The tilting tensor product theorem and decomposition numbers for symmetric groups

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Abstract. We show how the tilting tensor product theorem for algebraic groups implies a reduction formula for decomposition numbers of the symmetric group. We use this to prove generalisations of various theorems of Erdmann and of James and Williams.

Let G be a semisimple, connected, simply-connected algebraic group over an algebraically closed field k of characteristic $p > 0$. The simple modules $L(\lambda)$ are labelled by dominant weights, and by Steinberg's tensor product theorem [24] it is sufficient to determine only a finite number of simple modules (those labelled by p-restricted weights). For sufficiently large p these are known: by work of Andersen et al [2] they are given by the Lusztig conjecture [19].

In [6] Donkin has introduced the notion of a tilting module for G (following Ringel [21]). The indecomposable tilting modules $T(\lambda)$ are also labelled by dominant weights. There is also a tensor product theorem for tilting modules (which will be key in what follows), however it no longer reduces the problem to the study of a finite set of tilting modules. Indeed, the determination of tilting modules is considerably harder than that of simples; there are not even conjectural solutions.

The representation theory of the symmetric group Σ_d over k is also poorly understood. For each partition λ there is a Specht module S^λ, and determining decomposition numbers of these would be sufficient to determine the simples. However little is known except in the case of two part partitions [10], and certain three part partitions [13].

Erdmann has shown [8] that the determination of the decomposition numbers associated to Specht modules indexed by n part partitions is equivalent (by Ringel duality) to determining the good filtration multiplicities of tilting modules for Schur algebras associated to GL_n, provided that $p > n$. The principal aim of this note is to illustrate how tilting module results translate to the symmetric group setting, and in particular to prove a generalisation of results of James and Williams [13] and Erdmann [8, 9] by using the tensor product theorem for tilting modules.

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Implied in this tensor product theorem is a description of the good filtration multiplicities of a tilting module away from the boundaries of the dominant region in terms of those for tilting modules labelled by smaller weights. We will make this explicit, using the corresponding data for tilting modules for the corresponding quantum group over $\mathbb{C}$ at q, a pth root of unity (which are known by work of Soergel [23, 22]). This relies on the Lusztig conjecture, but gives nontrivial results even for SL_3 where the conjecture is known.

By Ringel duality this results in an explicit recursive formula for decomposition numbers for the symmetric group (for suitable weights) in terms of the (known) decomposition numbers for the Hecke algebra and decomposition numbers indexed by smaller weights.

The methods used to obtain our results on tilting modules can be regarded as generalising a filtration of Xanthopoulos [27] for SL_2 to any semisimple connected simply-connected algebraic group.

1. Preliminaries

We begin with a brief review of the basic results which we shall require. Further details can be found in [14, II, Chapters 1-7], except for the tilting module results which can be found in [14, II, Chapter E] or [6].

The dot action of the affine Weyl group W_p on $X(T)$ defines a system of facets. If p is at least as large as the Coxeter number h of G, then every alcove in $X(T)$ will be non-empty. In this case we denote by C_p the alcove containing the origin and by C_{p^2} the corresponding alcove for W_{p^2}.

We define modules $\nabla(\lambda) = \mathrm{ind}_B^G k_\lambda$ where k_λ is the one-dimensional B-module of weight λ. Then $\nabla(\lambda) \neq 0$ precisely when $\lambda \in X^+(T)$ (i.e. λ is dominant). Note that any weight λ can be uniquely written in the form $\lambda = \lambda' + p\lambda''$ where $\lambda' \in X_1(T)$ and $\lambda'' \in X^+(T)$. The Weyl module $\Delta(\lambda)$ is the contravariant dual of $\nabla(\lambda)$. Each $\nabla(\lambda)$ has simple socle $L(\lambda)$ and all simple modules are obtained in this manner.

A tilting module is a module which has both a good and a Weyl filtration. There exists a unique indecomposable tilting module $T(\lambda)$ for each dominant weight λ. The key result we will require concerning tilting modules is the following tensor product theorem [6, (2.1) Proposition and the following Example 1]. Let $\lambda \in (p-1)\rho + X^+(T)$, and write $\lambda = \tilde{\lambda} + p\lambda^+$, where $\tilde{\lambda} \in (p-1)\rho + X_1(T)$ and $\lambda^+ \in X^+(T)$. If $p \geq 2(h-1)$ then we have

$$T(\lambda) \cong T(\tilde{\lambda}) \otimes T(\lambda^+)^F \tag{1}$$

where F is the Frobenius morphism. There is a conjecture of Donkin [6, (2.2) Conjecture] which would allow us to remove the restriction on p.

A similar theory can be developed for quantum groups with q a pth root of unity in characteristic zero (a review this material can be found in [1]). Once again we can define modules $\nabla_q(\lambda)$ and tilting modules $T_q(\lambda)$, and we have a version of the tilting tensor product theorem: if we let λ and p be as in (1), then we have

$$T_q(\lambda) \cong T_q(\tilde{\lambda}) \otimes \nabla_{\mathbb{C}}(\lambda^+)^F \tag{2}$$

where F is the quantum Frobenius morphism, and $\nabla_{\mathbb{C}}(\lambda^+)$ is an induced module for the corresponding algebraic group in characteristic zero.

Andersen has conjectured [1] that if $p \geq h$ and $\lambda \in C_{p^2}$ then

$$\text{ch } T(\lambda) = \text{ch } T_q(\lambda). \tag{3}$$

If $p \geq 2(h-1)$ then $(p-1)\rho + X_1(T) \subset C_{p^2}$, and for such weights Andersen's conjecture follows from the Lusztig conjecture [1, 5.2(a)]. When our quantum group is simply laced (and with some mild restrictions on p) Soergel has given an algorithm [23, 22] for determining the tilting filtration multiplicities $(T_q(\lambda) : \nabla_q(\mu))$.

2. A formula for good filtration multiplicities

Clearly, given (1), we can in principle determine the multiplicities arising in a good filtration of $T(\lambda)$ (for $\lambda \in (p-1)\rho + X^+(T)$) from those for tilting modules labelled by smaller weights. Our main result gives an explicit description of this procedure (and thus can be regarded as analogous to the corresponding result on decomposition numbers for Weyl modules in [4]).

LEMMA 2.1. *Suppose that $p \geq 2(h-1)$. If $\tilde{\lambda} \in (p-1)\rho + X_1(T)$ and $\tau \in X^+(T)$ then $T(\tilde{\lambda}) \otimes \nabla(\tau)^F$ has a good filtration.*

Proof. We proceed by induction on τ using the dominance ordering. If $\tau \in C_p$ then $\nabla(\tau) \cong T(\tau)$ and we are done by (1). So assume the result holds for all $\mu < \tau$. By [14, II, 4.16] or [7, Proposition A.2.2] we may assume that $T(\tau)$ has a filtration of the form

$$0 = T_0 \subset T_1 \subset \ldots \subset T_k = T(\tau) \tag{4}$$

with $T_i/T_{i-1} \cong \nabla(\theta_i)$ such that $i > j$ implies that $\theta_i \geq \theta_j$ (where this latter inequality is strict when $i = k$). Hence we have a short exact sequence

$$0 \to T(\tilde{\lambda}) \otimes X^F \to T(\tilde{\lambda}) \otimes T(\tau)^F \to T(\tilde{\lambda}) \otimes \nabla(\tau)^F \to 0$$

where X has a good filtration whose quotients $\nabla(\theta)$ all satisfy $\theta < \tau$. By our inductive hypothesis and repeated applications of [14, II, 4.17] we see that the first two terms of this sequence both have a good filtration. Hence, by [14, II, 4.17] again, the right hand term does also. □

THEOREM 2.2. *Suppose that $p \geq 2(h-1)$ is such that the Lusztig conjecture holds for G. If $\lambda = \tilde{\lambda} + p\lambda^+$ with $\tilde{\lambda} \in (p-1)\rho + X_1(T)$ and $\lambda^+ \in X^+(T)$ then*

$$(T(\lambda) : \nabla(\mu)) = \sum_{\tau} (T(\lambda^+) : \nabla(\tau))(T_q(\tilde{\lambda} + p\tau) : \nabla_q(\mu)).$$

Proof. By (1) and Lemma 2.1 it is enough to show that

$$(T(\tilde{\lambda}) \otimes \nabla(\tau)^F : \nabla(\mu)) = (T_q(\tilde{\lambda} + p\tau) : \nabla_q(\mu)).$$

As the characters of $\nabla(\tau)$ and $\nabla_{\mathbb{C}}(\tau)$ are both given by Weyl's character formula we have by (3) and the remarks following it that both $T(\tilde{\lambda}) \otimes \nabla(\tau)^F$ and $T_q(\tilde{\lambda}) \otimes \nabla_{\mathbb{C}}(\tau)^F$ have the same character. But by (2) we have that $T_q(\tilde{\lambda}) \otimes \nabla_{\mathbb{C}}(\tau)^F \cong T_q(\tau)$ and the result follows as $\nabla(\mu)$ and $\nabla_{\mathbb{C}}(\mu)$ have the same character. □

In the simply laced case (with some further mild restrictions on p) we can determine which ∇-quotients occur in a filtration of this (quantum) tilting module using the results of Soergel [23, 22].

For SL_2, the good filtration described in Lemma 2.1 is given by a short exact sequence due to Xanthopoulos [27]. Lemma 2.1 can be regarded as a generalisation of this result; as in the SL_2 case we have

LEMMA 2.3. *If $p \geq 2(h-1)$, $\tilde{\lambda} \in (p-1)\rho + X_1(T)$ and $\lambda^+ \in X^+(T)$, then $T(\tilde{\lambda}) \otimes \nabla(\lambda^+)^F$ has a simple socle.*

Proof. First note that by our assumption on p we have that $T(\tilde{\lambda})$ has a simple socle $L(\mu)$ with $\mu \in X_1(T)$, and when considered as a module for the Frobenius kernel G_1 has simple socle $L_1(\mu)$ (see [6, Section 2, Example 1] and [14, II, 11.9(3) and (4)]). For any $\lambda \in X_1(T)$ we have that $L(\lambda) \cong L_1(\lambda)$ as G_1-modules. For any pair of G-modules M and N we have that

$$\mathrm{Hom}_G(M, N) \cong (N \otimes M^*)^G \cong ((N \otimes M^*)^{G_1})^{G/G_1} \cong (\mathrm{Hom}_{G_1}(M, N))^{G/G_1}$$

where M^* is the dual of M. Steinberg's tensor product theorem states that for any $\lambda = \lambda' + p\lambda''$ with $\lambda' \in X_1(T)$ and $\lambda'' \in X^+(T)$ we have

$L(\lambda) \cong L(\lambda') \otimes L(\lambda'')^F$. Hence for any $\tau \in X^+(T)$ we have

$$\begin{aligned}
&\mathrm{Hom}_G(L(\tau), T(\tilde{\lambda}) \otimes \nabla(\lambda^+)^F) \\
&\qquad \cong (((T(\tilde{\lambda}) \otimes \nabla(\lambda^+)^F) \otimes (L(\tau') \otimes L(\tau'')^F)^*)^{G_1})^{G/G_1} \\
&\qquad \cong ((\mathrm{Hom}_{G_1}(L(\tau'), T(\tilde{\lambda})) \otimes \nabla(\lambda^+)^F \otimes (L(\tau'')^*)^F)^{G/G_1} \\
&\qquad \cong \mathrm{Hom}_G(L(\tau)'', \nabla(\lambda^+)) \cong k\delta_{\tau'\mu}\delta_{\tau''\lambda^+}
\end{aligned}$$

as $\mathrm{Hom}_{G_1}(L(\tau'), T(\tilde{\lambda})) \cong \mathrm{Hom}_{G_1}(L(\tau'), L(\mu)) \cong k\delta_{\tau'\mu}$ and M^F is trivial as a G_1 module for any G-module M. Therefore $T(\tilde{\lambda}) \otimes \nabla(\lambda^+)^F$ has simple socle $L(\mu + p\lambda^+)$ as required. □

By arguing as in [3, Corollary 3.5] we can also see that as a G_1-module $T(\tilde{\lambda}) \otimes \nabla(\lambda^+)^F$ is the injective hull (or projective cover) of the restriction of a suitably chosen ∇ to G_1, again as in the SL_2 case.

3. A formula for decomposition numbers

We will now apply the results of the preceding section to derive a reduction theorem for decomposition numbers for the symmetric group. We begin by recalling some basic facts about symmetric groups and Hecke algebras which we will need; for symmetric groups these can be found in [12, Chapter 7], for Hecke algebras in [5].

Let Σ_d be the symmetric group on d letters, and $\mathcal{H}_d$ be the corresponding Hecke algebra over $\mathbb{C}$ at q, a pth root of unity. We denote the set of all partitions of d with at most n non-zero parts by $\Lambda^+(n, d)$. For each partition λ of d we may define a Specht module S^λ for $k\Sigma_d$, and for λ p-regular this module has unique simple quotient D^λ. These modules D^λ form a complete set of simple modules for $k\Sigma_d$. In exactly the same way we may define Specht modules S_q^λ and simples D_q^λ for $\mathcal{H}_d$; again the simple modules are labelled by p-regular partitions.

We are interested in working out the decomposition numbers $d(\lambda, \mu) = [S^\lambda : D^\mu]$ and $d_q(\lambda, \mu) = [S_q^\lambda : D_q^\mu]$. As $[S^\lambda : D^\mu] \neq 0$ only if μ has no more non-zero parts than λ, it makes sense to restrict our attention to the subset of Specht modules for $k\Sigma_d$ labelled by partitions with at most n parts.

By a theorem of Erdmann [8, 4.4 Theorem], and its quantum analogue due to Donkin [7, 4.7(7)] we have for $p > n$ that

$$d(\lambda, \mu) = (T(\mu) : \nabla(\lambda)) \qquad \text{and} \qquad d_q(\lambda, \mu) = (T_q(\mu) : \nabla_q(\lambda)) \quad (5)$$

where the right hand side is the good filtration multiplicity of a tilting module for (in the latter case quantum) GL_n. (Note that for $p > n$ all $\lambda \in \Lambda^+(n, d)$ are p-regular.) These tilting module multiplicities for GL_n

equal the corresponding multiplicities for SL_n via the usual conversion of partition notation into SL-notation: given a partition $\lambda \in \Lambda^+(n,d)$, we obtain an element $\bar{\lambda} \in X^+(T)$ for SL_n by setting $\bar{\lambda} = (\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n)$. With this labelling we have

$$(T(\mu) : \nabla(\lambda))_{\mathrm{GL}} = (T(\bar{\mu}) : \nabla(\bar{\lambda}))_{\mathrm{SL}}$$

where the subscripts denote the corresponding algebraic group. A similar result holds in the quantum case.

As $T(\lambda)_{\mathrm{GL}} \cong T(\bar{\lambda})_{\mathrm{SL}}$ and $\nabla(\lambda)_{\mathrm{GL}} \cong \nabla(\bar{\lambda})_{\mathrm{SL}}$ when regarded as SL_n-modules, if $(T(\bar{\mu}) : \nabla(\bar{\lambda}))_{\mathrm{SL}} \neq 0$ and $p > n$ then there exists some d and partitions λ and μ of d such that $(T(\bar{\mu}) : \nabla(\bar{\lambda}))_{\mathrm{SL}} = d(\lambda, \mu)$. *For this reason, when writing down decomposition numbers for symmetric groups or Hecke algebras we will adopt the convention (until stated otherwise) that these are labelled by the corresponding SL labels. Here we interpret $d(\lambda, \mu)$ for a pair of SL labels which cannot be converted into partitions of some common d as zero.*

Putting this all together with Theorem 2.2, and noting that $h = n$ for SL_n, we obtain:

THEOREM 3.1. *Suppose that $p \geq 2(n-1)$ (strictly greater if $n = 2$) is such that the Lusztig conjecture holds for GL_n. Let λ and μ be two partitions of d into at most n parts, written in SL_n notation. If $\lambda = \tilde{\lambda} + p\lambda^+$ with $\tilde{\lambda} \in (p-1)\rho + X_1(T)$ and $\lambda^+ \in X^+(T)$ then*

$$d(\mu, \lambda) = \sum_{\tau} d(\tau, \lambda^+) d_q(\mu, \tilde{\lambda} + p\tau).$$

REMARKS 3.2. (i) The decomposition numbers for the Hecke algebra arising in this Theorem can be determined from the results of Soergel by Ringel duality (in type A Soergel has no additional restriction on p), or using the LLT algorithm [17].

(ii) It is clear from the translation principle [14, II, 7.9 Proposition] that the tilting multiplicity $(T(\lambda) : \nabla(\mu))$ depends only on the facets containing λ and μ. From this we see that our result is a generalisation of [13, 4.13 Proposition] (which is the case $n = 3$, but without restriction on p). For $n = 2$ a version of our result was given by Erdmann [8] for arbitrary two part partitions, with a quantum analogue given by Donkin [7, 4.4 (6)].

(iii) The recursive procedure given in [13] is computationally more complicated than ours, as it does not reduce by a factor of p at every stage. Note that the complexity is not hidden in the verification of the Lusztig conjecture for SL_3 as this is almost trivial. By translation and block considerations it is only necessary to verify that a ∇ labelled by a p-restricted weight in the alcove above C_p has precisely two composition factors.

(iv) If $\tilde{\lambda} = (p-1)\rho$ then $T_q(\lambda) = \nabla_q(\lambda)$, and by block considerations we have $d(\mu, \lambda) \neq 0$ only if $\mu = (p-1)\rho + p\mu^+$. Thus in this case our result gives that $d(\mu, \lambda) = d(\mu^+, \lambda^+)$, and hence is also a generalisation of another theorem of Erdmann [9]. The quantum analogue in characteristic zero was proved by Leclerc [18].

(v) The Lusztig conjecture is known to hold for SL_2, SL_3, and SL_4 for all $p \geq n$, but for general G is known to be true only for sufficiently large values of p. (For SL_4 see [14, II, 8.20 (second edition only)].) However, it follows from (1) that for all $p \geq 2(n-2)$ there is still a reduction theorem for decomposition numbers of the symmetric group. If Donkin's conjecture [6, (2.2) Conjecture] holds then this would also hold for all $p > n$.

If Andersen's conjecture (3) holds, then $d(\mu, \lambda)$ is known for all $\lambda \in C_{p^2}$ (for $p > n$) by the work of Soergel. This conjecture is known to hold for SL_3, and in this case Jensen's tilting module calculations [15] determine all $d(\mu, \lambda)$ (for $p > 3$) with $\lambda \in C_{2p^2+2p}$ (using an obvious extension of notation). There are related results by Rasmussen [20, Theorem 4.3] and Jensen and Mathieu [16].

Finally, James and Williams [13, 3.12 Corollary] (see also [26]) have proved (in all characteristics) that the $d(\mu, \lambda)$ for $\mu = (\mu_1, \mu_2, \mu_3)$ (as a partition) with $\mu_3 < p$, can be determined from those with $\mu_3 \leq 1$. (These have been determined by James [11], for $p = 2$, and To Law [25], for $p > 2$.)

We will give a very short proof of this reduction theorem when $p > 3$. First note that if $\mu_3 = 0$ then λ and μ must both be 2-part partitions, and the result is known by SL_2-results. Thus we assume that $\mu_3 > 0$. Next suppose that $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\mu = (\mu_1, \mu_2, \mu_3)$ are partitions of d into at most 3 parts, with $0 < \mu_3 < p$ and λ greater than μ in the dominance ordering. Then it is elementary to verify that for $\bar{\tau}$ equal to either $\bar{\lambda}$ or $\bar{\mu}$ we must have

$$d - 3(p-1) \leq \bar{\tau}_1 + 2\bar{\tau}_2 \leq d.$$

Representing SL_3-weights in the usual manner this corresponds to $\bar{\lambda}$ and $\bar{\mu}$ lying in the shaded region in Figure 1(a).

By (5) we must show that the good filtration multiplicities $(T(\lambda) : \nabla(\mu))$ with $1 < \mu_3 < p$ can be determined from the $(T(\lambda) : \nabla(\mu))$ with $\mu_3 = 1$, and by Theorem 2.2 we may assume that λ is one of λ_a, λ_b, λ_c or λ_d as shown in Figure 1(b).

Suppose first that $\lambda = \lambda_a$. Then the only weights in the shaded region in Figure 1(a) which can occur as a μ with $d(\mu, \lambda) \neq 0$ are

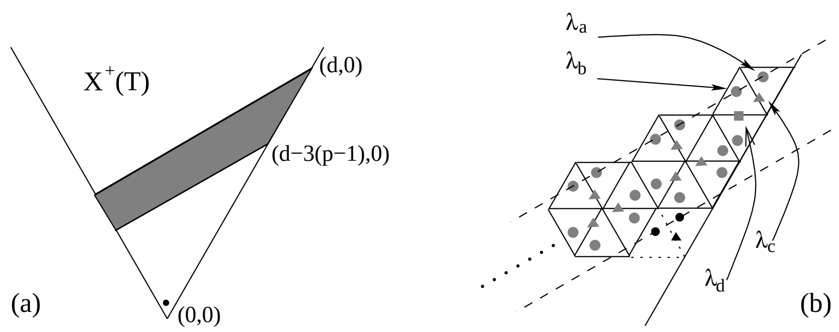

Figure 1.

those lying in one of the solid-walled alcoves in Figure 1(b) (where the sequence of hexagons continues down to the left). Here the region between the dotted lines (inclusive) contains all weights with third component less than *or equal to* p. To see that no weights in other alcoves which intersect the region between the dotted lines need be considered note that the block of SL_3 containing λ_a corresponds to the orbit $W_p.\lambda_a$. Thus the relevant weights in these extra alcoves are either not dominated by λ_a (for those above the line of hexagons), or lie on the lower dotted line (as shown for example by the dark circles) and hence have third component equal to p (which contradicts our assumption on μ). Thus for this choice of λ the only possible non-zero multiplicities can occur for μs labelled by circles in Figure 1(b). Similar arguments hold for the other choices of λ; for example the possible μs for λ_c are those labelled by triangles.

Using translation functors (see [14, II, Chapter 7]), we can translate $T(\lambda_c)$ off the wall to obtain $T(\lambda_a)$, and each $\nabla(\tau)$ occurring in a filtration of $T(\lambda_c)$ corresponding under translation to the pair of ∇s labelled by weights in $W.\lambda_a$ which lie in the two alcoves adjacent to τ. Thus to determine the good filtration multiplicities for $T(\lambda_a)$ it is enough to determine the good filtration multiplicities for $T(\lambda_c)$. (Similarly for $T(\lambda_b)$ we can reduce to considering $T(\lambda_d)$.)

By the translation principle we may assume that $\lambda_c = (e, 0)$ for some e; i.e. that λ_c is the nearest weight on its wall to the non-dominant vertex at the end of that wall. (Similarly we may assume that $\lambda_d = (f, 0)$.) Then the possible weights μ — labelled by shaded triangles — are all adjacent to one of the vertices at the centre of each hexagon, and hence lie in a band of width 3 about this central line of vertices. (Again the λ_d case is similar.) But we saw above that the width of such a band corresponds to the size of the third component of μ, and hence these multiplicities can all be realised as decomposition numbers of the form $d(\theta, \nu)$ where $\theta_3 = 1$.

References

1. Andersen, H. H.: 1998, 'Tilting modules for algebraic groups'. In: R. Carter and J. Saxl (eds.): *Algebraic Groups and their Representations*, Vol. 517 of *Nato ASI Series, Series C.* pp. 25–42.
2. Andersen, H. H., J. C. Jantzen, and W. Soergel: 1994, 'Representations of quantum groups at a p-th root of unity and of semisimple groups in characteristic p: independence of p'. *Astérisque* **220**.
3. Cox, A. G.: 1998, 'Ext^1 for Weyl modules for q-$\mathrm{GL}(2,k)$'. *Math. Proc. Camb. Phil. Soc.* **124**, 231–251.
4. Cox, A. G.: 2001, 'Decomposition numbers for distant Weyl modules'. *J. Algebra* **243**, 448–472.
5. Dipper, R. and G. D. James: 1986, 'Representations of Hecke algebras of general linear groups'. *Proc. London Math. Soc. (3)* **52**, 20–52.
6. Donkin, S.: 1993, 'On tilting modules for algebraic groups'. *Math. Z.* **212**, 39–60.
7. Donkin, S.: 1998, *The q-Schur algebra*, Vol. 253 of *LMS Lecture Notes Series.* Cambridge University Press.
8. Erdmann, K.: 1994, 'Symmetric groups and quasi-hereditary algebras'. In: V. Dlab and L. L. Scott (eds.): *Finite dimensional algebras and related topics.* Kluwer, pp. 123–161.
9. Erdmann, K.: 1996, 'Decomposition numbers for symmetric groups and composition factors of Weyl modules'. *J. Algebra* **180**, 316–320.
10. James, G. D.: 1976a, 'On the decomposition matrices of the symmetric groups, I'. *J. Algebra* **43**, 42–44.
11. James, G. D.: 1976b, 'Representations of the symmetric group over the field of order 2'. *J. Algebra* **38**, 280–308.
12. James, G. D. and A. Kerber: 1981, *The Representation Theory of the Symmetric Group*, Vol. 16 of *Encyclopedia of Mathematics and its Applications.* Addison-Wesley.
13. James, G. D. and A. L. Williams: 2000, 'Decomposition numbers of symmetric groups by induction'. *J. Algebra* **228**, 119–142.
14. Jantzen, J. C.: 2003, *Representations of algebraic groups*, Vol. 107 of *Mathematical Surveys and Monographs.* AMS, second edition. (note: all citations except those involving tilting modules can also be found in the first edition).
15. Jensen, J. G.: 2000, 'On the character of some modular indecomposable tilting modules for SL_3'. *J. Algebra* **232**, 397–419.
16. Jensen, J. G. and O. Mathieu, 'On three line representations of the symmetric group'. in preparation.
17. Lascoux, A., B. Leclerc, and J.-Y. Thibon: 1996, 'Hecke algebras at roots of unity and crystal bases of quantum affine algebras'. *Comm. Math. Phys.* **181**, 205–263.
18. Leclerc, B.: 2000, 'Decomposition numbers and canonical bases'. *Algebr. Representation Theory* **3**, 277–287.
19. Lusztig, G.: 1979, 'Some problems in the representation theory of finite Chevalley groups'. In: G. Mason and B. Cooperstein (eds.): *The Santa Cruz conference on finite groups*, Vol. 37. pp. 313–317.
20. Rasmussen, T. E., 'Multiplicities of second cell tilting modules'. preprint.
21. Ringel, C. M.: 1991, 'The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences'. *Math. Z.* **208**, 209–225.

22. Soergel, W.: 1997a, 'Charakterformeln für Kipp-Moduln über Kac-Moody-Algebren'. *Representation Theory* **1**, 115–132.
23. Soergel, W.: 1997b, 'Kazhdan-Lusztig polynomials and a combinatoric for tilting modules'. *Representation Theory* **1**, 83–114.
24. Steinberg, R.: 1957, 'Representations of algebraic groups'. *Nagoya Math. J.* **22**, 33–56.
25. To Law, K. W.: 1983, 'Results on decomposition matrices for the symmetric groups'. Ph.D. thesis, Cambridge.
26. Williams, A. L., 'Symmetric group decomposition numbers for some three-part partitions'. *Communications in Algebra.* to appear.
27. Xanthopoulos, S.: 1992, 'On a question of Verma about indecomposable representations of algebraic groups and of their Lie algebras'. Ph.D. thesis, London.