The blob algebra in positive characteristic

Anton Cox [a,1], John Graham [b] and Paul Martin [a]

[a] *Mathematics Department, City University, Northampton Square, London, EC1V 0HB, England*

[b] *School of Mathematics and Statistics, University of Birmingham, Edgbaston, Birmingham, B15 2TT, England*

1 Introduction

The blob algebra was originally introduced in [1] in the context of statistical mechanics, as a generalisation of the ordinary Temperley-Lieb algebra. In [2] the blocks of these algebras were determined, together with the structure of the standard modules (in the quasi-hereditary sense), over any field of characteristic zero. In this paper we shall determine corresponding results in positive characteristic.

The blob algebra can be defined as a quotient of the type B Hecke algebra $\mathcal{H}_B(n)$, much as the ordinary Temperley-Lieb algebra $\mathrm{TL}_A(n)$ can be identified with a quotient of the type A Hecke algebra $\mathcal{H}_A(n)$. For this reason we shall consider the blob algebra to be the type B analogue of $\mathrm{TL}_A(n)$, and denote it by $\mathrm{TL}_B(n)$. This algebra is closely related to the algebra TB_n defined by tom Dieck [3].

As $\mathcal{H}_B(n)$ is itself a quotient of the extended affine Hecke algebra of type A, the representations which we construct will also be representations of this algebra. The blob algebra was introduced as an algebra which, though of finite rank, underlies the finite dimensional representation theory of (infinite rank)

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Temperley-Lieb algebras on the cylinder or annulus [4] (confer [5] and [6]). In [7] the second author and Lehrer show that, on restriction, the standard modules for the blob algebra will be cell modules for the annular Temperley-Lieb algebra. Using this result and earlier work [8,9] they relate these modules to the standard modules of Bernstein and Zelevinsky [10] (confer Rogawski [11], Kazhdan and Lusztig [12] and Zelevinsky [13]) corresponding to two-step nilpotent matrices. It follows that the decomposition numbers for these standard modules can be determined from those for a corresponding family of blob algebras.

Our definition of the blob algebra differs slightly from that given in [1]. After an initial section of definitions, we shall relate these two definitions and show that the two algebras are isomorphic in all cases that we need consider. Thus the results in [2] can all be translated to our setting, a fact that we shall use repeatedly.

The blob algebra $\mathrm{TL}_B(n)$, defined over a field k, depends on two parameters $q \in k^\times$ and $y \in \mathbb{Z}$. In order to relate $\mathrm{TL}_B(n)$ to the original blob algebra, we shall assume that $[y] \neq 0$. For the remainder of this section we shall review the characteristic zero theory from [2], and indicate how the results must be modified in positive characteristic.

When $[2] \neq 0$ the algebra is quasi-hereditary, with standard modules $W_t(n)$ where $n+t$ is even with $-n \leq t \leq n$. We call the indexing set of these modules the set of weights for $\mathrm{TL}_B(n)$. By quasi-heredity, each $W_t(n)$ has simple head $D_t(n)$, and all irreducible modules arise in this manner. The hereditary order on the set of weights is given by $t \lhd u$ if and only if $|t| > |u|$. We can give an explicit basis of diagrams both for the algebra itself and for each of the standard modules.

From [2] we see that (in the cases of interest to us) the representation theory of $\mathrm{TL}_B(n)$ splits into three cases, depending on the choice of parameters. If $|y| \geq n$, and there is no y' such that $[y] = [y']$ and $|y'| < n$, then the algebra is semi-simple. Otherwise either q is not a root of unity (called the singly critical case), or q is a root of unity (the doubly critical case). For the semi-simple and singly critical cases, the methods and results in [2] all generalise unchanged to positive characteristic. Hence we shall assume that we are in the doubly critical case, with q a primitive lth root of unity. We also choose x a primitive $2l$th root of unity, such that $x^2 = q$. Under these assumptions we may (and shall) assume that $-l < y \leq l$. For simplicity, we shall also assume that $l > 2$.

We define an alcove structure on $\mathbb{R}$ by defining the elements of the form $y + al$ with $a \in \mathbb{Z}$ to be walls, and the connected components of non-wall elements to be alcoves. By regarding the set of weights for $\mathrm{TL}_B(n)$ as a subset of $\mathbb{R}$ in the obvious manner, we can thus refer to the alcove or wall in which a given weight lies. It will be convenient to draw the set of weights (embedded into $\mathbb{R}$) in the form shown in Figure 1, where the vertical lines denote the walls in $\mathbb{R}$. (In

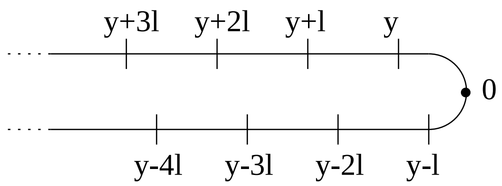

Fig. 1.

such diagrams we shall always assume that $0 \leq y$; however our results do not require this assumption.) Suppose that s and t are weights with $|s| > |t| \geq 0$, such that $s = w + m$ and $t = w - m$ with $w \in \mathbb{Z}$ on a wall. In this case we say that t is the reflection of s about the wall w. There is an injective map from $W_s(n)$ into $W_t(n)$ which we denote by ϕ_{st} (respectively ψ_{st}) if $|w| \geq |m|$ (respectively $|w| < |m|$). In [2] these maps were constructed indirectly in characteristic zero; in Section 6 we shall give an explicit description of them, and hence show that they exist in arbitrary characteristic.

Given a weight t, let s and u be minimal such that there exist maps ϕ_{st} and ψ_{-ut}. (If one or both of these maps does not exist because t is too close to $\pm n$ then we are in a degenerate case of the following; the reader will be able to make the appropriate modifications.) In characteristic zero with t in an alcove we have that the quotient $Y_t(n) = W_t(n)/(\mathrm{Im}\,\phi_{st})$ is indecomposable with composition factors $D_t(n)$ and $D_{-u}(n)$, while the quotient

$X_t^0(n) = W_t(n)/(\mathrm{Im}\,\phi_{st} + \mathrm{Im}\,\psi_{-ut})$ is isomorphic to $D_t(n)$. From this we can deduce that $W_t(n)$ has composition factors

$$\begin{array}{cc} & D_t(n) \\ D_{s_1}(n) & D_{-u_1}(n) \\ D_{s_2}(n) & D_{-u_2}(n) \\ D_{s_3}(n) & D_{-u_3}(n) \\ \vdots & \vdots \end{array} \qquad (1)$$

with elements s_i and $-u_i$ as in Figure 2

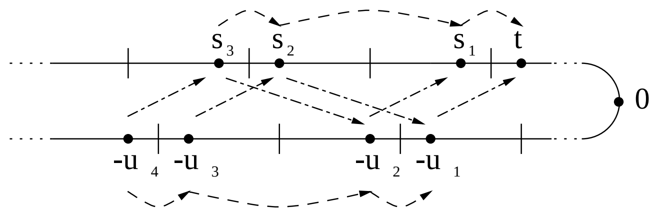

Fig. 2.

where the interior arrows denote ψ maps and the exterior denote ϕ maps. There is a similar (simpler) statement for weights on walls (see [2, (9.4)Theorem]). In positive characteristic the structure of the quotients $X_t^0(n)$ and $Y_t(n)$ is no longer so straightforward; however, for precisely one element in each alcove we can give an alternative description of these quotients that is independent of the characteristic.

To do this we need to consider an embedding of $\mathrm{TL}_A(n)$ into $\mathrm{TL}_B(n)$. When $t \equiv y - 1 \pmod{l}$ with $t \geq 0$ we have (Lemma 8.3) that

$$Y_t(n) \cong \Delta_t(n)$$

as $\mathrm{TL}_A(n)$-modules, where $\Delta_t(n)$ is the standard module for $\mathrm{TL}_A(n)$ associated to the weight t. Further, for this choice of t we shall show in Section 8 that each irreducible $\mathrm{TL}_A(n)$ composition factor of $X_t^0(n)$ has a $\mathrm{TL}_B(n)$-module structure, and hence that we can lift a composition series of $X_t^0(n)$ as a $\mathrm{TL}_A(n)$-module to a composition series as a $\mathrm{TL}_B(n)$-module. If $t < 0$ and $t \equiv y + 1 \pmod{l}$ we have an isomorphism

$$Y_t(n) \cong \Delta_{-t}(n)$$

and we can again lift a composition series of $X_t^0(n)$ from $\mathrm{TL}_A(n)$ to $\mathrm{TL}_B(n)$. It is for this reason that we chose to draw our set of weights in the form given above.

The composition factors of $\Delta_t(n)$ are well known. In Section 4 we determine them via Ringel duality from results about tilting modules for SL_2. (This is convenient in our context, as it allows us to state the result in alcove form.) Motivated by the submodule structure of the $\Delta_t(n)$ when $l = 1$ we next wish to define a series of modules $\Xi_t^i(n)$ for $\mathrm{TL}_A(n)$ such that $\Delta_t(n) \cong \Xi_t^{-1}(n)$ and we have for all $i \geq 0$ a short exact sequence

$$0 \to \Xi_u^i(n) \to \Xi_t^{i-1}(n) \to \Xi_t^i(n) \to 0$$

for suitably chosen u. We also require that there should be a simple combinatorial criterion for determining the compositions factors of the $\Xi_t^i(n)$ from those for the $\Delta_t(n)$. In order to define such modules we consider the global Temperley-Lieb algebra $\mathrm{TL}_A(\infty)$, and construct (Theorem 5.1) explicit homomorphisms $\theta^{rs} : \Delta^r(\infty) \to \Delta^s(\infty)$ between standard modules for this algebra. These maps give rise to all $\mathrm{TL}_A(n)$ homomorphisms between standard modules, and can be used to define the $\Xi_t^i(n)$ sought above.

Returning to our particular choice of weight for $\mathrm{TL}_B(n)$, we can show by comparing with the characteristic zero case (Proposition 8.4) that

$$X_t^0(n) \cong \Xi_t^0(n)$$

and (Proposition 8.6) that we have a short exact sequence

$$0 \to X_{-u}^0(n) \to Y_t(n) \to X_t^0(n) \to 0.$$

Hence we can determine the composition factors of $X_t(n)$ and $Y_t(n)$ for precisely one weight inside each alcove.

To deduce from this the composition factors of $W_t(n)$ for arbitrary weights, we use an analogue of the translation principle. From the maps ϕ and ψ, along with an explicit calculation of the action of certain elements of the algebra, we know the blocks of $\mathrm{TL}_B(n)$ (Theorem 7.3). If we define the exact functor pr_t from $\mathrm{TL}_B(n)$-mod to $\mathrm{TL}_B(n-1)$-mod to be restriction followed by projection onto the block containing $W_t(n)$, we can show for t in an alcove (Theorem 8.7) that $\mathrm{pr}_{t\pm1}D_t(n)$ is either 0 or $D_{t\pm1}(n-1)$, and determine precisely when each case can occur. In this way we can translate a composition series for $W_t(n)$ for our special choice of t to any other t' in the same alcove or on the wall just below, and hence determine the composition factors of the $W_t(n)$ by induction, using the injective maps ϕ (Theorem 8.8).

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2 Preliminaries

In this section we review the various algebras to be considered in what follows. We shall define these over $\mathcal{A} = \mathbb{Z}[x, x^{-1}, Q, Q^{-1}]$, but will concentrate mainly on their specialisations to a field k of characteristic $p > 0$. We set $q = x^2$, and fix an integer y. For any integer n we define the usual Gaussian coefficient $[n] = \frac{x^n - x^{-n}}{x - x^{-1}} \in \mathcal{A}$.

We begin by defining the various Hecke algebras that we shall require. The *Hecke algebra of type* B_n, denoted $\mathcal{H}_B(n)$, is the $\mathcal{A}$-algebra generated by elements T_i with $0 \leq i \leq n-1$ subject to the relations

$$\begin{aligned} (T_0 - Q)(T_0 + 1) &= 0 && \\ (T_i - q)(T_i + 1) &= 0 && \text{if } i \neq 0 \\ T_iT_j &= T_jT_i && \text{if } |i-j| \neq 1 \\ T_iT_jT_i &= T_jT_iT_j && \text{if } |i-j| = 1 \text{ and } i, j \neq 0 \\ T_0T_1T_0T_1 &= T_1T_0T_1T_0. && \end{aligned}$$

The subalgebra generated by the T_i with $1 \leq i \leq n-1$ is the *Hecke algebra of type* A_{n-1}, denoted $\mathcal{H}_A(n)$.

Next we define certain variants on the usual Temperley-Lieb algebra. The *blob algebra*, denoted here by $\mathrm{TL}_B^y(n)$ (or just $\mathrm{TL}_B(n)$ when this is unambiguous), is the $\mathcal{A}$-algebra generated by elements U_i with $0 \leq i \leq n-1$ satisfying the relations

$$\begin{aligned} U_0^2 &= -[y]U_0 && \\ U_i^2 &= -[2]U_i && \text{if } i > 0 \\ U_iU_j &= U_jU_i && \text{if } |i-j| \neq 1 \\ U_iU_jU_i &= U_i && \text{if } |i-j| = 1 \text{ and } i,j > 0 \\ U_1U_0U_1 &= [y+1]U_1. && \end{aligned} \tag{2}$$

Note that this is not the original definition of the blob algebra given in [1]; we shall consider the relationship between the two definitions in Section 3. The subalgebra of $\mathrm{TL}_B^y(n)$ generated by the U_i with $1 \leq i \leq n-1$ is the usual *Temperley-Lieb algebra of type* A_{n-1}, and will be denoted by $\mathrm{TL}_A(n)$.

It will be convenient in what follows to work with a diagrammatic construction of the blob algebra. In order to see that this coincides with our definition, it is convenient to use certain results about *projection algebras* from [14, Chapter 6]. The necessary background material has been included in an Appendix at the end of this paper.

We will construct a certain algebra of diagrams, and show that this is isomorphic to $\mathrm{TL}_B(n)$. Whenever we consider diagram constructions, we shall adopt the following conventions. By a *diagram* we mean a rectangular box containing non-intersecting line segments, possibly with decorations. We refer to the dotted boundary of a diagram as its *frame* and the interior line-segments as *lines*, and identify two diagrams if they differ by an (edgewise) frame-preserving ambient isotopy. Lines in a diagram are called *propagating lines* if they connect the northern and southern edges of the frame, and *northern* (respectively *southern*) *arcs* if they meet only the northern (respectively southern) edge of the frame. The endpoints of lines are called *nodes*. If the number of southern nodes in A equals the number of northern nodes in B then we define the product AB to be the concatenation of the diagram A above the diagram B. (In the product of two diagrams AB we assume that the southern nodes of A are identified with the corresponding northern nodes of B, and ignore the dotted line segment formed by their frames across the centre of the new diagram.)

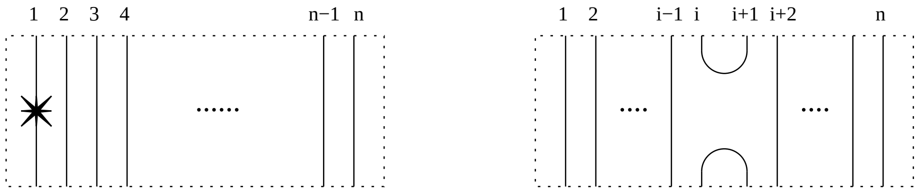


Fig. 3. U'_0 and U'_i

With these conventions, let U'_0 and U'_i (for $1 \leq i \leq n-1$) be the diagrams shown in Figure 3 and define $\mathrm{TL}'_B(n)$ to be the $\mathcal{A}$-algebra generated by these elements, in which we *(i)* identify any diagram with a double-starred line with $-[y]$ times the same diagram with a single star, and *(ii)* identify any diagram containing a closed internal loop with no decoration (respectively decorated by a star) with the same diagram without the loop, multiplied by $-[2]$ (respectively multiplied by $[y+1]$).

It is well known (see for example [15, Corollary 10.1]) that the subalgebra of $\mathrm{TL}'_B(n)$ generated by the U'_i with $1 \leq i \leq n-1$ is isomorphic to the ordinary Temperley-Lieb algebra $\mathrm{TL}_A(n)$. This subalgebra has a basis of those diagrams with n northern nodes and n southern nodes such that every line is either propagating or an arc, and no line is decorated with a star. We call such diagrams *Temperley-Lieb diagrams*. If we relax the condition that no line is decorated with a star then we call the diagrams that arise *generalised Temperley-Lieb diagrams*. The algebra $\mathrm{TL}'_B(n)$ has a basis consisting of those generalised Temperley-Lieb diagrams in which only the lines which can be deformed ambient isotopically to touch the western side of the frame may be decorated by a star, and each such line has at most one star.

It is easy to see that the algebra $\mathrm{TL}'_B(n)$ can be regarded as a quotient of $\mathrm{TL}_B(n)$ via the homomorphism which for each i takes U_i to U'_i. We wish to show that this is in fact an isomorphism. As noted in the Appendix, $\mathrm{TL}_B(n)$ is a projection algebra which satisfies the hypotheses of Theorem A.1 (i.e Theorem 6.20 of [14]), and hence is free. As $\mathrm{TL}'_B(n)$ is clearly free, it is enough to show that these two algebras have the same rank. In order to do this, we begin by considering a related algebra defined by tom Dieck [3].

The algebra TB_n is the $\mathcal{A}$-algebra generated by elements $\bar{U}_i$ with $0 \leq i \leq n-1$ satisfying the relations

$$\begin{aligned} \bar{U}_i^2 &= d\bar{U}_i \quad \text{if } i \geq 0 \\ \bar{U}_i\bar{U}_j &= \bar{U}_j\bar{U}_i \ \text{if } |i-j| \neq 1 \\ \bar{U}_i\bar{U}_j\bar{U}_i &= \bar{U}_i \quad\ \ \text{if } |i-j| = 1 \text{ and } i,j > 0 \\ \bar{U}_1\bar{U}_0\bar{U}_1 &= d\bar{U}_1 \end{aligned}$$

for some $d \in \mathcal{A}$. This algebra can also be realised as a projection algebra, and we show in the Appendix that it is free of the same rank as $\mathrm{TL}_B(n)$. In [3, (4.5) Satz] it is shown that this algebra has a basis consisting of those Temperley-Lieb diagrams with $2n$ northern nodes and $2n$ southern nodes that are symmetric under reflection about the vertical axis.

We will be done if we can show that the set of such diagrams can be put in bijective correspondence with our diagram basis for $\mathrm{TL}'_B(n)$. But this is straightforward as such symmetric diagrams are determined by their right hand halves, and these can be identified bijectively with our original diagram basis by replacing each consecutive pair of intersections of lines with the axis of symmetry with a connecting line decorated by a star, as illustrated in Figure 4.

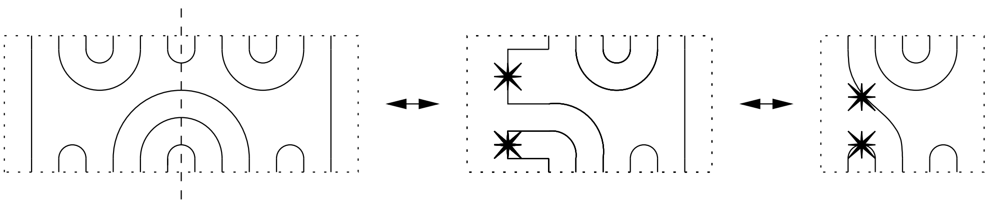


Fig. 4.

In summary, we have shown

Proposition 2.1 *The algebras* $\mathrm{TL}_B(n)$ *and* $\mathrm{TL}'_B(n)$ *are isomorphic as* $\mathcal{A}$-*algebras.*

This result can also be proved by mimicking the proof of [3, (4.5) Satz], or directly using projection algebra methods. (We outline the latter approach in the Appendix, leaving details to the reader.)

There is an injective algebra homomorphism from $\mathrm{TL}_A(n)$ into $\mathrm{TL}_A(n+1)$ obtained by adding a propagating line to the *left-hand side* of each diagram (there is a right-hand analogue, but we do not consider it here). We identify $\mathrm{TL}_A(n)$ with its image under this embedding. We can now define the (left-hand) *global Temperley-Lieb algebra* by setting $\mathrm{TL}_A(\infty) = \lim_{n\to\infty}\mathrm{TL}_A(n)$. This is an algebra consisting of all finite $\mathcal{A}$-linear combinations of Temperley-Lieb diagrams (where we identify such diagrams with those obtained by adding infinitely many propagating lines on the left-hand side). In a similar manner we can define an injective algebra homomorphism from $\mathrm{TL}_B(n)$ into $\mathrm{TL}_B(n+1)$ by adding a propagating line to the *right-hand side* of each diagram. (Note that there is no left-hand analogue in this case.)

Clearly, we also have embeddings of $\mathcal{H}_A(n)$ into $\mathcal{H}_B(n)$, and also of $\mathrm{TL}_A(n)$ into $\mathrm{TL}_B(n)$. By specialising Q to a unit in $\mathbb{Z}[x,x^{-1}]$ we may regard all of these algebras as $\mathbb{Z}[x,x^{-1}]$-algebras. With this identification it is a routine exercise to verify

Proposition 2.2 *Suppose that $Q = -x^{-2y}$. Then we have an algebra homomorphism $\tau : \mathcal{H}_B(n) \to TL_B(n)$ given on generators by*

$$T_0 \longmapsto (x - x^{-1})x^{-y}U_0 + Q \quad \text{and} \quad T_i \longmapsto xU_i + q \;\; (\text{for } i > 0).$$

This restricts to give a homomorphism from $\mathcal{H}_A(n)$ to $TL_A(n)$.

Henceforth we shall assume that $Q = -x^{-2y}$ so that the above proposition holds. For simplicity we will also assume that $q \neq \pm 1$.

3 Quasi-heredity and standard modules

The definition of the blob algebra given in Section 1 differs somewhat from the original definition in [1]. In this section we shall relate these two definitions, and recall certain basic results about representations of these algebras from [1] and [2].

We begin by recalling the original definition of the blob algebra. Let $\mathcal{B} = \mathbb{Z}[x, x^{-1}, y_-]$ and $b(n)$ be the $\mathcal{B}$-algebra generated by elements V_i (for $1 \leq i \leq n-1$), $\bullet$ and $\square$, which we shall represent diagrammatically as in Figure 5.

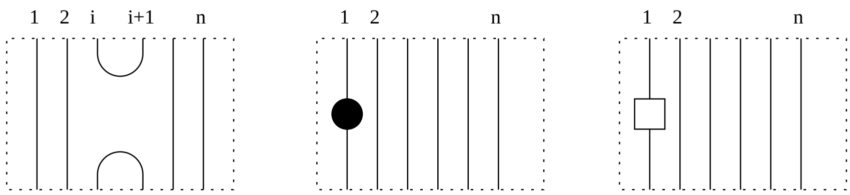


Fig. 5. V_i, $\bullet$ and $\square$

Multiplication of two diagrams corresponding to elements in $b(n)$ is given by concatenation of diagrams as for $\mathrm{TL}_B(n)$. We also impose the additional relations

$$\begin{aligned} \bullet + \square &= 1 \\ \bullet\,\square = \square\,\bullet &= 0 \qquad (3) \\ \bullet^2 &= \bullet \end{aligned}$$

and identify a diagram containing a closed internal loop with no decoration (respectively decorated by a blob) with the same diagram without the loop, multiplied by [2] (respectively by y_-). We set $y_+ = [2] - y_-$.

Consider the $\mathcal{B}$-algebra generated by elements V_i', for $0 \leq i \leq n-1$, with defining relations

$$\begin{aligned} V_0'^2 &= V_0' \\ V_i'^2 &= [2]V_i' \quad \text{if } i > 0 \\ V_i'V_j' &= V_j'V_i' \quad \text{if } |i-j| \neq 1 \\ V_i'V_j'V_i' &= V_i' \qquad \text{if } |i-j| = 1 \text{ and } i, j > 0 \\ V_1'V_0'V_1' &= y_- V_1'. \end{aligned}$$

This is a projection algebra with the same projection graph as $\mathrm{TL}_B(n)$, and exactly as for $\mathrm{TL}_B(n)$ we can show using Theorem A.1 that it can identified with $b(n)$ via the homomorphism which takes V_i' to V_i for $i \geq 1$ and V_0' to $\bullet$.

Over k, the algebra $b(n)$ is quasi-hereditary, as the construction of the heredity chains and corresponding standard modules given in [2, (3.2)] does not rely on the characteristic of the underlying field. (The proof of quasi-hereditary in [2] uses the well-known fact that if e is an idempotent in an algebra A such that AeA is a direct summand of A and eAe is semisimple then AeA is a hereditary ideal; confer [16, Example 1.5].) Further, the explicit construction of these standard modules in terms of the *ket diagram* basis given in [2, Section 4] also carries over. Just as for TL_B, there is a canonical inclusion $b(n-1) \hookrightarrow b(n)$ which adds an undecorated propagating line to the right-hand end of each diagram.

Arguing as is [1, Section 2.3] we see that the representation theory of $b(n)$ over k splits into three distinct cases depending on the number of integer values of a for which

$$[a]y_- = [a-1] \tag{4}$$

in k. If there is no solution to (4) then $b(n)$ is semisimple, by the arguments in [1]. When q is not a root of unity there is a unique solution, and we say that $b(n)$ is *singly critical*. If however q is a root of unity then there are infinitely many solutions, and we say that $b(n)$ is *doubly critical*. In this latter case we shall henceforth assume that q is a primitive lth root of unity. As noted in the introduction, we may then assume that $-l < y \leq l$, as in this case we have that $[t] = [t+2l]$ for all $t \in \mathbb{Z}$.

It is easy to verify that any solution of (4) must satisfy $[a] \neq 0$ in k. Thus in the singly critical case we have $y_- = [a-1]/[a]$ and $y_+ = [-a-1]/[-a]$ in k for some $a \in \mathbb{N}$. In the doubly critical case we have $y_- = [a'-1]/[a']$ and $y_+ = [b'-1]/[b']$ in k for some $a', b' \in \mathbb{N}$. In this latter case we shall choose a' and b' minimal with this property, and note that it is easy to verify that $a' + b' = 2l$. We choose a and b such that $-l < a, b \leq l$ with $a \equiv a' \pmod{(2l)}$ and $b \equiv b' \pmod{(2l)}$.

The algebras $b(n)$ and $\mathrm{TL}_B(n)$ are defined over different ground rings. However, for the cases of interest in this paper (when we consider specialisations to a field k) we have

Proposition 3.1 *In either critical case, with y_- and y_+ as above, we have an isomorphism of algebras $\theta : b(n) \to \mathrm{TL}^a_B(n)$ given by*

$$V_i \mapsto -U_i, \qquad \bullet \mapsto \frac{[a]1 + U_0}{[a]}, \qquad \Box \mapsto -\frac{U_0}{[a]}.$$

Further, we have another such isomorphism $\theta' : b(n) \to \mathrm{TL}^b_B(n)$ where we interchange the roles of $\bullet$ and $\Box$ above, and replace all occurrences of $[a]$ by $[-a]$ (respectively $[b]$) in the singly (respectively doubly) critical case.

PROOF. By our earlier remarks, these algebras are both free with the same rank. The proof is now an elementary exercise in checking that the morphism is compatible with the relations defining each of our algebras.

Recall the explicit description of the standard modules for the blob algebra given in [2, Section 4]. For $n + t$ even with $0 \leq t \leq n$, the *ket diagram* basis of $\tilde{W}_{\pm t}(n)$ is the subset of the set of diagrams with n northern and t southern nodes and no closed internal loops, with the following properties. Each diagram has t propagating lines (and hence no southern arcs) in which the leftmost propagating line (if any) is decorated with a blob (in the case of $-t$) or a box (in the case of $+t$). Further, every northern arc which can be deformed ambient isotopically to touch the western edge of the frame is decorated with either a blob or a box. The action of $b(n)$ on this basis is by composition of diagrams (acting from above), except that composites with fewer than t propagating lines are set to zero.

It will be convenient to introduce two other (isomorphic) realisations of these modules with bases consisting of *blob* (respectively *box*) *ket diagrams.* The blob ket diagrams are identical to the usual ket diagrams described above, except that every northern arc which can be deformed ambient isotopically to touch the western edge is now either undecorated or decorated with a blob. Notice that every element in $b(n)$ can be written as a linear combination of diagrams containing no boxes; the action of such diagrams on the blob ket basis is given by composition of diagrams (acting from above), except that composites with fewer than t propagating lines are set to zero. There is a corresponding description of the box ket diagrams interchanging the roles of blobs and boxes. It may help the reader to note that the blob ket diagrams for $\tilde{W}_{+t}(n)$ contain a box as well as blobs, but only on the leftmost propagating line. (And similarly box ket diagrams for $\tilde{W}_{-t}(n)$ contain a blob as well as boxes.) As an example, we give the blob ket diagram basis of $\tilde{W}_1(3)$ in Figure 6.

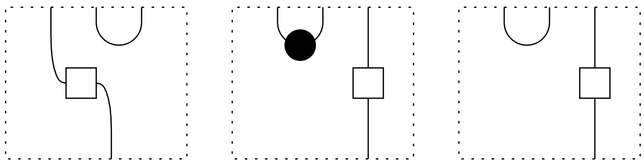


Fig. 6.

To see that the module defined by the blob ket diagrams is isomorphic to that arising from the usual ket diagrams consider the linear map that takes a ket diagram with s boxes on northern arcs to the signed sum of the 2^s blob ket diagrams obtained by replacing each box by either a blob or an undecorated arc. From the first relation in (3) we see that the sign of such a diagram is $(-1)^t$ where t is the number of new blobs in the diagram. It is easy to verify that this induces a module isomorphism. Again, there is a similar isomorphism in the box ket diagram case.

We next define certain representations of the algebra $\mathrm{TL}_B(n)$, in terms of our alternative presentation. For $n+t$ even with $0 \leq t \leq n$ we let $W_{\pm t}(n)$ be the $\mathrm{TL}_B(n)$-module with basis the set of diagrams with n northern and t southern nodes and no closed internal loops, with the following properties. Each diagram contains t propagating lines (and hence no southern arcs), all propagating lines are undecorated, and the only northern arcs that may possibly be decorated (in this case with a star) are those that can be deformed ambient isotopically to touch the western edge of the frame. In the case of $W_{+t}(n)$, the action of $\mathrm{TL}_B(n)$ on this basis is by composition of diagrams, except that composites with fewer than t propagating lines or a decorated propagating line are set to zero. In the case of $W_{-t}(n)$, the action of $\mathrm{TL}_B(n)$ on this basis is also by composition of diagrams, except that composites with fewer than t propagating lines are set to zero, while a diagram with a decorated propagating line is replaced with $-[y]$ times the corresponding undecorated diagram. Note that $W_{\pm n}(n)$ is one dimensional; we denote the corresponding basis element by $e_{\pm n}$ (or just e in the case $+n$).

Proposition 3.2 *Via the isomorphism θ between $b(n)$ and $\mathrm{TL}^a_B(n)$ of Proposition 3.1, we may identify $\tilde{W}_{\pm t}(n)$ with $W_{\mp t}(n)$. Similarly, via the isomorphism θ' between $b(n)$ and $\mathrm{TL}^b_B(n)$ we may identify $\tilde{W}_{\pm t}(n)$ with $W_{\pm t}(n)$.*

PROOF. We define a linear map from $\tilde{W}_{\pm t}(n)$ to $W_{\mp t}(n)$ on the box ket basis by replacing each box on a northern arc by a star, and removing the decoration from the leftmost propagating line. It is now an easy exercise to show that this is compatible with the map θ. For the θ' case the argument is similar, using the blob ket basis.

When q is a root of unity, we will find it useful to relate modules $W_t(n)$ for $\mathrm{TL}^a_B(n)$ and $\mathrm{TL}^b_B(n)$. We shall temporarily denote by $W^y_t(n)$ the module $W_t(n)$ defined for $\mathrm{TL}^y_B(n)$.

Corollary 3.3 *Suppose that q is a primitive lth root of unity and $a' \in \mathbb{N}$ is such that $0 < a' \leq 2l$ with $[a'] \neq 0$, and set $b' = 2l - a'$. Let a and b be the corresponding representatives mod $2l$ in the interval $(-l, l]$. Via the isomorphism $\theta'\theta^{-1}$ from $TL^a_B(n)$ to $TL^b_B(n)$ we may identify $W^a_t(n)$ with $W^b_{-t}(n)$ for all $-n \leq t \leq n$ with $n + t$ even.*

As our choice of notation implies, there is a close connection between $\mathrm{TL}_B(n)$ and the generalised Temperley-Lieb algebra of type B defined in [14]. The relationship between the two (in terms of the original definition of the blob algebra) is outlined in [17].

Motivated by Proposition 3.1, we shall henceforth assume that $[y] \neq 0$. As the arguments for the semi-simple and singly critical cases given in [2] go through unchanged in positive characteristic, we shall also assume that we are in the doubly critical case, with q a root of unity. *Thus for the remainder of this paper, unless explicitly stated otherwise, we shall assume that* $q \in \mathbb{N}$ *is a primitive* l*th root of unity with* $-l < y \leq l$ *and* $[y] \neq 0$ *(so in particular* $y \neq 0$*).* $\mathrm{TL}_B(n)$ *will always denote the algebra* $\mathrm{TL}_B^a(n)$ *where* a *is as in Corollary 3.3*

4 Ringel duality and $\mathbf{TL}_A$-modules

In this section we shall review the relationship between $\mathrm{TL}_A(n)$ and the q-Schur algebra $S_q(2,n)$ associated to the quantum general linear group q-GL(2) defined by Dipper and Donkin [18]. We follow the notation and conventions in [19], to which the reader is referred for the basic definitions and results concerning Schur algebras, quasi-hereditary algebras and Ringel duality that we shall require. Throughout this section we shall suppose that $l > 2$, or that $l = 1$ and $p > 2$.

We begin with a brief review of Specht module theory for $\mathcal{H}_A(n)$. Details can be found in [19, Section 4.7] and [20]. For each partition λ of n we may define a Specht module S^λ for $\mathcal{H}_A(n)$ (see [21] for details of their construction). Given a cosaturated subset of row-regular weights Donkin has determined [19, 4.7(5)] corresponding quotients $\mathcal{H}_A(n)/I(\pi)$ of $\mathcal{H}_A(n)$ such that S^λ is an $\mathcal{H}_A(n)/I(\pi)$-module for each $\lambda \in \pi$.

Taking π to be the set of 2-part partitions of n (which is always saturated, and row regular by our assumptions on l and p), and using [22] (see also [23]) we may identify $\mathrm{TL}_A(n)$ with the quotient $\mathcal{H}_A(n)/I(\pi)$. By Ringel duality, it is now easy to determine the composition multiplicities $[S^\mu : D^\lambda]$ of the simple modules D^λ in the Specht modules S^μ for TL_A. (These composition multiplicities are well-known, and were first obtained for $l = 1$ by James [24]. Further, when $l = 1$ the full submodule lattice of the Specht module is known [25].) These results will be used in Section 5, after we have given an explicit construction of Specht modules using diagram bases.

Let $S'(2,n)$ be the Ringel dual of $S(2,n)$. The main result that we shall require is

Theorem 4.1 *The algebra $S'(2,n)$ is Morita equivalent to $\mathrm{TL}_A(n)$, and under this equivalence, the standard modules for $S'(2,n)$ correspond to the Specht modules labelled by 2-part partitions of n.*

PROOF. This is a special case of [19, Section 4.7], which is a q-generalisation of [20, (4.3) and (4.4)].

We adopt the usual convention for labelling weights, so that the set of dominant polynomial weights $\Lambda^+(2,n)$ for q-GL(2) is identified with the set of 2-part partitions of n. Given a dominant polynomial weight μ, we let $\nabla(\mu)$ be the induced module of highest weight μ for q-GL(2), and $T(\mu)$ be the corresponding tilting module. Tilting modules have a filtration by induced modules, with multiplicities independent of the choice of filtration. We denote the multiplicity of $\nabla(\mu)$ in such a filtration of $T(\lambda)$ by $(T(\lambda) : \nabla(\mu))$. There are two labelling conventions for Specht modules commonly in use; we adopt that for which $[S^\mu : D^\lambda] \neq 0$ implies that $\lambda \geq \mu$ in the dominance order on weights. By Ringel duality we have

Proposition 4.2 *For all $\lambda, \mu \in \Lambda^+(2,n)$ we have*

$$[S^\mu : D^\lambda] = (T(\lambda) : \nabla(\mu)) \quad \textit{and} \quad \mathrm{Hom}(S^\lambda, S^\mu) \cong \mathrm{Hom}(\nabla(\lambda), \nabla(\mu)).$$

PROOF. See [19, A4.6 and A4.8(ii)].

Henceforth we shall identify a weight $\mu = (\mu_1, \mu_2)$ in $\Lambda^+(2,n)$ with $\mu_1 - \mu_2$, and thus regard $\Lambda^+(2,n)$ as a subset of $\mathbb{N}$. For $i \geq -1$ we call an integer μ an *lp^i-wall* if $\mu \equiv -1 \pmod{lp^i}$, where we interpret lp^{-1} as 1. When we refer to *the least lp^i-wall above μ* (or *the lp^i-wall immediately above μ*) we include the possibility that this wall is μ itself.

Corollary 4.3 *We have* $\mathrm{Hom}(S^\lambda, S^\mu)$ *isomorphic to* k *if* λ *is a reflection of* μ *about the least* lp^i*-wall above* μ *(for some* i*) and to zero otherwise.*

PROOF. This follows immediately from Proposition 4.2 and [26, Theorem 5.1].

It will be convenient to consider the (l,p)-adic expansion of various weights in what follows. We adopt the convention that in any expression of the form $a = \sum_{i\geq -1} a_i lp^i$ we shall interpret lp^{-1} as 1, and require (unless otherwise stated) that $0 \leq a_i \leq p-1$ (or $l-1$ if $i=-1$).

Theorem 4.4 *Suppose that* $\lambda \geq l-1$*, and let* m *be the unique non-negative integer such that* $lp^m - 1 \leq \lambda < lp^{m+1}-1$*. Writing* λ *in the form* $(lp^m-1)+a+lp^m b$ *with* $0 \leq a = \sum_{i\geq -1} a_i lp^i \leq lp^m - 1$*, we have*

$$(T(\lambda):\nabla(\mu)) = \begin{cases} 1 \text{ if } \mu = \lambda - 2a_J \text{ for some } J \subseteq [-1, m-1] \\ 0 \text{ otherwise} \end{cases}$$

where $a_J = \sum_{i \in J} a_i lp^i$.

PROOF. See [19, 3.4(3)].

We need to invert this formula in order to determine $[S^\mu : D^\lambda]$ for a given μ. In the notation of the theorem, all non-zero composition factors D^λ have labels of the form

$$\lambda + 1 = lp^m(b+1) + \sum_{i=-1}^{m-1} a_i lp^i \tag{5}$$

with $0 \leq b \leq p-2$. Thus we must have

$$\mu + 1 = lp^m(b+1) + \sum_{i=-1}^{m-1} J(i) a_i lp^i \tag{6}$$

where $J(i)$ equals -1 if $i \in J$ and $+1$ otherwise. Now any expression for $\mu+1$ of this form will arise from some λ, so we just need to determine all such expressions (subject to the constraint that $\lambda \leq n$). When $l = 1$ the following Proposition is a re-expression of a result of James [24].

Proposition 4.5 *The following algorithm gives the composition factors of* S^μ.
(i) Let M *be maximal such that* $lp^M \leq n$, *and set* $\mathrm{cf}(\mu) = \{\mu\}$ *and* $i = M$.
(ii) For each $\tau \in \mathrm{cf}(\mu)$ *from the preceding stage, if the least* lp^i*-wall* w *above* τ *is not an* lp^{i+1}*-wall then reflect* τ *about* w, *and if the result* $w\tau$ *satisfies* $\tau < w\tau \leq n$ *then add it to* $\mathrm{cf}(\mu)$.
(iii) If $i > 0$, *let* $i = i - 1$, *and repeat from step (ii).*

Before proving this we illustrate it with an example. Suppose that $l = 5$ and $p = 3$, and that μ is the weight represented by a in Figure 7. The horizontal line represents the set of weights (embedded into $\mathbb{R}$ and decreasing from left to right) while the vertical lines represent lp^i walls, where the length of a line increases with i. Thus in the diagram there are 3 lp^2-walls, 4 lp-walls that are not lp^2-walls, and 12 remaining l-walls.

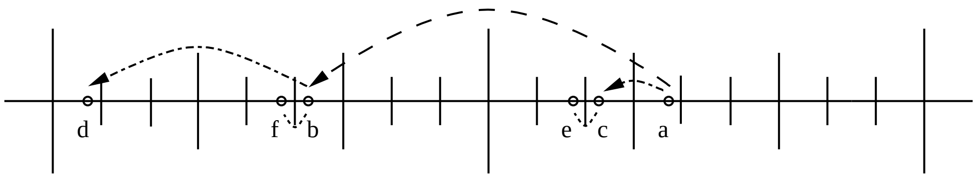


Fig. 7.

The first stage of the algorithm reflects our weight about the central lp^2 wall as indicated to give b. The next stage produces the two additional weights c and d obtained by the remaining reflections above the line of weights, while the final stage produces the final pair of weights e and f obtained by the reflections below the line of weights. Note that in this last step two potential new reflections are omitted, as they would involve l-walls that are either lp- or lp^2-walls.

Proof of Proposition 4.5 For fixed μ we must show that the set of composition factors obtained by our algorithm agrees with those elements λ of degree at most n arising from Theorem 4.4. First we show that any output λ from the algorithm satisfies $(T(\lambda) : \nabla(\mu)) = 1$.

Given a weight τ that is not an lp^j-wall, suppose that w is an lp^j-wall (but not an lp^{j+1}-wall) immediately above it. Let m be the unique non-negative integer such that $lp^m - 1 \leq w\tau < lp^{m+1} - 1$. Now writing $w\tau + 1$ in the form

$$(w\tau) + 1 = lp^m(b+1) + \sum_{i=-1}^{m-1} a_i lp^i \tag{7}$$

with $0 \leq b \leq p-2$, $0 \leq a_{-1} \leq l-1$ and $0 \leq a_i \leq p-1$ for $i \geq 0$, we have that

$$\begin{aligned} \tau + 1 &= lp^m(b+1) + \textstyle\sum_{i=-1}^{m-1} a_i lp^i - 2(\sum_{i=-1}^{j-1} a_i lp^i) \\ &= lp^m(b+1) + \textstyle\sum_{i=-1}^{m-1} J(w,i) a_i lp^i \end{aligned} \tag{8}$$

where $J(w,i) = +1$ if $i \geq j$ and -1 if $i < j$. Indeed, if we relax the restriction on the a_i in equation (7) to $0 \leq |a_{-1}| \leq l-1$ and $0 \leq |a_i| \leq p-1$ for $0 \leq i \leq j-1$ (but still require that $0 \leq a_i \leq p-1$ for $i \geq j$), then equation (8) still holds.

Suppose that $\lambda \in$cf(μ). By construction $\lambda = w_t(w_{t-1}(\ldots(w_1\mu))\ldots)$ where for each j we have that w_j is an lp^{f_j}-wall but not an lp^{f_j+1}-wall lying immediately above $w_{j-1}(w_{j-2}(\ldots(w_1\mu))\ldots)$, and $f_t < f_{t-1} < \cdots < f_1$. Now by induction using equations (7) and (8) we see that if $\lambda + 1$ is of the form in equation (5) then $\mu + 1$ is of the form in equation (6), and so by Theorem 4.4 we see that $(T(\lambda) : \nabla(\mu)) = 1$.

For the other direction, consider some λ which gives rise to an expression for $\mu + 1$ of the form (6). When $a_i = 0$, the element $J(i)$ may be chosen freely; in such cases we set $J(i) = J(i+1)$ if $i < m-1$ and $J(i) = 1$ otherwise. We also set $J(m) = 1$. Let K be the subset $\{k_1, \ldots k_t\}$ of $[-1, m-1]$ consisting of those i for which $J(i) \neq J(i+1)$, arranged so that $k_t < \cdots < k_1$. We now define weights w_i inductively (for $1 \leq i \leq t$) by setting w_i to be the lp^{k_i+1}-wall immediately above the weight $w_{i-1}(w_{i-2}(\ldots(w_1\mu))\ldots)$. It is now easy to see that the weight λ can be obtained from μ by using this set of walls in our algorithm.

In fact we have complete knowledge of the submodule structure of the Specht modules in the case $l = 1$ and $p > 2$. In order to state this result we need to introduce some more notation, based on [25]. Given a weight μ with $\mu + 1 = \sum_{i\geq 0} a_i p^i$ we set $\mathcal{B}^-_\mu = \{i \mid a_i \neq 0\}$ and $\mathcal{B}^+_\mu = \{i \mid a_i \neq p-1\}$. We then define $\hat{A}_\mu$ to be the family of sets of natural numbers comprising of the empty set along with any set I of the form $I = [i_1, i_2) \cup [i_3, i_4) \cup \ldots \cup [i_{2t-1}, i_{2t})$, with $i_1 < i_2 < \ldots < i_{2t}$ and $i_{2j-1} \in \mathcal{B}^-_\mu$ and $i_{2j} \in \mathcal{B}^+_\mu$ for $1 \leq j \leq t$. For such a set I we define $\delta_I = \sum_{i\in I}(p-1-a_i)p^i + \sum_{j=1}^{t} p^{i_{2j-1}}$ and set $A_\mu = \{I \in \hat{A}_\mu \mid \mu + 2\delta_I \leq n\}$. For $I \in A_\mu$ we set $\nu_I(\mu) = \mu + 2\delta_I$.

It is straightforward to verify that

$$\nu_I(\mu) + 1 = \sum_{i\notin I} \bar{a}_i p^i + \sum_{i\in I} \hat{a} p^i \tag{9}$$

where $\bar{a}_i = \begin{cases} a_i + 1 & \text{if } i = i_{2j} \text{ for some } j \\ a_i & \text{otherwise} \end{cases}$

and $\hat{a}_i = \begin{cases} p - a_i & \text{if } i = i_{2j-1} \text{ for some } j \\ p - a_i - 1 & \text{otherwise.} \end{cases}$

Note that the conditions on the sets $\mathcal{B}^-_\mu$ and $\mathcal{B}^+_\mu$ ensure that the coefficients in each summation in (9) lie between 0 and $p-1$. Further, it is easy to verify that

$$\mu + 1 = \sum_{i\notin I} \bar{a}_i p^i - \sum_{i\in I} \hat{a} p^i. \tag{10}$$

We can now state

Theorem 4.6 *When $l = 1$ and $p > 2$ the composition factors of S^μ are given by the set $\{D^{\nu_I(\mu)} \mid I \in A_\mu\}$, and there is a lattice isomorphism between the submodule lattice of S^μ and the lattice structure defined on A_μ by the relation $I \geq J$ if $I \subseteq J$.*

PROOF. See [25, Corollary 3.4]. (Note that the first part of the theorem follows from a comparison of equations (5) and (6) with (9) and (10), by the remarks before Proposition 4.5.)

The proof in [25] uses the 'truncated inverse Schur functor' defined in [27, Section 2], which gives an equivalence of categories between $k\Sigma_n$-Mod and the full subcategory of $S(n,n)$-modules with p-restricted socle and head. The result then follows from work of Adamovich [28,29] on the structure of certain Weyl modules for GL(n). An alternative proof of Theorem 4.6 can be found in [30], using Ringel duality and results of Doty [31] on the submodule structure of the symmetric powers of the natural representation for $\mathrm{GL}_n(k)$. These methods should extend to the case $l > 2$ (using Thams' generalisation [32] of the results of Doty), but we do not pursue this here.

Although we shall not use it directly in what follows, Theorem 4.6 motivates our definition of certain submodules $\Xi^i_\lambda(n)$ of S^λ in Section 5, which will play a key role in what follows.

5 Standard module morphisms for $\mathbf{TL}_A(\infty)$

In this section we define certain standard modules for the global Temperley-Lieb algebra (defined after Proposition 2.1), and homomorphisms between them. By restricting to the finite Temperley-Lieb algebras, we shall obtain all morphisms between standard modules for these algebras.

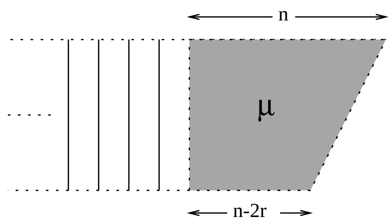


Fig. 8.

For $r \geq 0$ we define $\Delta^r(\infty)$ to be the k-module of arbitrary k-linear combinations of diagrams of the form in Figure 8, where the shaded region μ is a Temperley-Lieb diagram with n northern and $n-2r$ southern nodes containing no arcs on the southern edge, for some $n >> 0$. A diagram in $\mathrm{TL}_A(\infty)$ acts on this k-module via the usual composition of diagrams, except that composites containing a southern arc (and hence with more than r arcs on the northern edge) are set to zero. In this way $\Delta^r(\infty)$ becomes a left $\mathrm{TL}_A(\infty)$-module.

When $2r \leq n$, we have a similar module $\Delta^r(n)$ for $\mathrm{TL}_A(n)$, the free k-module with basis those diagrams having no closed loops, no arcs on the southern edge, $t = n - 2r$ propagating lines, and r arcs on the northern edge.

By restriction, $\Delta^r(\infty)$ is an (infinite dimensional) $\mathrm{TL}_A(n)$-module. We have an injective $\mathrm{TL}_A(n)$ homomorphism from $\Delta^r(n)$ into $\Delta^r(\infty)$ given by adding infinitely many vertical lines to the left-hand side of each diagram. We shall identify $\Delta^r(n)$ with its image under this morphism. Clearly, as a k-module, $\Delta^r(n)$ is a direct summand of $\Delta^r(\infty)$.

Given a diagram D in $\Delta^r(\infty)$, we may choose $n >> 0$ such that D is of the form shown in Figure 8. Any line $e \in \mu$ divides the interior of the frame into two regions (possibly containing other lines), and we define $h(e) \in \mathcal{A}$, by setting $h(e) = [a]$, where a is the number of lines (including e itself) in the region which does not include the western edge of μ. This is the same as defining a to be the number of lines to the right of (and including) e after we have deformed the diagram by a (*not necessarily edgewise*) frame preserving ambient isotopy so that e becomes a propagating line. As examples of this procedure, consider the two lines e and f in Figure 9(a). The line e is already propagating with $h(e) = [4]$. The diagram can be deformed so that f is a propagating line as shown in Figure 9(b), and hence $h(f) = [2]$.

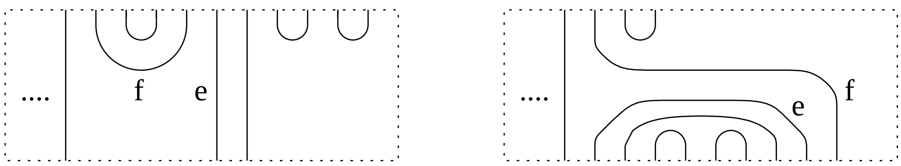


Fig. 9. (a) and (b)

We can now define the *hook product* $h(\mu)$ by setting

$$h(\mu) = \frac{[n-r]!}{\prod_{e \in \mu} h(e)}$$

where $[n]! = [n][n-1]\dots[1]$. This is independent of our choice of n, and so we may denote it just by $h(D)$. Note that this is a Laurent polynomial with integer coefficients by [33] (see [8, (3.3) Proposition]). The importance of the hook product arises from its role in Theorem 5.1, the proof of which will require the following notion of 'nipping'.

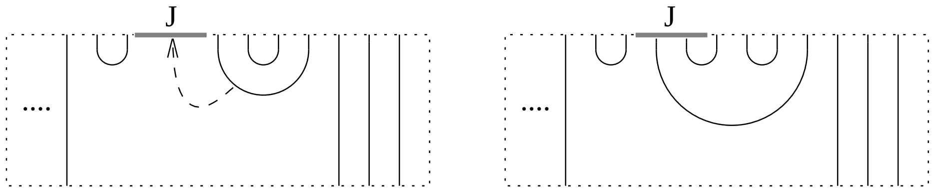


Fig. 10. (a) and (b)

Given an interval J on the northern edge of the frame of a diagram we say that a line in the diagram is *exposed* to J if it can be deformed ambient isotopically to touch it. If it is deformed to touch and cross J, and the segment outside the frame is then removed, this process is called *nipping* the line at J. Note that this procedure adds two new northern nodes to the diagram. For example, nipping the diagram in Figure 10(a) as shown produces the diagram in Figure 10(b).

Theorem 5.1 *For all $r \leq s$, there is a non-zero $\mathrm{TL}_A(\infty)$ homomorphism $\theta^{rs} : \Delta^r(\infty) \to \Delta^s(\infty)$ which acts on a diagram E by*

$$E \longmapsto \sum_{D \in \Delta^{s-r}(\infty)} h(D)ED \tag{11}$$

where ED denotes the composition of E with D, and the sum runs over those diagrams D in our basis of $\Delta^{s-r}(\infty)$. When $r = 0$ this is (up to scalars) the unique such morphism of standard modules.

PROOF. We begin by noting that this infinite summation makes sense, as only finitely many terms contribute to the coefficient of any given basis element of $\Delta^s(\infty)$. If $E \in \Delta^r(\infty)$, $D \in \Delta^{s-r}(\infty)$ and $ED \neq 0$, then this is equivalent to $D \in \Delta^{s-r}(n-2r)$ and $E \in \Delta^r(n)$ for large enough n.

To show that θ^{rs} is a homomorphism, we must verify that for all diagrams E in $\Delta^r(\infty)$ we have $\theta^{rs}(UE) = U\theta^{rs}(E)$, for any U which is the image in $\mathrm{TL}_A(\infty)$ of an element of some $\mathrm{TL}_A(n)$ of the form U_i. Clearly, if UE is a non-zero diagram in $\Delta^r(\infty)$ (i.e. has no southern arcs and no closed loops) we have

$$\theta^{rs}(UE) = \theta^{rs}(\textstyle\sum_{D\in\Delta^{s-r}(\infty)} h(D)UED) \tag{12}$$
$$= U(\textstyle\sum_{D\in\Delta^{s-r}(\infty)} h(D)ED) = U\theta^{rs}(E).$$

Also, any closed loop arising in the product UE will also occur in $U\theta^{rs}(E)$ by construction, and so it remains to check that $UE = 0$ implies that $U\theta^{rs}(E) = 0$. Note that this does not follow from (12), as it is possible for UE to have more than r northern and at least one southern arcs, while for some D in the sum the product UED has s northern and no southern arcs. Most of the remainder of the proof is devoted to showing that the contributions to our sum arising from such terms cancel.

Given such elements E and U we fix $n >> 0$ such that all but the $n-1$ rightmost northern nodes of both E and U have propagating lines. We then number the northern nodes $n, n-1, \ldots$ from right to left, and the southern nodes $n+1, n+2, \ldots$ from right to left. Having fixed n thus, we will number the nodes in any diagram in the same manner. Given a diagram D, any line e in D can now be uniquely specified by declaring the nodes at its endpoints, as in $e = (a, b)$, with $a < b$.

Having fixed n (and hence the numbering of the nodes) in this way, suppose that the unique northern arc in U is labelled $(i, i+1)$. We will label this element U as U_i (note that this depends on our choice of numbering of the nodes). As we are assuming that $U_iE = 0$, we must have that E has propagating lines at node i and $i+1$. For the purposes of comparing diagrams, we may thus suppress all arcs in E, which shall play no role in what follows. That is, we may assume that E consists entirely of propagating lines (i.e. $r = 0$).

Write $U_i\theta^{rs}(E) = \sum_{D\in\Delta^s(\infty)} C_D D$. We wish to show that $C_D = 0$ for all D. Clearly any D for which $C_D \neq 0$ must contain the line $(i, i+1)$ (as this occurs in U_i). If we write the image of E under θ^{rs} in the form $\sum_{F\in\Delta^s(\infty)} h(F)F$ (using our assumption that E only contains propagating lines) then there are in general several diagrams F which can contribute to the coefficient C_D.

Let D^- be the diagram obtained from D by removing $e = (i, i+1)$, and let

J be the interval of the frame of D^- which was of the form $[i, i+1]$ in D. Further, let e' be any line exposed to the interval J in D^-. Then nipping e' at J produces a diagram which contributes to C_D. Note that if a diagram F does contribute to C_D then all lines in F which do not end either at i or $i+1$ remain unchanged in $U_iF = D$. Thus F differs from D only in that the end of the lines at i and $i+1$ in F are connected in D, and a northern arc is added at $(i, i+1)$. Thus the diagrams F which can contribute to C_D are precisely those obtained by nipping as described above, and D itself.

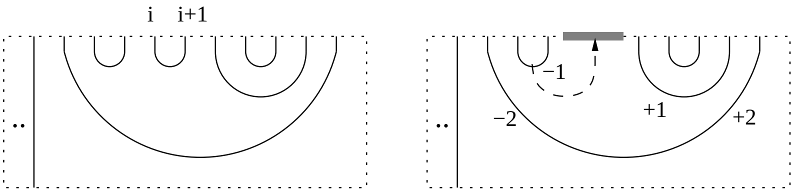


Fig. 11. (a) and (b)

As an example, consider the diagram D in Figure 11(a), with i and $i+1$ as indicated. In this case D^- is the diagram shown in Figure 11(b), and the lines that can be nipped at J are those numbered ± 1 and that numbered $-2/+2$. If we nip line -1 (as indicated in Figure 11(b)) we get the diagram shown in Figure 12(a), and clearly the product of U_i with this (as shown in Figure 12(b)) equals D.

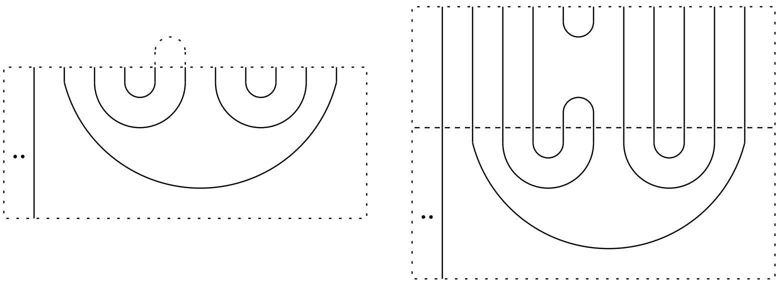


Fig. 12. (a) and (b)

It is convenient to number the lines e' in D^- exposed to J by $\pm j$ for the jth nearest exposed line to the right/left of J (as illustrated for the above example in Figure 11(b)). Note that at most one line can be given two such labels; such a line must consist of a northern arc whose nodes are on opposite sides of the interval J (such as the line labelled $-2/+2$ in Figure 11(b)). We now use this labelling to distinguish the various diagrams that can give rise to D. Given

$j \in \mathbb{Z}$, if there exists no line labelled by j, or such a line exists but there is some $m > j$ labelling the same line, then we set $D^j = 0$. Otherwise we let D^j denote the diagram obtained from D^- by nipping the corresponding line at J. (In our example $D^{-2} = 0$ and D^{-1} is the diagram in Figure 12(a).) With this notation we have

$$C_D = -[2]h(D) + \sum_j h(D^j) \tag{13}$$

where $h(0) = 0$.

In order to complete the proof of the theorem, we will need the following lemma. Suppose that the diagram D contains $e_1 = (a, b)$, $e_2 = (b + 1, c)$, and $(c + 1, c + 2)$, and that D' is the same, except for having $e = (a, c)$ and $m = (b, b+1)$ in place of e_1 and e_2. In other words, $D' = U_b D$. We will represent D and D' by the diagrams in Figures 13 (a) and (b) respectively (where the shaded areas represent collections of lines common to both diagrams, whose exact form will not concern us in what follows).

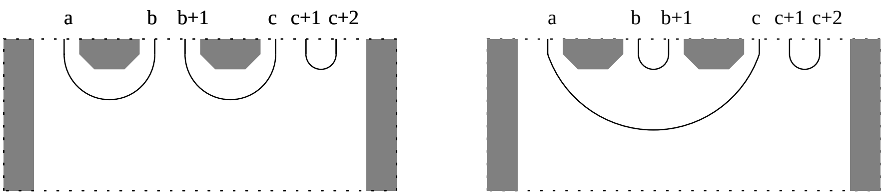


Fig. 13. (a) and (b)

Lemma 5.2 *Suppose that D and D' are as above, with $i = c + 1$. Then we have that $C_D = 0$ if and only if $C_{D'} = 0$.*

PROOF. We will show that $C_{D'} = 0$ implies $C_D = 0$ (the other case is similar). Given a line v (say), we will denote by the corresponding capital letter (in this case V) the integer satisfying $[V] = h(v)$. With this convention we have $h(e) = [E_1 + E_2]$. We will consider the expression for C_D given in (13), and the corresponding expression for $C_{D'}$ obtained by replacing each D by D' (which we will refer to as (13′)).

Every diagram X in the right-hand side of (13) except D^{-1} and D^{-2} corresponds to a diagram $\hat{X}$ in the right-hand side of (13′) which differs from X only in the replacement of the lines $(a,b),(b+1,c)$ by the lines $(a,c),(b,b+1)$ (in particular $\hat{D}=D'$). Consequently, for all such X we have

$$h(X)=\frac{[E_1+E_2]}{[E_1][E_2]}h(\hat{X}). \tag{14}$$

The only diagram in the right-hand side of (13′) not obtained under this correspondence is $(D')^{-1}$, and as $C_{D'}=0$ we have

$$h((D')^{-1})=[2]h(D')-\sum_{j\neq -1}h((D')^j). \tag{15}$$

Substituting (14) and (15) into (13) we see that

$$C_D=h(D^{-1})+h(D^{-2})-\frac{[E_1+E_2]}{[E_1][E_2]}h((D')^{-1}).$$

For convenience, the diagrams D^{-1}, D^{-2} and $(D')^{-1}$ are illustrated in Figures 14 (a), (b) and (c) respectively.

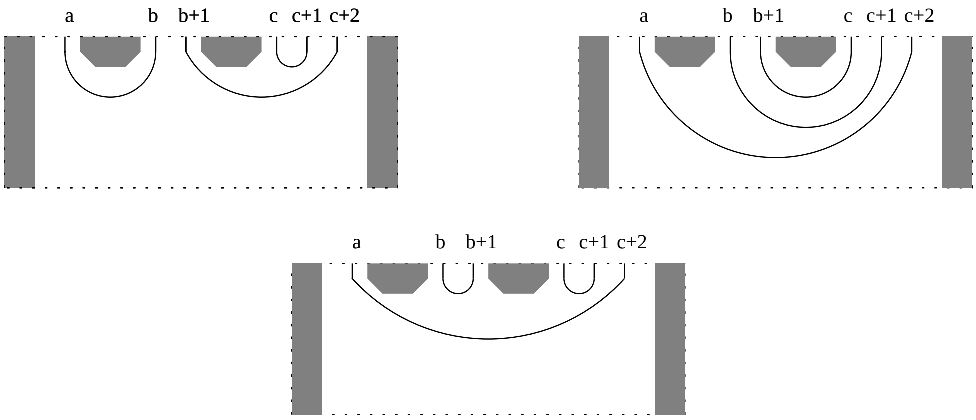


Fig. 14. (a), (b) and (c)

All shaded regions contribute equally to the respective hook products, and hence by considering the contributions to these products from the remaining lines we see that C_D is proportional to

$$\frac{1}{[E_1][E_2+1]}+\frac{1}{[E_2][E_2+1][E_1+E_2+1]}-\frac{[E_1+E_2]}{[E_1][E_2][E_1+E_2+1]}.$$

We may regard this expression (and similar ones later in the paper) as making sense even when some of the terms in the denominators are zero, by recalling that *all the hook products are elements of* $\mathcal{A}$, and hence that we may alter the constant of proportionality by a common factor which cancels any zeroes that occur. (This factor will be implicit in any such equation, and ignored in the discussions of them.) Thus it is enough to show that the numerator of

$$\frac{[E_2][E_1+E_2+1]+[E_1]-[E_2+1][E_1+E_2]}{[E_1][E_2][E_2+1][E_1+E_2+1]}$$

is zero, which is an easy exercise using the definition of Gaussian coefficients.

We return to the proof of Theorem 5.1. Recall that the value of $h(D)$ does not change if we deform D by moving southern points ambient isotopically anticlockwise round the frame to the northern side. Thus we may assume that D is of the form shown in Figure 15. Here we have omitted all lines which are not exposed to e from the diagram, as their effect will be accounted for in the labels. Each of the lines shown (after e is removed) may be nipped to produce a diagram contributing to C_D in $U_i\theta^{rs}(E)$.

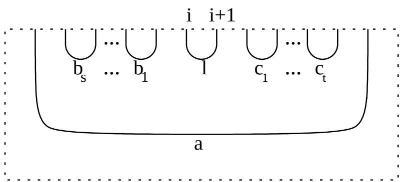


Fig. 15.

There are four kinds of contribution. Those from D itself, from nipping lines b_j, from nipping lines c_j, and those from nipping a. By repeated applications of Lemma 5.2 (and its analogue with $(i,i+1)$ on the other side of e_1 and e_2) we may assume that s and t are each at most one.

Let $h(b_1)=[B]$ and $h(c_1)=[C]$ (where these are taken to be 1 if b_1 or c_1 does not exist). Then $h(a)=[B+C+2]$ and the coefficient of D is proportional to

$$\begin{aligned}\frac{-[2]}{[B][C][B+C+2]}&+\frac{1}{[B+1][C+1][B][C]}\\&+\frac{1}{[B+1][C][B+C+2]}+\frac{1}{[B][C+1][B+C+2]}\\&=\frac{-[2][B+1][C+1]+[B+C+2]+[B][C+1]+[C][B+1]}{[B][C][B+1][C+1][B+C+2]}=0.\end{aligned}$$

as required. (Recall our convention introduced in the proof of Lemma 5.2 concerning the interpretation of such fractions.)

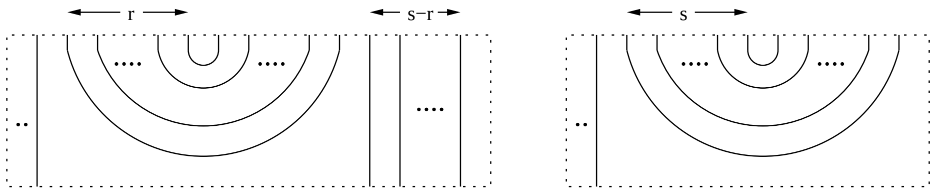


Fig. 16. (a) and (b)

We have shown that θ^{rs} is a homomorphism; to see that it is non-zero it is enough to consider the diagram E consisting of r nested northern arcs to the left of precisely $s-r$ propagating lines (as illustrated in Figure 16(a)). Consider the coefficient in $\theta^{rs}(E)$ of the diagram containing s nested northern arcs at the righthand end (illustrated in Figure 16(b)). It is easy to verify that this coefficient is 1, as the only diagram D in (11) contributing to this term is the diagram with $s-r$ nested northern loops at the righthand end, and this satisfies $h(D)=1$ as required.

For the final part of Theorem 5.1 suppose that $r=0$, and that we have a morphism θ from $\Delta^0(\infty)$ to $\Delta^t(\infty)$. We must show that this is a scalar multiple of θ^{0t}. As we wish to consider all diagrams in $\Delta^t(\infty)$, and hence cannot fix an $n >> 0$ beyond which all lines propagate, we will now label the northern nodes of diagrams in *increasing order from right to left*, starting at 1. As usual we will denote by U_i the image in $\mathrm{TL}_A(\infty)$ of some generator U_j of some $\mathrm{TL}_A(n)$ whose northern arc connects i and $i+1$.

With this convention we can now associate diagrams in our usual basis for $\Delta^t(\infty)$ with subsets $I \subset \mathbb{N}$ of size t, by associating a diagram to the set of labels of those nodes lying at the righthand end of an upper arc. To complete the proof of Theorem 5.1 it is enough to show that θ is determined by the coefficient of the diagram corresponding to $\{1,2,\ldots,t\}$.

We partially order the t-element subsets of $\mathbb{N}$ by setting $I \leq J$ if $i_m \leq j_m$ for $1 \leq m \leq t$. Here $I=\{i_1,\ldots,i_t\}$ and $J=\{j_1,\ldots,j_t\}$ are arranged so that the entries are in increasing order. Let J be such a subset, and suppose that

$J \neq \{1, \ldots, t\}$. Then there exists a unique minimal element $1 < i \in J$ such that $i - 1 \notin J$. Let K be the set obtained from J by replacing i by $i - 1$. Now the coefficient of K in $U_{i-1}L$ is non-zero only if $L = J$ or $L \leq K < J$ (as nipping always moves some $k \in K$ to the right). Inductively we know the coefficients of all $L < J$ in $\mathrm{Im}\,\theta$, and hence (as $U_{i-1}\mathrm{Im}\,\theta = 0$) we can determine the coefficient of J. This completes the proof of Theorem 5.1.

By the second part of the last Theorem, we see that each standard module $\Delta^u(\infty)$ contains a unique one-dimensional submodule annihilated by all the U_i. We fix a basis v^u for this submodule. Further, each morphism θ^{rs} acts on a diagram E by $E \longmapsto \lambda E v^{r-s}$ for some (fixed) scalar λ, where v^{r-s} is our fixed basis vector in $\Delta^{r-s}(\infty)$. It is now routine to verify that, given $\mathrm{TL}_A(\infty)$-homomorphisms

$$\begin{array}{ccc} \Delta^r(\infty) & \xrightarrow{\theta^{rs}} & \Delta^s(\infty) \\ & \searrow^{\theta^{rt}} & \downarrow{\scriptstyle \theta^{st}} \\ & & \Delta^t(\infty) \end{array}$$

we must have $\theta^{st}\theta^{rs} = a_{rst}\theta^{rt}$ for some $a_{rst} \in k$. Indeed, over $\mathcal{A}$, it is easy to compute a_{rst}. By considering the coefficient of the diagram with t nested northern arcs to the right of the first propagating line we see that

$$a_{rst} = \begin{bmatrix} t-r \\ t-s \end{bmatrix} = \begin{bmatrix} t-r \\ s-r \end{bmatrix} \qquad \text{where} \quad \begin{bmatrix} a \\ b \end{bmatrix} = \frac{[a]!}{[b]![a-b]!}. \tag{16}$$

Let $\theta^{rs}(n)$ be the linear map obtained by restricting θ^{rs} to $\Delta^r(n)$ and then projecting the image onto $\Delta^s(n)$. We shall refer to $\theta^{rs}(n)$ as the map *engendered* by θ^{rs}. Clearly the $\theta^{rs}(n)$ inherit the composition rules given above.

The importance of the θ^{rs} is that they engender all morphisms between finite standard modules that we shall require.

Theorem 5.3 *The map θ^{rs} engenders a non-zero $\mathrm{TL}_A(n)$-homomorphism $\theta^{rs}(n) : \Delta^r(n) \to \Delta^s(n)$ if and only if $n \equiv r + s - 1$ modulo lp^j for some j, with $0 \leq s - r < lp^j$.*

PROOF. By definition, as a linear map $\theta^{rs}(n)$ is of the desired form. So it just remains to verify that this is a $\mathrm{TL}_A(n)$-homomorphism. The only case in which the proof given for $\mathrm{TL}_A(\infty)$ above fails here is when (in the notation above) it is not possible to deform the finite diagram corresponding to (and also denoted by) D so that e is contained inside an arc a (as in Figure 15). This can only occur when D has no propagating lines to the left of $e = (i, i+1)$.

Here the discrepancy from vanishing in the coefficient of D in $U_i\theta^{rs}(n)(E)$ may be computed by thinking of it as (minus) the coefficient of the missing diagram (the diagram which would have contributed and cancelled the others in the global case). However, since this diagram, D' say, is that obtained from D by adding the requisite propagating line to the left of e, removing e, then nipping the propagating line in the interval vacated by e, the diagram has a total of $n-r-s+1$ lines and the coefficient is of the form

$$\frac{[n-r-s+1]!}{\prod_{f\in D'} h(f)}. \tag{17}$$

Thus $\theta^{rs}(n)$ will be a $\mathrm{TL}_A(n)$-homomorphism if and only if this expression is zero for all possible D' arising above.

Clearly for any $m = ab$ we have $x^m - x^{-m} = (x^a - x^{-a})(x^{(b-1)a} + x^{(b-3)a} + \ldots + x^{(1-b)a})$. From this it is easy to deduce that for any $u \geq v$ and $0 < c, d < p$ we have

$$\frac{[clp^u]}{[dlp^v]} = \begin{cases} 0 & \text{if } u > v \\ (-1)^{c-d}\left(\frac{c}{d}\right) & \text{if } u = v. \end{cases} \tag{18}$$

In general $[n-r-s+1]!$ contains a number of factors of the form $[hlp^u]$ with $0 < h < p$; factors which vanish in k. The value of (17) will be non-zero precisely when such factors can all be cancelled by factors in the denominator. If $n-r-s+1 \not\equiv 0 \pmod{l}$ then it is easy to construct a D giving rise to a D' with this property, and hence it only remains to consider the case when $n-r-s+1 = mlp^j$ for some j, with $0 < m < p$.

It will be evident that the greatest number of factors of the form $[hlp^u]$ in the denominator of (17) occurs when D' is a diagram with just two sets of nested northern arcs — one ending at i, one beginning at $i+1$. By (18), the critical case is when the position of i partitions the lines between the two factorials so that the greatest possible number of factors of form $[hlp^j]$ appear, as illustrated in Figure 17.

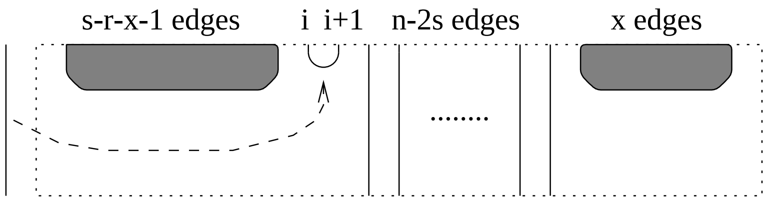


Fig. 17.

Here the region in the frame represents D, x is the excess of $s-r$ over lp^j, if any, and the missing diagram D' is that obtained by removing $e=(i,i+1)$ and nipping the additional propagating line as shown. In this case we have

$$h(D') = \begin{cases} \frac{[mlp^j]!}{[s-r]![n-2s+1]!} & \text{if } s-r < lp^j \\ \frac{[mlp^j]!}{[lp^j]![(m-1)lp^j]!} & \text{if } s-r \geq lp^j. \end{cases}$$

Thus the coefficient vanishes if $s-r<lp^j$, and not otherwise. This completes the proof of Theorem 5.3.

Henceforth, it will be more convenient to denote the modules $\Delta^r(n)$ by $\Delta_t(n)$, where $t=n-2r$, the number of propagating lines. This notation could not be introduced earlier, as it makes no sense in the context of $\mathrm{TL}_A(\infty)$-modules. In a similar manner, we shall denote the morphism $\theta^{rs}(n)$ by $\theta_{tu}(n)$, where $t=n-2r$ and $u=n-2s$. It is clear (for example by drawing the diagram of $\prod_{i=1}^r U_{2i-1}$) that

$$\Delta_t(n) \cong \frac{\mathrm{TL}_A(n)\left(\prod_{i=1}^r U_{2i-1}\right)}{\left(\mathrm{TL}_A(n)\left(\prod_{i=1}^r U_{2i-1}\right)\right) \cap \left(\mathrm{TL}_A(n)(\prod_{i=1}^{r+1} U_{2i-1})\mathrm{TL}_A(n)\right)}$$

and the latter module can be identified with the Specht module S^λ where $\lambda \in \Lambda^+(2,n)$ is given by $\lambda_1 - \lambda_2 = t$ (see [15, Chapters 6 and 9]). We denote the simple head of this by $L_t(n)$, and note that all simple $\mathrm{TL}_A(n)$-modules arise in this manner.

Remark 5.4 *By Corollary 4.3, it is clear that Theorem 5.3 constructs (up to scalars) all non-zero homomorphisms between standard modules.*

By Theorem 5.3, for fixed t we obtain for each $i \geq 0$ a non-zero $\mathrm{TL}_A(n)$-homomorphism of the form $\theta_{tt'}(n)$ from $\Delta_t(n)$ to $\Delta_{t'}(n)$, by reflection around the nearest lp^i wall below t (provided that a suitable element t' exists). We shall denote such a map by $\theta_t^i(n)$ and set

$$\Xi_t^i(n) = \begin{cases} \Delta_t(n)/\mathrm{Ker}\,\theta_t^i(n) & \text{if there exists such a } \theta_t^i(n) \\ L_t(n) & \text{otherwise.} \end{cases}$$

It will occasionally be convenient to denote $\Delta_t(n)$ by $\Xi_t^{-1}(n)$, and the identity morphism by θ_t^{-1}.

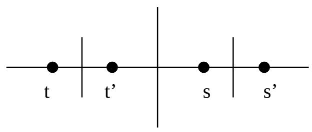


Fig. 18.

Our next aim is to construct certain homomorphisms between $\Xi_t^i(n)$ and $\Xi_s^i(n)$, for suitable s and t. For simplicity, we shall often suppress the argument n on morphisms where it is clear from context. Fix t and let t' (respectively s') be its reflection about the nearest lp^i-wall (respectively lp^{i+1}-wall) below t. Then s is chosen so that it is the reflection of s' about the nearest lp^i-wall above s'. (An example of such a quartet of weights is given in Figure 18.) By considering the maps engendered by our $\mathrm{TL}_A(\infty)$ morphisms on the various $\Delta(n)$'s, and verifying that $a_{tss'} = a_{tt's'} = 1$ using (16), we obtain the following commutative diagram of k-linear maps of $\mathrm{TL}_A(n)$-modules:

$$\begin{array}{ccc} \Delta_t(n) & \xrightarrow{\theta_{ts}(n)} & \Delta_s(n) \\ {\scriptstyle \theta_t^i(n)}\downarrow & \searrow^{\theta_t^{i+1}(n)} & \downarrow{\scriptstyle \theta_s^i(n)} \\ \Delta_{t'}(n) & \xrightarrow[\theta_{t's'}(n)]{} & \Delta_{s'}(n). \end{array} \tag{19}$$

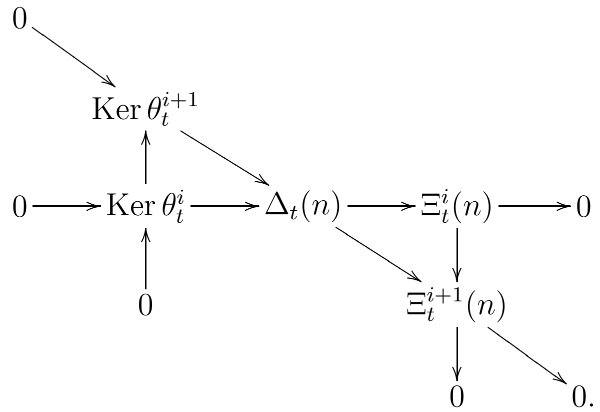
Here the horizontal maps will not in general be $\mathrm{TL}_A(n)$-homomorphisms. However, we have

$$\theta_t^{i+1}(\Delta_t(n)) = \theta_s^i(\theta_{ts}(\Delta_t(n))) \leq \theta_s^i(\Delta_s(n))$$

and hence we induce a $\mathrm{TL}_A(n)$-homomorphism from $\Delta_t(n)$ to $\Delta_s(n)/\mathrm{Ker}\,\theta_s^i(n)$. Further, if for $x \in \Delta_t(n)$ we have $\theta_t^i(x) = 0$, then

$$\theta_t^{i+1}(x) = \theta_{t's'}(\theta_t^i(x)) = 0$$

and hence $\theta_t^{i+1}(\mathrm{Ker}\,\theta_t^i) = 0$, and we have a commutative diagram



(20)

Thus we see from (19) that $\theta_{ts}(\mathrm{Ker}\,\theta_t^i) \leq \mathrm{Ker}\,\theta_s^i$, and so we obtain a non-zero $\mathrm{TL}_A(n)$-homomorphism

$$\theta_{ts}^i(n) : \Xi_t^i(n) \longrightarrow \Xi_s^i(n).$$

We shall call $L_v(n)$ a *combinatorial composition factor* (ccf) of $\Xi_t^i(n)$ if $L_v(n)$ is a composition factor of $\Delta_t(n)$, and $v \equiv t \pmod{2lp^i}$. Unsurprisingly, our main result about combinatorial composition factors is

Proposition 5.5 *The module $L_v(n)$ is a combinatorial composition factor of $\Xi_s^i(n)$ if and only if it is a composition factor of $\Xi_s^i(n)$.*

PROOF. We begin by showing that every ccf of $\Xi_s^i(n)$ is a composition factor of $\Xi_s^i(n)$. First note that for fixed s the result is clear for $i >> 0$, as then $\Xi_s^i(n) = L_s(n)$. Thus we shall proceed by descending induction on i.

Let s and t be as in the definition of $\theta_{ts}^i(n)$ (see Figure 18). If $L_v(n)$ is a ccf of $\Xi_s^i(n)$ then Proposition 4.5 implies that v is obtained from s by a series of reflections about lp^j-walls. Therefore either $v \equiv s \pmod{2lp^{i+1}}$ or $v \equiv t$

(mod $2lp^{i+1}$), as these are the only two equivalence classes of weights in $\Delta_s(n)$ which are congruent to s mod $2lp^i$. In the first case we immediately see that $L_v(n)$ is a ccf of $\Xi_s^{i+1}(n)$, but in the second we must also show that $L_v(n)$ is a composition factor of $\Delta_t(n)$ before we can deduce that it is a ccf of $\Xi_t^{i+1}(n)$.

Consider the second case: writing

$$v + 1 = lp^m(b+1) + \sum_{j=-1}^{m-1} a_j lp^j \tag{21}$$

as in (5), and recalling that $L_v(n)$ is a composition factor of $\Delta_s(n)$, we have that

$$s + 1 = lp^m(b+1) + \sum_{j=-1}^{m-1} J(j) a_j lp^j \tag{22}$$

for suitable $J(j) = \pm 1$ as in (6). Further, as $v \equiv t \pmod{2lp^i}$, we must have $J(j) = +1$ for $j < i$ and $J(i) = -1$. Now setting $J'(j) = \begin{cases} J(j) & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases}$ an easy calculation shows that

$$t + 1 = lp^m(b+1) + \sum_{j=-1}^{m-1} J'(j) a_j lp^j$$

and hence that $L_v(n)$ is a composition factor of $\Delta_t(n)$ (by the remarks after (6)). Thus we have shown that if $L_v(n)$ is a ccf of $\Xi_s^i(n)$ it must be a ccf of either $\Xi_s^{i+1}(n)$ or $\Xi_t^{i+1}(n)$. By induction we deduce that it is a composition factor of one of these two modules.

We have a commutative diagram

$$\begin{array}{ccc} \Delta_t(n) & \xrightarrow{\theta_{ts}(n)} & \Delta_s(n) \\ & \searrow^{\theta_t^{i+1}(n)} & \downarrow{\scriptstyle \theta_s^i(n)} \\ & & \Delta_{s'}(n). \end{array}$$

Note that the horizontal map is just a k-linear map, and that not all the modules are necessarily distinct. We deduce that $\Xi_t^{i+1}(n) \cong \operatorname{Im} \theta_t^{i+1}$ is a submodule of $\operatorname{Im} \theta_s^i$ (and hence of $\Xi_s^i(n)$), while from (20) we have that $\Xi_s^{i+1}(n)$ is a quotient of $\Xi_s^i(n)$. As $L_v(n)$ is a composition factor of $\Xi_s^{i+1}(n)$ or $\Xi_t^{i+1}(n)$ we deduce that it is also a composition factor of $\Xi_s^i(n)$ as required.

It remains to show that any composition factor of $\Xi^i_s(n)$ is a ccf of $\Xi^i_s(n)$. We proceed by induction on i. Given a composition factor $L_v(n)$ of $\Xi^i_s(n)$, either it must be a composition factor of both $\Delta_s(n)$ and $\Delta_{s'}(n)$, where s' is obtained from s by reflection about the nearest lp^i-wall below s, or if no such s' exists then $v = s$. Thus we may assume that s' exists.

As $L_v(n)$ is a composition factor of Ξ^{i-1}_s (by (20)) we know by induction that $v \equiv s \pmod{2lp^{i-1}}$. (For the base case $i = 0$ this is clear.) Writing v as in (21) and s as in (22) we deduce that $J(j) = +1$ for $j < i-1$. It remains to show that $J(i-1) = +1$. If $a_{i-1} = 0$ then we may choose $J(i-1)$ freely, so we may assume that $a_{i-1} \neq 0$.

Suppose that $J(i-1) = -1$. Now $L_v(n)$ is a composition factor of $\Delta_{s'}(n)$, and hence we have

$$s' + 1 = lp^m(b+1) + \sum_{j=-1}^{m-1} J'(j)a_j lp^j$$

for some $J'(j) = \pm 1$. But

$$s' + 1 = s + 1 - 2\sum_{j=1}^{i-2} J(j)a_j lp^j - 2a_{i-1}lp^{i-1}$$

which implies that

$$\sum_{j=1}^{m-1}(J(j) - J'(j))a_j lp^j = 2\sum_{j=1}^{i-1} J(j)a_j lp^j + 2a_{i-1}lp^{i-1}.$$

We deduce that $J'(j)a_j = -J(j)a_j$ for $j \leq i-2$ and

$$(-1 - J'(i-1))a_{i-1}lp^{i-1} \equiv 2a_{i-1}lp^{i-1} \pmod{2lp^i}.$$

But this latter equivalence is impossible, and so we are done.

The importance of the $\Xi^i_t(n)$ is clear from

Corollary 5.6 *Let $i \geq -1$ and suppose that $0 \leq s' \leq s < t' \leq t \leq n$ are such that we have the commutative diagram (19). Then there is a short exact sequence*

$$0 \to \Xi^{i+1}_t(n) \to \Xi^i_s(n) \to \Xi^{i+1}_s(n) \to 0.$$

If s is such that there does not exist a quadruple (s', s, t', t) satisfying the conditions above then we have $\Xi_s^i(n) = \Xi_s^{i+1}(n)$.

PROOF. First suppose that no such quadruple exists for a given s. Then in the quadruple arising from (19), either $s' = t'$, $s' < 0$ or $t > n$. In the first two cases we have $\Xi_s^i(n) = \Xi_s^{i+1}(n)$ by definition, while in the third the result follows from Proposition 5.5 and Proposition 4.5.

Now suppose that we do have such a quadruple of integers. As noted in Proposition 5.5 $\Xi_t^{i+1}(n)$ (respectively $\Xi_s^{i+1}(n)$) is a sub- (respectively quotient) module of $\Xi_s^i(n)$, and every composition factor of $\Xi_s^i(n)$ occurs in $\Xi_s^{i+1}(n)$ or $\Xi_t^{i+1}(n)$. The result now follows as these latter two modules clearly have no common composition factors (by the combinatorial composition factor condition), and $\Xi_s^i(n)$ has no composition factor repeated (being a quotient of such a module by Theorem 4.4).

By repeated application of this result, we obtain for each i a filtration of any given standard module by $\Xi_t^i(n)$'s, and hence (for i large enough) a composition series of our standard module.

To illustrate these results we return to the example considered in Figure 7. In this case $\Xi_a^0(n)$ has composition factors labelled by a, d, e, and f, while $\Xi_a^1(n)$ has composition factors labelled by a and d and $\Xi_1^2(n)$ is the simple module labelled by a. It is easy to verify that $\Xi_5^0(n)$ has composition factors labelled by b and c. There is a short exact sequence

$$0 \to \Xi_c^0(n) \to \Delta_a(n) \to \Xi_a^0(n) \to 0$$

and similar sequences for the other Ξ's.

6 Standard module morphisms for $\mathbf{TL}_B$

In this section we will construct morphisms between standard modules for $\mathrm{TL}_B(n)$, which will be our main tool in determining the composition factors of these standard modules. These morphisms were given in [2, (6.1) and (9.1) Theorem] over a field of characteristic zero. From our explicit description we shall see that these maps can also be constructed in positive characteristic. The proof of this will be similar to that for $\mathrm{TL}_A(n)$ standard module morphisms in the previous section.

For D a diagram in our standard basis of $W_t(n)$, we number the northern nodes from 1 to n, left to right, and the southern nodes $n+1$ to $n+t$ from right to left. Now a line e in D can be uniquely specified by declaring the nodes at its endpoints, as in $e = (a, a+2b+1)$, with $b > 0$. We assign to each line e in D an element $h(e)$ of $\mathcal{A}$ by setting

$$h(e) = \begin{cases} [b+1] & \text{if } e \text{ is undecorated} \\ [(a+2b+1)/2]\,[(n+t-a+1)/2] & \text{otherwise.} \end{cases}$$

Note that for undecorated lines e, this agrees with the value of $h(e)$ from the previous section. We also define

$$h'(e) = \begin{cases} -h(e) & \text{if } e \text{ is undecorated} \\ h(e) & \text{otherwise.} \end{cases}$$

We now define the *hook products* $h(D)$ and $h'(D)$ by setting

$$h(D) = \frac{[\frac{n+t}{2}]![\frac{n-t}{2}]!}{\prod_{e\in D} h(e)} \qquad \text{and} \qquad h'(D) = \frac{[\frac{n+t}{2}]![\frac{n-t}{2}]!}{\prod_{e\in D} h'(e)}.$$

(Note that $\frac{n+t}{2}$ and $\frac{n-t}{2}$ are the total number of lines and northern arcs respectively in D.)

Lemma 6.1 *We have $h(D), h'(D) \in \mathcal{A}$.*

PROOF. We show that $h(d) \in \mathcal{A}$, the other case is similar. Recall that a partially ordered set is called a *forest* [33] if, whenever $x \leq y$ and $x \leq z$, we have $y \leq z$ or $z \leq y$. We shall begin by associating a pair of forests to our given diagram D, corresponding to two related diagrams D' and D''.

We set D' to be the diagram obtained from D by replacing each decorated arc $e = (a, b)$ by an undecorated propagating line e' starting at a, with the node b deleted. We also set D'' to be the diagram on $n - t$ nodes obtained from D by taking the first $\frac{n-t}{2}$ lines (ordered by lowest numbered node), and replacing each decorated line $e = (a, b)$ with an undecorated line e'' containing all arcs to the west of b. If e is not a decorated arc, we denote the corresponding lines in D' and D'' by e' and e'' respectively (where the latter is taken to be zero if the corresponding line does not exist).

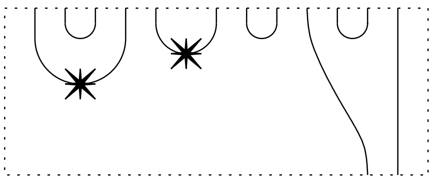


Fig. 19.

For example, if D is the diagram in Figure 19 then D' and D'' are given by the diagrams in Figures 20(a) and 20(b) respectively.

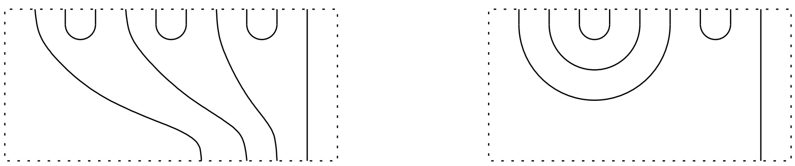


Fig. 20. (a) and (b)

Note that for all e we have that $h(e)$ divides $h(e')h(e'')$ (where we define $h(e'') = 1$ if e'' is zero), and that the (sets of lines in the) diagrams D' and D'' are both forests via the partial order inducing the h function (i.e. $e_1 \leq e_2$ if e_1 is to the right of e_2 after the diagram is deformed to make e_2 propagating). Clearly, it is enough to show that

$$\frac{[\frac{n+t}{2}]!}{\prod_{e \in E(D')} h(e)} \frac{[\frac{n-t}{2}]!}{\prod_{e \in E(D'')} h(e)} \in \mathcal{A}$$

but this is immediate from [8, (3.3) Proposition].

We shall again use nipping of diagrams to analyse various cases in the following theorem, just, as in the previous section. When nipping a decorated line, either one or three possible nippings can occur, as we may choose to decorate either (or both) of the new lines obtained by nipping (and must decorate at least one) provided that they are allowed to be decorated (i.e. are exposed to the western edge of the frame). Note that in the following Theorem we will drop our standing assumption that q is necessarily a root of unity.

Theorem 6.2 *Let $t = m+u \geq 0$ with $0 \leq u \leq m$ and either $m \equiv y \pmod{l}$ or $m = y$ if q is not a root of unity. Then there exists a non-zero homomorphism of $TL_B(n)$-modules $\phi : W_t(n) \to W_{t-2u}(n)$ given on diagrams by*

$$E \mapsto \sum_{D \in W_{t-2u}(t)} h(D)ED$$

if $m \equiv y \pmod{2l}$, and by

$$E \mapsto \sum_{D \in W_{t-2u}(t)} h'(D)ED$$

otherwise, where the sums run over those diagrams in our basis of $W_{t-2u}(t)$. When $t = n$ this is the unique (up to scalars) such morphism of standard modules.

PROOF. Note that if $m \equiv y \pmod{l}$ then we either have $[m] = [y]$ and $[m+1] = [y+1]$ (if $m \equiv y \pmod{2l}$), or $[m] = -[y]$ and $[m+1] = -[y+1]$ (otherwise).

Consider the case when q is a primitive lth root of unity and $m \equiv y \pmod{2l}$. (The other root of unity case is very similar, and left to the reader.) By considering the defining relations (2) we see that the algebras $\mathrm{TL}_B^y(n)$ and $\mathrm{TL}_B^m(n)$ may be identified. Thus we may assume that $m = y$.

We proceed much as in the proof of Theorems 5.1 and 5.3. As there we may suppress all arcs arising in E. We first verify that multiplication by U_0 annihilates $\phi(E)$. Terms in $\phi(E)$ in which the node 1 is connected to the south side are killed by U_0, while the remaining terms appear in pairs D, D', identical except for the decoration of the line from 1 in D'. Let $e = (1, 2a)$ be the line in question. The contribution to $h(D)$ from this line is effectively $[a]^{-1}$, and to $h(D')$ is $([a][y])^{-1}$. Thus the combined contribution to the coefficient of D' in $U_0\phi(E)$ is proportional to

$$\frac{1}{[a]} + \frac{-[y]}{[a][y]} = 0.$$

(Recall the convention by which we interpret such fractions introduced in the proof of Lemma 5.2.)

It remains to show that multiplication by U_i annihilates $\phi(E)$ for all $i > 0$. If we write $U_i\phi(E) = \sum_{D \in W_{t-2u}(t)} C_D D$ then we must show that $C_D = 0$ for all D. Clearly, any D for which $C_D \neq 0$ must contain an undecorated line $e = (i, i+1)$ (as this occurs in U_i). Let D be such a diagram and C_D be its coefficient in $U_i\phi(E)$. As for $\mathrm{TL}_A(n)$, in general several diagrams will contribute to C_D. These are just those diagrams obtained by nipping D^- (the diagram with two fewer northern nodes obtained from D by removing $e = (i, i+1)$), D itself and (if it exists) the diagram D^* obtained from D by decorating e.

Numbering the lines in D^- as in the $\mathrm{TL}_A(n)$ case, we let D^j denote the (sum of) the diagram(s) obtained from D^- by nipping the corresponding line at J, where J is the interval of the frame of D^- which was of the form $[i, i+1]$ in D. (As in the $\mathrm{TL}_A(n)$ case, we set $D^j = 0$ if there is no line labelled by j, or there exists some $x > j$ labelling the same line.) Note that D^j can consist of more than one diagram for at most one $j \in \mathbb{Z}$ (i.e when D can be pictured as in Figure 21(a) — where the shaded areas denote any suitable array of lines — and j is the decorated line). For such a j the three diagrams obtained by nipping the decorated line are of the form shown in Figure 21(b) where one or both of e_1 and e_2 are decorated. We will denote these by D^j_l, D^j_r and D^j_b where respectively the left line, the right line or both are decorated. We then have

$$C_D = \begin{cases} -[2]h(D) + \sum_f h(D^f) + [y+1]h(D^*) & \text{if } D^* \text{ exists} \\ -[2]h(D) + \sum_f h(D^f) & \text{otherwise.} \end{cases} \tag{23}$$

where if j is such that D^j consists of three diagrams we define $h(D^j)$ by $h(D^j) = h(D^j_l) + h(D^j_r) - [y]h(D^j_b)$.

We begin by considering the case where one of the D^j above consists of three diagrams. Let D' be the diagram obtained from D by removing the decoration from line j. We claim that $C_D = 0$ if $C_{D'} = 0$. (We will show that $C'_D = 0$ shortly.)

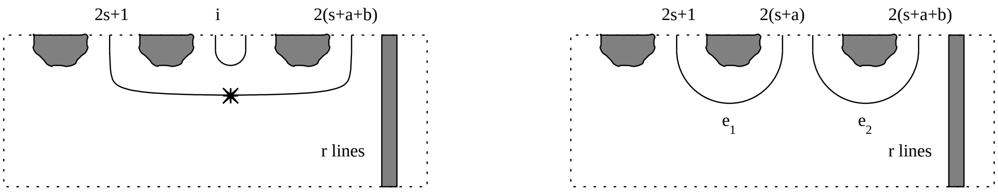


Fig. 21. (a) and (b)

To show this, we may assume that D is given by the diagram in Figure 21(a) where j is the decorated line (as noted above), and that D' is the same diagram without the decoration. The three diagrams obtained by nipping the decorated line are of the form shown in Figure 21(b) where one or both of e_1 and e_2 are decorated. The diagram obtained by nipping the corresponding line in D' is also of this form, but with neither e_1 nor e_2 decorated.

Arguing as in Lemma 5.2 we will compare the expression for C_D given in (23) with the corresponding expression for $C_{D'}$ obtained by replacing each D by D' (which we shall refer to as (23′)). Each diagram X in the right-hand side of (23) except those occuring in D^j corresponds to a diagram $\hat{X}$ in the right-hand side of (23′) which differs from X only in the removal of the decoration from line j (in particular $\hat{D} = D'$ and $(\hat{D^*}) = (D^*)'$). Consequently, for all such X we have

$$h(X) = \frac{[a+b]}{[s+a+b][a+b+r]} h(\hat{X}). \tag{24}$$

The only diagram in the right-hand side of (23′) not obtained under this correspondence is $(D')^j$, and as $C_{D'} = 0$ we have

$$h((D')^j) = [2]h(D') - \sum_{l \neq j} h((D')^j) - [y+1]h((D')^*). \tag{25}$$

Substituting (24) and (25) into (23) and using the definition of $h(D^j)$ we see that

$$C_D = h(D^j_l) + h(D^j_r) - [y]h(D^j_b) - \frac{[a+b]}{[s+a+b][a+b+r]} h((D')^j).$$

By considering the contributions to these hook products of lines not common to both D^j and $(D')^j$ (and noting that $n+t = 2(s+a+b+r)$) we see that C_D is proportional to

$$\frac{1}{[r+a+b][a+s][b]} + \frac{1}{[s+a+b][b+r][a]} - \frac{[y]}{[s+a+b][b+r][a+s][a+b+r]}$$
$$-\frac{[a+b]}{[s+a+b][a+b+r][b][a]}$$

Simplifying, using that $y = a+b+r+s$, this reduces to showing that

$$[s+a+b][a][b+r]+[a+b+r][b][s+a]-[a+b+s+r][a][b]-[a+b][s+a][b+r] = 0$$

an easily verified q-integer identity, and so the claim follows.

The analogue of Lemma 5.2 also holds in this case (with the same proof) provided we require that e_1 and e_2 are undecorated. With this, the proof proceeds just as in the $\mathrm{TL}_A(n)$ case if $e = (i, i+1)$ is not exposed to the west wall, so we may assume that e is so exposed.

Let D be a diagram with a decorated line exposed to e. By our assumption, and after ambiently deforming D, we may assume that D has no propagating lines. We will also assume that our decorated line is to the left of e — the other case is similar.

Suppose there is more than one such line to the left of e, and let f be the decorated line furthest from e to the left. Let D' be the diagram identical to D except for f being undecorated (and denote this undecorated line by $\bar{f}$). Note that the diagram obtained after removing e and nipping f (respectively $\bar{f}$) at the interval where e was does not contribute to C_D (respectively to $C_{D'}$). Thus $C_D = \frac{h(\bar{f})}{h(f)} C_{D'}$. As $h(f)$ is non-zero in $\mathcal{A}$, we infer that $C_D = 0$ in $\mathcal{A}$ if $C_{D'} = 0$ (in $\mathcal{A}$). We wish to show that this remains true in the specialisation that we are considering. However $h(f)$, being a product of q-integers, is polynomial in q and q^{-1}. Therefore the implication holds on an open subset of the choice of q parameter and hence — as $C_D \in \mathcal{A}$ — everywhere.

Now suppose that f is the only decorated line to the left of e. Repeating the arguments of the preceding paragraph (as none of the arcs to the left of f can be nipped at the interval formerly occupied by e) we see that $C_D = 0$ if and only if $C_{D_s} = 0$, where D_s is obtained from D by replacing all exposed arcs to the left of f by a single arc g (say). Thus D_s is of the form shown in Figure 22.

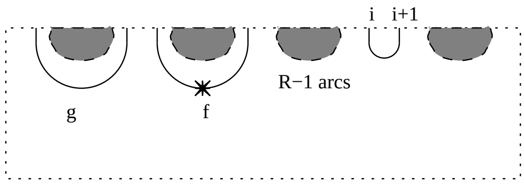


Fig. 22.

Here the exact structure of the areas in grey will not play a role in what follows. We claim that $C_{D_s} = 0$ if and only if $C_{D'} = 0$, where D' is obtained from D_s by removing the decoration on f. Once again, the argument is similar to that used in the proof of Lemma 5.2. We may assume that $h(g) = [G]$, and $h(f) = [F]$, and shall consider the case where $C_{D'} = 0$. As before we compare diagrams contributing to the coefficients C_{D_s} and $C_{D'}$: every diagram X contributing to C_{D_s} apart from that obtained by nipping f corresponds to a diagram $\hat{X}$ which differs from X only in the removal of the decoration from f. Consequently for all such X we have

$$h(X) = \frac{[F]}{[F+G][L-G]} h(\hat{X})$$

where L is the total number of lines in a diagram. The only diagrams contributing to $C_{D'}$ that do not arise under this correspondence are those obtained by nipping f or g. As before, by considering the different contributions to C_{D_s} and $C_{D'}$ obtained by nipping the remaining lines, we see that C_D is proportional to

$$\frac{1}{[R][F+G+R][G][L-G]} - \frac{[F]}{[F+G][L-G]} \left(\frac{1}{[G][R]} + \frac{1}{[F][F+G+R]} \right)$$

which equals zero by an easy q-integer identity.

After repeating the above argument for decorated arcs to the right of e, it only remains to show that $C_D = 0$ when D has no decorations, and e is exposed to the west. By Lemma 5.2 we can reduce to a D that differs from that displayed in the proof of Theorem 5.1 only by the lack of a line a. However, in this case we also have a diagram D^* contributing to C_D which differs from D in having e decorated (to become e^*). But $h(e^*)$ equals the product of the values of h coming from the two lines arising from nipping a, and so we are done by the arguments given at the end of the proof of Theorem 5.1.

Next consider the diagram E containing $n-t$ (undecorated) nested northern loops at the right-hand end (similar to that illustrated in Figure 16(b)). It is straightforward to verify that the coefficient in $\phi(E)$ of the product of E with the element in $W_{t-2u}(t)$ containing u decorated loops (similar to that illustrated in Figure 24) is 1, and hence ϕ is non-zero.

Finally, suppose that $t=n$, and that we have a morphism ϕ from $W_n(n)$ to $W_s(n)$. As in the proof of Theorem 5.1 we associate to each diagram in our basis for $W_s(n)$ a subset $I \subset \mathbb{N}$ of size $\frac{1}{2}(n-s)$. By regarding all diagrams as if they were undecorated, we may associate to each a subset of $\mathbb{N}$ (and hence induce a partial preorder on diagrams) just as in the proof of Theorem 5.1 on page 30. We denote the undecorated version of a diagram X by $\overline{X}$.

We proceed by induction with respect to this partial preorder. Let J be a subset of $\mathbb{N}$, and suppose that $J \neq \{1, \ldots, \frac{n-s}{2}\}$. Then there exists a unique $1 < i \in J$ such that $i-1 \notin J$. Let $D(J)$ be some diagram corresponding to J, and E be the diagram $U_{i-1}D(J)$. We label the line starting at 1 in E by a.

The coefficient of E in $U_{i-1}L$ is non-zero only if $\overline{L} = \overline{D(J)}$ or $\overline{L} \leq \overline{E} < D(J)$. If a is undecorated then the only diagram L with $\overline{L} = \overline{D(J)}$ that contributes is $D(J)$. By induction we know the coefficients of all $L < D(J)$ in $\operatorname{Im}\phi$, and hence (as $U_{i-1}\operatorname{Im}\phi = 0$) we can determine the coefficient of $D(J)$.

Now suppose that a is decorated. There are now three diagrams L with $\overline{L} = \overline{D(J)}$ which can contribute to the coefficient of E. These are obtained by removing the arc $(i-1, i)$, nipping a at this interval, and decorating one or both of the resulting arcs. We shall denote these three diagrams by L^l, L^r and L^b depending on whether the lefthand, righthand, or both arcs are decorated.

The diagram obtained by not decorating either arc we shall denote by L^0. Note that we have already determined the coefficient of L^0 by the argument above.

First consider L^l. The only diagrams that contribute to the coefficient of L^l in U_0L^l are L^l and L^0, and hence (as $U_0\mathrm{Im}\,\phi = 0$) we can determine the coefficient of L^l in $\mathrm{Im}\,\phi$. In a similar manner, the only diagrams contributing to the coefficient of L^b in U_0L^b are L^b and L^r, and hence we can write the coefficient of L^b in $\mathrm{Im}\,\phi$ in terms of the (as yet unknown) coefficient of L^r.

Now return to considering the coefficient of E in $U_{i-1}L$. By the arguments above (and induction) we know the coefficients of all such L in $\mathrm{Im}\,\phi$ except for L^r and L^b, and the latter coefficient can be written in terms of the former. Hence (as $U_{i-1}\mathrm{Im}\,\phi = 0$) we can determine the coefficient of L^r (and hence L^b) in $\mathrm{Im}\,\phi$.

This shows that the map ϕ is entirely determined by the coefficients of the undecorated diagram corresponding to $\{1, \ldots, \frac{n-s}{2}\}$, and hence is unique up to scalars as required.

It is now easy to give an explicit description of the other main class of standard morphisms for $\mathrm{TL}_B(n)$.

Theorem 6.3 *Let $t = m+u$ with $u \geq m \geq 0$ and $m \equiv y \pmod{l}$. Then there exists a non-zero homomorphism of $TL_B(n)$-modules $\psi : W_t(n) \to W_{t-2u}(n)$ given on diagrams by*

$$E \mapsto \sum_{D \in W_0(2m)} h(D) E A_{u-m}(D)$$

if $m \equiv y \pmod{2l}$, or by

$$E \mapsto \sum_{D \in W_0(2m)} h'(D) E A_{u-m}(D)$$

otherwise, where the sum runs over those diagrams in our basis of $W_0(2m)$ and $A_{u-m}(D)$ is the diagram in $W_{t-2u}(n)$ obtained by adding a further $u-m$

propagating lines to the right-hand side of D.

PROOF. We consider the case $m \equiv y \pmod{2l}$; the other case is similar. As in the last theorem, it is enough to consider the case where E has no arcs. By the last theorem, it is clear that $\psi(E)$ will be non-zero and annihilated by the elements of $\mathrm{TL}_B(2m)$. Clearly, it will also be annihilated by the U_i with $i > 2m$ by the definition of the basis of $W_{t-2u}(n)$. Thus it only remains to check the action of U_{2m}.

Fig. 23. (a) and (b)

A typical diagram in $\psi(e)$ is of the form shown in Figure 23(a), where the line from $2m$ may be decorated. The decorated and undecorated forms of this diagram are precisely those which contribute to the coefficient of the diagram in Figure 23(b) in $U_{2m}\Psi(e)$. Setting the length of the line from $2m$ to be b, we have that the coefficient is proportional to

$$\frac{1}{[b]} + \frac{-[y]}{[m][b]} = 0$$

(as $[y] = [m]$) and so we are done.

Recall that we have fixed $y = a$ for some a as in Corollary 3.3 to define $\mathrm{TL}_B(n)$, but that all of our results also hold if we replace a by $b = 2l - a$ in the defining relations. Thus by working with $\mathrm{TL}_B^b(n)$, and then using Corollary 3.3, we obtain the following analogue of Theorems 6.2 and 6.3.

Theorem 6.4 *(i) Let $t = m + u \geq 0$ with $0 \leq u \leq m$ and $m \equiv l - y \pmod{l}$. Then there exists a non-zero homomorphism of $TL_B(n)$-modules*

$$\phi : W_{-t}(n) \to W_{-(t-2u)}(n).$$

(ii) Let $t = m + u$ with $u \geq m \geq 0$ and $m \equiv l - y \pmod{l}$. Then there exists a non-zero homomorphism of $TL_B(n)$-modules

$$\psi : W_{-t}(n) \to W_{-(t-2u)}(n).$$

We denote a map $\phi : W_s(n) \to W_t(n)$ arising from Theorem 6.2 or Theorem 6.4(i) by ϕ_{st}, and similarly denote the maps ψ arising from Theorem 6.3 or Theorem 6.4(ii) by ψ_{st}.

Motivated by the type A case, we impose an alcove structure on $\mathbb{Z}$ by defining all elements of the form $y + al$ with $a \in \mathbb{Z}$ to be B*-walls*. Now the homomorphisms arising from Theorems 6.2, 6.3 and 6.4 correspond to reflections about some B-wall.

Proposition 6.5 *The maps ϕ_{st} and ψ_{-ut} are injective.*

PROOF. We consider the map ϕ_{st} (the other case is similar). By Corollary 3.3 we may assume that s and t are positive. Recall that the map ϕ_{st} on a diagram E is given by the sum over certain diagrams D of $h_D ED$. Let D_0 be the diagram D with all arcs decorated. It is easy to see that $h_{D_0} = 1$. We shall denote the number of decorated lines in a diagram D by $d(D)$. Note that for any diagram E, any composite $F = ED$ occurring in $\phi_{st}(E)$ is a diagram and satisfies

$$d(F) \leq d(E) + d(D_0)$$

with equality if and only if $D = D_0$.

Suppose that ϕ_{st} is not injective. Then there exists some set of non-zero scalars $\{\lambda_E\}$ such that

$$\phi_{st}(\sum_E \lambda_E E) = 0.$$

By the above remarks we must have

$$\sum_{E'} \lambda_{E'} E' D_0 = 0$$

where the sum runs over the set of diagrams E' such that $d(E')$ is maximal with $\lambda_{E'} \neq 0$. But E' can be recovered from the composite $E'D_0$ by replacing the rightmost $d(D_0)$ decorated lines with pairs of propagating lines, and so we must have $\lambda_{E'} = 0$ for all E', giving the desired contradiction.

7 The blocks of TL_B

In this section we shall show that the blocks of TL_B are just the equivalence classes given by the relation generated by $s \sim t$ if there exists ϕ_{st} or ψ_{st}. As all such maps are injections, it is clear that the blocks of TL_B must be unions of such classes; it remains to show that each class lies in a distinct block.

It is easy to verify that for all $t > 0$ and $n > 0$ we have the short exact sequence

$$0 \to W_{t-1}(n) \to \mathrm{res}_n^B W_t(n+1) \to W_{t+1}(n) \to 0 \tag{26}$$

obtained by identifying $W_{t-1}(n)$ with the set of diagrams in our usual basis for $W_t(n+1)$ whose rightmost line is propagating. We can also make the identification of the quotient module with $W_{t+1}(n)$ explicit. A basis of this quotient module can be identified with the set of diagrams in $W_t(n+1)$ containing an arc of the form $(e, n+1)$ for some e. Then the isomorphism is given by the map which takes this diagram to the element of $W_{t+1}(n)$ obtained by moving the endpoint of the arc at $n+1$ ambient isotopically to the southern edge to form a propagating line. If $t+1 > n$ we interpret $W_{t+1}(n)$ as the zero module. There is a similar result for $t < 0$ with the roles of $t-1$ and $t+1$ reversed.

For $t = 0$ and $n > 0$ we have the short exact sequence

$$0 \to W_{-1}(n) \to \mathrm{res}_n^B W_0(n+1) \to W_{+1}(n) \to 0. \tag{27}$$

In this case we identify diagrams in $W_{-1}(n)$ with those diagrams in $W_0(n+1)$ whose rightmost arc is decorated, via the map obtained by deforming the propagating line ambient isotopically into a decorated northern arc.

In $\mathcal{H}_B(n)$ we define elements L_i for $0 \leq i \leq n-1$ recursively by $L_0 = T_0$ and

$L_i = q^{-1}T_iL_{i-1}T_i$ for $i > 0$, and set $P_j = L_jL_{j-1}\dots L_0$. It is well-known (see for example [34, Lemma 3.3]) that all symmetric polynomials in the L_i (and hence in particular P_{n-1}) are central in $\mathcal{H}_B(n)$. It is also clear that for $i > 0$ we have $L_i = P_iP_{i-1}^{-1}$. We will abuse notation and denote by T_i, L_i and P_i the images of the corresponding elements in $\mathcal{H}_B(n)$ under the quotient map in Proposition 2.2. It will be convenient to define elements T'_i in $\mathrm{TL}_B(n)$ by

$$T'_i = x^{-2}T_i = 1 + x^{-1}U_i \quad \text{for } i > 0 \quad \text{and } T'_0 = -x^{2y}T_0 = 1 - (x - x^{-1})x^yU_0$$

and to set $L'_i = T'_iT'_{i-1}\dots T'_1T'_0T'_1\dots T'_{i-1}T'_i$ for $0 \le i \le n-1$. Note that $L'_j = -q^{y-j-1}L_j$. It will also be convenient to write $T'_0 = 1 + KU_0$.

We will need to consider certain special basis elements inside the $W_t(n)$. Let η_t be the element in $W_t(n)$ represented by the diagram with $\frac{(n-|t|)}{2}$ decorated northern arcs and $|t|$ propagating lines shown in Figure 24.

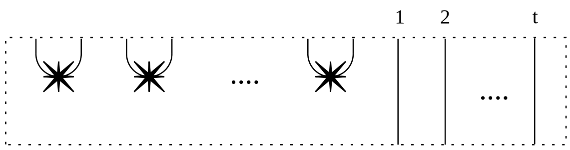


Fig. 24.

The key to our block calculation is

Lemma 7.1 *For all $0 \le i \le n-1$ and $TL_B(n)$-modules $W_t(n)$ the element L_i acts by a scalar on η_t.*

PROOF. As $L_i = P_i(P_{i-1})^{-1}$ for all $i > 0$, it is enough to show that P_i acts as a scalar on η_t for all $0 \le i \le n-1$. Consider the restriction of $W_t(n)$ to $\mathrm{TL}_B(n-1)$. By repeated applications of (26) and (27) we see that η_t lies inside a submodule isomorphic to some $W_s(i+1)$ as a $\mathrm{TL}_B(i+1)$-module. As P_i is central in $\mathrm{TL}_B(n+1)$, and $W_s(i+1)$ is generically irreducible, P_i must act generically on η_t as a scalar by Schur's lemma. Hence P_i acts on η_t as a scalar over $\mathcal{A}$ by restriction, and over our field k by specialisation.

We now wish to determine the value of this scalar. Consider the element

$$L'_j \eta_t = \underbrace{T'_j T'_{j-1} \dots T'_1}_{(*)} T'_0 T'_1 \dots T'_{j-1} T'_j \eta_t.$$

Any undecorated arc introduced by one of the terms arising in $(*)$ cannot be eliminated later, and so cannot contribute to the coefficient of η_t. Hence each of the factors in $(*)$ must act as 1, leaving the coefficient of η_t unchanged, and so it is enough to calculate the coefficient of η_t in $T'_0 T'_1 \dots T'_{j-1} T'_j \eta_t$.

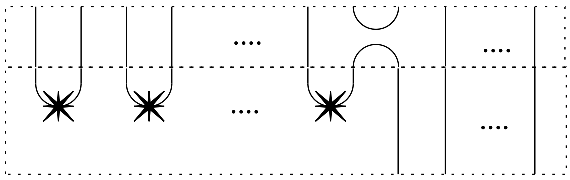


Fig. 25.

First suppose that $n > j > n - |t|$. In this case we have $U_j \eta_t = 0$, and hence $T'_{n-|t|+1} \dots T'_j \eta_t = \eta_t$. Thus if $n > j > n - |t|$ we only need to calculate the coefficient of η_t in $T'_0 T'_1 \dots T'_{j-1} T'_{n-|t|} \eta_t$. Consider the effect of $U_{n-|t|}$ on η_t as shown in Figure 25.

The only way to remove the undecorated arc using terms from $T'_0 \dots T'_{n-|t|-1}$ is via $U_0 U_1 \dots U_{n-|t|} \eta_t$. For example, with two decorated arcs this gives the diagram shown in Figure 26.

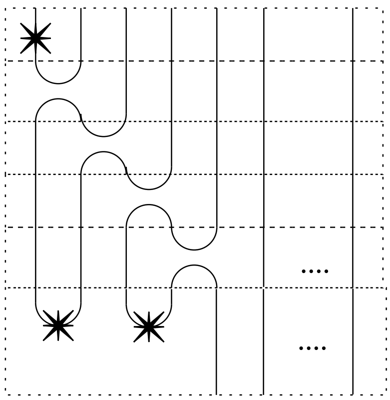


Fig. 26.

This diagram differs from η_t in the decoration of its leftmost propagating line, and hence

$$U_0U_1\dots U_{n-|t|}\eta_t = \begin{cases} -[y]\eta_t \text{ if } t<0 \\ \quad 0 \quad \text{otherwise.} \end{cases}$$

Similar arguments show that for $j < n - |t|$, the only way to remove an undecorated arc introduced by U_j is via a product of the form

$$U_0U_1\dots U_j\eta_t = \begin{cases} [y+1]\eta_t \text{ if } j \text{ odd} \\ \ -[y]\eta_t \quad \text{if } j \text{ even.} \end{cases}$$

(after using the defining relations in $\mathrm{TL}_B(n)$ to remove decorated loops or double decorations on arcs).

Combining the above results we can now calculate the value of $T'_0\dots T'_j\eta_t$ for $n > j \geq n - |t|$. The only terms that can contribute to this are those arising from the expressions $U_0\dots U_i\eta_t$ for $i \leq n - |t|$ and from $1\eta_t$. Summing the coefficients of each of these terms we see that the coefficient of η_t in the above expression is given by

$$1 + Kx^{|t|-n}(X(t) + x[y+1] - x^2[y] + x^3[y+1] - \cdots - [y]x^{n-|t|})$$

where $X(t)$ equals $-[y]$ (respectively 0) when $t < 0$ (respectively $t \geq 0$). After expanding and cancelling terms, it is routine to verify that when $n > j \geq n-|t|$ we have

$$L'_j\eta_t = T'_0\dots T'_j\eta_t = \begin{cases} \quad x^{t-n}\eta_t \quad \text{if } t>0 \\ x^{2y-t-n}\eta_t \text{ if } t<0. \end{cases}$$

(Note that this case cannot arise when $t = 0$). For $j < n-|t|$ similar arguments give

$$L'_j\eta_t = T'_0\dots T'_j\eta_t = \begin{cases} x^{-j-1}\eta_t \text{ if } j \text{ odd} \\ x^{2y-j}\eta_t \text{ if } j \text{ even.} \end{cases}$$

Setting $P'_{n-1} = L'_{n-1}L'_{n-2}\dots L'_0$ it is now easy to show that

$$P'_{n-1}\eta_t = x^{\frac{1}{2}(n-t)(2y-n-t)}\eta_t = x^{-\frac{1}{2}[(n-y)^2-v^2]}\eta_t$$

for all η_t with $t = y + v$ and $|t| \leq n$.

When q is not a root of unity, the element P'_{n-1} is sufficient to distinguish the blocks of $\mathrm{TL}_B(n)$. In general however, we will need to consider all symmetric polynomials in the L_j. As $T_0 = -x^{-2y}T'_0$, and $T_i = x^2T'_i$ for $i > 0$, it is easy to verify that

$$L_j\eta_t = \begin{cases} -x^{2j+2-2y+t-n}\eta_t & \text{if } j \geq n-|t| \text{ and } t > 0 \\ -x^{2j+2-t-n}\eta_t & \text{if } j \geq n-|t| \text{ and } t < 0 \\ -x^{j+1-2y}\eta_t & \text{if } j < n-|t| \text{ and } j \text{ odd} \\ -x^{j+2}\eta_t & \text{if } j < n-|t| \text{ and } j \text{ even.} \end{cases} \tag{28}$$

We shall write $L_j\eta_t = a_{jt}\eta_t$. Arguing as in [35, (5.9) Proposition] we see that if the multiset $A_s = \{a_{js} : 0 \leq j \leq n-1\}$ is not equal to the multiset $A_t = \{a_{jt} : 0 \leq j \leq n-1\}$ then there exists a symmetric polynomial S in the L_j such that $S\eta_s \neq S\eta_t$. Hence, as S is central by our earlier remarks, to show that s and t lie in different blocks it is sufficient to show that $A_s \neq A_t$.

Lemma 7.2 *Suppose that q is a primitive lth root of unity and $t = y + v$ with $t + n$ even and $|t| \leq n$. For any s with $s + n$ even and $|s| \leq n$ we have that $A_s = A_t$ if and only if $s \equiv y \pm v \pmod{2l}$.*

PROOF. Using (28) it is easy to verify that for all t the multiset A_t is of the form

$$\{-q^i : 1 \leq i \leq \frac{n-t}{2}\} \cup \{-q^{i-y} : 1 \leq i \leq \frac{n+t}{2}\}.$$

We may assume that $s \geq t$. First suppose that $s = t + 2lm$ for some $m > 0$. Then

$$\begin{aligned} A_s &= \{-q^i : 1 \leq i \leq \tfrac{n-t}{2} - lm\} \cup \{-q^{i-y} : 1 \leq i \leq \tfrac{n+t}{2} + lm\} \\ &= \{-q^i : 1 \leq i \leq \tfrac{n-t}{2} - lm\} \cup \{-q^{i-y} : \tfrac{n+t}{2} < i \leq \tfrac{n+t}{2} + lm\} \\ &\quad \cup \{-q^{i-y} : 1 \leq i \leq \tfrac{n+t}{2}\}. \end{aligned}$$

Clearly we have $\{-q^{i-y} : \frac{n+t}{2} < i \leq \frac{n+t}{2} + lm\} = \{-q^i : \frac{n-t}{2} - lm < i \leq \frac{n-t}{2}\}$ as each multiset consists of m copies of each distinct power of q. Therefore we have that $A_s = A_t$ when $s = t + 2lm$.

To complete the proof, it is enough to show that if $s = t + 2a$ with $0 < a < l$ and $t = y + v$ then $A_s = A_t$ if and only if $s \equiv y - v \pmod{2l}$. Arguing as above, we see that $A_s = A_t$ if and only if

$$\{-q^{i-y} : \frac{n+t}{2} < i \leq \frac{n+t}{2} + a\} = \{-q^i : \frac{n-t}{2} - a < i \leq \frac{n-t}{2}\}.$$

As each of these multisets contains a consecutive powers of q, and $a < l$, each is determined by its initial element. Thus these two sets are equal if and only if $-q^{\frac{n+t}{2}+1-y} = -q^{\frac{n-t}{2}+1-a}$. Rearranging we see that this is equivalent to $q^{t+a-y} = 1$, i.e. that $t + a \equiv y \pmod{l}$. Now $t = y + v$, so $A_s = A_t$ if and only if $a \equiv -v \pmod{l}$, i.e. $s = y + v + 2x \equiv y - v \pmod{2l}$. This completes the proof of Lemma 7.2.

Combining Lemma 7.2 with the partial block results arising from the existence of the various injective maps ϕ_{st} and ψ_{st} we obtain

Theorem 7.3 *For q a primitive lth root of unity, and $t = y + v$, the modules $D_s(n)$ and $D_t(n)$ lie in the same block of $\mathrm{TL}_B(n)$ if and only if $s \equiv y \pm v \pmod{2l}$.*

8 Decomposition numbers for TL_B

In this section we shall determine the decomposition numbers for the blob algebra. We begin by restricting our attention to the case when $t \geq 0$ with $t \equiv y - 1 \pmod{l}$. Our strategy will be to define certain quotients $X_t^0(n)$ of standard modules for $\mathrm{TL}_B(n)$, and identify these on restriction to $\mathrm{TL}_A(n)$ with the modules $\Xi_t^0(n)$ defined in Section 5. We will then show that a composition series of these as $\mathrm{TL}_A(n)$ modules can be lifted to $\mathrm{TL}_B(n)$. Similar results hold when $t \leq 0$ with $t \equiv y + 1 \pmod{l}$.

For the remaining values of t we are able to prove a version of the translation principle, which enables us to reduce the calculation to the case considered above. Combining these various results, we are then able to determine inductively the composition factors of the standard modules.

We will need an explicit description of $W_t(n)\!\uparrow$, where (as in [2]) we denote by $\uparrow$ the left adjoint to the restriction functor. For this we shall use certain *globalisation functors* $F(n)$, following [2, Sections 3 and 4]. The constructions given in [2] for the original blob algebra all transfer easily to our setting, so we shall recall the definitions given there.

We define an exact functor $\mathrm{pr}_u : \mathrm{Mod}\ (\mathrm{TL}_B(n)) \to \mathrm{Mod}\ (\mathrm{TL}_B(n-1))$. On modules this is given by restriction to $\mathrm{TL}_B(n-1)$ followed by projection onto the block containing $W_u(n-1)$, and on morphisms by restriction to the corresponding domain. We also define res_n^B to be restriction from Mod $(\mathrm{TL}_B(n+1))$ to $\mathrm{Mod}(\mathrm{TL}_B(n))$.

As $[2] \neq 0$ in k, we can consider for $n \geq 2$ the idempotent $x_n = U_{n-1}/[2]$. This lies in a hereditary chain for $\mathrm{TL}_B(n)$ (confer [2, Section 3.2]), and we have an algebra isomorphism $\mathrm{TL}_B(n-2) \cong x_n \mathrm{TL}_B(n) x_n$ which we view as an identification. The globalisation functor $F = F(n)$ from $\mathrm{TL}_B(n-2)$-mod to $\mathrm{TL}_B(n)$-mod is given by $\mathrm{TL}_B(n)x_n \otimes_{\mathrm{TL}_B(n-2)} -$. Using this we can prove

Proposition 8.1 *For all t we have*

$$W_t(n)\!\uparrow\, \cong W_{t+1}(n+1) + W_{t-1}(n+1)$$

where the sum is direct if t does not lie on a B-wall.

PROOF. By the results in [19, Section A3] we have $F(W_t(n)) \cong W_t(n+2)$ (see [36, Proposition 3]). Also, by (26) we have

$$\mathrm{res}_{n+1}^B F(W_t(n)) \cong W_{t+1}(n+1) + W_{t-1}(n+1)$$

and Theorem 7.3 implies that this sum is direct for t not on a B-wall. Thus it is enough to show that $W_t(n)\!\uparrow\, \cong \mathrm{res}_{n+1}^B F(W_t(n))$.

The modules $W_t(n)\!\uparrow$ and $F(W_t(n))$ consist of linear combinations of diagrams of the form shown in Figures 27(a) and 27(b) respectively, where A is a diagram in $\mathrm{TL}_B(n+1)$ and M is a diagram in $W_t(n)$.

Under restriction, the diagrams associated to $F(W_t(n))$ can be deformed am-

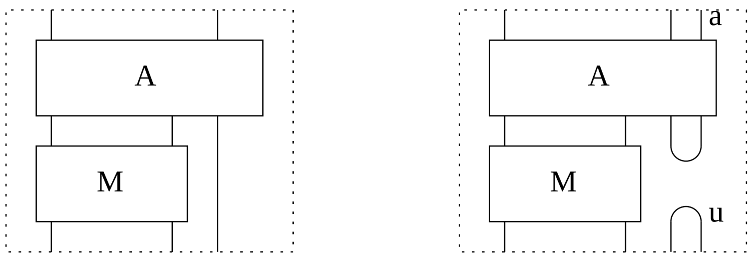


Fig. 27. (a) and (b)

bient isotopically without changing the module structure so that the extra line a becomes a propagating line. There is now an obvious bijection between diagrams in $W_t(n)\uparrow$ and diagrams in $\text{res}^B_{n+1}F(W_t(n))$ obtained by removing the loop u in the right-hand diagram. This induces the desired isomorphism.

A version of the above result was used implicitly in the proof of [2, Theorem 8.2(ii)].

Lemma 8.2 *For* $t \geq 0$ *the restriction of* $W_{\pm t}(n)$ *to* $TL_A(n)$ *has a filtration of* $TL_A(n)$*-modules*

$$0 = W^{-1}_{\pm t}(n) \subseteq W^0_{\pm t}(n) \subseteq \cdots \subseteq W^i_{\pm t}(n) \subseteq \cdots \subseteq W_{\pm t}(n)$$

such that $W^i_{\pm t}(n)/W^{i-1}_{\pm t}(n) \cong \Delta_{t+2i}(n)$.

PROOF. Recall that we denote the number of decorated lines in a diagram D by $d(D)$. Let $W^i_t(n)$ be the subspace of $W_t(n)$ spanned by all diagrams D in the basis such that $d(D) \leq i$. It is routine to verify that the map that takes the class of a maximally decorated diagram D in $W^i_t(n)/W^{i-1}_t(n)$ to the same diagram with the i decorated northern arcs replaced with $2i$ propagating lines, obtained by nipping these arcs at the southern edge of $W_t(n)$, is an isomorphism of $\text{TL}_A(n)$-modules.

Until the end of Proposition 8.6 we shall assume (unless explicitly stated otherwise) that $t \equiv y - 1 \pmod{l}$, and that $t \geq 0$. Then by the results in Section 6 we have injective $\text{TL}_B(n)$-homomorphisms $\phi_{st} : W_s(n) \to W_t(n)$ and $\psi_{-ut} : W_{-u}(n) \to W_t(n)$ with $s = t + 2$ and $u = t + 2(l - y)$ (respectively

$u = t - 2y$) if $y > 0$ (respectively $y < 0$). (The elements s and $-u$ correspond to those marked s_1 and $-u_1$ in Figure 2.) We set $Y_t(n) = W_t(n)/\phi_{st}(W_s(n))$ and $X_t^0(n) = W_t(n)/(\psi_{-ut}(W_{-u}(n)) + \phi_{st}(W_s(n)))$. If t is near to n then one or both of these maps may no longer exist — these are degenerate cases of the following, and we leave it to the reader to make the appropriate (easy) modifications, similar to those in Proposition 5.6.

Lemma 8.3 *Suppose that $t \geq 0$ with $t \equiv y - 1 \pmod{l}$. Then, considered as a $TL_A(n)$-module, we have*

$$Y_t(n) \cong \Delta_t(n).$$

PROOF. Recall that under the map ϕ_{st}, a diagram E gets mapped to ED_0 plus diagrams with fewer decorations. In this case $d(D_0) = 1$, and so it is clear that as a vector space we have $W_t(n) \cong \phi_{st}(W_s(n)) \oplus \Delta_t(n)$. But this is also a $\mathrm{TL}_A(n)$-module decomposition, and so we are done.

Proposition 8.4 *Suppose that $t \geq 0$ with $t \equiv y-1 \pmod{l}$. Then, considered as a $TL_A(n)$-module, we have*

$$X_t^0(n) \cong \Xi_t^0(n).$$

PROOF. We begin by noting that the modules $W_t(n)$ and $\Delta_t(n)$, along with the maps $\theta_{tt'}$, ϕ_{st} and ψ_{-ut} are all defined over $\mathbb{Z}[q, q^{-1}]/(q^l - 1)$. Thus we can consider specialisations of these maps to any field containing a primitive lth root of unity q. (Recall that we always assume that $l > 2$.) Now $Y_t(n)$ has simple head $D_t(n)$ by the quasi-heredity of $\mathrm{TL}_B(n)$, which cannot occur as a composition factor of $W_{-u}(n)$. We claim that we cannot have $\psi_{-ut}(W_{-u}(n)) \subseteq \phi_{st}(W_s(n))$, and hence that $X_t^0(n)$ is a proper quotient of $Y_t(n)$.

To see this, first note that both maps are injections, and hence we would induce a non-zero $\mathrm{TL}_B(n)$-homomorphism from $W_{-u}(n)$ into $W_s(n)$. By adjointness we have

$$\mathrm{Hom}(W_{-(u-1)}(n-1)\!\uparrow, W_s(n)) \cong \mathrm{Hom}(W_{-(u-1)}(n-1), \mathrm{res}_n^B W_s(n))$$

and Proposition 8.1, together with (26), (27), and our assumption imply that the left-hand side is non-zero. By Theorem 7.3 and Proposition 8.1, the right-hand side is isomorphic to

$$\mathrm{Hom}(W_{-(u-1)}(n-1), W_{s+1}(n-1)) \oplus \mathrm{Hom}(W_{-(u-1)}(n-1), W_{s-1}(n-1))$$

and by another application of Theorem 7.3 we see that the second of these Hom-spaces must be zero. Thus we deduce that $\mathrm{Hom}(W_{-(u-1)}(n-1), W_{s+1}(n-1))$ is non-zero.

Repeating the above argument we see that $\mathrm{Hom}(W_{-(u-i)}(n-i), W_{s+i}(n-i))$ is non-zero for all $0 \leq i \leq l-1$. But by quasi-heredity we know that there are no non-zero homomorphisms when $u-i \leq s+i$. We will show that there exists a $0 \leq i \leq l-1$ satisfying this latter inequality to deduce the desired contradiction. First suppose that $y > 0$. Then $u-i \leq s+i$ is equivalent to $l-y-1 \leq i$. As $y \neq 0$ we are done in this case. Next suppose that $y < 0$. Then $u-i \leq s+i$ is equivalent to $|y|-1 \leq i$, and as $y > -l$ we again deduce the desired contradiction, and our claim follows. (When $n-t$ is small this argument needs to be slightly modified, as $W_{s+i}(n-i)$ may not exist for i large. Again, we leave it to the reader to make the appropriate easy modifications.)

In characteristic zero, the only proper $\mathrm{TL}_A(n)$ quotient of $Y_t(n)$ is $\Xi_t^0(n)$. By the exactness of the $\theta_{tt'}^0$ maps in Corollary 5.6 we see by induction on t that the dimension of $\Xi_t^0(n)$ is independent of the ground field. By base change, the spanning set for $\mathrm{Im}\,(\phi_{st}) + \mathrm{Im}\,(\psi_{-ut})$ in characteristic zero induced from our usual bases must also give a spanning set in characteristic p, and hence the dimension of this space in characteristic p is at most that in characteristic zero. Thus we see that the dimension of $X_t^0(n)$ in characteristic p is at least that in characteristic zero.

We have a non-zero $\mathrm{TL}_B(n)$-morphism from $W_{-u}(n)$ to $Y_t(n)$, which restricts to a non-zero $\mathrm{TL}_A(n)$-morphism. The head of $W_{-u}(n)$ is $D_{-u}(n)$, and the restriction of this to $\mathrm{TL}_A(n)$ contains a copy of $L_u(n)$. Now as a $\mathrm{TL}_A(n)$-module, $Y_t(n)$ contains a unique composition factor $L_u(n)$, which occurs as the head of a copy of $\Xi_u^0(n)$. Hence this copy of $\Xi_u^0(n)$ must be annihilated in $X_t^0(n)$, and so we are done by the dimension estimate above.

Suppose t' is such that we have a non-zero $\mathrm{TL}_A(n)$-homomorphism $\Xi_t^0(n) \to \Xi_{t'}^0(n)$. Note that such a t' must also satisfy $t' \equiv y - 1 \pmod{l}$.

Lemma 8.5 *Let t and t' be as above. Via the identifications in Proposition 8.4 the $\mathrm{TL}_A(n)$-homomorphism $\theta_{tt'}^0$ induces a $\mathrm{TL}_B(n)$-homomorphism $X_t^0(n) \to X_{t'}^0(n)$.*

PROOF. We have an embedding of $\mathrm{TL}_A(n-1)$ into $\mathrm{TL}_A(n)$ obtained by adding a single propagating line to the *left* of each diagram; let res_{n-1}^A be the corresponding restriction functor from $\mathrm{TL}_A(n)$-mod to $\mathrm{TL}_A(n-1)$-mod. It is easy to verify that we have a short exact sequence of $\mathrm{TL}_A(n-1)$-modules

$$0 \to \Delta_{t-1}(n-1) \to \mathrm{res}_{n-1}^A \Delta_t(n) \to \Delta_{t+1}(n-1) \to 0 \qquad (29)$$

obtained by identifying $\Delta_{t-1}(n-1)$ with the set of diagrams in our usual basis for $\Delta_t(n)$ whose leftmost line is propagating. This is the $\mathrm{TL}_A(n)$ analogue of (26). Hence, as a $\mathrm{TL}_A(n-1)$-module, we have

$$Y_t(n) \cong \Delta_{t-1}(n-1) \oplus \Delta_{t+1}(n-1)$$

where the sum is direct because $t-1$ and $t+1$ lie in different $\mathrm{TL}_A(n-1)$ blocks by our assumption on t. (Clearly a corresponding isomorphism holds for $Y_{t'}(n)$ considered as a $\mathrm{TL}_A(n-1)$-module.) It is easy to verify using our identification of the diagram basis for $\Delta_{t-1}(n-1)$ inside $\Delta_t(n)$ that on $\Delta_{t-1}(n-1)$ the element U_0 acts as 0. Further, by identifying $\Delta_{t+1}(n-1)$ with the quotient of $\Delta_t(n)$ by the span of these basis elements, and considering the action of U_0 on elements representing basis elements of $\Delta_{t+1}(n-1)$, a simple diagram calculation shows that U_0 acts as $-[y]$ on $\Delta_{t+1}(n-1)$. Similar results hold for $Y_{t'}(n)$.

Now consider the map $\theta_{tt'}^0$ from $\Xi_t^0(n)$ to $\Xi_{t'}^0(n)$. As $\mathrm{TL}_A(n-1)$-modules these modules decompose into summands corresponding to the two blocks, which must be respected by the map $\theta^0_{tt'}$. By our calculations of the U_0 action and the relative positions of t and t' we see that the actions of $\theta^0_{tt'}$ and U_0 must commute, and hence $\theta^0_{tt'}$ induces a $\mathrm{TL}_B(n)$-homomorphism from $X^0_t(n)$ to $X^0_{t'}(n)$.

We now wish to show that all of the $\Xi^i_t(n)$ defined in Section 5, and the associated maps $\theta^i_{tt'}$, inherit a $\mathrm{TL}_B(n)$ structure. We proceed by induction on i. By Corollary 5.6 we see that either $\Xi^{i+1}_t(n)$ can be defined in terms of $\theta^i_{tt'}$ or θ^i_{ut} for some u and t', or $\Xi^{i+1}_t(n) = \Xi^i_t(n)$. Thus it is enough to show that the map θ^i_{rs} (for suitable r and s) becomes a $\mathrm{TL}_B(n)$ map. But this follows in exactly the same way as for θ^0_{rs} above, using (29). We denote the $\mathrm{TL}_B(n)$-module obtained in this way from $\Xi^i_t(n)$ by $X^i_t(n)$. Thus our $\mathrm{TL}_A(n)$ composition series for $\Xi^0_t(n)$ lifts to a $\mathrm{TL}_B(n)$ filtration for $X^0_t(n)$. It is also clear that the successive quotients in this filtration are irreducible $\mathrm{TL}_B(n)$-modules.

Note that the above arguments all extend to the case when $t \leq 0$ and $t \equiv y+1 \pmod{l}$, provided we extend the labelling of $\mathrm{TL}_A(n)$-modules to $\mathbb{Z}$ by defining $\Delta_t(n) = \Delta_{-t}(n)$ for $t < 0$ (and similarly for the Ξ^i_t). Thus for such values of t we can again define modules $X^0_t(n)$, and lift a $\mathrm{TL}_A(n)$ filtration to $\mathrm{TL}_B(n)$. We will now use this fact to determine the $\mathrm{TL}_B(n)$ composition factors of $Y_t(n)$ when $t \geq 0$ and $t \equiv y-1 \pmod{l}$.

Proposition 8.6 *Suppose that $t \geq 0$ with $t \equiv y-1 \pmod{l}$ and set $u = t+2(l-y)$ (respectively $u = t-2y$) if $y > 0$ (respectively $y < 0$). Then we have a short exact sequence of $\mathrm{TL}_B(n)$-modules*

$$0 \to X^0_{-u}(n) \to Y_t(n) \to X^0_t(n) \to 0.$$

As $-u \equiv y+1 \pmod{l}$ we thus obtain from the arguments above a composition series for $Y_t(n)$.

PROOF. Let $s = t+2$, and choose x and y minimal such that we have maps ϕ_{xs} and ϕ_{-y-u}. By our choice of t and u we have that $X^0_{-u}(n) =$

$W_{-u}(n)/(\psi_{x-u}(W_x(n))+\phi_{-y-u}(W_{-y}(n)))$. As ψ_{-ut} is an injection, it will thus be enough to show that

$$Y_t(n) \cong W_t(n)/(\phi_{st}(W_s(n)) + \psi_{-ut}(\psi_{x-u}(W_x(n)) + \phi_{-y-u}(W_{-y}(n)))) \quad (30)$$

and that for any $\psi_{x-u}(W_x(n)) + \phi_{-y-u}(W_{-y}(n)) \subset V \subseteq W_{-u}(n)$ we have

$$Y_t(n) \ncong W_t(n)/(\phi_{st}(W_s(n)) + \psi_{-ut}(V)). \quad (31)$$

As in Proposition 8.4 we shall use the fact that our modules $W_a(n)$ and $\Delta_a(n)$, along with the maps ϕ_{ab} and ψ_{ab} are defined over $\mathbb{Z}[q,q^{-1}]/(q^l-1)$. Specialising to characteristic zero, it is easy to see from [2, (9.4) Theorem] that (30) and (31) must hold in this case. However, the dimensions of the various $W_a(n)$, $Y_a(n)$ and $X_a^0(n)$ are independent of the characteristic of the field. Thus by a base change argument as in Proposition 8.4 we see that (30) and (31) must also hold in positive characteristic.

Again, we can obtain a similar result in the case $t \leq 0$ with $t \equiv y+1 \pmod{l}$. Thus in these two cases we have determined the composition factors (with multiplicities) of the $Y_t(n)$.

For arbitrary values of t we cannot apply the methods above, as it is no longer the case that the simple $\mathrm{TL}_A(n)$-modules can be endowed with a $\mathrm{TL}_B(n)$-module structure. However, for the remaining values of t it is possible to deduce the desired results from those above using an analogue of [2, (8.3)Proposition], which allows us to prove a version of the 'translation principle' for $\mathrm{TL}_B(n)$. We shall assume that $t \geq 0$; the corresponding results for $t < 0$ can be obtained in a similar manner.

We first prove an analogue of [2, (8.3) Proposition]. As usual we shall denote the simple head of $W_t(n)$ by $D_t(n)$.

Theorem 8.7 *Suppose that $t > 0$ does not lie on a B-wall, and let $u \in \{\pm 1\}$.*

Then we have

$$\mathrm{pr}_{t+u} D_t(n) \cong \begin{cases} 0 & \text{if } t+u \text{ lies on a } B\text{-wall and } u = +1 \\ 0 & \text{if } t+u > n \\ D_{t+u}(n-1) & \text{otherwise.} \end{cases}$$

PROOF. This can be shown by arguments as in [2, (8.3) Proposition]; however we shall give a slightly different proof. The case when $t = n$ is easy, and left to the reader. If $t + u \leq n$ then by Theorem 7.3 and the assumption on t we have that

$$\mathrm{pr}_{t+u} W_t(n) \cong W_{t+u}(n-1). \tag{32}$$

When $u = -1$ or $t + u$ does not lie on a B-wall, this implies that $D_{t+u}(n-1)$ must occur in the head of $\mathrm{pr}_{t+u} D_t(n)$, as by induction on $n-t$ and Theorem 7.3 we see that $D_t(n)$ is the only composition factor of $W_t(n)$ which can contribute such a factor in $W_{t+u}(n-1)$. In the remaining case we have that $D_t(n)$ and $D_{t+2}(n)$ are both composition factors of $W_t(n)$ by Theorems 6.2 and 6.5. Now if $\mathrm{pr}_{t+1} D_t(n)$ had a composition factor isomorphic to $D_{t+1}(n-1)$ then (32) and the above argument applied to $\mathrm{pr}_{t+1} D_{t+2}(n)$ would imply that $D_{t+1}(n-1)$ occurs in $W_{t+1}(n-1)$ with multiplicity at least two, a contradiction. Thus in all cases it is enough to show that no $D_s(n-1)$ can occur in the socle of $\mathrm{pr}_{t+u} D_t(n)$ for $s \neq t+u$.

Suppose that $D_s(n-1)$ is a composition factor of the socle of $\mathrm{pr}_{t+u} D_t(n)$. Then we must have

$$\mathrm{Hom}(W_s(n-1), \mathrm{res}^B_{n-1} D_t(n)) \neq 0$$

(as $D_s(n-1)$ is the head of $W_s(n-1)$) and hence by Proposition 8.1 we see that

$$\mathrm{Hom}(W_{s+1}(n) \oplus W_{s-1}(n), D_t(n)) \neq 0.$$

This implies that $s = t \pm 1$; by block considerations we see that $s = t + u$ as required.

Analogues of Proposition 8.1 and Theorem 8.7 can also be proved for the corresponding $\mathrm{TL}_A(n)$-modules in exactly the same way.

Combining our various results it is now easy to determine the composition factors of $W_t(n)$ for all t. We shall assume that $t \geq 0$ and proceed by induction on $n-t$. (The case $t < 0$ is similar.) When $n = t$ the result is trivial. By repeated applications of (26) and Theorem 8.7 with $u = -1$ (which does not change the value of $n-t$) we may assume that (for fixed n) the composition factors of $W_{t'}(n)$ are known for all $t' > t$, and that t lies immediately below a B-wall (i.e., $t \equiv y-1 \pmod{l}$). For such a t, there is an injection $\phi_{(t+2)t}$ from $W_{t+2}(n)$ into $W_t(n)$, and the quotient of $W_t(n)$ by the image of this map is precisely $Y_t(n)$, whose composition factors we have already determined in Proposition 8.6. As the composition factors of $W_{t+2}(n)$ are known by assumption, the composition factors of $W_t(n)$ can now be determined.

To give a more precise description of the composition factors of $W_t(n)$ we need a little more notation. For $-n \leq t \leq n$ with $t+n$ even we wish to define sets of simple modules $[X_t^0(n)]$. If $t \geq 0$ and $t \equiv y-1 \pmod{l}$, or $t < 0$ and $t \equiv y+1 \pmod{l}$ let $[X_t^0(n)]$ equal the set of composition factors (with multiplicities) of $X_t^0(n)$. In the remaining cases, for $\pm t \geq 0$ let $[X_t^0(n)]$ be either $\{D_t(n)\}$ if there is no B-wall between t and $\pm n$, or $\mathrm{pr}_t[X_{t\pm 1}^0(n+1)]$ otherwise, where pr_t acts on a set element-wise.

By Proposition 4.5, the composition factors of the $X_t^0(n)$ with t just below a B-wall are 'independent of n', in the sense that for $|s| \leq \min(n, n')$ the multiplicities $[W_t(n) : D_s(n)]$ and $[W_t(n') : D_s(n')]$ are equal. Using this, it is now easy to determine the composition factors arising from the above procedure:

Theorem 8.8 *For t not on a B-wall, the composition factors of $W_t(n)$ are* *given by the sets*

$$\begin{array}{cc} & [X^0_t(n)] \\ [X^0_{s_1}(n)] & [X^0_{-u_1}(n)] \\ [X^0_{s_2}(n)] & [X^0_{-u_2}(n)] \\ [X^0_{s_3}(n)] & [X^0_{-u_3}(n)] \\ \vdots & \vdots \end{array} \tag{33}$$

where the elements s_i and $-u_i$ are exactly as in (1). For weights on walls the weights s_i and s_{i-1} coincide for i odd (where $s_0 = t$), and similarly for the $-u_i$, with the roles of odd and even reversed. In this case the set of composition factors of $W_t(n)$ is given by (33) after the removal of the sets $[X^0_{s_i}(n)]$ with i odd and $[X^0_{-u_i}(n)]$ with i even.

9 Conclusion

We have determined the decomposition numbers of the standard modules for the blob algebra in all cases where the algebra is quasi-hereditary. This completely determines the decomposition numbers for the original blob algebra [1] in all cases except $[2] = 0$, by the remarks in Section 3.

In order to determine decomposition numbers for those standard modules over the extended affine Hecke algebra described in the introduction, the work of the second author and Lehrer [7] requires knowledge of decomposition numbers for the blob algebra without restriction on the defining parameters. Thus, to complete this analysis, it remains to determine decomposition numbers for the blob algebra in the non-quasi-hereditary cases.

For all choices of the parameters, the second author and Lehrer have shown [7] that the blob algebra is cellular in the sense of [35]. Most of the arguments in the present paper go through in this more general setting, with the exception of those given in Sections 4 and 8. Cellularity is a rather weaker property than quasi-heredity, and hence the modifications required in these sections would significantly lengthen this work. For this reason we shall present them in a later paper.

The blob algebra is amenable to a series of generalisations where higher-dimensional alcove geometry plays a similar role. This provides scope for significant generalisation of the methods used here. The basic machinery to do this has been constructed in [37].

Appendix: Projection Algebras

Projection algebras were introduced in [14] as a framework in which to study a large class of quotients of Hecke algebras. For the convenience of the reader we collect here those definitions and results required in the body of our paper, and indicate how they apply to the algebras $\mathrm{TL}_B(n)$ and TB_n defined in Section 2.

A *projection graph* $\mathbf{G} = (G, m, Z)$ is a finite graph G such that for each edge J (regarded as a subset of the vertex set $V(G)$ of G) we have an integer $m_J \geq 0$ (called the *multiplicity* of J) and a subset $Z_J \subseteq J$. Given such a projection graph $\mathbf{G}$ and a commutative ring A, let $\delta : V(G) \to A$ be some function, and for each edge J choose a monic polynomial $f_J(X) \in A[X]$ of degree m_J. Then the *projection algebra* $\mathcal{T}_{\mathbf{G}}(A, \delta, f)$ is the associative, unital A-algebra with generators B_s for $s \in V(G)$ and defining relations

(1) If $s \in V(G)$ then $B_s^2 = \delta(s) B_s$.
(2) If $r, s \in V(G)$ but $\{r, s\}$ is not an edge in G then $B_r B_s = B_s B_r$.
(3) If $J = \{r, s\}$ is an edge in G then
  (a) If $r \notin Z_J$ then $B_r f_J(B_s B_r) = 0$.
  (b) If $s \notin Z_J$ then $B_s f_J(B_r B_s) = 0$.
  (c) If $J = Z_J$ then $B_r B_s f_J(B_r B_s) = 0 = B_s B_r f_J(B_s B_r)$.

Consider the case where $\mathbf{G}$ is the graph on n vertices shown in Figure 28

Fig. 28.

with $m_J = 1$ for all J and $Z_J = \emptyset$ for $J \neq \{0,1\}$ and $Z_{\{0,1\}} = \{0\}$, and let A be a field k, containing a non-zero element x used to define our usual Gaussian coefficient $[m]$. If we set $f_J(X) = X - 1$ for $J \neq \{0,1\}$ and $f_{\{0,1\}}(X) = X - [y+1]$, and $\delta(i) = -[2]$ for $1 \leq i \leq n-1$ and $\delta(0) = -[y]$, then the resulting projection algebra is just the algebra $\mathrm{TL}_B(n)$. Thus the results in [14, Chapter 6] can be applied in this case. However, in order to state them, we first need some more notation.

Let a and b be elements of a partially ordered set $(P, \leq)$. We say that a *covers* b if $a > b$ and there is no $c \in P$ with $a > c > b$, and that a and b are *incomparable* if $a \not\leq b$ and $b \not\leq a$. A subset D of P is called *convex* if $a > c > b$ with a and b in D implies that $c \in D$ also. Given a projection graph $\mathbf{G}$, a $\mathbf{G}$-*sequence* is a finite poset $(P, \leq)$ and a function $h : P \to G$ such that if a and b are incomparable then $h(a)$ and $h(b)$ are not adjacent (i.e. do not form an edge in G). A $\mathbf{G}$-*morphism* θ from a $\mathbf{G}$-sequence $(P, \leq_P, h_P)$ to a $\mathbf{G}$-sequence $(Q, \leq_Q, h_Q)$ is a map between the underlying sets such that the image is convex in Q, $h_P = h_Q \circ \theta$, and $\theta(a) <_Q \theta(b)$ implies that $a <_P b$ for $a, b \in P$. If θ is a bijective $\mathbf{G}$-morphism whose inverse is also a $\mathbf{G}$-morphism, then θ is a $\mathbf{G}$-*isomorphism*.

A $\mathbf{G}$-*set* is a $\mathbf{G}$-sequence $(P, \leq, h)$ such that every surjective $\mathbf{G}$-morphism with domain P is a $\mathbf{G}$-isomorphism. By [14, Proposition 6.13], a $\mathbf{G}$-sequence $(P, \leq, h)$ is a $\mathbf{G}$-set if and only if for each pair $a, b \in P$ we have

(1) If a and b are incomparable then $h(a) \neq h(b)$.
(2) If a covers b then $h(a)$ and $h(b)$ are adjacent.

Note that if $(D, \leq_D, h_D)$ is a $\mathbf{G}$-set such that for some edge J in G we have $h(d) \in J$ for all $d \in D$, then this $\mathbf{G}$-set must be totally ordered, and hence contain a unique minimal element. If $(D, \leq_D, h_D)$ and $(P, \leq_P, h_P)$ are $\mathbf{G}$-sets, D is a subset of P, and the inclusion map is a $\mathbf{G}$-morphism, then we say that $(D, \leq_D, h_D)$ is a $\mathbf{G}$-*subset* of $(P, \leq_P, h_P)$. Finally, a $\mathbf{G}$-set $(P, \leq_P, h_P)$ is called *complex* if there exists an edge J in G and a $\mathbf{G}$-subset $(D, \leq_D, h_D)$ of $(P, \leq_P, h_P)$ such that $h(d) \in J$ for all $d \in D$ and either $|D| \geq 2m_J + 2$, or

$|D| = 2m_J + 1$ and $s \notin Z_J$, where s is the image under h_D of the unique minimal element of D. (We call such an s the *divisor* of D.) We denote by $M_{\mathbf{G}}$ the set of $\mathbf{G}$-isomorphism classes of non-complex $\mathbf{G}$-sets.

We can now state the main result from [14, Chapter 6].

Theorem A.1 *Let $\mathcal{T}_{\mathbf{G}}(A, \delta, f)$ be a projection algebra. If $\{r, s\}$ is an edge of multiplicity zero then assume both that $Z_{\{r,s\}} = \{s, t\}$ and that if t is a vertex adjacent to s and $t \notin Z_{\{s,t\}}$ then t is also adjacent to r. If $\{r, s\}$ and $\{s, t\}$ are edges of multiplicity one with $r \notin Z_{\{r,s\}}$ and $t \notin Z_{\{t,s\}}$, and r and s are not adjacent, then futher assume that*

$$\delta(r) f_{ts}(0) = \delta(t) f_{rs}(0).$$

Under these assumptions, the projection algebra has a unique basis $\{B_p : p \in M_{\mathbf{G}}\}$ such that $B_{b_s} = B_s$ if $s \in V(G)$ and

$$B_p B_q = B_{pq} \quad \text{if } p, q \text{ and } pq \in M_{\mathbf{G}} \text{ with } l(p) + l(q) = l(pq)$$

where $l(p)$ denotes the cardinality of a $\mathbf{G}$-set in the class p.

PROOF. See [14, Theorem 6.20].

It is clear that $\mathrm{TL}_B(n)$ satisfies the hypotheses of this theorem. In this case a $\mathbf{G}$-set is complex if there exists a convex subset D and an edge J such that $h(d) \in J$ for all $d \in D$ and either $|D| \geq 4$ or $|D| = 3$ and $s \neq 0$, where s is the divisor of D.

Note that TB_n can also be realised as a projection algebra with the same underlying projection graph as for $\mathrm{TL}_B(n)$; it satisfies the conditions of Theorem A.1 and hence (as the conditions for a $\mathbf{G}$-set to be complex only depend on the projection graph) is free of the same rank as $\mathrm{TL}_B(n)$.

As noted after Proposition 2.1, it is possible to construct directly a bijection between our diagram basis for $\mathrm{TL}_B(n)$ and non-complex $\mathbf{G}$-sets. We conclude

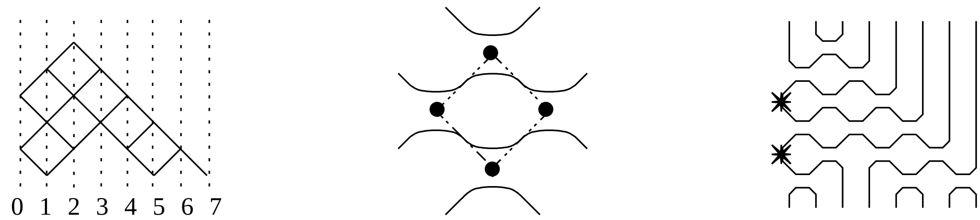


Fig. 29. (a), (b) and (c)

by sketching this correspondence. Any non-complex **G**-set (for our **G**) can be represented as a planar graph of the form shown in Figure 29(a) (where we ignore all dotted lines). Here we represent the elements of the set as vertices in the graph, with the relation induced by $a \leq b$ if a and b are connected by an edge with b more northerly than a. The values of the function h are indicated by the dotted lines.

We convert any such graph into a diagram basis element by replacing each vertex a with either a matching northern and southern arc (if $h(a) \neq 0$) or by a star (if $h(a) = 0$), and adding a boundary frame. We illustrate the change from nodes into arcs for a graph on four vertices (none of which correspond to the vertex 0 in our projection graph) in Figure 29(b), and for the graph in Figure 29(a) in Figure 29(c). For this latter example the resulting diagram is shown in Figure 30.

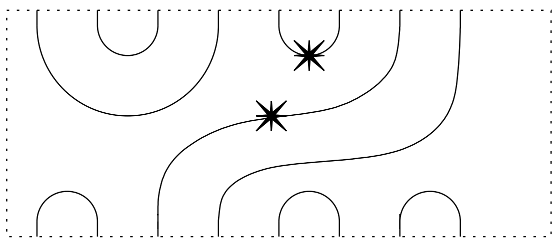


Fig. 30.

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