The Representation of Biharmonic Functions

in terms of Harmonic Functions

by

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ABSTRACT

In this thesis we investigate the Almansi representation of biharmonic functions in terms of two harmonic functions. Our results extend a previous investigation by Bergman and Schiffer. Also they have an interesting application to the Papkovich-Neuber representation of linear elastostatic displacement vectors in terms of harmonic functions.

INTRODUCTION

A biharmonic function χ in a domain B is continuous and differentiable to the fourth order in B and satisfies the equation

$$\nabla^4\chi = \nabla^2(\nabla^2\chi) = 0 \qquad (0\text{-}1)$$

in B. It was first demonstrated by Almansi (1897) that a biharmonic function could always be represented in the terms of two harmonic functions. Thus, if χ is a biharmonic function of the independent variables x,y, in a domain B, then we may write

$$\chi = x\phi + \psi \; ; \; \nabla^2\phi = 0, \;\; \nabla^2\psi = 0, \qquad (0\text{-}2)$$

where ϕ,ψ are harmonic functions in B. Similarly we may write

$$\chi = y\phi + \psi; \; \nabla^2\phi = 0, \;\; \nabla^2\psi = 0. \qquad (0\text{-}3)$$

This work was interesting and important, particularly from the point of view of possible applications to two-dimensional theory of elastic systems. An account of it was given by Goursat (1959-64). However these representations became largely ignored, principally because of the subsequent development of Muskhelishivili's (1953b) complex variable approach. This approach is extremely powerful but is limited to domains which can be mapped analytically onto the unit ~~circle~~ disc. It does not work well for domains which cannot be so mapped. Accordingly, attention has recently been directed to Almansi's original approach, in the hope that it might be effective for more general domains. This field has been particularly explored since 1965 by Professor M.A. Jaswon and his colleagues.

A difficulty arises with the representations (O-2), (O-3) because the harmonic functions ϕ, ψ are not unique for a given χ. Thus the homogenuous equation

$$x\phi + \psi = 0; \quad \nabla^2\phi = 0, \quad \nabla^2\psi = 0 \qquad \text{(O-4)}$$

has the two independent non-trivial solutions

$$\phi = 1, y \quad ; \quad \psi = -x, -xy. \qquad \text{(O-5)}$$

This difficulty might not be serious from an analytical point of view. However it could be a severe limitation in attempting to achieve numerical solutions of boundary-value problems. An alternative approach is to utilise the representation

$$\chi = r^2\,\phi + \psi \quad ; \quad \nabla^2\,\phi = 0, \quad \nabla^2\psi = 0 \qquad \text{(O-6)}$$

where $r^2 = x^2 + y^2$ and ϕ, ψ are harmonic functions in B.

The representation (0-6) is essentially equivalent to (0-1) or (0-2), but it has the advantage that ϕ, ψ are unique for a given χ in an important class of domain. Bergman and Schiffer (1953) gave a simple differential equation, the Bergman-Schiffer equation, for the construction of ϕ in terms of χ. Also they gave a particular solution of this equation in the form of an integral. However the Bergman-Schiffer integral is not always a harmonic function. In Part I of this thesis we give a complete analysis of the 2-dimensional Bergman-Schiffer integral, and we also examine rigorously the uniqueness of the representation (0-6). In two dimensions it is possible to formulate the Bergman-Schiffer equation and the Bergman-Schiffer integral by means of complex variables. This provides an elegant and powerful method for constructing the harmonic function ϕ in cases where the original Bergman-Schiffer integral fails.

Biharmonic function may also exist in three dimensions. There is no difficulty in proving that χ has representations of the form

$$\chi = x\phi + \psi \ , \ y\phi + \psi \ , \ z\phi + \psi \qquad (0\text{-}7)$$

where ϕ, ψ are harmonic functions. As before ϕ, ψ are not unique for a given χ. Thus the homogeneous equation

$$x\,\phi + \psi = 0 \ ; \quad \nabla^2 \phi = 0, \quad \nabla^2 \psi = 0 \qquad (0\text{-}8)$$

has the class of the non-trivial solutions

$$\phi = \eta(y,z), \ \psi = -x\,\eta(y,z) \qquad (0\text{-}9)$$

where $\eta(y,z)$ is any two-dimensional harmonic function. The representation

$$\chi = r^2\phi + \psi \quad ; \ \nabla^2\phi = 0, \ \nabla^2\psi = 0 \qquad (0\text{-}10)$$

where $r^2 = x^2 + y^2 + z^2$, and ϕ, ψ are harmonic functions, is essentially equivalent to the representations (0-7). However ϕ, ψ are unique for a given χ in an important class of domain. This requires a systematic analysis of the Bergman-Schiffer equation and the Bergman-Schiffer integral in three dimensions, which we provide in Part II of the thesis. Three-dimensional biharmonic functions have no direct physical significance. However the 3-dimensional Bergman-Schiffer equation has an important application to the Papkovich-Neuber (1932, 1934) representation of linear elastostatic displacement fields. Briefly this involves a harmonic vector function and harmonic scalar function. The latter can generally be ignored, but Eubanks and Strenberg (1956) proved that its presence is necessary for certain values of Poisson's ratio in certain domains. An improved theory of this appears in Part II.

A given biharmonic function χ may sometimes be more conveniently represented by one representation than by another. For instance in two dimensions we may see directly that

$$2x \log r = r^2 \left(\frac{x \log r + y\theta}{r^2} \right) + (x \log r - y\theta) \qquad (0\text{-}11)$$

$$2y \log r = r^2 \left(\frac{y \log r - x\theta}{r^2} \right) + (y \log r + x\theta), \qquad (0\text{-}12)$$

where the bracketed terms in (0-11), (0-12) are 2-dimensional harmonic functions. However they are multi-valued, in contrast to the single-valued harmonic function log r which appears on the left-hand side. These and other 2-dimensional transformations are discussed in Chapter 1. Transformations between three-dimensional representations involve a more difficult theory, which appears in Part III. To summarise, this thesis makes a clear step forward in the theory of the representation of biharmonic functions by means of harmonic functions.

PART I

TWO-DIMENSIONAL ANALYSIS

CHAPTER I

TWO-DIMENSIONAL REPRESENTATIONS

As mentioned in (0-2), an arbitrary harmonic function χ has the representation

$$\chi = xh + \Psi \; ; \quad \nabla^2 h = \nabla^2 \Psi = 0, \tag{1-1}$$

where h, Ψ, χ are functions of x and y. To determine h from χ, we note that

$$\nabla^2 \chi = 2 \frac{\partial h}{\partial x} \; ; \; \nabla^2 h = 0. \tag{1-2}$$

Therefore

$$h = \int^x \frac{\nabla^2 \chi}{2} \, d\xi + \eta(y) \; ; \; \chi = \chi(\xi, y), \tag{1-3}$$

where we require

$$0 = \nabla^2 h = \nabla^2 \int^x \frac{\nabla^2 \chi}{2} \, d\xi + \nabla^2 \eta(y), \tag{1-4}$$

$$\text{i.e.} \quad \frac{d^2}{dy^2} \eta(y) = - \nabla^2 \int^x \frac{\nabla^2 \chi}{2} \, d\xi \, . \tag{1-5}$$

It may be verified by direct analysis that the right-hand side of (1-5) is independent of x. Equation (1-5) for $\eta(y)$

always has a solution, and therefore a particular harmonic function h_p always exists. Clearly we may always add to this the complementary harmonic solution a + by, where a,b are arbitrary constants. This makes a contribution $x(a + by)$ to χ, which is seen to be a harmonic function.

Often it is possible to compute h_p by direct arguments. For instance if

$$\nabla^2\chi = 2r^\nu \cos \nu\theta,\ 2r^\nu \sin \nu\theta \tag{1-6}$$

where ν is a given real number, then

$$h_p = \frac{1}{\nu+1} r^{\nu+1} \cos(\nu + 1)\theta \quad ; \ \nu \neq -1 \tag{1-7}$$

$$h_p = \frac{1}{\nu+1} r^{\nu+1} \sin(\nu + 1)\theta \quad ; \ \nu \neq -1 \tag{1-8}$$

respectively, since

$$(\nu + 1)r^\nu \cos\nu\theta = \frac{\partial}{\partial x} r^{\nu+1} \cos(\nu + 1)\theta \ , \tag{1-9}$$

$$(\nu + 1)r^\nu \sin\nu\theta = \frac{\partial}{\partial x} r^{\nu+1} \sin(\nu + 1)\theta \ , \tag{1-10}$$

respectively. Therefore χ always has representations of the form

$$\chi = xh_p = \frac{1}{\nu+1} r^{\nu+2} \cos\theta \cos(\nu+1)\theta \quad ; \quad \nu \neq -1 \tag{1-11}$$

$$\chi = xh_p = \frac{1}{\nu+1} r^{\nu+2} \cos\theta \sin(\nu+1)\theta \quad ; \quad \nu \neq -1 \tag{1-12}$$

respectively, to which we must generally add harmonic functions.

As regards the failing cases

$$\nabla^2\chi = 2r^{-1}\cos\theta \ , \ \ 2r^{-1} \sin\theta \ ; \quad \nu \neq -1 \tag{1-13}$$

of (1-7), (1-8) respectively, we have from (1-3)

$$h = \int^x \rho^{-1}\cos\theta \, d\xi + \eta(y) = \log r + \eta(y), \tag{1-14}$$

$$h = \int^x \rho^{-1}\sin\theta \, d\xi + \eta(y) = -\theta + \eta(y), \tag{1-15}$$

respectively, bearing in mind $\rho^{-1}\cos\theta = \xi(\xi^2 + y^2)^{-1}$, $\rho^{-1} \sin\theta = y(\xi^2 + y^2)^{-1}$. Since $\nabla^2(\log r) = 0$, $\nabla^2\theta = 0$, it follows that we may put $\eta = 0$ in choosing h_p. Accordingly

$$\chi = xh_p = x \log r \tag{1-16}$$

$$\chi = xh_p = -x\theta \tag{1-17}$$

respectively, to which we must generally add harmonic functions.

A similar analysis holds for the representation

$$\chi = yh + \Psi \ ; \ \nabla^2 h = \nabla^2 \Psi = 0. \tag{1-18}$$

For instance, given (1-6), we see that

$$h_p = \frac{1}{\nu + 1} r^{\nu+1} \sin(\nu+1)\theta \quad ; \quad \nu \neq -1 \tag{1-19}$$

$$h_p = -\frac{1}{\nu + 1} r^{\nu+1} \cos(\nu+1)\theta \ ; \ \nu \neq -1 \tag{1-20}$$

respectively, since

$$(\nu + 1)r^{\nu}\cos\nu\theta = \frac{\partial}{\partial y} r^{\nu+1} \sin(\nu+1)\theta \tag{1-21}$$

$$(\nu + 1)r^{\nu}\sin\nu\theta = -\frac{\partial}{\partial y} r^{\nu+1} \sin(\nu+1)\theta \tag{1-22}$$

respectively. Therefore χ always has representations of the form

$$\chi = y\,h_p = \frac{1}{\nu + 1} r^{\nu+2} \sin\theta \sin(\nu + 1)\theta \ ; \ \nu \neq -1 \tag{1-23}$$

$$\chi = y\,h_p = -\frac{1}{\nu + 1} r^{\nu+2} \sin\theta \cos(\nu + 1)\theta \ ; \ \nu \neq -1 \tag{1-24}$$

respectively, omitting harmonic functions. In the failing cases (1-13),

$$\chi = yh_p = y\theta \qquad (1\text{-}25)$$

$$\chi = yh_p = y \log r \qquad (1\text{-}26)$$

respectively. These forms can only differ from (1-16), (1-17) by harmonic functions, and we note that

$$y\theta = x \log r + (y\theta - x\log r) \qquad (1\text{-}27)$$

$$y \log r = -x\theta + (y \log r + x\theta). \qquad (1\text{-}28)$$

Conversely

$$x\log r = y\theta + (x \log r - y\theta) \qquad (1\text{-}29)$$

$$-x\theta = y \log r + (-y \log r - x\theta). \qquad (1\text{-}30)$$

These transformations between the x & y - forms are important in the theory of two-dimensional Volterra dislocations (see Jaswon & Symm (1977)).

Now χ also has a representation of the form

$$\chi = r^2\phi + \psi \; ; \; \nabla^2 \phi = \nabla^2\psi = 0, \qquad (1\text{-}31)$$

in which case ϕ satisfies the equation

$$\nabla^2 \chi = \nabla^2 (r^2 \phi) + \nabla^2 \psi = \nabla^2 (r^2 \phi)$$

$$= 4\phi + 4\left(x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y}\right) \; ; \quad \text{2 dimensions} \qquad (1\text{-}32)$$

$$\nabla^2 f = 6\phi + 4\left(x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} + z \frac{\partial \phi}{\partial z}\right) \; ; \quad \text{3 dimensions.} \qquad (1\text{-}33)$$

These are differential equations for ϕ in terms of $\nabla^2 \chi$. Bergman and Schiffer (1953) pointed out that equations (1-32), (1-33) can be transformed to

$$4\phi + 4r \frac{\partial \phi}{\partial r} = \nabla^2 \chi \; ; \quad \text{2 dimensions} \qquad (1\text{-}34)$$

$$6\phi + 4r \frac{\partial \phi}{\partial r} = \nabla^2 \chi \; ; \quad \text{3 dimensions} \qquad (1\text{-}35)$$

where

$x, y \rightarrow r, \theta$ (plane polar co-ordinates),

$x, y, z \rightarrow r, \theta, \psi$ (spherical polar co-ordinates).

It is convenient to replace equations (1-34) and (1-35) by the more general equation

$$\kappa \phi + r \frac{\partial \phi}{\partial r} = f \quad ; \quad \nabla^2 f = 0 \tag{1-36}$$

where f is a harmonic function and κ is any real number. We refer to equation (1-36) as the Bergman-Schiffer equation. An immediate particular solution of (1-36) is the Bergman-Schiffer integral

$$\phi_p = r^{-\kappa} \int^r \rho^{\kappa-1} f \, d\rho \, ; \quad \begin{array}{l} f = f(\rho,\theta,\psi) \text{ in 3D} \\ f = f(\rho,\theta) \text{ in 2D.} \end{array} \tag{1-37}$$

Thus, for instance, substituting from (1-6) into (1-37) with $\kappa = 1$ and $f = \frac{\nabla^2 \chi}{2}$, we find

$$\phi_p = \frac{r^\nu}{2(\nu + 1)} \cos \nu\theta, \ \frac{r^\nu}{2(\nu + 1)} \sin \nu\theta \quad ; \quad \nu \neq -1 \tag{1-38}$$

from which (omitting harmonic functions)

$$\chi = r^2 \phi_p = \frac{r^{\nu+2}}{2(\nu + 1)} \cos \nu\theta \quad ; \quad \nu \neq -1 \tag{1-39}$$

$$\chi = r^2 \phi_p = \frac{r^{\nu+2}}{2(\nu + 1)} \sin \nu\theta \quad ; \quad \nu \neq -1. \tag{1-40}$$

The differences between the forms χ in (1-11) and (1-39), and between (1-12) and (1-40), are the harmonic functions

$$\Psi = \frac{-1}{2(\nu + 1)} r^{\nu+2} \cos(\nu + 2)\theta, \ \frac{-1}{2(\nu + 1)} r^{\nu+2} \sin(\nu + 2)\theta \tag{1-41}$$

respectively. Also the difference between the forms χ in

(1-23) and (1-39), and between (1-24) and (1-40), are the harmonic functions

$$\Psi = \frac{1}{2(\nu+1)} r^{\nu+2} \cos(\nu+2) \ , \ \frac{1}{2(\nu+1)} r^{\nu+2} \sin(\nu+2)\theta \tag{1-42}$$

respectively. As regards the breakdown cases $\nu = -1$, which are the same breakdown cases as previously, it is necessary to determine ϕ_p from theory of Chapter 3. The results are

$$\phi_p = \frac{r^{-2}}{2}(x \log r + y\theta \ ), \ \frac{r^{-2}}{2}(y \log r - x\theta \ ) \tag{1-43}$$

$$\chi = r^2 \phi_p = \frac{1}{2}(x \log r + y\theta \ ), \tag{1-44}$$

$$\chi = r^2 \phi_p = \frac{1}{2}(y \log r - x\theta \ ), \tag{1-45}$$

respectively. The difference between the forms χ in (1-16) and (1-44), and between (1-17) and (1-45), are harmonic functions and we note that

$$x \log r = r^2 \ (\frac{x \log r + y\theta}{2r^2}) + \frac{1}{2}(x \log r - y\theta \ ) \tag{1-46}$$

$$y \log r = r^2 \ (\frac{y \log r - x\theta}{2r^2}) + \frac{1}{2}(y \log r - x\theta \ ) \tag{1-47}$$

as anticipated in (0-11), (0-12).

The results (1-46), (1-47) show that x log r, y log r cannot be conveniently represented in the r^2-form. Conversely the biharmonic function $r^2 \log r$ cannot be conveniently expressed in the x-or y-forms. Thus, if

$$\chi = r^2 \log r, \tag{1-48}$$

then

$$\nabla^2 \chi = 4 \log r + 4. \tag{1-49}$$

By applying (1-3) we obtain

$$h_p = 2\int^x \log\rho \, d\xi + 2x = 2\left\{(x \log r - y\theta) - x\right\} + 2x = 2(x \log r - y\theta) \tag{1-50}$$

from which, omitting a harmonic function,

$$\chi = 2x(x \log r - y\theta). \tag{1-51}$$

The difference between the forms χ in (1-51) and (1-48) is a harmonic function, and we note that

$$r^2 \log r = x\left\{2(x \log r - y\theta)\right\} + \left\{(y^2 - x^2)\log r + 2xy\theta\right\}. \tag{1-52}$$

Similarly

$$r^2 \log r = y\left\{2(x \log r - y\theta)\right\} + \left\{(x^2 - y^2)\log r - 2xy\theta\right\}. \tag{1-53}$$

As before, these transformations are important in the theory of two-dimensional Volterra dislocations.

An elegant and powerful method for evaluating h in (1-3) is provided by complex variable theory. Thus, if $\overline{\nabla^2\chi}$, $\bar{h}$ are the harmonic conjugate of $\nabla^2\chi$, h respectively, then (1-3) in the complex variable ~~domain~~ symbolism becomes

$$H = \int^z \frac{1}{2} F \, d\zeta \, , \tag{1-54}$$

where

$$H = h + i\bar{h}, \; F = \nabla^2\chi + i \, \overline{\nabla^2\chi}, \; z = x + iy.$$

Accordingly

$$h_p = \text{Re } H = \text{Re} \left\{ \int^z \frac{1}{2} F \, d\zeta \right\}. \tag{1-55}$$

For example, if

$$\nabla^2\chi = 2 \log r, \text{ then } \overline{\nabla^2\chi} = 2\theta \text{ and } F = 2 \log z,$$

from which

$$H = \int^z \log\zeta \, d\zeta = z \log z - z \tag{1-56}$$

$$h_p = \text{Re } H = (x \log r - y\theta) - x \tag{1-57}$$

as utilised in (1-50).

As a second example, if $\nabla^2\chi = 2\,r^{-1}\cos\theta$, then

$$F = \nabla^2\chi + i\,\overline{\nabla^2\chi} = z^{-1}, \tag{1-58}$$

from which

$$H = \int^z \frac{d\zeta}{\zeta} = \log z, \tag{1-59}$$

$$h_p = \operatorname{Re} H = \log r. \tag{1-60}$$

CHAPTER 2

BERGMAN-SCHIFFER INTEGRAL: NON-FAILING CASES

Given a harmonic function f in a two dimensions domain B, we seek a second harmonic function ϕ in B which satisfies the Bergman-Schiffer equation

$$\kappa \phi + r \frac{\partial \phi}{\partial r} = f \tag{2-1}$$

where $r^2 = x^2 + y^2$ and κ is any constant. The choice $\kappa = 1$ corresponds to (1-34). Equation (2-1) has the family of solutions

$$\phi = r^{-\kappa} g(\theta) + r^{-\kappa} \int^{r} \rho^{\kappa-1} f \, d\rho \tag{2-2}$$

where $g(\theta)$ is an arbitrary function of θ. It is necessary to determine $g(\theta)$ by the requirement that ϕ is a harmonic function, i.e. $\nabla^2\phi = 0$, from which it follows that

$$\nabla^2 \left\{ r^{-\kappa} g(\theta) \right\} = - \nabla^2 \left\{ r^{-\kappa} \int^{r} \rho^{\kappa-1} f d\rho \right\}, \tag{2-3}$$

i.e. by straightforward manipulations

$$r^{-\kappa-2} \left(\kappa^2 g + \frac{\partial^2 g}{\partial \theta^2}\right) = -\nabla^2 \left\{ r^{-\kappa} \int^{r} \rho^{\kappa-1} f d\rho \right\}, \tag{2-4}$$

so that g satisfies the equation

$$\kappa^2 g + \frac{\partial^2 g}{\partial\theta^2} = - r^{\kappa+2} \nabla^2 \left\{ r^{-\kappa} \int_{\rho}^{r} \rho^{\kappa-1} f d\rho \right\}. \tag{2-5}$$

The expression on the left is only a function of θ , therefore the expression on the right must only be a function of θ. This may be verified by a detailed analysis. Accordingly, denoting the right-hand side of (2-5) by $h(\theta)$, we re-write (2-5) as

$$\frac{\partial^2 g}{\partial\theta^2} + \kappa^2 g = h(\theta). \tag{2-6}$$

There are now two distinct possibilities for h, i.e. $h = 0$, $h \neq 0$. If $h = 0$, we obtain at once

$$g(\theta) = \alpha \cos \kappa\theta + \beta \sin \kappa\theta \tag{2-7}$$

where α, β be arbitrary constants. Accordingly

$$\phi = r^{-\kappa}(\alpha \cos\kappa\theta + \beta\sin\kappa\theta) + r^{-\kappa}\int^{r} f\rho^{\kappa-1} d\rho \, . \tag{2-8}$$

This result can also be seen directly, since $h = 0$ corresponds with the Bergman-Schiffer integral being a harmonic function: $r^{-\kappa}g(\theta)$ must then be a harmonic function and therefore g must have the form (2-7).

For $\kappa > 0$, the harmonic function $r^{-\kappa}(\alpha\cos\kappa\theta + \beta\sin\kappa\theta)$ becomes singular at $r = 0$. Accordingly it is inadmissible in any simply-connected domain which includes $r = 0$. Therefore the Bergman-Schiffer integral provides a unique harmonic solution of equation (2-1) in such domains.

Within a ring-shaped domain which excludes $r = 0$, the functions $r^{-\kappa}\cos\kappa\theta$, $r^{-\kappa}\sin\kappa\theta$ would be quite admissible. For $\kappa = 1$ it may be verified that these satisfy the homogeneous differential equation

$$4\phi + 4r\frac{\partial\phi}{\partial r} = 0 \qquad (2\text{-}9)$$

Equivalently they provide two independent non-trivial solutions of the functional equation

$$r^2\phi + \psi = 0\ ; \quad \nabla^2\phi = \nabla^2\psi = 0, \qquad (2\text{-}10)$$

which are respectively compensated by the harmonic functions

$$\left.\begin{aligned} \psi &= -r^2(r^{-1}\cos\theta) = -r\cos\theta = -x, \\ \psi &= -r^2(r^{-1}\sin\theta) = -r\sin\theta = -y. \end{aligned}\right\} \qquad (2\text{-}11)$$

This can be seen directly, but our analysis proves that they are the only possible independent non-trivial solutions.

Consequently, in order to ensure the uniqueness of the representation (1-31) within a ring-shaped domain, we must introduce two side conditions which eliminate the components $r^{-1}\cos\theta$, $r^{-1}\sin\theta$ from ϕ. There remains the problem of covering the biharmonic functions $x \log r$, $y \log r$ by the representation (1-31). These could exist within a ring-shaped domain, but ϕ, ψ would then be multi-valued functions as shown by (1-46) and (1-47). It is necessary to remove multi-valued functions from our analysis as far as possible. Accordingly Jaswon and Symm (1977) introduce the representation

$$\chi = r^2\phi + \psi + ax \log r + by \log r \qquad (2\text{-}12)$$

inside a ring-shaped domain, where a,b are constants to be determined and ϕ,ψ are single-valued harmonic functions. These two extra unknown coefficients exactly balance the two side conditions previously mentioned, so providing a well-posed formulation of the biharmonic boundary-value problems for such domains.

Within an infinite exterior domain we must introduce the restriction

$$\chi = O(r) \quad \text{as} \quad r \to \infty \; . \qquad (2\text{-}13)$$

This implies $\phi = O\left(\frac{1}{r}\right)$ as $r \to \infty$. Also, in two dimensions it is necessary that $\psi = O(1)$ as $r \to \infty$.

Accordingly Jaswon and Symm (1977) utilise the representation

$$\chi = r^2\phi + \psi + ax + by \tag{2-14}$$

where ψ is restricted so that $\psi = O(1)$ as $r \to \infty$ and where a,b are constants to be determined. The coefficients a,b are balanced by two side conditions, as previously, which eliminate the components $r^{-1}\cos\theta$, $r^{-1}\sin\theta$ from ϕ.

For $\kappa = 0$, it follows that $r^{-\kappa}(\alpha\cos\kappa\theta + \beta\sin\kappa\theta) = \text{constant}$. This would be acceptable in any two-dimensional domain . For $\kappa < 0$, the harmonic function $r^{-\kappa}\cos\kappa\theta$, $r^{-\kappa}\sin\kappa\theta$ would not be admissible for infinite exterior domains. Therefore the Bergman-Schiffer integral provides a unique harmonic solution of equation (2-1) for such domains.

CHAPTER 3

BERGMAN-SCHIFFER INTEGRAL: FAILING CASES

The Bergman-Schiffer integral may or may not be a harmonic function. For example, choosing

$$f = r^{\nu}\cos \nu\theta, \text{ for an arbitrary choice of } \nu, \tag{3-1}$$

we find

$$r^{-\kappa}\int^{r}\rho^{\kappa-1}f\,d\rho = r^{-\kappa}\int^{r}\rho^{\nu+\kappa-1}\cos\nu\theta\,d\rho = \frac{r^{\nu}}{\nu+\kappa}\cos\nu\theta. \tag{3-2}$$

Clearly the integral (3-2) is a harmonic function when $\nu \neq -\kappa$. However the integral is not a harmonic function when $\nu = -\kappa$. In this case we write

$$f = r^{-\kappa}\cos\kappa\theta , \tag{3-3}$$

which yields

$$r^{-\kappa}\int^{r}\rho^{-1}\cos\kappa\theta\,d\rho = r^{-\kappa}\cos\kappa\theta\log r. \tag{3-4}$$

Clearly this corresponds to the case $h \neq 0$, i.e.

$$\nabla^2\left\{r^{-\kappa}\cos\kappa\theta\log r\right\} = -2\kappa\, r^{-\kappa-2}\cos\kappa\theta, \tag{3-5}$$

and therefore from (2-5) we have

$$h(\theta) = -r^{\kappa+2} \nabla^2 \left\{ r^{-\kappa} \cos\kappa\theta \log r \right\} = 2\kappa \cos\kappa\theta \ . \tag{3-6}$$

Equation (2-6) now becomes

$$\frac{\partial^2 g}{\partial\theta^2} + \kappa^2 g = 2 \kappa\cos \kappa\theta, \tag{3-7}$$

with the particular solution

$$g_p(\theta) = \theta \sin \kappa\theta. \tag{3-8}$$

Accordingly equation (2-1) has the particular harmonic solution

$$\phi_p = r^{-\kappa}(\cos\kappa\theta \log r + \theta \sin \kappa\theta) \tag{3-9}$$

in this case. For $\kappa = 1$, the expression (3-9) becomes

$$\phi_p = r^{-1}(\cos\theta \log r + \theta\sin \theta) = r^{-2}(x \log r + y\theta) \tag{3-10}$$

which is one of the bracketed multi-valued functions (0-11). Similarly, when

$$f = r^{-\kappa}\sin \kappa\theta, \tag{3-11}$$

we have the particular solution

$$\phi_p = r^{-\kappa}(\sin\kappa\theta \log r - \theta \sin\kappa\theta), \tag{3-12}$$

which becomes

$$\phi_p = r^{-2}(y \log r - x\theta) \tag{3-13}$$

when $\kappa = 1$, in agreement with the second bracketed multi-valued functions (O-12).

A second example is

$$f = \log r. \tag{3-14}$$

Here the Bergman-Schiffer integral becomes

$$r^{-\kappa}\int^{r} f\rho^{\kappa-1}d\rho = r^{-\kappa}\int^{r}\log\rho\,\rho^{\kappa-1}d\rho = \frac{1}{\kappa}\left\{\log r - \frac{1}{\kappa}\right\} \;;$$

$$\kappa \neq 0\,, \tag{3-15}$$

from which it follows that

$$h(\theta) = - r^{\kappa+2}\,\nabla^2\left\{\frac{1}{\kappa}\log r - \frac{1}{\kappa^2}\right\} = 0\;; \quad \kappa \neq 0. \tag{3-16}$$

Therefore equation (2-1) has the particular solution

$$\phi_p = \frac{1}{\kappa}\left\{\log r - \frac{1}{\kappa}\right\} \;; \quad \kappa \neq 0 \tag{3-17}$$

in this case. On the other hand, if $\kappa = 0$, then

$$\int^{r}\rho^{-1}\log\rho\; d\rho = \tfrac{1}{2}\log^2 r$$

$$h(\theta) = -r^2\nabla^2 \left\{\tfrac{1}{2} \log^2 r\right\} = -1.$$

Therefore equation (2-6) becomes

$$\frac{\partial^2 g}{\partial\theta^2} = 1 \tag{3-18}$$

with the particular solution

$$g_p = - \frac{\theta^2}{2} . \tag{3-19}$$

Accordingly

$$\phi_p = \tfrac{1}{2} (\log^2 r - \theta^2) \tag{3-20}$$

is a particular harmonic solution of equation (2-1) in this case.

We complete this chapter by proving directly that

$r^{\kappa+2} \nabla^2 \left\{r^{-\kappa}\int^r \rho^{\kappa-1} f d\rho \right\}$, i.e. h, is independent of r.

Thus

$$\nabla^2 \left\{r^{-\kappa} \int^r \rho^{\kappa-1} f \, d \rho\right\} = \kappa(\kappa - 1)r^{-\kappa-2} \int^r \rho^{\kappa-1} f \, d \rho$$

$$-r^{-2} f - \kappa r^{-2} f + r^{-1} \frac{\partial f}{\partial r} - \kappa r^{-\kappa-2} \int^r \rho^{\kappa-1} f d\rho$$

$$+ r^{-2} f + r^{-\kappa-2} \int^r \rho^{\kappa-1} \frac{\partial^2 f}{\partial\theta^2} d\rho . \tag{3.21}$$

Since f is harmonic and therefore satisfies the equation

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0, \qquad (3\text{-}22)$$

expression (3-21) becomes

$$\kappa^2 r^{-\kappa-2} \int^r \rho^{\kappa-1} f \, d\rho - \kappa r^{-2} f + r^{-1} \frac{\partial f}{\partial r}$$

$$-r^{-\kappa-2} \int^r \rho^{\kappa-1} \left(\rho^2 \frac{\partial^2 f}{\partial \rho^2} + \rho \frac{\partial f}{\partial \rho}\right) d\rho \; . \qquad (3\text{-}23)$$

Integration by parts yields

$$\int^r \rho^{\kappa+1} \frac{\partial^2 f}{\partial \rho^2} d\rho = r^{\kappa+1} \frac{\partial f}{\partial r} - (\kappa + 1) \int^r \rho^{\kappa} \frac{\partial f}{\partial \rho} \, d\rho \; , \qquad (3\text{-}24)$$

$$\int^r \rho^{\kappa} \frac{\partial f}{\partial \rho} d\rho = r^{\kappa} f - \kappa \int^r \rho^{\kappa-1} f \, d\rho, \qquad (3\text{-}25)$$

in which case (3-23) becomes

$$\kappa^2 r^{-\kappa-2} \int^r \rho^{\kappa-1} f \, d\rho - \kappa r^{-2} f + r^{-1} \frac{\partial f}{\partial r}$$

$$-r^{-1} \frac{\partial f}{\partial r} + \kappa r^{-2} f - \kappa^2 r^{-\kappa-2} \int^r \rho^{\kappa-1} f \, d\rho - r^{-\kappa-2} h(\theta),$$

$$= -r^{-\kappa-2} h(\theta) \tag{3-26}$$

where $h(\theta)$ arises because of the indefinite integrals in (3-24) and (3-25).

CHAPTER 4

BERGMAN-SCHIFFER EQUATION: COMPLEX ANALYSIS

In this chapter, we analyse the preceding results by regard to complex variable theory.

Given a harmonic function f in B, suppose equation (1-36) has a harmonic solution ϕ in B. Taking the conjugate of each side of this equation we note that (utilising the x,y form)

$$\kappa \overline{\phi} + x \frac{\partial \overline{\phi}}{\partial x} + y \frac{\partial \overline{\phi}}{\partial y} = \overline{f}, \tag{4-1}$$

where $\overline{f}$, $\overline{\phi}$ are the conjugate functions to f, ϕ respectively. Multiplying both sides of (4-1) by i and adding to (1-36) we find

$$(\phi + i\overline{\phi}) + x \frac{\partial}{\partial x} (\phi + i\overline{\phi}) + y \frac{\partial}{\partial y} (\phi + i\overline{\phi}) = f + i\overline{f} \tag{4-2}$$

$$\text{i.e.} \quad \kappa\Phi + \frac{\partial \Phi}{\partial x} + y \frac{\partial \Phi}{\partial y} = F, \tag{4-3}$$

where $\Phi = \phi + i\overline{\phi}$, $F = f + i\overline{f}$. Writing $z = x + iy$, $\overline{z} = x - iy$ and noting that $\Phi(x,y) \equiv \Phi(z,\overline{z})$, it follows that

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial \Phi}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial x} = \frac{\partial \Phi}{\partial z} + \frac{\partial \Phi}{\partial \overline{z}},$$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial \Phi}{\partial \overline{z}}\frac{\partial \overline{z}}{\partial y} = i\left(\frac{\partial \Phi}{\partial z} - \frac{\partial \Phi}{\partial \overline{z}}\right).$$

Therefore

$$x \frac{\partial\Phi}{\partial x} + y \frac{\partial\Phi}{\partial y} = x \left(\frac{\partial\Phi}{\partial z} + \frac{\partial\Phi}{\partial \overline{z}}\right) + iy \left(\frac{\partial\Phi}{\partial z} - \frac{\partial\Phi}{\partial \overline{z}}\right)$$

$$= z \frac{\partial\Phi}{\partial z} + \overline{z} \frac{\partial\Phi}{\partial \overline{z}} . \tag{4-4}$$

Substituting from (4-4) into (4-5) yields

$$\kappa\Phi + z \frac{\partial\Phi}{\partial z} + \overline{z} \frac{\partial\Phi}{\partial \overline{z}} = F . \tag{4-5}$$

However the analyticity of Φ implies $\frac{\partial\Phi}{\partial \overline{z}} = 0$, so that (4-5) becomes

$$\kappa\Phi + z \frac{\partial\Phi}{\partial z} = F. \tag{4-6}$$

By reference to equation (2-1) and the general solution (2-2), equation (4-6) has the general solution

$$\Phi(z) = Cz^{-\kappa} + z^{-\kappa} \int^{z} \xi^{\kappa-1} F(\xi)d\xi , \tag{4-7}$$

where C is an arbitrary complex number. We concentrate upon the particular complex solution

$$\Phi_p(z) = z^{-\kappa} \int^{z} \xi^{\kappa-1} F(\xi)d\xi , \tag{4-8}$$

which yields the particular real solution

$$\phi_p = \text{Re}\ \Phi_p(z) \tag{4-9}$$

of equation (2-1). This is a particular solution of equation (2-1), which is harmonic in either the whole or part of the domain B where f is harmonic.

Since f is harmonic in B, it follows that F is analytic in B. However it does not necessarily follow that Φ_p is analytic in B. For instance, if $\kappa = 0$ and $f = c$, then $F = C$ (C is a complex constant) and

$$\Phi_p(z) = \int^z C\,\frac{d\xi}{\xi} = C\log z \tag{4-10}$$

which is not analytic at the origin. It can be seen directly that the equation $z\,\frac{\partial\Phi}{\partial z} = C$ has a particular solution $\Phi_p = C\log z$.

As a second example of (4-9), we note that

$f = r^\nu \cos\nu\theta$ for an arbitrary choice of ν. (4-11)

This has the harmonic conjugate $\overline{f} = r^\nu \sin\nu\theta$, so that

$F = f + i\overline{f} = z^\nu$. (4-12)

Substituting from (4-11) into (4-7), we find

$$\Phi_p(z) = z^{-\kappa}\int^z \xi^{\nu+\kappa-1}\,d\xi = \frac{z^\nu}{\nu+\kappa}\ ;\quad \nu \neq -\kappa\ , \tag{4-13}$$

so that

$$\phi_p = \text{Re}\ (\frac{z^\nu}{\nu + \kappa}) = \frac{r^\nu}{\nu+\kappa} \cos \nu\theta \ ; \ \nu \neq -\kappa \ . \tag{4-14}$$

Clearly ϕ_p is a harmonic function identical with (1-38). If $\nu = -\kappa$, then

$$\Phi_p = z^{-\kappa} \int^z \frac{d\xi}{\xi} = z^{-\kappa} \log z, \tag{4-15}$$

so that

$$\phi_p = \text{Re}\,\Phi_p = r^{-\kappa}(\cos \kappa\theta \log r + \theta \sin \kappa\theta). \tag{4-16}$$

This is identical with (3-9). We note that the complex Bergman-Schiffer integral provides a harmonic solution even though the corresponding real integral does not.

A third example of (4-9) is

$$F = \log z, \tag{4-17}$$

from which we obtain

$$\phi_p = z^{-\kappa} \int^z \xi^{\kappa-1} \log\xi \ d\xi = \frac{1}{\kappa} \left\{ \log z - \frac{1}{\kappa} \right\} \ ; \quad \kappa \neq 0. \tag{4-18}$$

Accordingly

$$f = \mathrm{Re}\,\log z = \log r \tag{4-19}$$

implies

$$\phi_p = \mathrm{Re}\ \Phi_p = \mathrm{Re}\,\frac{1}{\kappa}\left\{\log z - \frac{1}{\kappa}\right\} = \frac{1}{\kappa}\ \left\{\log - \frac{1}{\kappa}\right\}\ ;\quad \kappa \neq 0. \tag{4-20}$$

If $\kappa = 0$, then

$$\Phi_p = \int^z \frac{\log\xi}{\xi}\,d\xi = \frac{1}{2}\log^2 z, \tag{4-21}$$

$$\phi_p = \frac{1}{2}\,\mathrm{Re}\,\log^2 z = \frac{1}{2}\,(\log^2 r - \theta^2). \tag{4-22}$$

Similarly

$$f = \mathrm{Im}\,\log z = \theta \tag{4-23}$$

implies

$$\left.\begin{aligned} \phi_p &= \mathrm{Im}\ \Phi_p = \mathrm{Im}\,\frac{1}{\kappa}\left\{\log z - \frac{1}{\kappa}\right\} = \frac{1}{\kappa}(\theta - \frac{1}{\kappa})\ ; && \kappa \neq 0 \\ &= \frac{1}{2}\,\mathrm{Im}\,\log^2 z = \theta\log r\ ; && \kappa = 0 \end{aligned}\right\} \tag{4-24}$$

These examples show that the complex variable approach to the Bergman-Schiffer integral provides a more powerful approach than that of the real variable, since both the cases $h = 0$, $h \neq 0$ are covered by the same formulae (4-8), (4-9)

As a final example, we choose the analytic function

$$F(z) = z^{\nu}\log z \quad ; \quad \nu \text{ is any constant.} \tag{4-25}$$

Then

$$\Phi_p = z^{-\kappa}\int^{z} \xi^{\kappa+\nu-1}\log\xi d\xi = \frac{z^{\nu}}{\nu+\kappa}\,(\log z - \frac{1}{\nu+\kappa});\ \nu \neq -\kappa. \tag{4-26}$$

Accordingly

$$f = \operatorname{Re} z^{\nu}\log z = r^{\nu}(\cos\nu\theta \log r - \theta\sin\theta), \tag{4-27}$$

implies

$$\phi_p = \operatorname{Re}\Phi_p = \frac{r^{\nu}}{\nu+\kappa}\,[(\log r - \frac{1}{\nu+\kappa})\cos\nu\theta - \theta\sin\nu\theta]. \tag{4-28}$$

In the special case $\nu = -\kappa$, i.e. if

$$F(z) = z^{-\kappa}\log z, \tag{4-29}$$

we find

$$\Phi_p = z^{-\kappa}\int^{z}\xi^{-1}\log\xi\, d\xi = \frac{z^{-\kappa}}{2}\log^2 z. \tag{4-30}$$

Accordingly

$$f = \text{Re } z^{-\kappa} \log z = r^{-\kappa}(\cos\kappa\theta \log r + \theta \sin\kappa\theta), \qquad (4\text{-}31)$$

implies

$$\phi_p = \text{Re } \frac{z^{-\kappa}}{2} \log z = - \frac{r^{-\kappa}}{2} [\cos\kappa\theta(\log^2 r - \theta^2) + 2\theta \sin\kappa\theta \log r]. \qquad (4\text{-}32)$$

It would be extremely difficult to deduce (4-28) and (4-32) by real variable methods. Thus substituting from (4-27) into formula (2-5), we find

$$h(\theta) = -r^{\kappa+2} \nabla^2 r^{-\kappa} \int^r \rho^{\nu+\kappa-1} (\cos\nu\theta \log\rho - \theta\sin\nu\theta) d\rho$$

$$= -r^{\kappa+2} \nabla^2 [\frac{r^\nu}{\nu+\kappa} (\cos\nu\theta (\log r - \frac{1}{\nu+\kappa} - \theta\sin\nu\theta]; \nu \neq -\kappa$$

$$= 0; \qquad \nu \neq -\kappa$$

$$= -r^{\kappa+2} \nabla^2 [\frac{r^{-\kappa}}{2} \log r (\cos\kappa\theta \log r + 2\theta \sin\kappa\theta)]; \nu = - \kappa.$$

i.e. $\quad = 2\kappa\theta\sin\kappa\theta - \cos\kappa\theta; \qquad \nu \neq -\kappa.$

Accordingly equation (2-6) becomes

$$\left.\begin{aligned} &\frac{\partial^2 g}{\partial\theta^2} + \kappa^2 g = 0\ ; && \nu \neq -\kappa \\ &\frac{\partial^2 g}{\partial\theta^2} + \kappa^2 g = 2\kappa\ \sin\kappa\theta - \cos\kappa\theta ; && \nu = -\kappa . \end{aligned}\right\} \qquad (4\text{-}33)$$

Equations (4-33) have the particular solutions

$$\left.\begin{aligned} &g_p(\theta) = 0 \quad ; && \nu \neq -\kappa \\ &g_p(\theta) = -\ \frac{\theta^2}{2}\ ; && \nu = -\kappa . \end{aligned}\right\} \qquad (4\text{-}34)$$

Accordingly

$$\phi_p = \frac{r^\nu}{\nu+\kappa}\,[\cos\nu\theta\,(\log r - \frac{1}{\nu+\kappa}\,) - \theta\sin\nu\theta];\ \nu \neq -\kappa \qquad (4\text{-}35)$$

$$\phi_p = -\frac{r^{-\kappa}}{2}\,(\cos\kappa\theta\log^2 r + 2\theta\ \sin\kappa\theta\log r - \frac{\theta^2}{2}\cos\kappa\theta);$$

$$\nu = -\kappa. \qquad (4\text{-}36)$$

These solutions are identical with (4-28) and (4-32), respectively.

PART I : CONCLUSION

NON-EXISTENCE OF ϕ (TWO DIMENSIONS)

Our analysis shows that, for any κ, we may always choose a harmonic function f so that the Bergman-Schiffer integral fails to give a harmonic function. The integral contributes to a particular function ϕ_p which satisfies the Bergman-Schiffer equation. In this case ϕ_p may not exist, or may be multi-valued, in some domain B even though f exists and is single-valued in B. For instance,

$$\text{if } f = r^{-\kappa}\cos\kappa\theta, \text{ then } \phi_p = r^{-\kappa}(\cos\kappa\theta \log r + \theta \sin\kappa\theta)$$

as shown in Chapter 3. If κ is a positive integer then f would be single-valued and ϕ_p would be multi-valued in any ring-shaped domain surrounding r = 0. If κ is a negative integer then f exists but ϕ_p does not exist in any domain which includes r = 0, e.g. putting κ = 0 above we see that ϕ_p = log r when f = 1.

PART II

THREE-DIMENSIONAL ANALYSIS

CHAPTER 5

THREE DIMENSIONS : GENERAL CASE

Given a harmonic function f in a three-dimensional domain B, we seek a second harmonic function ϕ in B which satisfies the Bergman-Schiffer equation

$$\kappa \phi + r \frac{\partial \phi}{\partial r} = f, \tag{5-1}$$

where $r^2 = x^2 + y^2 + z^2$ and κ is any constant. The choices $\kappa = \frac{3}{2}$ corresponds to equation (1-35) and $-4 < \kappa \lesssim -2$ corresponds to the Papkovich and Neuber problem mentioned in the Introduction (see App IV). Equation (5-1) has the family of solutions

$$\phi = r^{-\kappa} g(\theta,\psi) + r^{-\kappa} \int^{r} \rho^{\kappa-1} f \, d\rho \tag{5-2}$$

where $g(\theta,\psi)$ is an arbitrary function of the angular variables θ, ψ. We must restrict $g(\theta,\psi)$ so that ϕ becomes a harmonic function, i.e. $\nabla^2 \phi = 0$, from which it follows that

$$\nabla^2 \left\{ r^{-\kappa} g(\theta\ ,\psi) \right\} = - \nabla^2 \left\{ r^{-\kappa} \int^{r} \rho^{\kappa-1} f \, d\rho \right\}. \tag{5-3}$$

Now the right-hand side of (5-3) always exists in ~~at least~~ some domain. Therefore equation (5-3) is a Poisson's equation for $r^{-\kappa} g$, which always has a solution in this domain. Accordingly there always exists a harmonic function ϕ of the form (5-2).

A direct argument for the existence of ϕ is given in App. V.

Since

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\psi^2} \tag{5-4}$$

in spherical polar co-ordinates, we find

$$\nabla^2\left\{r^{-\kappa}g(\theta,\psi)\right\} = -r^{-\kappa-2}\left\{\kappa(\kappa-1)g + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial g}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\psi^2}\right\} \tag{5-5}$$

whence, from (5-3), g satisfies the equation

$$\kappa(\kappa-1)g + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial g}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\psi^2} = - r^{\kappa+2}\nabla^2\left\{r^{-\kappa}\int^r \rho^{\kappa-1} f\, d\rho\right\} \equiv h(\theta,\psi). \tag{5-6}$$

The expression on the right-hand side of (5-6) is at most a function of θ, ψ, which, following chapters I and II, we term $h(\theta,\psi)$. Accordingly, we write

$$\kappa(\kappa-1)g + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\psi^2} = h\ (\theta,\psi). \tag{5-7}$$

There are now two distinct possibilities for h: h = 0, $h \neq 0$, i.e. the Bergman-Schiffer integral is a harmonic function or it is not. The first possibility is now discussed.

If $h(\theta,\psi) = 0$, then g satisfies the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\psi^2} + \kappa(\kappa-1)g = 0, \tag{5-8}$$

which has solutions of the form

$$\left.\begin{aligned} g(\theta,\psi) &= P_{-\kappa}^{m}(\cos\theta)\, e^{im\psi} \\ &= P_{\kappa-1}^{m}(\cos\theta)\, e^{im\psi} \end{aligned}\right\} \tag{5-9}$$

using standard symbolism (Lebedev, 1972) where $P_{-\kappa}^{m}(\cos\theta)$ is a Legendre associated function of the first kind, and we restrict the index m to be a positive or negative integer or zero subject to the inequalities

$$|m| \leq |\kappa| \ \text{ if } \kappa \leq 0, \quad |m| < |\kappa-1| \ \text{ if } \kappa > 0. \tag{5-10}$$

The conclusion (5-9) may be seen directly from (5-2), since $r^{-\kappa}g(\theta,\psi)$ must be a harmonic function if the Bergman-Schiffer integral is a harmonic function. Accordingly equation (5-1) has harmonic solutions of the form

$$\phi = r^{-\kappa}P_{-\kappa}^{\;m}(\cos\theta)e^{im\psi} + r^{-\kappa}\int^{r}\rho^{\kappa-1}\, f\, d\,\rho. \tag{5-11}$$

For $\kappa > 0$, the harmonic function $r^{-\kappa}P_{-\kappa}^{\;m}(\cos\theta)e^{im\psi}$ becomes singular at $r = 0$. Accordingly it would not be admissible in any finite domain which includes $r = 0$. Therefore the Bergman-Schiffer integral provides a unique harmonic solution of equation (5-1) in such a domain. If $\kappa = 0$, then $r^{-\kappa}P_{-\kappa}^{\;m}(\cos\theta)e^{im\psi}$ = constant. If $\kappa < 0$, the function $r^{-\kappa}P_{-\kappa}^{m}(\cos\theta)\, e^{im\psi}$ would be acceptable only for integer values of κ in a finite domain which includes $r = 0$ (Rabenstein, 1966); since otherwise $P_{-\kappa}^{\;m}(\cos\theta)\, e^{im\psi}$ becomes singular for $\theta = \pi$ (ie. along negative z-axis). In an infinite exterior domain which excludes $r = 0$, $r^{-\kappa}P_{-\kappa}^{\;m}(\cos\theta)e^{im\psi}$ would be acceptable for $\kappa \geqslant 1$ and provided that it is an integer.

As a first example of $h = 0$, we choose the harmonic function

$$f = r^{\nu}\, P_{\nu}^{\;\mu}(\cos\theta)e^{i\mu\psi}\; ; \quad \nu \neq -\kappa \tag{5-12}$$

where μ is a given positive integer or zero and ν an arbitrary constant, in which case

$$r^{-\kappa}\int^{r}\rho^{\kappa+\nu-1}P_{\nu}^{\mu}(\cos\theta)e^{i\mu\psi}d\rho = \frac{r^{\nu}}{\nu+\kappa}P_{\nu}^{\mu}(\cos\theta)e^{i\mu\psi}\ ;\ \nu \neq -\kappa.$$

Therefore equation (5-1) has harmonic solutions of the form

$$\phi = \frac{r^{\nu}}{\nu+\kappa}P_{\nu}^{\mu}(\cos\theta)e^{i\mu\psi}+r^{-\kappa}P_{\kappa-1}^{m}(\cos\theta)e^{im\psi}\ ;\ \nu \neq -\kappa\ , \qquad (5\text{-}13)$$

subject to the inequalities (5-10) on m. In view of the restrictions mentioned in the previous paragraph, we may define the unique harmonic solution

$$\left.\begin{array}{l} \phi_p = \dfrac{r^{\nu}}{\nu+\kappa}P_{\nu}^{\mu}(\cos\theta)e^{i\mu\psi}\ ;\ \kappa > 0,\ \nu \neq -\kappa,\quad \nu \geq 0 \\ \qquad\qquad\qquad \text{or a non-integer} \quad \kappa < 0 \end{array}\right\} \qquad (5\text{-}14)$$

in any finite domain which includes $r = 0$. Also we may define the unique harmonic solutions

$$\left.\begin{array}{l} \phi_p = \dfrac{r^{\nu}}{\nu+\kappa}\ P_{-(\nu+1)}^{\mu}\ (\cos\theta)\ e^{i\mu\psi}\quad ;\ \kappa < 0, \nu \neq -\kappa, \nu \leq -1 \\ \qquad\qquad\qquad \text{or a non-integer} \quad \kappa \geq 1, \end{array}\right\} \qquad (5\text{-}15)$$

in any infinite domain which excludes $r = 0$ and is intersected by the negative z-axis. For other choices of κ, the solution would not necessarily be unique as previously explained.

As a more specialised example of h = 0, we choose the function

$$f = \log(r \sin\theta). \tag{5-16}$$

This is a harmonic function which becomes singular along the z-axis.

We find

$$r^{-\kappa}\int^{r} f\rho^{\kappa-1} d\rho = r^{-\kappa}\int^{r}_{\rho}{}^{\kappa-1}\log(\rho\sin\theta)d\rho = \frac{1}{\kappa}\,[\log(r\ \sin\theta\ ) - \frac{1}{\kappa}\,];$$

$$\kappa \neq 0 \tag{5-17}$$

from which it follows that

$$h(\theta) = -r^{\kappa+2}\,\nabla^2 \left\{\frac{1}{2}\log(r\ \sin\theta) - \frac{1}{\kappa^2}\right\} = 0\ ;\quad \kappa \neq 0. \tag{5-18}$$

Accordingly equation (5-1) has the particular solution

$$\phi_p = \frac{1}{\kappa}\,[\log(r\ \sin\theta) - \frac{1}{\kappa}\,]\ ;\quad \kappa \neq 0 \tag{5-19}$$

in this case.

The failing cases $\nu = -\kappa$ of (5-14) and $\kappa = 0$ of (5-19) will be examined in the following chapters.

CHAPTER 6

THREE DIMENSIONS: INTEGER FAILING CASE ($\mu = 0$)

The solution (5-13) clearly breaks down when $\nu = -\kappa$. This is the case where the Bergman-Schiffer integral is not a harmonic function, i.e. $h \neq 0$. Thus, when

$$f = r^{-\kappa} P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi}, \qquad (6\text{-}1)$$

where μ is an arbitrary positive integer or zero, we find

$$r^{-\kappa}\int^{r} \rho^{-1} P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi}\, d\rho = r^{-\kappa} P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi}\log r, \qquad (6\text{-}2)$$

$$\nabla^2 \left\{r^{-\kappa}P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi}\log r\right\} = -(2\kappa - 1)r^{-\kappa-2}P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi}, \qquad (6\text{-}3)$$

$$h(\theta,\psi) = r^{\kappa+2}\nabla^2 \left\{r^{-\kappa}P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi} \log r\right\}$$

$$= (2\kappa - 1)P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi} . \qquad (6\text{-}4)$$

Equation (5-7) now becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial g}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 g}{\partial\psi^2} + \kappa(\kappa-1)g = (2\kappa-1)P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi}. \qquad (6\text{-}5)$$

Putting

$$g(\theta,\psi) = w(\theta)e^{i\mu\psi}, \tag{6-6}$$

this becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial w}{\partial\theta}\right) + \left[\kappa(\kappa-1) - \frac{\mu^2}{\sin^2\theta}\right] w = (2\kappa-1)P^{\mu}_{\kappa-1}(\cos\theta). \tag{6-7}$$

The simplest possible choice of μ in (6-7) is $\mu = 0$.

In this case equation (6-7) becomes

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial w}{\partial\theta}\right) + \kappa(\kappa-1)\,w = (2\kappa-1)P_{\kappa-1}(\cos\theta). \tag{6-8}$$

To make further progress we restrict κ to be a positive integer and look for a particular solution of equation (6-8) of the form

$$w_p(\theta) = -P_{\kappa-1}(\cos\theta)\log\sin\theta + S(\theta)\ ; \quad 0 < \theta < \pi\ , \tag{6-9}$$

where $S(\theta)$ is a function of θ to be determined. Substituting from (6-9) into (6-8), we find

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial S}{\partial\theta}\right) + \kappa(\kappa-1)S = 2(\kappa-1)P_{\kappa-1}(\cos\theta) + 2\cot\theta\,\frac{\partial}{\partial\theta}P_{\kappa-1}(\cos\theta)$$

$$= \frac{2}{\sin\theta}\,\frac{\partial}{\partial\theta}P_{\kappa-2}(\cos\theta). \tag{6-10}$$

Equation (6-10) has a polynomial particular solution of the form

$$S(\theta) = \sum_{r=1}^{p} \alpha_r \cos^{\kappa-2r}\theta, \qquad (6\text{-}11)$$

where

$p = \frac{\kappa-1}{2}$ if $(\kappa-1)$ is even, $p = \frac{(\kappa-2)}{2}$ if $(\kappa-1)$ is odd

and the coefficients a_r satisfy the recurrence formula

$$\left.\begin{aligned} a_1 &= -\frac{(2\kappa-4)!}{2^{\kappa-2}(\kappa-2)!\,(\kappa-3)!\,(2\kappa-3)} \quad ; \quad \kappa \geq 3 \\ &(\kappa-2r+1)(\kappa-2r)a_{r-1}+2r(2\kappa-2r-1)a_r \\ &= (-1)^r \frac{(2\kappa-2r-2)!}{2^{\kappa-3}(r-1)!(\kappa-r-1)!(\kappa-2r-1)!} \quad ; \quad r \geq 2 \end{aligned}\right\} \qquad (6\text{-}12)$$

as can be verified by substituting (6-11) into (6-10).
It may be remarked that $S(\theta) = 0$ for $\kappa = 1,2$. Accordingly equation (6-5) has the particular solution

$$g_p = -P_{\kappa-1}(\cos\theta)\log\sin\theta + \sum_{r=1}^{p} a_r \cos^{\kappa-2r-1}\theta \qquad (6\text{-}13)$$

in the case of positive integer values of κ, when $\mu = 0$.
To the particular solution (6-13) we supperpose complementary solutions of the form

$$a\,P^m_{\kappa-1}(\cos\theta)e^{im\psi} + b\,Q^m_{\kappa-1}(\cos\theta)e^{im\psi} \quad , \qquad (6\text{-}14)$$

where a,b are arbitrary constants and $P^m_{\kappa-1}(\cos\theta)$, $Q^m_{\kappa-1}(\cos\theta)$ denote the associated Legendre functions of the first and second kind respectively. The index m is restricted to be an arbitrary integer to ensure that the solution remains single-valued for any circuit around the z-axis. We note that $P^m_{\kappa-1}(\cos\theta) = 0$ for $|m| > \kappa - 1$ as follows from the definition of the associated Legendre function.
Accordingly, when $\mu = 0$, equation (6-5) has solutions of the form

$$g(\theta,\psi) = -P_{\kappa-1}(\cos\theta)\log\sin\theta + [aP^m_{\kappa-1}(\cos\theta)+b\,Q^m_{\kappa-1})]e^{im\psi};$$

$$(6\text{-}15)$$

$$\kappa = 1,2$$

$$\left.\begin{aligned} g(\theta,\psi) = {} & -P_{\kappa-1}(\cos\theta)\log\sin\theta + \sum_{r=1}^{p} a_r \cos^{\kappa-2r}\theta \\ & + [a\, P^m_{\kappa-1}(\cos\theta) + b\, Q^m_{\kappa-1}(\cos\theta)]\, e^{im\psi} \end{aligned}\right\} \quad ; \ \kappa \geq 3. \tag{6-16}$$

Therefore equation (5-1) has harmonic solutions of the form

$$\phi = r^{-\kappa}\, g(\theta,\psi) + r^{-\kappa}\, P_{\kappa-1}(\cos\theta)\log r \tag{6-17}$$

i.e.

$$\phi = r^{-\kappa}P_{\kappa-1}(\cos\theta)\log\frac{r}{\sin\theta} + r^{-\kappa}[aP^m_{\kappa-1}(\cos\theta)+bQ^m_{\kappa-1}(\cos\theta)]e^{im\psi} \ ; \ \kappa = 1,2. \tag{6-18}$$

$$\left.\begin{aligned} \phi = {} & r^{-\kappa}P_{\kappa-1}(\cos\theta)\log\frac{r}{\sin\theta} + r^{-\kappa}\sum_{r=1}^{p} a_r \cos^{\kappa-2r-1}\theta \\ & + r^{-\kappa}[a\, P^m_{\kappa-1}(\cos\theta) + b\, Q^m_{\kappa-1}(\cos\theta)]\ e^{im\psi} \end{aligned}\right\} \quad ; \ \kappa \geq 3 \tag{6-19}$$

It will be seen that the solutions (6-19) have singularities along the z-axis. Accordingly for any positive integer value of κ, equation (5-1) has no harmonic solutions for the particular choice $f = r^{-\kappa}P_{\kappa-1}(\cos\theta)$, inside any domain which is intersected by the z-axis.

If $\kappa = 1$, the solution (6-15) reduces to the much simpler form

$$g(\theta,\psi) = -\log \sin \theta + aP_0^{\,m}(\cos\theta)e^{im\psi} + b\, Q_0^{\,m}(\cos\theta)e^{im\psi}. \qquad (6\text{-}20)$$

Now $P_0^{\,m}(\cos\theta) = 0$ for $m \neq 0$, $P_0^{\,0}(\cos\theta) = 1$, as follows from the definition of the associated Legendre functions. Also

$$Q_0^{\,m} = \tfrac{1}{2}(1 - u^2)^{\frac{1}{2}} \frac{d^m}{du^m} [\log \frac{1 + u}{1 - u}] \quad ; \quad u = \cos \theta , \qquad (6\text{-}21)$$

using Kellogg's symbolism (1954). Accordingly, we obtain

$$g(\theta,\psi) = \log \sin\theta + a + b\, Q_0^{\,m}(\cos\theta)e^{im\psi} , \qquad (6\text{-}22)$$

$$\begin{aligned} \phi &= r^{-1}g(\theta,\psi) + r^{-1} \log r \\ &= r^{-1}\log (\frac{r}{\sin\theta}) + r^{-1} [a + b\, Q_0^{\,m}(\cos\theta)]\, e^{im\psi} . \end{aligned} \qquad (6\text{-}23)$$

As before this has singularities along the z-axis. Clearly therefore when $\kappa = 1$ equation (5-1) has no harmonic solutions for the particular choice $f = r^{-1}$ inside any domain which is intersected by the z-axis.

For any negative integer or zero value of κ, it may be verified that equation (6-8) has a particular solution of the form

$$w_p(\theta) = P_{-\kappa}(\cos\theta)\log\sin\theta + S(\theta), \tag{6-24}$$

$$S(\theta) = -\sum_{r=1}^{p} a_r \cos^{-\kappa-2r}\theta, \tag{6-25}$$

where

$$p = -\frac{\kappa}{2} \text{ if } -\kappa \text{ is even, } p = -\frac{(\kappa+1)}{2} \text{ if } -(\kappa+1) \text{ is odd}$$

and the coefficients a_r satisfy the recurrence formula (6-12) with $-\kappa$ in the place of κ. It may be remarked that $S(\theta) = 0$ for $\kappa = 0, -1$. Therefore equation (6-5) has solutions of the form

$$g(\theta,\psi) = P_{-\kappa}(\cos\theta)\log\sin\theta + [a\, P_{-\kappa}^{m}(\cos\theta) + b\, Q_{-\kappa}^{m}(\cos\theta)]e^{im\psi};$$

$$\kappa = 0, -1 \tag{6-26}$$

$$g(\theta,\psi) = P_{-\kappa}\log\sin\theta - \sum_{r=1}^{p} a_r \cos^{-\kappa-2r}\theta + [a\,P_{-\kappa}^{m}(\cos\theta) + b\,Q_{-\kappa}^{m}(\cos\theta)]\,e^{im\psi} \quad ; \ \kappa \leq -2 \qquad (6\text{-}27)$$

where a,b are arbitrary constants. Accordingly equation (5-1) now has harmonic solutions

$$\phi = r^{-\kappa}g(\theta,\psi) + r^{-\kappa}P_{-\kappa}(\cos\theta)\log r \qquad (6\text{-}28)$$

i.e. $\phi = r^{-\kappa}P_{-\kappa}(\cos\theta)\log(r\sin\theta) + r^{-\kappa}[aP_{-\kappa}^{m}(\cos\theta) + b\,Q_{-\kappa}^{m}(\cos\theta)]\,e^{im\psi}$;

$$\kappa = 0, -1 \qquad (6\text{-}29)$$

$$\phi = r^{-\kappa}P_{-\kappa}(\cos\theta)\log(r\sin\theta) - r^{-\kappa}\sum_{r=1}^{p} a_r \cos^{-\kappa-2r}\theta + r^{-\kappa}[a\,P_{-\kappa}^{m}(\cos\theta) + b\,Q_{-}^{m}(\cos\theta)]\,e^{im\psi} \quad ; \ \kappa \leq -2. \qquad (6\text{-}30)$$

As before this has singularities along the z-axis. Accordingly for any negative integer or zero value of κ, equation (5-1) has no harmonic solutions for the particular choice

$f = r^{-\kappa} P_{-\kappa}(\cos\theta)$ inside any finite domain which is intersected by the z-axis.

If $\kappa = 0$, the solution (6-26) reduces to

$$g(\theta,\psi) = \log \sin\theta + a + b\, Q_0^{\,m}(\cos\theta)e^{im\psi}. \tag{6-31}$$

Accordingly equation (5-1) now has harmonic solutions of the form

$$\phi = g(\theta,\psi) + \log r\,,$$

$$= \log(r \sin\theta) + a + b\, Q_0^m\,(\cos\theta)e^{im\psi}\;. \tag{6-32}$$

As before this has singularities along the z-axis. Clearly therefore, when $\kappa = 0$, equation (5-1) has no harmonic solutions for the particular choice f = constant inside any finite domain which is intersected by the z-axis.

CHAPTER 7

THREE DIMENSIONS: INTEGER FAILING CASE ($\mu \geq 1$)

For any integer value of $\mu \geq 1$, we divide the discussion into (a) positive integer values of κ and (b) zero and negative integer values of κ.

A. positive integer value of κ

In this case we look for a particular solution of equation (6-7) of the form

$$W_p(\theta) = - P^{\mu}_{\kappa-1}(\cos\theta)\log\sin\theta + S(\theta) \; ; \quad 0 < \theta < \pi \tag{7-1}$$

where $S(\theta)$ is a function of θ to be determined. Substituting from (7-1) into (6-7), we find

$$\sin\theta \frac{\partial}{\partial\theta}(\sin\theta \frac{\partial S}{\partial\theta}) + [\kappa(\kappa-1)-\mu^2]\, S$$

$$= 2(\kappa-1)P^{\mu}_{\kappa-1}(\cos\theta)\sin^2\theta + \sin 2\theta \frac{\partial}{\partial\theta} P^{\mu}_{\kappa-1}(\cos\theta). \tag{7-2}$$

This equation does not generally have a finite polynomial solution, but closed-form solutions can be achieved in some special cases. For instance, if $\mu = 1$, equation (7-2) becomes

$$\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial S}{\partial\theta}\right) + \left[\kappa(\kappa-1)\sin^2\theta - 1\right] S$$

$$= 2(\kappa-1)\sin^2\theta \, P_{\kappa-1}^{1}(\cos\theta) + \sin 2\theta \frac{\partial}{\partial\theta} P_{\kappa-1}^{1}(\cos\theta). \qquad (7\text{-}3)$$

Choosing $\kappa = 1$, the expression on the right-hand side of equation (7-3) vanishes, since $P_o^{\,1}(\cos\theta) = 0$, in which case (7-3) has the particular solution $S_p(\theta) = 0$. If so

$$W_p(\theta) = P_o^{\,1}(\cos\theta)\log \sin\theta + S_p(\theta) = 0, \qquad (7\text{-}4)$$

$$g_p(\theta,\psi) = w_p(\theta)e^{i\psi} = 0, \qquad (7\text{-}5)$$

$$\phi_p(r,\theta,\psi) = r^{-1}g(\theta,\psi)+r^{-1}P_o^{\,1}(\cos\theta)\log r = 0. \qquad (7\text{-}6)$$

Here equation (5-1) becomes

$$\phi + r\frac{\partial\phi}{\partial r} = r^{-1} P_o^{\,1}(\cos\theta)e^{i\psi} = 0, \qquad (7\text{-}7)$$

with the harmonic solution $\phi = c\, r^{-1}$, where c is an arbitrary constant. This solution arises entirely from the complementary solution, and it is seen to be a particular case of the general complementary solution (5-9), i.e.

$$g(\theta,\psi) = c\, P_{\kappa-1}^{\ m} e^{im\psi} = c\, P_o^{\ m}(\cos\theta)e^{im\psi} = c$$

since $P_o^{\ o}(\cos\theta) = 1$, $P_o^{\ m}(\cos\theta) = 0$ for $m \neq 0$,

$$\phi(r,\theta,\psi) = r^{-1}g(\theta,\psi) = c\, r^{-1}. \tag{7-8}$$

Choosing $\kappa = 2$, equation (7-3) becomes

$$\sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial S}{\partial\theta}\right) + (2\sin^2\theta - 1) = 2\sin\theta\ , \tag{7-9}$$

which has the particular solution

$$S_p(\theta) = (\sin\theta)^{-1}. \tag{7-10}$$

Accordingly

$$W_p(\theta) = -P_1^{\ 1}(\cos\theta)\log\sin\theta + S_p(\theta)$$

$$= -\sin\theta\log\sin\theta + (\sin\theta)^{-1}, \tag{7-11}$$

$$g_p(\theta,\psi) = W_p(\theta)e^{i\psi}$$

$$= [-\sin\theta \log \sin\theta + (\sin\theta)^{-1}]\, e^{i\psi} \;; \tag{7-12}$$

$$\phi_p(r,\theta,\psi) = r^{-2}g(\theta,\psi)+r^{-2}\, P_1^{\,1}(\cos\theta) \log r$$

$$= r^{-2}\, [\sin\theta \log \frac{r}{\sin\theta} + (\sin\theta)^{-1}]\, e^{i\psi}. \tag{7-13}$$

We note that $\sin\theta \log \sin\theta \to 0$ as $\theta \to 0$ or π, but a singularity arises from $(\sin\theta)^{-1}$. Choosing $\kappa = 3$, equation (7-3) has the particular solution

$$S_p(\theta) = \cot\theta \;, \tag{7-14}$$

and therefore

$$W_p(\theta) = -\sin\theta \log \sin\theta + \cot\theta \;, \tag{7-15}$$

$$g_p(\theta,\psi) = [-\sin\theta \log \sin\theta + \cot\theta]\, e^{i\psi}, \tag{7-16}$$

$$\phi_p(r,\theta,\psi) = r^{-3}\, [\, \sin\theta \log \frac{r}{\sin\theta} + \cot\theta \,]\; e^{i\psi}. \tag{7-17}$$

It will be seen that ϕ_p has a singularity of $O(\sin\theta)^{-1}$ as $\theta \to 0$.

Equation (7-2), for $\mu = 2$, becomes

$$\sin\theta \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial S}{\partial\theta}) + [\kappa(\kappa-1)\sin^2\theta - 4] \quad S$$

$$= 2(\kappa-1)P_{\kappa-1}^{2} (\cos\theta)\sin^2\theta + \sin 2\theta\frac{\partial}{\partial\theta} P_{\kappa-1}^{2}(\cos\theta). \qquad (7\text{-}18)$$

choosing $\kappa = 1, 2$, the expression on the right-hand side of equation (7-18) vanishes, since $P_o^{\ 2}(\cos\theta) = 0$, $P_1^{\ 2}(\cos\theta) = 0$, in which case (7-18) has the particular solution $S_p(\theta) = 0$. If so

$$w_p\ (\theta) = 0, \quad g_p\ (\theta,\psi) = 0, \quad \phi_p(r,\theta,\psi) = 0$$

following from (7-4), (7-5) and (7-6). Here equation (5-1) becomes ($\kappa\ = 2$)

$$2\phi + r \frac{\partial\phi}{\partial r} = r^{-2} P_1^{\ 2}(\cos\theta)e^{2i\psi} = 0 \qquad (7\text{-}19)$$

with solution $\phi = gr^{-2}$, where g is an arbitrary functions of θ,ψ. We may always choose g such that ϕ is a harmonic function. This solution of equation (7-19) is a particular case of the general complemtary solution (5-10), i.e.

$$g(\theta,\psi) = P_{\kappa-1}^{\ m}(\cos\theta)e^{im\psi} = P_1^{\ m}(\cos\theta)e^{im\psi} = \cos\theta + \sin\theta e^{i\psi}, \qquad (7\text{-}20)$$

since $P_1^{\ o}(\cos\theta) = \cos\theta$, $P_1^{\ 1}(\cos\theta) = \sin\theta$, $P_1^{\ m}(\cos\theta) = 0$ for $m \geq 2$,

$$\phi(r,\theta,\psi) = r^{-2}(\cos\theta + \sin\theta\, e^{i\psi}) \tag{7-21}$$

Choosing $\kappa = 3$, equation (7-18) becomes

$$\sin\theta \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial S}{\partial\theta}) + (6 \sin^2\theta - 4)\, S = 12 \sin^2\theta \tag{7-22}$$

which has the particular solution

$$S_p(\theta) = 2 \sin^{-2}\theta + 2. \tag{7-23}$$

Accordingly

$$w_p(\theta) = -3\sin^2\theta \log\sin\theta + 2\sin^{-2}\theta + 2 \tag{7-24}$$

$$g_p(\theta,\psi) = [-3\sin^2\theta \log\sin\theta + 2\sin^{-2}\theta + 2]\, e^{2i\psi}, \tag{7-25}$$

$$\phi_p(r,\theta,\psi) = r^{-2}[-3\sin^2\theta \log\sin\theta + 2\sin^{-2}\theta + 2]\, e^{2i\psi}. \tag{7-26}$$

As before it will be seen that ϕ_p has a singularity of $O(\sin\theta)^{-2}$ as $\theta \to 0$.

For any positive integer value of κ and any integer value of $\mu > \kappa-1$, we see that $f = 0$ since $P^{\mu}_{\kappa-1} = 0$ in this case. For any positive integer value of κ and any integer value of $\mu \leq \kappa - 1$, we may prove (App. II) that the solution of equation (7-3) has a singularity of $O(\sin\theta)^{-\kappa}$ as $\theta \to 0$.

Some examples have already been given in this Chapter.

B. zero and negative integer values of κ

In this case we seek a particular solution of equation (6-7) of the form

$$w_p(\theta) = P_{-\kappa}^{\mu}(\cos\theta)\log\sin\theta + S(\theta), \tag{7-27}$$

where $S(\theta)$ is a function of θ to be determined. Substituting from (7-27) into (5-6), we find

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + [\kappa(\kappa-1)\sin^2\theta - \mu^2]S$$

$$= 2\kappa P_{-\kappa}^{\mu}(\cos\theta) - \sin 2\theta\frac{\partial}{\partial\theta}P_{\kappa-1}^{\mu}(\cos\theta). \tag{7-28}$$

As before this equation does not generally have a finite polynomial solution, but closed-form solutions can be achieved in some special cases. For instance, if $\mu = 1$, equation (7-28) becomes

$$\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + [\kappa(\kappa-1)\sin^2\theta - 1]S$$

$$= 2\kappa P_{-\kappa}^{1}(\cos\theta)\sin^2\theta - \sin 2\theta\frac{\partial}{\partial\theta}P_{-\kappa}^{1}(\cos\theta). \tag{7-29}$$

Choosing $\kappa = 0$, the expression on the right-hand side of equation (6-29) vanishes, since $P_o^{\,1}(\cos\theta) = 0$, in which case (7-29) has the particular solution $s_p(\theta) = 0$. If so

$$w_p(\theta) = 0,\ g_p(\theta,\psi) = 0,\ \phi_p(r,\theta,\psi) = 0 \tag{7-30}$$

following from (7-4), (7-5) and (7-6). Here equation (5-1) becomes

$$r\,\frac{\partial\phi}{\partial r} = P_o^{\,1}(\cos\theta) = 0 \tag{7-31}$$

with the harmonic solution ϕ = cons. This solution arises entirely from the complementary solution, and it is seen to be a particular case of the general complementary solution (5-9), i.e.

$$g(\theta,\psi) = c\,P_{-\kappa}^{\,m}\,e^{im\psi} = c\,P_o^{m}\,e^{im\psi} = c\,P_o^{\,o}(\cos\theta) = c$$

$$\phi(r,\theta,\psi) = g(\theta,\psi) = c. \tag{7-32}$$

Choosing $\kappa = -1$, equation (7-29) has the particular solution

$$S_p(\theta) = -(\sin\theta)^{-1}\ . \tag{7-33}$$

Accordingly

$$w_p(\theta) = \sin\theta \log \sin\theta \; - \; (\sin\theta)^{-1} \qquad (7\text{-}34)$$

$$g_p(\theta,\psi) = [\sin\theta \; \log \sin\theta \; - \; (\sin\theta)^{-1}] \; e^{i} \qquad (7\text{-}35)$$

$$\phi_p(r,\theta,\psi) = r \; [\sin\theta \; \log(r \sin\theta) \; - \; (\sin\theta)^{-1}] \; e^{i\psi}. \qquad (7\text{-}36)$$

It will be seen that ϕ_p has a singularity of $O(\sin\theta)^{-1}$ as $\theta \rightarrow 0$.

For any zero or negative integer value of κ and any integer value of $\mu > -\kappa$, we see that $f = 0$ since $P_{-\kappa}^{\mu} = 0$ in this case. For any zero or negative integer value of κ and any integer value $\mu \leq -\kappa$, the solution of equation (7-3) has a singularity of $O(\sin\theta)^{-\kappa}$ as $\theta \rightarrow 0$. Some examples have already been given in this Chapter.

CHAPTER 8

THREE DIMENSIONS : NON-INTEGER FAILING CASE

Now we look for a solution of equation (6-7) for any positive real number κ, of the form

$$w(\theta) = W(\theta)\, P_{\kappa-1}^{\mu}(\cos\theta); \quad \mu \text{ is any positive integer or zero,} \tag{8-1}$$

where $W(\theta)$ is a function of θ to be determined. Substituting from (8-1) into (6-7), we find

$$P_{\kappa-1}^{\mu}(\cos\theta)\ \frac{\partial^2 W}{\partial\theta^2} + \left[2\frac{\partial}{\partial\theta} P_{\kappa-1}^{\mu}(\cos\theta) + \cot\theta\ P_{\kappa-1}^{\mu}(\cos\theta)\right]\frac{\partial W}{\partial\theta}$$

$$= (2\kappa - 1) P_{\kappa-1}^{\mu}(\cos\theta). \tag{8-2}$$

This equation can be written in the form

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\ \frac{\partial W}{\partial\theta}\, P_{\kappa-1}^{\mu}(\cos\theta)\right] + \frac{\partial W}{\partial\theta}\frac{\partial}{\partial\theta}\, P_{\kappa-1}^{\mu}(\cos\theta)$$

$$= (2\kappa - 1)\ P_{\kappa-1}^{\mu}(\cos\theta). \tag{8-3}$$

multiplying both sides of this by $\sin\theta\, P^{\mu}_{\kappa-1}(\cos\theta)$, and putting

$$U = \sin\theta \frac{\partial W}{\partial \theta} P^{\mu}_{\kappa-1}(\cos\theta), \tag{8-4}$$

gives

$$\frac{\partial}{\partial\theta} [U\, P^{\mu}_{\kappa-1}(\cos\theta)] = (2\kappa - 1)\, [P^{\mu}_{\kappa-1}(\cos\theta)]^2 \sin\theta, \tag{8-5}$$

i.e.

$$U\, P^{\mu}_{\kappa-1}(\cos\theta) = (2\kappa-1) \int^{\theta} [P^{\mu}_{\kappa-1}(\cos\gamma)]^2 \sin\gamma\, d\gamma. \tag{8-6}$$

Substituting from (8-6) into (8-4) yields

$$W_p(\theta) = \int^{\theta} \frac{U\, d\lambda}{\sin\lambda [P^{\mu}_{\kappa-1}(\cos\lambda)]^2}$$

$$= (2\kappa-1)\int^{\theta} \frac{d\lambda}{\sin\lambda [P^{\mu}_{\kappa-1}(\cos\lambda)]^2} \int^{\lambda} [P^{\mu}_{\kappa-1}(\cos\gamma)]^2 \sin\gamma\, d\gamma. \tag{8-7}$$

Accordingly equation (6-7) has a particular solution

$$w_p(\theta) = (2\kappa-1)P_{\kappa-1}^{\mu}(\cos\theta) \int^{\theta} \frac{d\lambda}{\sin\lambda[P_{\kappa-1}^{\mu}(\cos\lambda)]^2} \times$$

$$\times \int^{\lambda} [P_{\kappa-1}^{\mu}(\cos\gamma)]^2 \sin\gamma d\gamma. \tag{8-8}$$

It will be noted that this approach yields $w_p(\theta)$ directly without introducing $S(\theta)$, in contrast to (6-9) where we write

$$w_p(\theta) = -P_{\kappa-1}(\cos\theta)\log\sin\theta + S(\theta) \; ; \quad \left.\begin{array}{l} 0 < \theta < \pi \\ \mu = 0 \end{array}\right\}. \tag{8-9}$$

By virtue of (6-6), it follows that

$$g_p(\theta,\psi) = (2\kappa-1)P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi}\int^{\theta} \frac{d\lambda}{\sin\lambda[P_{\kappa-1}^{\mu}(\cos\lambda)]^2} \times$$

$$\times \int^{\lambda} [P_{\kappa-1}^{\mu}(\cos\gamma)]^2 \sin\gamma d\gamma \; , \tag{8-10}$$

from which we arrive at the particular solution

$$\phi_p = r^{-\kappa} P_{\kappa-1}^{\mu}(\cos\theta) e^{i\mu\psi} \left\{ \log r + \right.$$

$$\left. (2\kappa-1) \int^{\theta} \frac{d\lambda}{\sin\lambda [P_{\kappa-1}^{\mu}(\cos\lambda)]^2} \int^{\lambda} [P_{\kappa-1}^{\mu}(\cos\gamma)]^2 \sin\gamma \, d\gamma \right\}; \ \kappa > 0. \qquad (8\text{-}11)$$

For zero or negative real values of κ, the corresponding particular solution is

$$\phi_p = r^{-\kappa} P_{-\kappa}^{\mu}(\cos\theta) e^{i\mu\psi} \left\{ \log r + \right.$$

$$\left. (2\kappa-1) \int^{\theta} \frac{d\lambda}{\sin\lambda [P_{-\kappa}^{\mu}(\cos\lambda)]^2} \int^{\lambda} [P_{-\kappa}^{\mu}(\cos\gamma)]^2 \sin\gamma \, d\gamma \right\}; \ \kappa \leq 0 \qquad (8\text{-}12)$$

For $\mu = 0$, and integer values of κ, we recover the results of Chapter 6. Thus, if κ is any positive integer, the solution (8-11) reduces to the solutions (6-18), (6-19) with $a = b = 0$, as is proved in App. I. Similarly if κ is zero or any negative integer, the solution (8-12) reduces to the solutions (6-29), (6-30) with $a = b = 0$, as is proved in App. I. It is interesting to verify these reductions for some special choices of κ. Thus, putting $\kappa = 0$ in (8-12), we find

$$\phi_p = P_o{}^o(\cos\theta) \left\{\log r - \int^{\theta} \frac{d\lambda}{\sin\lambda[P_o{}^o(\cos\lambda)]^2} \int^{\lambda} [P_o{}^o(\cos\gamma)]^2 \sin\gamma\, d\gamma,\right.$$

$$= \left\{\log r - \int^{\theta} \frac{d\lambda}{\lambda} \int^{\lambda} \sin\gamma d\gamma \right\} = \log(r \sin\theta) , \qquad (8\text{-}13)$$

is essential agreement with (6-32) with $a = b = 0$. Also putting $\kappa = 1$ in (8-11), we find

$$\phi_p = r^{-1} \left\{\log r - \log \sin\theta\right\} = r^{-1} \log \frac{r}{\sin\theta} , \qquad (8\text{-}14)$$

is essential agreement with (6-23).

When $\mu = 1$, the solutions (8-11) and (8-12) respectively become

$$\phi_p = r^{-\kappa} P_{\kappa-1}^{1} (\cos\theta) e^{i\psi} \left\{\log r + \right.$$

$$\left. (2\kappa-1) \int^{\theta} \frac{d\lambda}{\sin\lambda[P_{\kappa-1}^{1}(\cos\lambda)]^2} \int^{\lambda} [P_{\kappa-1}^{1} (\cos\gamma)]^2 \sin\gamma\, d\gamma\right\} ; \kappa > 0, \qquad (8\text{-}15)$$

$$\phi_p = r^{-\kappa} P_{-\kappa}^{1}(\cos\theta) e^{i\psi} \left\{ \log r + \right.$$

$$\left. (2\kappa -1) \int^{\theta} \frac{d\lambda}{\sin\lambda[P_{-\kappa}^{1}(\cos\lambda)]^2} \int^{\lambda} [P_{-\kappa}^{1}(\cos\gamma)]^2 \sin\gamma d\gamma \right\}; \kappa \leqslant 0. \qquad (8\text{-}16)$$

For certain specialised integer values of κ, these yield explicit results already obtained in Chapter 7. Thus, putting $\kappa = 0$ in (8-16) gives $\phi_p = 0$, since $P_o^{\,1}(\cos\theta) = 0$. Similarly, putting $\kappa = 1$ in (8-15) gives $\phi_p = 0$ since $P_o^{\,1}(\cos\theta) = 0$. Also putting $\kappa = 2$ in (8-15), the solution (8-15) becomes

$$\phi_p = r^{-2}P_1^{\,1}(\cos\theta)e^{i\psi}\left\{\log r + 3\int^{\theta}\frac{d\lambda}{\sin\lambda[P_1^{\,1}(\cos\lambda)]^2}\int^{\lambda}[P_1^{\,1}(\cos\gamma)]^2\sin\gamma d\gamma\right\},$$

$$= r^{-2}\sin\theta e^{i\psi}\left\{\log r + 3\int^{\theta}\frac{d\lambda}{\sin^3\lambda}\int^{\lambda}\sin^3\gamma d\gamma\right\},$$

$$= r^{-2}\sin\theta\, e^{i\psi}\left\{\log r + \sin^{-2}\theta - \log\sin\theta\right\}$$

$$= r^{-2}\left\{\sin\theta\log\frac{r}{\sin\theta} + (\sin\theta)^{-1}\right\}e^{i\psi}, \tag{8-17}$$

which identically equals (7-13). For $\kappa = 3$, the solution (8-15) becomes

$$\phi_p = r^{-3}\,[\sin\theta\log\frac{r}{\sin\theta} + \cot\theta]\,e^{i\psi}, \tag{8-18}$$

which identically equals (7-17). Finally putting $\kappa = -1$ in (8-16) gives

$$\phi_p = r \sin\theta \left\{ \log r - 3\int^{\theta} \frac{d\,\lambda}{\sin^3\lambda} \int^{\lambda} \sin^3 \gamma \, d\,\gamma \right\}$$

$$= r \left\{\sin\theta \log(r \sin\theta) - (\sin\theta)^{-1}\right\} e^{i\psi}, \qquad (8\text{-}19)$$

which identically equals (7-36).

Similarly for $\mu = 2$; $\kappa = 0,1,2,3$ we recover the explicit results of Chapter 7.

For any choice of κ, the solutions (8-11), (8-12) have a singularity of the form $O(\sin\theta)^{-\mu}$ as $\theta \to 0$; $\mu = 1,2,\ldots$ and of the form $O(\log \sin\theta)$ as $\theta \to 0$; $\mu = 0$. Thus, from the definition of $P_{\kappa-1}^{\mu}(\cos\theta)$,

$$[P_{\kappa-1}(\cos\theta)]^2 = \sin^{2\mu}\theta \left[\frac{d^{\mu}}{du^{\mu}} P_{\kappa-1}(\kappa)\right]^2 ; \quad u = \cos\theta , \qquad (8\text{-}20)$$

where $\left[\frac{d^{\mu}}{du^{\mu}} P_{\kappa-1}(\kappa)\right]_{\kappa=1} \neq 0$; $\mu = 0,1,2,\ldots$. Therefore the constant term in the Taylor's expansion of this function around $u = 1$ is non-zero. It follows that the solution (8-11) includes a term of the form

$$r^{-\kappa}P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi}\int^{\theta}\frac{d\,\lambda}{\sin\lambda\,[P_{\kappa-1}^{\mu}(\cos\lambda)]^2}\int^{\lambda}\sin^{2\mu+1}\gamma\;\;d\gamma. \tag{8-21}$$

This expression includes a term of the form

$$r^{-\kappa}P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi}\int^{\theta}\frac{d\,\lambda}{\sin\lambda\,[P_{\kappa-1}^{\mu}(\cos\lambda)]^2}\int^{\lambda}\sin\gamma\,d\,\gamma,$$

$$= -\,r^{-\kappa}P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi}\int^{\theta}\frac{\cos\lambda\,d\,\lambda}{\sin\lambda\,[P_{\kappa-1}^{\mu}(\cos\lambda)]^2}, \tag{8-22}$$

as can be seen by writing $\sin^{2\mu}\gamma$ as a polynomial in $\cos\theta$.
Now the integral on the right-hand side of (8-22) includes a term of the form

$$r^{-\kappa}P_{\kappa-1}^{\mu}(\cos\theta)e^{i\mu\psi}\int^{\theta}\frac{\cos\lambda\,d\,\lambda}{\sin^{2\mu+1}\lambda} = \frac{1}{2\mu}\,r^{-\kappa}P_{\kappa-1}^{\mu}(\cos\theta)\sin^{-2\mu}\theta\,e^{i\mu\psi},$$

$$= r^{-\kappa}O(\sin\theta)^{-\mu}\text{ as }\theta\to 0;\;\mu = 1,2,\ldots \tag{8-23}$$

$$r^{-\kappa}P_{\kappa-1}(\cos\theta)\int^{\theta}\frac{\cos\lambda}{\sin\lambda}\,d\lambda = r^{-\kappa}P_{\kappa-1}(\cos\theta)\log\sin\theta\quad\text{as }\theta\to 0;\;\mu = 0, \tag{8-24}$$

so proving the theorem stated above. It may be noted that these results hold for any real value of κ, integer or non-integer.

The behaviour (8-23), (8-24) hold as $\theta \to \pi$, for integer values of κ. For non-integer values of κ, f is not defined.

The analysis of Chapters 5-8 can be applied to negative integer values μ by virtue of the relation

$$P_\nu^{\mu}(\cos\theta) = (-1)^{\mu} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(\cos\theta) \tag{8-25}$$

As a final example we consider the special harmonic function (5-16), i.e. $f = \log(r \sin\theta)$, which proved to be a failing case of the Bergman-Schiffer integral for $\kappa = 0$. This integral becomes

$$\int^{r} \rho^{-1} \log(\rho\sin\theta)d\rho = \frac{1}{2} \log r \,(\log r + 2\log \sin\theta), \tag{8-26}$$

from which it follows that

$$h(\theta,\psi) = -r^{\kappa+2}\nabla^2 \int^{r} \rho^{\kappa-1} f d\rho \,,$$

$$= -r^2\nabla^2 \left\{\frac{1}{2}\log^2 r + \log r \log \sin\theta \right\} = -1-\log \sin\theta. \tag{8-27}$$

Accordingly equation (5-7) becomes

$$\frac{1}{\sin}\ \frac{\partial}{\partial\theta}\left(\sin\ \frac{\partial g}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\ \frac{\partial^2 g}{\partial\psi^2} = -\ 1 - \log\ \sin\theta \qquad (8\text{-}28)$$

with the particular solution

$$g_p(\theta,\psi) = \frac{1}{2}\left(\log^2\sin\theta\ - \log^2\tan\frac{\theta}{2}\right). \qquad (8\text{-}29)$$

Therefore equation (5-1) has the particular harmonic solution

$$\phi_p = r^{-\kappa}g(\theta,\psi) + r^{-\kappa}\int^{r}\rho^{\kappa-1}\ f\ d\ \rho$$

$$= \frac{1}{2}\ \left\{\log^2(r\ \sin\theta) - \log^2\ \tan\frac{\theta}{2}\right\}$$

$$= \log\ (2r\ \cos^2\frac{\theta}{2})\ \log(2r\ \sin^2\frac{\theta}{2}) \qquad (8\text{-}30)$$

PART II : CONCLUSION

NON-EXISTENCE OF ϕ (THREE DIMENSIONS)

Our analysis shows that, for any κ, we may always choose a harmonic function f so that the Bergman-Schiffer integral fails to give a harmonic function. The integral contributes to a particular harmonic function ϕ_p which satisfies the Bergman-Schiffer equation. In this case ϕ_p may not exist in some domain B even though f exists in B. For instance,

if $\quad f = r^{-\kappa} P_{\kappa-1}(\cos\theta)$,

$$\text{then } \phi_p = r^{-\kappa} P_{\kappa-1}(\cos\theta) \left\{ \log r + (2\kappa-1) \int^{\theta} \frac{d\lambda}{\sin\lambda [P_{\kappa-1}(\cos\lambda)]^2} \int^{\lambda} [P_{\kappa-1}(\cos\gamma)]^2 \sin\gamma \; d\gamma \right\}.$$

As shown in Chapter 6 and Chapter 8, ϕ_p has a singularity along the z-axis and it therefore does not exist in any domain B which is intersected by the z-axis. On the other hand, for any positive integer value of κ, f exists in any domain B which excludes $r = 0$; also for zero or any negative integer value of κ, f exists in any finite domain which includes $r = 0$. For instance, putting $\kappa = 0$ above we see that $\phi_p = \log(r \sin\theta)$ when $f = 1$, therefore ϕ does not exist in any domain which is intersected by the z-axis.

PART III

TRANSFORMATION OF REPRESENTATIONS

CHAPTER 9

TRANSFORMATION OF A BIHARMONIC FUNCTION FROM r^2-FORM TO z-FORM AND VICE VERSA

As mentioned in (O-7), an arbitrary biharmonic function χ has the representation

$$\chi = z\,h + \Psi \quad ; \quad \nabla^2 h = \nabla^2 \Psi = 0. \tag{9-1}$$

To determine h from χ, we note that

$$\nabla^2 \chi = 2\,\frac{\partial h}{\partial z} \quad ; \quad \nabla^2 h = 0. \tag{9-2}$$

Therefore

$$h = \int^{z} \frac{\nabla^2 \chi}{2}\, d\zeta \quad + \eta(x,y)\,;\ \chi = \chi(x,y,\zeta). \tag{9-3}$$

where we require

$$0 = \nabla^2 h = \nabla^2 \int^{z} \frac{\nabla^2 \chi}{2}\, d\zeta \; + \; \nabla^2\, \eta(x,y) \tag{9-4}$$

i.e. $$\nabla^2 \eta(x,y) = - \nabla^2 \int^{z} \frac{\nabla^2 \chi}{2}\; d\,\zeta\,. \tag{9-5}$$

It may be verified by direct analysis that the right-hand side of (9-5) is independent of z. Equation (9-5) is a Poisson's equation for $\eta(x,y)$ to which a solution always exists, and therefore the harmonic function h always exists. Often it is possible to compute h by direct arguments.

For instance, if

$$\nabla^2\chi = 2r^\nu P_\nu^{\mu}(\cos\theta)\cos\mu\psi, \qquad (9\text{-}6)$$

where ν is a given real number and μ is a positive integer or zero, then

$$h = \frac{1}{\nu+\mu+1} r^{\nu+1} P_{\nu+1}^{\mu}(\cos\theta)\cos\mu\psi \quad ; \quad \nu + \mu \neq -1 \qquad (9\text{-}7)$$

since (Hobson, 1965)

$$\frac{\partial}{\partial z} r^{\nu+1} P_{\nu+1}^{\mu}(\cos\theta)\cos\mu\psi = (\nu+\mu+1)r^\nu P_\nu^{\mu}(\cos\theta)\cos\mu\psi \; ;$$

$$\nu + \mu \neq -1. \qquad (9\text{-}8)$$

Therefore χ always has a representation of the form

$$\chi = zh = \frac{1}{\nu+\mu+1} r^{\nu+2} \cos\theta\, P_{\nu+1}^{\mu}(\cos\theta)\cos\mu\psi \; ; \quad \nu + \mu \neq -1 \qquad (9\text{-}9)$$

to which we must generally add a harmonic function.

Since μ is an integer, a failing case of (9-7) arises if ν is an integer. In this case, putting $\nu = -\mu - 1$ in (9-6), we must make use of (9-3), i.e.

$$h = \int^{z} \rho^{-(\mu+1)} P_{\mu}{}^{\mu}(\cos\theta)\cos \mu\psi \, d\zeta + \eta(x,y). \tag{9-10}$$

Notice that the indefinite integral on the right-hand side (9-10) can be computed by reference to the polynomial expansions of $P_{\mu}{}^{\mu}(\cos\theta)\cos\mu\psi$ in terms of $\cos\theta$, $\sin\theta$ and $\cos\psi$. Accordingly χ has the representation

$$\chi = zh = z\int^{z} \rho^{-(\mu+1)} P_{\mu}{}^{\mu}(\cos\theta)\cos \mu\psi d\zeta + z\eta(x,y), \tag{9-11}$$

in this case, to which we may add an arbitrary harmonic function. For instance, if

$$\nabla^2\chi = 2r^{-2} P_1{}^{1}(\cos\theta) \; ; \quad \mu = 1 \tag{9-12}$$

we find from (9-10) that

$$h = \frac{xz}{r(x^2+y^2)} + \eta(x,y) \; , \quad \eta(x,y) = 0, \tag{9-13}$$

where $\eta(x,y) = 0$ since $\nabla^2 \frac{xz}{r(x^2+y^2)} = 0$. Accordingly, in

this case

$$\chi = zh = \frac{x\, z^2}{r(x^2+y^2)} \tag{9-14}$$

In the special failing case $\mu = 0$ of (9-17), i.e.

$$\nabla^2\chi = 2r^{-1} P_o(\cos\theta) = 2r^{-1}, \tag{9-15}$$

we obtain from (9-10) that

$$h = \int^z \rho^{-1} d\rho + \eta(x,y) = \log(z+r) + \eta(x,y) \tag{9-16}$$

where $\eta(x,y) = 0$ since $\nabla^2 \log(z+r) = 0$. Accordingly

$$\chi = zh = z \log\,(z+r) \tag{9-17}$$

No additional analysis is required to cover negative integer values μ, in view of the relation

$$P_\nu^{\ \mu}(\cos\theta) = (-1)^\mu \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{\ -\mu}(\cos\theta). \tag{9-18}$$

For instance, if

$$\nabla^2\chi = 2r^{\mu-1} \; P_{-\mu}{}^{\mu}(\cos\theta) \cos \mu\psi; \quad \mu < 0, \tag{9-19}$$

we may write

$$\nabla^2\chi = (-1)^{\mu} \; \frac{2}{(-2\mu)!} \, r^{\mu-1} \quad P_{-\mu}^{-\mu}(\cos\theta)\cos \mu\psi \; ; \; -\mu > 0 \tag{9-20}$$

Now χ also has a representation of the form

$$\chi = r^2\phi + \psi \; ; \; \nabla^2\phi = \nabla^2\psi = 0, \tag{9-21}$$

where, given (9-6), ϕ may be determined from the Bergman-Schiffer equation (5-1) with

$\kappa = \frac{3}{2}$ and $f = \frac{\nabla^2\chi}{2}$. By virtue of (5-14)

$$\phi_p = \frac{r^{\nu}}{2\,\nu + 3} \; P_{\nu}{}^{\mu}(\cos\theta)\cos\mu\psi \; ; \quad \nu \neq -\frac{3}{2}, \quad \nu \geqslant 0 \tag{9-22}$$

from which (apart from a harmonic function)

$$\chi = r^2\phi = \frac{r^{\nu+2}}{2\nu+3} P_\nu^{\ \mu}(\cos\theta) \cos \mu\psi \ ; \qquad \nu \neq -\frac{3}{2} . \tag{9-23}$$

The difference between the forms of χ in (9-9) and (9-23) is the harmonic function

$$\Psi = \frac{r^{\nu+2}}{(\nu+\mu+1)(2\nu+3)} P_{\nu+2}^{\ \mu}(\cos\theta)\cos\mu\psi \ ; \ \nu + \mu \neq -1, \ \nu \neq -\frac{3}{2} , \tag{9-24}$$

which does not apply in the failing cases $\nu + \mu = -1$, $\nu = -\frac{3}{2}$ of (9-9), (9-22) respectively.

The choice

$$\nabla^2\chi = 2r^{-\mu-1} P_\mu^{\ \mu}(\cos\theta) \cos\mu\psi , \tag{9-25}$$

corresponding to the failing case $\nu + \mu = -1$ of (9-7), yields

$$\phi_p = \frac{r^{-\mu-1}}{1 - 2\mu} P_\mu^{\ \mu}(\cos \ )\cos \mu\psi , \tag{9-26}$$

by an application of (9-22). If so, χ has the representation

$$\chi = r^2\phi = \frac{r^{-\mu+1}}{1 - 2\mu} P_\nu^{\ \mu}(\cos\theta)\cos \mu\psi. \tag{9-27}$$

The difference between the forms of χ in (9-27) and (9-11) must necessarily be harmonic function. Clearly a failing case of (9-9) is more conveniently represented by (9-27) than by (9-11).

In the special failing cases $\nu = -1$, $\nu = -2$, corresponding to (9-15), (9-12) respectively, we have

$$\phi_p = \frac{r^{-1}}{-2+3} P_0(\cos\theta) = r^{-1}, -r^{-2}\sin\theta \cos\psi \qquad (9\text{-}28)$$

$$\chi = r^2 r^{-1} = r, -\sin\theta \cos\psi \qquad (9\text{-}29)$$

As a corollary, the difference between the forms of χ in (9-17), (9-14) and (9-29), are the harmonic functions.

$$\Psi = z \log (z + r) - r, \cos\psi(\sin\theta)^{-1} \text{ respectively.} \qquad (9\text{-}30)$$

The choice

$$\nabla^2\chi = 2r^{-\frac{3}{2}} P_{-\frac{1}{2}}{}^{\mu}(\cos\theta) \qquad (9\text{-}31)$$

corresponding to the failing case $\nu = -\frac{3}{2}$ of (9-22), yields

$$h = \frac{2r^{-\frac{1}{2}}}{2\mu - 1} P_{-\frac{1}{2}}(\cos\theta)\cos \mu\psi, \qquad (9\text{-}32)$$

by an application of (9-7). If so, χ has the representation

$$\chi = zh = \frac{2r^{\frac{1}{2}}}{2\mu - 1} \cos\theta \, P_{-\frac{1}{2}}^{\mu}(\cos\theta)\cos \mu\psi. \tag{9-33}$$

The difference between the forms of χ is (9-33) and (9-21) must necessarily be harmonic function. Clearly a failing case of (9-23) is more conviently represented by (9-33) that by (9-21), as can be seen by putting $\kappa = \frac{3}{2}$ in (8-11).

CHAPTER 10

TRANSFORMATION OF A BIHARMONIC FUNCTION FROM r^2-FORM TO x & y-FORMS AND VICE VERSA

As mentioned in (0-7), an arbitrary harmonic function has the representation

$$\chi = xh + \Psi \quad ; \quad \nabla^2 h = \nabla^2 \Psi = 0, \tag{10-1}$$

from which

$$\nabla^2 \chi = 2\frac{\partial h}{\partial x} \quad ; \quad \nabla^2 h = 0, \tag{10-2}$$

$$h = \int^x \frac{\nabla^2 \chi}{2} \, d\,\xi + \eta(y,z). \quad \chi = \chi\,(\xi,y,z) \tag{10-3}$$

We require

$$0 = \nabla^2 h = \nabla^2 \int^x \frac{\nabla^2 \chi}{2} \, d\,\xi + \nabla^2 \eta(y,z), \tag{10-4}$$

$$\text{i.e.} \quad \nabla^2 \eta(y,z) = -\nabla^2 \int^x \frac{\nabla^2 \chi}{2} \, d\,\xi\,, \tag{10-5}$$

and it may be verified by direct analysis that the right-hand side (10-5) is independent of x. Equation (10-5) is a Poisson's equation for $\eta(y,z)$, to which a solution always exists, and therefore the harmonic function h always exists. Often it is possible to compute h by direct arguments. Thus it can be shown that

$$(\nu+\mu+1)(\nu+\mu+2)r^{\nu}P_{\nu}^{\;\mu}(\cos\theta)\cos\mu\psi = 2\,\frac{\partial}{\partial x}\,r^{\nu+1}P_{\nu+1}^{\mu+1}\;(\cos\theta)\cos(\mu+1)\psi$$

$$+\; r^{\nu}P_{\nu}^{\;\mu+2}(\cos\theta)\cos(\mu+2)\psi \tag{10-6}$$

where ν is a given real number and μ is a positive integer or zero. This result is essentially in Bateman (1932) for integer values of ν. We re-write relation (10-6) as

$$(\nu+\mu+1)(\nu+\mu+2)r^{\nu}\;P_{\nu}^{\;\mu}(\cos\theta)\cos\mu\psi = 2\,\frac{\partial}{\partial x}\,[r^{\nu+1}P_{\nu+1}^{\mu+1}(\cos\theta)\cos(\mu+1)\psi$$

$$+\;\tfrac{1}{2}\int^{x}\rho^{\nu}P_{\nu}^{\mu+2}(\cos\theta)\cos(\mu+2)\psi d\xi], \tag{10-7}$$

Accordingly, if for instance

$$\nabla^2\chi = 2r^{\nu}\;P_{\nu}^{\;\mu}(\cos\theta)\cos\;\mu\psi, \tag{10-8}$$

then

$$h = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\mu+3)} \quad [2r^{\nu+1} \; P_{\nu+1}^{\mu+1}(\cos\theta)\cos(\mu+1)\psi + \int_{\rho}^{x} \rho^{\nu} \; P_{\nu}^{\;\mu+2}(\cos\theta)\cos(\mu+2)\,\psi d\xi] ;$$

$$\nu \neq - \mu - 1, \; - \mu - 2 \qquad (10\text{-}9)$$

where $\rho^2 = \xi^2 + y^2 + z^2$, $r^2 = x^2 + y^2 + z^2$, $\Gamma(\nu+1) = \nu\Gamma(\nu)$.

By successive applications of (10-9), we obtain the harmonic function

$$h = 2r^{\nu+1} \sum_{j=0}^{\infty} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\mu+2j+3)} \; P_{\nu+1}^{\mu+2j+1}(\cos\theta) \; \cos(\mu+2j+1)\psi \quad ;$$

$$\nu \neq - \mu - n \; ; \quad n = 1,2,... \qquad (10\text{-}10)$$

which is a finite summation for integer values of ν.

For $\mu = 0$, $\nu = 1$, i.e. $\nabla^2\chi = 2rP_1(\cos\theta)$, formula (10-10) gives $h = \frac{r^2}{3} P_2^{\;1}(\cos\theta)\cos\psi = zx$

as expected. Also for $\mu = 1$, $\nu = 1$, i.e. $\nabla^2\chi = 2r \; P_1^{\;1}(\cos\theta) \; \cos\psi$, formula (10-10) gives

$$h = \frac{1}{3!} \; r^2 P_2^{\;2}(\cos\theta)\cos 2\psi \; = \frac{1}{2} \; (x^2 - y^2)$$

as expected. A similar analysis holds for

$$\nabla^2\chi = 2r^{\nu} \; P_{\nu}^{\;\mu}(\cos\theta) \; \sin \mu\psi \qquad (10\text{-}11)$$

and we obtain

$$h = 2r^{\nu+1} \sum_{j=0}^{\infty} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\mu+2j+1)} P_{\nu+1}^{\mu+2j+1}(\cos\theta)\sin(\mu+2j+1)\psi \ ;$$

$$\nu \neq -\mu - n \ ; \quad n = 1,2,\ldots \ . \qquad (10\text{-}12)$$

For $\mu = 1$, $\nu = 1$, i.e. $\nabla^2\chi = 2rP_1{}^1(\cos\theta)\sin\psi$, formula (10-12) gives

$$h = \frac{1}{3!} r^2 P_2{}^2(\cos\theta)\sin 2\psi \ = yx$$

as expected.

Since μ is an integer, a failing case of (10-10) arises if ν is a negative integer. In this case, putting $\nu = -\mu - n$; $n = 1,2,\ldots$ in (10-8), we must make use of (10-3), i.e.

$$h = \int^x \rho^{-(\mu+n)} P_{\mu+n-1}^{\mu}(\cos\theta)\cos\mu\psi \, d\xi + \eta(y,z). \qquad (10\text{-}13)$$

Notice that the indefinite integral on the right-hand side of (10-13) can be computed by reference to the polynomial expansion of $P^{\mu}_{\mu+n-1}(\cos\theta)\cos\mu\psi$ in terms of $\cos\theta$, $\sin\theta$ and $\cos\psi$. For instance, if

$$\nabla^2\chi = 2r^{-2}P_1(\cos\theta) \quad ; \qquad \mu = 0, \ n = 2 \qquad (10\text{-}14)$$

we find from (10-13) that

$$h = \int^{x} \rho^{-2} P_1(\cos\theta)\,d\xi \;+ \eta(y,z) = \frac{zx}{r(y^2+z^2)} + \eta(y,z), \qquad (10\text{-}15)$$

where $\eta(y,z) = 0$ since $\nabla^2 \quad \dfrac{zx}{r(y^2+z^2)} = 0.$

In the special failing case $\mu = 0$, $n = 1$, i.e. $\nabla^2\chi = 2r^{-1}$ of formula (10-10), we obtain the harmonic function

$$h = \log(x+r) \qquad (10\text{-}16)$$

by analogy with (9-16).

A simplification occurs for

$$\nabla^2\chi \;= 2r^{-n-1}P_n{}^1(\cos\theta)\cos\psi \;;\;\; \mu = 1,\, \nu = -\,n\,-1 \qquad (10\text{-}17)$$

which also forms a failing case of (10-10), where n = 1,2,... .
This is because (10-6) now takes the form

$$r^{-n-1}P_n{}^1(\cos\theta)\cos\psi \;\; = -\frac{\partial}{\partial x}\, r^{-n}P_{n-1}(\cos\theta) \qquad (10\text{-}18)$$

as can be readily verified, from which we obtain

$$h = -r^{-n}\, P_{n-1}(\cos\theta). \qquad (10\text{-}19)$$

In particular, for n = 1, i.e. $\nabla^2\chi = 2r^{-2}P_1^{\,1}(\cos\theta)\cos\psi$, formula (10-19) gives

$$h = - r^{-1} \tag{10-20}$$

as expected.

No additional analysis is required to cover negative integer value of μ in view of the relation

$$P_\nu^{\,m}(\cos\theta) = (-1)^m \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} P_\nu^{\,-m}(\cos\theta). \tag{10-21}$$

For instance, if

$$\nabla^2\chi = 2r^{-n-1} P_n^{\,-1}(\cos\theta)\cos\psi \,, \tag{10-22}$$

we obtain from (10-19) and (10-21) that

$$h = \frac{r^{-n}}{n(n+1)} P_{n-1}(\cos\theta). \tag{10-23}$$

There is no difficulty in adapting the preceding analysis to the problem of determining h from

$$\nabla^2\chi = 2\,\frac{\partial h}{\partial y}\,; \quad \nabla^2 h = 0. \tag{10-24}$$

Details are provided in App. III.

APPENDIX I : REDUCTION OF AN INTEGRAL TO A POLYNOMIAL FOR INTEGER VALUES OF κ

The following functions are essentially equal:

$$G(\theta) = (2\kappa-1)P_{\kappa-1}(\cos\theta)\int^{\theta} \frac{d\ \lambda}{\sin\lambda[P_{\kappa-1}(\cos\lambda)]^2} \int^{\lambda} [P_{\kappa-1}(\cos\gamma)]^2 \sin\gamma\ d\ \gamma;$$

$$\kappa = 1,2,3,\ldots. \qquad \text{(A1-1)}$$

$$g(\theta) = -P_{\kappa-1}(\cos\theta)\log\sin\theta\ ; \quad \kappa = 1,2$$

$$g(\theta) = -P_{\kappa-1}(\cos\theta)\log\sin\theta + \sum_{r=1}^{p} a_r \cos^{\kappa-2r}\theta \quad ; \quad \kappa \geq 3, \qquad \text{(A1-2)}$$

where κ is a positive integer and the coefficients a_r satisfy the recurrence formula (6-12).

By mathematical induction, we have

$$(2\kappa-1)\int^{u} [P_{\kappa-1}(\xi)]^2 d\xi = u(P_{\kappa-1})^2 + 2u\left\{(P_1)^2+(P_2)^2 + \ldots+(P_{\kappa-2})^2\right\}$$

$$-2(P_1P_2+P_2P_3+\ldots+P_{\kappa-2}P_{\kappa-1}), \qquad \text{(A1-3)}$$

Since (Hobson 1965)

$$(2\kappa+1)\int^{u} [P_{\kappa}(\xi)]^2 d\xi - (2\kappa-1)\int^{u}[P_{\kappa-1}(\xi)]^2 d\xi = u\left\{(P_{\kappa})^2+(P_{\kappa-1})^2\right\}$$

$$- 2\ P_{\kappa}\ P_{\kappa-1}\ . \qquad \text{(A1-4)}$$

Therefore the expression on the right-hand side of (Al-1) becomes

$$-P_{\kappa-1}(\cos\theta)\log\sin\theta + 2P_{\kappa-1}(\cos\theta)\int^{u}\frac{\zeta\left\{(P_1)^2+\ldots+(P_{\kappa-2})^2\right\} - (P_1P_2+\ldots+P_{\kappa-2}P_{\kappa-1})}{(1-\zeta^2)\ (P_{\kappa-1})^2}d\xi;$$

$$u = \cos\theta\ ,\quad \zeta = \cos\lambda \qquad \text{(Al-5)}$$

If we write

$$s(\theta) = G(\theta) + P_{\kappa-1}(\cos\theta)\log\sin\theta\ , \qquad \text{(Al-6)}$$

then $s(\theta)$ is a particular solution of equation (6-10), i.e. of the equation

$$\frac{1}{\sin\theta}\ \frac{\partial}{\partial\theta}\ (\sin\theta\ \frac{\partial S}{\partial\theta}\ ) + \kappa(\kappa-1)S = \frac{2}{\sin\theta}\frac{\partial}{\partial\theta}\ P_{\kappa-2}(\cos\theta) \qquad \text{(Al-7)}$$

Since, substituting (Al-6) into left-hand side (Al-7) and using (6-5) with $\mu = 0$, we find

$$2[(\kappa-1)P_{\kappa-1}(u) - u\frac{\partial P_{\kappa-1}(u)}{\partial u}],\ \text{i.e.}-\frac{\partial P_{\kappa-1}(u)}{\partial u}\quad \text{i.e.}\ \frac{2}{\sin\theta}\ \frac{\partial P_{\kappa-1}}{\partial\theta}\ (\cos\theta).$$

Accordingly

$$s(\theta) = AQ_{\kappa-1}(\cos\theta) + B\ P_{\kappa-1}(\cos\theta) + S(\theta), \qquad \text{(Al-8)}$$

where $S(\theta)$ is a particular solution of (Al-7), defined by (6-11), and the coefficients A,B are constants to be determined. Our argument now proceeds as follows. Substituting for $G(\theta)$ in (Al-6), we note that

$$s(\theta) = 2P_{\kappa-1}(\cos\theta)\int^{u}\frac{\zeta\left\{(P_1)^2+\ldots+(P_{\kappa-2})^2\right\} - (P_1P_2+\ldots+P_{\kappa-2}P_{\kappa-1})}{(1-\zeta^2)\ (P_{\kappa-1})^2}d\xi \qquad \text{(Al-9)}$$

where (Whittaker and Watson 1973)

$$\frac{u\left\{(P_1)^2+\ldots+(P_{\kappa-2})^2 - (P_1P_2+\ldots+P_{\kappa-2}P_{\kappa-1})\right\}}{(1-u^2)\ (P_{\kappa-1})^2}$$

$$= \frac{\tfrac{1}{2}\, P_1P_1 \ + \ldots + \ {}^{1}/_{\kappa-1}\ P_{\kappa-2}\ P_{\kappa-2}}{(P_{\kappa-1})^2} , \qquad \text{(A1-10)}$$

which shows that there are no singularities in $s(\theta)$ at $\theta = 0,\pi$. Therefore $A = 0$ and equation (A1-8) becomes

$$s(\theta) = B\ P_{\kappa-1}(\cos\theta) + S(\theta). \qquad \text{(A1-11)}$$

Suppose $\sum_{j=1}^{\kappa-1} \left[\frac{a_g}{\zeta-u_j} + \frac{b_y}{(\zeta-u_j)^2} \right]$ is the partial fraction corresponding the integrand on the right-hand side (A1-9) where u_j ; $j= 1,\ldots, \kappa- 1$ are the zeros of $P_{\kappa-1}(u)$, then

$$s(\theta) = 2\ P_{\kappa-1}(\cos\theta) \sum_{j=1}^{\kappa-1} \left[a_j \log(u-u_j) - \frac{b_j}{(u-u_j)} \right]\Bigg\} . \qquad \text{(A1-12)}$$

The coefficients a_j must vanish, since these cannot be singularity at u_j by virtue of (A1-11) and (6-11). Accordingly we may write

$$s(\theta) = -2P_{\kappa-1}(\cos\theta) \sum_{j=1}^{\kappa-1} \frac{b_y}{u-u_j} , \qquad \text{(A1-13)}$$

which is a polynomial of degree $(\kappa-2)$. However by (A1-11) $s(\theta)$

is of degree $(\kappa-1)$ which implies $B = 0$. Accordingly $s(\theta) = S(\theta)$, i.e. $g(\theta) = G(\theta)$. Therefore the solution (8-11) reduces to the solutions (6-18), (6-19) with $a = b = 0$. As a corollary we can assert that (A1-2) is a primitive integral of (A1-1).

The above analysis can be used for negative integer or zero value of κ, by using the equality $P_{\kappa-1} = P_{-\kappa}$ to show that the solution (8-12) reduces to the solutions (6-29), (6-30) with $a = b = 0$.

APPENDIX II : SINGULARITIES OF AN INTEGRAL

The equation

$$\sin\theta \frac{\partial}{\partial\theta}(\sin\theta \frac{\partial S}{\partial\theta}) + [\kappa(\kappa-1)\sin^2\theta - \mu^2] S$$

$$= 2(\kappa-1)P_{\kappa-1}^{\mu}(\cos\theta)\sin^2\theta + \sin 2\theta \frac{\partial}{\partial\theta} P_{\kappa-1}^{\mu}(\cos\theta), \qquad \text{(A2-1)}$$

where κ is a positive integer and μ is a positive integer or zero, has a solution with a singularity of the form $O(\sin\theta)^{-\mu}$ as $\theta \to 0$; $\mu = 0,1,\ldots$. To find a solution of equation (A2-1), we put

$$S(\theta) = s(\theta) P_{\kappa-1}^{\mu}(\cos\theta) \qquad \text{(A2-2)}$$

where $s(\theta)$ is a function of θ to be determined. Substituting from (A2-2) into (A2-1), we find

$$P_{\kappa-1}^{\mu}(\cos\theta) \frac{\partial^2 s}{\partial\theta^2} + [2 \frac{\partial}{\partial\theta} P_{\kappa-1}^{\mu}(\cos\theta)+\cot\theta\, P_{\kappa-1}^{\mu}(\cos\theta)] \frac{\partial s}{\partial\theta} =$$

$$2(\kappa-1)P_{\kappa-1}^{\mu}(\cos\theta) + 2 \cot\theta \frac{\partial}{\partial\theta} P_{\kappa-1}^{\mu}(\cos\theta), \qquad \text{(A2-3)}$$

which can be written in the form

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} [\sin\theta \frac{\partial s}{\partial\theta} P_{\kappa-1}^{\mu}(\cos\theta)] + \frac{\partial s}{\partial\theta} \frac{\partial}{\partial\theta} P_{\kappa-1}^{\mu}(\cos\theta) =$$

$$2(\kappa-1)P_{\kappa-1}^{\mu}(\cos\theta) + 2 \cot\theta \frac{\partial}{\partial\theta} P_{\kappa-1}^{\mu}(\cos\theta). \qquad \text{(A2-4)}$$

multiplying both sides of equation (A2-4) by $\sin\theta P_{\kappa-1}^{\mu}(\cos\theta)$, and putting

$$U = \sin\theta \frac{\partial s}{\partial\theta} P_{\kappa-1}^{\mu}(\cos\theta), \tag{A2-5}$$

gives

$$U P_{\kappa-1}^{\mu}(\cos\theta) =$$

$$\int^{\theta} [2(\kappa-1)P_{\kappa-1}^{\mu}(\cos\theta)+2\cot\theta\frac{\partial}{\partial\theta} P_{\kappa-1}^{\mu}(\cos\theta)] \sin\theta P_{\kappa-1}^{\mu}(\cos\theta). \tag{A2-6}$$

Substituting from (A2-6) into (A2-5) yields

$$s(\theta) = \int^{\theta} \frac{d\lambda}{[P_{\kappa-1}^{\mu}(\cos\lambda)]^2\sin\lambda} \int^{\lambda}[2(\kappa-1)P_{\kappa-1}^{\mu}(\cos\gamma)+2\cot\gamma\frac{\partial}{\partial\gamma} P_{\kappa-1}^{\mu}(\cos\gamma)] \times$$

$$\times \sin\gamma P_{\kappa-1}^{\mu}(\cos\gamma)d\gamma \ . \tag{A2-7}$$

Accordingly equation (A2-1) has the solution

$$S(\theta) = P_{\kappa-1}^{\mu}(\cos\theta)\int^{\theta}\frac{d\lambda}{[P_{\kappa-1}^{\mu}(\cos\theta)]^2\sin\lambda}\int^{\lambda}[2(\kappa-1)P_{\kappa-1}^{\mu}(\cos\gamma)+2\cot\gamma\frac{d}{d\gamma} P_{\kappa-1}^{\mu}(\cos\gamma)] \times$$

$$\times \sin\gamma\, P_{\kappa-1}^{\mu}(\cos\gamma)d\gamma, \tag{A2-8}$$

with a singularity of the form

$O(\sin\theta)^{-\mu}$ as $\theta \to 0$; $\mu = 0,1,2,\ldots$. Thus, from the definition of $P_{\kappa-1}^{\mu}(\cos\theta)$, in ~~(8-20)~~ (8-20), the second integrand on the right-hand side (A2-8) includes a term of the form $\sin^{2\mu-1}\gamma$; $\mu = 1,2,\ldots$. Therefore (A2-8) includes a term of the form

$$P_{\kappa-1}^{\mu}(\cos\theta) \int^{\lambda} \frac{\cos\lambda \; d\lambda}{[P_{\kappa-1}^{\mu}(\cos\theta)]^2 \sin\lambda} \; ; \quad \mu = 1,2,\ldots .$$

As mentioned in Chapter 8, this has a singularity of $O\,(\sin\theta)^{-\mu}$ as $\theta \to 0$; $\mu = 1,2,\ldots$. In the case $\mu = 0$ the solution (A2-7) identically equals the function (A1-1), which essentially equals (A1-2). Accordingly $s(\theta)$ is a finite polynomial in this case.

APPENDIX III : TRANSFORMATION OF A BIHARMONIC FUNCTION FROM r^2-FORM TO y-FORM AND VICE VERSA

As mentioned in (O-7), an arbitrary harmonic function χ has the representation

$$\chi = yh + \Psi \ ; \quad \nabla^2 h = \nabla^2 \Psi = 0, \tag{A3-1}$$

from which

$$\nabla^2 \chi = 2 \ \frac{\partial h}{\partial y} \ ; \ \nabla^2 h = 0, \tag{A3-2}$$

$$h = \int^y \frac{\nabla^2 \chi}{2} \, d\beta \ + \eta(x,z); \ \ \chi = \chi(x,\beta,z). \tag{A3-3}$$

We require

$$0 = \nabla^2 h = \nabla^2 \int^y \frac{\nabla^2 \chi}{2} \, d\beta \ + \eta(x,z), \tag{A3-4}$$

$$\text{i.e. } \nabla^2 \eta(x,z) = - \nabla^2 \int^y \frac{\nabla^2 \chi}{2} \, d\beta, \tag{A3-5}$$

and it may be verified by direct analysis that the right-hand side of (A3-5) is independent of y. Equation (A3-5) is a Poisson's equation for $\eta(y,z)$, to which a solution always exists, and therefore the harmonic function h always exists. Often it is possible to compute h by direct arguments. Thus it can be shown that

$$(\nu+\mu+1)(\nu+\mu+2)r^{\nu}P_{\nu}^{\ \mu}(\cos\theta)\sin\mu\psi = -2\,\frac{\partial}{\partial y}r^{\nu+1}P_{\nu+1}^{\mu+1}(\cos\theta)\cos(\mu+1)\psi$$

$$- r^{\nu}P_{\nu}^{\mu+2}(\cos\theta)\sin(\mu+2)\phi \qquad \text{(A3-6)}$$

where ν is a given real number and μ is a positive integer. We re-write (A3-6) as

$$(\nu+\mu+1)(\nu+\mu+2)r^{\nu}P_{\nu}^{\ \mu}(\cos\theta)\sin\mu\psi = -2\,\frac{\partial}{\partial y}\,[r^{\nu+1}P_{\nu+1}^{\ \mu+1}(\cos\theta)\cos(\mu+1)\psi$$

$$+ \tfrac{1}{2}\int^{y}\rho^{\nu}P_{\nu}^{\ \mu+2}(\cos\theta)\sin(\mu+2)\psi\,d\beta]. \qquad \text{(A3-7)}$$

Accordingly, if for instance

$$\nabla^2\chi = 2r^{\nu}\,P_{\nu}^{\ \mu}(\cos\theta)\sin\mu\psi\,, \qquad \text{(A3-8)}$$

then

$$h = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\mu+3)}\,[-2r^{\nu+1}P_{\nu+1}^{\mu+1}(\cos\theta)\cos(\mu+1)\psi - \int^{y}\rho^{\nu}P^{\mu+2}(\cos\theta)\sin(\mu+2)\psi d\beta];$$

$$\nu \neq -\mu-1 \quad -\mu-2. \qquad \text{(A3-9)}$$

By successive applications of (A3-9), we obtain the harmonic function

$$h = 2r^{\nu+1} \sum_{j=0}^{\infty} (-1)^{j+1} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\mu+2j+3)} P_{\nu+1}^{\mu+2j+1}(\cos\theta)\cos(\mu+2j+1)\psi \quad ;$$

$$\nu \neq -\mu - n \; ; \quad n = 1,2,\ldots \qquad \text{(A3-10)}$$

which is a finite summation for integer values of ν.

For $\mu = 1$, $\nu = 1$, i.e. $\nabla^2\chi = 2r\, P_1^1(\cos\theta)\sin\psi$, formula (A3-9) gives

$$h = -\frac{1}{3!}\, r^2 P_2{}^2(\cos\theta)\cos 2\psi = \frac{1}{2}\,(y^2 - x^2)$$

as expected. A similar analysis holds for

$$\nabla^2\chi = 2r^{\nu} P_{\nu}{}^{\mu}(\cos\theta)\cos\mu\psi \qquad \text{(A3-11)}$$

and we obtain

$$h = 2r^{\nu+1} \sum_{j=0}^{\infty} (-1)^{j} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\mu+2j+3)} P_{\nu+1}^{\mu+2j+1}(\cos\theta)\sin(\mu+2j+1)\psi \quad ;$$

$$\nu \neq -\mu - n \; ; \quad n = 1,2, \ldots \; . \qquad \text{(A3-12)}$$

Since μ is an integer, a failing case of (A3-10) arises if ν is a negative integer. In this case, putting $\nu = -\mu - n$, $n = 1,2,\ldots$ in (A3-8), we must make use of (A3-3), i.e.

$$h = \int^{y} r^{-(\mu+n)} P_{\mu+n-1}{}^{\mu}(\cos\theta)\sin\mu\psi\, d\beta + \eta(x,z). \qquad \text{(A3-13)}$$

In the special failing case $\mu = 0$, $n = 1$, i.e. $\nabla^2\chi = 2r^{-1}$ of formula (A3-11), we obtain the harmonic function

$$h = \log(y+r). \tag{A3-14}$$

by analogy with (9-16).

A simplification occurs for

$$\nabla^2\chi = 2r^{-n-1}\, P_n^{\ 1}(\cos\theta)\, \sin\psi \ ; \quad \mu = 1, \quad \nu = -n-1 \tag{A3-15}$$

which also forms a failing case of (A3-10), where $n = 1,2,\ldots$.
This is because (A3-6) now takes the form

$$r^{-n-1}P_n^{\ 1}(\cos\theta)\sin\psi = -\frac{\partial}{\partial y}P_{n-1}(\cos\theta) \tag{A3-16}$$

as can be verified, fron which we obtain

$$h = -r^{-n}P_{n-1}(\cos\theta).$$

APPENDIX IV: PAPKOVICH-NEUBER REPRESENTATION

Papkovich (1932) and Neuber (1934) proved that an arbitrary elastostatic displacement vector $\underset{\sim}{\phi}$ always has the representation

$$\underset{\sim}{\phi}(\underset{\sim}{p}) = \underset{\sim}{h} - k\nabla(\underset{\sim}{p}.\underset{\sim}{h}+f);\ \nabla^2\underset{\sim}{h}(\underset{\sim}{p}) = 0,\ \nabla^2 f(\underset{\sim}{p}) = 0, \tag{A4-1}$$

where $\underset{\sim}{h}$ is a harmonic vector and f is a harmonic scalar. Also

$$k^{-1} = 4(1-\nu);\quad 0 < \nu \leq \frac{1}{2},\ \frac{1}{4} < k \leq \frac{1}{2}\ . \tag{A4-2}$$

This vector automatically satisfies the Cauchy-Navier equation

$$\mu\nabla^2\underset{\sim}{\phi} + (\lambda+\mu)\nabla(\nabla.\underset{\sim}{\phi}) = 0, \tag{A4-3}$$

$$\text{i.e.}\quad \nabla^2\underset{\sim}{\phi} + \frac{1}{(1-2\nu)}\nabla(\nabla.\underset{\sim}{\phi}) = 0\ ;\qquad \nu = \frac{\lambda}{2(\lambda+\mu)}, \tag{A4-4}$$

where μ, λ are Lamé's elastic constants. Eubanks and Sternberg (1956) have examined whether the scalar harmonic function f in (A4-1) is necessary. An alternative investigation has been given by Jaswon and Symm (1977), who suggested writing

$$-k\nabla f = \underset{\sim}{h} - k\nabla(\underset{\sim}{p}.\underset{\sim}{h});\ \nabla^2\underset{\sim}{h} = 0,\ \nabla^2 f = 0 \tag{A4-5}$$

where $\underset{\sim}{h}$ must be determined in terms of f. If $\underset{\sim}{h}$ exists, then clearly it may be written in the form $\underset{\sim}{h} = \nabla S$ where S is a scalar harmonic function. Substituting for $\underset{\sim}{h}$ in (A4-5) we find

$$-k\nabla f = \nabla S - k\nabla(\underset{\sim}{p}.\nabla S) = \nabla(S-k\underset{\sim}{p}.\nabla S), \qquad \text{(A4-6)}$$

where the expressions in brackets are harmonic functions. Accordingly S satisfies the equation

$$S - k\,\underset{\sim}{p}\,.\nabla S = -kf, \qquad \text{(A4-7)}$$

which becomes

$$\kappa S + r\frac{\partial S}{\partial r} = f \qquad \text{(A4-8)}$$

using the equality $\underset{\sim}{p}.\ \nabla S = r\frac{\partial S}{\partial r}$ and writing

$$\kappa = -k^{-1}\ ; \qquad -4<\kappa \leq -2. \qquad \text{(A4-9)}$$

This is the Bergman-Schiffer equation (5-1). According to Part II: Conclusion, we see that S, and therefore $\underset{\sim}{h} = \nabla S$, does not exist in any finite domain intersected by the z-axis if

$$f = r^{\mu}P_{\mu}(\cos\theta); \qquad 2 \leq \mu < 4. \qquad \text{(A4-10)}$$

This means that the scalar harmonic function (A4-10) cannot be eliminated from the representation (A4-1).

APPENDIX V: EXISTENCE OF A HARMONIC SOLUTION TO BERGMAN-SCHIFFER EQUATION

Operating by ∇^2 upon both sides of the Bergman-Schiffer equation

$$\kappa\phi + r\frac{\partial\phi}{\partial r} = f, \tag{A5-1}$$

i.e. $$\kappa\phi + (x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + z\frac{\partial\phi}{\partial z}) = f, \tag{A5-2}$$

we obtain

$$\kappa\nabla^2\phi + (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})\nabla^2\phi + 2(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}) = 0. \tag{A5-3}$$

Using $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} = r\frac{\partial}{\partial r}$, this gives

$$(\kappa+2)\nabla^2\phi + r\frac{\partial}{\partial r}(\nabla^2\phi) = 0, \tag{A5-4}$$

which has the family of solutions

$$\nabla^2\phi = Dr^{-(\kappa+2)} \tag{A5-5}$$

where D is an arbitrary function of θ,ψ. The choice D = 0 in (A5-5) implies the existence of at least one solution of the Bergman-Schiffer equation which is a harmonic function. If ϕ stands for Bergman-Schiffer integral introduced in Chapters 2,5 then we may identify D = h.

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