QUIVER PRESENTATIONS AND SCHUR–WEYL DUALITY FOR KHOVANOV ARC ALGEBRAS

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Abstract. We provide an Ext-quiver and relations presentation of the Khovanov arc algebras and as an application we prove a precise analogue of the Kleshchev–Martin conjecture in this setting.

1. Introduction

The Khovanov arc algebras, H_n^m, were first introduced by Khovanov (in the case $m = n$) in his pioneering construction of homological knot invariants for tangles [Kho00, Kho02]. These homological knot invariants have subsequently been developed by Rasmussen and put to use in Piccirillo's proof that the Conway knot is not slice [Ras10, Pic20]. The Khovanov arc algebras and their quasi-hereditary covers, K_n^m, have been studied from the point of view of their cohomological and representation theoretic structure [BS10, BS11b, BS12a, BS12b, BW], symplectic geometry [MS22], and they provide the exciting possibility of constructing algebraic invariants suitable for Crane–Frenkel's approach to the smooth 4-dimensional Poincaré conjecture [Man].

In previous work, we gave a quadratic presentation of the algebras K_n^m as $\mathbb{Z}$-algebras in terms of Dyck path combinatorics, which upon base change to any field $\Bbbk$ specialised to be the Ext-quiver and relations for the $\Bbbk$-algebra K_n^m [BDHS]; we hence completely determined the representation theoretic structure of K_n^m in an entirely characteristic-free manner. In [BDV$^+$] we further proved that over any field the algebra K_n^m is an $(|n-m|-1)$-faithful cover of H_n^m (in the sense of Rouquier [Rou08]) thus establishing a strong cohomological connection between these algebras. In this paper, we push the quadratic presentation of K_n^m through this Schur–Weyl duality to provide complete Ext-quiver and relations presentations of H_n^m, hence completely determining the representation theoretic structure of the Khovanov arc algebras. We will see that the structure of the algebras H_n^m is not as rigid as that of their quasi-hereditary covers: the Khovanov arc algebras are cubic and their structure does depend, to some extent, on the characteristic, p, of the underlying field $\Bbbk$. We assume, without loss of generality, that $m \leqslant n$.

Theorem A. *The* Ext*-quiver of H_n^m has vertices labelled by the partitions λ in an $(m \times n)$-rectangle such that $(m, m-1, \dots, 2, 1) \subseteq \lambda$. The edges are labelled by addable and removable Dyck paths: for $m = n$ and $p \neq 2$ there are m additional loops at $\lambda = (m^a, (m-a)^{m-a})$ for $1 \leqslant a \leqslant m$; for $m = n$ and $p = 2$ every vertex has an additional loop. The $\Bbbk$-algebra H_n^m is the quotient of the path algebra of its* Ext*-quiver modulo relations* (6.1) *to* (6.7)*.*

Examples of these Ext-quivers are given in Figures 1 to 3.

Frobenius algebras and categorification. The algebras H_n^m are symmetric (and therefore Frobenius) algebras; they are also graded Morita equivalent to certain level 2 cyclotomic quiver Hecke algebras (see [BDV$^+$, Remark 8.7]). In this wider Hecke algebra context, our Theorem A provides the first examples of new quiver presentations in the literature in over thirty years [EM94] — this three-decade gap underscores just how formidable it is to extract complete submodule structures for natural algebraic objects in the literature. Such presentations are hugely powerful if we seek to understand associated extension algebras as in [MT13, BE22, ES]. Quiver presentations of Frobenius algebras in particular are of wide interest as they serve as local models that feed into larger categorical frameworks, as in the work of many authors [CL12, KM19, Sav20, RS20, BSW21, BSW21, LNX24]. A particularly striking illustration of their power is the zigzag case, the unique prior instance of Theorem A (for m or n equal to 1) which played a pivotal role in Evseev–Kleshchev's construction of RoCK blocks, ultimately leading to the resolution of Turner's conjecture [EK18]. We expect our

Theorem A to play a central role in extending Evseev–Kleshchev's work to RoCK blocks of level 2 quiver Hecke algebras— this was our original motivation for this paper.

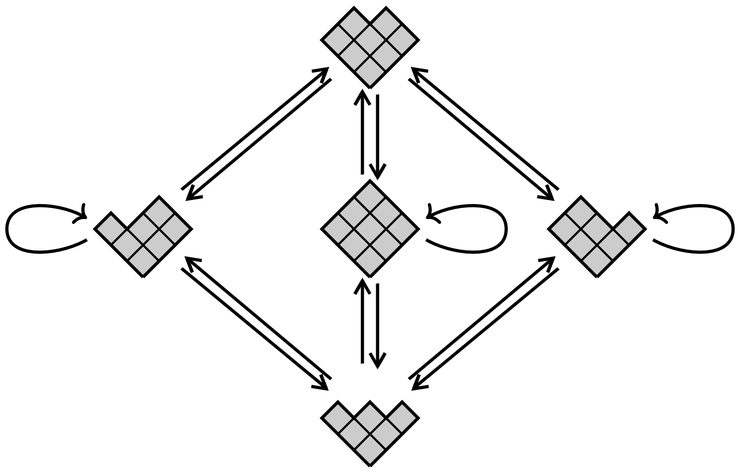


FIGURE 1. The Ext-quiver of H^3_3 for $p \neq 2$.

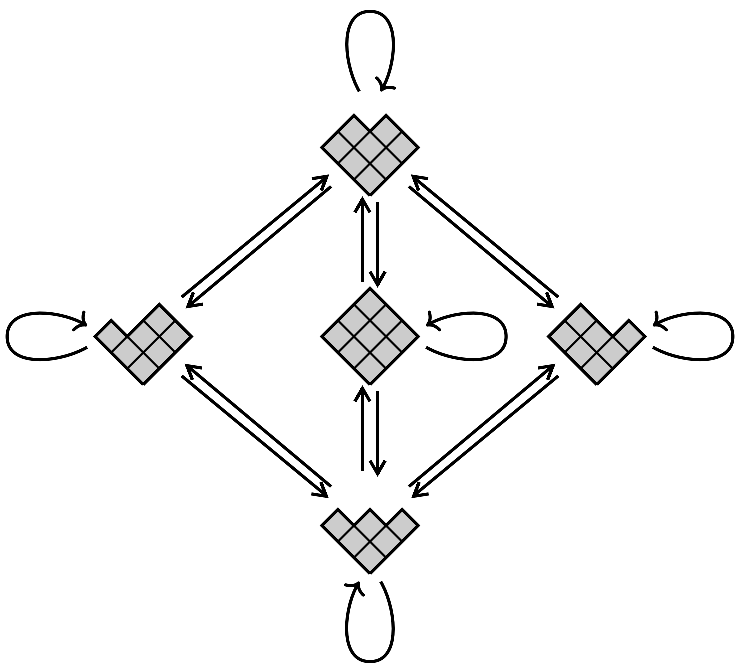


FIGURE 2. The Ext-quiver of H^3_3 for $p = 2$.

Schur–Weyl duality. Schur–Weyl duality is a more general phenomenon that relates Lie theoretic objects (such as categories $\mathcal{O}$ of Lie algebras, or (quantum) reductive algebraic groups) to interesting finite dimensional algebras (such as the group algebras of symmetric groups, Iwahori–Hecke algebras, and the Brauer and walled Brauer algebras). Representation theorists have long endeavoured to pass cohomological information back-and-forth by way of these dualities. A benchmark for our understanding (or lack thereof) for this back-and-forth process has been provided by the famous Kleshchev–Martin conjecture: this states that simple modules for the symmetric group do not admit self-extensions (providing that the Schur algebra is at least a 0-faithful cover). We prove that the exact analogue of this statement holds in our context.

Theorem B. *Let $\Bbbk$ be a field. The* Ext*-quiver of H^m_n is loop-free if and only if K^m_n is an i-faithful quasi-hereditary cover for some $i \geqslant 0$ if and only if $m \neq n$.*

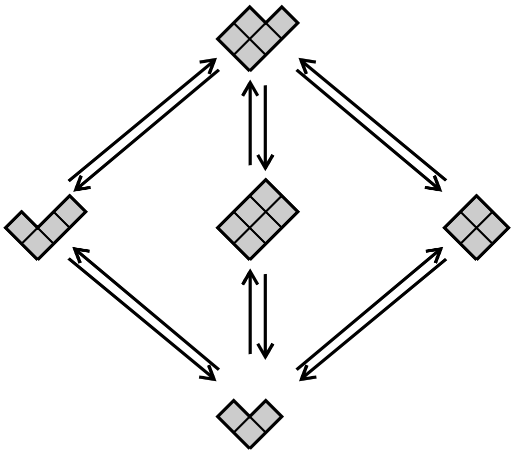


FIGURE 3. The Ext-quiver of H_3^2 for all $p \geqslant 2$.

The paper is structured as follows. Section 2 recalls the combinatorics of partitions and Dyck paths necessary for the statements of our results. Sections 3 and 4 develops this Dyck combinatorics further, providing the language needed to discuss the simple heads of cell modules of Khovanov arc algebras. Section 5 recalls the definition of the Khovanov arc algebras, their quasi-hereditary covers *the extended Khovanov arc algebras* and discusses their cellular structures; here we prove that every cell module has a simple head and deduce the full submodule structures of these cell modules. In Section 6 we define an abstract $\mathbb{Z}$-algebra via Dyck combinatorial generators and relations and prove that this algebra projects surjectively onto the Khovanov arc algebra. In Section 7 we show that this map is an isomorphism of algebras, therefore the arc algebras inherit these Dyck combinatorial presentations. Finally, in Section 8 we make the few tweaks necessary to refine these presentations into Ext-quiver and relations presentations over a field $\Bbbk$; as our algebras are basic, this simply requires that we find a minimal set of generators. In Section 8 we also recouch the Kleshchev–Martin conjecture in our language and propose a vast generalisation of this conjecture to all anti-spherical Hecke categories.

2. Partitions, cup and cap diagrams, and p-Kazhdan-Lusztig polynomials

We begin by reviewing and unifying the combinatorics of Khovanov arc algebras [BS10, BS11b, BS12a, BS12b] and the Hecke categories of interest in this paper [BDHN, BDH$^+$].

Let S_n denote the symmetric group of degree n. Throughout this paper, we will work with the parabolic Coxeter system $(W, P) = (S_{m+n}, S_m \times S_n)$. For the entire paper we assume, without loss of generality, that $m \leqslant n$. We label the simple reflections with the slightly unusual subscripts s_i, $-m+1 \leqslant i \leqslant n-1$ so that $P = \langle s_i \,|\, i \neq 0 \rangle \leqslant W$. We view W as the group of permutations of the $n + m$ points on a horizontal strip numbered by the half integers $i \pm \frac{1}{2}$ where the simple reflection s_i swaps the points $i - \frac{1}{2}$ and $i + \frac{1}{2}$ and fixes every other point. The right cosets of P in W can then be identified by labelled horizontal strips called weights, where each point $i \pm \frac{1}{2}$ is labelled by either $\wedge$ or $\vee$ in such a way that the total number of $\wedge$ is equal to m (and so the total number of $\vee$ is equal to n). We let $\Lambda_{m,n}$ denote the set of all weights with m points labelled by an $\wedge$ and with n points labelled by a $\vee$. Specifically, the trivial coset P is represented by the weight with negative points labelled by $\wedge$ and positive points labelled by $\vee$. The other cosets are obtained by permuting the labels of the identity weight. An example is given in 4. For more details on this combinatorics, see [BDHS, Section 2].

Formally, a partition λ of ℓ is defined to be a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ which sum to ℓ. We call $\ell(\lambda) := \ell = \sum_i \lambda_i$ the length of the partition λ. We define the Young diagram of a partition to be the collection of tiles

$$[\lambda] = \{[r, c] \mid 1 \leqslant c \leqslant \lambda_r\}$$

depicted in Russian style with rows at 135° and columns at 45°. We identify a partition with its Young diagram. We let λ^t denote the transpose partition given by reflection of the Russian Young

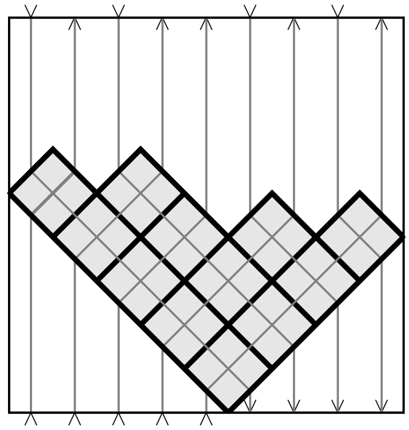


FIGURE 4. Along the top we picture the weight $\vee \wedge \vee \wedge \wedge \vee \wedge \vee \wedge \in \Lambda_{5,4}$, which corresponds to the partition $(5,4,2,1) \in \mathscr{P}_{5,4}$. Filling each box in the partition with an s_i generator constructs the corresponding coset (which when applied to the minimal element of $\Lambda_{5,4}$ permutes the $\wedge$s and $\vee$s so as to arrive at the given weight).

diagram through the vertical axis. Given $m, n \in \mathbb{N}$ we let $\mathscr{P}_{m,n}$ denote the set of all partitions which fit into an $m \times n$ rectangle, that is

$$\mathscr{P}_{m,n} = \{\lambda \mid \lambda_1 \leqslant m, \lambda_1^t \leqslant n\}.$$

For $\lambda \in \mathscr{P}_{m,n}$, the x-coordinate of a tile $[r,c] \in \lambda$ is equal to $r - c \in \{-m+1, -m+2, \dots, n-2, n-1\}$ and we define this x-coordinate to be the content of the tile and we write $\mathsf{cont}[r,c] = r - c$. Given $\lambda \in \mathscr{P}_{m,n}$, we define the set $\mathrm{Add}(\lambda)$ to be the set of all tiles $[r,c] \notin \lambda$ such that $\lambda \cup [r,c] \in \mathscr{P}_{m,n}$. Similarly, we define the set $\mathrm{Rem}(\lambda)$ to be the set of all tiles $[r,c] \in \lambda$ such that $\lambda \setminus [r,c] \in \mathscr{P}_{m,n}$.

The following definitions come from [BS11a].

Definition 2.1. ∘ *To each weight λ we associate a* cup diagram $\underline{\lambda}$ *and a* cap diagram $\overline{\lambda}$. *To construct $\underline{\lambda}$, repeatedly find a pair of vertices labeled $\vee\ \wedge$ in order from left to right that are neighbours in the sense that there are only vertices already joined by cups in between. Join these new vertices together with a cup. Then repeat the process until there are no more such $\vee\ \wedge$ pairs. Finally draw south-westerly rays at all the remaining $\wedge$ vertices and south-easterly rays at all the remaining $\vee$ vertices. The cap diagram $\overline{\lambda}$ is obtained by flipping $\underline{\lambda}$ horizontally. We stress that the vertices of the cup and cap diagrams are not labeled.*

- *Let λ and μ be weights. We can glue $\underline{\mu}$ under λ to obtain a new diagram $\underline{\mu}\lambda$. We say that $\underline{\mu}\lambda$ is* oriented *if (i) the vertices at the ends of each cup in $\underline{\mu}$ are labelled by exactly one $\vee$ and one $\wedge$ in the weight λ and (ii) it is impossible to find two rays in $\underline{\mu}$ whose top vertices are labeled $\vee\ \wedge$ in that order from left to right in the weight λ. Similarly, we obtain a new diagram $\lambda\overline{\mu}$ by gluing $\overline{\mu}$ on top of λ. We say that $\lambda\overline{\mu}$ is oriented if $\underline{\mu}\lambda$ is oriented.*
- *Let λ, μ be weights such that $\underline{\mu}\lambda$ is oriented. We set the* degree *of the diagram $\underline{\mu}\lambda$ (respectively $\lambda\overline{\mu}$) to the the number of clockwise oriented cups (respectively caps) in the diagram.*
- *Let λ, μ, ν be weights such that $\underline{\mu}\lambda$ and $\lambda\overline{\nu}$ are oriented. Then we form a new diagram $\underline{\mu}\lambda\overline{\nu}$ by gluing $\underline{\mu}$ under and $\overline{\nu}$ on top of λ. We set* $\deg(\underline{\mu}\lambda\overline{\nu}) = \deg(\underline{\mu}\lambda) + \deg(\lambda\overline{\nu})$.

An example is provided in Figure 5.

For the purposes of this paper, for $p \geqslant 0$, we can define the p-Kazhdan–Lusztig polynomials of type $(W,P) = (S_{n+m}, S_m \times S_n)$ as follows. For $\lambda, \mu \in \mathscr{P}_{m,n}$ we set

$${}^p n_{\lambda,\mu}(q) = \begin{cases} q^{\deg(\underline{\mu}\lambda)} & \text{if } \underline{\mu}\lambda \text{ is oriented;} \\ 0 & \text{otherwise.} \end{cases}$$

We refer to [BDH+, Theorem 7.3] and [BDHN, Theorem A] for a justification of this definition and to [BS11a] for the origins of this combinatorics.

FIGURE 5. The construction of the cup diagram $\underline{\lambda}$ for $\lambda = (5, 4, 2^2) \in \mathscr{P}_{5,6}$.

It is clear that for a fixed $\mu \in \mathscr{P}_{m,n}$, the diagram $\underline{\mu}\lambda$ is oriented if and only if the weight λ is obtained from the weight μ by swapping the labels on some of the pairs of vertices connected by a cup in $\underline{\mu}$. Moreover, in this case the degree of $\underline{\mu}\lambda$ is precisely the number of such swapped pairs.

We define the defect of $\lambda \in \mathscr{P}_{m,n}$, to be $d(\lambda) = d - m \in \mathbb{Z}_{\leqslant 0}$ if $(d, d-1, d-2, \dots, 1) \subseteq \lambda$ but $(d+1, d, d-1, \dots, 1) \not\subseteq \lambda$. In other words, the defect records the largest staircase partition sitting inside of λ (relative to the largest staircase partition $(m, m-1, m-2, \dots, 1) \in \mathscr{P}_{m,n}$). We say that a partition $\lambda \in \mathscr{P}_{m,n}$ is regular if $d(\lambda) = 0$. We let $\mathscr{R}_{m,n} \subseteq \mathscr{P}_{m,n}$ denote the subset of regular partitions. It follows from definitions that $\lambda \in \mathscr{P}_{m,n}$ has defect $d(\lambda) \leqslant 0$ if and only if $\underline{\lambda}\lambda$ has precisely $|d(\lambda)|$ vertices labelled by $\wedge$ connected to south-westerly rays. For example the partitions $\lambda = (5, 4, 2^2)$ has defect -1, as seen in Figure 5.

3. Dyck combinatorics

We have defined the p-Kazhdan–Lusztig polynomials via counting of certain oriented diagrams. For the purposes of this paper, we require richer combinatorial objects which *refine* the diagrammatic construction: these are provided by tilings by Dyck paths.

Let us start with a simple example to see how these Dyck paths come from the oriented diagrams. Consider the partitions $\mu = (5^3, 4, 1)$ and $\lambda = (4^2, 3, 1^2)$. The oriented diagrams $\underline{\mu}\mu$ and $\underline{\lambda}\lambda$ are illustrated in Figure 6. We see that λ is obtained from μ by swapping the labels of the vertices of one cup, $p \in \underline{\mu}$. The partition λ is obtained from the partition μ by removing a corresponding Dyck path $P \subseteq \mu$. More generally, if $\lambda, \mu \in \mathscr{P}_{m,n}$ with $\underline{\mu}\lambda$ oriented of degree k, then we will see that the partition λ is obtained from the partition μ by removing k Dyck paths.

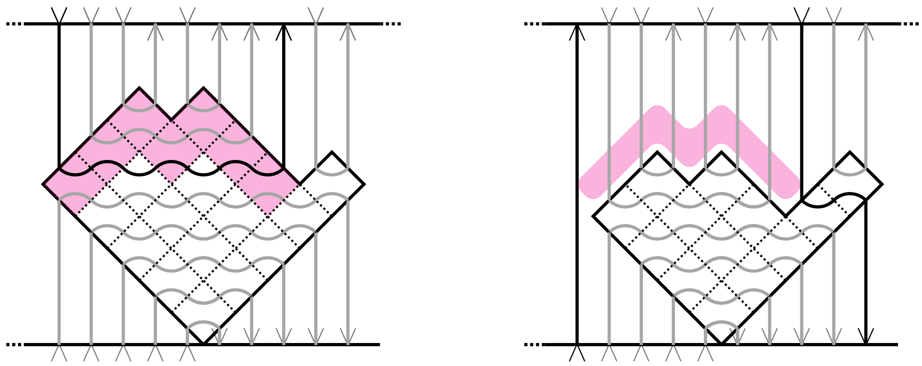


FIGURE 6. On the left we picture the partition/cup diagram for $(5^3, 4, 1)$ and we highlight the arc, p, and the corresponding removable Dyck path P. On the right we have the partition/cup diagram obtained by removing P.

3.1. **Dyck paths.** We define a path on the $m \times n$ tiled rectangle to be a finite non-empty set P of tiles that are ordered $[r_1, c_1], \dots, [r_s, c_s]$ for some $s \geqslant 1$ such that for each $1 \leqslant i \leqslant s-1$ we have $[r_{i+1}, c_{i+1}] = [r_i + 1, c_i]$ or $[r_i, c_i - 1]$. Note that the set $\underline{\mathsf{cont}}(P)$ of contents of the tiles in a path P form an interval of integers. We say that P is a Dyck path if

$$\min\{r_i + c_i - 1 : 1 \leqslant i \leqslant s\} = r_1 + c_1 - 1 = r_s + c_s - 1,$$

that is, the minimal height of the path is achieved at the start and end of the path. We will write

$$\mathsf{first}(P) = \mathsf{cont}([r_1, c_1]) \qquad \text{and} \qquad \mathsf{last}(P) = \mathsf{cont}([r_s, c_s]).$$

We designate the height $\mathsf{ht}(P)$ and breadth $b(P)$ as:

$$\mathsf{ht}(P) = r_1 + c_1 - 1 - m = r_s + c_s - 1 - m \qquad \text{and} \qquad b(P) = \tfrac{1}{2}(|P| + 1)$$

so that $\mathsf{ht}(P)$ records the vertical position of the lowest nodes in P and $b(P)$ records the horizontal distance covered by P.

Definition 3.1. *Let P and Q be Dyck paths.*

- *We say that P and Q are* adjacent *if and only if the multiset given by the disjoint union $\mathsf{cont}(P) \sqcup \mathsf{cont}(Q)$ is an interval.*
- *We say that P and Q are* distant *if and only if*

$$\min\{|\mathsf{cont}[r,c] - \mathsf{cont}[x,y]| \,:\, [r,c] \in P, [x,y] \in Q\} \geqslant 2.$$

- *We say that P* covers *Q and write $Q \prec P$ if and only if*

$$\mathsf{first}(Q) > \mathsf{first}(P) \text{ and } \mathsf{last}(Q) < \mathsf{last}(P).$$

Examples of such Dyck paths P and Q are given in Figure 7.

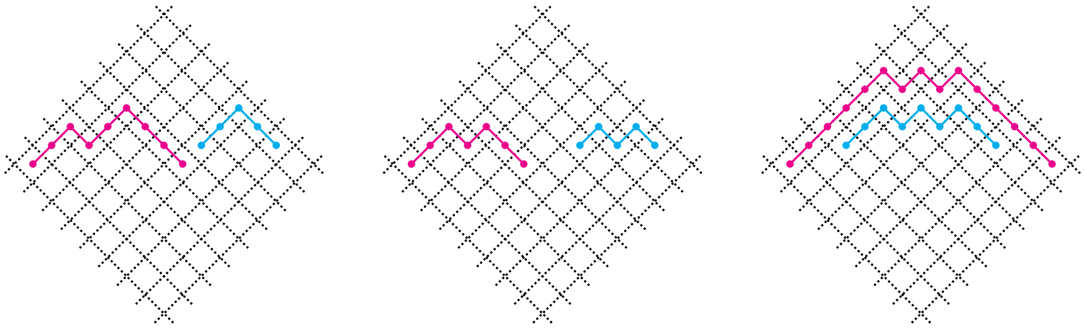


FIGURE 7. Examples of P and Q adjacent, distant, and $Q \prec P$ respectively.

Now we fix a partition $\mu \in \mathscr{P}_{m,n}$. Throughout the paper, we will identify all Dyck paths P having the same content interval $\underline{\mathsf{cont}}(P)$. There are a few places where we will need to fix a particular representative for a Dyck path P and in that case we will use subscripts, such as P_b.

Definition 3.2. *Let $\mu \in \mathscr{P}_{m,n}$ and P be a Dyck path. We say that P is a* removable Dyck path *from μ if there is a representative P_b of P such that $\lambda := \mu \setminus P_b \in \mathscr{P}_{m,n}$. In this case we will write $\lambda = \mu - P$, and $\mathsf{ht}^\mu(P) := \mathsf{ht}(P_b)$. We define the set $\mathrm{DRem}(\mu)$ to be the set of all removable Dyck paths from μ.*

We say that P is an addable Dyck path *of μ if there is a representative P_b of P such that $\lambda := \mu \sqcup P_b \in \mathscr{P}_{m,n}$. In this case we will write $\lambda = \mu + P$ and $\mathsf{ht}^\lambda(P) := h(P_b)$. We define the set $\mathrm{DAdd}(\mu)$ to be the set of all addable Dyck paths of μ.*

We let $\mathrm{DRem}_k(\mu)$ denote the set of all removable Dyck paths of μ of height k and similarly $\mathrm{DRem}_{\geqslant k}(\mu)$, $\mathrm{DRem}_{\leqslant k}(\mu)$ (and we define similarly $\mathrm{DAdd}_k(\mu)$ et cetera).

A removable Dyck path $P \in \mathrm{DRem}(\mu)$ corresponds to an anti-clockwise arc in the diagram $\underline{\mu}\mu$ with endpoints at $\mathsf{first}(P)$ and $\mathsf{last}(P)$. In particular, if P_b is the removable representative of $P \in \mathrm{DRem}(\mu)$, then its height $\mathsf{ht}^\mu(P)$ is given by

$$\mathsf{ht}^\mu(P) = \#\{\vee \text{ to the left of } \mathsf{first}(P) \text{ in } \mu\} - \#\{\wedge \text{ to the left of } \mathsf{first}(P) \text{ in } \mu\}.$$

Definition 3.3. *Let $\mu \in \mathscr{P}_{m,n}$ and $P, Q \in \mathrm{DRem}(\mu)$. We say that P and Q* commute *if $P \in \mathrm{DRem}(\mu - Q)$ and $Q \in \mathrm{DRem}(\mu - P)$.*

We wish to consider the effect of adding two Dyck paths, or subtracting one from the other.

Definition 3.4. *Let $P \in \mathrm{DRem}(\mu)$ and $Q \in \mathrm{DRem}(\mu - P)$ be adjacent. We define the* merge *of P and Q, denoted $\langle P \cup Q \rangle_\mu$, if it exists, to be the smallest removable Dyck path of μ containing $P \cup Q$.*

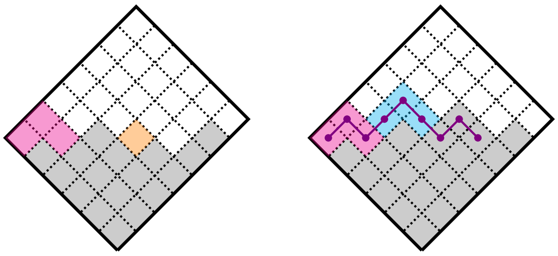


FIGURE 8. On the left we depict two commuting Dyck paths $P, Q \in \mathrm{DRem}(\mu)$ for $\mu = (6^2, 4, 3, 1^2)$. On the right we depict an example of two adjacent Dyck paths $P \in \mathrm{DRem}(\mu)$ and $Q \in \mathrm{DRem}(\mu - P)$ for $\mu = (6^2, 5^2, 3, 1)$ with the merge $R = \langle P, Q\rangle_\mu \in \mathrm{DRem}(\mu)$.

Definition 3.5. *Let $P, Q \in \mathrm{DRem}(\mu)$ be adjacent, with $P \prec Q$. Then we define the* split *of Q by P, denoted* $\mathsf{split}_Q(P)$*, to be the Dyck tiling $Q \setminus P = Q^1 \sqcup Q^2$ where $Q^1, Q^2 \in \mathrm{DRem}(\mu - P)$.*

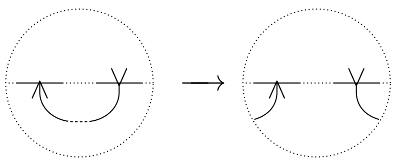
Definition 3.6. *Let $\lambda \subseteq \mu \in \mathscr{P}_{m,n}$. A* Dyck tiling *of the skew partition $\mu \setminus \lambda$ is a set $\{P^1, \dots, P^k\}$ of Dyck paths such that*

$$\mu \setminus \lambda = \bigsqcup_{i=1}^{k} P^i$$

and for each $i \neq j$ we have either P^i covers P^j (or vice versa), or P^i and P^j are distant. We call (λ, μ) a Dyck pair of degree k *if $\mu \setminus \lambda$ has a Dyck tiling with k Dyck paths.*

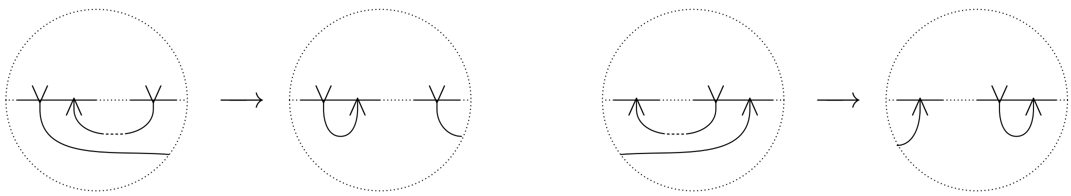
The Dyck tiling $\{P^1, \dots, P^k\}$ for $\mu \backslash \lambda$ (if it exists) is unique, though there are generally distinct choices of representatives $\{P^1_b, \dots, P^k_b\}$ for $\{P^1, \dots, P^k\}$ (we will consider these choices in the next subsection). We will freely associate the skew partition $\mu \backslash \lambda$ with its Dyck tiling, and abuse notation in using $\mu \backslash \lambda$ to refer to the Dyck tiling as well.

3.2. **Dyck tableaux.** A good move is a map $\underline{\mu}\lambda \to \underline{\nu}\lambda$ such that $\deg(\underline{\nu}\lambda) = \deg(\underline{\mu}\lambda) - 1$ and is of one of the following four possible forms. The first case to consider is when $\nu = \mu - P$ where $P \in \mathrm{DRem}(\mu)$ and $d(\nu) = d(\mu) - 1$, in which case we can break the arc p into 2 strands as follows:



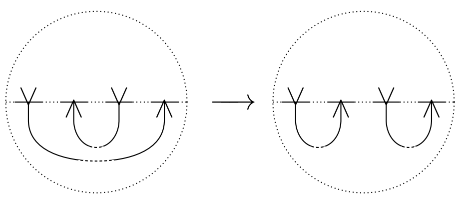
(G1)

The second case is that $\nu = \mu - P$ where $P \in \mathrm{DRem}(\mu)$ and $d(\mu) = d(\nu)$. We first suppose that there does not exist $P \prec Q \in \mathrm{DRem}(\lambda)$, in which case we can do precisely one of the following good moves



(G2)

We now suppose that there does exist $P \prec Q \in \mathrm{DRem}(\lambda)$ and suppose $b(Q)$ is minimal with respect to this property. We can further assume that there is no $P \prec R \prec Q$ such that $R \in \mu/\lambda$. In which case we do the following good move



(G3)

Proposition 3.7. *For any Dyck pair (λ, μ) of degree k, there clearly exists a sequence of good moves*

$$\underline{\mu}\lambda = \underline{\mu_k}\lambda \to \underline{\mu_{k-1}}\lambda \to \cdots \to \underline{\mu_0}\lambda = \underline{\lambda}\lambda.$$

Proof. We simply note that the only *apparent* restriction for this process is that the (G3) good move requires there not exist $P \prec R \prec Q$; however, if there does exist such an R then we can consider the pair (R, Q) first. □

Definition 3.8. *Let (λ, μ) be a Dyck pair of degree k. A* Dyck tableau *for (λ, μ) is any sequence of good moves*

$$\underline{\mu}\lambda = \underline{\mu_k}\lambda \to \underline{\mu_{k-1}}\lambda \to \cdots \to \underline{\mu_0}\lambda = \underline{\lambda}\lambda.$$

3.3. **Height of paths in a tiling.** Let (λ, μ) be a Dyck pair. Let P be a clockwise arc in $\underline{\mu}\lambda$. We define the set of Dyck paths supported by $P \in \mu/\lambda$ (and denote this set by $\mathsf{supp}^\mu_\lambda(P)$) as follows. Extend a line downward from the lowest point of the arc P in $\underline{\mu}\lambda$ which terminates as soon as it has intersected as many clockwise arcs (including P) as anti-clockwise arcs/half-arcs, or else never terminates. Then $\mathsf{supp}^\mu_\lambda(P)$ is the set of clockwise arcs in $\underline{\mu}\lambda$ intersected by this line. We define the height of a Dyck path $P \in \mu/\lambda$ as follows

$$\mathsf{ht}^\mu_\lambda(P) = \min\{\mathsf{ht}^\mu(P), \mathsf{ht}^\mu(Q) - 1 \mid Q \in \mathsf{supp}^\mu_\lambda(P)\backslash P\}. \tag{3.1}$$

Note that if $P \in \mathrm{DRem}(\mu)$ then $\mathsf{ht}^\mu(P) = \mathsf{ht}^\mu_{\mu-P}(P)$.

Proposition 3.9. *Let $(P^1, \dots, P^k)$ be the unique ordering of the clockwise arcs in $\underline{\mu}\lambda$ such that*

$$\mathsf{ht}^\mu_\lambda(P^i) < \mathsf{ht}^\mu_\lambda(P^{i+1}) \qquad \text{or} \qquad \mathsf{ht}^\mu_\lambda(P^i) = \mathsf{ht}^\mu_\lambda(P^{i+1}) \text{ and } \mathsf{last}(P^i) < \mathsf{first}(P^{i+1}).$$

Then, setting $\mu_i = \mu_{i+1} - P^{i+1}$, we have that

$$\underline{\mu}\lambda = \underline{\mu_k}\lambda \to \underline{\mu_{k-1}}\lambda \to \cdots \to \underline{\mu_0}\lambda = \underline{\lambda}\lambda \tag{3.2}$$

is a Dyck tableau.

Proof. Let $P \prec Q \in \mathrm{DRem}(\lambda)$ and assume $b(Q)$ is minimal with respect to this property. If $P \prec R \prec Q$ for $R \in \mu/\lambda$ then $R \in \mathsf{supp}^\mu_\lambda(P)$ and so $\mathsf{ht}^\mu_\lambda(P) < \mathsf{ht}^\mu_\lambda(R)$. Therefore we can always remove any maximal height Dyck path using a good move. □

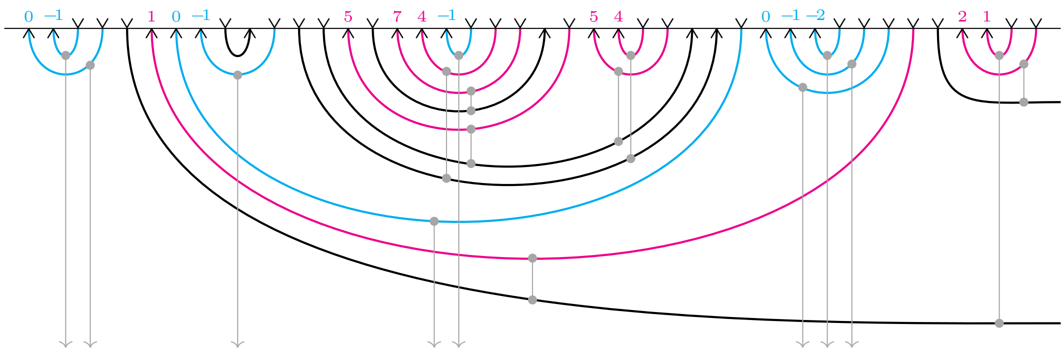


FIGURE 9. The arc diagram of $\underline{\mu}\lambda$, for the partitions $\mu = (20^2, 18^5, 16^7, 11^2, 6^3, 2^3)$ and $\lambda = (18^3, 15, 14^3, 13, 10^3, 9, 7^2, 5, 2^5)$. For each coloured clockwise arc P, the support set $\mathsf{supp}^\mu_\lambda(P)$ is indicated by the set of coloured arcs intersected by the grey line emanating downward from P. We record the height $\mathsf{ht}^\mu_\lambda(P)$ (in the sense of equation (3.1)) at the upward vertex of P (and we colour pink/blue if this height is positive/non-positive). Compare with Figure 10.

Definition 3.10. *We refer to the tableau of (3.2) as the* canonical Dyck tableau. *For $k \in \mathbb{Z}$ we define*

$$(\mu \setminus \lambda)_k = \{P \mid P \in \mu \setminus \lambda \text{ and } \mathsf{ht}^\mu_\lambda(P) = k\}$$

and we define $(\mu \setminus \lambda)_{\leqslant k}$ and $(\mu \setminus \lambda)_{\geqslant k}$ in a similar fashion. See Figure 10 for an example.

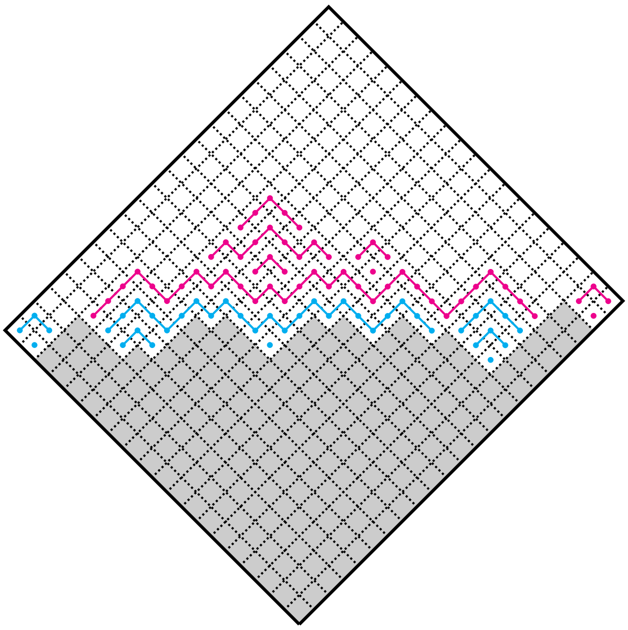


FIGURE 10. A Dyck tiling for the Dyck pair (λ, μ) as in Figure 9 chosen so as to highlight the heights. The canonical tableau is the tableau which adds these specific Dyck paths in order from bottom-to-top and left-to-right. The Dyck paths of strictly positive height are coloured pink and all others are coloured blue. Compare with Figure 9.

Remark 3.11. *The canonical Dyck tiling corresponds to the unique choice of representatives* $\{P_b^1, \dots, P_b^k\}$ *for the Dyck tiles* $\{P^1, \dots, P^k\}$ *with the following property: for all* i, j *with* $P^i \prec P^j$ *we have that* P_b^i *is below* P_b^j *only if there is no other choice of representatives* $\{P_{b'}^1, \dots, P_{b'}^k\}$ *with* P_b^i *above* P_b^j.

One may arrive at the canonical tiling for $\mu \backslash \lambda$ *from any choice of representatives for the Dyck tiling of* $\mu \backslash \lambda$ *by iteratively commuting shorter Dyck tiles upwards past longer Dyck tiles whenever possible. The acceptable arrangements for Dyck paths* $P \prec Q$ *in a canonical tiling are captured in Figure 11.*

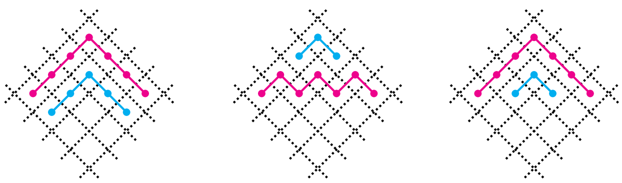


FIGURE 11. The first two arrangements of Dyck paths are good and the final arrangement is bad. This is because the small blue Dyck path in the third diagram can be commuted upwards to obtained the second diagram (thus turning the bad arrangement into a good one). See Figure 9 for an example.

4. THE REGULARISATION MAP

We now define a regularisation map from non-regular partitions to regular partitions. For any Dyck pair (α, μ), we then construct a sequence from $\mathsf{reg}(\alpha)$ to μ given by adding and splitting Dyck paths. This is the combinatorial shadow of a representation theoretic result: in Section 5 we will use this map to show that the H_n^m-cell-module $S_{m,n}(\alpha)$ has simple head $D_{m,n}(\mathsf{reg}(\alpha))$.

Definition 4.1. *Given* $\alpha \in \mathscr{P}_{m,n}$ *with* $d(\alpha) = d < 0$, *we define the regularisation of* α, *denoted* $\mathsf{reg}(\alpha)$, *as follows. We define a sequence of partitions*

$$\alpha = \mathsf{reg}_d(\alpha) \subset \mathsf{reg}_{d+1}(\alpha) \subset \cdots \subset \mathsf{reg}_{-1}(\alpha) \subset \mathsf{reg}_0(\alpha) = \mathsf{reg}(\alpha) \tag{4.1}$$

such that $P^k = \mathsf{reg}_k(\alpha) \setminus \mathsf{reg}_{k-1}(\alpha)$ *is the maximal breadth addable Dyck path of height* k*. In other words,* P^k *is the Dyck path obtained by connecting all the nodes* $[r, c] \notin \mathsf{reg}_{k-1}(\alpha)$ *such that* $r + c - 1 - m = k$*.*

From the arc diagram point of view, $\overline{\mathsf{reg}(\alpha)}\alpha$ *is the diagram obtained from* $\underline{\alpha}\alpha$ *by applying (the reverse of) as many (G1) good moves as possible, thus replacing all* $|d(\alpha)|$ *south-westerly strands (and therefore a corresponding* $|d(\alpha)|$ *south-easterly strands) with cups.*

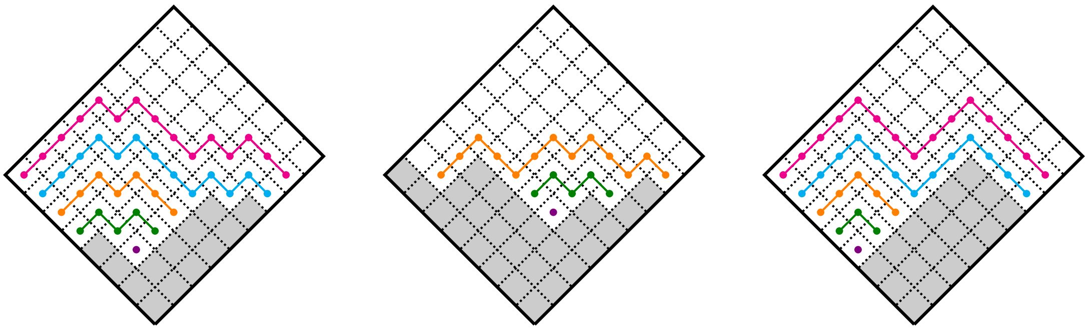


FIGURE 12. Three partitions of negative defect and their regularisations.

Remark 4.2. *The regularisation process produces the canonical tiling*

$$\mathsf{reg}(\alpha) \setminus \alpha = \bigsqcup_{d(\alpha)+1 \leqslant k \leqslant 0} P^k$$

of $\mathsf{reg}(\alpha) \setminus \alpha$ *and with* $\mathsf{ht}(P^k) = k \leqslant 0$ *for all* $d(\alpha) + 1 \leqslant k \leqslant 0$*. Moreover, this is the unique tiling of* $\mathsf{reg}(\alpha) \setminus \alpha$ *and is of degree* $|d(\alpha)|$*.*

Example 4.3. *Consider the partition* $\alpha = (8, 6^2, 2^2, 1^2)$ *whose regularisation is the one in the centre of Figure 12. We have that* $d(\alpha) = -3$ *and the sequence of partitions produced by the regularisation is:*

$$\alpha = \mathsf{reg}_{-3}(\alpha) \subset \mathsf{reg}_{-2}(\alpha) \subset \mathsf{reg}_{-1}(\alpha) \subset \mathsf{reg}_0(\alpha) = \mathsf{reg}(\alpha)$$

where $\mathsf{reg}_{-2}(\alpha) = \mathsf{reg}_{-3}(\alpha) + P^{-2}$, $\mathsf{reg}_{-1}(\alpha) = \mathsf{reg}_{-2}(\alpha) + P^{-1}$*, and* $\mathsf{reg}_0(\alpha) = \mathsf{reg}_{-1}(\alpha) + P^0$*. Each* P^k *for* $-2 \leqslant k \leqslant 0$ *corresponds to the the Dyck path of the same colour in the Figure 14. The corresponding sequence of oriented diagrams is pictured in Figure 13*

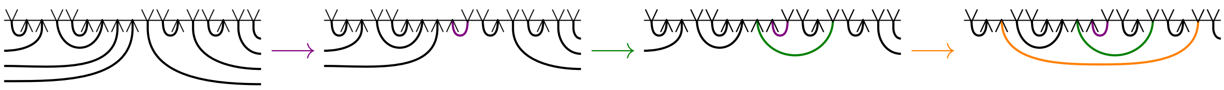


FIGURE 13. On the left we picture $\underline{\alpha}\alpha$ and on the right $\overline{\mathsf{reg}(\alpha)}\alpha$ for $\alpha = (8, 6^2, 2^2, 1^2)$. Reading the diagrams from right to left we are doing good move (G1) three times.

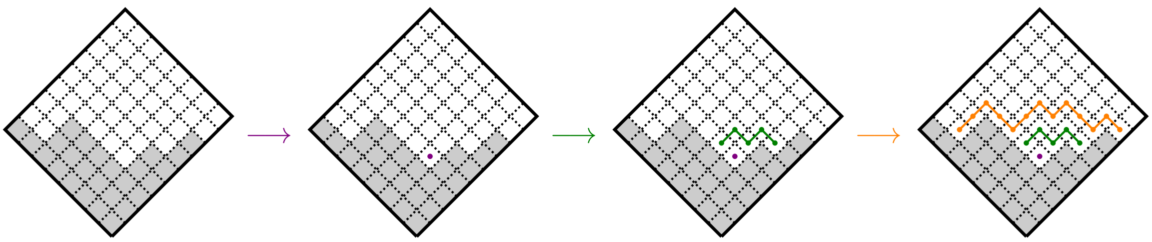


FIGURE 14. The sequence of partitions produced by the regularisation of $\alpha = (8, 6^2, 2^2, 1^2)$.

Proposition 4.4. *Let $\alpha \in \mathscr{P}_{m,n}, \mu \in \mathscr{R}_{m,n}$. Assume $\mu\backslash\alpha = (\mu\backslash\alpha)_{\leqslant 0}$ is a Dyck pair of degree k. Then $(\mu, \mathsf{reg}(\alpha))$ is a Dyck pair of degree $k + d(\alpha)$.*

Proof. Consider the diagrams $\underline{\alpha}\alpha$ and $\underline{\mu}\alpha$. We let $\mathcal{A}'$ denote the set of arcs and strands common to both diagrams $\underline{\alpha}\alpha$ and $\underline{\mu}\alpha$ and we set $\mathcal{A}$ to the set of arcs in $\underline{\mu}\alpha$ not belonging to $\mathcal{A}'$. Our assumption that $(\mu\backslash\alpha)_{\leqslant 0} = \mu\backslash\alpha$ implies that $\underline{\alpha}\alpha$ is obtained from $\underline{\mu}\alpha$ by a sequence of good moves of the form (G1) and (G2). In particular, no end point of an arc in $\mathcal{A}$ lies within a region above an arc or strand from $\mathcal{A}'$.

We let $\mathcal{B}$ denote the set of arcs in $\underline{\alpha}\alpha$ and which are not in $\mathcal{A}'$. We let $\mathcal{C}$ denote the set of strands in $\underline{\alpha}\alpha$ which do not belong to $\mathcal{A}'$. Since μ is of defect zero, the end points of the arcs in $\mathcal{A}$ are in bijection with the end points of the arcs and strands in $\mathcal{B} \sqcup \mathcal{C}$. We have that $\mathcal{C}$ consists of $2|d(\alpha)|$ strands (half south-easterly and half south-westerly) and that $\mathcal{B}$ consists solely of anti-clockwise oriented cups.

We observe that $\underline{\mathsf{reg}(\alpha)}$ is obtained from $\underline{\alpha}$ by replacing all the strands in $\mathcal{C}$ with cups. Therefore $\mathsf{reg}(\alpha)$ is obtained from α by flipping the labels of the vertices at the end points of the strands from $\mathcal{C}$. On the other hand, μ is obtained from $\mathsf{reg}(\alpha)$ by flipping all the labels of the vertices at the end points of the arcs in $\mathcal{B}$.

Therefore $\underline{\mathsf{reg}(\alpha)}\mu$ is an oriented diagram of degree equal to the number of arcs in $\mathcal{B}$ (which is equal to $k + d(\alpha)$) as required. ☐

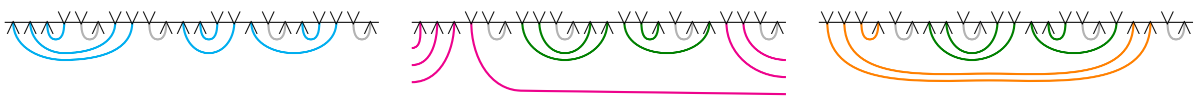


FIGURE 15. From left to right we picture $\underline{\mu}\alpha$, $\underline{\alpha}\alpha$, $\underline{\mathsf{reg}(\alpha)}\mu$. The sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are coloured.

Instead of adding Dyck paths to get from λ to $\mathsf{reg}(\alpha)$, we can remove Dyck paths to get from $\mathsf{reg}(\alpha)$ to λ. In fact, since μ/α is a Dyck tiling, these removals can be thought of as splitting Dyck paths of $\mathsf{reg}(\alpha)/\alpha$. In particular, we have the following.

Corollary 4.5. *Let $(\mu \setminus \alpha)_{\leqslant 0}$ be a Dyck tiling with $\alpha \notin \mathscr{R}_{m,n}$. Then there is a sequence of partitions of the form*

$$\mathsf{reg}(\alpha) = \mathsf{split}_{d(\alpha)}(\mu \setminus \alpha) \supseteq \mathsf{split}_{d(\alpha)+1}(\mu \setminus \alpha) \cdots \supseteq \mathsf{split}_0(\mu \setminus \alpha) = \alpha \sqcup (\mu \setminus \alpha)_{\leqslant 0} \tag{4.2}$$

with $(\mathsf{split}_k(\mu \setminus \alpha)) \setminus (\mathsf{split}_{k-1}(\mu \setminus \alpha)) = R_1^k \sqcup R_2^k \sqcup \cdots \sqcup R_K^k$ a disjoint union of commuting Dyck paths such that

$$P^k - R_1^k - R_2^k - \cdots - R_K^k = (\mu \setminus \alpha)_k.$$

In particular, $R_1^k, R_2^k, \ldots, R_K^k$ are ordered from left to right.

Remark 4.6. *It's worth noting that if R_i^k and R_j^ℓ are such that $k < \ell$, then either R_i^k and R_j^ℓ commute or $R_i^k \succ R_j^\ell$.*

Example 4.7. *Consider the Dyck tiling $\mu \setminus \alpha$ on the left of Figure 16 where α is the partition of Example 4.3. We know the regularisation of α from Example 4.3. Following Corollary 4.5, we have that our "favourite path" from $\mathsf{reg}(\alpha) \setminus \alpha$ to $(\mu \setminus \alpha)_{\leqslant 0}$ is depicted in Figure 17. With the notation introduced in Corollary 4.5, we have that*

$$\mathsf{reg}(\alpha) = \mathsf{split}_{-3}(\mu \setminus \alpha) \supseteq \mathsf{split}_{-1}(\mu \setminus \alpha) \supseteq \mathsf{split}_0(\mu \setminus \alpha) = \alpha \sqcup (\mu \setminus \alpha)_{\leqslant 0}$$

and we have that

- $\mathsf{reg}(\alpha) = \alpha \sqcup P^{-2} \sqcup P^{-1} \sqcup P^0$;
- *there are no splitting at height -2, so $\mathsf{split}_{-2}(\mu \setminus \alpha) = \mathsf{split}_{-3}(\mu \setminus \alpha)$;*
- $\mathsf{split}_{-1}(\mu \setminus \alpha) \setminus \mathsf{split}_{-2}(\mu \setminus \alpha) = R^{-1}$ *and so* $(\mu \setminus \alpha)_{-1} = P^{-1} - R^{-1}$;
- $\mathsf{split}_0(\mu \setminus \alpha) \setminus \mathsf{split}_{-1}(\mu \setminus \alpha) = R_1^0 \sqcup R_2^0$ *and so* $(\mu \setminus \alpha)_0 = P^0 - R_1^0 - R_2^0$.

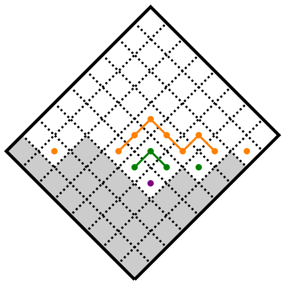


FIGURE 16. The Dyck tiling of $(\mu \setminus \alpha)_{\leqslant 0}$ and $\mu \setminus \alpha$ with $\mu = (8^3, 7, 6^3, 4, 2)$ and $\alpha = (8, 6^2, 2^2, 1^2)$

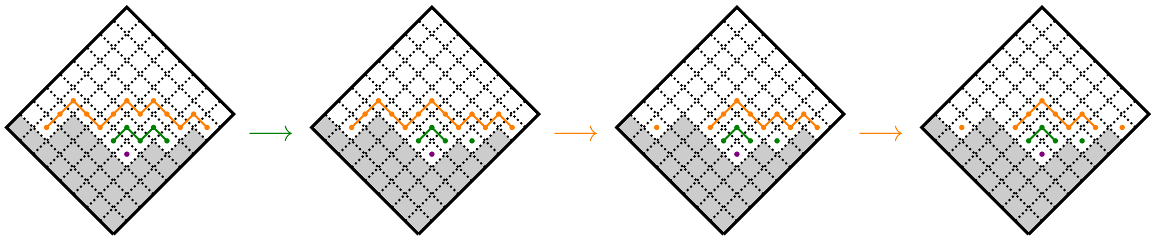


FIGURE 17. Our favourite path as in example 4.7.

Remark 4.8. *Let $\mu \setminus \alpha$ be a Dyck tiling with $\alpha \notin \mathscr{R}_{m,n}$. Then there is a sequence of partitions of the form*

$$(\mu \setminus \alpha)_{\leqslant 0} = \mathsf{add}_0(\mu \setminus \alpha) \subseteq \mathsf{add}_1(\mu \setminus \alpha) \cdots \subseteq \mathsf{add}_{k-1}(\mu \setminus \alpha) \subseteq \mathsf{add}_k(\mu \setminus \alpha) = \mu \tag{4.3}$$

where

$$\mathsf{add}_k(\mu \setminus \alpha) \setminus \mathsf{add}_{k-1}(\mu \setminus \alpha) = A_1^k \sqcup A_2^k \sqcup \cdots \sqcup A_K^k$$

is a union of addable Dyck paths of height $k > 0$.

Example 4.9. *Consider the Dyck tiling $\mu \setminus \alpha = (8^3, 7, 6^3, 4, 2) \setminus \alpha = (8, 6^2, 2^2, 1^2)$ depicted in Figure 16. Notice that the Dyck tiling of $\mu \setminus \alpha$ has a Dyck path of height 1 and that $(\mu \setminus \alpha)_{\leqslant 0}$ is the Dyck tiling consider in Example 4.7.*

Algorithm 4.1. *Let (α, μ) be a Dyck pair. We construct a canonical sequence of add/splits from $\alpha \to \mu$ by first adding Dyck paths as in (4.1), then splitting Dyck paths as in (4.2), then adding Dyck paths as in (4.3).*

An example of the three "big steps" in this algorithm is given in Figure 18, the smaller steps are given in Figures 14 and 17.

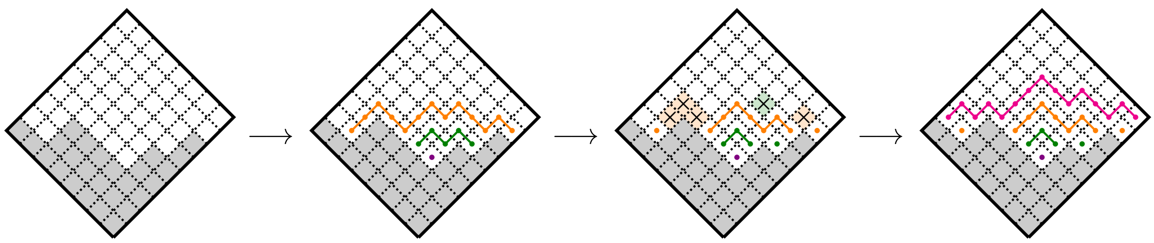


FIGURE 18. The sequence $\alpha \to \mathsf{reg}(\alpha) \to \alpha \sqcup (\mu \setminus \alpha)_{\leqslant 0} \to \mu \setminus \alpha$ where the steps are regularise, split, add as in (4.1), (4.2), and (4.3). In the third diagram the boxes marked with an $\times$ are the ones deleted in the splitting step.

5. The (extended) Khovanov arc algebras, cellularity, and the Schur functor

We are now ready to introduce the algebras of interest in this paper and recall the facts we will require about their representation theory. The new result of this section is that *all* cell modules of Khovanov arc algebras have simple heads (this makes use of the combinatorics developed in Section 4). This allows us to give a new construction of the cellular basis of the Khovanov arc algebra; this will provide the backbone of our proof of Theorem A from the introduction.

5.1. The original definitions of the (extended) Khovanov arc algebras. We now recall the definition of the extended Khovanov arc algebras studied in [BS10, BS11b, BS12a, BS12b]. We define K_n^m to be the algebra spanned by diagrams

$$\{\underline{\lambda}\mu\overline{\nu} \mid \lambda, \mu, \nu \in \mathscr{P}_{m,n} \text{ such that } \mu\overline{\nu}, \underline{\lambda}\mu \text{ are oriented}\}$$

with the multiplication defined as follows. First set

$$(\underline{\lambda}\mu\overline{\nu})(\underline{\alpha}\beta\overline{\gamma}) = 0 \quad \text{unless } \nu = \alpha.$$

To compute $(\underline{\lambda}\mu\overline{\nu})(\underline{\nu}\beta\overline{\gamma})$ place $(\underline{\lambda}\mu\overline{\nu})$ under $(\underline{\nu}\beta\overline{\gamma})$ then follow the 'surgery' procedure. This surgery combines two circles into one or splits one circle into two using the following rules for re-orientation (where we use the notation 1 = anti-clockwise circle, x = clockwise circle, y = oriented strand). We have the splitting rules

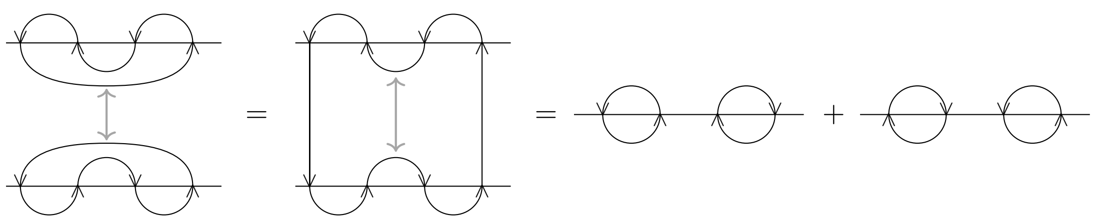
$$1 \mapsto 1 \otimes x + x \otimes 1, \quad x \mapsto x \otimes x, \quad y \mapsto x \otimes y.$$

and the merging rules

$$1 \otimes 1 \mapsto 1, \quad 1 \otimes x \mapsto x, \quad x \otimes 1 \mapsto x, \quad x \otimes x \mapsto 0, \quad 1 \otimes y \mapsto y, \quad x \otimes y \mapsto 0,$$

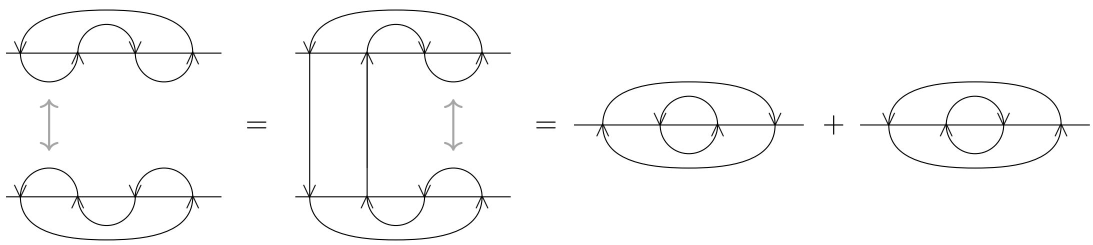
$$y \otimes y \mapsto \begin{cases} y \otimes y & \text{if both strands are propagating, one is} \\ & \wedge\text{-oriented and the other is } \vee\text{-oriented;} \\ 0 & \text{otherwise.} \end{cases}$$

Example 5.1. *We have the following product of Khovanov diagrams*



where we highlight with arrows the pair of arcs on which we are about to perform surgery. The first equality follows from the merging rule for $1 \otimes 1 \mapsto 1$ and the second equality follows from the merging rule $1 \mapsto 1 \otimes x + x \otimes 1$.

Example 5.2. *We have the following product of Khovanov diagrams*



where we highlight with arrows the pair of arcs on which we are about to perform surgery. This is similar to Example 5.1.

5.2. **Quiver presentations and cellularity of the extended Khovanov arc algebra.** In [BDHS] we proved that the *extended* Khovanov arc algebras are isomorphic to the basic algebras of the category algebras of Hecke categories of type $(S_{m+n}, S_m \times S_n)$, see also [BDHS24]. This isomorphism was constructed in a similar spirit to that of [BCH23, BCHM22]. These isomorphisms allowed us to prove an integral presentation of the extended Khovanov arc algebras in [BDHS, Theorem B], which we now recall.

Definition 5.3. *The algebra K^m_n is the associative $\Bbbk$-algebra generated by the elements*

$$\{D^\lambda_\mu, D^\mu_\lambda \mid \lambda,\mu \in \mathscr{P}_{m,n} \text{ with } \lambda = \mu - P \text{ for some } P \in \mathrm{DRem}(\mu)\} \cup \{1_\mu \mid \mu \in \mathscr{P}_{m,n}\} \tag{5.1}$$

subject to the following relations and their duals.

The idempotent relations: *For all $\lambda,\mu \in \mathscr{P}_{m,n}$, we have that*

$$1_\mu 1_\lambda = \delta_{\lambda,\mu} 1_\lambda \qquad\qquad 1_\lambda D^\lambda_\mu 1_\mu = D^\lambda_\mu. \tag{5.2}$$

The self-dual relation: *Let $P \in \mathrm{DRem}(\mu)$ and $\lambda = \mu - P$. Then we have*

$$D^\lambda_\mu D^\mu_\lambda = (-1)^{b(P)-1}\Bigg(2\sum_{\substack{Q\in\mathrm{DRem}(\lambda)\\ P\prec Q}} (-1)^{b(Q)} D^\lambda_{\lambda-Q} D^{\lambda-Q}_\lambda + \sum_{\substack{Q\in\mathrm{DRem}(\lambda)\\ Q \text{ adj. } P}} (-1)^{b(Q)} D^\lambda_{\lambda-Q} D^{\lambda-Q}_\lambda\Bigg) \tag{5.3}$$

where we abbreviate "adjacent to" simply as "adj."

The commuting relations: *Let $P, Q \in \mathrm{DRem}(\mu)$ which commute. Then we have*

$$D^{\mu-P-Q}_{\mu-P} D^{\mu-P}_\mu = D^{\mu-P-Q}_{\mu-Q} D^{\mu-Q}_\mu \qquad D^{\mu-P}_\mu D^\mu_{\mu-Q} = D^{\mu-P}_{\mu-P-Q} D^{\mu-P-Q}_{\mu-Q}. \tag{5.4}$$

The non-commuting relation: *Let $P, Q \in \mathrm{DRem}(\mu)$ with $P \prec Q$ which do not commute. Then $Q \setminus P = Q^1 \sqcup Q^2$ where $Q^1, Q^2 \in \mathrm{DRem}(\mu - P)$ and we have*

$$D^{\mu-Q}_\mu D^\mu_{\mu-P} = D^{\mu-Q}_{\mu-P-Q^1} D^{\mu-P-Q^1}_{\mu-P} = D^{\mu-Q}_{\mu-P-Q^2} D^{\mu-P-Q^2}_{\mu-P}. \tag{5.5}$$

The adjacent relation: *Let $P \in \mathrm{DRem}(\mu)$ and $Q \in \mathrm{DRem}(\mu - P)$ be adjacent. Recall that $\langle P\cup Q\rangle_\mu$, if it exists, denotes the smallest removable Dyck path of μ containing $P \cup Q$. Then we have*

$$D^{\mu-P-Q}_{\mu-P} D^{\mu-P}_\mu = \begin{cases} (-1)^{b(\langle P\cup Q\rangle_\mu)-b(Q)} D^{\mu-P-Q}_{\mu-\langle P\cup Q\rangle_\mu} D^{\mu-\langle P\cup Q\rangle_\mu}_\mu & \text{if } \langle P\cup Q\rangle_\mu \text{ exists;} \\ 0 & \text{otherwise.}\end{cases} \tag{5.6}$$

Definition 5.4. *Given a Dyck tiling $\mu \setminus \lambda = \bigsqcup_{i=1}^k P^i$ we define an associated diagram*

$$D^\lambda_\mu = D^\lambda_{\lambda+P^1} D^{\lambda+P^1}_{\lambda+P^1+P^2} \cdots D^{\lambda+P^1+\dots+P^{k-1}}_{\lambda+P^1+\dots+P^{k-1}+P^k}$$

which is independent of the ordering in which the Dyck paths are added.

The following construction is due to Libedinsky–Williamson [LW], but is couched in our language in [BDHS, Theorem 5.4]. Recall that a Dyck pair (λ,μ) consists of two partitions such that $\mu\setminus\lambda$ has a Dyck tiling.

Theorem 5.5 ([BDHS, Theorem 5.4] and [LW])**.** *The algebra K^m_n is a graded cellular (in fact quasi-hereditary) algebra with graded cellular basis given by*

$$\{D^\mu_\lambda D^\lambda_\nu \mid \lambda,\mu,\nu \in \mathscr{P}_{m,n} \text{ with } (\lambda,\mu),(\lambda,\nu) \text{ Dyck pairs}\} \tag{5.7}$$

with

$$\deg(D^\mu_\lambda D^\lambda_\nu) = \deg(\lambda,\mu) + \deg(\lambda,\nu),$$

with respect to the involution $*$ and the partial order on $\mathscr{P}_{m,n}$ given by inclusion.

Definition 5.6. *For $\lambda \in \mathscr{P}_{m,n}$, we define the* standard module *$\Delta_{m,n}(\lambda)$ to be the right K^m_n-module*

$$\Delta_{m,n}(\lambda) = 1_\lambda K^m_n / (K^m_n (\textstyle\sum_{\mu\subset\lambda} 1_\mu) K^m_n).$$

This module has basis $\{D^\lambda_\nu \mid (\lambda,\nu) \text{ is a Dyck pair}\}$ with action given by right concatenation (modulo the ideal spanned by diagrams which factor through $\mu \subset \lambda$).

Since K^m_n is a quasi-hereditary cellular algebra, we have the following (see [GL96, Remark 3.10]).

Proposition 5.7. *A complete and irredundant set of non-isomorphic simple K^m_n-modules is given by*

$$\{L_{m,n}(\lambda) \mid L_{m,n}(\lambda) = \Delta_{m,n}(\lambda)/\mathrm{rad}(\Delta_{m,n}(\lambda)),\ \textit{for}\ \lambda \in \mathscr{P}_{m,n}\}.$$

We now describe the full submodule structure of the standard modules. As K^m_n is positively graded, the grading provides a submodule filtration of $\Delta_{m,n}(\lambda)$. Decompose $\mathrm{DP}(\lambda)$ as

$$\mathrm{DP}(\lambda) = \bigsqcup_{k\geqslant 0} \mathrm{DP}_k(\lambda) \quad \text{where} \quad \mathrm{DP}_k(\lambda) = \{\mu \in \mathrm{DP}(\lambda) \,:\, \deg(\lambda,\mu) = k\}.$$

Theorem 5.8. *Let $\lambda \in \mathscr{P}_{m,n}$. The Alperin diagram of the standard module $\Delta_{m,n}(\lambda)$ has vertex set labelled by the set $\{L_{m,n}(\mu) \,:\, \mu \in \mathrm{DP}(\lambda)\}$ and edges*

$$L_{m,n}(\mu) \longrightarrow L_{m,n}(\nu)$$

whenever $\mu \in \mathrm{DP}_k(\lambda)$, $\nu \in \mathrm{DP}_{k+1}(\lambda)$ for some $k \geqslant 0$ and $\nu = \mu \pm P$ for some $P \in \mathrm{DAdd}(\mu)$ or $P \in \mathrm{DRem}(\mu)$ respectively.

An example is depicted in Figure 19, below.

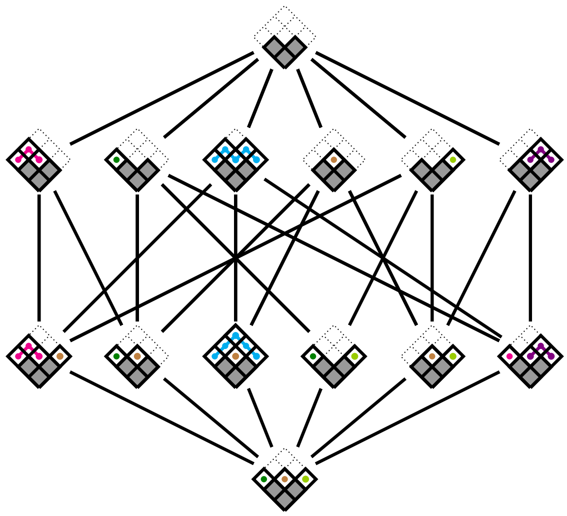


FIGURE 19. The full submodule lattice of the Verma module $\Delta_{3,3}(2,1)$ for $K_{3,3}$ and $\Bbbk$ any field. We represent each simple module by the corresponding partition (in Russian notation) and highlight the 3×3 rectangle in which the partition exists. This module has simple head $L_{3,3}(2,1)$ and simple socle $L_{3,3}(3,2,1)$.

5.3. **The Schur functor.** We set $\mathscr{R}_{m,n} = \{\lambda \in \mathscr{P}_{m,n} \mid \lambda \text{ has defect zero}\}$. We define the Schur idempotent to be the element

$$e = \sum_{\lambda\in\mathscr{R}_{m,n}} 1_\lambda \in K^m_n.$$

The main object of study in this paper will be the subalgebras

$$eK^m_n e \subseteq K^m_n$$

for $m, n \in \mathbb{N}$. We have the following immediate results:

Corollary 5.9. *The algebra $eK^m_n e$ is a graded cellular algebra with graded cellular basis given by*

$$\{D^\mu_\lambda D^\lambda_\nu \mid \lambda \in \mathscr{P}_{m,n}, \mu,\nu \in \mathscr{R}_{m,n} \textit{ with } (\lambda,\mu),(\lambda,\nu) \textit{ Dyck pairs}\} \tag{5.8}$$

with $\deg(D^\mu_\lambda D^\lambda_\nu) = \deg(\lambda,\mu) + \deg(\lambda,\nu)$, with respect to the involution $*$ and the partial order on $\mathscr{P}_{m,n}$ given by inclusion.

Proof. This follows from Theorem 5.5 and the definition of these algebras by idempotent truncation. ☐

Definition 5.10. *For $\lambda \in \mathscr{P}_{m,n}$, we define the* Specht module $S_{m,n}(\lambda)$ *to be the right H_n^m-module*

$$S_{m,n}(\lambda) = \Delta_{m,n}(\lambda)e.$$

This module has basis $\{D_\nu^\lambda \mid (\lambda,\nu) \text{ is a Dyck pair}, \nu \in \mathscr{R}_{m,n}\}$ with action given by right concatenation (modulo the ideal spanned by diagrams which factor through $\mu \subset \lambda$).

Proposition 5.11. *A complete and irredundant set of non-isomorphic simple H_n^m-modules is given by*

$$\{D_{m,n}(\lambda) \mid D_{m,n}(\lambda) = L_{m,n}(\lambda)e, \text{ for } \lambda \in \mathscr{R}_{m,n} \subset \mathscr{P}_{m,n}\}.$$

Proof. By Theorem 5.5 (which implies that $\mathscr{P}_{m,n}$ labels the simple K_n^m-modules) we need only show that $L_{m,n}(\lambda)e = 0$ if and only if $\lambda \notin \mathscr{R}_{m,n}$. This claim follows immediately from the fact that the algebra K_n^m is basic (and so all simple modules are 1-dimensional and generated by weight idempotents) and the definition of $e \in K_n^m$ as the sum over all weight idempotents from $\mathscr{R}_{m,n}$. □

Proposition 5.12. *Let $\lambda \in \mathscr{P}_{m,n}$. The Alperin diagram of the Specht module $S_{m,n}(\lambda)$ has vertex set labelled by the set $\{D_{m,n}(\mu) : \mu \in \mathrm{DP}(\lambda) \cap \mathscr{R}_{m,n}\}$ and edges*

$$D_{m,n}(\mu) \longrightarrow D_{m,n}(\nu)$$

whenever $\mu \in \mathrm{DP}_k(\lambda)$, $\nu \in \mathrm{DP}_{k+1}(\lambda)$ for some $k \geqslant 0$ and $\nu = \mu \pm P$ for some $P \in \mathrm{DAdd}(\mu)$ or $P \in \mathrm{DRem}(\mu)$ respectively.

Proof. This follows from Theorem 5.8 and the definition of these algebras by idempotent truncation. □

An example is depicted in Figure 20, below.

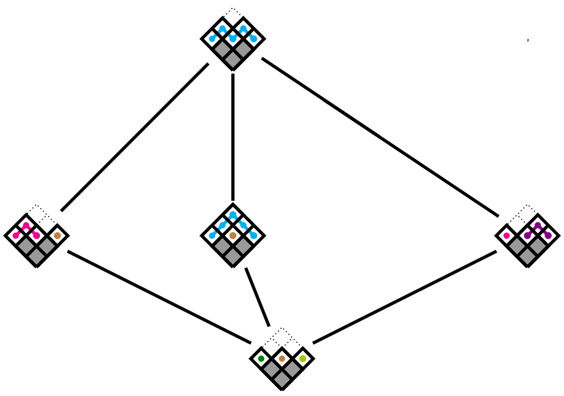


FIGURE 20. The full submodule lattice of the Specht module $S_{3,3}(2,1)$ for $eK_{3,3}e$ and $\Bbbk$ any field. This is obtained from the diagram in Figure 19 by simply deleting all composition factors labelled by $\lambda \notin \mathscr{R}_{m,n}$ (and edges from these vertices).

Thus we have complete submodule structure of the Specht modules, for free. In particular, we have the following:

Proposition 5.13. *The Specht module $S_{m,n}(\alpha)$ has simple head isomorphic to $D(\mathsf{reg}(\alpha))\langle d(\alpha)\rangle$.*

Proof. This follows from Proposition 5.12 and algorithm 4.1. □

Proposition 5.14. *The algebra $eK_n^m e$ is a graded cellular algebra with graded cellular basis given by*

$$\{D^\mu_{\mathsf{reg}(\alpha)} D^{\mathsf{reg}(\alpha)}_\alpha D^\alpha_{\mathsf{reg}(\alpha)} D^{\mathsf{reg}(\alpha)}_\nu \mid \alpha \in \mathscr{P}_{m,n}, \mu,\nu \in \mathscr{R}_{m,n} \text{ with } (\alpha,\mu),(\alpha,\nu) \text{ Dyck pairs}\} \qquad (5.9)$$

with $\deg(D^\mu_\alpha D^\alpha_\nu) = \deg(\alpha,\mu) + \deg(\alpha,\nu)$, with respect to the involution $*$ and the partial order on $\mathscr{P}_{m,n}$ given by inclusion.

Proof. This follows from Proposition 5.12 and algorithm 4.1. □

6. A symmetric algebra defined via Dyck combinatorics

We now define an abstract algebra, $\mathcal{A}_{m,n}$ via Dyck-combinatorial generators and relations. The main result of this paper will be that this algebra is isomorphic as a $\mathbb{Z}$-graded $\Bbbk$-algebra to $eK_n^m e$. We will prove this result in two steps over the next two chapters: we will first provide a spanning set of $\mathcal{A}_{m,n}$ in terms of pairs of Dyck paths; we will then construct a surjective $\Bbbk$-algebra homomorphism $\mathcal{A}_{m,n} \to H_n^m$. Putting these two facts together, we will deduce that the spanning set is in fact a basis and the homomorphism is injective, as required. However, before we can get on with the task of defining $\mathcal{A}_{m,n}$, we first require some additional combinatorics.

Definition 6.1. *We define the* regular quiver $Q_{m,n}$ *with vertex set* $\{\mathbb{1}_\lambda \mid \lambda \in \mathscr{R}_{m,n}\}$ *and arrows*

- $\mathbb{D}_\mu^\lambda : \lambda \to \mu$ *and* $\mathbb{D}_\lambda^\mu : \mu \to \lambda$ *for every* $\lambda = \mu - P$ *with* $P \in \mathrm{DRem}_{>0}(\mu)$*;*
- *for* $m = n$ *we have additional "loops" of degree 2,* $\mathbb{L}_\lambda^\lambda : \lambda \to \lambda$ *for every* $\lambda \in \mathscr{R}_{m,m}$.

An example is depicted in Figure 21 below.

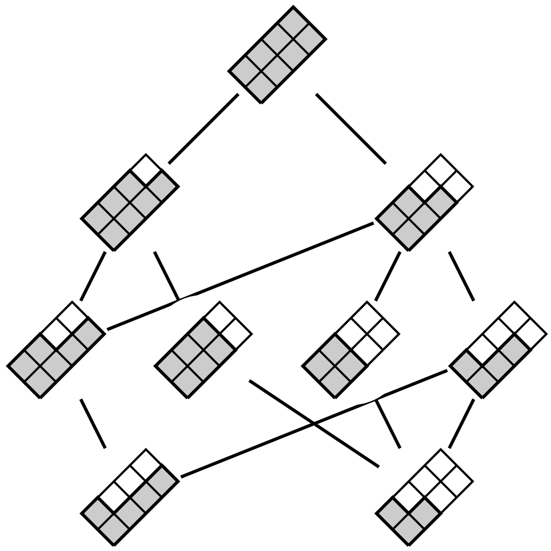


Figure 21. The regular quiver $Q_{2,4}$ is obtained by "doubling up" the above graph.

Definition 6.2. *Given* P, Q *two Dyck paths, we say that* P *is* (right) dominated by Q, *and write* $P \lhd Q$ *if* $\mathsf{last}(P) < \mathsf{first}(Q)$. *Given* $P \in \mathrm{Rem}_0(\lambda)$, *we let* $\mathsf{rt}(P)$ *denote the element of* $\{Q \mid P \lhd Q, Q \in \mathrm{Add}_1(\lambda)\}$ *which is of maximal breadth.*

We define certain generalised loop elements of the path algebra $\Bbbk Q_{m,n}$ as follows. We first associate the loop generator to a canonical Dyck path as follows:

Definition 6.3. *For* $\lambda \in \mathscr{R}_{m,n}$ *and* $P \in \mathrm{DRem}(\lambda)$, *we define an element* $\mathbb{L}_\lambda^\lambda(-P)$ *as a product of the generators of the path algebra as follows*

$$\mathbb{L}_\lambda^\lambda(-P) = \begin{cases} (-1)^{b(P)} \mathbb{D}_{\lambda-P}^\lambda \mathbb{D}_\lambda^{\lambda-P} & \text{if } \mathsf{level}(P) > 0; \\ (-1)^{b(\mathsf{rt}(P))+1} \mathbb{D}_{\lambda+\mathsf{rt}(P)}^\lambda \mathbb{D}_\lambda^{\lambda+\mathsf{rt}(P)} & \text{if } \mathsf{level}(P) = 0 \text{ and } m < n; \\ -\mathbb{L}_\lambda^\lambda + (-1)^{b(\mathsf{rt}(P))+1} \mathbb{D}_{\lambda+\mathsf{rt}(P)}^\lambda \mathbb{D}_\lambda^{\lambda+\mathsf{rt}(P)} & \text{if } \mathsf{level}(P) = 0 \text{ and } m = n; \\ \mathbb{L}_\lambda^\lambda & \text{if } \mathsf{level}(P) = 0,\ m = n,\ \mathsf{last}(P) = m-1. \end{cases}$$

Definition 6.4. *We define the symmetric Dyck algebra,* $\mathcal{A}_{m,n}$, *to be the* $\mathbb{Z}$*-graded* $\Bbbk$*-algebra given by the path algebra* $\Bbbk Q_{m,n}$ *modulo the following relations and their duals:*

The idempotent relations: *For all* $\lambda, \mu \in \mathscr{R}_{m,n}$, *we have that*

$$\mathbb{1}_\mu \mathbb{1}_\lambda = \delta_{\lambda,\mu} \mathbb{1}_\lambda \qquad \mathbb{1}_\lambda \mathbb{D}_\mu^\lambda \mathbb{1}_\mu = \mathbb{D}_\mu^\lambda \qquad \mathbb{1}_\lambda \mathbb{L}_\lambda^\lambda \mathbb{1}_\lambda = \mathbb{L}_\lambda^\lambda \tag{6.1}$$

(where the final relation is only defined for the $m = n$ case).

The self-dual relation: *Let $\lambda \in \mathscr{R}_{m,n}$ and $P \in \mathrm{DAdd}(\lambda)$. Then we have*

$$\mathbb{D}^{\lambda}_{\lambda+P}\mathbb{D}^{\lambda+P}_{\lambda} = (-1)^{b(P)-1}\Bigg(2\sum_{\substack{Q\in\mathrm{DRem}(\lambda)\\ P\prec Q}} \mathbb{L}^{\lambda}_{\lambda}(-Q) + \sum_{\substack{Q\in\mathrm{DRem}(\lambda)\\ Q\ adj.\ P}} \mathbb{L}^{\lambda}_{\lambda}(-Q)\Bigg) \tag{6.2}$$

where the elements on the righthand-side are linear combinations of the generators, as in Definition 6.3.

The commuting relations: *Let $P, Q \in \mathrm{DRem}(\lambda)$ which commute. Then we have*

$$\mathbb{D}^{\lambda-P-Q}_{\lambda-P}\mathbb{D}^{\lambda-P}_{\lambda} = \mathbb{D}^{\lambda-P-Q}_{\lambda-Q}\mathbb{D}^{\lambda-Q}_{\lambda} \qquad \mathbb{D}^{\lambda-P}_{\lambda}\mathbb{D}^{\lambda}_{\lambda-Q} = \mathbb{D}^{\lambda-P}_{\lambda-P-Q}\mathbb{D}^{\lambda-P-Q}_{\lambda-Q}. \tag{6.3}$$

The non-commuting relation: *Let $P, Q \in \mathrm{DRem}(\mu)$ with $P \prec Q$ which do not commute. Then $Q \setminus P = Q^1 \sqcup Q^2$ where $Q^1, Q^2 \in \mathrm{DRem}(\mu - P)$ and we have*

$$\mathbb{D}^{\mu-Q}_{\mu}\mathbb{D}^{\mu}_{\mu-P} = \mathbb{D}^{\mu-Q}_{\mu-P-Q^1}\mathbb{D}^{\mu-P-Q^1}_{\mu-P} = \mathbb{D}^{\mu-Q}_{\mu-P-Q^2}\mathbb{D}^{\mu-P-Q^2}_{\mu-P}. \tag{6.4}$$

The adjacent relation: *Let $P \in \mathrm{DRem}(\mu)$ and $Q \in \mathrm{DRem}(\mu-P)$ be adjacent. Recall that $\langle P\cup Q\rangle_\mu$, if it exists, denotes the smallest removable Dyck path of μ containing $P \cup Q$. Then we have*

$$\mathbb{D}^{\mu-P-Q}_{\mu-P}\mathbb{D}^{\mu-P}_{\mu} = \begin{cases} (-1)^{b(\langle P\cup Q\rangle_\mu)-b(Q)}\mathbb{D}^{\mu-P-Q}_{\mu-\langle P\cup Q\rangle_\mu}\mathbb{D}^{\mu-\langle P\cup Q\rangle_\mu}_{\mu} & \text{if } \langle P\cup Q\rangle_\mu \text{ exists;}\\ 0 & \text{otherwise.}\end{cases} \tag{6.5}$$

The cubic relation: *Let $P \in \mathrm{DAdd}_1(\mu)$ be such $\mathsf{last}(P)$ is maximal with respect to this property. Then*

$$\mathbb{D}^{\mu+P}_{\mu}\mathbb{D}^{\mu}_{\mu+P}\mathbb{D}^{\mu}_{\mu+P} = \begin{cases}(-1)^{b(P)+1}2\mathbb{L}^{\mu+P}_{\mu+P}\mathbb{D}^{\mu+P}_{\mu} & \text{if } m = n;\\ 0 & \text{if } m < n.\end{cases} \tag{6.6}$$

The additional $m = n$ relations: *We have the loop-nilpotency and loop-commutation relations: namely, for all $\lambda, \mu \in \mathscr{R}_{m,m}$ the following holds*

$$(\mathbb{L}^{\mu}_{\mu})^2 = 0 \qquad \mathbb{D}^{\lambda}_{\mu}\mathbb{L}^{\mu}_{\mu} = \mathbb{L}^{\lambda}_{\lambda}\mathbb{D}^{\lambda}_{\mu}. \tag{6.7}$$

7. The isomorphism theorem

In this section, we prove the main result of this paper: that the algebras $\mathcal{A}_{m,n}$ and K^m_n are isomorphic as $\mathbb{Z}$-graded $\Bbbk$-algebras. We first fix some notation as follows:

$$\mathbb{D}(-Q) := \sum_{\substack{\lambda\in\mathscr{R}_{m,n}\\ Q\in\mathrm{DRem}(\lambda)}} \mathbb{1}_\lambda\mathbb{D}^{\lambda}_{\lambda-Q} \qquad \mathbb{D}(+Q) := \sum_{\substack{\lambda\in\mathscr{R}_{m,n}\\ Q\in\mathrm{DAdd}(\lambda)}} \mathbb{1}_\lambda\mathbb{D}^{\lambda}_{\lambda+Q} \qquad \mathbb{L}(-Q) := \sum_{\substack{\lambda\in\mathscr{R}_{m,n}\\ Q\in\mathrm{DRem}(\lambda)}} \mathbb{1}_\lambda\mathbb{L}^{\lambda}_{\lambda}(-Q).$$

7.1. **Implied relations in the symmetric Dyck path algebra.** In this section we go through the not inconsiderable effort of establishing a number of lemmas and propositions regarding the products of elements from $\mathcal{A}_{m,n}$.

Lemma 7.1. *Let $m < n$. For $Q \in \mathrm{DRem}_0(\lambda)$ and $P \in \mathrm{DRem}_{>0}(\lambda)$ we have that either P and Q commute or $Q \setminus P = Q^1 \sqcup Q^2$ where $Q^1, Q^2 \in \mathrm{DRem}(\lambda - P)$. In the former case we have that*

$$\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda-P} = \mathbb{D}^{\lambda}_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q)$$

and in the latter case we have that

$$\mathbb{D}^{\lambda}_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q^1) = \mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda-P} = \mathbb{D}^{\lambda}_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q^2). \tag{7.1}$$

Proof. We recall from Definition 6.3 that

$$\mathbb{L}^{\lambda}_{\lambda}(-Q) = (-1)^{b(\mathsf{rt}(Q))}\mathbb{D}^{\lambda}_{\lambda+\mathsf{rt}(Q)}\mathbb{D}^{\lambda+\mathsf{rt}(Q)}_{\lambda}.$$

If P, Q commute, then $P, \mathsf{rt}(Q)$ commute as well. This is straightforward if P is to the left of Q. If P is to the right of Q, then our assumptions $Q \in \mathrm{DRem}_0(\lambda)$ and $P \in \mathrm{DRem}_{>0}(\lambda)$ together imply that $\mathsf{first}(P) \geqslant \mathsf{last}(Q) + 2$. By definition $\mathsf{rt}(Q) \in \mathrm{DAdd}_1(\lambda)$ and so $\mathsf{first}(\mathsf{rt}(Q)) \leqslant \mathsf{first}(P) - 2$ and either $\mathsf{last}(\mathsf{rt}(Q)) \leqslant \mathsf{first}(P) - 2$ or $\mathsf{last}(\mathsf{rt}(Q)) \geqslant \mathsf{last}(P) + 2$; in either case, P and $\mathsf{rt}(Q)$ commute. Examples of these two cases are depicted in Figure 22. Therefore we have that

$$\begin{aligned}\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda-P} &= (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}(-P)\\ &= (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-P)\mathbb{D}(-\mathsf{rt}(Q))\\ &= (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(-P)\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\\ &= \mathbb{D}^{\lambda}_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q)\end{aligned}$$

where only the second and third equalities follow from applying the commuting relation (6.3) to the Dyck paths P and $\mathsf{rt}(Q)$; the first and fourth equalities follow by Definition 6.3.

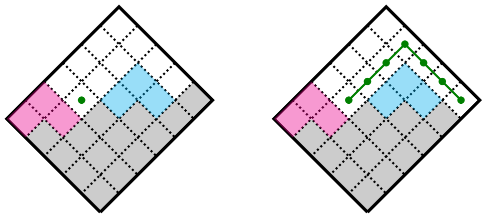


FIGURE 22. Examples of commuting paths P, Q and $\mathsf{rt}(Q)$ such that $Q \in \mathrm{DRem}_0(\lambda)$ and $P \in \mathrm{DRem}_{>0}(\lambda)$ for $\lambda = (5^2, 3^3, 1)$ and $\lambda = (5^2, 3^3)$ respectively.

We first consider the latter equality in equation (7.1). We now suppose that $Q \setminus P = Q^1 \sqcup Q^2$ where $Q^1, Q^2 \in \mathrm{DRem}(\lambda - P)$ and without loss of generality, we assume that Q^1 is to the left of P and Q^2 is to the right of P. In which case, $\mathsf{rt}(Q) = \mathsf{rt}(Q^2)$ and this Dyck path commutes with P. We have that

$$\begin{aligned}\mathbb{D}^{\lambda}_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q) &= (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(-P)\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\\ &= (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-P)\mathbb{D}(-\mathsf{rt}(Q))\\ &= (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}(-P)\\ &= \mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda-P}\end{aligned}$$

where the first and final equalities follow from Definition 6.3 and our observation that $\mathsf{rt}(Q) = \mathsf{rt}(Q^2)$; the second and third equalities follow from applying the commuting relation (6.3) to the Dyck paths P and $\mathsf{rt}(Q)$.

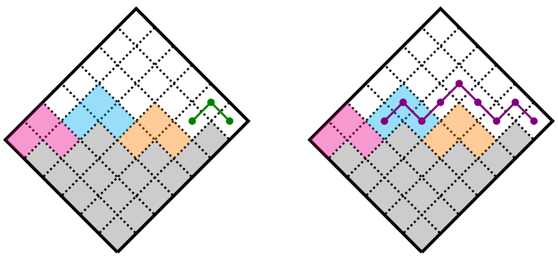


FIGURE 23. An example of $Q, P \in \mathrm{DRem}_1(\lambda)$ for $\lambda = (6^2, 5^2, 3, 1)$ such that $Q - P = Q^1 \sqcup Q^2$. On the left we depict $\mathsf{rt}(Q) = \mathsf{rt}(Q^2)$. On the right we depict $\mathsf{rt}(Q^1)$ which we note is the smallest Dyck path in $\mathrm{DAdd}(\lambda - P)$ containing both $\mathsf{rt}(Q^2)$ and P.

We now consider the former equality in equation (7.1). We have that $\mathsf{rt}(Q^1)$ is the smallest Dyck path in $\mathrm{Add}_1(\lambda - P)$ such that $P \prec \mathsf{rt}(Q^1)$ and $\mathsf{rt}(Q) \prec \mathsf{rt}(Q^1)$. We further note that $Q^2 \prec \mathsf{rt}(Q^1)$ and that $b(\mathsf{rt}(Q^1)) = b(\mathsf{rt}(Q)) + b(Q^2) + b(P)$. See Figure 23 for examples. We have that

$$\mathbb{D}^{\lambda}_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q^1) = (-1)^{b(\mathsf{rt}(Q^1))}\mathbb{1}_{\lambda}\mathbb{D}(-P)\mathbb{D}(+\mathsf{rt}(Q^1))\mathbb{D}(-\mathsf{rt}(Q^1))$$

$$\begin{aligned}
&= (-1)^{2b(\mathsf{rt}(Q^1))-b(\mathsf{rt}(Q^2))}\mathbb{1}_\lambda\mathbb{D}(+\mathsf{rt}(Q^2))\mathbb{D}(+Q^2)\mathbb{D}(-\mathsf{rt}(Q^1))\\
&= (-1)^{b(\mathsf{rt}(Q^2))}\mathbb{1}_\lambda\mathbb{D}(+\mathsf{rt}(Q^2))\mathbb{D}(-\mathsf{rt}(Q^2))\mathbb{D}(-P)\\
&= \mathbb{L}^\lambda_\lambda(-Q)\mathbb{D}^\lambda_{\lambda-P}
\end{aligned}$$

where the second equality follows from the adjacency relation (6.5) applied to $\mathsf{rt}(Q^2) \in \mathrm{Add}(\lambda)$ and $Q^2 \in \mathrm{Add}(\lambda + \mathsf{rt}(Q^1))$; the third equality follows from relation (6.4) applied to the non-commuting pair $Q^2 \prec \mathsf{rt}(Q^1)$; the first and fourth equalities follow from Definition 6.3. $\square$

Lemma 7.2. *Let $m = n$. For $Q \in \mathrm{DRem}_0(\lambda)$ and $P \in \mathrm{DRem}_{>0}(\lambda)$ we have that either P and Q commute or $Q \setminus P = Q^1 \sqcup Q^2$ where $Q^1, Q^2 \in \mathrm{DRem}(\lambda - P)$. In the former case we have that*

$$\mathbb{L}^\lambda_\lambda(-Q)\mathbb{D}^\lambda_{\lambda-P} = \mathbb{D}^\lambda_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q) \tag{7.2}$$

and in the latter case we have that

$$\mathbb{D}^\lambda_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q^1) = \mathbb{L}^\lambda_\lambda(-Q)\mathbb{D}^\lambda_{\lambda-P} = \mathbb{D}^\lambda_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q^2). \tag{7.3}$$

Proof. We first consider the case that $\mathsf{last}(Q) = m - 1$, that is, the case that $\mathbb{L}^\lambda_\lambda(-Q) = \mathbb{L}^\lambda_\lambda$. We first consider the subcase in which P and Q commute. We have that $\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q) = \mathbb{L}^{\lambda-P}_{\lambda-P}$ and by relation (6.7) it follows that

$$\mathbb{L}^\lambda_\lambda(-Q)\mathbb{D}^\lambda_{\lambda-P} = \mathbb{L}^\lambda_\lambda\mathbb{D}^\lambda_{\lambda-P} = \mathbb{D}^\lambda_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P} = \mathbb{D}^\lambda_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q).$$

Now we consider the subcase in which $P \in \mathrm{DRem}(\lambda)$ is such that $Q \setminus P = Q^1 \sqcup Q^2$ where $Q^1, Q^2 \in \mathrm{DRem}(\lambda - P)$. Without loss of generality, we suppose that Q^1 is to the left of Q^2 and therefore $\mathsf{last}(Q^2) = m - 1$. Thus, $\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q^2) = \mathbb{L}^{\lambda-P}_{\lambda-P}$ and by relation (6.7) it follows that

$$\mathbb{L}^\lambda_\lambda(-Q)\mathbb{D}^\lambda_{\lambda-P} = \mathbb{L}^\lambda_\lambda\mathbb{D}^\lambda_{\lambda-P} = \mathbb{D}^\lambda_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P} = \mathbb{D}^\lambda_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q^2).$$

Now, we observe that our assumptions on P and Q imply that $\mathsf{rt}(Q^1) = P$. Thus by Definition 6.3 we have that

$$\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q^1) = -\mathbb{L}^{\lambda-P}_{\lambda-P} - (-1)^{b(P)}\mathbb{D}^{\lambda-P}_\lambda\mathbb{D}^\lambda_{\lambda-P},$$

which we input into the lefthand-side of equation (7.3) and hence obtain

$$\begin{aligned}
\mathbb{D}^\lambda_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q^1) &= \mathbb{1}_\lambda\mathbb{D}(-P)(-\mathbb{L}^{\lambda-P}_{\lambda-P} - (-1)^{b(P)}\mathbb{D}(+P)\mathbb{D}(-P))\\
&= -\mathbb{1}_\lambda\mathbb{D}(-P)\mathbb{L}^{\lambda-P}_{\lambda-P} - 2(-1)^{2b(P)+1}\mathbb{1}_\lambda\mathbb{L}^\lambda_\lambda\mathbb{D}(-P)\\
&= -\mathbb{1}_\lambda\mathbb{L}^\lambda_\lambda\mathbb{D}(-P) + 2\mathbb{1}_\lambda\mathbb{L}^\lambda_\lambda\mathbb{D}(-P)\\
&= \mathbb{L}^\lambda_\lambda\mathbb{D}^\lambda_{\lambda-P}\\
&= \mathbb{L}^\lambda_\lambda(-Q)\mathbb{D}^\lambda_{\lambda-P}
\end{aligned}$$

where the first and final equalities follow from Definition 6.3; the second equality follows from relation (6.6); the third follows from applying relation (6.7) to the lefthand term and tidying-up the signs for the righthand term; the fourth equality is trivial.

It remains to consider the case in which $\mathsf{last}(Q) < m - 1$. By Definition 6.3 we can express these loops in terms of $\mathsf{rt}(Q)$ and the case already considered above, as follows:

$$\mathbb{L}^\lambda_\lambda(-Q) = -\mathbb{L}^\lambda_\lambda - (-1)^{b(\mathsf{rt}(Q))}\mathbb{D}^\lambda_{\lambda+\mathsf{rt}(Q)}\mathbb{D}^{\lambda+\mathsf{rt}(Q)}_\lambda.$$

Notice that $\mathsf{rt}(Q) \in \mathrm{DAdd}_1(\lambda)$ and $P \in \mathrm{DRem}_{>0}(\lambda)$ and so they cannot be adjacent. Thus, the Dyck paths $\mathsf{rt}(Q)$ and P commute. We have that

$$\begin{aligned}
\mathbb{L}^\lambda_\lambda(-Q)\mathbb{D}^\lambda_{\lambda-P} &= 1_\lambda\big(-\mathbb{L}^\lambda_\lambda - (-1)^{b(\mathsf{rt}(Q))}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\big)\mathbb{D}(-P)\\
&= -\mathbb{1}_\lambda\mathbb{D}(-P)\mathbb{L}^{\lambda-P}_{\lambda-P} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_\lambda\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}(-P)\\
&= -\mathbb{1}_\lambda\mathbb{D}(-P)\mathbb{L}^{\lambda-P}_{\lambda-P} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_\lambda\mathbb{D}(-P)\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\\
&= -1_\lambda\mathbb{D}(-P)\big(\mathbb{L}^{\lambda-P}_{\lambda-P} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\big)
\end{aligned}$$

where the first equality follows from Definition 6.3; the second follows from the loop-commutation relation (6.7); the third follows from the commuting relation applied to the Dyck paths $\mathsf{rt}(Q)$ and P; the fourth equality follows from re-bracketing. Finally, we observe that if $P \prec Q$, then $\mathsf{rt}(Q^2) = \mathsf{rt}(Q)$

for $Q^2 \in \mathrm{DRem}(\lambda - P)$; whereas if P and Q commute we have that $Q \in \mathrm{DRem}(\lambda - P)$ (and so we do not need to rewrite anything). Therefore we conclude that

$$\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda-P} = \begin{cases} \mathbb{1}_{\lambda}\mathbb{D}(-P)\mathbb{L}(-Q) & \text{if } Q, P \text{ commute;} \\ \mathbb{1}_{\lambda}\mathbb{D}(-P)\mathbb{L}(-Q^2) & \text{otherwise.} \end{cases}$$

We hence deduce that equation (7.2) and the righthand equality of equation (7.3) both hold.

It remains to verify the lefthand equality in equation (7.3). We have that $\mathsf{rt}(Q^1)$ is the smallest Dyck path in $\mathrm{Add}_1(\lambda - P)$ such that $P \prec \mathsf{rt}(Q^1)$ and $\mathsf{rt}(Q) \prec \mathsf{rt}(Q^1)$. We further note that $Q^2 \prec \mathsf{rt}(Q^1)$ and that $b(\mathsf{rt}(Q^1)) = b(\mathsf{rt}(Q)) + b(Q^2) + b(P)$. See Figure 24 for examples. We have that

$$\begin{aligned}
\mathbb{D}^{\lambda}_{\lambda-P}\mathbb{L}^{\lambda-P}_{\lambda-P}(-Q^1) &= 1_{\lambda}\mathbb{D}(-P)\big(-\mathbb{L}^{\lambda-P}_{\lambda-P} - (-1)^{b(\mathsf{rt}(Q^1))}\mathbb{D}(+\mathsf{rt}(Q^1))\mathbb{D}(-\mathsf{rt}(Q^1))\big) \\
&= -\mathbb{L}^{\lambda}_{\lambda}\mathbb{D}^{\lambda}_{\lambda-P} - (-1)^{b(\mathsf{rt}(Q^1))}\mathbb{D}(-P)\mathbb{D}(+\mathsf{rt}(Q^1))\mathbb{D}(-\mathsf{rt}(Q^1)) \\
&= -\mathbb{L}^{\lambda}_{\lambda}\mathbb{D}^{\lambda}_{\lambda-P} - (-1)^{2b(\mathsf{rt}(Q^1))-b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(+Q^2)\mathbb{D}(-\mathsf{rt}(Q^1)) \\
&= -\mathbb{L}^{\lambda}_{\lambda}\mathbb{D}^{\lambda}_{\lambda-P} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}(-P) \\
&= (-\mathbb{L}^{\lambda}_{\lambda} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}^{\lambda}_{\lambda-P} \\
&= \mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda-P}
\end{aligned}$$

where the first and final equalities follow by Definition 6.3 and the second and third equalities follow from relations (6.5) and (6.4) (exactly as in the $m \neq n$ case). □

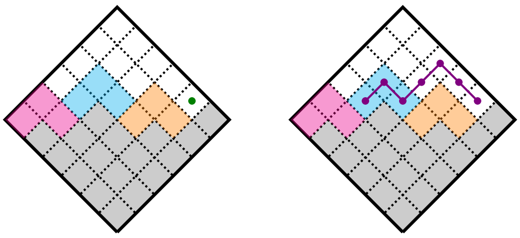


FIGURE 24. An example of $Q, P \in \mathrm{DRem}_1(\lambda)$ for $\lambda = (6^2, 5^2, 3, 1)$ such that $Q - P = Q^1 \sqcup Q^2$. On the left we depict $\mathsf{rt}(Q) = \mathsf{rt}(Q^2)$. On the right we depict $\mathsf{rt}(Q^1)$ which we note is the smallest Dyck path in $\mathrm{DAdd}(\lambda - P)$ containing both $\mathsf{rt}(Q^2)$ and P.

We observe that the previous two lemmas covered the $m = n$ and $m \neq n$ cases separately. Both proofs were very similar, but it was the $m = n$ case that was more intricate. In what follows, we prove the results for the $m = n$ case and leave adapting these arguments to the easier $m \neq n$ case as an exercise for the reader.

Proposition 7.3. *Let $Q \in \mathrm{DRem}_0(\lambda)$ and $P \in \mathrm{DAdd}_{>0}(\lambda)$. If P and Q commute, we have that*

$$\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda+P} = \mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P}(-Q). \tag{7.4}$$

If P and Q are adjacent, we have that

$$\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda+P} = \mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P}(-\langle P \cup Q\rangle_{\lambda+P}). \tag{7.5}$$

Proof. We focus on the $m = n$ case (the $m \neq n$ case is similar, but easier). As in the proof of Lemma 7.2, we must separate out according to the two distinct cases for $\mathbb{L}^{\lambda}_{\lambda}(-Q)$ in Definition 6.3.

Case 1. We first suppose that $\mathsf{last}(Q) = m-1$ where, by Definition 6.3, we have that $\mathbb{L}^{\lambda}_{\lambda}(-Q) = \mathbb{L}^{\lambda}_{\lambda}$. We first consider (7.4), in which P and Q are commuting Dyck paths. In this case $Q \in \mathrm{DRem}(\lambda - P)$. Then

$$\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda+P} = \mathbb{L}^{\lambda}_{\lambda}\mathbb{D}^{\lambda}_{\lambda+P} = \mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P} = \mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P}(-Q)$$

by the loop-commutation relation (6.7) and Definition 6.3.

Now, suppose that P, Q are adjacent. In which case, $\langle P \cup Q\rangle_{\lambda+P}$ is the rightmost removable Dyck path of $\lambda + P$ (that is $\mathsf{last}(\langle P\cup Q\rangle_{\lambda+P}) = m-1$). We hence have that

$$\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda+P} = \mathbb{L}^{\lambda}_{\lambda}\mathbb{D}^{\lambda}_{\lambda+P} = \mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P} = \mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P}(-\langle P\cup Q\rangle_{\lambda+P})$$

by the loop-commutation relation (6.7).

Case 2. For the remainder of the proof, we can assume that $\mathsf{last}(Q) < m-1$ and so

$$\mathbb{L}^{\lambda}_{\lambda}(-Q) = -\mathbb{L}^{\lambda}_{\lambda} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))$$

by Definition 6.3. Then

$$\begin{aligned}\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda+P} &= \mathbb{1}_{\lambda}(-\mathbb{L}^{\lambda}_{\lambda} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q)))\mathbb{D}(+P)\\ &= -\mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{L}^{\lambda+P}_{\lambda+P} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}(+P) \qquad (7.6)\end{aligned}$$

where the second equality follows from the loop-commutation relation (6.7). We now need to consider the commuting and adjacent case separately (and focus on the latter term on the righthand-side of (7.6)).

Case 2: Commuting subcase. The first case is that in which P, Q commute. We must refine this further into two subcases: that in which P and $\mathsf{rt}(Q)$ are a commuting pair of Dyck paths, and that in which P and $\mathsf{rt}(Q)$ do not commute.

We first suppose that P and $\mathsf{rt}(Q)$ commute (as in the leftmost example in Figure 25). In this case the latter term on the righthand-side of (7.6) is as follows

$$\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}(+P) = \mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q)) \qquad (7.7)$$

by two applications of the commuting relation (6.3). Substituting (7.7) into (7.6) we obtain

$$\begin{aligned}\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda+P} &= -\mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{L}^{\lambda+P}_{\lambda+P} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\\ &= \mathbb{1}_{\lambda}\mathbb{D}(+P)(-\mathbb{L}^{\lambda+P}_{\lambda+P} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q)))\\ &= \mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P}(-Q)\end{aligned}$$

as required.

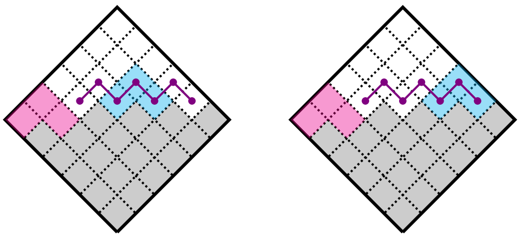


FIGURE 25. Examples of commuting Dyck paths $Q \in \mathrm{DRem}_0(\lambda)$ and $P \in \mathrm{DAdd}_1(\lambda)$ for $\lambda = (6^2, 4, 3, 2, 1)$. In the former case $\mathsf{rt}(Q) \in \mathrm{DAdd}_1(\lambda)$ and $P \in \mathrm{DAdd}_1(\lambda)$ commute, in the latter case they do not.

We continue with our assumption that P and Q commute, but now suppose that $P, \mathsf{rt}(Q)$ do not commute. This implies that $\mathsf{last}(P) = \mathsf{last}(\mathsf{rt}(Q))$ and such that $P \prec \mathsf{rt}(Q)$ as illustrated in the rightmost diagram in Figure 25. In which case, $P \in \mathrm{DAdd}_1(\lambda)$ and there exists $S \in \mathrm{DRem}_2(\lambda+\mathsf{rt}(Q))$ such that

$$\mathsf{split}_S(\mathsf{rt}(Q)) = T \sqcup P$$

for some $T \in \mathrm{DAdd}_1(\lambda)$. We have that

$$\begin{aligned}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}(+P) &= (-1)^{b(\mathsf{rt}(Q))-b(T)}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-S)\mathbb{D}(-T)\\ &= (-1)^{b(\mathsf{rt}(Q))-b(T)}\mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{D}(+T)\mathbb{D}(-T) \qquad (7.8)\end{aligned}$$

where the first equality follows from applying relation (6.5) to the pair of adjacent Dyck paths S, T and the second equality follows by applying relation (6.4) to the pair of non-commuting Dyck paths $S \prec \mathsf{rt}(Q)$. Substituting (7.8) into (7.6) we obtain the following

$$\begin{aligned}\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda+P} &= -\mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P} - (-1)^{2b(\mathsf{rt}(Q))-b(T)}\mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{D}(+T)\mathbb{D}(-T)\\ &= \mathbb{1}_{\lambda}\mathbb{D}(+P)(-\mathbb{L}^{\lambda+P}_{\lambda+P} - (-1)^{b(T)}\mathbb{D}(+T)\mathbb{D}(-T))\\ &= \mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P}(-Q)\end{aligned}$$

where the final equality follows since $T = \mathsf{rt}(Q)$ in $\lambda + P$.

Case 2: Adjacent subcase. For the remainder of the proof we suppose that P, Q are a pair of adjacent Dyck paths (in particular $P \in \mathrm{DAdd}_1(\lambda)$). There are three subcases to consider: (i) the Dyck paths P and $\mathsf{rt}(Q)$ do not commute (ii) the Dyck paths P and $\mathsf{rt}(Q)$ do commute (iii) the case $P = \mathsf{rt}(Q)$. These are depicted in Figure 26.

Subcase (i). We first assume that the pair P and $\mathsf{rt}(Q)$ do not commute. Or equivalently, that $\mathsf{first}(P) = \mathsf{last}(Q) + 1$ and $b(P)$ is not maximal with respect to this property (when $b(P)$ maximal we are in case (iii)). An example of this is depicted in the leftmost diagram in Figure 26. In this case, there exists $S \in \mathrm{DRem}_2(\lambda + \mathsf{rt}(Q))$ such that

$$\mathsf{split}_S(\mathsf{rt}(Q)) = T \sqcup P$$

for some $T \in \mathrm{DAdd}_1(\lambda)$. We have that

$$\begin{aligned}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}(+P) &= (-1)^{b(\mathsf{rt}(Q))-b(T)}\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-S)\mathbb{D}(-T)\\ &= (-1)^{b(\mathsf{rt}(Q))-b(T)}\mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{D}(+T)\mathbb{D}(-T)\end{aligned} \tag{7.9}$$

where the first equality follows by applying relation (6.5) to the adjacent pair S, T and the second follows by relation (6.4) to the non-commuting pair $S \prec \mathsf{rt}(Q)$. Substituting (7.9) into (7.6) we obtain the following

$$\begin{aligned}\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda+P} &= -\mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{L}^{\lambda+P}_{\lambda+P} - (-1)^{2b(\mathsf{rt}(Q))-b(T)}\mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{D}(+T)\mathbb{D}(-T)\\ &= \mathbb{D}^{\lambda}_{\lambda+P}(-\mathbb{L}^{\lambda+P}_{\lambda+P} - (-1)^{b(T)}\mathbb{D}(+T)\mathbb{D}(-T))\\ &= \mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P}(-\langle P \sqcup Q\rangle_{\lambda+P})\end{aligned}$$

where the final equality follows since $T = \mathsf{rt}(\langle P \sqcup Q\rangle_{\lambda+P})$.

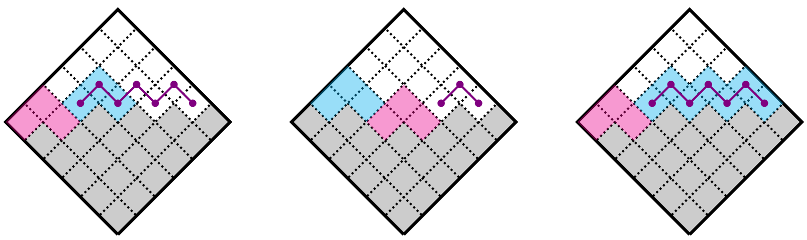


FIGURE 26. Examples of adjacent paths $P \in \mathrm{DAdd}_1(\lambda)$, $Q \in \mathrm{DRem}_0(\lambda)$ and $\mathsf{rt}(Q) \in \mathrm{DAdd}_1(\lambda)$. The three subcases depicted from left-to-right are (i) $P \prec \mathsf{rt}(Q)$ are non-commuting (ii) P and $\mathsf{rt}(Q)$ are commuting and (iii) $P = \mathsf{rt}(Q)$.

Subcase (ii). We now assume that P and $\mathsf{rt}(Q)$ commute. Or equivalently, that P is to the left of Q as in the central diagram in Figure 26. We have that

$$\mathbb{1}_{\lambda}\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}(+P) = \mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q)) \tag{7.10}$$

by two applications of the commuting relation (6.3). Substituting (7.10) into (7.6) we obtain the following

$$\begin{aligned}\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda+P} &= -\mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{L}^{\lambda+P}_{\lambda+P} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{1}_{\lambda}\mathbb{D}(+P)\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\\ &= \mathbb{D}^{\lambda}_{\lambda+P}(-\mathbb{L}^{\lambda+P}_{\lambda+P} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{D}^{\lambda}_{\lambda+P}\mathbb{D}^{\lambda+P}_{\lambda+\mathsf{rt}(Q)+P}\mathbb{D}^{\lambda+\mathsf{rt}(Q)+P}_{\lambda+P})\end{aligned}$$

$$= \mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P}(-\langle P \sqcup Q\rangle_{\lambda+P})$$

where the final equality follows as $\mathsf{rt}(Q) = \mathsf{rt}(\langle P \sqcup Q\rangle_{\lambda+P})$.

Subcase (iii). We now assume that $P = \mathsf{rt}(Q)$, as in the rightmost diagram in Figure 26. We have that

$$\mathbb{1}_\lambda \mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}(+P) = 2(-1)^{b(P)+1}\mathbb{1}_\lambda\mathbb{D}(+P)\mathbb{L}^{\lambda+P}_{\lambda+P} \tag{7.11}$$

by cubic relation (6.6). Substituting (7.11) into (7.6) we obtain the following

$$\begin{aligned}\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{D}^{\lambda}_{\lambda+P} &= -\mathbb{1}_\lambda\mathbb{D}(+P)\mathbb{L}^{\lambda+P}_{\lambda+P} - 2(-1)^{2b(\mathsf{rt}(Q))+1}\mathbb{1}_\lambda\mathbb{D}(+P)\mathbb{L}^{\lambda+P}_{\lambda+P}\\ &= (-1+2)\mathbb{1}_\lambda\mathbb{D}(+P)\mathbb{L}^{\lambda+P}_{\lambda+P}\\ &= \mathbb{D}^{\lambda}_{\lambda+P}\mathbb{L}^{\lambda+P}_{\lambda+P}(-\langle P\sqcup Q\rangle_{\lambda+P})\end{aligned}$$

where the first two equalities follow as $b(\mathsf{rt}(Q)) = b(P)$ and so the signs cancel; the third equality follows as $\langle P \sqcup Q\rangle_{\lambda+P} \in \mathrm{DRem}_0(\lambda)$ is the rightmost such removable Dyck path. ☐

Proposition 7.4. *For $P, Q \in \mathrm{DRem}(\lambda)$ we have that*

$$\mathbb{L}^{\lambda}_{\lambda}(-P)\mathbb{L}^{\lambda}_{\lambda}(-Q) = \mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{L}^{\lambda}_{\lambda}(-P) \tag{7.12}$$

and moreover, if $P \in \mathrm{DRem}_0(\lambda)$ then we have that

$$(\mathbb{L}^{\lambda}_{\lambda}(-P))^2 = 0.$$

Proof. As in previous proofs, we assume that $m = n$ (the $m \neq n$ case is similar, but easier). We first suppose that $\mathsf{last}(Q) = m-1$. In which case, we have that $\mathbb{L}^{\lambda}_{\lambda}(-Q) = \mathbb{L}^{\lambda}_{\lambda}$ by Definition 6.3. For any $Q \neq P \in \mathrm{DRem}(\lambda)$ one can check that

$$\mathbb{L}^{\lambda}_{\lambda}(-P)\mathbb{L}^{\lambda}_{\lambda} = \mathbb{L}^{\lambda}_{\lambda}\mathbb{L}^{\lambda}_{\lambda}(-P)$$

by expanding out $\mathbb{L}^{\lambda}_{\lambda}(-P)$ as prescribed by Definition 6.3 and then noticing that $\mathbb{L}^{\lambda}_{\lambda}$ commutes past every single term in the expansion, by relation (6.7). On the other hand if $P = Q$ with $\mathsf{last}(Q) = m-1$, then $\mathbb{L}^{\lambda}_{\lambda}(-Q)^2 = 0$ by relation (6.7). Therefore we can assume for the remainder of the proof that $\mathsf{last}(P) \leqslant \mathsf{last}(Q) < m$.

Case 1. For $P, Q \in \mathrm{DRem}_0(\lambda)$, we claim that

$$\mathbb{L}^{\lambda}_{\lambda}(-P)\mathbb{L}^{\lambda}_{\lambda}(-Q) = (1-\delta_{PQ})\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{L}^{\lambda}_{\lambda}(-P). \tag{7.13}$$

We first consider the case that $P = Q \in \mathrm{DRem}_0(\lambda)$. We have that

$$\begin{aligned}&\mathbb{L}^{\lambda}_{\lambda}(-P)^2\\ =&(-\mathbb{L}^{\lambda}_{\lambda} - (-1)^{b(\mathsf{rt}(P))}\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P)))^2\\ =&(\mathbb{L}^{\lambda}_{\lambda})^2 + 2(-1)^{b(\mathsf{rt}(P))}\mathbb{L}^{\lambda}_{\lambda}\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P)) + \mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P))\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P))\\ =&2(-1)^{b(\mathsf{rt}(P))}\mathbb{L}^{\lambda}_{\lambda}\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P)) + \mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P))\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P)).\\ =&2(-1)^{b(\mathsf{rt}(P))}\mathbb{L}^{\lambda}_{\lambda}\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P)) + 2(-1)^{b(\mathsf{rt}(P))+1}\mathbb{D}(+\mathsf{rt}(P))\mathbb{L}^{\lambda+\mathsf{rt}(P)}_{\lambda+\mathsf{rt}(P)}\mathbb{D}(-\mathsf{rt}(P))\\ =&2(-1)^{b(\mathsf{rt}(P))}\mathbb{L}^{\lambda}_{\lambda}\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P)) + 2(-1)^{b(\mathsf{rt}(P))+1}\mathbb{L}^{\lambda}_{\lambda}\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P))\\ =&0\end{aligned}$$

where the first equality follows from Definition 6.3; the second is expanding out the brackets; the third and fifth equalities follow from (6.7) and rearranging terms; the fourth follows from the cubic relation (6.6).

We can now assume that $P \neq Q$ and so $\mathsf{last}(P) \leqslant \mathsf{first}(Q) - 2$. We can expand out the product $\mathbb{L}^{\lambda}_{\lambda}(-P)\mathbb{L}^{\lambda}_{\lambda}(-Q)$ as follows

$$(-\mathbb{L}^{\lambda}_{\lambda} - (-1)^{b(\mathsf{rt}(P))}\mathbb{D}^{\lambda}_{\lambda+\mathsf{rt}(P)}\mathbb{D}^{\lambda+\mathsf{rt}(P)}_{\lambda})(-\mathbb{L}^{\lambda}_{\lambda} - (-1)^{b(\mathsf{rt}(Q))}\mathbb{D}^{\lambda}_{\lambda+\mathsf{rt}(Q)}\mathbb{D}^{\lambda+\mathsf{rt}(Q)}_{\lambda})$$

and one can expand out $\mathbb{L}^{\lambda}_{\lambda}(-Q)\mathbb{L}^{\lambda}_{\lambda}(-P)$ similarly; all of the products involving $\mathbb{L}^{\lambda}_{\lambda}$ can be easily rearranged using relation (6.7); thus, in order to deduce equation (7.13) it will suffice to verify the following claim:

$$(\mathbb{D}^{\lambda}_{\lambda+\mathsf{rt}(P)}\mathbb{D}^{\lambda+\mathsf{rt}(P)}_{\lambda})(\mathbb{D}^{\lambda}_{\lambda+\mathsf{rt}(Q)}\mathbb{D}^{\lambda+\mathsf{rt}(Q)}_{\lambda}) = (\mathbb{D}^{\lambda}_{\lambda+\mathsf{rt}(Q)}\mathbb{D}^{\lambda+\mathsf{rt}(Q)}_{\lambda})(\mathbb{D}^{\lambda}_{\lambda+\mathsf{rt}(P)}\mathbb{D}^{\lambda+\mathsf{rt}(P)}_{\lambda}).$$

We now set about proving this claim. Recall that $P, Q \in \mathrm{DRem}_0(\lambda)$ and P is to the left of Q; therefore there exists a Dyck path $T \in \mathrm{DRem}_1(\lambda + \mathsf{rt}(P))$ such that

$$\mathsf{split}_T(\mathsf{rt}(P)) = S \sqcup \mathsf{rt}(Q)$$

for some Dyck path $S \in \mathrm{DRem}(\lambda + \mathsf{rt}(P) - T)$ which is adjacent to T. In which case $\langle T \cup S\rangle_{\lambda+\mathsf{rt}(P)} = \mathsf{rt}(P)$. This is pictured in Figure 27. We have that

$$\begin{aligned}
&\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P))\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\\
=&(-1)^{b(\mathsf{rt}(P))+b(S)}\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-T)\mathbb{D}(-S)\mathbb{D}(-\mathsf{rt}(Q))\\
=&(-1)^{b(\mathsf{rt}(P))+b(S)}\Big(\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-T)\Big)\Big(\mathbb{D}(-S)\mathbb{D}(-\mathsf{rt}(Q))\Big)\\
=&(-1)^{b(\mathsf{rt}(P))+b(S)}\Big(\mathbb{D}(+\mathsf{rt}(Q)\mathbb{D}(+S))\Big)\Big(\mathbb{D}(+T)\mathbb{D}(-\mathsf{rt}(P))\Big)\\
=&\mathbb{D}(+\mathsf{rt}(Q))\mathbb{D}(-\mathsf{rt}(Q))\mathbb{D}(+\mathsf{rt}(P))\mathbb{D}(-\mathsf{rt}(P))
\end{aligned}$$

where the first and final equations follow from applying relation (6.5) to the pair of adjacent Dyck paths S, T; the second equation is merely re-bracketing; the third equation follows by applying relation (6.4) for the non-commuting pair $T \prec \mathsf{right}(P)$ for both the left and right bracketed term.

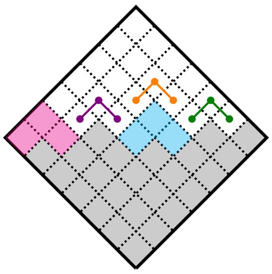


FIGURE 27. We picture $P, Q \in \mathrm{DRem}(\lambda)$ for $\lambda = (7^2, 5, 4^2, 2, 1)$. We have also pictured $\mathsf{rt}(Q), S \in \mathrm{DAdd}_1(\lambda)$ and $T \in \mathrm{DRem}(\lambda + \mathsf{rt}(P))$. We note that $\mathsf{rt}(P) = \mathsf{rt}(Q) \sqcup S \sqcup T$.

Case 2. For $P, Q \in \mathrm{DRem}_{>0}(\lambda)$, we claim that

$$\mathbb{L}^\lambda_\lambda(-P)\mathbb{L}^\lambda_\lambda(-Q) := (\mathbb{D}^\lambda_{\lambda-P}\mathbb{D}^{\lambda-P}_\lambda)(\mathbb{D}^\lambda_{\lambda-Q}\mathbb{D}^{\lambda-Q}_\lambda) = (\mathbb{D}^\lambda_{\lambda-Q}\mathbb{D}^{\lambda-Q}_\lambda)(\mathbb{D}^\lambda_{\lambda-P}\mathbb{D}^{\lambda-P}_\lambda) =: \mathbb{L}^\lambda_\lambda(-Q)\mathbb{L}^\lambda_\lambda(-P).$$

If P and Q commute, then the claim follows by applying the commuting relations (6.3) and if $P = Q$ then the statement is obvious. Then without loss of generality, we can assume that $\mathsf{ht}(P) = \mathsf{ht}(Q)+1$ and $Q \setminus P = Q^1 \sqcup Q^2$ where $Q^1, Q^2 \in \mathrm{DRem}(\lambda - P)$. In which case we have that

$$\begin{aligned}
\mathbb{L}^\lambda_\lambda(-P)\mathbb{L}^\lambda_\lambda(-Q) &= \mathbb{1}_\lambda\mathbb{D}(-P)\mathbb{D}(+P)\mathbb{D}(-Q)\mathbb{D}(+Q)\\
&= \mathbb{1}_\lambda\mathbb{D}(-P)\mathbb{D}(-Q^1)\mathbb{D}(-Q^2)\mathbb{D}(+Q)\\
&= \mathbb{1}_\lambda\Big(\mathbb{D}(-P)\mathbb{D}(-Q^1)\Big)\Big(\mathbb{D}(-Q^2)\mathbb{D}(+Q)\Big)\\
&= \mathbb{1}_\lambda\Big(\mathbb{D}(-Q)\mathbb{D}(+Q^2)\Big)\Big(\mathbb{D}(+Q^1)\mathbb{D}(+P)\Big)\\
&= \mathbb{1}_\lambda\mathbb{D}(-Q)\mathbb{D}(+Q)\mathbb{D}(-P)\mathbb{D}(+P)\\
&= \mathbb{L}^\lambda_\lambda(-P)\mathbb{L}^\lambda_\lambda(-Q)
\end{aligned}$$

where the first and final equalities are by Definition 6.3; the second and penultimate inequalities follow by applying relation (6.4) to the non-commuting pair $P \prec Q$; the third equality is merely suggestive rebracketing; the fourth equality follows by applying relation (6.5) for the adjacent Dyck paths Q^1 and P to *both* bracketed pairs.

Case 3. For $P \in \mathrm{DRem}(\lambda)_0$, $Q \in \mathrm{DRem}(\lambda)_{>0}$, we claim that

$$\mathbb{L}^\lambda_\lambda(-P)\mathbb{L}^\lambda_\lambda(-Q) := \mathbb{L}^\lambda_\lambda(-P)(\mathbb{D}^\lambda_{\lambda-Q}\mathbb{D}^{\lambda-Q}_\lambda) = (\mathbb{D}^\lambda_{\lambda-Q}\mathbb{D}^{\lambda-Q}_\lambda)\mathbb{L}^\lambda_\lambda(-P) =: \mathbb{L}^\lambda_\lambda(-Q)\mathbb{L}^\lambda_\lambda(-P).$$

If P and Q commute, then the claim follows from Lemma 7.2 and Proposition 7.3. So we can assume that P, Q are adjacent. In particular, we have that $P \setminus Q = P^1 \sqcup P^2$ where $P^1, P^2 \in \mathrm{DRem}(\lambda - Q)$. Under this assumption we have that

$$\begin{aligned}\mathbb{L}^\lambda_\lambda(-P)\mathbb{L}^\lambda_\lambda(-Q) &= (-1)^{b(Q)}\mathbb{1}_\lambda\mathbb{L}(-P)\mathbb{D}(-Q)\mathbb{D}(+Q)\\ &= (-1)^{b(Q)}\mathbb{D}(-Q)\mathbb{L}(-P^2)\mathbb{D}(+Q)\\ &= (-1)^{b(Q)}\mathbb{D}(-Q)\mathbb{D}(+Q)\mathbb{L}(-\langle P^2 \sqcup Q\rangle_\lambda)\\ &= \mathbb{L}^\lambda_\lambda(-Q)\mathbb{L}^\lambda_\lambda(-P)\end{aligned}$$

where the second equality follows from Lemma 7.2 and the third equality follows from Proposition 7.3 □

Notice that every product we have considered thus far has been a product of a pair of elements of degree at most 2 (each element corresponding to adding/removing a Dyck path of degree 1, or a loop of degree 2). We are now ready to consider more complicated products — in particular, products of arbitrarily high degree. In particular we will consider loops of arbitrarily high degree as follows. Given $\lambda \setminus \alpha = (\lambda \setminus \alpha)_{\leqslant 0} = \sqcup_{1\leqslant i\leqslant k} P^i$ a Dyck pair, we define

$$\mathbb{L}^\lambda_\lambda(\alpha) = \prod_{1\leqslant i\leqslant k} \mathbb{L}^\lambda_\lambda(-P^i) \tag{7.14}$$

which we observe is independent of the ordering on the Dyck paths in the tiling, by Proposition 7.4. With this notation in place, we can now look at the effect of multiplying loop tilings together.

Proposition 7.5. *Let $(\lambda \setminus \alpha) = (\lambda \setminus \alpha)_{\leqslant 0} = \sqcup_{i\in I} P^i$, and $(\lambda \setminus \beta) = (\lambda \setminus \beta)_{\leqslant 0} = \sqcup_{j\in J} Q^j$. We have that*

$$\mathbb{L}^\lambda_\lambda(\alpha)\mathbb{L}^\lambda_\lambda(\beta) = \begin{cases}\mathbb{L}^\lambda_\lambda(\alpha\cap\beta) & \text{if } P^i \neq Q^j \text{ for } i \in I, j \in J;\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We first consider the easy case in which $(\lambda\setminus\alpha)\cap(\lambda\setminus\beta) = \emptyset$. In this case $\lambda\setminus(\alpha\cap\beta)$ is a Dyck pair with tiling

$$\lambda \setminus (\alpha\cap\beta) = \bigsqcup_{i\in I} P^i \sqcup \bigsqcup_{j\in J} Q^j = (\lambda\setminus\alpha)\sqcup(\lambda\setminus\beta).$$

In the notation of (7.14) we have that $\mathbb{L}^\lambda_\lambda(\alpha)\mathbb{L}^\lambda_\lambda(\beta) =: \mathbb{L}^\lambda_\lambda(\alpha\cap\beta) \neq 0$ (and we remind the reader that this product is entirely independent of any ordering on Dyck paths, by Proposition 7.4).

Now suppose that there exists $T = P^{i_0} \in \mathrm{DRem}_h(\lambda)$ and $T = Q^{j_0} \in \mathrm{DRem}_h(\lambda)$ for $i_0 \in I, j_0 \in J$ and we can assume that h is minimal with respect to this property. We claim that $h = 0$. Suppose not, then our assumption that $(\lambda\setminus\alpha) = (\lambda\setminus\alpha)_{\leqslant 0}$ implies that $T = P^{i_0} \prec P^{i'}$ for some $i' \in I$ such that $P^{i'} \in \mathrm{DRem}_0(\lambda)$. Arguing similarly for $T = Q^{j_0}$ we deduce the claim. Therefore we can rewrite the product as follows

$$\mathbb{L}^\lambda_\lambda(\alpha)\mathbb{L}^\lambda_\lambda(\beta) = \mathbb{L}^\lambda_\lambda(-T)\mathbb{L}^\lambda_\lambda(-T)\prod_{i\in I\setminus\{i_0\}}\mathbb{L}^\lambda_\lambda(-P^i)\prod_{j\in J\setminus\{j_0\}}\mathbb{L}^\lambda_\lambda(-Q^j) = 0 \tag{7.15}$$

where the latter equality follows by (7.4) and the fact that $T \in \mathrm{DRem}_0(\lambda)$. □

Proposition 7.6. *Let $(\lambda\setminus\alpha) = (\lambda\setminus\alpha)_{\leqslant 0} = \sqcup_{i=0}^k T^i$ such that $T^k \prec T^{k-1} \prec \cdots \prec T^1 \prec T^0$ with $\mathsf{ht}^\lambda_\alpha(T^i) = -i \in \mathbb{Z}_{\leqslant 0}$. Then*

$$\Big(\prod_{i=0}^k \mathbb{L}^\lambda_\lambda(-T^i)\Big)\mathbb{D}^\lambda_{\lambda-T^k} = 0.$$

Proof. For ease of notation, we set $b_i = b(T^i)$ for all $i \geqslant 0$. We proceed by induction on $k \geqslant 1$. We first consider the base case in which $k = 1$. Then we have $T^0\setminus T^1 = P \sqcup Q$ for some $P, Q \in \mathrm{DRem}_0(\lambda - T^1)$. Since $T^1 \in \mathrm{DRem}_1(\lambda)$, we have that

$$\mathbb{1}_\lambda\mathbb{L}(-T^0)\mathbb{L}(-T^1)\mathbb{D}(-T^1)\mathbb{1}_{\lambda-T^1} = (-1)^{b_1}\mathbb{1}_\lambda\mathbb{L}(-T^0)\mathbb{D}(-T^1)\mathbb{D}(+T^1)\mathbb{D}(-T^1)\mathbb{1}_{\lambda-T^1}$$

simply by Definition 6.3. We now apply to the subproduct $\mathbb{D}(+T^1)\mathbb{D}(-T^1)\mathbb{1}_{\lambda-T^1}$ the self-dual relation (6.2) and hence we obtain

$$(-1)^{b_1}\mathbb{1}_\lambda\mathbb{L}(-T^0)\mathbb{D}(-T^1)\Big((-1)^{b(P)-1}\mathbb{L}(-P)+(-1)^{b(Q)-1}\mathbb{L}(-Q)\Big)$$

and, applying Lemma 7.1 and 7.2, we obtain

$$(-1)^{b_1}\mathbb{1}_\lambda\mathbb{L}(-T^0)\Big((-1)^{b(P)-1}\mathbb{L}(-T^0)+(-1)^{b(Q)-1}\mathbb{L}(-T^0)\Big)\mathbb{D}(-T^1)$$

and hence, applying Proposition 7.4 to the subproduct $\mathbb{L}(-T^0)\mathbb{L}(-T^0)$ we obtain that the overall product is zero, thus completing the base case of our induction.

We now turn to the inductive step. We suppose that

$$\Big(\prod_{i=0}^{j-1}\mathbb{L}^\lambda_\lambda(-T^i)\Big)\Big(\mathbb{1}_\lambda\mathbb{L}(-T^j)\mathbb{D}(-T^j)\Big)=0 \tag{7.16}$$

for all $j<k$. We want to show that (7.16) holds for $j=k$ (using essentially the same argument as in the base case). Then we have $T^{k-1}\setminus T^k=P\sqcup Q$ for some $P,Q\in\mathrm{DRem}_{k-1}(\lambda-T^k)$. Within (7.16) with $j=k$, we consider the subproduct of the form

$$\mathbb{1}_\lambda\mathbb{L}(-T^k)\mathbb{D}(-T^k)=(-1)^{b_k}\mathbb{D}(-T^k)\mathbb{D}(+T^k)\mathbb{D}(-T^k)$$

simply by Definition 6.3. We now apply to the subproduct $\mathbb{D}(+T^k)\mathbb{D}(-T^k)\mathbb{1}_{\lambda-T^k}$ the self-dual relation (6.2) and we hence obtain

$$-\mathbb{1}_\lambda\mathbb{D}(-T^k)\Big((-1)^{b(P)}\mathbb{L}(-P)+(-1)^{b(Q)}\mathbb{L}(-Q)+2(-1)^{b_{k-2}}\mathbb{L}(-T^{k-2})+\cdots+2(-1)^{b_0}\mathbb{L}(-T^0)\Big).$$

Applying Lemma 7.1 and 7.2, we get

$$-\mathbb{1}_\lambda\Big((-1)^{b(P)}\mathbb{L}(-T^{k-1})+(-1)^{b(Q)}\mathbb{L}(-T^{k-1})+2(-1)^{b_{k-2}}\mathbb{L}(-T^{k-2})+\cdots+2(-1)^{b_0}\mathbb{L}(-T^0)\Big)\mathbb{D}(-T^k)$$

and we observe that every summand is a scalar multiplied by some $\mathbb{1}_\lambda\mathbb{L}(-T^i)\mathbb{D}(-T^k)$ for $0\leqslant i<k$; therefore when we substitute these terms into the rightmost bracketed term of (7.16) the resulting product is zero by the loop-nilpotency relation (6.7) and by Proposition 7.4. The result follows. □

Lemma 7.7. *Let $T^k\prec T^{k-1}\prec\cdots\prec T^1\prec T^0$ be elements of $\mathrm{DRem}(\lambda)$ with $\mathsf{ht}(T^i)=i\in\mathbb{Z}_{\geqslant 0}$ and set $\alpha=\lambda-T^0-T^1-\cdots-T^k$. Let $S\in\mathrm{DAdd}(\lambda)$ be adjacent to T^k. We have that*

$$\mathbb{1}_\lambda\mathbb{L}(\alpha)\mathbb{D}(+S)=\begin{cases}\mathbb{1}_\lambda\mathbb{D}(+S)\mathbb{L}(\alpha+S+T^k)\mathbb{L}(-\langle T^k\cup S\rangle_{\lambda+S}) & \textit{if } \langle T^k\cup S\rangle_{\lambda+S} \textit{ exists;}\\ 0 & \textit{otherwise.}\end{cases}$$

An example of the Dyck tilings involved in Lemma 7.7 is given in Figure 28.

Proof. Recall that the loop generators commute and so we can rewrite the product as follows

$$\begin{aligned}\mathbb{1}_\lambda\mathbb{L}(\alpha)\mathbb{D}(+S)&=\mathbb{1}_\lambda\mathbb{L}(\alpha+T^k)\mathbb{L}(-T^k)\mathbb{D}(+S)\\&=\begin{cases}\mathbb{1}_\lambda\mathbb{L}(\alpha+T^k)\mathbb{D}(+S)\mathbb{L}(-\langle T^k\cup S\rangle_{\lambda+S}) & \text{if } \langle T^k\cup S\rangle_{\lambda+S} \text{ exists;}\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

by equation (7.5). Now S necessarily commutes with the T^j for $0\leqslant j<k$ and so the result follows by equation (7.4). □

Proposition 7.8. *Let $Q^1\prec Q^2\prec Q^3$ be removable Dyck paths of $\lambda\in\mathscr{R}_{m,n}$. Suppose that $Q^3\in\mathrm{DRem}_0(\lambda)$ and that Q^1 does not commute with Q^2 and that Q^2 does not commute with Q^3. We have that*

$$\mathbb{L}^\lambda_\lambda(-Q^3)\mathbb{D}^\lambda_{\lambda-Q^2}\mathbb{L}^{\lambda-Q^2}_{\lambda-Q^2}(-Q^1)=\mathbb{L}^\lambda_\lambda(-Q^3)\mathbb{L}^\lambda_\lambda(-Q^1)\mathbb{D}^\lambda_{\lambda-Q^2}.$$

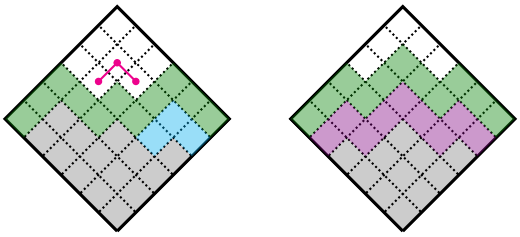


FIGURE 28. On the left we depict a pair $T^1 \prec T^0$ in DRem(λ) and $S \in$ DAdd(λ) for $\lambda = (6^3, 4, 3^3)$ as in Lemma 7.7. We note that S and T^1 are adjacent. On the right we depict the paths T^0 and $\langle T^1 \cup S\rangle_{\lambda+S}$.

Proof. We focus on the $m = n$ case (the $m \neq n$ case is similar, but easier). By our assumptions $Q^2 \in \mathrm{DRem}_1(\lambda)$, $Q^1 \in \mathrm{DRem}_2(\lambda)$ and $Q^1 \in \mathrm{DRem}_0(\lambda - Q^2)$ is such that $\mathsf{last}(Q^2) < m-1$. By Definition 6.3 we have that

$$\mathbb{L}^{\lambda-Q^2}_{\lambda-Q^2}(-Q^1) = -\mathbb{L}^{\lambda-Q^2}_{\lambda-Q^2} - (-1)^{b(\mathsf{rt}(Q^1))}\mathbb{1}_{\lambda-Q^2}\mathbb{D}(+\mathsf{rt}(Q^1))\mathbb{D}(-\mathsf{rt}(Q^1));$$
$$\mathbb{L}^{\lambda}_{\lambda}(-Q^1) = (-1)^{b(Q^1)}\mathbb{1}_{\lambda}\mathbb{D}(-Q^1)\mathbb{D}(+Q^1).$$

Moreover, since Q^1, Q^2 do not commute and $Q^1 \prec Q^2$, we have that $Q^2 \setminus Q^1 = P \sqcup T$ for some $P, T \in \mathrm{DRem}_1(\lambda - Q^1)$ and we can assume that P is to the left of T.

We first consider the case that $Q^3 \in \mathrm{DRem}_0(\lambda)$ is the rightmost removable Dyck path, in which case $\mathbb{L}^\lambda_\lambda(-Q^3) = \mathbb{L}^\lambda_\lambda$. Now, since Q^1, Q^2 do not commute, our assumption on Q^3 implies that $T = \mathsf{rt}(Q^1)$. We have that

$$\begin{aligned}
\mathbb{1}_\lambda\mathbb{D}(-Q^2)\mathbb{L}(-Q^1) &= \mathbb{1}_\lambda\mathbb{D}(-Q^2)(-\mathbb{L}^{\lambda-Q^2}_{\lambda-Q^2} - (-1)^{b(\mathsf{rt}(Q^1))}\mathbb{D}(+\mathsf{rt}(Q^1))\mathbb{D}(-\mathsf{rt}(Q^1)))\\
&= -\mathbb{L}^\lambda_\lambda\mathbb{D}(-Q^2) - (-1)^{b(\mathsf{rt}(Q^1))}\mathbb{1}_\lambda\mathbb{D}(-Q^2)\mathbb{D}(+\mathsf{rt}(Q^1))\mathbb{D}(-\mathsf{rt}(Q^1))\\
&= -\mathbb{1}_\lambda\mathbb{L}(-Q^3)\mathbb{D}(-Q^2) - (-1)^{b(\mathsf{rt}(Q^1))}\mathbb{1}_\lambda\mathbb{D}(-Q^2)\mathbb{D}(+\mathsf{rt}(Q^1))\mathbb{D}(-\mathsf{rt}(Q^1))\\
&= -\mathbb{1}_\lambda\mathbb{L}(-Q^3)\mathbb{D}(-Q^2) - (-1)^{b(\mathsf{rt}(Q^1))+b(Q^2)-b(P)}\mathbb{1}_\lambda\mathbb{D}(-Q^1)\mathbb{D}(-P)\mathbb{D}(-\mathsf{rt}(Q^1))\\
&= -\mathbb{1}_\lambda\mathbb{L}(-Q^3)\mathbb{D}(-Q^2) - (-1)^{b(\mathsf{rt}(Q^1))+b(Q^2)-b(P)}\mathbb{1}_\lambda\mathbb{D}(-Q^1)\mathbb{D}(+Q^1)\mathbb{D}(-Q^2)
\end{aligned}$$

where the second equality follows from relation (6.7); the third from Definition 6.3; the fourth equality follows by applying the relation (6.5) to the adjacent pair P, Q^1; the fifth equality follows by applying the non-commuting relation to the pair $Q^1 \prec Q^2$. We therefore have that

$$\begin{aligned}
&\mathbb{1}_\lambda\mathbb{L}(-Q^3)\mathbb{D}(-Q^2)\mathbb{L}(-Q^1)\\
=&- \mathbb{1}_\lambda\mathbb{L}(-Q^3)^2\mathbb{D}(-Q^2) - (-1)^{b(\mathsf{rt}(Q^1))+b(Q^2)-b(P)}\mathbb{1}_\lambda\mathbb{L}(-Q^3)\mathbb{D}(-Q^1)\mathbb{D}(+Q^1)\mathbb{D}(-Q^2)\\
=&0 - (-1)^{b(\mathsf{rt}(Q^1))+b(Q^2)-b(P)}\mathbb{1}_\lambda\mathbb{L}(-Q^3)\mathbb{D}(-Q^1)\mathbb{D}(+Q^1)\mathbb{D}(-Q^2)\\
=&\mathbb{1}_\lambda\mathbb{L}(-Q^3)\mathbb{L}(-Q^1)\mathbb{D}(-Q^2)
\end{aligned}$$

where the first equality follows from substituting our expansion of $\mathbb{D}(-Q^2)\mathbb{L}(-Q^1)$ from above; the second follows from Proposition 7.4; the third by Definition 6.3 and the fact that $b(Q^2) = b(Q^1) + b(P) + b(T) = b(Q^1) + b(P) + b(\mathsf{rt}(Q^1))$.

It remains to consider the case in which $\mathsf{last}(Q^3) < m-1$. By Definition 6.3 we can express these loops in terms of $\mathsf{rt}(Q^3)$ and the case already considered above, as follows

$$\mathbb{L}^\lambda_\lambda(-Q^3) = -\mathbb{L}^\lambda_\lambda - (-1)^{b(\mathsf{rt}(Q^3))}\mathbb{1}_\lambda\mathbb{D}(+\mathsf{rt}(Q^3))\mathbb{D}(-\mathsf{rt}(Q^3)).$$

Our assumptions allow us to fix the following notation

$$Q^3 \setminus Q^2 = P_1 \sqcup P_2, \quad Q^2 \setminus Q^1 = S_1 \sqcup S_2$$

where $P_1, P_2 \in \mathrm{DRem}(\lambda - Q^2)$ and $S_1, S_2 \in \mathrm{DRem}(\lambda - Q^1)$. We assume that P_1 and S_1 are to the left of P_2 and S_2 respectively. Moreover, notice that $\mathsf{rt}(Q^3) = \mathsf{rt}(P_2)$. We have that

$$\mathbb{1}_\lambda\mathbb{L}(-Q^3)\mathbb{L}(-Q^1)\mathbb{D}(-Q^2) = (-1)^{b(Q^1)}\mathbb{1}_\lambda\mathbb{L}(-Q^3)\mathbb{D}(-Q^1)\mathbb{D}(+Q^1)\mathbb{D}(-Q^2)$$

$$
\begin{aligned}
&= (-1)^{b(Q^1)} \mathbb{1}_\lambda \mathbb{L}(-Q^3)\mathbb{D}(-Q^1)\mathbb{D}(-S_1)\mathbb{D}(-S_2)\\
&= (-1)^{2b(Q^1)+b(S_2)} \mathbb{1}_\lambda \mathbb{L}(-Q^3)\mathbb{D}(-Q^2)\mathbb{D}(+S_2)\mathbb{D}(-S_2)\\
&= \mathbb{1}_\lambda \mathbb{L}(-Q^3)\mathbb{D}(-Q^2)\big(\mathbb{L}(-Q^1)+\mathbb{L}(-P_2)\big)
\end{aligned}
$$

where the first equality follows from Definition 6.3; the second from (6.4) for the non-commuting pair $Q^1 \prec Q^2$; the third from (6.5) for the adjacent pair Q^1, S_1 and the fact that $b(Q^1)+b(Q^2)-b(S_1) = 2b(Q^1)+b(S_2)$; the fourth from the self-dual relation for S_2 and the fact that S_2 is adjacent to $P_2, Q^1 \in \mathrm{DRem}(\lambda - Q^2)$. Thus it suffices to prove $\mathbb{L}(-Q^3)\mathbb{D}(-Q^2)\mathbb{L}(-P_2) = 0$ and the result will follow. We have that

$$\mathbb{1}_\lambda \mathbb{L}(-Q^3)\mathbb{D}(-Q^2)\mathbb{L}(-P_2) = \mathbb{1}_\lambda \mathbb{L}(-Q^3)\mathbb{D}(-Q^2)(-\mathbb{L}^{\lambda-Q^2}_{\lambda-Q^2} - (-1)^{b(\mathsf{rt}(Q^3))}\mathbb{D}(+\mathsf{rt}(Q^3))\mathbb{D}(-\mathsf{rt}(Q^3)))$$

by Definition 6.3 and the fact that $\mathsf{rt}(P_2) = \mathsf{rt}(Q^3)$. Thus it will suffice to show that

$$(-1)^{b(\mathsf{rt}(Q^3))}\mathbb{L}(-Q^3)\mathbb{D}(-Q^2)\mathbb{D}(+\mathsf{rt}(Q^3))\mathbb{D}(-\mathsf{rt}(Q^3))) = -\mathbb{1}_\lambda \mathbb{L}(-Q^3)\mathbb{L}^\lambda_\lambda \mathbb{D}(-Q^2).$$

We have that

$$
\begin{aligned}
&(-1)^{b(\mathsf{rt}(Q^3))}\mathbb{L}(-Q^3)\mathbb{D}(-Q^2)\mathbb{D}(+\mathsf{rt}(Q^3))\mathbb{D}(-\mathsf{rt}(Q^3))\\
=&\mathbb{L}(-Q^3)\mathbb{D}(+\mathsf{rt}(Q^3))\mathbb{D}(-\mathsf{rt}(Q^3))\mathbb{D}(-Q^2)\\
=&\mathbb{L}(-Q^3)(-\mathbb{L}^\lambda_\lambda - \mathbb{L}(-Q^3))\mathbb{D}(-Q^2)\\
=&-\mathbb{L}(-Q^3)\mathbb{L}^\lambda_\lambda\mathbb{D}(-Q^2) - \mathbb{L}(-Q^3)^2\mathbb{D}(-Q^2)\\
=&-\mathbb{L}(-Q^3)\mathbb{L}^\lambda_\lambda\mathbb{D}(-Q^2)
\end{aligned}
$$

as required; where the first equality follows from the commutativity relation (6.3); the second follows from the self-dual relation for Q^3; the third equality expands out the brackets; the fourth equality follows from relation (6.7). The result follows. □

Lemma 7.9. *Let $Q^1 \prec Q^2 \prec Q^3$ be removable Dyck paths of $\lambda \in \mathscr{R}_{m,n}$. Suppose that $Q^3 \in \mathrm{DRem}_{>0}(\lambda)$ and that Q^1 does not commute with Q^2 and that Q^2 does not commute with Q^3. We set $Q^3 \setminus Q^2 = P^1 \sqcup P^2$. We have that*

$$\mathbb{D}(-Q^3)\mathbb{D}(+Q^3)\mathbb{D}(-Q^2)\mathbb{D}(-Q^1) = \mathbb{D}(-Q^1)\mathbb{D}(-Q^3)\mathbb{D}(+P^1)\mathbb{D}(+P^2).$$

Proof. This follows simply using the "old relations" and is left as an exercise for the reader. □

7.2. **A spanning set of the symmetric Dyck path algebra.** Let $\mu \setminus \alpha$ be a Dyck tiling. We recall our favourite path from α to μ is given as follows: we first regularise

$$\alpha = \mathsf{reg}_d(\alpha) \subseteq \mathsf{reg}_{d+1}(\alpha) \cdots \subseteq \mathsf{reg}_{-1}(\alpha) \subseteq \mathsf{reg}_0(\alpha) = \mathsf{reg}(\alpha)$$

for $d(\alpha) = d$ and $P^k = (\mathsf{reg}_k(\alpha) \setminus \mathsf{reg}_{k-1}(\alpha))$ the maximal breadth addable Dyck path of height k. After regularising, we then split

$$\mathsf{reg}(\alpha) = \mathsf{split}_d(\mu \setminus \alpha) \supseteq \mathsf{split}_{1-d}(\mu \setminus \alpha) \cdots \supseteq \mathsf{split}_{-1}(\mu \setminus \alpha) \supseteq \mathsf{split}_0(\mu \setminus \alpha) = \alpha \sqcup (\mu \setminus \alpha)_{\leqslant 0}$$

with $(\mathsf{split}_k(\mu \setminus \alpha)) \setminus (\mathsf{split}_{k+1}(\mu \setminus \alpha)) = R^k_1 \sqcup R^k_2 \sqcup \cdots \sqcup R^k_K$ a disjoint union of commuting Dyck paths such that

$$P^k - R^k_1 - R^k_2 - \cdots - R^k_K = (\mu \setminus \alpha)_k. \tag{7.17}$$

After splitting, we then add as follows

$$\alpha \sqcup (\mu \setminus \alpha)_{\leqslant 0} = \mathsf{add}_0(\mu \setminus \alpha) \subseteq \mathsf{add}_1(\mu \setminus \alpha) \cdots \subseteq \mathsf{add}_{a-1}(\mu \setminus \alpha) \subseteq \mathsf{add}_a(\mu \setminus \alpha) = \mu$$

where

$$\mathsf{add}_k(\mu \setminus \alpha) \setminus \mathsf{add}_{k-1}(\mu \setminus \alpha) = A^k_1 \sqcup A^k_2 \sqcup \cdots \sqcup A^k_K \tag{7.18}$$

is a union of addable Dyck paths of height $1 \leqslant k \leqslant a$. We now lift these paths to elements of $\mathcal{A}_{m,n}$. For $d(\alpha) \leqslant k \leqslant 0$ we lift the kth step in equation (4.1) to the element

$$\mathbb{L}^k(\alpha) = \mathbb{L}^{\mathsf{reg}(\alpha)}_{\mathsf{reg}(\alpha)}(-P^k)$$

and for $d(\alpha) \leqslant k \leqslant 0$ we lift the kth step in equation (4.2) to the element

$$\mathbb{R}^k_{\mu\backslash\alpha} = \mathbb{D}^{\mathsf{split}_{k-1}(\mu\backslash\alpha)}_{\mathsf{split}_k(\mu\backslash\alpha)} = \mathbb{D}^{\mathsf{split}_{k-1}(\mu\backslash\alpha)}_{\mathsf{split}_{k-1}(\mu\backslash\alpha)-R^k_1}\mathbb{D}^{\mathsf{split}_{k-1}(\mu\backslash\alpha)-R^k_1}_{\mathsf{split}_{k-1}(\mu\backslash\alpha)-R^k_1-R^k_2}\cdots \tag{7.19}$$

and for $1 \leqslant k \leqslant a$ we lift the kth step in equation (4.3) to the element

$$\mathbb{A}^k_{\mu\backslash\alpha} = \mathbb{D}^{\mathsf{add}_{k-1}(\mu\backslash\alpha)}_{\mathsf{add}_k(\mu\backslash\alpha)} = \mathbb{D}^{\mathsf{add}_{k-1}(\mu\backslash\alpha)}_{\mathsf{add}_{k-1}(\mu\backslash\alpha)+A^k_1}\mathbb{D}^{\mathsf{add}_{k-1}(\mu\backslash\alpha)+A^k_1}_{\mathsf{add}_{k-1}(\mu\backslash\alpha)+A^k_1+A^k_2}\cdots \tag{7.20}$$

We note that the ordering in (7.19) and (7.20) does not matter, as all these paths commute. Finally, we put all this together

$$\mathbb{L}(\alpha) = \prod_{d(\alpha)\leqslant k\leqslant 0} \mathbb{L}^k(\alpha) \qquad \mathbb{R}_{\mu\backslash\alpha} = \prod_{d(\alpha)\leqslant k\leqslant 0} \mathbb{R}^k_{\mu\backslash\alpha} \qquad \mathbb{A}_{\mu\backslash\alpha} = \prod_{k\geqslant 1}\mathbb{A}^k_{\mu\backslash\alpha}$$

(and we extend this to duals in the usual fashion). We define the elements

$$\mathbb{R}^{<h}(\mu\setminus\alpha) = \prod_{d(\alpha)\leqslant k<h} \mathbb{R}^k_{\mu\backslash\alpha} \qquad \mathbb{R}^{>h}(\mu\setminus\alpha) = \prod_{h<k\leqslant 0} \mathbb{R}^k_{\mu\backslash\alpha}$$

and extend this to weak inequalities and similar subproducts of $\mathbb{A}(\mu\setminus\alpha)$ in the obvious fashion. With this notation in place, we are able to state the main result of this subsection.

Theorem 7.10. *The symmetric Dyck algebra $\mathcal{A}_{m,n}$ has spanning set*

$$\{\mathbb{A}^*_{\lambda\backslash\alpha}\mathbb{R}^*_{\lambda\backslash\alpha}\mathbb{L}(\alpha)\mathbb{R}_{\mu\backslash\alpha}\mathbb{A}_{\mu\backslash\alpha} \mid \alpha\in\mathscr{P}_{m,n}, \lambda,\mu\in\mathscr{R}_{m,n} \text{ and } \mu\setminus\alpha, \lambda\setminus\alpha \text{ are Dyck pairs}\}. \tag{7.21}$$

Proof. We will prove that the 2-sided ideal $\mathcal{A}^{\leqslant\alpha}_{m,n} = \langle \mathbb{1}_\sigma, \mathbb{L}^\sigma_\sigma(\tau) \mid \tau\leqslant\sigma\leqslant\alpha\rangle$ has spanning set given by

$$\mathrm{Dyck}_{\leqslant\alpha} = \{\mathbb{A}^*_{\lambda\backslash\beta}\mathbb{R}^*_{\lambda\backslash\beta}\mathbb{L}(\beta)\mathbb{R}_{\mu\backslash\beta}\mathbb{A}_{\mu\backslash\beta} \mid \lambda,\mu\in\mathscr{R}_{m,n} \text{ and } \mu\setminus\beta, \lambda\setminus\beta \text{ are Dyck pairs for } \beta\leqslant\alpha\}.$$

We proceed by induction on the Bruhat ordering on α, refining the induction by the degree of the element

$$\mathbb{A}^*_{\lambda\backslash\beta}\mathbb{R}^*_{\lambda\backslash\beta}\mathbb{L}(\beta)\mathbb{R}_{\mu\backslash\beta}\mathbb{A}_{\mu\backslash\beta}$$

for all $\beta\leqslant\alpha$. Our aim is to show the following:

$$(\mathbb{L}(\alpha)\mathbb{R}_{\mu\backslash\alpha}\mathbb{A}_{\mu\backslash\alpha})X = \begin{cases} \mathbb{L}(\alpha)\mathbb{R}_{\mu\backslash\alpha}\mathbb{A}_{\mu\backslash\alpha} & \text{if } X=\mathbb{1}_\mu;\\ \mathbb{L}(\alpha)\mathbb{R}_{\nu\backslash\alpha}\mathbb{A}_{\nu\backslash\alpha} & \text{if } X=\mathbb{D}(+Q) \text{ and } (\mu+Q)\setminus\alpha \text{ a Dyck pair};\\ \pm\mathbb{L}(\alpha)\mathbb{R}_{\nu\backslash\alpha}\mathbb{A}_{\nu\backslash\alpha} & \text{if } X=\mathbb{D}(-Q) \text{ for } Q\notin\mu\setminus\alpha \text{ and } (\mu-Q)\setminus\alpha \text{ a Dyck pair};\\ Y\in\mathcal{A}^{<\alpha}_{m,n} & \text{otherwise,}\end{cases}$$

for any $\mu\setminus\alpha$ such that $\deg(\mu\setminus\alpha) = 2|d(\alpha)|+k$ and will refine our induction by the degree $k\geqslant 0$. In the $\alpha=\varnothing$ case, we first observe that $\mathsf{reg}(\varnothing) = (m^m)\in\mathcal{P}_{m,n}$. Now, we note that the unique non-zero element of (7.21) is given by

$$\mathbb{L}(\varnothing) = \textstyle\prod_{P\in\mathrm{DRem}(m^m)}\mathbb{L}^{(m^m)}_{(m^m)}(-P)$$

and is of degree $2m$ (note that this varies over the m distinct removable Dyck paths, of breadth $b(P)=p\in\{1,3,\dots,2m-1\}$). Thus it will suffice to show that

$$\mathbb{L}(\varnothing)X = \begin{cases}\mathbb{L}(\varnothing) & \text{for } X=\mathbb{1}_{(m^m)},\\ 0 & \text{otherwise.}\end{cases}$$

Let $Q\in\mathrm{DRem}(m^m)$. For $X=\mathbb{L}(-Q)$ the desired equality follows from Proposition 7.5. For $X=\mathbb{D}(-Q)$ the desired equality follows from Proposition 7.6. For $X=\mathbb{D}(+Q)$ the desired equality follows from Lemma 7.7. The case that X is an idempotent is trivial.

Thus we can assume the result holds for all $\alpha'<\alpha$ and proceed to the α case. We consider each type of generator $X\in K^m_n$ in turn: loops, adding generators, and removing generators (idempotent generators are trivial).

Loops. Consider the product $(\mathbb{L}(\alpha)\mathbb{R}_{\mu\backslash\alpha}\mathbb{A}_{\mu\backslash\alpha})X$ with $X=\mathbb{L}^\mu_\mu(-Q)$ with $Q\in\mathrm{DRem}(\mu)$. There are four cases to consider: (i) $Q=H\in\mu\setminus\alpha$ or (ii) $Q\notin\mu\setminus\alpha$ and Q commutes with $\mu\setminus\alpha$ (iii) $Q\notin\mu\setminus\alpha$ but $Q\prec H\in\mu\setminus\alpha$ for some H which does not commute with Q (iv) $Q\notin\mu\setminus\alpha$ but $Q\succ H\in\mu\setminus\alpha$ for some H with $\mathsf{ht}^\mu_\alpha(H)>0$ which does not commute with Q.

Case (ii) is not as simple as one might first think. By assumption Q commutes with all of $\mu \setminus \alpha$. Thus Q commutes with the A_i^k for $i \geqslant 1$ and $1 \leqslant k \leqslant a$ as in (7.18) for the Dyck pair $\mu \setminus \alpha$ and we claim that $\mathbb{L}_\mu^\mu(-Q)$ commutes with $\mathbb{A}_{\mu\setminus\alpha}$. The claim follows by applying Lemmas 7.1 and 7.2 in the case that $\mathsf{ht}^\mu(Q) = 0$ or by using the factorisation $\mathbb{L}_\mu^\mu(-Q) = (-1)^{b(Q)}\mathbb{D}_{\lambda-Q}^{\lambda}\mathbb{D}_{\lambda}^{\lambda-Q}$ and (6.3) in the case that $\mathsf{ht}^\mu(Q) > 0$. We thus obtain

$$(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{L}_\mu^\mu(-Q) = \mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{L}(-Q)\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu.$$

If Q commutes with all the Dyck paths R_i^k for all $i \geqslant 1$ and $d(\alpha) \leqslant k \leqslant 0$ in equation (7.17) then (again, applying the commutation relations of Lemmas 7.1 and 7.2 or (6.3) as above) we obtain

$$\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{L}(-Q)\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu = \mathbb{L}(\alpha)\mathbb{L}(-Q)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu \in \mathcal{A}_{m,n}^{\leqslant\alpha-Q}$$

by Proposition 7.5. Now assume that Q does not commute with all of the Dyck paths R_i^k in equation (7.17) (but recall our ongoing assumption that Q commutes with all Dyck paths in $\mu \setminus \alpha$). In which case, Q commutes with a pair of Dyck paths $H \in \mu \setminus \alpha$ (to the left of Q) and $H' \in \mu \setminus \alpha$ (to the right of Q) such that $\mathsf{ht}_\alpha^\mu(H) = \mathsf{ht}_\alpha^\mu(H') = h \in \mathbb{Z}_{\leqslant 0}$ and such that

$$H \sqcup H' = \mathrm{split}_{R_i^h}(H'') \tag{7.22}$$

for some $i \geqslant 1$ and some (necessarily unique) R_i^h which does not commute with Q. We remark that in such a case the Dyck path R_i^h, in turn, does not commute with H''. An example of how this can happen is given in Figure 29.

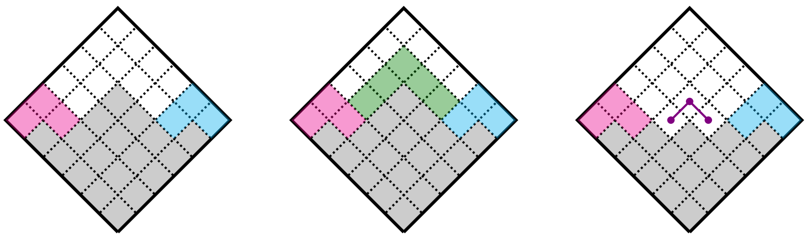


FIGURE 29. An example of case (ii) for loop generators. On the left we depict $\mu \setminus \alpha = H \sqcup H'$. In the middle we depict $\mathsf{reg}(\alpha) \setminus \alpha$ and we highlight the Dyck path R_1^h. On the right we depict $\nu \setminus \alpha$ where $\nu = \mu - Q$. Notice that Q commutes with the tiling $\mu \setminus \alpha = H \sqcup H'$ but does not commute with R_1^h.

For Dyck paths as in equation (7.22) we have that

$$\begin{aligned}
\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{L}(-Q)\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu &= (\textstyle\prod_{k\neq h}\mathbb{L}(-P^k))\mathbb{R}_{\mu\setminus\alpha}^{<h}\mathbb{L}(-P^h)\mathbb{R}_{\mu\setminus\alpha}^{h}\mathbb{L}(-Q)\mathbb{R}_{\mu\setminus\alpha}^{>h}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu \\
&= (\textstyle\prod_{k\neq h}\mathbb{L}(-P^k))\mathbb{R}_{\mu\setminus\alpha}^{<h}\mathbb{L}(-P^h)\mathbb{L}(-Q)\mathbb{R}_{\mu\setminus\alpha}^{h}\mathbb{R}_{\mu\setminus\alpha}^{>h}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu \\
&= (\textstyle\prod_{k\neq h}\mathbb{L}(-P^k))\mathbb{L}(-P^h)\mathbb{L}(-Q)\mathbb{R}_{\mu\setminus\alpha}^{<h}\mathbb{R}_{\mu\setminus\alpha}^{h}\mathbb{R}_{\mu\setminus\alpha}^{>h}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu \\
&= \mathbb{L}(\alpha)\mathbb{L}(-Q)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu
\end{aligned}$$

where first and third equalities follow from the commuting relations or Lemmas 7.1 and 7.2; the second equalities from applying Proposition 7.8 for $h = 0$ (or Lemma 7.9 if $h > 0$) to the pair R_i^h and Q and the commutation relations (6.3) to the pairs R_j^h and Q for $j \neq i$; the fourth equality follows from the definitions of the products. We have that $\mathbb{L}(\alpha)\mathbb{L}(-Q) \in \mathcal{A}_{m,n}^{\leqslant\alpha-Q}$ by Proposition 7.5.

Case (iii). We now suppose that $Q \notin \mu \setminus \alpha$, but $Q \prec H \in \mu \setminus \alpha$ for some H which does not commute with Q. By assumption $\mathsf{ht}^\mu(Q) > 0$ and so $\mathbb{L}_\mu^\mu(-Q) = (-1)^{b(Q)}\mathbb{D}_{\mu-Q}^{\mu}\mathbb{D}_\mu^{\mu-Q}$. First, we suppose that $\mathsf{ht}_\alpha^\mu(H) \leqslant 0$. In which case Q commutes with every Dyck path A_k^j, R_l^i in (7.17) and (7.18) (see Figure 30 for an example). By the commutation relations (6.3), we have that

$$(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{L}_\mu^\mu(-Q) = \mathbb{L}(\alpha)\mathbb{L}(-Q)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu \in \mathcal{A}_{m,n}^{\leqslant\alpha-Q}$$

by Proposition 7.5.

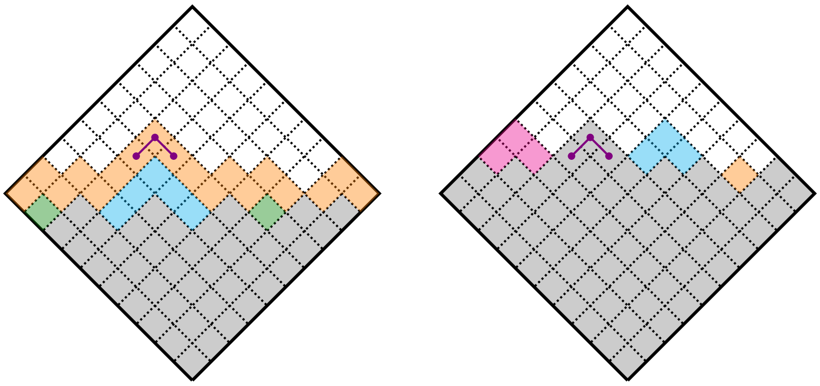


FIGURE 30. An example of case (iii) for loop generators. On the left we depict the Dyck tiling of $\mu \setminus \alpha$ and the Dyck path X which commutes with all of $\mu \setminus \alpha$ except for $X \prec H$ which is of height -1. On the right we depict the Dyck tiling of $\mathsf{reg}(\alpha) \setminus \mu$ (consisting of the Dyck paths R_1^{-1}, R_2^{-1}, and R_1^0) and the Dyck path X which commutes with all of $\mathsf{reg}(\alpha) \setminus \mu$.

We now consider the case that $\mathsf{ht}^\mu_\alpha(H) = h > 0$ and Q does not commute with some $Q \prec H = A_1^h$ in (7.18). We note that $H \setminus Q = H^1 \sqcup H^2$. We have that

$$\begin{aligned}(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{L}^\mu_\mu(-Q) =& \mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+H)\mathbb{L}(-Q)(\textstyle\prod_{j\neq 1}\mathbb{D}(+A_j^h))\mathbb{A}^{>h}_{\mu\setminus\alpha}\mathbb{1}_\mu\\ =& \mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+H^1)\mathbb{D}(+H^2)\mathbb{D}(+Q)(\textstyle\prod_{j\neq 1}\mathbb{D}(+A_j^h))\mathbb{A}^{>h}_{\mu\setminus\alpha}\mathbb{1}_\mu\\ =& \mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+H^1)\mathbb{D}(-H^1)\mathbb{D}(+H)(\textstyle\prod_{j\neq 1}\mathbb{D}(+A_j^h))\mathbb{A}^{>h}_{\mu\setminus\alpha}\mathbb{1}_\mu\\ =& \mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+H^1)\mathbb{D}(-H^1)\mathbb{A}^{\geqslant h}_{\mu\setminus\alpha}\mathbb{1}_\mu\\ =& \sum_{P\in \mathrm{DRem}(\nu)} c_P\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{L}(-P)\mathbb{A}^{\geqslant h}_{\mu\setminus\alpha}\mathbb{1}_\mu\end{aligned}$$

for $\nu = \alpha \cup (\mu \setminus \alpha)_{<h}$ and $c_P \in \Bbbk$ some coefficients which can be calculated explicitly using the self-dual relation (6.2); the second and third equalities follow by the non-commuting relation (6.4) and adjacency relations (6.5). Now, we observe that

$$\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{L}(-P) = \mathbb{L}(\alpha)\mathbb{R}_{\nu\setminus\alpha}\mathbb{A}_{\nu\setminus\alpha}\mathbb{L}(-P) \in \mathcal{A}^{<\alpha}_{m,n}$$

by our inductive assumption on the degree; therefore $(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{L}^\mu_\mu(-Q) \in \mathcal{A}^{<\alpha}_{m,n}$.

Case (i). We now suppose that $Q = H \in \mu \setminus \alpha$. We first assume that $\mathsf{ht}^\mu_\alpha(H) = h \leqslant 0$. There are two distinct subcases to consider. First suppose that H is the unique element of $(\mu \setminus \alpha)_h$. In this case, Q commutes with all the Dyck paths A_k^j, R_l^i in (7.17) and (7.18) and so we have that

$$(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{L}(-Q) = \mathbb{L}(\alpha)\mathbb{L}(-Q)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu \in \mathcal{A}^{\leqslant\alpha-Q}_{m,n}$$

by the commutation relations (6.3) and Proposition 7.5. We must now consider the case that H is not the unique Dyck path in $(\mu \setminus \alpha)_h$ for $h \leqslant 0$. Our assumptions imply that Q is adjacent to $H' := R_1^h$ and possibly $H'' := R_2^h$ (if the latter exists) removed in the h step of (7.17); moreover, Q commutes with every other Dyck path A_k^j, R_l^i in (7.17) and (7.18) not equal to H' or H''. Our assumptions further imply that $\mathsf{ht}^\mu(Q) > 0$ and so $\mathbb{L}(-Q) = (-1)^{b(Q)}\mathbb{1}_\mu\mathbb{D}(-Q)\mathbb{D}(+Q)$ by Definition 6.3.

We first consider the case that $H' = R_1^h$ is the unique Dyck path in (7.17) which is adjacent to Q. We have that

$$\begin{aligned}(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{L}^\mu_\mu(-Q) =& \mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(-H')\mathbb{L}(-Q)(\textstyle\prod_{j\neq 1}\mathbb{D}(-R_j^h))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu\\ =& \pm\,\mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{L}(-\langle H'\cup Q\rangle_\nu)\mathbb{D}(-H')(\textstyle\prod_{j\neq 1}\mathbb{D}(-R_j^h))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu\end{aligned}$$

for $\nu = \alpha \cup (\mu \setminus \alpha)_{<h}$; the first equality follows from the commutativity relations (6.3); the second equality follows from the adjacent relation (6.5). Therefore $(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{L}^\mu_\mu(-Q) \in \mathcal{A}^{<\alpha}_{m,n}$ by induction on degree.

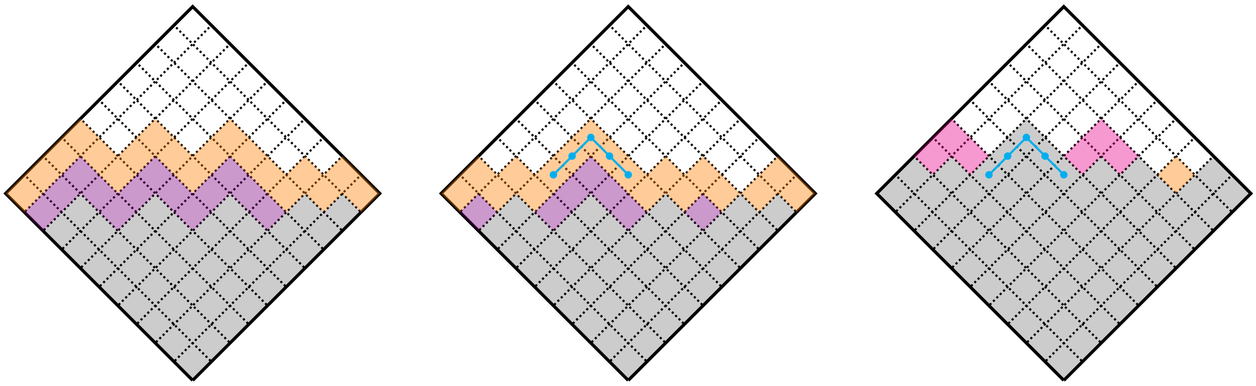


FIGURE 31. An example of case (i) for loop generators. We depict the Dyck tilings of $\mathsf{reg}(\alpha) \setminus \alpha$, $\mu \setminus \alpha$, and $\mathsf{reg}(\alpha) \setminus \mu$ respectively. In the two rightmost pictures, we also depict the Dyck path X which is equal to a $H \in \mu \setminus \alpha$ which is of height -1.

We now consider the case that $H' = R_1^h$ and $H'' = R_2^h$ are the two Dyck paths in (7.17) which are adjacent to Q. We have that

$$\begin{aligned}&(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{L}(-Q)\\ =&\mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(-H'')\mathbb{D}(-H')\mathbb{L}^\mu_\mu(-Q)(\textstyle\prod_{j\neq 1,2}\mathbb{D}(-R_j^h))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu\\ =&\pm\mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{1}_\nu\mathbb{D}(-H'')\mathbb{L}(-\langle H'\cup Q\rangle_\nu)\mathbb{D}(-H')(\textstyle\prod_{j\neq 1,2}\mathbb{D}(-R_j^h))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu\\ =&\pm\mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{1}_\nu\mathbb{L}(-\langle H'\cup H''\cup Q\rangle_\nu)\mathbb{D}(-H'')\mathbb{D}(-H')(\textstyle\prod_{j\neq 1,2}\mathbb{D}(-R_j^h))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu\end{aligned}$$

for $\nu = \alpha\cup(\mu\setminus\alpha)_{<h}$; the first equality follows from the commutativity relations (6.3); the second and third equalities follow from the adjacent relation (6.5). Therefore $(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{L}^\mu_\mu(-Q) \in \mathcal{A}^{<\alpha}_{m,n}$ by induction on degree.

Now assume that $\mathsf{ht}^\mu_\alpha(H) > 0$. In this case Q is equal to precisely one Dyck path, $Q = H = A_1^h$ say, in (7.18) and Q commutes with all other paths in A_k^j, R_l^i in (7.18) and (7.17) not equal to H. We have that

$$\begin{aligned}(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{L}(-Q) &= \mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+Q)\mathbb{L}(-Q)(\textstyle\prod_{j\neq 1}\mathbb{D}(+A_j^h))\mathbb{A}^{>h}_{\mu\setminus\alpha}\mathbb{1}_\mu\\ &= \mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+Q)\mathbb{D}(-Q)\mathbb{D}(+Q)(\textstyle\prod_{j\neq 1}\mathbb{D}(+A_j^h))\mathbb{A}^{>h}_{\mu\setminus\alpha}\mathbb{1}_\mu\\ &= \textstyle\sum_{P\in\mathrm{DRem}(\nu)} c_P\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{L}(-P)\mathbb{A}^{\geqslant h}_{\mu\setminus\alpha}\mathbb{1}_\mu\end{aligned}$$

for $\nu = \alpha\cup(\mu\setminus\alpha)_{<h}$ and $c_P \in \Bbbk$ some coefficients which can be calculated explicitly using the self-dual relation (6.2); the second equality follows by Definition 6.3. Therefore $(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{L}(-Q) \in \mathcal{A}^{<\alpha}_{m,n}$ by induction on degree.

Finally, suppose we are in case (iv) and we assume that Q is not as in case (ii) or (iii), an example is depicted in Figure 32. Since $\mathsf{ht}^\mu(Q) \in \mathbb{Z}_{\geqslant 0}$, this implies that $\mathsf{ht}^\mu_\alpha(H) = h \in \mathbb{Z}_{>0}$. Our assumptions imply that $Q \setminus H = H^1 \sqcup H^2$. and that H^1, H^2 commute with $\nu\setminus\alpha$ for $\nu = \mu - H$. We have that

$$\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu\mathbb{L}(-Q) = \mathbb{A}_{\nu\setminus\alpha}\mathbb{D}(+H)\mathbb{L}(-Q)\mathbb{1}_\mu = \mathbb{A}_{\nu\setminus\alpha}\mathbb{L}(-H^1)\mathbb{D}(+H)\mathbb{1}_\mu$$

by either Proposition 7.3 if $\mathsf{ht}^\mu(Q) = 0$ or application of the adjacent relation (6.5) if $\mathsf{ht}^\mu(Q) > 0$ (using the factorisation of Definition 6.3). In either case, the result again follows by induction on the degree.

Adding a Dyck Path. We now consider product where $X = \mathbb{D}(+Q)$ corresponds to adding a Dyck path $Q \in \mathrm{DAdd}(\mu)$ (to obtain $\nu = \mu \cup Q$). There are three cases to consider: Q is adjacent to $i = 0, 1$, or 2 Dyck paths in $\mu \setminus \alpha$.

The $i = 0$ case. Assume that $Q \in \mathrm{DAdd}(\mu)$ is adjacent to zero Dyck paths in the Dyck tiling of $\mu \setminus \alpha$. In which case, $(\mu\setminus\alpha)\cup Q$ is itself a Dyck tiling and

$$(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(+Q) = \mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}(\mathbb{A}_{\mu\setminus\alpha}\mathbb{D}(+Q))\mathbb{1}_\nu = \mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\nu\setminus\alpha}\mathbb{1}_\nu = \mathbb{L}(\alpha)\mathbb{R}_{\nu\setminus\alpha}\mathbb{A}_{\nu\setminus\alpha}\mathbb{1}_\nu$$

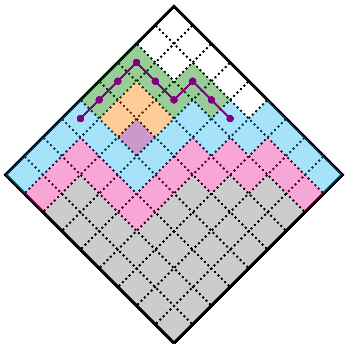


FIGURE 32. Case (iv) of the loop generator. Notice that Q commutes with the big pink and cyan Dyck paths (as we are not in case (iii) by assumption). The Dyck paths $Q \setminus H = H^1 \sqcup H^2$ commute with the rest of the Dyck tiling (again as we are not in case (iii)).

is an element of the claimed spanning set (the third equality follows immediately from the definition of the elements, the second equality might also involve some application of the commutativity relations (6.3)).

The $i = 1$ case. We now assume that Q is adjacent to one Dyck path $H \in \mu \setminus \alpha$ with $\mathsf{ht}^\mu_\alpha(H) = h \in \mathbb{Z}$. We first suppose that $\langle H \cup Q \rangle_{\mu+Q}$ does not exist, in which case

$$(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(+Q) = 0$$

using many applications of the commuting relation (6.3) and one application of either the adjacency relation 6.5 or Lemma 7.7 for $h > 0$ or $h \leqslant 0$ respectively. We can now assume that $\langle H \cup Q \rangle_{\mu+Q}$ does exist. Our assumption that H is the *unique* Dyck path in $\mu \setminus \alpha$ that is adjacent to Q implies that $\langle H \cup Q \rangle_{\mu+Q} - Q - H = H' \in \mathrm{DRem}(\alpha)$. We consider the case that $h \leqslant 0$ (the case that $h > 0$ is almost identical) where we have that

$$\begin{aligned}(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(+Q) =& \mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(-H)\mathbb{D}(+Q)(\textstyle\prod_{j\neq 1}\mathbb{D}(-R^j_h))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu+Q}\\ =& \mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(-H')\mathbb{D}(+(H\cup H'\cup Q))(\textstyle\prod_{j\neq 1}\mathbb{D}(-R^j_h))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu+Q}\\ =& \mathbb{L}(\alpha)\mathbb{D}(-H')\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+(H\cup H'\cup Q))(\textstyle\prod_{j\neq 1}\mathbb{D}(-R^j_h))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu+Q}\\ =& \mathbb{D}(-H')\mathbb{L}(\alpha-H')\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+(H\cup H'\cup Q))(\textstyle\prod_{j\neq 1}\mathbb{D}(-R^j_h))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu+Q}\end{aligned}$$

where the second equality follows from the adjacency relation (6.5) for $h \neq 0$ and Lemma 7.7 for $h = 0$; the first and third equalities follow by the commuting relations (6.3); the final equality follows by Lemmas 7.1 and 7.2. Thus we conclude that $(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(+Q) \in \mathcal{A}^{\leqslant\alpha-H'}_{m,n}$.

The $i = 2$ case. It remains to consider the case that there are precisely two Dyck paths adjacent to Q; these must be of the same height $h \in \mathbb{Z}$ and we label them by $H' = A^h_1$ and $H'' = A^h_2$. If $h \in \mathbb{Z}_{>0}$, we apply the commutativity relation (6.3) and the adjacent relation (6.5) and hence obtain the following

$$\begin{aligned}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu\mathbb{D}(+Q) =& \mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{A}^{h}_{\mu\setminus\alpha}\mathbb{D}(+Q)\mathbb{A}^{>h}_{\mu\setminus\alpha}\mathbb{1}_{\mu+Q}\\ =& \mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+H')\mathbb{D}(+H'')\mathbb{D}(+Q)(\textstyle\prod_{j\neq 1,2}\mathbb{D}(+A^h_j))\mathbb{A}^{>h}_{\mu\setminus\alpha}\mathbb{1}_{\mu+Q}\\ =& \mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+H')\mathbb{D}(-H')\mathbb{D}(+Q\cup H\cup H')(\textstyle\prod_{j\neq 1,2}\mathbb{D}(+A^h_j))\mathbb{A}^{>h}_{\mu\setminus\alpha}\mathbb{1}_{\mu+Q}\\ =& \textstyle\sum_{P\in\mathrm{DRem}(\nu)} c_P\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{L}(-P)\mathbb{D}(+Q\cup H\cup H')(\prod_{j\neq 1,2}\mathbb{D}(+A^h_j))\mathbb{A}^{>h}_{\mu\setminus\alpha}\mathbb{1}_{\mu+Q}\end{aligned}$$

for $\nu = \alpha \cup (\mu\setminus\alpha)_{<h}$ and $c_P \in \Bbbk$ some coefficients which can be calculated explicitly using the self-dual relation (6.2). Substituting the above into the overall product, we deduce that $(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(+Q) \in \mathcal{A}^{<\alpha}_{m,n}$ by induction on degree.

We now assume that $h \in \mathbb{Z}_{\leqslant 0}$ (and continue with out assumption that Q is adjacent to two Dyck paths in $\mu \setminus \alpha$). We note that Q commutes past all Dyck paths A^j_k in (7.18) but is actually *equal to* some path $Q = R^h_1$ in (7.17). This is because Q is adjacent to two Dyck paths H' and H'' of height

$h \in \mathbb{Z}_{\leqslant 0}$ and these paths were formed by first doing a loop and then splitting with the path R_1^h. We hence obtain the following

$$\begin{aligned}(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(+Q) =&\mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{R}^{h}_{\mu\setminus\alpha}\mathbb{D}(+Q)\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu+Q}\\ =&\mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(-Q)\mathbb{D}(+Q)\textstyle\prod_{j\neq 1}\mathbb{D}(-R^j_h)\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu+Q}\\ =&\mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{L}(-Q)\textstyle\prod_{j\neq 1}\mathbb{D}(-R^j_h)\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu+Q}\end{aligned}$$

where the first and second equalities follow from the commutation relations (6.3); and the third equality follows from Definition 6.3. Again, we deduce that $(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(+Q) \in \mathcal{A}^{<\alpha}_{m,n}$ by induction on degree.

Removing a Dyck Path. We now consider product where $X = \mathbb{D}(-Q)$ corresponds to removing a Dyck path Q. There are three cases to consider: (i) $Q = H \in \mu\setminus\alpha$ or (ii) $Q \notin \mu\setminus\alpha$ and Q commutes with $\mu\setminus\alpha$ or (iii) $Q \notin \mu\setminus\alpha$ and $Q \prec H \in \mu\setminus\alpha$ for some H which does not commute with Q (iv) $Q \notin \mu\setminus\alpha$ but $Q \succ H \in \mu\setminus\alpha$ which does not commute with Q.

Case (ii) is (again, as in the loop case) not as simple as one might first think. By assumption Q commutes with all of $\mu\setminus\alpha$. Thus Q commutes with the A^j_l in equation (7.18) and (as in the loop case) $\mathbb{D}(-Q)$ commutes with $\mathbb{A}_{\mu\setminus\alpha}$. Now, either $\mathbb{D}(-Q)$ commutes with all the of the Dyck paths R^i_k in equation (7.17) or there exists a pair H and H' as in equation (7.22). We have that

$$(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(-Q) = \mathbb{L}(\alpha)\mathbb{D}(-Q)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu = \mathbb{D}(-Q)\mathbb{L}(\alpha-Q)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu \in \mathcal{A}^{<\alpha}_{m,n}$$

where the first equality follows from an identical argument to that used in case (i) of the loop case (with $\mathsf{ht}^\mu_\alpha(H) = h > 0$); the second equality follows from Lemmas 7.1 and 7.2 as Q commutes with $\mathsf{reg}(\alpha)\setminus\alpha$.

Case (iii). Suppose that $Q \notin \mu\setminus\alpha$ but $Q \prec H \in \mu\setminus\alpha$ of height $h \in \mathbb{Z}$. In this case, we have that $H \setminus Q = Q^1 \sqcup Q^2$ for some Q^1 and Q^2 and moreover that $\nu\setminus\alpha$ for $\nu = \mu - Q$ is a Dyck tiling. If $\mathsf{ht}^\mu_\alpha(H) > 0$ then we have that

$$(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(-Q) = \mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{(\mu-Q)\setminus\alpha}\mathbb{1}_{\mu-Q} = \mathbb{L}(\alpha)\mathbb{R}_{(\mu-Q)\setminus\alpha}\mathbb{A}_{(\mu-Q)\setminus\alpha}$$

where the first equality follows by the non-commuting relation (6.4) for the Dyck paths H and Q (together with the usual commutation relations (6.3)); the second equality follows as $\mathbb{R}_{\mu\setminus\alpha} = \mathbb{R}_{(\mu-Q)\setminus\alpha}$. If $\mathsf{ht}^\mu_\alpha(H) \leqslant 0$ we have that

$$(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(-Q) = \mathbb{L}(\alpha)(\mathbb{R}_{\mu\setminus\alpha}\mathbb{D}(-Q))\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu-Q} = \mathbb{L}(\alpha)\mathbb{R}_{(\mu-Q)\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu-Q}$$

where the first equality follows by the commutation relations (6.3); the second equality follows as $\mathbb{R}_{(\mu-Q)\setminus\alpha} = \mathbb{R}_{\mu\setminus\alpha}\mathbb{D}(-Q)$ (and the commutation relations (6.3)) and $\mathbb{A}_{\mu\setminus\alpha} = \mathbb{A}_{(\mu-Q)\setminus\alpha}$.

Case (i). We suppose that $Q = H \in \mu\setminus\alpha$ for some $\mathsf{ht}^\mu_\alpha(H) = h \in \mathbb{Z}$. We first assume that $\mathsf{ht}^\mu_\alpha(H) \leqslant 0$. First suppose that H is the unique element of $(\mu\setminus\alpha)_h$. In this case, Q commutes with all the Dyck paths A^j_k, R^i_l in (7.17) and (7.18) and so we have that

$$(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(-Q) = \mathbb{L}(\alpha)\mathbb{D}(-Q)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu-Q} = 0$$

where the final equality follows by Proposition 7.6 and the fact that $Q \in \mu\setminus\alpha$. We must now consider the case that H is not the unique Dyck path in $(\mu\setminus\alpha)_h$. In which case, there exist Dyck path(s) $H' := R^h_1$ and possibly $H'' := R^h_2$ removed in the h step (an example is depicted in Figure 31). Our assumptions further imply that Q is adjacent to H' and H'' (if the latter exists) and that Q commutes with every other Dyck path A^j_k, R^i_l in (7.17) and (7.18). We set P^h to be the unique Dyck path in $\mathsf{reg}(\alpha)\setminus\alpha$ of height h. We note that P^h commutes with all Dyck paths R^i_k for $i < h$.

We first consider the case that $H' = R^h_1$ is the unique Dyck path in (7.17) which is adjacent to Q. We note that $H' \prec P^h$ is a non-commuting pair with $P^h\setminus H' = Q \sqcup Q'$ for some Q' of height h. We have that

$$\begin{aligned}(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(-Q) =&\mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+H')\mathbb{D}(-Q)(\textstyle\prod_{j\neq 1}\mathbb{D}(+R^h_j))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu-Q}\\ =&\pm\mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(-P^h)\mathbb{D}(+Q')(\textstyle\prod_{j\neq 1}\mathbb{D}(+R^h_j))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu-Q}\\ =&\pm\mathbb{L}(\alpha)\mathbb{D}(-P^h)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+Q')(\textstyle\prod_{j\neq 1}\mathbb{D}(+R^h_j))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu-Q}\end{aligned}$$

$$=0$$

where the second equality follows from applying relation (6.5) to the pair H', Q of adjacent Dyck paths; the final equality follows from Proposition 7.6 (as $P^h \in \mathsf{reg}(\alpha) \setminus \alpha$); all other equalities follow from the commuting relations (6.3).

Continuing with the case that $h \in \mathbb{Z}_{\leqslant 0}$, we now suppose that $H' = R^h_1$ and $H'' = R^h_2$ are the two Dyck paths in (7.17) which are adjacent to Q. We note that $H', H'' \prec P^h$ are non-commuting pairs with $P^h \setminus H' = Q \sqcup Q'$ and $P^h \setminus H'' = Q' \sqcup Q''$ for some Q', Q'' of height h. We have that

$$\begin{aligned}(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(-Q) =&\mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+H')\mathbb{D}(+H'')\mathbb{D}(-Q)(\textstyle\prod_{j\neq 1,2}\mathbb{D}(+R^h_j))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu-Q}\\ =&\pm\,\mathbb{L}(\alpha)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(-P^h)\mathbb{D}(+Q')\mathbb{D}(+Q'')(\textstyle\prod_{j\neq 1,2}\mathbb{D}(+R^h_j))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu-Q}\\ =&\pm\,\mathbb{L}(\alpha)\mathbb{D}(-P^h)\mathbb{R}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+Q')\mathbb{D}(+Q'')(\textstyle\prod_{j\neq 1,2}\mathbb{D}(+R^h_j))\mathbb{R}^{>h}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_{\mu-Q}\\ =&0\end{aligned}$$

where the second equality follows from two applications of the adjacent relation (6.5); the final equation follows from Proposition 7.6; all others follow from the commuting relations (6.3).

If $h > 0$ then $H = A^1_h$ for some A^1_h as in equation (7.18). Therefore we have that

$$\begin{aligned}(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(-Q) &= \mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{D}(+Q)\mathbb{D}(-Q)(\textstyle\prod_{j\neq 1}\mathbb{D}(+A^j_h))\mathbb{A}^{>h}_{\mu\setminus\alpha}\mathbb{1}_{\mu-Q}\\ &= \textstyle\sum_{P\in\mathrm{DRem}(\nu)} c_P\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}^{<h}_{\mu\setminus\alpha}\mathbb{L}(-P)(\prod_{j\neq 1}\mathbb{D}(+A^j_h))\mathbb{A}^{>h}_{\mu\setminus\alpha}\mathbb{1}_{\mu-Q}\end{aligned}$$

for $\nu = \alpha \cup (\mu \setminus \alpha)_{<h}$ and $c_P \in \Bbbk$ coefficients which can be calculated explicitly using the self-dual relation (6.2). Thus $(\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\mu\setminus\alpha}\mathbb{1}_\mu)\mathbb{D}(-Q) \in \mathcal{A}^{<\alpha}_{m,n}$, by induction on degree.

Finally, suppose we are in case (iv) and we assume that Q is not as in case (ii) or (iii), an example is depicted in Figure 32. Our assumptions imply that $Q \setminus H = H^1 \sqcup H^2$. Recall our assumption that $H \prec Q \not\succ H' \in \nu \setminus \alpha$ unless H' commutes with Q; this assumption further implies that H^1 and H^2 commute with all of the Dyck paths R^i_k, A^j_l in (7.17) and (7.18) for the pair $\nu \setminus \alpha$. Therefore

$$\mathbb{L}(\alpha)\mathbb{R}_{\mu\setminus\alpha}\mathbb{A}_{\nu\setminus\alpha}\mathbb{D}(-H^1)\mathbb{D}(-H^2)\mathbb{1}_\nu = \mathbb{D}(-H^1)\mathbb{D}(-H^2)\mathbb{L}(\alpha - H^1 - H^2)\mathbb{R}_{\nu\setminus\alpha}\mathbb{A}_{\nu\setminus\alpha} \in \mathcal{A}^{<\alpha}_{m,n}$$

by applying the non-commuting relation (6.4) to $\mathbb{D}(+H)\mathbb{D}(-Q)$ in the relevant place and then the commuting relations (6.3) and Lemmas 7.1 and 7.2 for all other products. ☐

7.3. **The isomorphism theorem.** Having established the spanning set of the algebra $\mathcal{A}_{m,n}$ in the previous section, we are now ready to prove the main result of the paper. It will be convenient to set

$$D(-Q) := \sum_{\substack{\lambda\in\mathscr{R}_{m,n}\\ Q\in\mathrm{DRem}(\lambda)}} 1_\lambda D^\lambda_{\lambda-Q}; \qquad D(+Q) := \sum_{\substack{\lambda\in\mathscr{R}_{m,n}\\ Q\in\mathrm{DAdd}(\lambda)}} 1_\lambda D^\lambda_{\lambda+Q}.$$

Theorem 7.11. *For $m \leqslant n$ the map $\varphi : \mathcal{A}_{m,n} \to H^m_n$ given by*

$$\varphi(\mathbb{1}_\mu) = 1_\mu \qquad \varphi(\mathbb{D}^\lambda_\mu) = D^\lambda_\mu \qquad \varphi(\mathbb{L}^\lambda_\lambda(-P)) = (-1)^{b(P)}1_\lambda D(-P)D(+P)$$

for $\lambda, \mu \in \mathcal{R}_{m,n}$ is a $\mathbb{Z}$-graded $\Bbbk$-algebra isomorphism.

Proof. We first verify that the map is a $\Bbbk$-algebra homomorphism. Clearly (6.1), (6.2), (6.3), (6.4), (6.5) hold in K^m_n as they are verbatim the same (just change the font!). We must check that the cubic relation (6.6) and loop relations (6.7) hold.

We first check the cubic relation. Let $P \in \mathrm{DAdd}_1(\mu)$ be such $\mathsf{last}(P)$ is maximal with respect to this property. Since $P \in \mathrm{DAdd}_1(\mu)$, we have that

$$\{Q \mid Q \in \mathrm{DRem}(\mu) \text{ and } P \prec Q\} = \emptyset.$$

If $\mathsf{last}(P) \leqslant m-2$, then we let $Q \in \mathrm{DRem}_0(\mu + P)$ be the unique element such that $P \prec Q$ and we set $Q \setminus P = Q^1 \sqcup Q^2$. Otherwise, $\mathsf{last}(P) = m$ and there does not exist any $P \prec Q$ and we set $Q^1 \in \mathrm{DRem}_0(\mu)$ to be the unique element that is adjacent to P. We have that

$$\begin{aligned}&\varphi(\mathbb{D}^{\mu+P}_\mu\mathbb{D}^\mu_{\mu+P}\mathbb{D}^{\mu+P}_\mu)\\ &= D(-P)D(+P)D(-P)1_\mu\end{aligned}$$

$$= \begin{cases} (-1)^{b(P)+1} D(-P)(D(-Q^1)D(+Q^1) + D(-Q^2)D(+Q^2))1_\mu & \text{if } \mathsf{last}(P) \leqslant m-2; \\ (-1)^{b(P)+1} D(-P)D(-Q^1)D(+Q^1)1_\mu & \text{if } \mathsf{last}(P) = m, \end{cases} \tag{7.23}$$

where the second equality follows from the self-dual relation (5.3). Note that $\mu \in \mathscr{R}_{m,n}$, $P \in \mathrm{DAdd}_1(\mu)$, and $\mathsf{last}(P)$ being maximal with respect to this property together imply that (i) $\mathsf{last}(P) \leqslant m-2$ if and only if $m = n$ (ii) $\mathsf{last}(P) = m$ if and only if $m < n$ as stated in the cubic relation.

We consider the first case of (7.23) where (by the above) $m = n$. For $m = n$ we observe that $\langle P \cup Q^1 \rangle_{\mu+P} = \langle P \cup Q^2 \rangle_{\mu+P}$ and we denote this Dyck path by Q. We have that

$$D(-P)(D(-Q^1)D(+Q^1) + D(-Q^2)D(+Q^2))1_\mu = \pm 2D(-Q)D(+Q)D(-P)1_\mu$$

by applying the adjacent relation (5.6) first to the pairs (P, Q^1) and (P, Q^2) followed by the commuting relation to Q^1, Q^2. We note that $\mathsf{last}(Q) = m-1$ and so relation (6.5) is preserved by φ for $m = n$.

We consider the second case of (7.23) where (by the above) $m < n$. For $m < n$ we have that $\langle P \cup Q^1 \rangle_{\mu+P}$ does not exist and so $D(-P)D(-Q^1)D(+Q^1)1_\mu = 0$ and so relation (6.5) is preserved by φ for $m < n$.

It remains to verify that the loop relations (6.7) are preserved by φ for $m = n$. Let $P \in \mathrm{DRem}_0(\mu)$ with $\mathsf{last}(P) = m-1$. We first check the loop nilpotency relation. We have that

$$\varphi(\mathbb{L}^\mu_\mu(-P)\mathbb{L}^\mu_\mu(-P)) = 1_\mu D(-P)D(+P)D(-P)D(+P) = 0$$

by applying the self-dual relation (5.3) to the innermost pair and observing that since $P \in \mathrm{Add}_0(\mu - P)$ and $\mu \in \mathcal{R}_{m,m}$ this implies that there are no $Q \in \mathrm{DRem}_{<0}(\mu - P)$ in order to provide non-zero terms on the righthand-side of the self-dual relation. Thus the loop nilpotency relation holds.

Finally, we must verify the loop-commutation relation. We continue to let $P \in \mathrm{DRem}_0(\mu)$ with $\mathsf{last}(P) = m-1$. We note that if $\lambda = \mu \pm Q$ where Q commutes with P then φ preserves the loop relation trivially. Thus it remains to consider the cases (i) $\lambda = \mu - Q$ with $Q \prec P$ a non-commuting pair and (ii) $\lambda = \mu + Q$ with P, Q an adjacent pair. In case (i) we note that $P \setminus Q = Q^1 \sqcup Q^2$ where $\mathsf{last}(Q^2) = m-1$. Therefore we have that

$$\begin{aligned} \varphi(\mathbb{D}^\lambda_\mu \mathbb{L}^\mu_\mu) &= \varphi(\mathbb{D}^\lambda_\mu \mathbb{L}^\mu_\mu(-P)) \\ &= (-1)^{b(P)} 1_\lambda D(+Q)D(-P)D(+P) \\ &= (-1)^{b(P)} 1_\lambda D(-Q^2)D(-Q^1)D(+P) \\ &= (-1)^{b(Q^2)} 1_\lambda D(-Q^2)D(+Q^2)D(+Q) \\ &= \varphi(\mathbb{L}^\lambda_\lambda(-Q^2)\mathbb{D}^\lambda_\mu) \\ &= \varphi(\mathbb{L}^\lambda_\lambda \mathbb{D}^\lambda_\mu) \end{aligned}$$

where the third equality follows from the non-commuting relation (5.5) applied to Q, P, the fourth equality follows from the adjacent relation (5.6) applied to Q^2, Q. Case (ii) follows from a similar argument.

We must now check that the homomorphism is surjective. This will immediately imply that the spanning set of Theorem 7.10 is actually a basis (as the spanning set of $\mathcal{A}_{m,n}$ has the same size as the basis of K^m_n) and we hence will deduce that φ is an isomorphism. We need only show that the algebra K^m_n is generated by the elements 1_μ, D^λ_μ and $D^\lambda_{\lambda-P} D^{\lambda-P}_\lambda$ for $\lambda, \mu \in \mathcal{R}_{m,n}$ and $P \in \mathrm{DRem}(\lambda)$. Thus it suffices to write the cellular basis in terms of these claimed generators which we have already done in equation (5.9). The result follows. □

8. The Ext-quiver and relations of $eK^m_n e$ and the Kleshchev–Martin conjecture.

We have shown that the algebra $H^m_n = eK^m_n e$ is a basic algebra generated in degrees 0, 1, and 2. Thus for $\lambda \subseteq \mu$, we have that

$$\dim_\Bbbk(\mathrm{Ext}^1_{H^m_n}(D(\lambda), D(\mu))) = \dim_\Bbbk(\mathrm{Hom}_{H^m_n}(\mathrm{rad}_1(P(\lambda)e), D(\mu)\langle k \rangle)) = 0 \tag{8.1}$$

unless $k = 1, 2$. By the cellular self-duality, we have that

$$\dim_\Bbbk(\mathrm{Ext}^1_{H^m_n}(D(\lambda), D(\mu)\langle k \rangle)) = \dim_\Bbbk(\mathrm{Ext}^1_{H^m_n}(D(\mu), D(\lambda)\langle k \rangle)) \tag{8.2}$$

and so we will be able to focus solely on the first and second grading layers of the projective H^m_n-module $P(\lambda)e$ for each $\lambda \in \mathscr{R}_{m,n}$.

Lemma 8.1. *For $\lambda \neq \mu \in \mathscr{R}_{m,n}$, we have that*

$$\dim_{\Bbbk}(\mathrm{Ext}^1_{H^m_n}(D(\lambda), D(\mu))) = \begin{cases} 1 & \text{if } \lambda = \mu \pm P \text{ for } P \text{ a Dyck path;} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\lambda = \mu \pm P$, then $\dim_{\Bbbk}(\mathrm{Hom}_{H^m_n}(\mathrm{rad}_1(P(\lambda)e), D(\mu)\langle 1\rangle)) = 1$ and by the parity on the grading, we have that $\dim_{\Bbbk}(\mathrm{Hom}_{H^m_n}(\mathrm{rad}_1(P(\lambda)e), D(\mu)\langle 2\rangle)) = 0$. Now suppose that $\mu \neq \lambda \pm P$. By equation (8.1), we can assume that $\lambda \subseteq \mu$ is such that (i) $\mu = \nu + P$ for $\nu = \lambda - Q$ and $\nu \in \mathscr{P}_{m,n}$ or (ii) $\mu = \nu + P$ for $\nu = \lambda + Q$ and $\nu \in \mathscr{R}_{m,n}$. Case (ii) is trivial. Case (i) is non-trivial only when $\nu \notin \mathscr{R}_{m,n}$ (equivalently $Q \in \mathrm{DRem}_0(\lambda)$) in which case our assumption that $\mu \in \mathscr{R}_{m,n}$ implies that $P = Q$ and so $\lambda = \mu$, as required. ☐

This is already enough to deduce the Ext-quiver for $m \neq n$.

Theorem 8.2. *Let $m \neq n$. The* Ext*-quiver of H^m_n has vertex set $\{\mathbb{1}_\lambda \mid \lambda \in \mathscr{R}_{m,n}\}$ and arrows $\mathbb{D}^\lambda_\mu : \lambda \to \mu$ and $\mathbb{D}^\mu_\lambda : \mu \to \lambda$ for every $\lambda = \mu - P$ with $P \in \mathrm{DRem}_{>0}(\mu)$. The symmetric algebra H^m_n is the path algebra of this quiver modulo relations (6.1), (6.2), (6.3), (6.4), (6.5) and (6.6).*

For the proofs of the last two theorems, we will first need to enumerate the removable Dyck paths of height zero from left to right as follows

$$P^1, P^2, \dots, P^r \in \mathrm{DRem}_0(\lambda).$$

For each pair $1 \leqslant j < k \leqslant r$ we let $Q_{j,k} \in \mathrm{DAdd}_1(\lambda)$ denote the addable Dyck path which is adjacent to P^j and P^k. Examples are depicted in Figure 33.

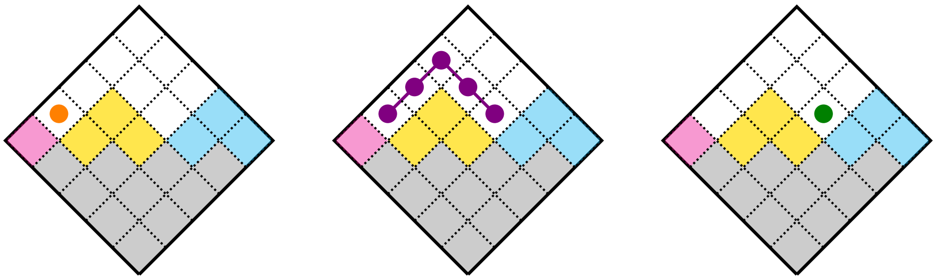


FIGURE 33. We picture the three removable Dyck paths $P^1, P^2, P^3 \in \mathrm{DRem}_0(\lambda)$. The corresponding $\binom{3}{2}$ addable Dyck paths $Q_{1,2}$, $Q_{1,3}$, and $Q_{1,2}$ in $\mathrm{DAdd}_1(\lambda)$ are also pictured.

With this notation in place, we can rewrite the self-dual relation for these Dyck paths as follows:

$$\mathbb{D}^\lambda_{\lambda+Q_{j,k}} \mathbb{D}^{\lambda+Q_{j,k}}_\lambda = \mathbb{L}^\lambda_\lambda(-P^j) + \mathbb{L}^\lambda_\lambda(-P^k). \tag{8.3}$$

We notice that the righthand-side consists of degree 2 terms which factor through an idempotent labelled by a non-regular partition. If we can rewrite the terms on the righthand-side using solely products of the form on the lefthand-side, then these loops will not appear in our Ext-quiver. This is best illustrated via examples.

Example 8.3. *Let $p > 2$ and $\lambda = (5, 4^2, 2^3)$ as in Figure 33. We have that*

$$\mathbb{D}^\lambda_{\lambda+Q_{1,2}} \mathbb{D}^{\lambda+Q_{1,2}}_\lambda = \mathbb{L}^\lambda_\lambda(-P^1) + \mathbb{L}^\lambda_\lambda(-P^2)$$
$$\mathbb{D}^\lambda_{\lambda+Q_{1,3}} \mathbb{D}^{\lambda+Q_{1,3}}_\lambda = \mathbb{L}^\lambda_\lambda(-P^1) + \mathbb{L}^\lambda_\lambda(-P^3)$$
$$\mathbb{D}^\lambda_{\lambda+Q_{2,3}} \mathbb{D}^{\lambda+Q_{2,3}}_\lambda = \mathbb{L}^\lambda_\lambda(-P^2) + \mathbb{L}^\lambda_\lambda(-P^3)$$

and inverting this we obtain

$$\mathbb{L}^\lambda_\lambda(-P^1) = \tfrac{1}{2}\left(\mathbb{D}^\lambda_{\lambda+Q_{1,2}} \mathbb{D}^{\lambda+Q_{1,2}}_\lambda + \mathbb{D}^\lambda_{\lambda+Q_{1,3}} \mathbb{D}^{\lambda+Q_{1,3}}_\lambda - \mathbb{D}^\lambda_{\lambda+Q_{2,3}} \mathbb{D}^{\lambda+Q_{2,3}}_\lambda\right)$$

$$\mathbb{L}_\lambda^\lambda(-P^2) = \tfrac{1}{2}\left(\mathbb{D}^\lambda_{\lambda+Q_{1,2}}\mathbb{D}_\lambda^{\lambda+Q_{1,2}} - \mathbb{D}^\lambda_{\lambda+Q_{1,3}}\mathbb{D}_\lambda^{\lambda+Q_{1,3}} + \mathbb{D}^\lambda_{\lambda+Q_{2,3}}\mathbb{D}_\lambda^{\lambda+Q_{2,3}}\right)$$
$$\mathbb{L}_\lambda^\lambda(-P^3) = \tfrac{1}{2}\left(-\mathbb{D}^\lambda_{\lambda+Q_{1,2}}\mathbb{D}_\lambda^{\lambda+Q_{1,2}} + \mathbb{D}^\lambda_{\lambda+Q_{1,3}}\mathbb{D}_\lambda^{\lambda+Q_{1,3}} + \mathbb{D}^\lambda_{\lambda+Q_{2,3}}\mathbb{D}_\lambda^{\lambda+Q_{2,3}}\right).$$

Therefore we deduce that none of the loops labelled by P^1, P^2, P^3 are required in the Ext*-quiver of $eK^m_n e$, as they can be written as linear combinations of other paths.*

In fact, this is the most complicated thing we have to deal with in constructing these Ext-quivers. Therefore the following lemma will be used repeatedly during the upcoming proofs.

Lemma 8.4. *Let $\Bbbk$ be a field of characteristic $p > 2$. Then*

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

and otherwise the matrix on the left is not invertible.

Theorem 8.5. *Let $p \neq 2$. The* Ext*-quiver of H^m_m has vertex set $\{\mathbb{1}_\lambda \mid \lambda \in \mathscr{R}_{m,m}\}$ and arrows $\mathbb{D}^\lambda_\mu : \lambda \to \mu$ and $\mathbb{D}^\mu_\lambda : \mu \to \lambda$ for every $\lambda = \mu - P$ with $P \in \mathrm{DRem}_{>0}(\mu)$ together with the loops $\mathbb{L}^\lambda_\lambda : \lambda \to \lambda$ for any $\lambda = (m^a, (m-a)^{m-a})$ for $1 \leqslant a \leqslant m$. The symmetric algebra H^m_m is given by the path algebra of this quiver modulo relations (6.1), (6.2), (6.3), (6.4), (6.5), (6.6), and (6.7).*

Proof. We first suppose that $\lambda \in \mathscr{R}_{m,m}$ is such that $\lambda \neq (m^a, (m-a)^{m-a})$ for some $1 \leqslant a \leqslant m$. There are two subcases to consider, either $|\mathrm{DAdd}_{>1}(\lambda)| > 0$ or $|\mathrm{DAdd}_1(\lambda)| > 1$. We first suppose that there exists $Q \in \mathrm{DAdd}_{>1}(\lambda)$, in which case there exists a unique $Q \prec P \in \mathrm{DRem}_0(\lambda)$ and

$$\mathbb{L}^\lambda_\lambda(-P) = \tfrac{1}{2}\mathbb{D}^\lambda_{\lambda+Q}\mathbb{D}^{\lambda+Q}_\lambda + \sum_{Q \prec R \prec P} \alpha_R D^\lambda_{\lambda-R}\mathbb{D}^{\lambda-R}_\lambda$$

where the $\alpha_R \in \Bbbk$ can be determined explicitly with the self-dual relation. If P is the unique element of $\mathrm{DRem}_0(\lambda)$ then we are done. Otherwise, let $P^1, P^2, \dots, P^r$ denote all removable Dyck paths of height zero with $P^j = P$ for some $1 \leqslant j \leqslant r$. For $i < j < k$, we have that

$$\mathbb{L}^\lambda_\lambda(-P^i) = \mathbb{D}^\lambda_{\lambda+Q_{i,j}}\mathbb{D}^{\lambda+Q_{i,j}}_\lambda - \mathbb{L}^\lambda_\lambda(-P^j) \qquad \mathbb{L}^\lambda_\lambda(-P^k) = \mathbb{D}^\lambda_{\lambda+Q_{j,k}}\mathbb{D}^{\lambda+Q_{j,k}}_\lambda - \mathbb{L}^\lambda_\lambda(-P^j)$$

and so we can rewrite every loop as a linear combination of other paths in the quiver (hence the loop at λ can be deleted from the quiver).

We now suppose that $|\mathrm{DAdd}_1(\lambda)| > 1$. Therefore $|\mathrm{DRem}_0(\lambda)| > 2$ and we let $P^1, P^2, \dots, P^r$ denote these removable Dyck paths of height zero. For each $1 \leqslant i \leqslant j \leqslant l \leqslant r$ we have that have that

$$\begin{aligned}
\mathbb{D}^\lambda_{\lambda+Q_{i,j}}\mathbb{D}^{\lambda+Q_{i,j}}_\lambda &= \mathbb{L}^\lambda_\lambda(-P^i) + \mathbb{L}^\lambda_\lambda(-P^j)\\
\mathbb{D}^\lambda_{\lambda+Q_{i,k}}\mathbb{D}^{\lambda+Q_{i,k}}_\lambda &= \mathbb{L}^\lambda_\lambda(-P^i) + \mathbb{L}^\lambda_\lambda(-P^k)\\
\mathbb{D}^\lambda_{\lambda+Q_{j,k}}\mathbb{D}^{\lambda+Q_{j,k}}_\lambda &= \mathbb{L}^\lambda_\lambda(-P^j) + \mathbb{L}^\lambda_\lambda(-P^k).
\end{aligned}$$

We can invert this system of equations using Lemma 8.4 and hence rewrite this loop as a linear combination of other paths in the quiver (hence this loop can be deleted from the quiver).

For the remainder of the proof, we let $Q_{m,m}$ denote the quiver with vertex set $\{\mathbb{1}_\lambda \mid \lambda \in \mathscr{P}_{m,m}\}$ and arrows

- $\mathbb{D}^\lambda_\mu : \lambda \to \mu$ and $\mathbb{D}^\mu_\lambda : \mu \to \lambda$ for every $\lambda = \mu - P$ with $P \in \mathrm{DRem}(\mu)$;
- for $m = n$ we have additional "loops" of degree 2, $\mathbb{L}^\lambda_\lambda : \lambda \to \lambda$ for every $\lambda \in \mathscr{R}_{m,m}$.

The algebra K^m_m is the quotient of the path algebra $\Bbbk Q_{m,m}$ by the relations in Definitions 6.3 and 6.4. We will detail bases of $\Bbbk Q_{m,m}$-modules and determine the linear dependencies arising from the relations in Definitions 6.3 and 6.4 and hence determine the generators needed for the Ext-quiver of H^m_n.

We first suppose that $\lambda = (m^m)$. The degree 2 subspace of $\mathbb{1}_{(m^m)}\Bbbk Q_{m,m}\mathbb{1}_{(m^m)}$ is m-dimensional with basis

$$\{\mathbb{D}^{(m^m)}_{(m^m)-P^j}\mathbb{D}^{(m^m)-P^j}_{(m^m)} \mid P^j \in \mathrm{DRem}_j(m^m) \text{ for } 0 \leqslant j \leqslant m\} \cup \{\mathbb{L}^\lambda_\lambda\}.$$

When we project onto the quotient $\mathbb{1}_{(m^m)}H^m_n\mathbb{1}_{(m^m)}$ the unique relation we apply is

$$\mathbb{D}^{(m^m)}_{(m^m)-P^j}\mathbb{D}^{(m^m)-P^j}_{(m^m)} = \mathbb{L}^{(m^m)}_{(m^m)}$$

where the lefthand-side cannot be factorised as a product of elements in H^m_n and so we cannot delete the loop from the regular quiver.

We now suppose that $\lambda = (m^a, (m-a)^{m-a})$ for $1 \leqslant a \leqslant m$, then we set $P^1, P^2 \in \mathrm{DRem}_0(\lambda)$. We suppose that the sets $\{Q \mid Q \prec P^1\}$ and $\{Q \mid Q \prec P^2\}$ have size $p_1, p_2 \in \mathbb{Z}_{>0}$ respectively. In which case the degree 2 subspace of $\mathbb{1}_{(m^m)}\Bbbk Q_{m,m}\mathbb{1}_{(m^m)}$ is (p_1+p_2+2)-dimensional with basis

$$\{\mathbb{D}^\lambda_{\lambda-Q}\mathbb{D}^{\lambda-Q}_\lambda \mid Q \preceq P^1 \text{ or } Q \preceq P^2\} \cup \{\mathbb{D}^\lambda_{\lambda+Q_{1,2}}\mathbb{D}^{\lambda+Q_{1,2}}_\lambda\} \cup \{\mathbb{L}^\lambda_\lambda\}.$$

When we project onto the quotient $\mathbb{1}_{(m^m)}H^m_n\mathbb{1}_{(m^m)}$ the two relations we apply are

$$\mathbb{D}^\lambda_{\lambda+Q_{1,2}}\mathbb{D}^{\lambda+Q_{1,2}}_\lambda = \mathbb{D}^\lambda_{\lambda-P^1}\mathbb{D}^{\lambda-P^1}_\lambda + \mathbb{D}^\lambda_{\lambda-P^2}\mathbb{D}^{\lambda-P^2}_\lambda, \qquad \mathbb{L}^\lambda_\lambda = \mathbb{D}^\lambda_{\lambda-P^2}\mathbb{D}^{\lambda-P^2}_\lambda,$$

thus projecting onto a 2-dimensional space. We have that $\lambda - P^1, \lambda - P^2 \notin \mathcal{R}_{m,m}$ and so we cannot delete the loop generator in this case, as above. □

Theorem 8.6. *Let $p = 2$. The* Ext*-quiver of H^m_m has vertex set $\{\mathbb{1}_\lambda \mid \lambda \in \mathscr{R}_{m,m}\}$ and arrows $\mathbb{D}^\lambda_\mu : \lambda \to \mu$ and $\mathbb{D}^\mu_\lambda : \mu \to \lambda$ for every $\lambda = \mu - P$ with $P \in \mathrm{DRem}_{>0}(\mu)$ together with all possible loops $\mathbb{L}^\lambda_\lambda : \lambda \to \lambda$ for every $\lambda \in \mathscr{R}_{m,m}$. The symmetric algebra H^m_m is given by the path algebra of this quiver modulo relations* (6.1), (6.2), (6.3), (6.4), (6.5), (6.6), *and* (6.7).

Proof. The case $|\mathrm{DRem}_0(\lambda)| = 1, 2$ are identical to the proof for the $p \neq 2$ case. For $|\mathrm{DRem}_0(\lambda)| > 2$ the argument for $p \neq 2$ cannot be passed through because the matrix of Lemma 8.4 is no longer invertible. In fact, the linear dependencies listed between paths in the proof of Theorem 8.5 are exhaustive (by inspection of the relations in the presentation) and it is impossible to rewrite the loops as linear combinations of paths, and so we cannot delete any of these loops from the quiver. □

We now wish to discuss the existence of self-extensions of simple modules (in other words, the existence of loops in the Ext-quiver) in the context of faithfulness of quasi-hereditary covers. This necessitates us recalling the main result of [BDV+].

Theorem 8.7. *The extended arc algebras K^m_n are $(|m-n|-1)$-faithful covers of the Khovanov arc algebras H^m_n for $m, n \in \mathbb{N}$ such that $n \neq m$. In other words,*

$$\mathrm{Ext}^i_{K^m_n}(M, N) \cong \mathrm{Ext}^i_{H^m_n}(eM, eN)$$

for M, N a pair of standard-filtered modules and $0 \leqslant i \leqslant |m-n|-1$. Moreover, the H^m_n-modules eM and eN both have filtrations by Specht modules, in the sense of Definition 5.10.

Corollary 8.8. *The extended arc algebras K^m_n are 0-faithful covers of the Khovanov arc algebras H^m_n if and only if $m \neq n$.*

Proof. One direction is immediate from Theorem 8.7. To see that the $m = n$ case cannot have a 0-faithful cover, we observe that $S_{m,m}(m^m) \cong D_{m,m}(m^m) \cong S_{m,m}(\varnothing)$. □

Putting together the results of this section and those from our previous work, we obtain the following.

Theorem 8.9. *The* Ext*-quiver of the symmetric algebra H^m_n is loop-free if and only if K^m_n is an i-faithful quasi-hereditary cover for some $i \geqslant 0$.*

Finally, we recall that this behaviour was predicted in another context and a slightly different language. Firstly, the conjecture is somewhat folkloreish and so the best citation we have for its formulation is the very recent work of Geranios–Morotti–Kleshchev. In their statement the condition for the existence of loops is that $p = 2$, but this can reformulated in terms of faithfulness of quasi-hereditary covers using [HN04, Corollary 3.9.1] or [Don07] to obtain the following.

Conjecture 8.10 (The Kleshchev–Martin conjecture [GKM22, Introduction])**.** *The* Ext*-quiver of the group algebra of the symmetric group $\Bbbk S_r$ is loop-free if and only if the classical Schur algebra $S_\Bbbk(r, r)$ is an i-faithful quasi-hereditary cover for some $i \geqslant 0$.*

Remark 8.11. *For some historical context, we remark that the first results on this conjecture were implicit in* [Erd94] *and later made explicit and extended in* [KS99]*. While the conjecture is well-known and often referenced in the literature, progress in this direction has been incredibly limited. Arguably the first major step towards resolving this conjecture was recently taken in* [GKM22]*, where the authors "generically" verify this conjecture by proving that it holds for all RoCK blocks (which constitute "most blocks").*

The original Kleshchev–Martin context concerns truncation from a quasi-hereditary algebra to a symmetric algebra using a highest weight idempotent 1_ω for $\omega = (1^r, 0, 0, \dots)$. Our truncation is from a quasi-hereditary algebra to a symmetric algebra by a highest weight idempotent $\sum_{\rho \subseteq \lambda} e_\lambda$ where $\rho = (m, m-1, \dots, 2, 1) \in \mathscr{P}_{m,n}$. Such ("co-saturated") truncations appear in many different contexts in Lie theory, for example they were the subject of conjectures of Khovanov [Kho04] proven in [MS08]. All of the quasi-hereditary algebras discussed above are Morita equivalent to (singular) anti-spherical Hecke categories. We propose the following vast generalisation of the above self-extension conjecture.

Let (W, P) denote a parabolic Coxeter system with generators $s_i \in S_W$. Let $\mathcal{H}_{(W,P)}$ denote the category algebra of the Elias–Libedinsky–Williamson anti-spherical Hecke category (see [BHN22] for the definition of this category algebra and [LW22, EW16] for the original definition of the anti-spherical Hecke category). Given a reduced word $\underline{w} = s_{i_1} s_{i_2} \dots s_{i_k}$ for some $w \in {}^P W$, we have an idempotent $1_{\underline{w}} = 1_{s_{i_1}} \otimes 1_{s_{i_2}} \otimes \cdots \otimes 1_{s_{i_k}} \in \mathcal{H}_{(W,P)}$.

Conjecture 8.12 (Generalised Kleshchev–Martin conjecture)**.** *Let $x, w \in {}^P W$ and consider the subalgebra*

$$(\textstyle\sum_{x<y\leqslant w} 1_{\underline{y}})\mathcal{H}_{(W,P)}(\sum_{x<y\leqslant w} 1_{\underline{y}}) \subseteq (\sum_{y\leqslant w} 1_{\underline{y}})\mathcal{H}_{(W,P)}(\sum_{y\leqslant w} 1_{\underline{y}})$$

where the latter is a quasi-hereditary cover of the former, by construction. If the subalgebra is a symmetric algebra, then its Ext*-quiver is loop-free if and only if the quasi-hereditary cover is i-faithful, for some $i \geqslant 0$.*

In order to truly generalise the classical Kleshchev–Martin conjecture, we should state the above conjecture for *singular* Hecke categories (and indeed we do believe this holds). However these objects are in relative infancy and even defining reduced words in this context is a difficult problem and the subject of very recent work of [Wil08, EK23].

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