \tag{4.6.15}$$

The non-linear solutions for U_4, U_5, U_6, Φ are tabulated in TABLE XXIX for D_1 = 126.702 only. Due to the similarity with the results of η = 0.5 no graphs of these solutions are given.

The non-linear values of $f_{20,k}(x)$ and $f_{22,k}(x)$ can be found in TABLE XXVIII; these can be used along with other tables to find various solutions for different values of ε and δ .

The terminating value of $T_{Lcrit,N}$ for $\varepsilon = 0.1$ and D_1 = 126.702 is

$$T_{Lcrit,N} = 1743.03 . \tag{4.6.16}$$

η_L	T_{Lc}	λ_{Lc}
0.5	3099.78	3.16242
0.51	3034.68	3.1607
0.52	2972.83	3.1591
$0.5\dot{3}$	2895.02	3.1571
0.65[a]	2383.96	3.1425
0.75[a]	2102.17	3.1355
0.85[a]	1902.40	3.1302
0.95	1754.96	3.12735

[a] Results obtained from Roberts [3]

TABLE IV Critical values of T_{Lc}, λ_{Lc}.

j	n	ω	T_2^*	$\psi_{nj}(z^*)$
1	0	1.9354	1649.25	A sech z^*
2	0	1.1174	1099.50	A $\text{sech}^2 \omega z^*$
	1		2748.76	A sech ωz^* tanh ωz^*
3	0	0.7901	824.63	A $\text{sech}^3 \omega z^*$
	1		2199.00	A $\text{sech}^2 \omega z^*$ tanh ωz^*
	2		3023.63	A sech $\omega z^*[\tanh^2 \omega z^* - \frac{1}{5}]$

TABLE V Eigenvalues and eigensolutions for $\eta = 0.5$ and fixed j and n.

j	n	ω	T_2^*
1	0	2.2425	1289.90
2	0	1.2947	859.93
	1		2149.83
3	0	0.9155	644.95
	1		1719.87
	2		2364.81

TABLE VI Eigenvalues for $\eta = 0.95$ and fixed j and n.

x	$u^{(r)}_{11,4}$	$u^{(r)}_{11,5}$	$u^{(i)}_{11,6}$	$g^{(i)}_{21,4}$	$g^{(i)}_{21,5}$	$g^{(r)}_{21,6}$
-0.5	0	0	0	0	0	0
-0.4	-1.1964	0.0959	-6.6810	0.4627	-0.0006	-0.3610
-0.3	-3.4104	0.1811	-7.8681	1.2261	-0.0023	-0.0665
-0.2	-5.2953	0.2452	-5.9281	1.7787	-0.0033	0.2490
-0.1	-6.2380	0.2791	-2.6785	1.9711	-0.0026	0.3536
0	-6.1209	0.2793	0.6161	1.8385	-0.0003	0.2456
0.1	-5.1275	0.2490	3.1792	1.4868	0.0026	0.0364
0.2	-3.5975	0.1961	4.5998	1.0289	0.0046	-0.1357
0.3	-1.9354	0.1311	4.6744	0.5610	0.0050	-0.1733
0.4	-0.5758	0.0637	3.2381	0.1743	0.0039	-0.0760
0.5	0	0	0	0	0	0

TABLE VII Summary of results for the velocity components for $\eta = 0.5$ and $x = -0.5$ to 0.5.

x	$g^{(i)}_{21,4} / u^{(r)}_{11,4}$	$g^{(i)}_{21,5} / u^{(r)}_{11,5}$	$g^{(r)}_{21,6} / u^{(i)}_{11,6}$
-0.5	-0.4178	0	0.1016
0.5	-0.3256	0.0598	0.0094

TABLE VIII Values of $g_{21,j}/u_{11,j}$ ($j = 4,5,6$) for $\eta = 0.5$ evaluated at $x = -0.5$ and 0.5 .

z^*	$\psi_{01}(z^*)$	z^*	$\psi_{01}(z^*)$
0	0.5000	1.6	0.0451
0.2	0.4647	1.8	0.0307
0.4	0.3802	2.0	0.0208
0.6	0.2851	2.2	0.0141
0.8	0.2034	2.4	0.0096
1.0	0.1414	2.6	0.0065
1.2	0.0971	2.8	0.0044
1.4	0.0663	3.0	0.0030

TABLE IX Values of $\psi_{01}(z^*)$ for the case $\eta = 0.5$ and $j = 1$.

z^*	η_L	T_{Lcrit}	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$
0	0.5	3116.27	3.17627	3.18812	3.17690
0.5	0.5007	3093.17	3.16603	3.17125	3.16631
1.0	0.5012	3078.25	3.15941	3.16036	3.15947
1.5	0.5012	3075.44	3.15817	3.15831	3.15818

TABLE X Values of η_L, T_{Lcrit} and λ_{Lcrit} as given by equations (4.3.21), (4.3.20) and (4.3.19) for the case $\eta = 0.5$, $j = 1$, $\varepsilon = 0.1$ and $x = 0$.

z^*	η_L	T_{Lcrit}	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$
0	0.5	3209.85	3.50876	3.80488	3.52449
0.5	0.5181	2632.20	3.25265	3.38322	3.25959
1.0	0.5305	2259.24	3.08729	3.11098	3.08855
1.5	0.5329	2188.95	3.05612	3.05966	3.05631

TABLE XI Values of η_L, T_{Lcrit} and λ_{Lcrit} as given by the same equations above for the case $\eta = 0.5$, $j = 1$, $\varepsilon = 0.5$ and $x = 0$.

	$z^* = 0$			$z^* = 1.2$		
x	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$
-0.5	3.18067	3.19252	3.17687	3.15881	3.15925	3.15866
-0.4	3.17951	3.19136	3.17709	3.15876	3.15921	3.15867
-0.3	3.17849	3.19034	3.17734	3.15872	3.15917	3.15868
-0.2	3.17760	3.18945	3.17738	3.15869	3.15914	3.15868
-0.1	3.17686	3.18870	3.17722	3.15866	3.15911	3.15868
0	3.17627	3.18812	3.17690	3.15864	3.15909	3.15866
0.1	3.17588	3.18773	3.17648	3.15863	3.15907	3.15865
0.2	3.17574	3.18758	3.17598	3.15862	3.15907	3.15863
0.3	3.17588	3.18772	3.17542	3.15863	3.15907	3.15861
0.4	3.17636	3.18821	3.17488	3.15864	3.15909	3.15859
0.5	3.17722	3.18906	3.17463	3.15868	3.15912	3.15858

TABLE XII Variation in x for wavenumbers of Φ, U_4 and U_5 given by equations (4.3.19) and (4.3.17) for the case $\eta = 0.5$, $j = 1$, $\varepsilon = 0.1$ with $z^* = 0$ and $z^* = 1.2$.

	$z^* = 0$			$z^* = 1.0$		
x	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$
-0.5	3.61873	3.91485	3.52361	3.09609	3.11977	3.08848
-0.4	3.58968	3.88581	3.52913	3.09376	3.11745	3.08892
-0.3	3.56417	3.86030	3.53530	3.09172	3.11541	3.08941
-0.2	3.54203	3.83816	3.53634	3.08995	3.11364	3.08949
-0.1	3.52339	3.81951	3.53231	3.08846	3.11215	3.08917
0	3.50876	3.80488	3.52449	3.08729	3.11098	3.08855
0.1	3.49902	3.79515	3.51397	3.08651	3.11020	3.08771
0.2	3.49532	3.79144	3.50144	3.08621	3.10990	3.08670
0.3	3.49890	3.79503	3.48754	3.08650	3.11019	3.08559
0.4	3.51093	3.80706	3.47383	3.08746	3.11115	3.08449
0.5	3.53240	3.82852	3.46765	3.08918	3.11287	3.08400

TABLE XIII Variation in x for wavenumbers of Φ, U_4 and U_5 for the case $\eta = 0.5$, $j = 1$, $\varepsilon = 0.5$ with $z^* = 0$ and $z^* = 1.0$.

z^*	Φ	$\Phi^{(p)}$	U_4	$U_4^{(p)}$	U_5	$U_5^{(p)}$	U_6	$U_6^{(p)}$
0	0	0	-6.1209	-6.1209	0.2793	0.2793	0	0
0.05	-1.9262	-1.9354	0.1416	0.0637	-0.0049	-0.0029	-0.6131	-0.6161
0.1	0.0658	0.0403	6.0035	6.1196	-0.2740	-0.2793	0.0302	0.0128
0.15	1.8546	1.9346	-0.4062	-0.1912	0.0142	0.0087	0.5899	0.6158
0.2	-0.1243	-0.0806	-5.6725	-6.1156	0.2591	0.2791	-0.0565	-0.0257
0.25	-1.7237	-1.9329	0.6214	0.3186	-0.0219	-0.0145	-0.5477	-0.6153
1.0	-0.1806	-0.4002	-1.6277	-5.9886	0.0752	0.2733	-0.0706	-0.1274
1.05	-0.4726	-1.8894	0.6359	1.3279	-0.0251	-0.0606	-0.1478	-0.6014
1.1	0.1639	0.4396	1.3342	5.9610	-0.0617	-0.2720	0.0631	0.1399
1.15	0.3874	1.8803	-0.5676	-1.4521	0.0226	0.0663	0.1209	0.5985
1.2	-0.1472	-0.4787	-1.0903	-5.9307	0.0505	0.2706	-0.0560	-0.1524
1.25	-0.3166	-1.8703	0.5025	1.5756	-0.0202	-0.0719	-0.0986	-0.5953
2.5	0.0231	0.9629	0.0668	5.3098	-0.0032	-0.2423	0.0080	0.3065
2.55	0.0197	1.6689	-0.0721	-3.1001	0.0031	0.1415	0.0059	0.5312
2.6	-0.0197	-0.9976	-0.0533	-5.2452	0.0025	0.2394	-0.0068	-0.3176
2.65	-0.0158	-1.6481	0.0612	3.2094	-0.0026	-0.1465	-0.0047	-0.5246
2.7	0.0167	1.0320	0.0423	5.1784	-0.0020	-0.2363	0.0058	0.3285
2.75	0.0126	1.6266	-0.0518	-3.3172	0.0022	0.1514	0.0037	0.5178

TABLE XIV Solutions of U_4, U_5, U_6 and Φ, and the corresponding parallel wall solutions, for the case $\eta = 0.5$, $j = 1$, $\varepsilon = 0.1$ and $x = 0$.

z^*	Φ	$\Phi^{(p)}$	U_4	$U_4^{(p)}$	U_5	$U_5^{(p)}$	U_6	$U_6^{(p)}$
0	0	0	-6.1209	-6.1209	0.2793	0.2793	0	0
0.1	-1.2290	-1.1442	-4.3926	-4.9369	0.2094	0.2253	-0.4286	-0.3642
0.2	-1.7900	-1.8457	-0.3554	-1.8429	0.0427	0.0841	-0.5953	-0.5875
0.3	-1.4543	-1.8332	3.4730	1.9641	-0.1243	-0.0896	-0.4276	-0.5835
0.4	-0.5211	-1.1114	5.0610	5.0112	-0.2080	-0.2287	-0.0660	-0.3538
0.5	0.4535	0.0403	4.0161	6.1196	-0.1841	-0.2793	0.2670	0.0128
1.0	-0.3670	-0.0806	-1.6631	-6.1156	0.0768	0.2791	-0.1838	-0.0257
1.1	-0.5271	-1.2082	-0.2395	-4.7819	0.0225	0.2182	-0.2118	-0.3846
1.2	-0.4358	-1.8684	0.8879	-1.5982	-0.0256	0.0729	-0.1510	-0.5947
1.3	-0.2002	-1.8057	1.3649	2.2038	-0.0503	-0.1006	-0.0496	-0.5748
1.4	0.0469	-1.0445	1.1915	5.1532	-0.0486	-0.2352	0.0423	-0.3325
1.5	0.1415	-0.4859	0.6172	6.1090	-0.0297	-0.2788	0.0736	-0.1547
2.5	0.0484	0.2012	0.0784	6.0878	-0.0037	-0.2778	0.0193	0.0640
2.6	0.0453	1.3003	-0.0398	4.5340	0.0011	-0.2069	0.0170	0.4139
2.7	0.0270	1.8963	-0.1095	1.2262	0.0041	-0.0560	0.0091	0.6036
2.8	0.0037	1.7587	-0.1192	-2.5560	0.0048	0.1166	0.0003	0.5598
2.9	-0.0142	0.9407	-0.0828	-5.3494	0.0035	0.2441	-0.0061	0.2994
3.0	-0.0219	-0.2412	-0.0270	-6.0732	0.0013	0.2771	-0.0084	-0.0768

TABLE XV Solutions of U_4, U_5, U_6 and Φ etc., for $\eta = 0.5$, $j = 1, \varepsilon = 0.5$ and $x = 0$.

x	$u^{(r)}_{11,4}$	$u^{(r)}_{11,5}$	$u^{(i)}_{11,6}$	$g^{(i)}_{21,4}$	$g^{(i)}_{21,5}$	$g^{(r)}_{21,6}$
-0.5	0	0	0	0	0	0
-0.4	-1.1402	0.1003	-6.3586	0.4435	-0.0006	-0.3426
-0.3	-3.4611	0.1975	-7.8949	1.2623	-0.0028	-0.1119
-0.2	-5.7094	0.2790	-6.2321	1.9693	-0.0051	0.1663
-0.1	-7.1221	0.3307	-2.8419	2.3452	-0.0057	0.2811
0	-7.3688	0.3441	1.0068	2.3430	-0.0042	0.2111
0.1	-6.4768	0.3181	4.3089	2.0173	-0.0013	- 0.0431
0.2	-4.7423	0.2591	6.3413	1.4734	0.0018	-0.0950
0.3	-2.6480	0.1786	6.6026	0.8390	0.0036	-0.1040
0.4	-0.8133	0.0892	4.6724	0.2693	0.0030	0.0043
0.5	0	0	0	0	0	0

TABLE XVI Summary of results for the velocity components for η = 0.95 and x = -0.5 to 0.5.

x	$g^{(i)}_{21,4}/u^{(r)}_{11,4}$	$g^{(i)}_{21,5}/u^{(r)}_{11,5}$	$g^{(r)}_{21,6}/u^{(i)}_{11,6}$
-0.5	-0.4182	0	0.0985
0.5	-0.3546	0.0402	0.0349

TABLE XVII Values of $g_{21,j}/u_{11,j}$ (j = 4,5,6) for η= 0.95 at x = -0.5 and x = 0.5.

z^*	$\psi_{01}(z^*)$	z^*	$\psi_{01}(z^*)$
0	0.5000	1.6	0.0276
0.2	0.4536	1.8	0.0177
0.4	0.3496	2.0	0.0113
0.6	0.2439	2.2	0.0072
0.8	0.1618	2.4	0.0046
1.0	0.1050	2.6	0.0029
1.2	0.0675	2.8	0.0019
1.4	0.0432	3.0	0.0012

TABLE XVIII Values of $\psi_{01}(z^*)$ for the case $\eta = 0.95$ and $j = 1$.

z^*	η_L	T_{Lcrit}	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$
0	0.9500	1767.86	3.13516	3.15124	3.13587
0.5	0.9502	1750.82	3.12635	3.13193	3.12659
1.0	0.9502	1742.91	3.12226	3.12297	3.12229
1.5	0.9502	1741.88	3.12173	3.12180	3.12173

TABLE XIX Values of η_L, T_{Lcrit} and λ_{Lcrit} for the case $\eta = 0.95$, $j = 1, \varepsilon = 0.1$ and $x = 0$.

	$z^* = 0$			$z^* = 1.0$		
x	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$
-0.5	3.14020	3.15628	3.13525	3.12248	3.12319	3.12226
-0.4	3.13873	3.15481	3.13556	3.12242	3.12313	3.12228
-0.3	3.13751	3.15359	3.13598	3.12236	3.12307	3.12229
-0.2	3.13652	3.15260	3.13617	3.12232	3.12303	3.12230
-0.1	3.13573	3.15181	3.13612	3.12228	3.12299	3.12230
0	3.13516	3.15124	3.13587	3.12226	3.12297	3.12229
0.1	3.13483	3.15091	3.13545	3.12224	3.12295	3.12227
0.2	3.13480	3.15088	3.13490	3.12224	3.12295	3.12225
0.3	3.13510	3.15118	3.13424	3.12226	3.12297	3.12222
0.4	3.13582	3.15190	3.13355	3.12229	3.12300	3.12219
0.5	3.13701	3.15309	3.13323	3.12234	3.12305	3.12217

TABLE XX Variation in x for wavenumbers of Φ, U_4 and U_5 for $\eta = 0.95$, $j = 1$, $\varepsilon = 0.1$ with $z^* = 0$ and $z^* = 1$.

z^*	Φ	$\Phi^{(p)}$	U_4	$U_4^{(p)}$	U_5	$U_5^{(p)}$	U_6	$U_6^{(p)}$
0	0	0	-7.3688	-7.3688	0.3441	0.3441	0	0
0.05	-2.3415	-2.3562	0.0351	-0.0525	0.0010	0.0025	-1.0005	-1.0067
0.1	-0.0147	-0.0336	7.1890	7.3680	-0.3356	-0.3440	0.0040	-0.0143
0.15	2.2290	2.3557	-0.0956	0.1574	-0.0028	-0.0074	0.9527	1.0065
0.2	0.0269	0.0671	-6.6912	-7.3658	0.3122	0.3439	-0.0067	0.0287
0.25	-2.0291	-2.3547	0.1323	-0.2623	0.0041	0.0122	-0.8677	-1.0061
1.0	0.0261	0.3345	-1.5671	-7.2942	0.0725	0.3406	0.0014	0.1429
1.05	-0.4447	-2.3299	0.0204	-1.0979	0.0034	0.0513	-0.1913	-0.9955
1.1	-0.0227	-0.3676	1.2596	7.2785	-0.0582	-0.3399	-0.0019	-0.1571
1.15	0.3568	2.3247	-0.0116	1.2015	-0.0029	-0.0561	0.1536	0.9933
1.2	0.0196	0.4008	-1.0110	-7.2614	0.0467	0.3391	0.0021	0.1712
1.25	-0.2859	-2.3190	0.0054	-1.3050	0.0025	0.0609	-0.1232	-0.9908
2.5	-0.0022	-0.8214	0.0566	6.9066	-0.0026	-0.3225	-0.0006	-0.3509
2.55	0.0158	2.2025	0.0029	2.6178	-0.0003	-0.1222	0.0069	0.9411
2.6	0.0018	0.8527	-0.0454	-6.8693	0.0021	0.3208	0.0005	0.3643
2.65	-0.0127	-2.1904	-0.0025	-2.7156	0.0002	0.1268	-0.0055	-0.9359
2.7	-0.0015	-0.8839	0.0364	6.8306	-0.0017	-0.3189	-0.0004	-0.3777
2.75	0.0101	2.1778	0.0022	2.8129	-0.0002	-0.1313	0.0044	0.9305

TABLE XXI Solutions of U_4, U_5, U_6 and Φ, and the corresponding parallel wall solutions for $\eta = 0.95$, $j = 1$, $\varepsilon = 0.1$ and $x = 0$.

x	$f^{(r)}_{22,4}$	$f^{(r)}_{22,5}$	$f^{(i)}_{22,6}$	$f^{(r)}_{20,5}$
-0.5	0	0	0	0
-0.4	-0.8207	0.0367	-2.3157	-0.3006
-0.3	-2.4049	0.0673	-2.8936	-0.5053
-0.2	-3.8469	0.0843	-2.3143	-0.5387
-0.1	-4.6349	0.0862	-1.1329	-0.4020
0	-4.6268	0.0766	0.1081	-0.1784
0.1	-3.9566	0.0603	1.0625	0.0239
0.2	-2.8782	0.0420	1.6205	0.1335
0.3	-1.6504	0.0251	1.7827	0.1405
0.4	-0.5418	0.0111	1.4235	0.0806
0.5	0	0	0	0

TABLE XXII Results for the velocity components for $\eta = 0.5$ and $x = -0.5$ to 0.5.

z^*	$\psi_N(z^*)_{D=200}$	$\psi_N(z^*)_{D=600}$	$\psi_N(z^*)_{D=1500}$
0	0.2222	0.3822	0.5948
0.2	0.2074	0.3600	0.5718
0.4	0.1718	0.3053	0.5102
0.6	0.1310	0.2405	0.4282
0.8	0.0953	0.1812	0.3436
1.0	0.0677	0.1332	0.2674
1.2	0.0475	0.0966	0.2039
1.4	0.0331	0.0695	0.1532
1.6	0.0230	0.0497	0.1139
1.8	0.0159	0.0355	0.0840
2.0	0.0110	0.0252	0.0616

TABLE XXIII Values of $\psi_N(z^*)$ obtained by the perturbation expansion for $\eta = 0.5$, $j = 1$, $D = 200$, 600 and 1500.

z^*	$\psi_N(z^*)_{D=200}$	$\psi_N(z^*)_{D=600}$	$\psi_N(z^*)_{D=1500}$	$\psi_N(z^*)_{D=2000}$	$\psi_N(z^*)_{D=3000}$
0	0.2222	0.3820	0.5932	0.6770	0.8076
0.2	0.2074	0.3595	0.5670	0.6513	0.7830
0.4	0.1718	0.3046	0.5027	0.5887	0.7237
0.6	0.1311	0.2406	0.4267	0.5162	0.6570
0.8	0.0955	0.1827	0.3564	0.4511	0.5999
1.0	0.0679	0.1361	0.2976	0.3993	0.5572
1.2	0.0477	0.1005	0.2501	0.3598	0.5277
1.4	0.0333	0.0739	0.2117	0.3304	0.5079
1.6	0.0233	0.0542	0.1803	0.3086	0.4951
1.8	0.0162	0.0397	0.1541	0.2923	0.4869
2.0	0.0113	0.0290	0.1319	0.2802	0.4816

TABLE XXIV Values of $\psi_N(z^*)$ given by the numerical solution of the full equation for $\eta = 0.5$, $j = 1$, $D = 200, 600, 1500, 2000$ and 3000.

D	$\psi_{0.5}(0)$	D_1	$\psi_{0.95}(0)$
100	0.1575	78.21	0.3966
162	0.2	126.70	0.5037
200	0.2222	156.42	0.5596
367	0.3	287.03	0.7555
600	0.382	469.27	0.9620
659	0.4	515.41	1.0073
1000	0.4895	782.11	1.2327
1045	0.5	817.31	1.2592
1500	0.5932	1173.17	1.4939
2000	0.6770	1564.22	1.7048
3000	0.8076	2346.33	2.0339
4000	0.9140	3128.44	2.3019
5000	1.0067	3910.55	2.5353

TABLE XXV Comparison of D and $\psi_N(0)$ from the numerical solution for $\eta = 0.5$ and $\eta = 0.95$.

	D = 200				D = 600			
z*	Φ	U_4	U_5	U_6	Φ	U_4	U_5	U_6
0	0.0000	-0.2766	0.0124	0.0000	0.0000	-0.4817	0.0213	0.0000
0.05	-0.0856	0.0109	-0.0005	-0.0272	-0.1472	0.0354	-0.0016	-0.0469
0.1	0.0030	0.2627	-0.0122	0.0019	0.0119	0.4467	-0.0210	0.0053
0.15	0.0827	-0.0141	0.0009	0.0263	0.1428	-0.0493	0.0027	0.0453
0.2	-0.0058	-0.2572	0.0115	-0.0036	-0.0228	-0.4497	0.0199	-0.0098
0.25	-0.0770	0.0317	-0.0017	-0.0244	-0.1333	0.1047	-0.0051	-0.0424
1.0	-0.0087	-0.0783	0.0035	-0.0042	-0.0264	-0.1484	0.0067	-0.0109
1.05	-0.0227	0.0306	-0.0016	-0.0069	-0.0431	0.0869	-0.0043	-0.0132
1.1	0.0079	0.0644	-0.0029	0.0037	0.0234	0.1231	-0.0056	0.0097
1.15	0.0189	-0.0271	0.0014	0.0057	0.0364	-0.0760	0.0037	0.0111
1.2	-0.0072	-0.0535	0.0024	-0.0033	-0.0210	-0.1041	0.0047	-0.0086
1.25	-0.0155	0.0246	-0.0012	-0.0047	-0.0303	0.0681	-0.0033	-0.0092
2.5	0.0013	0.0038	-0.0002	0.0005	0.0035	0.0083	-0.0004	0.0013
2.55	0.0011	-0.0040	0.0002	0.0003	0.0025	-0.0109	0.0005	0.0007
2.6	-0.0011	-0.0030	0.0001	-0.0004	-0.0030	-0.0067	0.0003	-0.0011
2.65	-0.0009	0.0034	-0.0002	-0.0002	-0.0020	0.0093	-0.0004	-0.0005
2.7	0.0009	0.0024	-0.0001	0.0004	0.0026	0.0054	-0.0002	0.0010
2.75	0.0007	-0.0029	0.0001	0.0002	0.0016	-0.0080	0.0004	0.0004

TABLE XXVI Non-linear solutions of U_4, U_5, U_6 and Φ, for the case $\eta = 0.5$, $j = 1, \varepsilon = 0.1$, $x = 0$, D = 200 and D = 600.

	D = 200				D = 600			
z*	Φ	U_4	U_5	U_6	Φ	U_4	U_5	U_6
0	0.0000	-1.4751	0.0618	0.0000	0.0000	-2.6844	0.1058	0.0000
0.5	-0.1698	-1.3513	0.0563	-0.0604	-0.4031	-2.3592	0.0923	-0.1300
0.1	-0.3089	-1.0085	0.0410	-0.1093	-0.7137	-1.4697	0.0549	-0.2314
0.15	-0.3948	-0.5228	0.0191	-0.1379	-0.8674	-0.2427	0.0023	-0.2829
0.2	-0.4189	0.0071	-0.0054	-0.1420	-0.8441	1.0314	-0.0541	-0.2753
0.25	-0.3855	0.4875	-0.0282	-0.1223	-0.6676	2.0882	-0.1030	-0.2121
1.0	-0.0882	-0.4074	0.0179	-0.0650	-0.4097	-0.7811	0.0339	-0.1960
1.05	-0.1174	-0.2367	0.0089	-0.0690	-0.4370	-0.2435	0.0073	-0.1954
1.1	-0.1291	-0.0636	0.0001	-0.0652	-0.4145	0.2490	-0.0166	-0.1751
1.15	-0.1246	0.0916	-0.0075	-0.0552	-0.3516	0.6482	-0.0356	-0.1399
1.2	-0.1070	0.2141	-0.0132	-0.0408	-0.2612	0.9240	-0.0484	-0.0955
1.25	-0.0804	0.2954	-0.0167	-0.0243	-0.1571	1.0654	-0.0546	-0.0477
2.5	0.0130	0.0220	-0.0010	0.0067	0.0440	0.0491	-0.0022	0.0198
2.55	0.0135	0.0049	-0.0001	0.0064	0.0433	-0.0024	0.0004	0.0186
2.6	0.0126	-0.0102	0.0007	0.0056	0.0384	-0.0461	0.0025	0.0158
2.65	0.0105	-0.0219	0.0012	0.0043	0.0303	-0.0788	0.0041	0.0119
2.7	0.0076	-0.0297	0.0016	0.0028	0.0205	-0.0989	0.0051	0.0073
2.75	0.0044	-0.0333	0.0017	0.0012	0.0100	-0.1062	0.0053	0.0028

TABLE XXVII Non-linear solutions of U_4, U_5, U_6 and Φ, for the case $\eta = 0.5$, $j = 1$, $\varepsilon = 0.5$, $x = 0$, D = 200 and D = 600.

x	$f^{(r)}_{22,4}$	$f^{(r)}_{22,5}$	$f^{(i)}_{22,6}$	$f^{(r)}_{20,5}$
-0.5	0	0	0	0
-0.4	-0.3346	0.0179	-0.9511	-0.1198
-0.3	-1.0622	0.0347	-1.2979	-0.2043
-0.2	-1.8457	0.0459	-1.1598	-0.2067
-0.1	-2.4218	0.0492	-0.6754	-0.1149
0	-2.6349	0.0453	-0.0387	0.0309
0.1	-2.4502	0.0367	0.5711	0.1620
0.2	-1.9266	0.0261	0.0363	0.2222
0.3	-1.1824	0.0157	0.2699	0.1965
0.4	-0.4102	0.0071	0.0894	0.1095
0.5	0	0	0	0

TABLE XXVIII Results for the velocity components for $\eta = 0.95$ and $x = -0.5$ to 0.5.

	D = 126.702			
z*	Φ	U_4	U_5	U_6
0	0.0000	-0.7561	0.0350	0.0000
0.05	-0.2361	0.0091	-0.0002	-0.1008
0.1	-0.0060	0.7124	-0.0336	0.0013
0.15	0.2250	0.0239	0.0000	0.0963
0.2	0.0112	-0.6879	0.0319	-0.0023
0.25	-0.2059	-0.0075	-0.0003	-0.0882
1.0	0.0076	-0.1684	0.0079	-0.0008
1.05	-0.0479	-0.0112	0.0001	-0.0210
1.1	-0.0063	0.1357	-0.0064	0.0006
1.15	0.0387	0.0103	-0.0001	0.0170
1.2	0.0053	-0.1104	0.0052	-0.0004
1.25	-0.0313	0.0084	0.0001	-0.0138
2.5	-0.0005	0.0069	-0.0003	-0.0000
2.55	0.0019	0.0009	-0.0000	0.0009
2.6	0.0004	-0.0055	0.0003	0.0000
2.65	-0.0016	-0.0008	0.0000	-0.0007
2.7	-0.0003	0.0045	-0.0002	-0.0000
2.75	0.0013	0.0006	-0.0000	0.0006

TABLE XXIX Non-linear solutions of U_4, U_5, U_6 and Φ for the case $\eta = 0.95$, $j = 1$, $\varepsilon = 0.1$, $x = 0$ and $D = 126.702$.

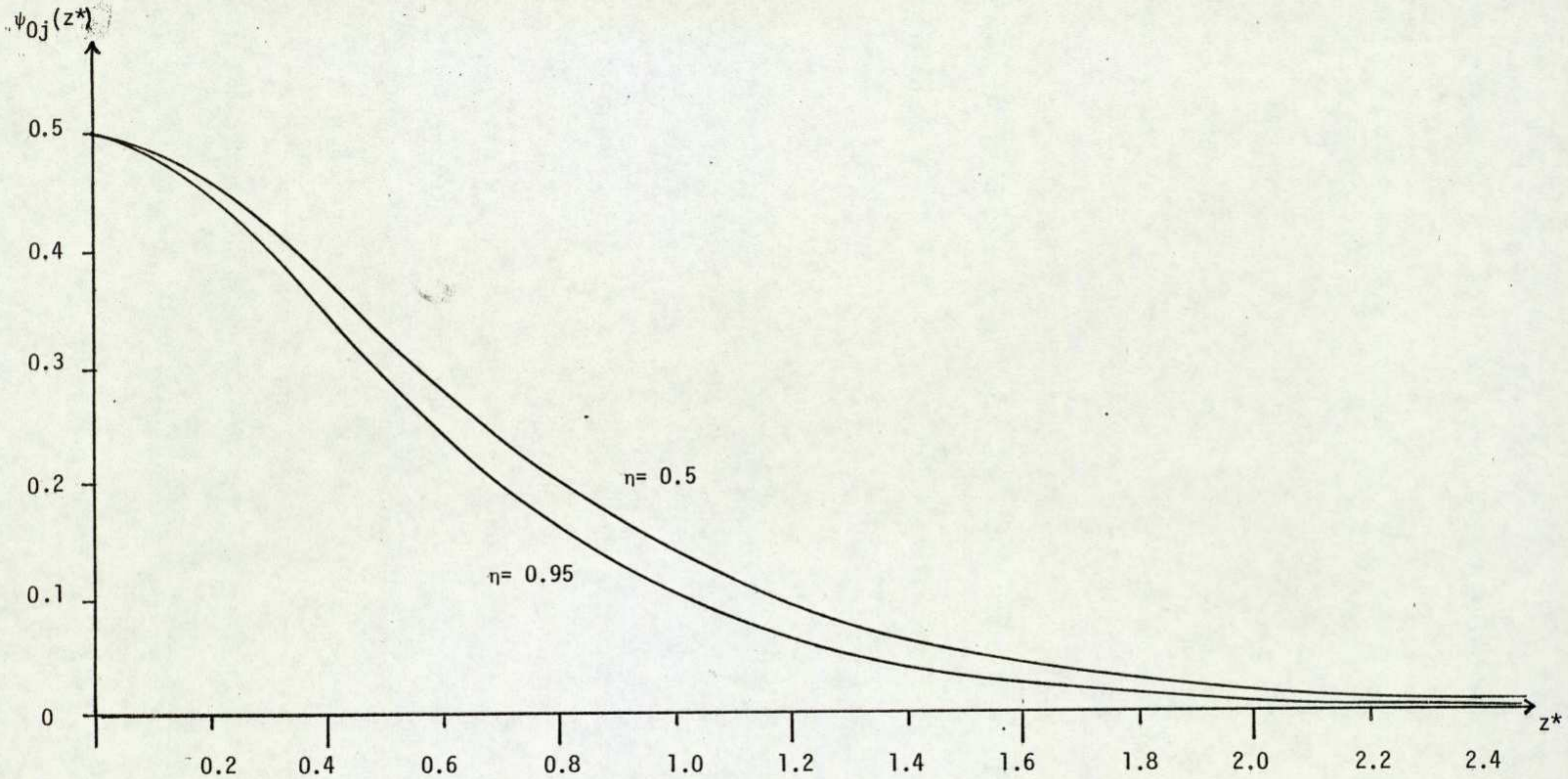

FIG V Graph of $\psi_{01}(z^*)$ for $\eta = 0.95$ and $\eta = 0.5$ with $j = 1$.

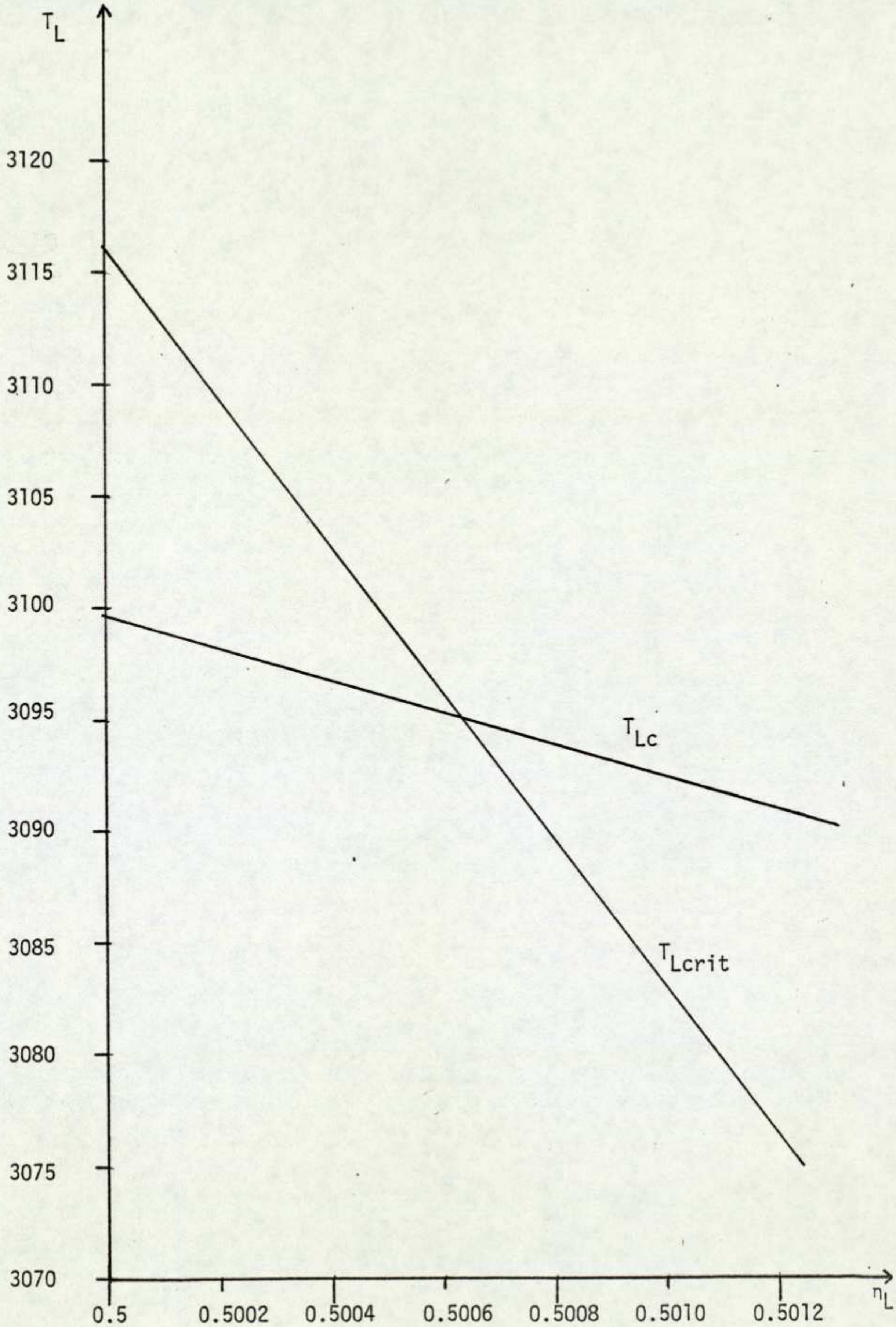

FIG VI Graph of T_{Lc}, T_{Lcrit} against η_L for $\eta = 0.5$, $j = 1$ and $\varepsilon = 0.1$.

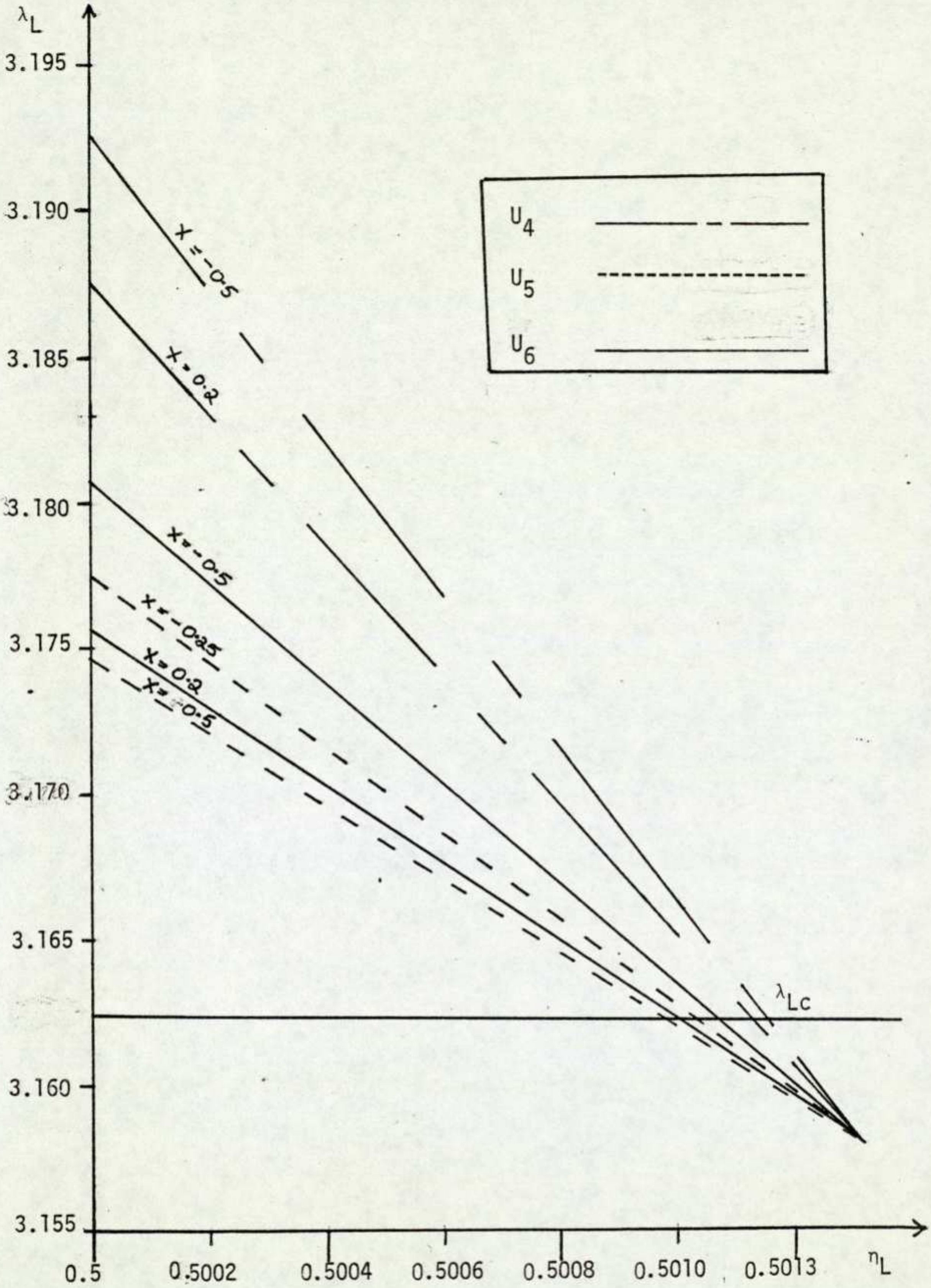

FIG VII Graph of λ_L, λ_{Lcrit} for U_4, U_5 and Φ against η_L, for $\eta = 0.5$, selected x values, $j = 1$ and $\varepsilon = 0.1$.

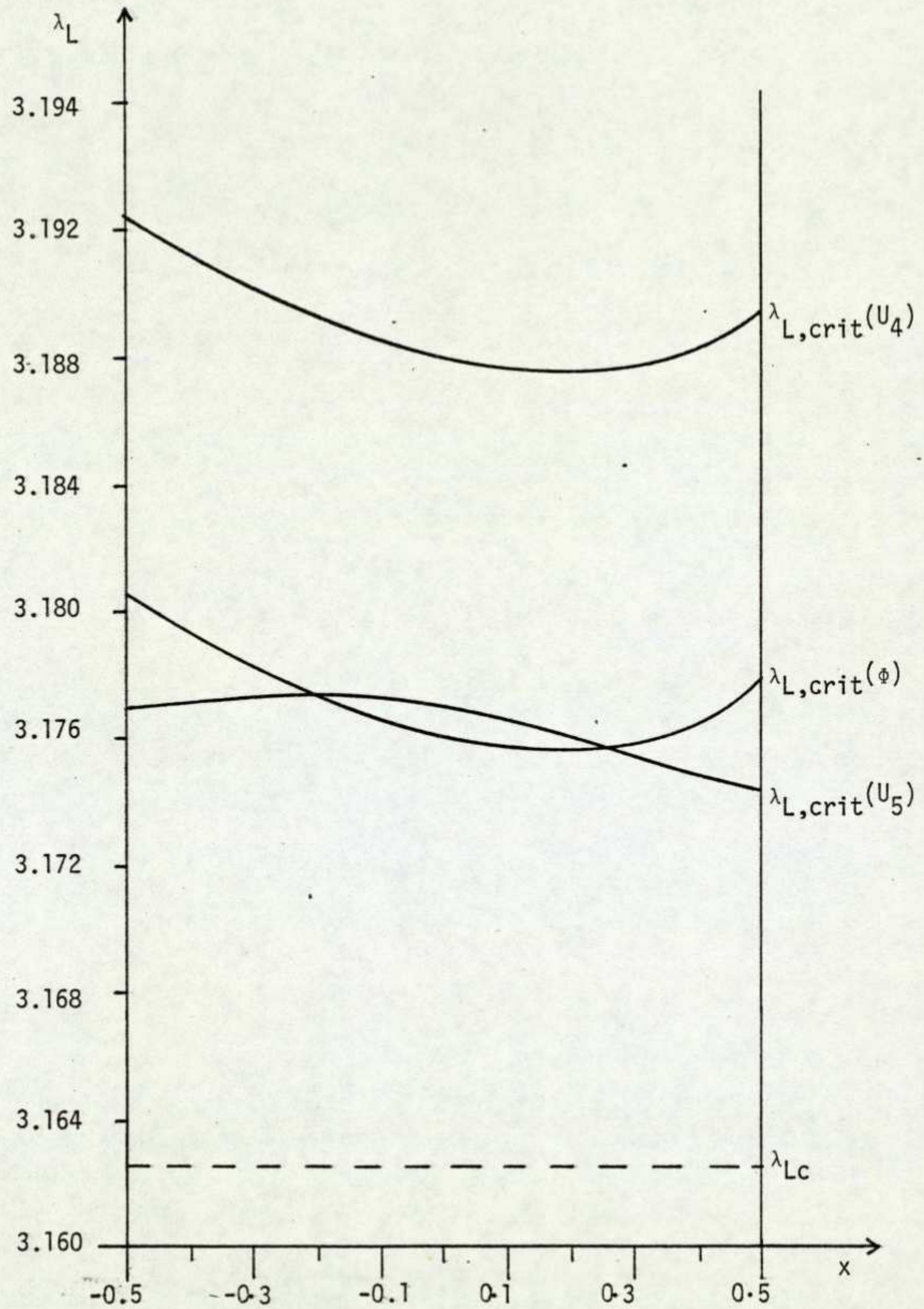

FIG VIII Variation in $\lambda_{L,crit}$ for U_4, U_5 and Φ against x for $\eta = 0.5$, $z^* = 0$, $j = 1$ and $\varepsilon = 0.1$.

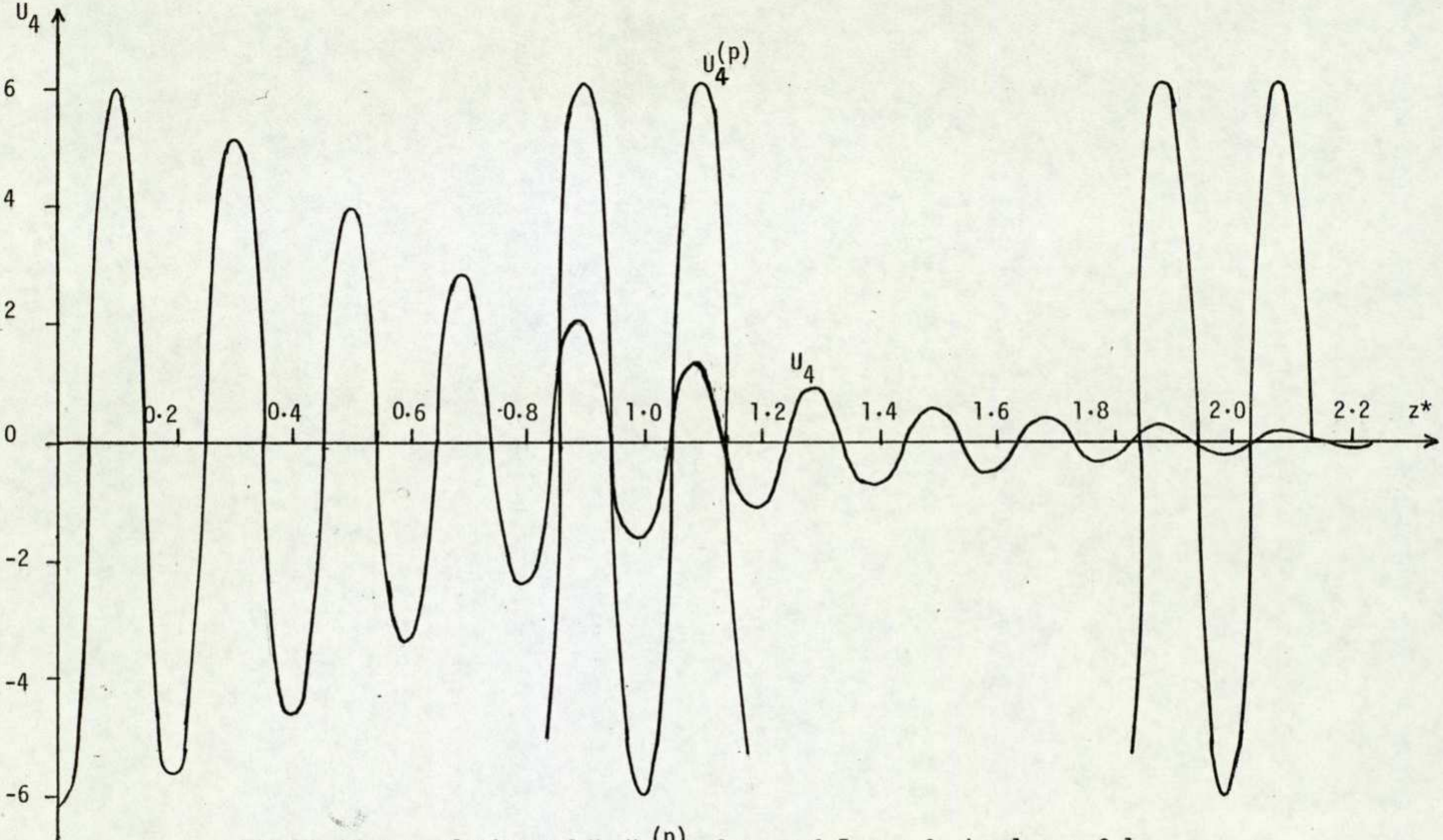

FIG IX Linear solutions of $U_4, U_4^{(p)}$ for $\eta = 0.5$, $x = 0$, $j = 1$, $\varepsilon = 0.1$.

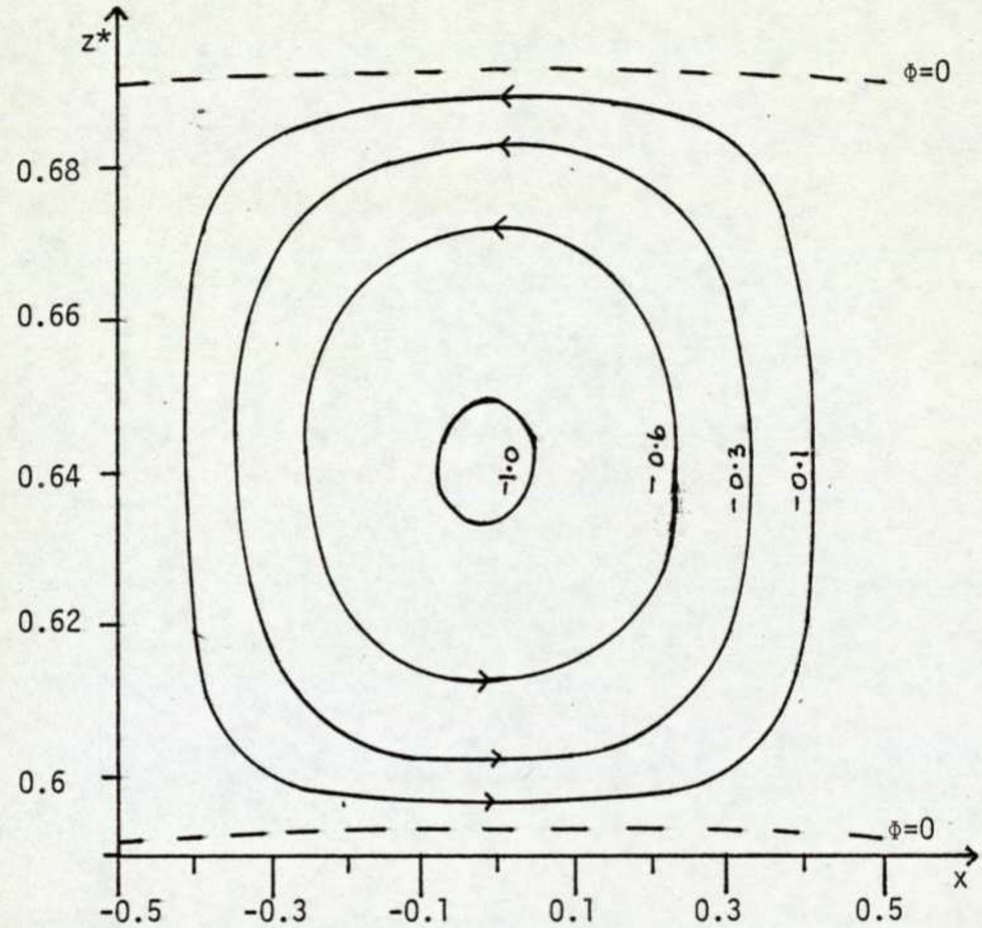

FIG Xa Seventh vortex streamlines for η = 0.5, j = 1 and ε = 0.1 .

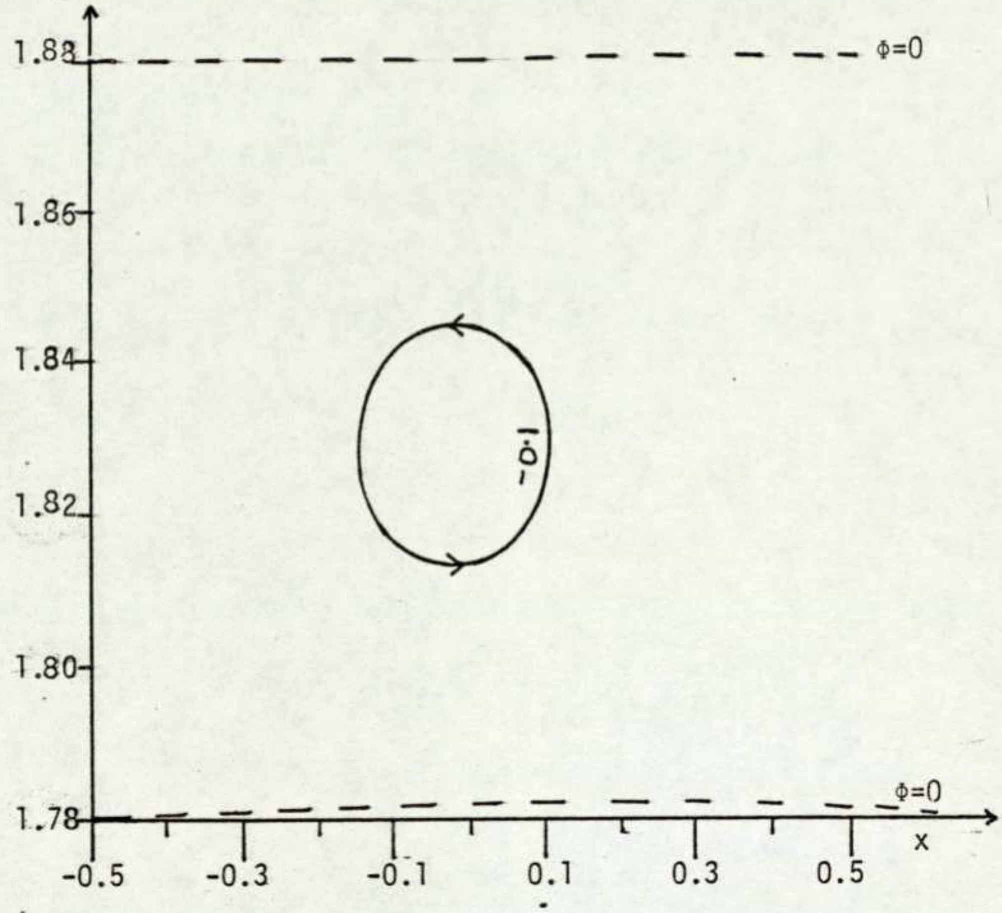

FIG Xc Nineteenth vortex streamlines for $\eta = 0.5$, $j = 1, \varepsilon = 0.1$ and various values of c.

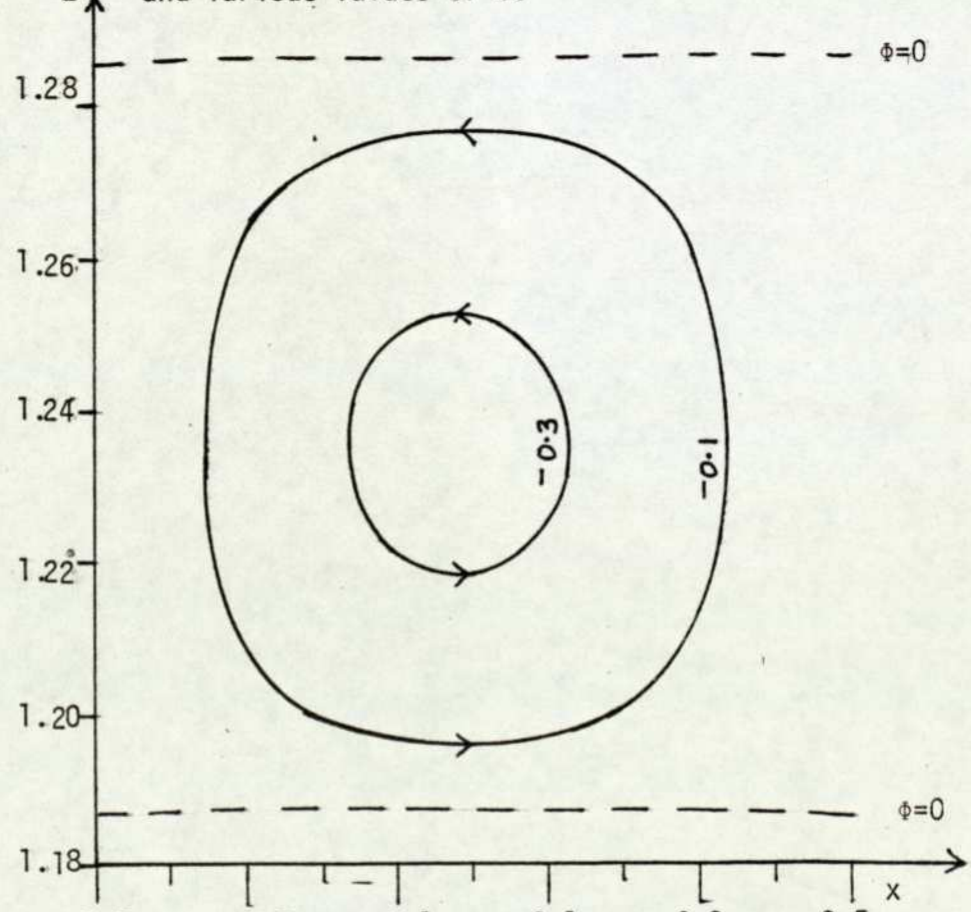

FIG Xb Thirteenth vortex streamlines for $\eta = 0.5$, $j = 1$ and $\varepsilon = 0.1$.

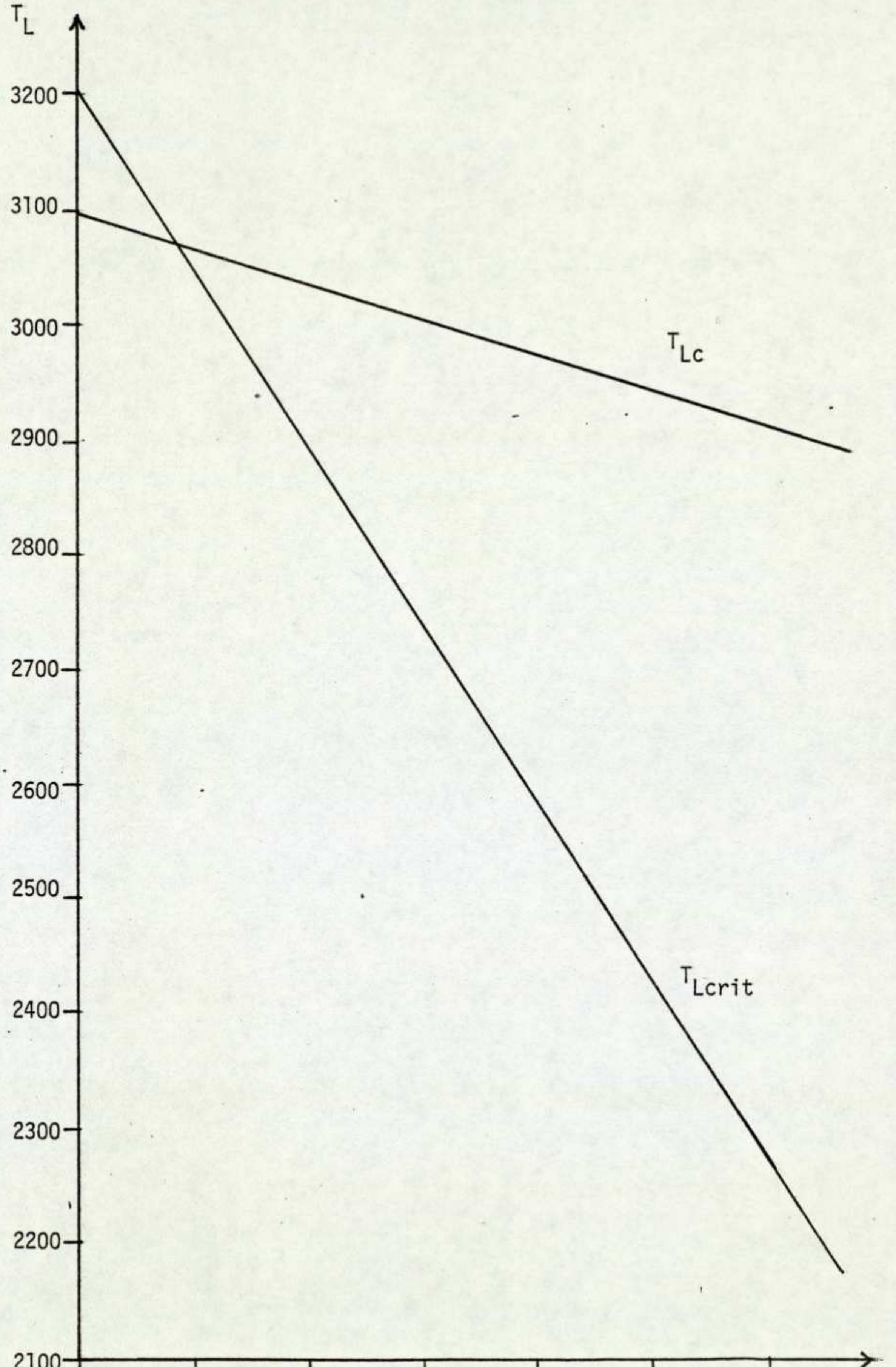

FIG XI Graph of T_{Lc} , T_{Lcrit} against η_L for = 0.5, j = 1 and ε= 0.5.

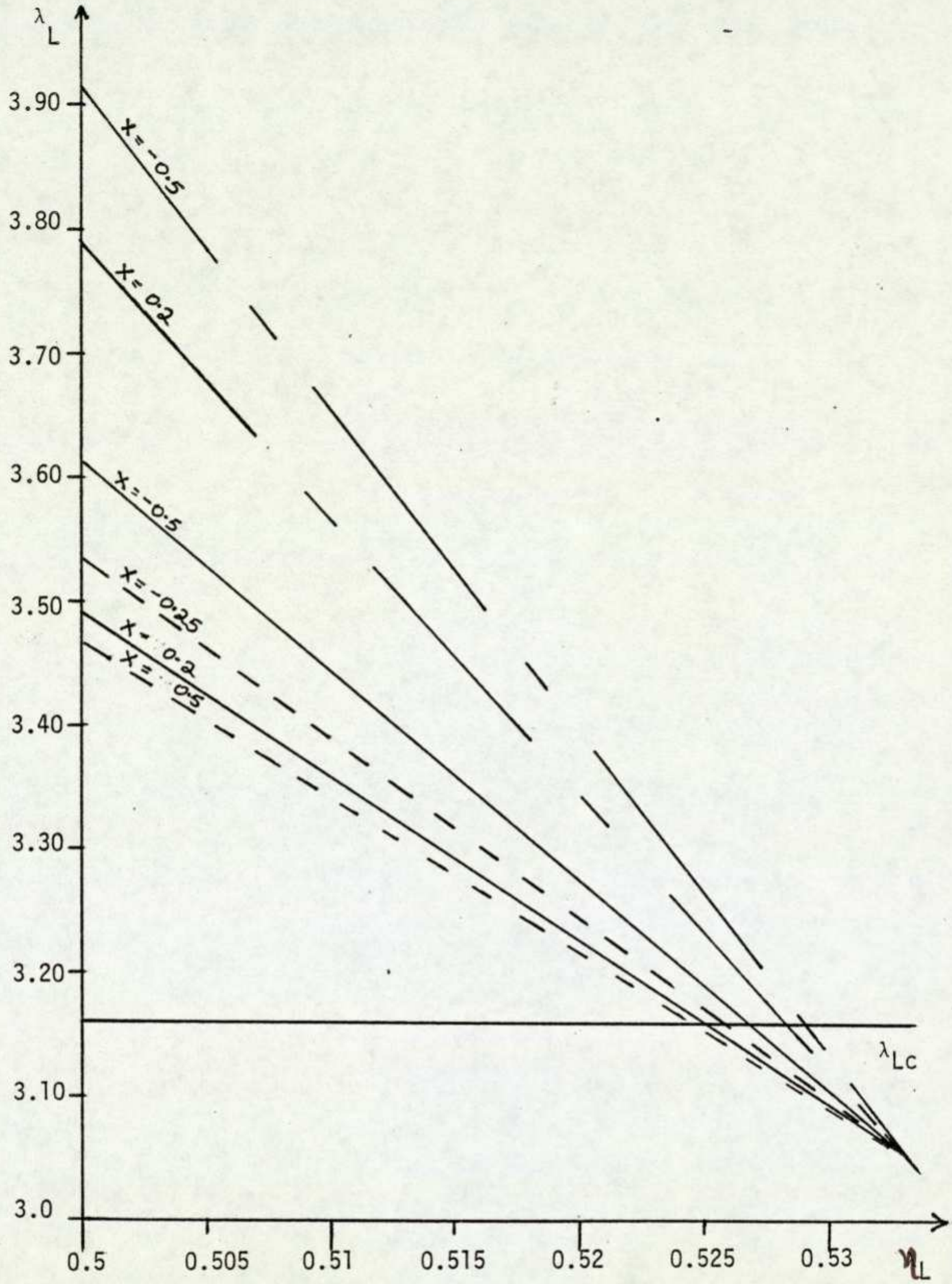

FIG XII Graph of λ_{Lc}, λ_{Lcrit} for U_4, U_5 and Φ against η_L, for $\eta = 0.5$, selected x values, $j = 1$ and $\varepsilon = 0.5$. See FIG VII.

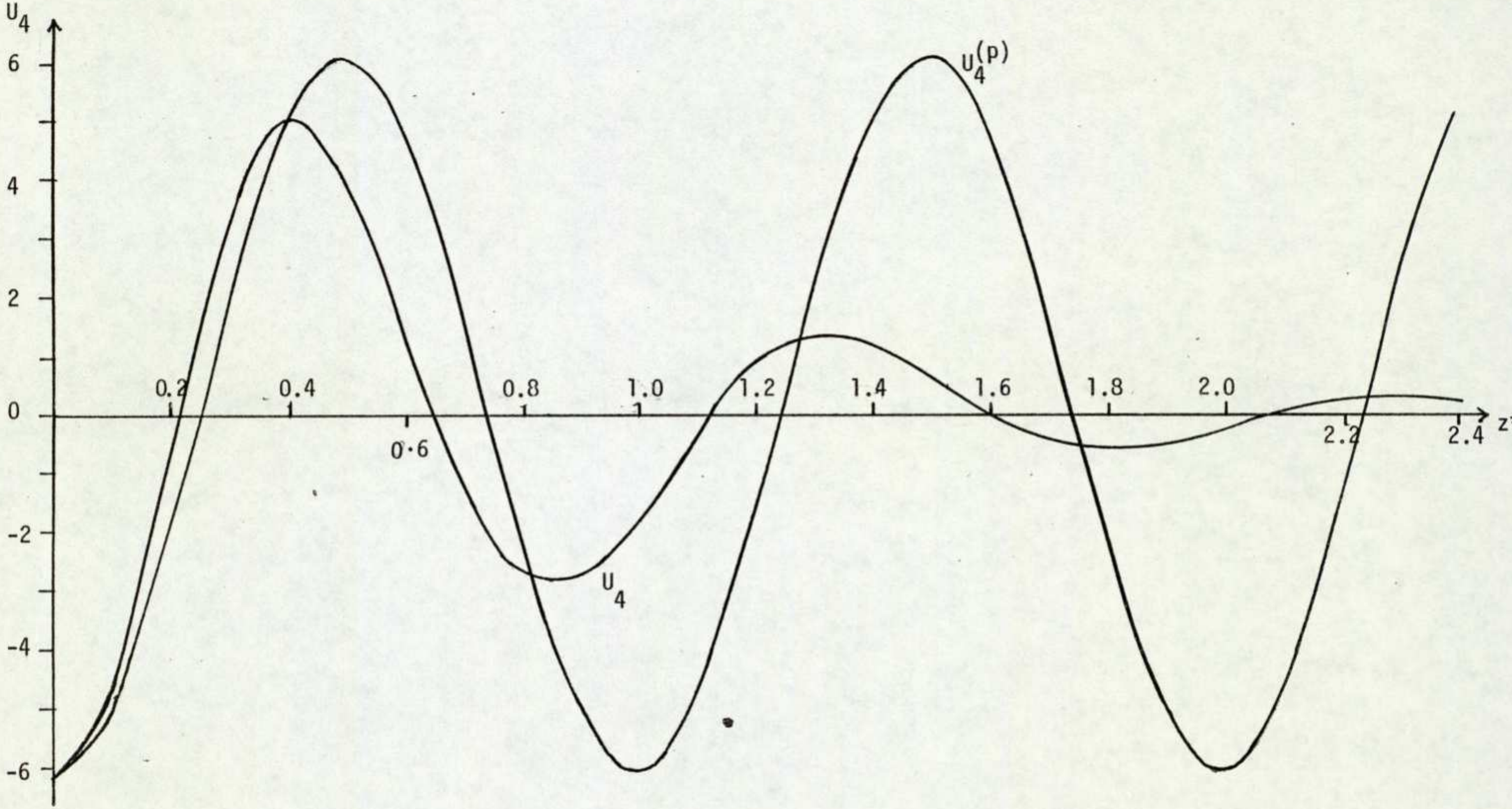

FIG XIII Linear solutions of U_4, $U_4^{(p)}$ for $\eta = 0.5$, $x = 0$, $j = 1$ and $\varepsilon = 0.5$.

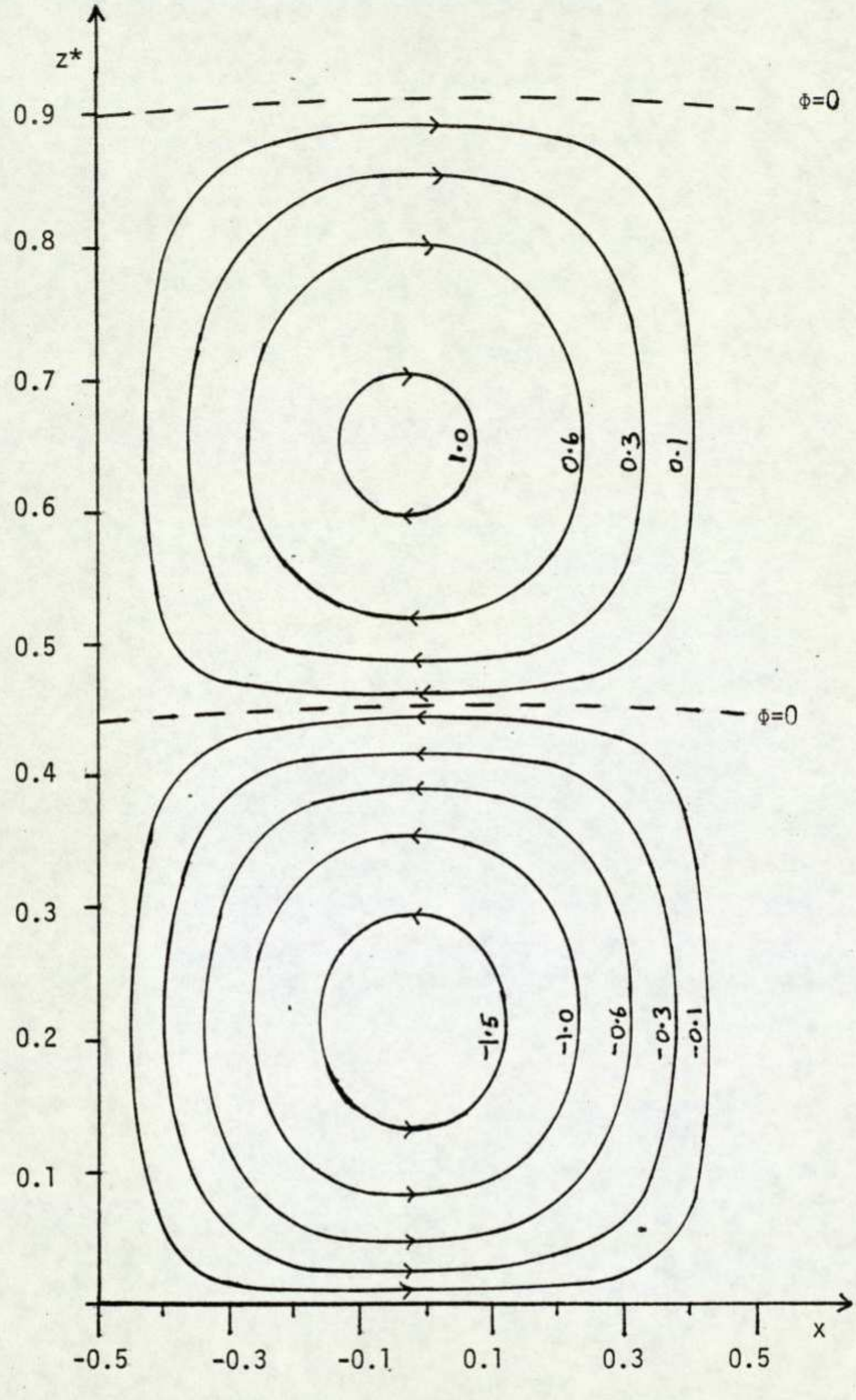

FIG XIVa First and second vortex streamlines for $\eta = 0.5$, $j = 1$ and $\varepsilon = 0.5$.

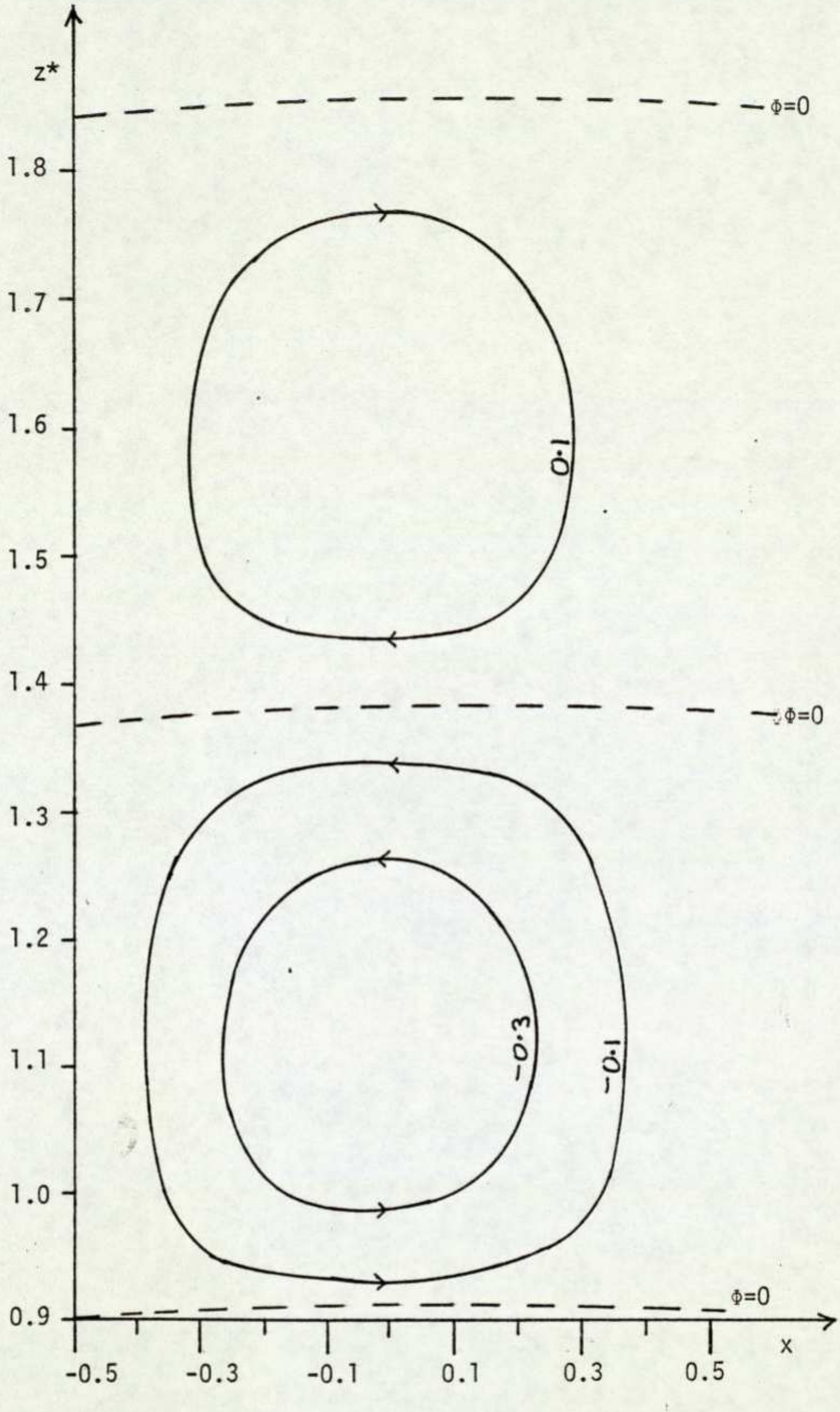

FIG XIVb Third and fourth vortex streamlines for $\eta = 0.5$, $j = 1$ and $\varepsilon = 0.5$.

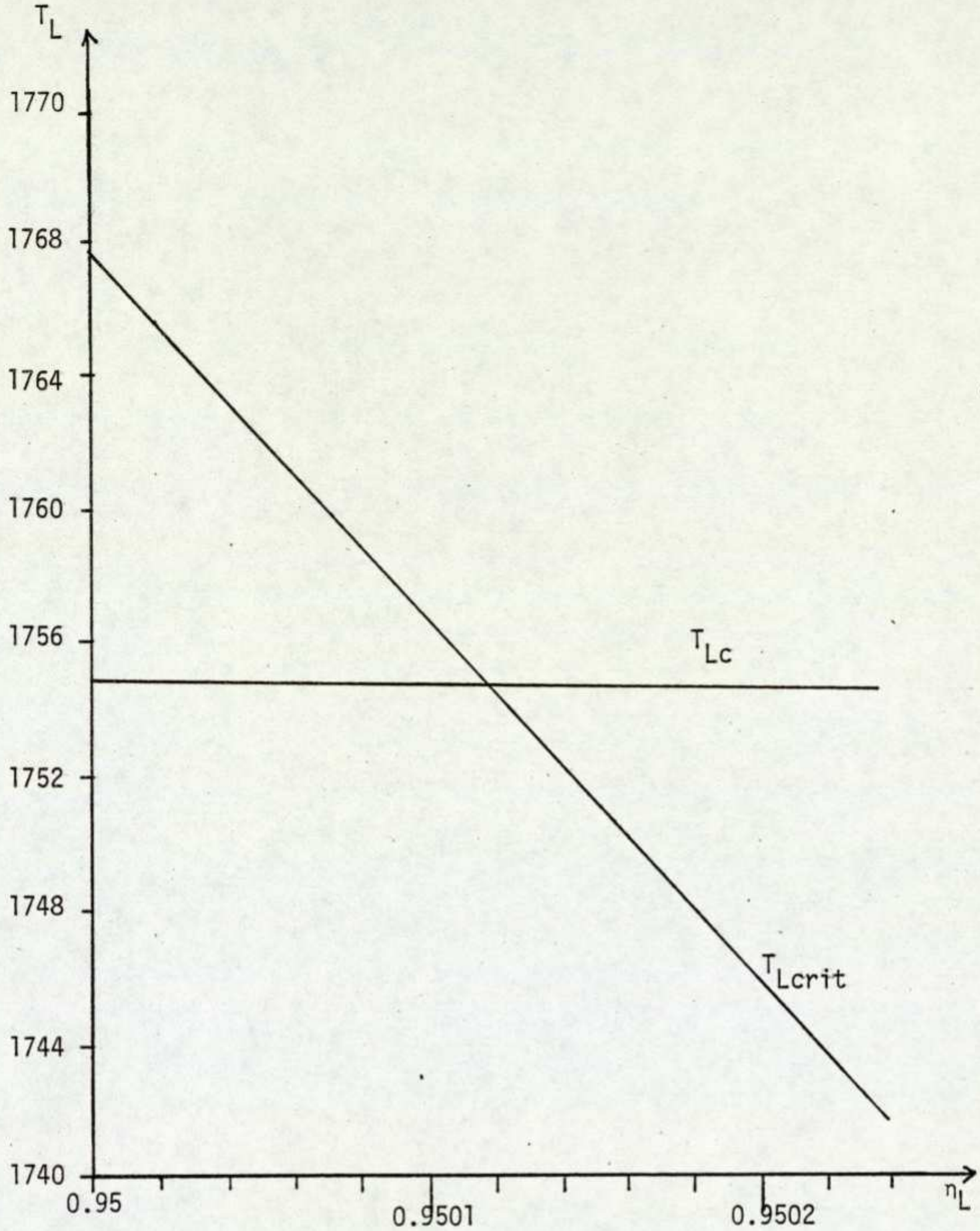

FIG XV Graph of T_{Lc}, T_{Lcrit} against η_L for $\eta = 0.95$, $j = 1$ and $\varepsilon = 0.1$.

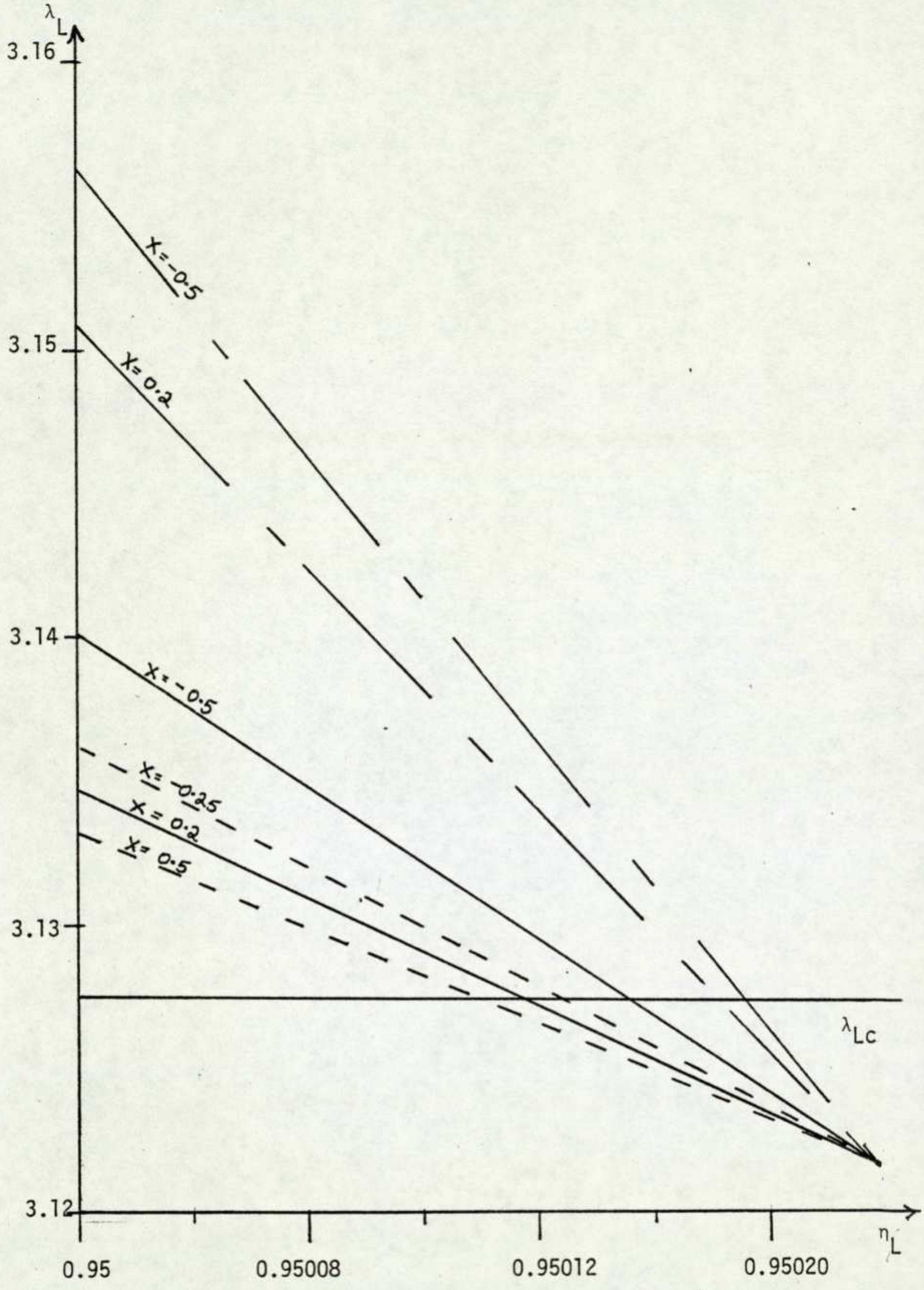

FIG XVI Graph of λ_{Lc}, λ_{Lcrit} for U_4, U_5 and Φ against η_L, for $\eta = 0.95$, selected x values, $j = 1$ and $\varepsilon = 0.1$. See FIG VII.

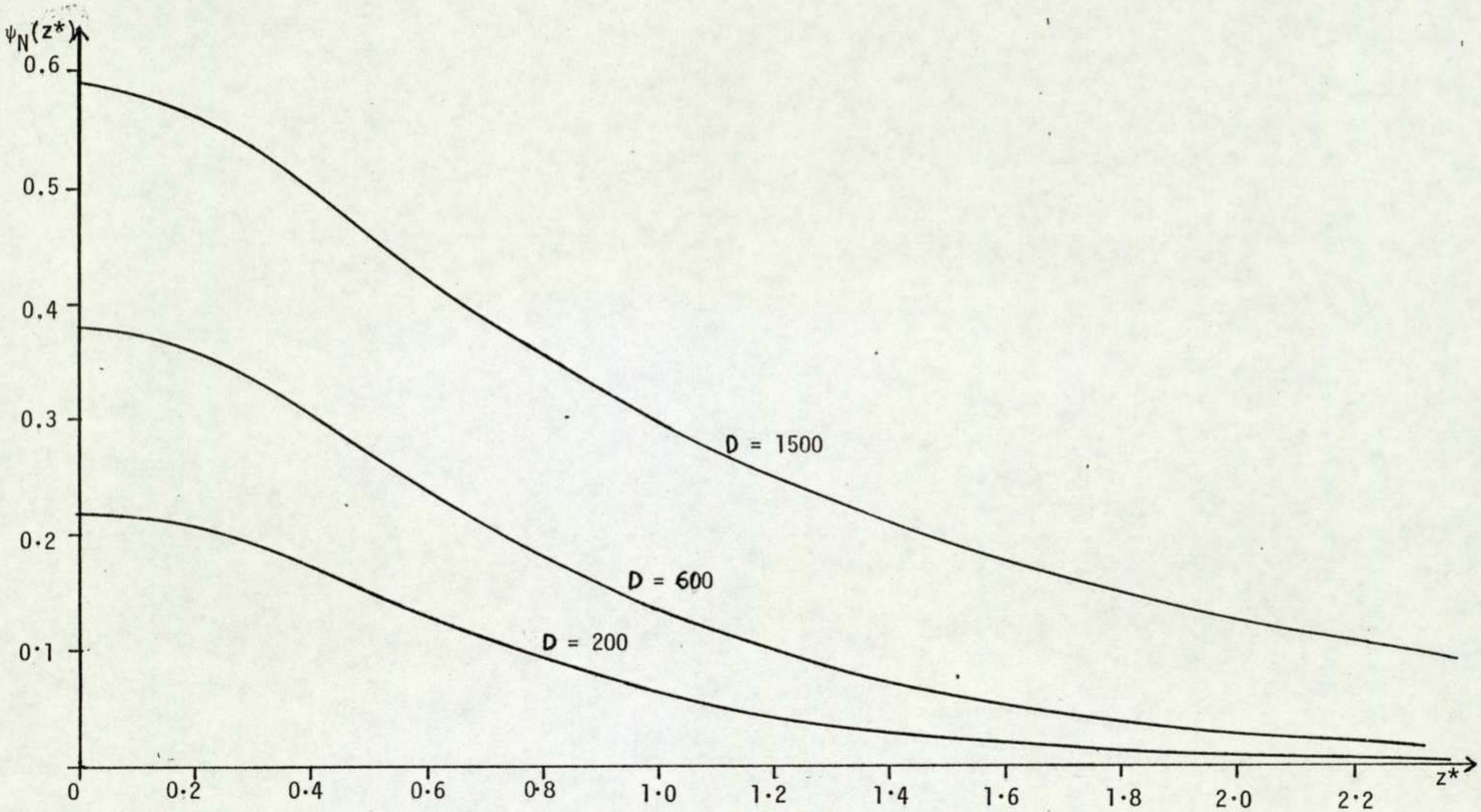

FIG XVII Graph of numerical solution of $\psi_N(z^*)$ for η = 0,5, j = 1, D = 200, D = 600 and D = 1500.

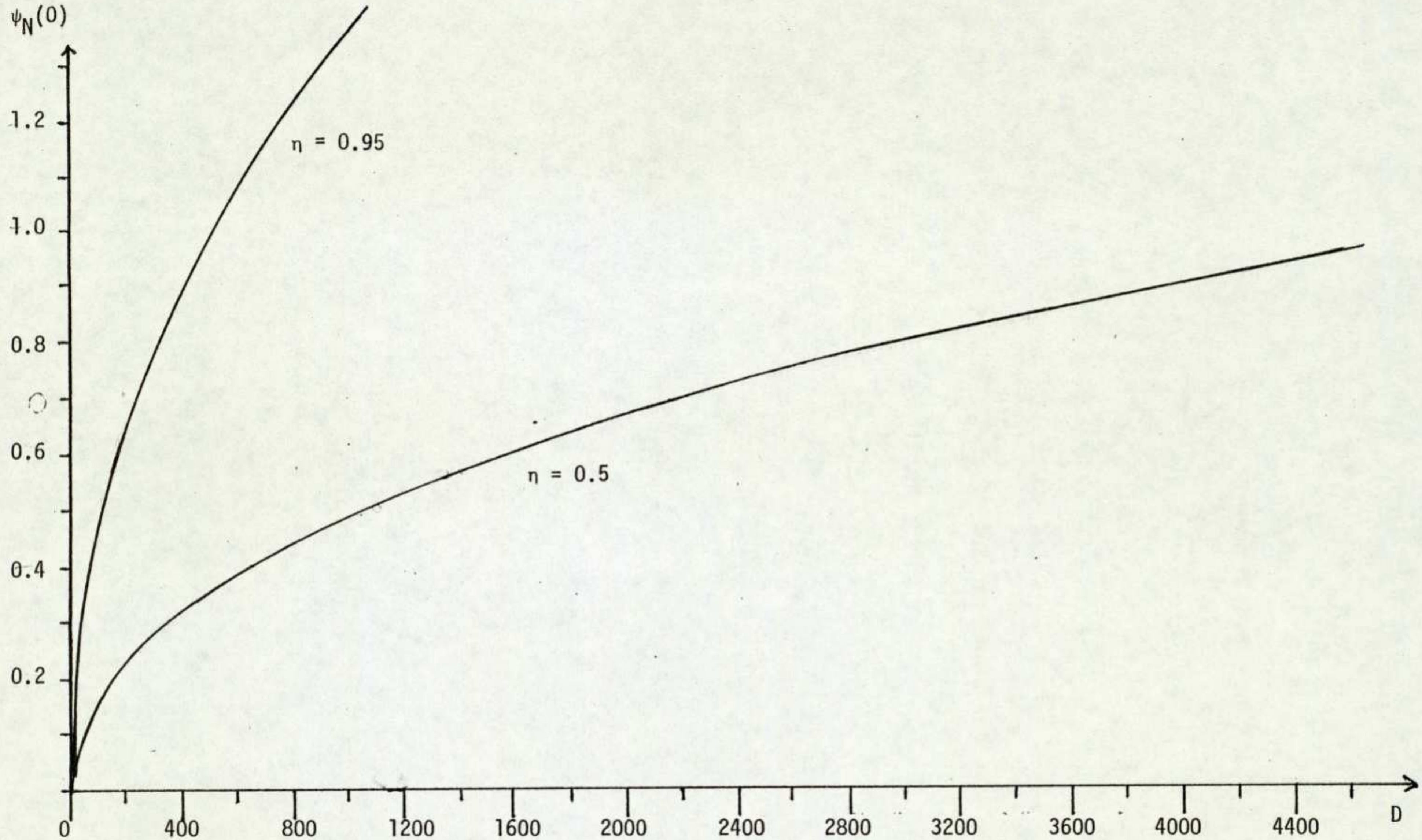

FIG XVIII Graph of $\psi_N(0)$ against D for $\eta = 0.5$ and $\eta = 0.95$ with $j = 1$.

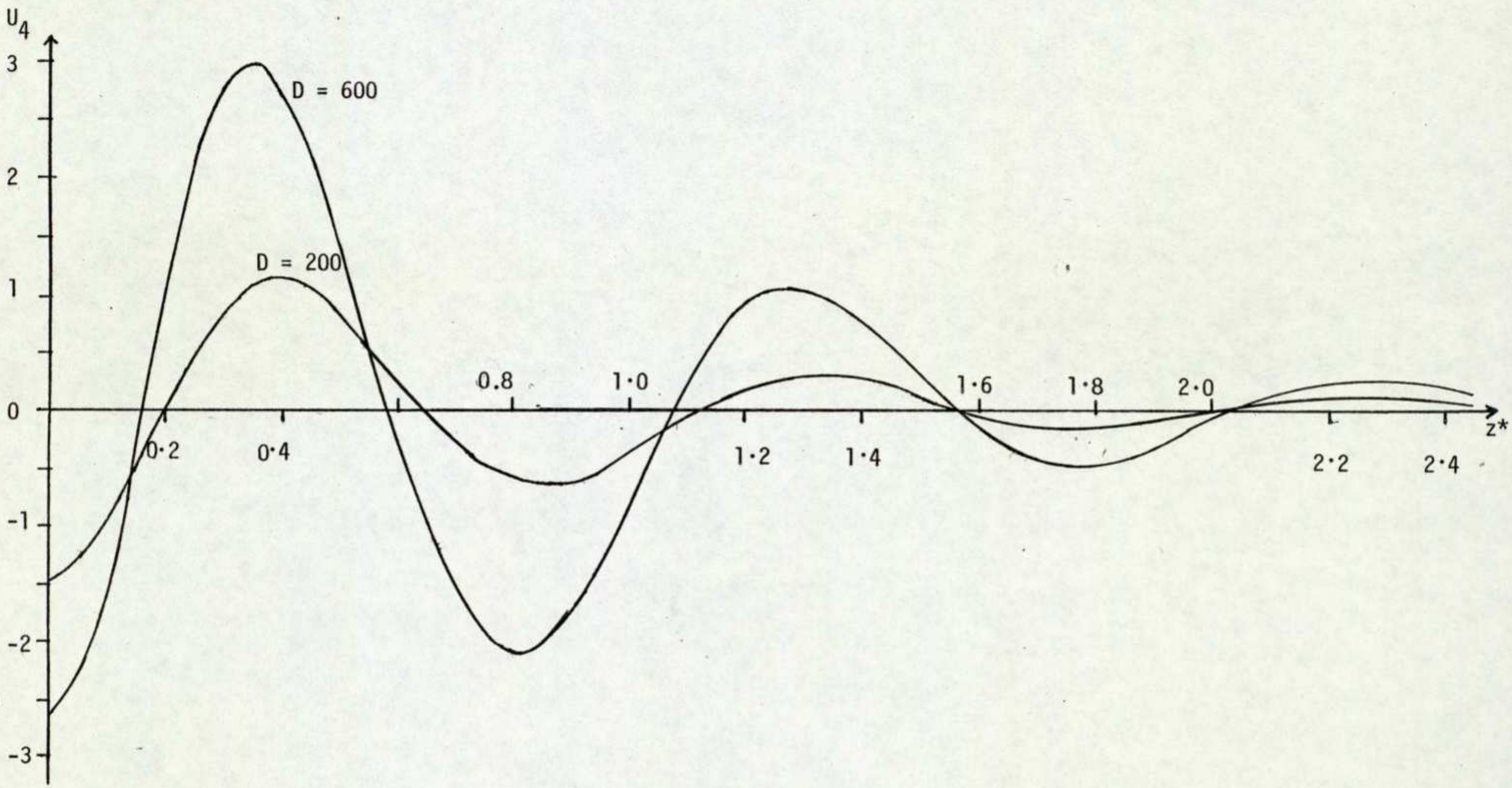

FIG XIX Non-linear solutions for U_4 for $\eta = 0.5$, $x = 0$, $j = 1$, $\varepsilon = 0.5$, $D = 200$ and $D = 600$.

5. Slowly Varying Outer Wall With Order One Total Variation

5.1 Introduction

We now consider the case where the outer wall has equation

$$r = Y(Z) = R_2 + (R_2 - R_1)H(Z)/2 \qquad (5.1.1)$$

with $H(0) = 0$. Here R_1 and R_2 are the inner and outer radii at $z^* = 0$. We shall use $Y(Z)$ as the equation of the outer wall for convenience. The non-dimensional equation of the outer wall is

$$x = [1+F(z^*)]/2 \qquad (z^* = Z/d) \quad . \qquad (5.1.2)$$

To obtain this function $F(z^*)$ we put

$$H(dz^*) = F(z^*) \; . \qquad (5.1.3)$$

We shall first find a steady state series for the flow when the inner cylinder is rotating and then perturb this to obtain a Taylor-vortex like flow for linear perturbations.

It seems that the Taylor-vortex like flow is not obtainable in terms of the variable z^*. We shall need to introduce rates of change with respect to ζ of $O(\varepsilon^{\frac{1}{2}})$ to solve the problem ($\varepsilon\zeta = z^*$).

The function $F(z^*)$, exemplified by

$$F(z^*) \;=\; -\tanh^2 \omega z^* \; , \qquad (5.1.4)$$

is chosen so that the base flow is locally more unstable near $z^* = 0$ than near $z^* = \pm\infty$. We shall find an overall critical Taylor number T_{crit} which will give the lowest value of T for which neutrally stable perturbations of the base flow can be found.

5.2 Assumptions made about the outer surface

The function $F(z^*)$ defined in (5.1.3) is taken to have the following properties :

(i) $|F(z^*)| < 1$ for all z^* , (5.2.1)

(ii) $F(0) = 0$, (5.2.2)

(iii) $\lim_{z^* \to \infty} F(z^*)$ exists, and $\lim_{z^* \to -\infty} F(z^*)$ exists . (5.2.3)

These properties are all that are required (apart from existence of derivatives of all orders) to find a base flow . Later we shall limit $F(z^*)$ further by requiring

(iv) $F(z^*) = F(-z^*)$, (5.2.4)

(v) $F(z^*) < 0$ for $z^* \neq 0$ (5.2.5)

for the perturbation problem.

5.3 Equations for the base flow

We again denote the velocity components and pressure of the base flow by u_s, v_s, w_s and p_s . Using cylindrical polar co-ordinates , the boundary conditions are

$$u_s = v_s = w_s = 0 \text{ on } r = Y(Z), \; (Z=\varepsilon z) \qquad (5.3.1)$$

and

$$u_s = v_s = 0 \; , \; v_s = \Omega_1 R_1 \text{ on } r = R_1 \qquad (5.3.2)$$

where $r = Y(Z)$ is the equation of the stationary outer wall.

For ε small we expand

$$\left.\begin{aligned} u_s &= \varepsilon u_1 + \varepsilon^2 u_2 + \ldots \\ v_s &= v_o + \varepsilon v_1 + \ldots \\ w_s &= \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \\ p_s &= p_o + \varepsilon p_1 + \ldots \quad . \end{aligned}\right\} \qquad (5.3.3)$$

All the functions are considered to be functions just of r and Z. The functions must satisfy

$$v_o = u_i = v_i = w_i = 0 \quad \text{on} \quad r = Y(Z) \text{ for } i = 1,2,3,\ldots \qquad (5.3.4)$$

and

$$v_o = \Omega_1 R_1 \ , \ u_i = v_i = w_i = 0 \text{ on } r = R_1 \text{ for } i = 1,2,3,\ldots \qquad (5.3.5)$$

We note that any $\partial/\partial z$ in the basic equations becomes $\varepsilon\partial/\partial Z$. Therefore when we substitute (5.3.3) into the steady Navier-Stokes and continuity equations we obtain the same set of partial differential equations (3.3.14) to (3.3.16), (3.3.18) to (3.3.21) and (3.3.26) to (3.3.28). The solution for $u_1(r,Z)$ is again

$$u_1(r,Z) = 0 \ . \qquad (5.3.6)$$

The second momentum equation gives the new equations

$$\frac{\partial^2 v_o}{\partial r^2} + \frac{1}{r}\frac{\partial v_o}{\partial r} - \frac{v_o}{r^2} = 0 \ , \qquad (5.3.7)$$

$$\frac{\partial^2 v_1}{\partial r^2} + \frac{1}{r}\frac{\partial v_1}{\partial r} - \frac{v_1}{r^2} = 0 \ , \qquad (5.3.8)$$

$$\frac{u_2}{r}\frac{\partial}{\partial r}(rv_o) + w_1\frac{\partial v_o}{\partial z} = \nu\left(\frac{\partial^2 v_2}{\partial r^2} + \frac{1}{r}\frac{\partial^2 v_2}{\partial r} - \frac{v_2}{r} + \frac{\partial^2 v_o}{\partial z^2}\right) \ . \qquad (5.3.9)$$

We shall also assume that at $Z = \pm\infty$ the flow reduces to purely circumferential flow so that

$$u_j \to 0 \text{ and } w_j \to 0 \text{ as } Z \to \pm\infty \ , \text{ for } j \geq 1. \qquad (5.3.10)$$

5.4 Solutions for the base flow

Equation (5.3.7) and boundary conditions give

$$v_0(r,Z) = A(Z)r + B(Z)/r \tag{5.4.1}$$

where

$$A(Z) = -\frac{R_1^2\Omega_1}{Y^2(Z)-R_1^2}, \quad B(Z) = -Y^2(Z)A(Z) , \tag{5.4.2}$$

and from (3.3.18) we find

$$p_0(r,Z) = \rho[A^2(Z)r^2/2 + 2A(Z)B(Z)\log r - \tfrac{1}{2} B^2(Z)/r^2 + C(Z)] , \tag{5.4.3}$$

where $C(Z)$ is still unknown.

From (5.3.8) and boundary conditions we see

$$v_1(r,Z) = 0 \tag{5.4.4}$$

and from (3.3.19) and (5.4.4) we see that p_1 is a function of Z only. The solution of (3.3.27) is therefore

$$w_2(r,Z) = \frac{1}{4\nu\rho}\frac{dp_1}{dZ}\left[r^2 + \frac{R_1^2\log(r/Y(Z)) - Y^2(Z)\log(r/R_1)}{\log(Y(Z)/R_1)}\right] \tag{5.4.5}$$

$$= \frac{1}{4\nu\rho}\frac{dp_1}{dZ} W_2(r,Z) \text{ , say.} \tag{5.4.6}$$

Using (3.3.16) and the fact $u_3 = 0$ on $r = R_1$, it was found that

$$-ru_3(r,Z) = \frac{1}{4\nu\rho}\left[\frac{d^2p_1}{dZ^2}\int_{R_1}^{r} rW_2\,dr + \frac{dp_1}{dZ}\int_{R_1}^{r} r\frac{\partial W_2}{\partial Z}\,dr\right]. \tag{5.4.7}$$

The condition that $u_3 = 0$ at $r = Y(Z)$ implies

$$\frac{d^2p_1}{dZ^2}\int_{R_1}^{Y(Z)} rW_2\,dr + \frac{dp_1}{dZ}\int_{R_1}^{Y(Z)} r\frac{\partial W_2}{\partial Z}\,dr = 0. \tag{5.4.8}$$

Remembering that $W_2(Y(Z),Z) = 0$ this can be written as

$$\frac{d}{dZ}\left[\frac{dp_1}{dZ}\int_{R_1}^{Y(Z)} rW_2\,dr\right] = 0 \tag{5.4.9}$$

and hence

$$dp_1/dZ = K \Bigg/ \left[\int_{R_1}^{Y(Z)} rW_2\,dr\right] \tag{5.4.10}$$

where K is a constant.

We require $w_2 \to 0$ as $Z \to \pm\infty$ for all r in the range and this is only possible (from (5.4.5)) if $dp_1/dZ \to 0$ as $Z \to \pm\infty$. Since the denominator of (5.4.10) is not infinite at $Z = \pm\infty$ therefore $K = 0$ and $dp_1/dZ = 0$ for all Z. Hence

$$w_2(r,Z) = 0 \tag{5.4.11}$$

and

$$u_3(r,Z) = 0. \tag{5.4.12}$$

We shall now attempt the more difficult task of solving for $w_1(r,Z)$. We use the differential equation for $w_1(r,Z)$ and the expression for $p_o(r,Z)$ given respectively by (3.3.26) and (5.4.3), and note

$$\frac{1}{\rho}\frac{\partial p_o}{\partial Z} = \frac{dC}{dZ} - \frac{4\Omega_1^2 R_1^4 Y(Z)}{(Y^2(Z)-R_1^2)^3}\frac{dY}{dZ}\left(\frac{r^2}{2} - 2Y^2(Z)\log r - \frac{Y(Z)^4}{2r^2}\right) - \frac{4\Omega_1^2 R_1^4 Y(Z)}{(Y^2(Z)-R_1^2)^2}\frac{dY}{dZ}\left(\log r + \frac{Y^2(Z)}{2r^2}\right). \tag{5.4.13}$$

The above equation can be re-written as

$$\frac{1}{\rho}\frac{\partial p_o}{\partial Z} = \frac{2\Omega_1^2 R_1^4 Y(Z)}{(Y^2(Z)-R_1^2)^3}\frac{dY}{dZ}\left[-r^2 + 2(Y^2(Z)+R_1^2)\log r + \frac{R_1^2 Y^2(Z)}{r^2} +\right.$$

$$2(Y^2(Z)+R_1^2)\,(b_o(Z) - \log R_2)\Big], \tag{5.4.14}$$

where for easier analysis we wrote

$$\frac{dC}{dZ} = \frac{4\Omega_1^2 R_1^4 Y(Z)}{(Y^2(Z)-R_1^2)^3} \frac{dY}{dZ} (Y^2(Z)+R_1^2)(b_o(Z)-\log R_2) \tag{5.4.15}$$

and $b_o(Z)$ is an unknown non-dimensional function at this stage and depends only on η and Z.

Therefore from (3.3.26) we see

$$\frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_1}{\partial r}\right) = \frac{2\Omega_1^2 R_1^4 Y(Z)}{(Y^2(Z)-R_1^2)^3}\frac{dY}{dZ}\Big[-r^2+2(Y^2(Z)+R_1^2)\log r + \frac{R_1^2 Y^2(Z)}{r^2}$$

$$+ \quad 2(Y^2(Z)+R_1^2)(b_o(Z)-\log R_2)\Big]. \tag{5.4.16}$$

We shall assume a solution of (5.4.16) of the form

$$w_1(r,Z) = \frac{2\Omega_1^2 R_1^4 Y(Z)}{(Y^2(Z)-R_1^2)^3\nu}\frac{dY}{dZ} w^{(1)}(r,Z) \tag{5.4.17}$$

and if we follow a similar line of approach as in (3.4.27) onwards we will eventually obtain

$$w^{(1)}(r,Z) = -\frac{1}{16}\left[r^4 + \frac{R_1^4 \log(r/Y(Z) - Y^4(Z)\log(r/R_1)}{\log(Y(Z)/R_1)}\right] +$$

$$+\frac{(R_1^2+Y^2(Z))}{2}\Big[r^2\log r + \frac{R_1^2\log R_1\log(r/Y(Z))-Y^2(Z)\log Y(Z)\log(r/R_1)}{\log(Y(Z)/R_1)}$$

$$+ (b_o(Z)-1-\log R_2)\Big(r^2 + \frac{R_1^2\log(r/Y(Z))-Y^2(Z)\log(r/R_1)}{\log(Y(Z)/R_1)}\Big)\Big] \quad +$$

$$\frac{R_1^{\,2} Y^2(Z)}{2} \log(r/R_1)\log(r/Y(Z)) \; . \tag{5.4.18}$$

To find the function $b_o(Z)$, we shall use the continuity equation

$$\frac{1}{r}\,\frac{\partial}{\partial r}\,(r\,u_2) \;+\; \frac{\partial w_1}{\partial Z} \;=\; 0 \tag{5.4.19}$$

and write the solution for $u_2(r,Z)$ as

$$r\,u_2\,(r,Z) \;=\; -\int_{R_1}^{r} r\,\frac{\partial w_1}{\partial Z}\,dr \; . \tag{5.4.20}$$

We can obtain $b_o(Z)$ explicitly by solving the following extremely complicated integral relationship

$$\int_{R_1}^{Y(Z)} r\,\frac{\partial w_1}{\partial Z}\,dr \;=\; 0. \tag{5.4.21}$$

This is equivalent to solving

$$\int_{R_1}^{Y(Z)} r\,w_1(r,Z)dr = 0 \tag{5.4.22}$$

when the boundary condition $w_1 \to 0$ as $Z \to \pm\infty$ is used.
With the value of $b_o(Z)$ known we can find the velocity component $w_1(r,Z)$ completely.

In our expansion procedure later we expand $b_o(Z)$ as

$$b_o(Z) = b_o(0) + Zb'(0) + Z^2b''(0)/2 + \ldots , \tag{5.4.23}$$

of which only the first term will be needed. The value of $b_o(0)$ is seen to be the same value as b_o given in (3.4.37) for $\eta = 0.5$ and (3.4.38) for $\eta = 0.95$, and these values were calculated using a computer program.

Therefore upon neglecting powers of ε^n for $n \geq 2$, the base velocities are represented by

$$u_s = 0, \; v_s = v_o(r,Z) \; , \; w_s = \varepsilon w_1(r,Z) \tag{5.4.24}$$

where the subscript s denotes that the base velocities are steady. The pressure distribution is given by

$$p_s = p_o(r,Z) \; . \tag{5.4.25}$$

5.5 The problem for the Taylor-vortex like flow

We look for a steady state perturbation to the base flow described in §5.4. We first state the problem for the perturbation in terms of the variable $z^* = \varepsilon\zeta$. Non-dimensionalizing in the usual way, the problem for the linear perturbation may be written as

$$\frac{\partial \underline{U}}{\partial x} - \underline{A}^*\underline{U} + \varepsilon\frac{dF}{dz^*} \; \underline{D}^*\underline{U} + O(\varepsilon^2) = 0. \tag{5.5.1}$$

Here

$$\underline{A}^* = \begin{bmatrix} 0 & 0 & -\partial/\partial\zeta & \partial^2/\partial\zeta^2 & -T\Omega^*_{0,s} & 0 \\ 0 & -\delta G & 0 & \dfrac{1-\eta^2}{E^2-\eta^2} & -\partial^2/\partial\zeta^2 + \delta^2 G^2 & 0 \\ \partial/\partial\zeta & 0 & -\delta G & 0 & 0 & -\partial^2/\partial\zeta^2 \\ 0 & 0 & 0 & -\delta G & 0 & -\partial/\partial\zeta \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{5.5.2}$$

$$
\underline{D}^* = \begin{bmatrix} & TW^*_{1,s}\partial/\partial\zeta & 0 & 0 \\ \underset{\sim}{0} & 0 & -TW^*_{1,s}\partial/\partial\zeta & V^*_2 \\ & -T\dfrac{\partial W^*_{1,s}}{\partial x} & 0 & -TW^*_{1,s}\ \partial/\partial\zeta \\ \underset{\sim}{0} & & \underset{\sim}{0} & \end{bmatrix}. \quad (5.5.3)
$$

The various functions involved are defined below. We remind the reader that $G(x)$ is $1/(1+\delta x)$ and $\eta = R_1/R_2$ and that $F(z^*)$ is defined by (5.1.3). Then

$$
E(z^*) = 1 + (1-\eta)F(z^*)/2 , \quad (5.5.4)
$$

$$
\Omega^*_{0,s}(x,z^*,\eta) = \frac{-2\eta^2}{E^2(z^*)-\eta^2} + \frac{8E^2(z^*)\eta^2G^2(x)}{\{E^2(z^*)-\eta^2\}(1+\eta)^2} , \quad (5.5.5)
$$

and

$$
V^*_2\ (x,z^*,\eta) = \frac{(1-\eta^2)(1+\eta)E(z^*)}{\{E^2(z^*)-\eta^2\}^2}\left[\frac{1}{4G(x)} - \frac{\eta^2}{(1+\eta)^2}G(x)\right]. \quad (5.5.6)
$$

The function $W^*_{1,s}$ contains the dimensionless function $\sigma(x)$ where

$$
\sigma(x) = (1+\eta)/2 + (1-\eta)x \quad (5.5.7)
$$

and we have

$$
\begin{aligned}
W^*_{1,s}\ (x,z^*,\eta) = {} & \frac{\eta^2E(z^*)(1-\eta^2)}{2(1-\eta)^3\{E^2(z^*)-\eta^2\}}\left[\frac{\eta^2+E^2(z^*)}{2\log(\eta/E(z^*))}\left\{(\sigma^2-\eta^2)\log\sigma\right.\right. \\
& \log\eta + (E^2(z^*)-\sigma^2)\log E(z^*)\log\sigma + (\eta^2-E^2(z^*))\log E(z^*)\log\eta + \\
& (B_o(z^*)-1)((\sigma^2-\eta^2)\log\sigma\log\eta + (E^2(z^*)-\sigma^2)\log E(z^*)\log\sigma + \\
& \left.(\eta^2-E^2(z^*))\log E(z^*)\log\eta)\right\} + \frac{\eta^2E^2(z^*)}{2}(\log\sigma-\log\eta)(\log\sigma-\log E(z^*))
\end{aligned}
$$

$$- \frac{1}{16\ \log(\eta/E(z^*))} \{(\eta^4 - \sigma^4)\log E(z^*) + (\sigma^4 - E^4(z^*))\log\eta + (E^4(z^*) - \eta^4)\log\sigma\}\Bigg] , \qquad (5.5.8)$$

where $B_0(z^*) = b_0(Z)$. The function $\sigma(x)$ is merely the expression of r/R_2.

The six-vector $\underline{U}$ is just the usual extended velocity vector and the boundary conditions are

$$\beta_3^* \ : \ \text{the last 3 components of } \underline{U} \text{ are zero at } x = -\frac{1}{2} \text{ and at } x = \{1 + F(z^*)\}/2 , \qquad (5.5.9)$$

and

$$\underline{U} \to 0 \quad \text{as} \quad z^* \to \pm\ \infty. \qquad (5.5.10)$$

The function $F(z^*)$ satisfies the conditions (5.2.1) to (5.2.3) .

It should be explained that

$$\Omega_{0,s}^* (x,z^*,\eta) = 2v_0(r,Z)/\ \Omega,r , \qquad (5.5.11)$$

$$\frac{dF}{dz^*} V_2^* (x,z^*,\eta) = \frac{2d}{\Omega_1 R_0\alpha} \frac{\partial v_0(r,Z)}{\partial Z} , \qquad (5.5.12)$$

and

$$T \frac{dF}{dz^*} W_{1,s}^*(x,z^*,\eta) = \frac{d}{\nu} w_1(r,Z) . \qquad (5.5.13)$$

5.6 Attempt at a solution in terms of z^*

Attempting a formal solution of (5.5.1) with $T = T_c + \varepsilon T_2^* + \ldots$ and with

$$\underline{U} = e^{iS} \{\underline{V}_0 (x,z^*) + \varepsilon \underline{V}_1(x,z^*) + \ldots\} + \text{c.c.} \qquad (5.6.1)$$

where the W.K.B. method suggests

$$\frac{dS}{d\zeta} = K(z^*) \qquad (5.6.2)$$

we find the problem for $\underline{V}_o$ is

$$\frac{\partial \underline{V}_o}{\partial x} - \underline{A}_c^{*(1)} \underline{V}_o = 0 \quad ; \quad \beta_3^* \tag{5.6.3}$$

where $\underline{A}_c^{*(1)}$ is defined by

$$\underline{A}_c^{*(1)} = \begin{cases} \underline{A}^* \text{ with T replaced by } T_c \text{ and} \\ \partial/\partial\zeta \text{ replaced by } iK(z^*) \end{cases} \tag{5.6.4}$$

Thus $K(z^*)$ is an eigenvalue, if T_c and η are fixed.

If we perturb the problem (5.6.3) about $z^* = 0$, expand $\underline{V}_o = \underline{v}_1(x) + z^*\underline{v}_2(x) + \ldots$ and write $K(z^*) = \lambda_c + \lambda_1 z^* + \lambda_2 z^{*2} + \ldots$, it is fairly easy to see by methods which have been used throughout this work that the consistency condition at order z^{*2} leads to

$$\lambda_1^2 = aF''(0)/2 \tag{5.6.5}$$

where a is defined in (3.7.20) and is positive, while $F''(0)$ is negative. Thus λ_1 is purely imaginary near $z^* = 0$ and there are two complex values of $K(z^*)$, which both reduce to λ_c at $z^* = 0$. Also, (5.6.3) may be transformed into the standard linear perturbation problem for the parallel wall case (see Eagles [10]) but with T replaced by

$$T_{L,0} = \frac{[1 + F(z^*)/2]^3}{[1 + \delta F(z^*)/4]} T_c \ , \tag{5.6.6}$$

η replaced by η_L the 'local' ratio of the radii and λ replaced by $(d_L/d)K(z^*)$ where d_L is the local gap width. Hence for sufficiently large $|z^*|$ we have $T_{L,0}$ less than the local parallel wall critical Taylor number and with $\partial/\partial\tau = 0$ we will expect complex values of $K(z^*)$. We assume without proof, then, that there exists one root $K(z^*)$ such that

$$K(z^*) \sim \lambda_c + i\sqrt{-aF''(0)/2}\,.\,z^* \quad \text{for} \quad z^* \to 0 \tag{5.6.7}$$

and

$$\mathrm{Im}(K(z^*)) \gtrless 0 \quad \text{for} \quad z^* \gtrless 0. \tag{5.6.8}$$

If this is so then, using that root, we can see that since

$$S(z^*) = \int_0^{z^*} K(z^*)dz^*/\,\varepsilon \tag{5.6.9}$$

then $|e^{iS}| \to 0$ as $z^* \to \pm\infty$ with ε fixed, and we can presumably satisfy the boundary conditions $\underline{U} \to 0$ as $z^* \to \pm\infty$. However, substituting (5.6.1) into (5.5.1) we find that

$$\underline{V}_o(x,z^*) = \psi^*(z^*)\underline{u}_{11}(x,z^*) \tag{5.6.10}$$

and a consistency condition at $O(\varepsilon)$ leads to

$$\ell_1(z^*)\frac{d\psi^*}{dz^*} + \ell_2(z^*)\psi^* = 0 \;. \tag{5.6.11}$$

But $\ell_1(0) = 0$ by virtue of (3.7.6) and hence the expansion may fail near $z^* = 0$. This leads us to consider the region near $z^* = 0$ by using an inner series allowing rates of change with respect to ζ to be $O(\varepsilon^{\frac{1}{2}})$, instead of $O(\varepsilon)$ implied by the use of the outer series.

5.7 The problem in terms of $\varepsilon^{-\frac{1}{2}}z^*$: inner series

By trial it is found that an appropriate 'inner' variable is q where

$$\varepsilon^{\frac{1}{2}}q = z^* = \varepsilon\zeta \tag{5.7.1}$$

and that we must expand

$$T = T_c + \varepsilon^{\frac{1}{2}}T_1^* + \varepsilon T_2^* + \ldots \tag{5.7.2}$$

and

$$\underline{U} = e^{i\lambda_c\zeta}\{\underline{u}_1(x,q) + \varepsilon^{\frac{1}{2}}\underline{u}_2(x,q) + \varepsilon\underline{u}_3(x,q) + \ldots\} + \text{c.c.}\;. \tag{5.7.3}$$

All the terms involving z* in (5.5.1) are expanded in terms of q, using for example

$$F(z^*) = \varepsilon F''(0)q^2 + \ldots , \qquad (5.7.4)$$

since $F(0) = F'(0) = 0$. By substituting (5.7.2) and (5.7.3) into (5.5.1) and equating powers of $\varepsilon^{\frac{1}{2}}$ we find the following sequence of problems for the $\underline{u}_j$'s:

$$\frac{\partial \underline{u}_1}{\partial x} - \underline{A}_c^{(1)} \underline{u}_1 = 0 , \qquad (5.7.5)$$

$$\frac{\partial \underline{u}_2}{\partial x} - \underline{A}_c^{(1)} \underline{u}_2 = \underline{B}_{c1}^{(1)} \frac{\partial \underline{u}_1}{\partial q} + T_1^* \underline{A}_2 \underline{u}_1 , \qquad (5.7.6)$$

$$\frac{\partial \underline{u}_3}{\partial x} - \underline{A}_c^{(1)} \underline{u}_3 = \underline{B}_{c1}^{(1)} \frac{\partial \underline{u}_2}{\partial q} + T_1^* \underline{A}_2 \underline{u}_2 + T_2^* \underline{A}_2 \underline{u}_1 + \underline{B}_{c2} \frac{\partial^2 \underline{u}_1}{\partial q^2} - \frac{F''(0)}{2} q^2 \underline{C}_c \underline{u}_1 , \qquad (5.7.7)$$

$$\frac{\partial \underline{u}_4}{\partial x} - \underline{A}_c^{(1)} \underline{u}_4 = \underline{B}_{c1}^{(1)} \frac{\partial \underline{u}_3}{\partial q} + T_1^* \underline{A}_2 \underline{u}_3 + T_2^* \underline{A}_2 \underline{u}_2 + \underline{B}_{c2} \frac{\partial^2 \underline{u}_2}{\partial q^2} - \frac{F''(0)}{2} q^2 \underline{C}_c \underline{u}_2 + T_3^* \underline{A}_2 \underline{u}_1 - qF''(0) \underline{D}_c^{(1)} \underline{u}_1 . \qquad (5.7.8)$$

The matrix $\underline{A}_c^{(1)}$ is defined in (2.4.6) and $\underline{A}_2$ is defined in (2.4.7) while $\underline{C}_c$, $\underline{B}_{c1}^{(1)}$ and $\underline{B}_{c2}$ are given in (3.5.23)and(3.6.14),the suffix c denoting the substitution $T = T_c$ or $\lambda = \lambda_c$ as appropriate. Also $\underline{D}_c^{(1)}$ is defined in (3.6.14) and contains the constant $b_o(0)$ through the function $W_{3,S}$ which is identical to $W_{1,s}^*(x,0,\eta)$.

The boundary conditions due to the outer wall are obtained by expanding about $x = \frac{1}{2}$ and are, for j = 4,5 and 6,

$$\left.\begin{aligned} &u_{ij} = 0 \quad , u_{2j} = 0 \\ &u_{3j} + F''(0)q^2 \; u_{1x,j}/4 = 0 \\ &u_{4j} + F''(0)q^2 \; u_{2x,j}/4 = 0 \end{aligned}\right\} \begin{array}{l} \text{at } x = \frac{1}{2} \text{ for} \\ \text{all } q \end{array} \qquad (5.7.9)$$

where u_{Kj} denotes the j^{th} component of $\underline{u}_K$. The inner wall conditions are, for $j = 4,5$ and 6,

$$u_{1j} = u_{2j} = u_{3j} = u_{4j} = 0 \quad \text{at} \quad x = -1/2 \; . \qquad (5.7.10)$$

The analysis is similar to §3.7 and we therefore merely summarize the results. A solution of (5.7.5) is

$$\underline{u}_1(x,q) = \psi(q)\underline{u}_{11}(x) \qquad (5.7.11)$$

where $\underline{u}_{11}$ is the usual linear critical eigenfunction.

Then (5.7.6) gives $T_1^* = 0$ and

$$\underline{u}_2 = \frac{d\psi}{dq}\underline{g}_{21}(x) + S(q)\underline{u}_{11}(x) \qquad (5.7.12)$$

where $\underline{g}_{21}$ is identical with the function used in § 3.7 and satisfies equation (3.7.10).

Substituting (5.7.12) into (5.7.7) and using the usual existence condition gives the amplitude equation

$$\frac{d^2\psi}{dq^2} + \psi\left[\frac{T_2^*}{T_2} + \frac{aF''(0)}{2}\; q^2\right] = 0 \qquad (5.7.13)$$

where a and T_2 are defined in (3.7.20) and (2.6.10) . We make the assumption that

$$\psi \to 0 \quad \text{as } q \to \pm\, \infty \; . \qquad (5.7.13a)$$

When (5.7.7) is solved subject to the given boundary conditions it can be shown that

$$\underline{u}_3(x,q) = \frac{\psi F''(0)}{2} q^2 \underline{h}_{31}(x) + \psi \underline{g}_{31}(x) + \frac{dS}{dq}\underline{g}_{21}(x) + P(q)\underline{u}_{11}(x), \qquad (5.7.14)$$

where $\underline{h}_{31}$ and $\underline{g}_{31}$ are the functions defined by equations (3.9.3) and (3.9.4) and boundary conditions (3.9.5) and (3.9.6).

We substitute (5.7.11), (5.7.12) and (5.7.14) into the right hand side of (5.7.8) and use the derivative of (5.7.13) to eliminate $d^3\psi/dq^3$, and by applying the existence condition we find

$$\frac{d^2S}{dq^2} + S(q)\left[\frac{T_2^*}{T_2} + a\frac{F''(0)}{2}q^2\right] = r_1\frac{F''(0)}{2}q^2\frac{d\psi}{dq} + r_2F''(0)q\psi + r_3\frac{d\psi}{dq} - \psi\frac{T_3^*}{T_2} \qquad (5.7.15)$$

and where we assume

$$S \to 0 \text{ as } q \to \pm\infty . \qquad (5.7.15a)$$

The values r_1, r_2 and r_3 are defined in (3.9.13), (3.9.14) and (3.9.22) and are purely imaginary constants.

5.8 Solutions of the equations

In the amplitude equation (5.7.13) we know that $a > 0$ and also $F''(0) < 0$. Introducing the variable

$$\xi = q\sqrt{\alpha_0}\,, \quad (\alpha_0^2 = -aF''(0)/2) \qquad (5.8.1)$$

we find

$$\frac{d^2\psi}{d\xi^2} + \psi\left[\frac{T_2^*}{T_2\alpha_0} - \xi^2\right] = 0\,, \qquad (5.8.2)$$

and

$$\psi \to 0 \quad \text{as} \quad \xi \to \pm\infty . \qquad (5.8.2a)$$

This is a well known problem (see e.g. Landau & Lifschitz [31]) with solutions only when T_2^* takes the values $T_{2,n}^*$ where

$$T_{2,n}^* = \alpha_o(2n+1)T_2 \tag{5.8.3}$$

and

$$\alpha_o = +\sqrt{-aF''(0)/2} \quad , \tag{5.8.4}$$

for n = 0,1,2,... . Then the appropriate solutions are

$$\psi = \psi_n = \text{const} \,.\, e^{-\xi^2/2} H_n(\xi) \tag{5.8.5}$$

where $H_n(\xi)$ is the Hermite polynomial of order n. In terms of q these are equivalent to

$$\psi_n = \text{const} \,.\, e^{-\alpha_o q^2/2} H_n(q\sqrt{\alpha_o}). \tag{5.8.6}$$

Since the physically most likely flow is given by the lowest eigenvalue we consider the detailed solutions only for n = 0, though other values may be solved.

With the overall normalization $U_2(-\frac{1}{2}, 0) = 1$ and with $u_{11,2}(-\frac{1}{2}, 0) = 1$ as usual, we have $\psi(0) = \frac{1}{2}$ and

$$\psi_o = \frac{1}{2} e^{-\alpha_o q^2/2} \tag{5.8.7}$$

and by considering the equation (5.7.15) for S(q) in the same manner as in § 3.9 etc., we find $T_3^* = 0$ and S(q) is purely imaginary with differential equation

$$\frac{d^2 S_i}{dq^2} + S_i\left[\frac{T_2^*}{T_2} + a\frac{F''(0)}{2}q^2\right] = r_{1i}\frac{F''(0)}{2}q^2\frac{d\psi}{dq} + r_{2i}F''(0)q\psi + r_{3i}\frac{d\psi}{dq} \quad . \tag{5.8.8}$$

The method of solution is to let

$$S_i(q) = \psi(q)R(q) \,, \tag{5.8.9}$$

to obtain

$$\frac{dR}{dq} = \frac{1}{2}\left[r_{3i} + r_{1i}\frac{F''(0)}{2}\ q^{2} + \frac{F''(0)(2r_{2i}-r_{1i})}{\psi^{2}(q)}\int_{-\infty}^{q} q\,\psi^{2}dq\right] \tag{5.8.10}$$

from which we can find $S_i(q)$.

The solutions for U_1, U_4 and U_5 are now written as

$$U_K = 2\psi(q)u_{11,k}^{(r)}\cos\lambda_c\zeta - 2\varepsilon^{\frac{1}{2}}\sin\lambda_c\zeta\left[S_i(q)u_{11,k}^{(r)} + \frac{d\psi}{dq}\,g_{21,k}^{(i)}\right] \tag{5.8.11}$$

for $K = 1,4,5$.

The solutions for U_6 and the non-dimensional Stokes Stream function Φ are

$$U_6 = -2\psi(q)u_{11,6}^{(i)}\sin\lambda_c\zeta + 2\varepsilon^{\frac{1}{2}}\cos\lambda_c\zeta\left[g_{21,6}^{(r)}\frac{d\psi}{dq} - S_i(q)u_{11,6}^{(i)}\right] \tag{5.8.12}$$

and

$$\Phi = \frac{2(1+\delta x)}{\lambda_c}\left[u_{11,4}^{(r)}\psi(q)\sin\lambda_c\zeta + \varepsilon^{\frac{1}{2}}\cos\lambda_c\zeta\,(u_{11,4}^{(r)}S_i(q) + g_{21,4}^{(i)}\frac{d\psi}{dq} + \frac{u_{11,4}^{(r)}}{\lambda_c}\frac{d\psi}{dq})\right]. \tag{5.8.13}$$

We can use these solutions to plot the various velocity components along the q-axis for particular values of x and ε.

For the value of $T = T_{crit}$, for which we obtain a neutral solution such that $\psi \to 0$ as $q \to \pm\infty$, there is a local critical Taylor number. This is given by

$$T_{Lcrit} = T_c + \varepsilon\left[T_2^* + \frac{T_cF''(0)}{2}\left(\frac{3}{2} - \frac{\delta}{4}\right)q^{2}\right]. \tag{5.8.14}$$

The observed or physical wavenumbers for the velocity components U_i and the Stokes Stream function are

$$N_\zeta(U_k) = \lambda_c + \varepsilon\left[\frac{dR}{dq} + \frac{g_{21,k}^{(i)}}{u_{11,k}^{(r)}}\frac{d}{dq}\left(\frac{d\psi}{dq}\Big/\psi\right)\right] \tag{5.8.15}$$

for $k = 1,4,5$ and

$$N_\zeta(\Phi) = N_\zeta(U_4) + \frac{\varepsilon}{\lambda_c}\frac{d}{dq}\left(\frac{d\psi}{dq} \middle| \psi\right) . \tag{5.8.16}$$

The 'local' wavenumber in the parallel wall case that is comparable with our non-parallel wall wavenumber as a correction term

$$\lambda_{Lcrit} = \left(1 + \frac{\varepsilon F''(0)q^2}{4}\right) N_\zeta \tag{5.8.17}$$

from which only terms of $O(\varepsilon)$ are found and kept, all higher terms are ignored.

The 'local' value of η is

$$\eta_L = \frac{\eta}{1+(1-\eta)F(z^*)/2} \tag{5.8.18}$$

which on expansion is equal to

$$\eta_L = \eta - \varepsilon(1-\eta)\eta F''(0)q^2/4 + O(\varepsilon^2) . \tag{5.8.19}$$

5.9 Matching of inner and outer Series

We can only show the plausibility of the matching here since we have not obtained enough details of the outer series.

We first note that the outer series has a factor

$$e^{-\int_0^{z^*} K_i(z^*)dz^*/\varepsilon} \tag{5.9.1}$$

and for $\varepsilon \to 0$ with z^* fixed this is exponentially small if we assume (5.6.8). Also the inner series written in terms of z^* has a factor

$$e^{-\alpha_0 z^{*2}/2\varepsilon} \tag{5.9.2}$$

and so this series has the correct behaviour for $\varepsilon \to 0$ with z^* fixed. To perform a proper matching we would need to write the inner series to order ε^n in terms of an appropriate intermediate variable $\hat{q}$ where

$$\varepsilon^a \hat{q} \quad = \quad z^* \quad , \quad 0 < a < \frac{1}{2} \tag{5.9.3}$$

and the outer series to order ε^n in terms of the same variable $\hat{q}$ and show that the modulus of the difference as $\varepsilon \to 0$ with $\hat{q}$ fixed was $O(\varepsilon^n)$.

Using the inner series to order ε we see that since $\exp\{-\alpha_o \varepsilon^{2a-1} \hat{q}^2/2\}$ is a factor this is exponentially small as $\varepsilon \to 0$ with $\hat{q}$ fixed. Also using (5.6.2) and (5.6.7) we see that the expression (5.9.1) $\sim \exp\{-\alpha_o \varepsilon^{2a-1} \hat{q}^2/2\}$, so that both series have the same exponentially small factor and it is plausible that they match. But to complete the argument we would need to examine in more detail the behaviour of $\psi^*(z^*)$ and $\underline{u}_1(x,z^*)$ as $z^* \to 0$ (see (5.6.10)). In fact, to complete a formal matching in the overlapping sense as described above with n = 0 one needs to prove only that

$$\left| e^{i \int_o^{z^*} K(z^*)dz^*/\varepsilon} \psi^*(z^*) \right| \to 0 \quad \text{as } \varepsilon \to 0 \text{ with } \hat{q} \text{ fixed.} \tag{5.9.4}$$

5.10 The linear case with $\eta = 0.5$

The function $F(z^*)$ is given in (5.1.4) and the value of ω is chosen with the value given in TABLE V for j=1 and n=0.

We shall only consider the cases with $\eta = 0.5$,$\varepsilon = 0.1$ and $\varepsilon = 0.01$ for small q, since the inner solutions are only valid near $z^* = 0$ and not for large q.

The values of λ_c, T_c etc., are given in §4.4a in the appropriate sections, only the value of r_{3i} is different since this depends on the eigenvalue T_2^* yet to be chosen.

The eigenvalue and eigenfunction for n = 0 are

$$T_2^* = a\, T_2 \sqrt{2}/2 = 2332.40 \tag{5.10.1}$$

and

$$\psi(q) = \frac{1}{2} e^{-2.6487q^2} . \tag{5.10.2}$$

The value of r_{3i} in this case is now

$$r_{3i} = 1.2025 . \tag{5.10.3}$$

The graph of $\psi_0(q)$ is plotted in FIG XX and the table of values is given in TABLE XXX. The changing values of η_L, T_{Lcrit} and λ_{Lcrit}, given by (5.8.14), (5.8.17) and (5.8.19), between q = 0 and q = 1 for both values of ε are in TABLE XXXI and TABLE XXXII. A graph of η_L against T_{Lc} and T_{Lcrit} is plotted for each case in FIG XXI and FIG XXIV. These graphs are similar to those of Chapter 4 (see FIG VI and FIG XI).

A similar graph of η_L against λ_{Lc} and λ_{Lcrit} is also plotted in FIG XXII and FIG XXV. There is once again a dependence on the x-co-ordinate, so only the maximum and minimum values of λ_{Lcrit} for each component are plotted. None of the λ_{Lcrit}'s converge as in FIG VII and FIG XII, due to the presence of q^2 in the formulae and therefore we have no lower bound for our solutions. In fact, the distance between $\lambda_{Lcrit,max}$ and $\lambda_{Lcrit,min}$ is constant for all η_L; this can be seen in FIG XXII and FIG XXV where the lines are parallel to each other. The reason why these lines are parallel is due to $\frac{d}{dq}\left(\frac{d\psi}{dq}\Big/\psi\right)$ being a constant.

The variation with x of λ_{Lcrit} for each component is given in TABLE XXXIII and TABLE XXXIV for two fixed values of q. The graphs of this variation are not plotted since these are similar to those of FIG VIII for all values of q.

The solutions for U_4, U_5, U_6 and Φ are summarized for various q values in TABLE XXXV for $\varepsilon = 0.01$ and TABLE XXXVI for $\varepsilon = 0.1$. These solutions are compared with those of the appropriate parallel wall case. Since all these solutions depend on the x-co-ordinate we shall only plot and tabulate for x = 0. The graphs of U_4 are drawn in FIG XXIII and XXVI for each value of ε . The reader should note the similarity with those of FIG IX and FIG XIII.

The reader can find other solutions and results using the appropriate formulae and tables of this chapter and the previous chapter.

q	$\psi_o(q)$	q	$\psi_o(q)$
0	0.5000	1.1	0.0203
0.1	0.4869	1.2	0.0110
0.2	0.4497	1.3	0.0057
0.3	0.3939	1.4	0.0028
0.4	0.3273	1.5	0.0013
0.5	0.2579	1.6	0.0006
0.6	0.1927	1.7	0.0002
0.7	0.1366	1.8	0.0001
0.8	0.0918	1.9	0.0000
0.9	0.0585	2.0	0.0000
1.0	0.0354		

TABLE XXX Values of $\psi_o(q)$ for the case $\eta = 0.5$ and $n = 0$.

q	η_L	T_{Lcrit}	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$_{Lcrit}(U_5)$
0	0.5000	3123.10	3.18201	3.19876	3.18290
0.1	0.5000	3121.56	3.18169	3.19844	3.18258
0.2	0.5002	3116.91	3.18070	3.19746	3.18160
0.3	0.5004	3109.17	3.17907	3.19582	3.17996
0.4	0.5007	3098.33	3.17678	3.19354	3.17767
0.5	0.5012	3084.40	3.17384	3.19059	3.17473
0.6	0.5017	3067.37	3.17025	3.18700	3.17114
0.7	0.5023	3047.24	3.16600	3.18275	3.16689
0.8	0.5030	3024.02	3.16110	3.17785	3.16199
0.9	0.5038	2997.70	3.15555	3.17230	3.15644
1.0	0.5047	2968.28	3.14934	3.16609	3.15023

TABLE XXXI Values of η_L, T_{Lcrit} and λ_{Lcrit} as given by equations (5.8.19), (5.8.14) and (5.8.17) for the case $\eta = 0.5$, $n = 0$, $\varepsilon = 0.01$ and $x = 0$.

q	η_L	T_{Lcrit}	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$
0	0.5000	3333.02	3.35834	3.52585	3.36724
0.1	0.5005	3317.54	3.35507	3.52258	3.36397
0.2	0.5019	3271.09	3.34527	3.51278	3.35417
0.3	0.5042	3193.68	3.32893	3.49644	3.33783
0.4	0.5075	3085.31	3.30606	3.47357	3.31496
0.5	0.5117	2945.97	3.27665	3.44417	3.28555
0.6	0.5169	2775.67	3.24071	3.40822	3.24961
0.7	0.5229	2574.41	3.19823	3.36575	3.20714
0.8	0.5300	2342.18	3.14922	3.31674	3.15812
0.9	0.5379	2078.98	3.09368	3.26119	3.10258
1.0	0.5468	1784.83	3.03159	3.19911	3.04050

TABLE XXXII Values of η_L, T_{Lcrit} and λ_{Lcrit} as given by the same equations as in TABLE XXXI for the case $\eta = 0.5$, $n = 0$, $\varepsilon = 0.1$ and $x = 0$.

	$q = 0$			$q = 1$		
x	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$
-0.5	3.18823	3.20498	3.18285	3.15556	3.17231	3.15018
-0.4	3.18659	3.20334	3.18316	3.15392	3.17067	3.15049
-0.3	3.18515	3.20190	3.18351	3.15247	3.16922	3.15084
-0.2	3.18389	3.20065	3.18357	3.15122	3.16797	3.15090
-0.1	3.18284	3.19959	3.18334	3.15017	3.16692	3.15067
0	3.18201	3.19876	3.18290	3.14934	3.16609	3.15023
0.1	3.18146	3.19821	3.18231	3.14879	3.16554	3.14963
0.2	3.18125	3.19800	3.18160	3.14858	3.16533	3.14892
0.3	3.18145	3.19821	3.18081	3.14878	3.16553	3.14814
0.4	3.18213	3.19889	3.18004	3.14946	3.16621	3.14736
0.5	3.18335	3.20010	3.17969	3.15067	3.16743	3.14701

TABLE XXXIII Variation in x for wavenumbers of Φ, U_4 and U_5 given by equations (5.8.15) and (5.8.17) for the case $\eta = 0.5$, $n = 0$, $\varepsilon = 0.01$ with $q = 0$ and $q = 1$.

	$q = 0$			$q = 0.5$		
x	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$	$\lambda_{Lcrit}(\Phi)$	$\lambda_{Lcrit}(U_4)$	$\lambda_{Lcrit}(U_5)$
-0.5	3.42055	3.58806	3.36674	3.33886	3.50637	3.28505
-0.4	3.40412	3.57163	3.36986	3.32243	3.48995	3.28817
-0.3	3.89690	3.55720	3.37335	3.30800	3.47551	3.29167
-0.2	3.37716	3.54468	3.37394	3.29548	3.46299	3.29225
-0.1	3.36661	3.53413	3.37166	3.28493	3.45244	3.28998
0	3.35834	3.52585	3.36724	3.27665	3.44417	3.28555
0.1	3.35283	3.52034	3.36129	3.27114	3.43866	3.27960
0.2	3.35074	3.51825	3.35420	3.26905	3.43656	3.27251
0.3	3.35276	3.52028	3.34634	3.27108	3.43859	3.26465
0.4	3.35957	3.52708	3.33858	3.27788	3.44540	3.25689
0.5	3.37171	3.53922	3.33508	3.29002	3.45754	3.25340

TABLE XXXIV Variation in x for wavenumbers of Φ , U_4 and U_5 for the case $\eta = 0.5$, $n = 0$, $\varepsilon = 0.1$ with $q = 0$ and $q = 0.5$.

q	Φ	$\Phi^{(p)}$	U_4	$U_4^{(p)}$	U_5	$U_5^{(p)}$	U_6	$U_6^{(p)}$
0	0.0000	0.0000	-6.1209	-6.1209	0.2793	0.2793	0.0000	0.0000
0.1	0.0763	0.0403	5.9551	6.1196	-0.2718	-0.2793	0.0375	0.0128
0.2	-0.1419	-0.0806	-5.4840	-6.1156	0.2506	0.2791	-0.0695	-0.0257
0.3	0.1883	0.1209	4.7797	6.1090	-0.2188	-0.2788	0.0919	0.0385
0.4	-0.2115	-0.1611	-3.9422	-6.0997	0.1809	0.2783	-0.1026	-0.0513
0.5	0.2120	0.2012	3.0763	6.0877	-0.1416	-0.2778	0.1022	0.0640
0.6	-0.1941	-0.2412	-2.2707	-6.0732	0.1049	0.2771	-0.0928	-0.0768
0.7	0.1644	0.2812	1.5848	6.0560	-0.0735	-0.2764	0.0779	0.0895
0.8	-0.1297	-0.3210	-1.0454	-6.0361	0.0488	0.2755	-0.0609	-0.1022
0.9	0.0958	0.3607	0.6514	6.0137	-0.0306	-0.2744	0.0445	0.1148
1.0	-0.0664	-0.4002	-0.3832	-5.9886	0.0181	0.2733	-0.0305	-0.1274
1.1	0.0433	0.4396	0.2126	5.9610	-0.0102	-0.2720	0.0197	0.1399
1.2	-0.0266	-0.4787	-0.1111	-5.9307	0.0054	0.2706	-0.0119	-0.1524

TABLE XXXV Solutions of U_4, U_5, U_6 and Φ , and the corresponding parallel wall solutions, for the case $\eta = 0.5$, $n = 0$, $x = 0$ and $\varepsilon = 0.01$.

q	Φ	$\Phi^{(p)}$	U_4	$U_4^{(p)}$	U_5	$U_5^{(p)}$	U_6	$U_6^{(p)}$
0	0.0000	0.0000	-6.1209	-6.1209	0.2793	0.2793	0.0000	0.0000
0.1	-1.6495	-1.6287	-2.6426	-3.3069	0.1321	0.1509	-0.5476	-0.5185
0.2	-1.4916	-1.7599	3.4534	2.5477	-0.1347	-0.1163	-0.4428	-0.5602
0.3	0.0770	-0.2729	5.0140	6.0598	-0.2241	-0.2765	0.1246	-0.0869
0.4	1.1790	1.4650	1.1698	4.0001	-0.0811	-0.1825	0.4485	0.4663
0.5	0.8594	1.8559	-2.7413	-1.7376	0.0905	0.0793	0.2423	0.5908
0.6	-0.1013	0.5403	-2.7589	-5.8776	0.1169	0.2682	-0.1272	0.1720
0.7	-0.5584	-1.2721	-0.2706	-4.6133	0.0300	0.2105	-0.2394	-0.4049
0.8	-0.3184	-1.9148	1.3452	0.8928	-0.0410	-0.0407	-0.0922	-0.6095
0.9	0.0639	-0.7969	1.0179	5.5780	-0.0404	-0.2545	0.0614	-0.2537
1.0	0.1778	1.0537	0.0263	5.1344	-0.0066	-0.2343	0.0820	0.3354
1.1	0.0781	1.9355	-0.4076	-0.0301	0.0123	0.0014	0.0248	0.6161
1.2	-0.0213	1.0377	-0.2488	-5.1669	0.0094	0.2358	-0.0163	0.3303

TABLE XXXVI Solutions of U_4, U_5, U_6 and Φ, and the corresponding parallel wall solutions, for the case $\eta = 0.5$, $n = 0$, $x = 0$ and $\varepsilon = 0.1$.

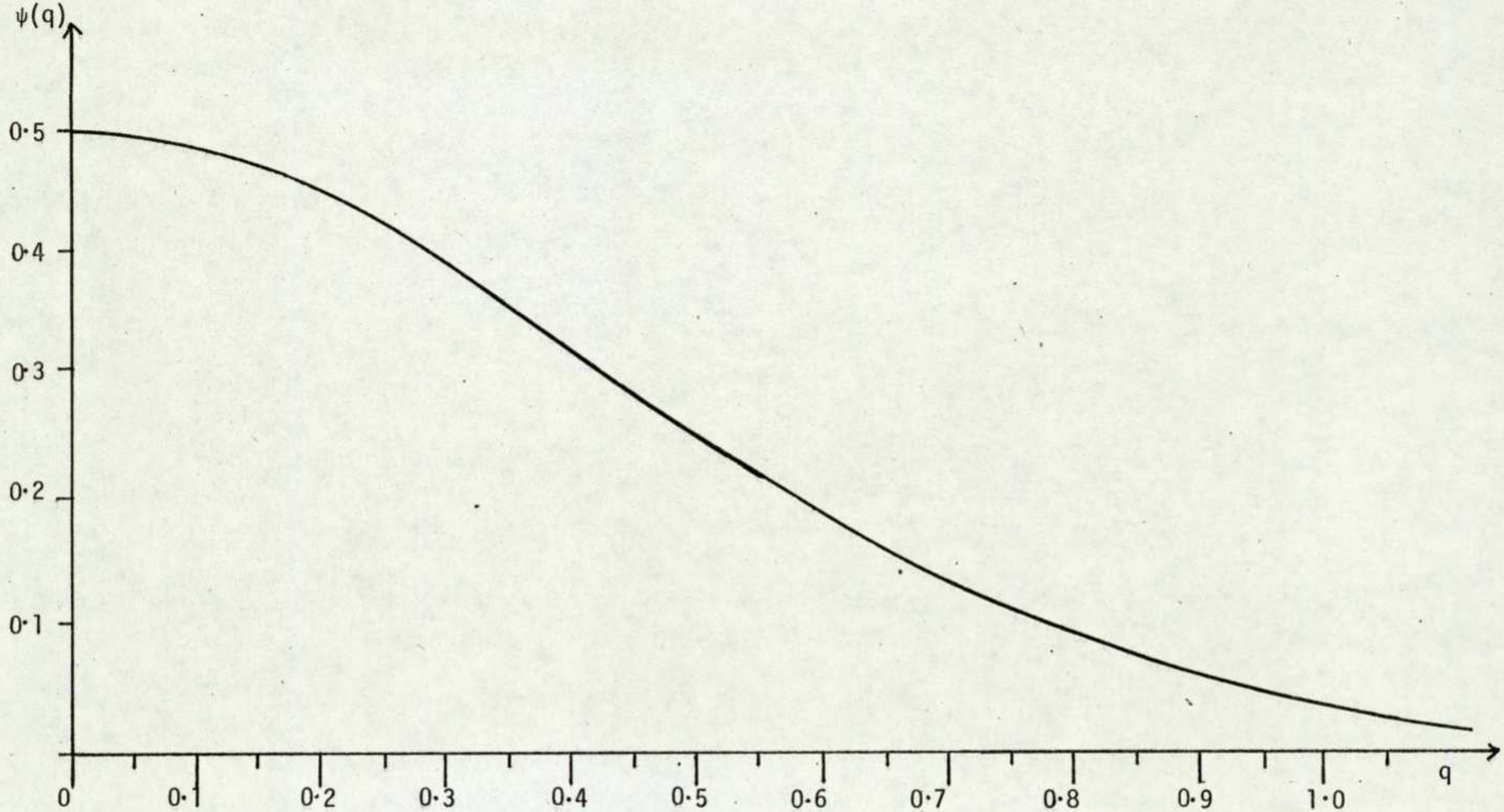

FIG XX Graph of $\psi_o(q)$ for $\eta = 0.5$ and $n = 0$.

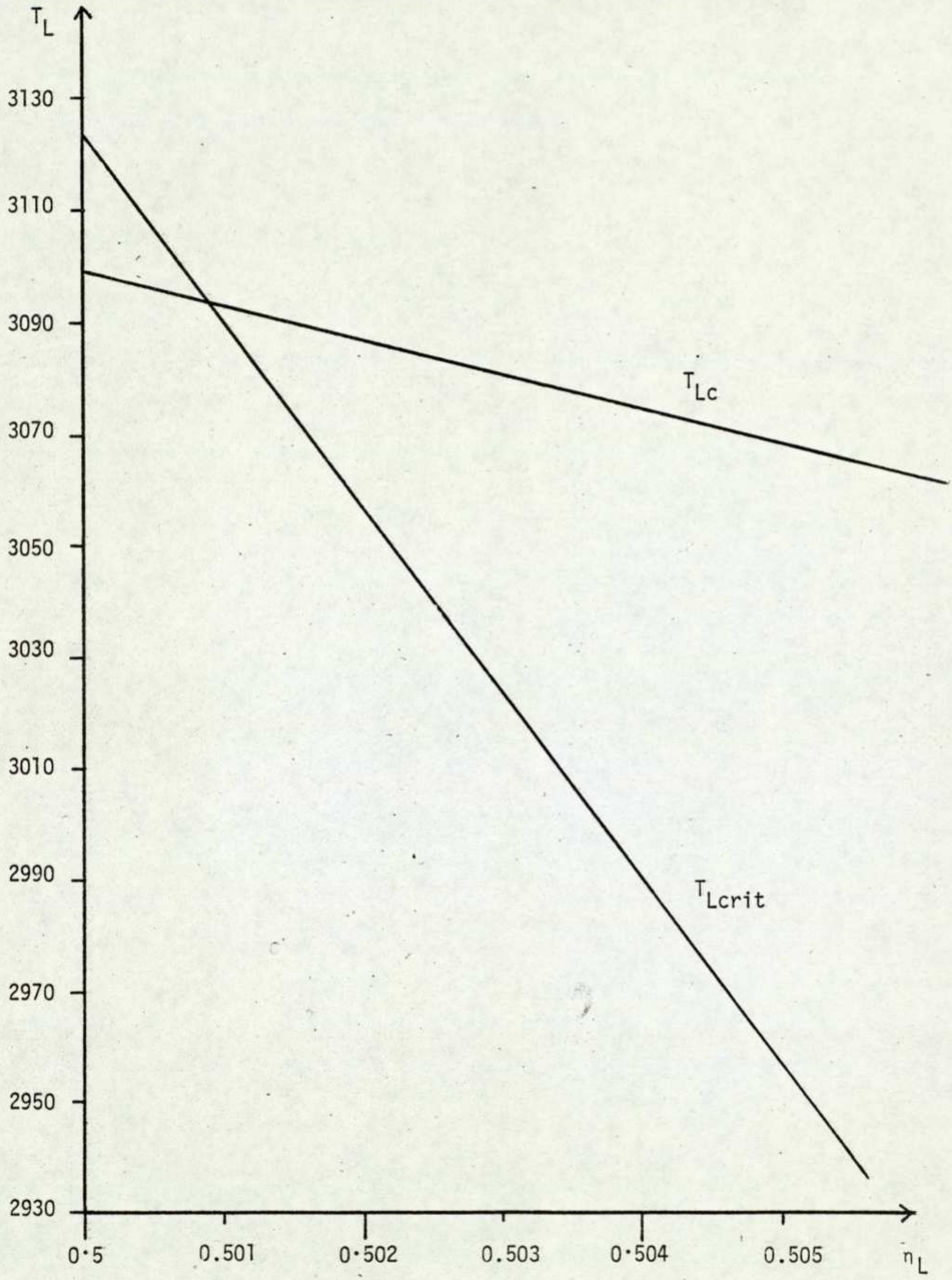

FIG XXI Graph of T_{Lc}, T_{Lcrit} against n_L for $\eta = 0.5$, $n = 0$ and $\varepsilon = 0.01$

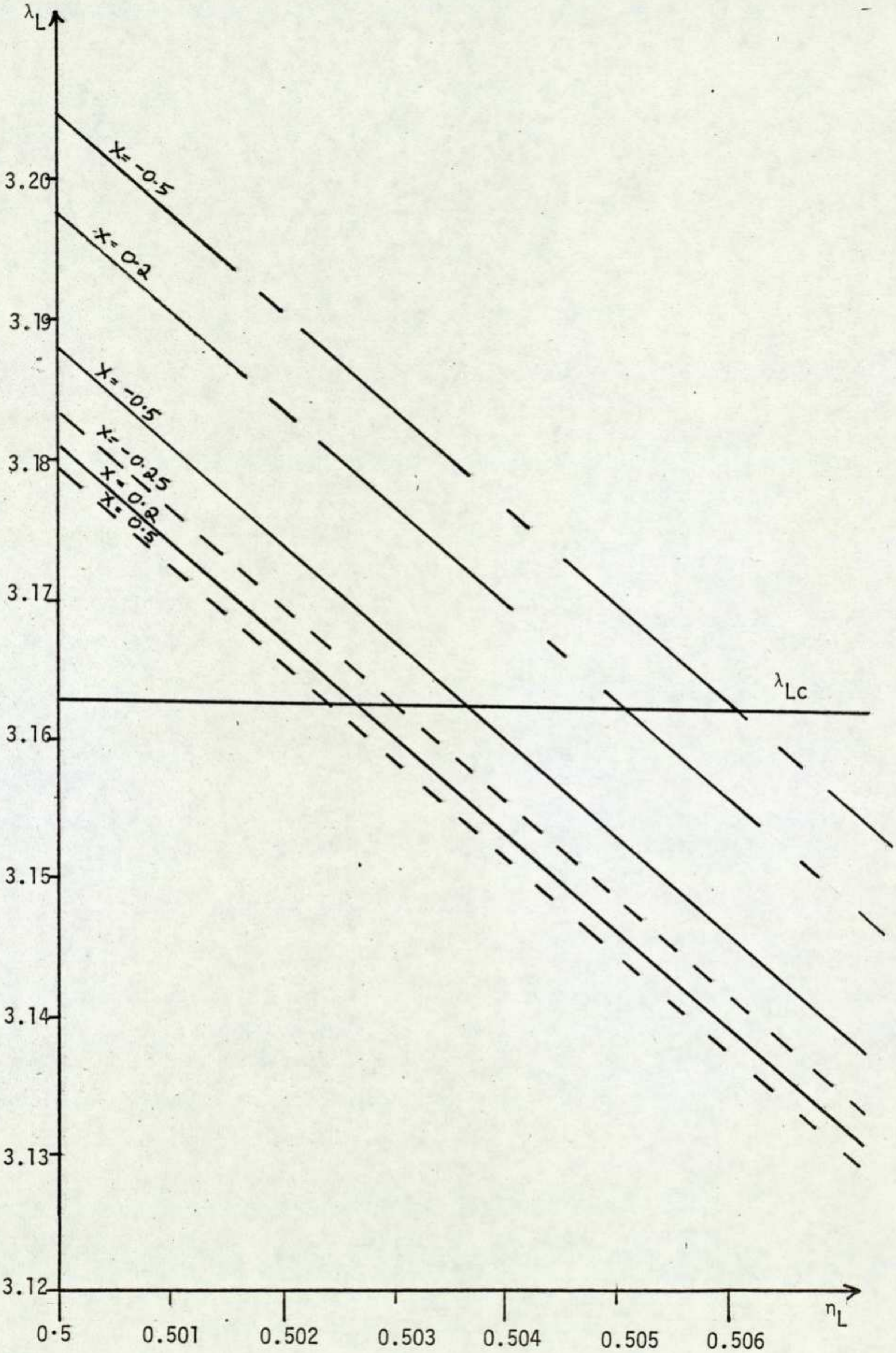

FIG XXII Graph of λ_{Lc}, λ_{Lcrit} for U_4, U_5 and Φ against η_L, for $\eta = 0.5$, selected x values, n = 0 and $\varepsilon = 0.01$. See FIG VII .

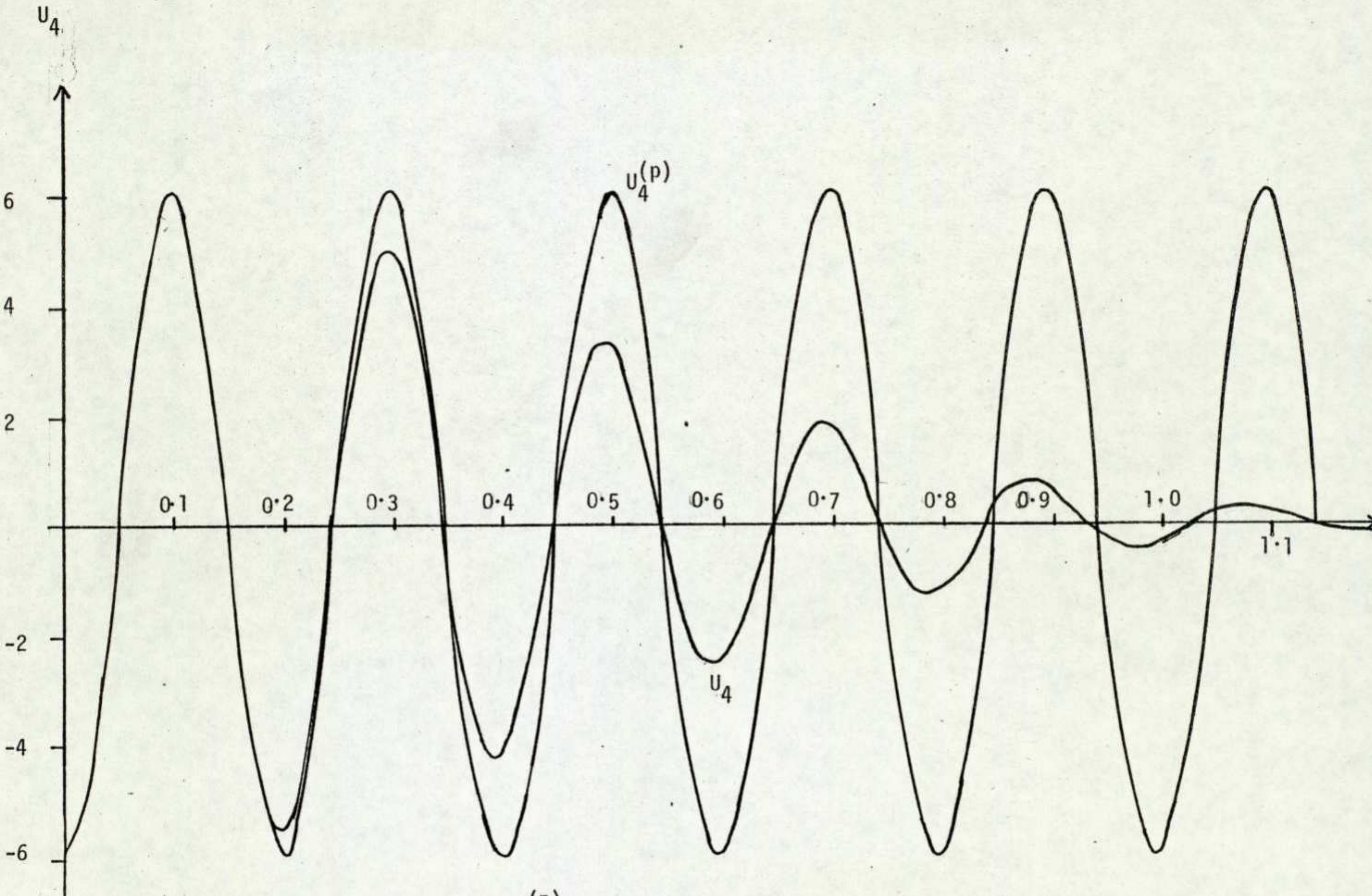

FIG XXIII Linear solutions for U_4, $U_4^{(p)}$ for $\eta = 0.5$, $x = 0$, $n = 0$ and $\varepsilon = 0.01$.

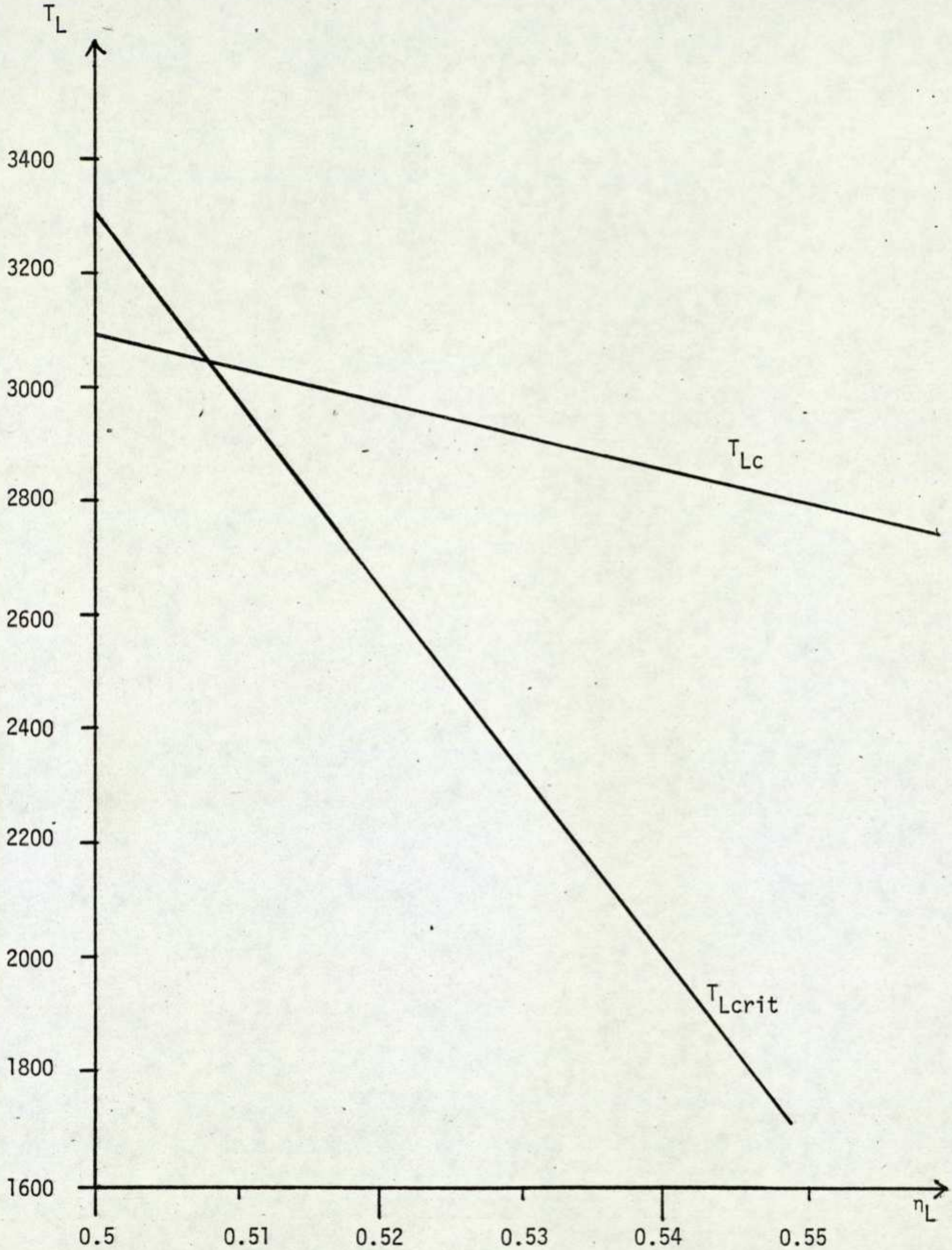

FIG XXIV Graph of T_{Lc}, T_{Lcrit} against η_L for $\eta = 0.5$, $n = 0$ and $\varepsilon = 0.1$.

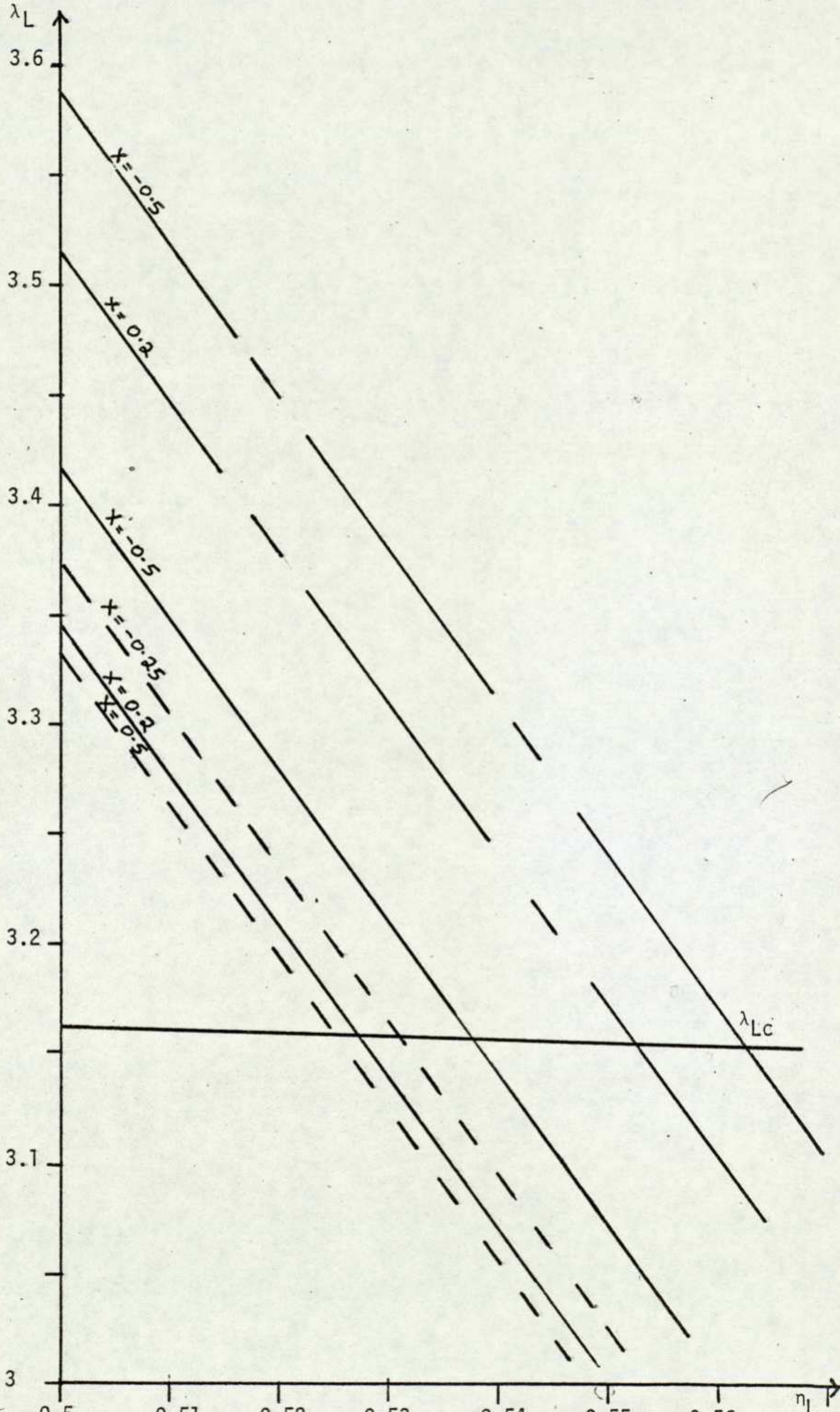

FIG XXV Graph of λ_{Lc}, λ_{Lcrit} for U_4, U_5 and Φ against η_L, for $\eta = 0.5$, various x values, $n = 0$ and $\varepsilon = 0.1$. See FIG VII.

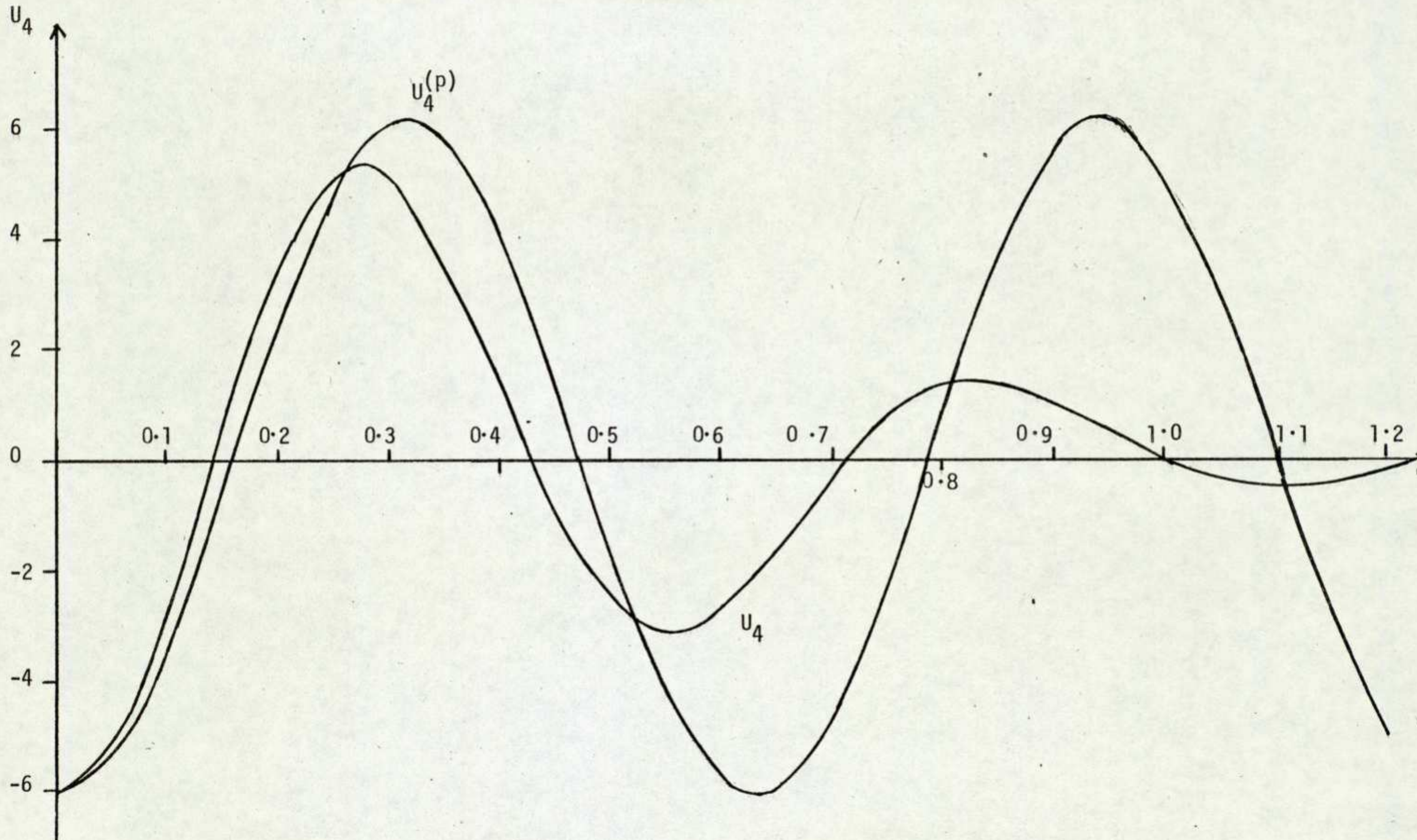

FIG XXVI Linear solutions for U_4, $U_4^{(p)}$ for $\eta = 0.05$, $x = 0$, $n = 0$ and $\varepsilon = 0.1$.

6. Some Miscellaneous Results

6.1 Introduction

This chapter involves trying to explain some anomalies that arose in the previous chapters. These being :-

(1) Why no solutions could be found if we took $\lambda = \lambda_o$ and $T = T_o$ initially in the expansion procedure with $\psi \to 0$ as $z^* \to \pm \infty$.

(2) Why no solutions could be found near $\lambda = \lambda_c$ and $T = T_c$ with $\psi \to 0$ as $z^* \to \pm \infty$.

(3) Also the solutions of the amplitude equations for $\psi(z^*)$ and $S(z^*)$ with the boundary condition, ψ and $S \to A$(constant) as $z^* \to \pm \infty$ are found.

6.2 An attempt at a solution with $\lambda = \lambda_o$ and $T = T_o$

We again assume that η and $f(z^*)$ are fixed and expand the disturbance velocity $\underline{U}$, and look for solutions of the form

$$\underline{U} = e^{i\lambda_o\zeta}\underline{u}(x,z^*,\varepsilon) + e^{-i\lambda_o\zeta}\underline{\tilde{u}}(x,z^*,\varepsilon) \qquad (6.2.1)$$

where λ_o is a real constant. We now search for the eigenvalue T such that $\underline{U}$ satisfies the boundary conditions (3.5.25) and (3.5.26).

We are led to expand T as

$$T = T_o + \varepsilon T_1^* + \varepsilon^2 T_2^* + \dots , \qquad (6.2.2)$$

remembering that (λ_o, T_o) lie on the neutral curve for a fixed η .

Following the same sort of procedure as in Chapter 3 we attempt to find a solution for $\underline{u}_1(x,z^*)$ as

$$\underline{u}_1(x,z^*) = \psi(z^*)\underline{u}_{11}(x) \tag{6.2.3}$$

where $\underline{u}_{11}(x)$ is the eigensolution of the parallel wall problem with parameters η , T_o and λ_o. The boundary condition on $\psi(z^*)$ is

$$\psi(z^*) \to 0 \quad \text{as} \quad z^* \to \pm\infty . \tag{6.2.4}$$

The reader is reminded that any matrix mentioned in Chapter 3 containing λ_c, T_c is now replaced by λ_o, T_o respectively. These 'new' matrices will be denoted by $\underline{A}_{01}^{(1)}$, $\underline{A}_0^{(1)}$ etc.

We find that for $\underline{u}_2\ (x,z^*)$ to have a solution we must satisfy

$$\frac{d\psi}{dz^*} \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\, \underline{B}_{01}^{(1)}\, \underline{u}_{11}\, dx + \psi T_1^* \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{A}_2 \underline{u}_{11}\, dx = 0, \tag{6.2.5}$$

where from the parallel wall case we know

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\, \underline{B}_{01}^{(1)}\, \underline{u}_{11}\, dx \neq 0 . \tag{6.2.6}$$

If we use the result of (3.6.15) and the following parallel wall result

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t}\, \underline{A}_{01}^{(1)}\, \underline{f}_{11}\, dx + T_1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\sim}{f}^{a,t} \underline{A}_2 \underline{f}_{11}\, dx = 0, \tag{6.2.7}$$

equation (6.2.5) can be simplified to give

$$\frac{d\psi}{dz^*} - i\,\frac{T_1^*}{T_1}\,\psi = 0. \tag{6.2.8}$$

The solution of (6.2.8) is

$$\psi(z^*) = A\, e^{i\frac{T_1^*}{T_1} z^*} \tag{6.2.9}$$

and the solution which satisfies our boundary condition (6.2.4) can be seen to be given by

$$\psi(z^*) = 0 \text{ for all } z^* , \tag{6.2.10}$$

which in turn implies we obtain the trivial solution of

$$\underline{U} = 0 \tag{6.2.11}$$

for our problem.

This result implies we can only obtain a solution with $\underline{U} \to 0$ as $z^* \to \pm\infty$ if the parameters chosen from the local neutral curve for fixed η are λ_c, T_c.

This result however eliminates the problem of which λ to choose given a particular T, remembering that for any value of T, except for $T = T_c$, there exists two values of λ which lie on the neutral curve.

6.3 An attempt at a solution near $\lambda = \lambda_c$ and $T = T_c$

Following the result of §6.2 it was thought that if we modify the form of expansion (3.6.8) and use

$$\underline{U} = e^{i(\lambda_c+q)\zeta}\, \underline{u}(x,z^*,\varepsilon,q) + \text{c.c.} \tag{6.3.1}$$

with a wavenumber $\lambda_c + q$ close to λ_c (small q) we would obtain a solution with $\underline{U} \to 0$ as $z^* \to \pm\infty$ close to the one already obtained in Chapter 3, and a value of T close to $T_{crit} = T_c + \varepsilon^2 T_2^*$, and thus we would obtain other solutions close to our first solution by varying q. However this attempt leads to an inconsistency.

We expand T as

$$T = T_c + \varepsilon T_\varepsilon + q\,T_q + \varepsilon^2 T_{\varepsilon\varepsilon} + \varepsilon q\,T_{\varepsilon q} + q^2 T_{qq} + \ldots \quad (6.3.2)$$

and the velocity vector $\underline{u}$ as

$$\underline{u} = \underline{u}_1 + \varepsilon\underline{u}_\varepsilon + q\underline{u}_q + \varepsilon^2\underline{u}_{\varepsilon\varepsilon} + \varepsilon q\underline{u}_{\varepsilon q} + q^2\underline{u}_{qq} + \ldots, \quad (6.3.3)$$

and substitute these equations along with (6.3.1) into (3.5.22). When we equate powers of $\varepsilon^n q^m$ we obtain a set of partial differential equations.

The boundary conditions on $\underline{u}_{\varepsilon\varepsilon}$ are, for j = 4,5,6,

$$u_{\varepsilon\varepsilon,j} = -f(z^*)u_{1x,j}/2 \quad \text{and} \quad u_{\varepsilon\varepsilon,j} = 0 \quad (6.3.4)$$

on the outer and inner wall respectively, $u_{\varepsilon\varepsilon,j}$ denoting the j^{th} component of $\underline{u}_{\varepsilon\varepsilon}$. The boundary conditions on the rest of the vectors are given by β_2 defined in (2.4.1).

The analysis is similar to §2.6 and §3.7 and we therefore only summarize the results.

From our equations we find that

$$\underline{u}_1 = \psi(z^*)\underline{u}_{11}(x) \quad (6.3.5)$$

where $\underline{u}_{11}(x)$ satisfies (3.7.2). The existence conditions on the partial differential equations at $O(\varepsilon)$ and $O(q)$ lead to

$$T_\varepsilon = T_q = 0. \quad (6.3.6)$$

We find the equations at $O(\varepsilon)$ and $O(q)$ can be solved subject to the boundary conditions β_2 and written as

$$\underline{u}_{\varepsilon\varepsilon} = \frac{d\psi}{dz^*}\underline{g}_{21}(x) + S_\varepsilon(z^*)\underline{u}_{11}(x) \tag{6.3.7}$$

and

$$\underline{u}_q = i\psi(z^*)\underline{g}_{21} + S_q(z^*)\underline{u}_{11} \tag{6.3.8}$$

where $\underline{g}_{21}(x)$ satisfies (3.7.10).

The equation at order εq now becomes

$$\frac{\partial \underline{u}_{\varepsilon q}}{\partial x} - \underline{A}_c^{(1)}\underline{u}_{\varepsilon q} = \psi T_{\varepsilon q}\underline{A}_2\underline{u}_{11} + 2i\frac{d\psi}{dz^*}\underline{B}_{c2}\underline{u}_{11} + 2i\frac{d\psi}{dz^*}\underline{B}_{c1}^{(1)}\underline{g}_{21}$$
$$+ \frac{dS_q}{dz^*}\underline{B}_{c1}^{(1)}\underline{u}_{11} + iS_\varepsilon(z^*)\underline{B}_{c1}^{(1)}\underline{u}_{11};\beta_2. \tag{6.3.9}$$

The existence condition for $\underline{u}_{\varepsilon q}$ to have a solution leads to

$$2iT_2\frac{d\psi}{dz^*} + \psi T_{\varepsilon q} = 0 \tag{6.3.10}$$

or

$$\frac{d^2\psi}{dz^{*2}} + \left(\frac{T_{\varepsilon q}}{2T_2}\right)^2\psi = 0, \tag{6.3.10a}$$

where T_2 is defined in (2.6.10). Proceeding to the term of $O(\varepsilon^2)$ we found the amplitude equation (3.7.19) is obtained without alteration. But this is inconsistent with (6.3.10a).

6.4 The solutions with ψ and $S \to$ constant as $z^* \to \pm\infty$

In Chapters 3 and 4 we considered the case with $\underline{U} \to 0$ as $z^* \to \pm\infty$. This boundary condition forced the discrete spectrum of eigenvalues for T_2^* in the amplitude equation for $\psi(z^*)$ to have an upper-bound, given by

$$T_2^* < a\,T_2(-f_\infty) \tag{6.4.1}$$

where $f_\infty = \lim_{z^* \to \pm\infty} f(z^*)$ which in our case is negative.

If however we choose

$$T_2^* = aT_2(-f_\infty) \tag{6.4.2}$$

this implies that the asymptotic behaviour of ψ is

$$\psi \to A \quad \text{as} \quad z^* \to \pm\infty \tag{6.4.3}$$

where A is a real constant, see §3.8.

These solutions for ψ with the eigenvalue given by (6.4.2) lead to different solutions for T_{Lcrit} and λ_{Lcrit} . These numerical results were of some interest when evaluated at $z^* = \pm\infty$, though the proposed theoretical answer was not proved.

We look for solutions of (3.7.19) with $f(z^*)$ and T_2^* replaced by

$$f(z^*) = \operatorname{sech}^2 \omega z^* - 1 \tag{6.4.4}$$

and

$$T_2^* = a\, T_2 \tag{6.4.5}$$

respectively in (3.7.19). The equation for ψ becomes

$$\frac{d^2\psi}{dz^{*2}} + a\,\psi \operatorname{sech}^2 \omega z^* = 0 \tag{6.4.6}$$

subject to the boundary condition given in (6.4.3).

We can look for a series solution of (6.4.6) in the form

$$\psi = A + \sum_{k=1}^{\infty} c_k \operatorname{sech}^{k+1} \omega z^* . \tag{6.4.7}$$

The analysis that would follow shows for even values of j defined in (4.3.2) and (4.3.3) the series terminates.

For example for j = 2, the solution is

$$\psi = A[1 - \frac{3}{2} \operatorname{sech}^2 \omega z^*] \tag{6.4.8}$$

while for j = 4, the solution is

$$\psi = A[1 + \frac{35}{8} \operatorname{sech}^4 \omega z^* - 5 \operatorname{sech}^2 \omega z^*] \quad . \tag{6.4.9}$$

The solutions for ψ for odd values of j show the series does not terminate but remains a series of the form

$$\psi = A + \sum_{k=1}^{\infty} c_{2k-1} \operatorname{sech}^{2k} \omega z^*. \tag{6.4.10}$$

For example for j = 1, the solution can be written as

$$\psi = A - \frac{A}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{\Gamma(k-\frac{1}{2})}{\Gamma(k+1)} \operatorname{sech}^{2k} \omega z^* \; . \tag{6.4.11}$$

The amplitude equation for $S_i(z^*)$ with $T_2^* = aT_2$ is given by

$$\frac{d^2 S_i}{dz^{*2}} + S_i \, a \operatorname{sech}^2 \omega z^* = r_{1i} f(z^*) \frac{d\psi}{dz^*} + r_{2i} \psi \frac{df}{dz^*} + r_{3i} \frac{d\psi}{dz^*} \tag{6.4.12}$$

subject to the boundary condition

$$S_i \to A \quad \text{as} \quad z^* \to \pm \infty, \tag{6.4.13}$$

with ψ given in (6.4.11), $f(z^*)$ given in (6.4.4) and r_{3i} dependent on the eigenvalue T_2^* .

The solution for $S_i(z^*)$ is found by placing $S = \psi R$ and we obtain

$$\frac{dR}{dz^*} = \frac{1}{2} \left[r_{3i} + r_{1i} f(z^*) + \frac{2r_{2i} - r_{1i}}{\psi^2} \int_{-\infty}^{z^*} \psi^2 \frac{df}{dz^*} \, dz^* \right]. \tag{6.4.14}$$

We are only interested in the value of λ_{Lcrit} evaluated at $z^* = \pm \infty$ and so able to write

$$\lambda_{Lcrit,\infty} = \lim_{z^*\to\pm\infty} \left\{\lambda_c + \varepsilon^2\left[\frac{dR}{dz^*} + \frac{g^{(i)}_{21,k}}{u^{(r)}_{11,k}}\frac{d}{dz^*}\left(\frac{d\psi}{dz^*}\Big/\psi\right) - \lambda_c\frac{\tanh^2\omega z^*}{2}\right]\right\} \tag{6.4.15}$$

see §4.3.

When we use the form of $\frac{dR}{dz^*}$ given in (6.4.14) and evaluate this at $z^* = \infty$ it can be shown that

$\left[\frac{d}{dz^*}\left(\frac{d\psi}{dz^*}\Big/\psi\right)\right] \to 0$ as $z^* \to \pm\infty$, and we note

$$\lambda_{Lcrit,\infty} = \lambda_c + \frac{\varepsilon^2}{2}\,[r_{3i} - r_{1i} - \lambda_c] \tag{6.4.16}$$

irrespective of the velocity components U_i and Φ. The integrand $\psi^2\frac{df}{dz^*}$ is odd and therefore when integrated over $(-\infty,\infty)$ its value is zero.

Using the formula for T_{Lcrit} given in (4.3.17) evaluated at $z^* = \pm\infty$, with $T_2^* = a\,T_2$ we have

$$T_{Lcrit,\infty} = T_c + \varepsilon^2[a\,T_2 - T_c(\tfrac{3}{2} - \tfrac{\delta}{4})]\,. \tag{6.4.17}$$

The numerical results seem to indicate given an ε and therefore fixing η_L at $z^* = \pm\infty$ to be $\eta_{L,\infty}$ that

$$\lambda_{Lcrit,\infty} \triangleq \lambda_{Lc,\infty} \quad\text{and}\quad T_{Lcrit,\infty} \triangleq T_{Lc,\infty}\,. \tag{6.4.18}$$

However, no proof of this is given as the theory is extremely difficult and was not completed.

6.5 Results for §6.4 with $\eta = 0.5$

From our computing methods the following constants were found to be

$$T_2^* = 3298.51 \quad \text{and} \quad r_{3i} = 1.7006. \tag{6.5.1}$$

We carefully choose ε given by

$$\varepsilon^2 = \frac{2(\eta_L - \eta)}{\eta_L(1-\eta)} \quad . \tag{6.5.2}$$

This equation is obtainable from (3.8.21). We are able, by selecting ε, to fix η_L at $z^* = \infty$ to be any value we choose.

With our fixed value for $\eta_{L,\infty}$ and ε we can calculate $\lambda_{Lcrit,\infty}$ and $T_{Lcrit,\infty}$, given in (6.4.16) and (6.4.17), and so compare these values with those of λ_{Lc} and T_{Lc} evaluated from the parallel wall case with $\eta = \eta_{L,\infty}$.

The results are given in TABLE XXXVII .

$\eta_L = \eta_{L,\infty}$	T_{Lc}	λ_{Lc}	ε^2	$T_{Lcrit,\infty}$	$\lambda_{Lcrit,\infty}$
0.5	3099.78	3.1624	-	-	-
0.51	3034.68	3.1607	0.0784	3034.32	3.1607
0.52	2972.83	3.1591	0.1538	2971.39	3.1590
0.53	2895.02	3.1571	0.2500	2891.15	3.1569

TABLE XXXVII. Values of T_{Lc} , λ_{Lc} , $T_{Lcrit,\infty}$ and $\lambda_{Lcrit,\infty}$

Appendix

Computation and checks

The eigenvalue problem of (2.4.4) was solved using the method fully explained by Eagles [10]. In solving for the eigenfunction of (2.4.8) and later similar differential equations the method of Eagles [10] was used, except that his adjoint eigenfunction $\underline{f}_o$ is replaced by $\tilde{f}^a$ in this thesis. The non-homogeneous differential equations were also solved using the method of Eagles [10].

It should be noted that equations containing $\mathcal{L}^{(0)}(\ \ )$ on the left hand side of a differential equation, for example (2.11.4) and (3.16.18), require special treatment because there exists an eigenfunction with its first components = constant and all the other components equal to zero. The solutions for $\underline{f}_{20}(x)$, $\underline{g}_{30}(x)$ etc., may be taken as normalized with the first component equal to zero at $x = -1/2$. Any other normalization does not alter subsequent functions because the column vectors $\underline{R}_o(\ \ )$ do not contain any first component of $\underline{f}_{20}(x)$, $\underline{g}_{30}(x)$ etc.

Computing checks were made by varying the normalization of the functions $\underline{g}_{21}(x)$ and $\underline{g}_{31}(x)$, see §2.7. We verified the equations (2.7.11) and (2.7.15) were true and that the values of $T_1, T_2, T_3, \ldots$ remain invariant under different normalizations.

The results of Chapter 2 concerning the neutral curve and the torque calculations agree with those of Di Prima & Eagles [4].

In Chapter 3 the constant b_0, given by solving the relationship (3.4.36) was calculated both by specifying values of R_1 and R_2 separately, and also by integrating with respect to the non-dimensional x variable where the integrand contains just η. The results in both cases agreed to four decimal places.

A number of comparisons were made between Chapter 2 and Chapter 3 which gave indirect checks on the calculations and computing, for example (3.7.12) and (3.9.7). We knew the results of Chapter 2 were more or less accurate because these were compared with those of Eagles & Di Prima [4].

An unexpected result was that a different normalization of $\underline{g}_{21}(x)$ etc., in Chapter 3 gave a different numerical value for only the integral r_2, given in (3.9.14), for a fixed value of T_2^*. However, the resultant theory showed why, see (3.10.18). All the relationships between $\underline{h}_{31}$, $\hat{\underline{h}}_{31}$ etc., in Chapter 3, see (3.10.8), (3.10.12) and (3.10.13), along with further integral checks (changing the value of T_2^* to verify (3.9.22)) were all used as numerical checks for the computing. Similar checks were carried out for the non-linear theory of Chapter 3, see (3.16.27), (3.17.6) and (3.17.15).

In Chapter 4 the analytic and numerical solutions for the slowly varying linear amplitude function $\psi(z^*)$ agreed for a fixed value of T_2^*. The numericalsolution was obtained using a Runga Kutta routine. The program was then modified to solve the non-linear function $\psi_N(z^*)$ by first fixing T_2', and by a trial and error process found $\psi_N(0)$ such that $\psi_N(z^*) \to 0$ as $z^* \to \pm\infty$.

Although we have not emphasised the computing side of the work, and it is theoretically easy, the details and checking of results have consituted a considerable task. Since this thesis is already long enough we do not give any further details, except to say the programming was done in Fortran, using the maximop terminal system with the help of the City University Computer advisory service.

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