A STUDY OF SINGLE LATIN-SQUARE DESIGNS IN CHANGEOVER EXPERIMENTS

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ABSTRACT

When it is desired to measure the residual effects and the number of treatments is odd, it is recommended that a combination of at least two Latin-square designs should be used; an extra refinement is to introduce an extra period in order to arrive at a design balanced for residual effects. The final design, therefore, becomes cumbersome and may prove to be a great hindrance in the practical execution of the experiment.

Here we consider single Latin-squares with the object of determining the most efficient of these designs, which when applied on their own produce optimum results.

A number of Latin-square designs having different patterns have been studied with the assumption that the residual effects do not persist beyond the following period. A method for the estimation of parameter effects, i.e. Period, Subject, Treatment direct and Treatment residual effects, including the cases when residual effects persist beyond the following period, has been developed. The efficiency of these designs based on minimum average variance of the differences between two estimates and minimum variance of linear components of treatment as well as residual effects has been investigated.

Further study on the optimality of the designs has been made by applying the D-optimality test, which states that the most efficient design has the maximum determinant of $\underline{X}'\underline{X}$, where $\underline{X}$ is the matrix of independent variables at which a response is measured from the model, or equivalently has its generalised variance determinant $\left| (\underline{X}'\underline{X})^{-1}\sigma^2 \right|$ minimum.

The pattern of the most efficient Latin-square design has been established along with a method to generate such designs.

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Chapter 1

Litrature survey.

1.1 Introduction.

Often we are concerned with the situation that treatments applied in sequence to a number of subjects produce residual effects persisting at least for one following period; as in the cases of feeding effects on cows yielding milk, medicine effects on patients, chemical effects in dyeing fabrics, fertilizers on land, intelligence tests on students, etc, etc.

If it is not desired to measure the residual effects it may be possible to eliminate these effects by widely separating the periods in which the treatments are applied to the subjects. For example it may be possible to eliminate the residual or carry-over effects of certain drugs by leaving an interval between the periods in which the drugs are administered. In the case when the drugs under investigation are given in close succession one might expect carry-over effects from the previous drugs. Similarly in psychological research when subjects are given a number of treatments successively one may expect the presence of residual or carry-over effects. Early treatments may produce a feeling of anxiety, fear or encouragement in a subject and such feelings can influence the subjects' performance or behaviour during subsequent treatments. In such cases it may not be possible to eliminate residual effects completely even though there is a long interval between successive treatments.

Therefore we seek such designs which could be highly efficient in the estimation of treatment as well as residual effects without becomming cumbersome or complicated from the point of view of both practicality and analysis.

In 1941 Cochran described certain Latin-square designs and their analysis for the estimation of residual effects. These designs require that each treatment be preceded by every other treatment the same number of times. For example, for three treatments we require two Latin-squares, to fulfil this condition. These two designs are to be conducted side by side as follows:-

	Subjects					
	1	2	3	1	2	3
Periods	2	3	1	3	1	2
	3	1	2	2	3	1

This type of balance can be obtained with a minimum number of two Latin-squares when the number of treatments is odd and with one Latin-square in the case of an even number of treatments as shown by Williams (1949).

Williams (1949) has also studied special Latin-square designs with the aim of estimating residual effects lasting for more than one period, particularly to carry over into two periods.

Patterson (1952) has published series of incomplete Latin-squares which are balanced for one period residual effects.

Lucas (1956,57) provided designs, balanced for one period residual effects, simply by repeating the last row of the Latin-square design.

Lucas also points out that the advantages of the extra-period designs decrease as the number of treatments is increased, compared with the regular Latin-square designs. He further mentions that the extra-period designs are less efficient than the Latin-squares, if residual effects can be assumed not to exist. He thereby suggests that the Latin-squares might be preferred to the extra-period designs, even when studying few treatments, when it can be assumed that the residual effects are negligible.

Berenblut (1964) presented a family of designs in which direct and residual effects are orthogonal. These designs, for testing n treatments, require 2n periods and n^2 subjects.

We notice that the efforts so far made for constructing balanced designs, for carry-over effects, in which each treatment follows every other treatment the same number of times, have been directed towards either combining two or more regular designs as suggested by Cochran, Williams and Berenblut or by truncating and then combining the designs as given by Patterson or by introducing an extra period as by Lucas. These conditions often become a great hindrance in the practical execution of such designs.

Therefore, the importance of the study of single Latin-square designs, which can provide equally good estimates of the parameter effects when residual effects are assumed to exist, has led to the search for the most efficient pattern of designs.

Chapter 2

Parameter Estimates in the presence of First Residual Effects and the Variance of the Estimates.

2.1 Aim.

The designs to be presented are free from the restrictions mentioned in the introduction such as an extra period, a combination of two or more designs, etc. The designs chosen for the study are Latin-square designs.

The analysis has been performed under the assumption that residual effects exist from one treatment to the next and persist only for one period.

The effect of the extra assumption on the analysis of the designs has been investigated.

2.2 The Design.

The smallest design that can be chosen for the study is a 4x4 Latin-square design. A single 3x3 Latin-square is not suitable for estimating the residual effects as it does not provide any degree of freedom for the error term in the analysis of variance and hence cannot provide a test for the efficiency of the design for estimating the parameter effects. Similarly a single 2x2 Latin-square is totally unsuitable for this type of study.

2.2.1 4x4 Latin-square.

		Subjects			
		S_1	S_2	S_3	S_4
Periods	P_1	1	2	3	4
	P_2	2	3	4	1
	P_3	3	4	1	2
	P_4	4	1	2	3

The assumptions about this design are as follows:-

(a) Treatments are assigned to digits 1,2,3 and 4.

(b) Subjects are placed along columns. Each subject is thus regarded as a block or column.

(c) The result of one test contains a residual part of the effect of the previous test.

(d) The effects of the treatments do not persist beyond one step.

2.3 The Model.

We start with the linear model useful for the analysis of designs with residual effects limited to the period immediately following the application of a treatment.

This model is as follows:-

Observation = General mean (μ)
+ Period effect (p)
+ Subject effect (s)
+ Treatment direct effect (t)
+ Treatment residual effect (r)
+ Error (e).

Or

$$Y_{ij(kl)} = \mu + p_i + s_j + t_k + r_l + e_{ij(kl)}$$

Where

μ = General mean

p = Effect due to occasions or periods which occur along the rows.

s = Effect due to subjects which are placed along the columns.

t = Treatment direct effect.

r = Treatment residual effect-----effect of the previous treatment.

e = Experimental error.

Each cell of the design is composed of the general mean, the period effect, the subject effect, the treatment direct effect, the treatment residual effect (apart from cells in the first row) from the preceding treatment and a random variable representing experimental error.

The effect of treatment k is defined as the difference of the mean of the group of observations having treatment k and the general mean of the population. For example, the effect of treatment k, $t_k = \mu_k - \mu$ where t_k stands for the effect of any single treatment k.

Since the general mean μ is also the mean of all the treatment group means it follows that the sum of all the treatment effects must be zero.

$$\sum_{k=1}^{n} t_k = \sum_{k=1}^{n} (\mu_k - \mu)$$

$$= \sum \mu_k - n \mu$$

$$= n \mu - n \mu = 0$$

Similarly we define other parameter effects.

$$\sum t_k = \sum s_j = \sum p_i = \sum r_l = 0$$

i, j, k and l = 1, 2, 3, 4.

The error term is distributed independently and normally with mean zero and variance σ^2.

2.4 Latin-square Designs.

The following four 4x4 Latin-square designs (Cochran and Cox page 145) have been chosen for detailed study with respect to residual or carry-over effects. The first design has a cyclic pattern and thus each treatment has been preceded by the same treatment on all occasions. The second is the Williams' design where each treatment has been preceded by every other treatment once only. The third and fourth designs have the pattern where each treatment has been preceded twice by one treatment and once by another.

2.4.1 Columns

	1	2	3	4
	2	3	4	1
Rows	3	4	1	2
	4	1	2	3

2.4.2	1	2	3	4
	2	4	1	3
Rows	3	1	4	2
	4	3	2	1

2.4.3	1	2	3	4
	2	1	4	3
Rows	3	4	2	1
	4	3	1	2

2.4.4	1	2	3	4
	2	1	4	3
Rows	3	4	1	2
	4	3	2	1

2.5 Derivation of the Estimates.

We find the estimates of the parameters by minimising

$$\sum \left\{ Y_{ij(kl)} - \hat{\mu} - \hat{p}_i - \hat{s}_j - \hat{t}_k - \hat{r}_l \right\}^2$$

subject to

$$\Sigma\hat{p}_i = 0 \text{ , } \Sigma\hat{s}_j = 0 \text{ , } \Sigma\hat{t}_k = 0 \text{ , } \Sigma\hat{r}_l = 0$$

The normal equation for any unknown is obtained by equating the observed total to the expected total over all units containing the constant.

Therefore the normal equations for estimating the period, subject, treatment and residual effects of design 2.4.1 are as follows:-

2.5.1 Period effects------------placed along rows.

$$P_1 = 4\hat{\mu} + 4\hat{p}_1$$
$$P_2 = 4\hat{\mu} + 4\hat{p}_2$$
$$P_3 = 4\hat{\mu} + 4\hat{p}_3$$
$$P_4 = 4\hat{\mu} + 4\hat{p}_4$$

2.5.2 Subject effects----------measured along columns.

$$S_1 = 4\hat{\mu} + 4\hat{s}_1 - \hat{r}_4$$
$$S_2 = 4\hat{\mu} + 4\hat{s}_2 - \hat{r}_1$$
$$S_3 = 4\hat{\mu} + 4\hat{s}_3 - \hat{r}_2$$
$$S_4 = 4\hat{\mu} + 4\hat{s}_4 - \hat{r}_3$$

2.5.3 Treatment direct effects.

$$T_1 = 4\hat{\mu} + 4\hat{t}_1 + 3\hat{r}_4$$
$$T_2 = 4\hat{\mu} + 4\hat{t}_2 + 3\hat{r}_1$$
$$T_3 = 4\hat{\mu} + 4\hat{t}_3 + 3\hat{r}_2$$
$$T_4 = 4\hat{\mu} + 4\hat{t}_4 + 3\hat{r}_3$$

2.5.4 Treatment residual effects.

$$R_1 = 3\hat{\mu} + 3\hat{r}_1 + 3\hat{t}_2 - \hat{s}_2 - \hat{p}_1$$
$$R_2 = 3\hat{\mu} + 3\hat{r}_2 + 3\hat{t}_3 - \hat{s}_3 - \hat{p}_1$$
$$R_3 = 3\hat{\mu} + 3\hat{r}_3 + 3\hat{t}_4 - \hat{s}_4 - \hat{p}_1$$
$$R_4 = 3\hat{\mu} + 3\hat{r}_4 + 3\hat{t}_1 - \hat{s}_1 - \hat{p}_1$$

where

P_i denotes the row total representing the sum of the observations on period i.

S_j denotes the column total representing the sum of the observations on subject j.

T_k denotes the total of the observations on treatment k.

R_l denotes the total of the observations immediately following the treatment l or total of the observations involving the residual effect of treatment l.

2.6 Parameter Estimates.

The parameter estimates have been obtained by solving simultaneously the normal equations given under 2.5.1 to 2.5.4.

2.6.1 Period effect estimates.

$$\hat{p}_1 = \frac{P_1}{4} - \frac{G}{16}$$

$$\hat{p}_2 = \frac{P_2}{4} - \frac{G}{16}$$

$$\hat{p}_3 = \frac{P_3}{4} - \frac{G}{16}$$

$$\hat{p}_4 = \frac{P_4}{4} - \frac{G}{16}$$

2.6.2 Subject effect estimates.

$$\hat{s}_1 = \frac{3}{8} S_1 + \frac{1}{2} R_4 - \frac{3}{8} T_1 + \frac{1}{8} P_1 - \frac{1}{8} G$$

$$\hat{s}_2 = \frac{3}{8} S_2 + \frac{1}{2} R_1 - \frac{3}{8} T_2 + \frac{1}{8} P_1 - \frac{1}{8} G$$

$$\hat{s}_3 = \frac{3}{8} S_3 + \frac{1}{2} R_2 - \frac{3}{8} T_3 + \frac{1}{8} P_1 - \frac{1}{8} G$$

$$\hat{s}_4 = \frac{3}{8} S_4 + \frac{1}{2} R_3 - \frac{3}{8} T_4 + \frac{1}{8} P_1 - \frac{1}{8} G$$

2.6.3 Treatment direct effect estimates.

$$\hat{t}_1 = \frac{11}{8} T_1 - \frac{3}{2} R_4 - \frac{3}{8} P_1 - \frac{3}{8} S_1 + \frac{1}{8} G$$

$$\hat{t}_2 = \frac{11}{8} T_2 - \frac{3}{2} R_1 - \frac{3}{8} P_1 - \frac{3}{8} S_2 + \frac{1}{8} G$$

$$\hat{t}_3 = \frac{11}{8} T_3 - \frac{3}{2} R_2 - \frac{3}{8} P_1 - \frac{3}{8} S_3 + \frac{1}{8} G$$

$$\hat{t}_4 = \frac{11}{8} T_4 - \frac{3}{2} R_3 - \frac{3}{8} P_1 - \frac{3}{8} S_4 + \frac{1}{8} G$$

2.6.4 Treatment residual effect estimates.

$$\hat{r}_1 = 2 R_1 - \frac{3}{2} T_2 + \frac{1}{2} P_1 + \frac{1}{2} S_2 - \frac{1}{4} G$$

$$\hat{r}_2 = 2 R_2 - \frac{3}{2} T_3 + \frac{1}{2} P_1 + \frac{1}{2} S_3 - \frac{1}{4} G$$

$$\hat{r}_3 = 2 R_3 - \frac{3}{2} T_4 + \frac{1}{2} P_1 + \frac{1}{2} S_4 - \frac{1}{4} G$$

$$\hat{r}_4 = 2 R_4 - \frac{3}{2} T_1 + \frac{1}{2} P_1 + \frac{1}{2} S_1 - \frac{1}{4} G$$

$\hat{\mu} = \frac{G}{16}$ and G = Grand total.

2.7 Matrix Approach.

Considering 2.4.1 for illustration:

		Subjects			
		S_1	S_2	S_3	S_4
Periods	P_1	1	2	3	4
	P_2	2	3	4	1
	P_3	3	4	1	2
	P_4	4	1	2	3

For the model:

$$Y_{ij(kl)} = \mu + p_i + s_j + t_k + r_l + e_{ij(kl)}$$

let

$$\underline{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$ period parameters subject to $\underline{h}' \underline{p} = 0$

$$\underline{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$$ subject parameters subject to $\underline{h}' \underline{s} = 0$

$$\underline{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$$ treatment parameters subject to $\underline{h}' \underline{t} = 0$

$$\underline{r} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$ residual parameters subject to $\underline{h}' \underline{r} = 0$

where $\underline{h}$ is an nx1 vector with all elements = 1

The $e_{ij(kl)}$ are mutually independent with $E(e_{ij(kl)}) = 0$ and $E(e^2_{ij(kl)}) = \sigma^2$

Let $\underline{\ell}$ be a permutation matrix (n x n) such that

$$\underline{\ell} \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

where a_1, a_2, ... a_n are the index numbers of the treatments in the last row of the Latin-square. For example for this design the above expression can be given as:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

Note that $\underline{\ell}'\underline{\ell} = \underline{\ell}\underline{\ell}' = \underline{I}_n$

Let $\bar{Y}$ be the general mean of all n^2 observations and

$$\bar{Y} = \frac{1}{n^2}\left[\underline{h}' \ \underline{h}' \ \underline{h}' \ \cdots\cdots \ \underline{h}'\right] \underline{Y}$$

where

$$\underline{Y} = \begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \\ \vdots \\ \underline{Y}_n \end{bmatrix} \qquad \underline{Y}_1 = \begin{bmatrix} Y_{11} \\ Y_{21} \\ \vdots \\ Y_{n1} \end{bmatrix}$$

and similarly $\underline{Y}_2$,....... $\underline{Y}_n$ which are column vectors of the observations.

Let $\underline{P}$ be vector of totals for periods (rows) so that

$$\underline{P} = \left[\underline{I} \ \underline{I} \ \underline{I} \ \cdots\cdots\cdots \ \underline{I}\right] \underline{Y}$$

Let $\underline{S}$ be vector of totals for subjects (columns) so that

$$\underline{S} = \left[\underline{h}\,\underline{e}_1' \;\; \underline{h}\,\underline{e}_2' \cdots\cdots\cdots \underline{h}\,\underline{e}_n' \right]' \underline{Y}$$

$\underline{e}_i$ being a column vector with ith element equal to 1 and all other elements are zero.

Let $\underline{T}$ be vector of totals for treatments (entries) so that

$$\underline{T} = \left[\underline{T}_1 \;\; \underline{T}_2 \cdots\cdots\cdots\cdots \underline{T}_n \right]' \underline{Y}$$

where $\underline{T}_1$, $\underline{T}_2$, ... $\underline{T}_n$ are row vectors having element 1 in the positions of their corresponding treatments and other elements being zero. For example for design (2.4.1) under consideration

$$\underline{T}_1 = \left[1\ 0\ 0\ 0\quad 0\ 0\ 0\ 1\quad 0\ 0\ 1\ 0\quad 0\ 1\ 0\ 0 \right]$$

$$\underline{T}_2 = \left[0\ 1\ 0\ 0\quad 1\ 0\ 0\ 0\quad 0\ 0\ 0\ 1\quad 0\ 0\ 1\ 0 \right]$$

and so on.

Let $\underline{R}$ be vector of totals for observations having residual effects so that

$$\underline{R} = \left[\underline{R}_1 \;\; \underline{R}_2 \cdots\cdots\cdots\cdots \underline{R}_n \right]' \underline{Y}$$

where $\underline{R}_1$, $\underline{R}_2$, ... $\underline{R}_n$ are row vectors having element 1 in the positions of the treatments having residual effects of their corresponding treatments while other elements are zero. For 2.4.1 we have

$$\underline{R}_1 = \left[0\ 1\ 0\ 0\quad 0\ 0\ 0\ 0\quad 0\ 0\ 0\ 1\quad 0\ 0\ 1\ 0 \right]$$

$$\underline{R}_2 = \left[0\ 0\ 1\ 0\quad 0\ 1\ 0\ 0\quad 0\ 0\ 0\ 0\quad 0\ 0\ 0\ 1 \right]$$

and similarly other vectors.

$\underline{P}$, $\underline{S}$, and $\underline{T}$ are totals of n observations for each element while $\underline{R}$ is a total of (n-1) observations for each element since there is no residual effect present at period 1.

2.7.1 Least Square Equations.

The least-square equations may be obtained in different ways. We require the quantities which minimise

$$D = (\underline{Y} - \underline{M}\,\underline{\theta})' (\underline{Y} - \underline{M}\,\underline{\theta})$$

There are 4 constraints and 4n + 1 parameters for this design. Here $\underline{M}$ is the coefficient matrix and $\underline{\theta}$ is the parameter vector.

$$\therefore \quad D = \sum_i \sum_j (Y_{ij(kl)} - \eta_{ij(kl)})^2$$

where $Y_{ij(kl)} = \mu + p_i + s_j + t_k + r_l + e_{ij(kl)}$

and $\eta_{ij(kl)} = E(Y_{ij(kl)})$

To minimise D we equate its partial derivatives with respect to p_i, s_j, t_k and r_l to zero.

$\frac{\partial D}{\partial p_i} = 0$ gives $\sum_j (Y_{ij} - \eta_{ij}) = 0$ where the summation is over all elements in the ith row and so on.

We obtain:

2.7.2 $$T_{..(..)} = n^2 \hat{\mu} + \sum_1^n \hat{p}_i + \sum_1^n \hat{s}_j + \sum_1^n \hat{t}_k + \sum_l \hat{r}_l$$

$$= n^2 \hat{\mu} - \hat{r}_{a1} - \hat{r}_{a2} \cdots\cdots - \hat{r}_{an}$$

$$= n^2 \hat{\mu}$$

where the index numbers of treatments in the last row are a1, a2, ... an.

The constraints $\sum \hat{p}_i = \sum \hat{s}_j = \sum \hat{t}_k = 0$

but $\sum_l \hat{r}_l = \sum_1^n \hat{r}_l - \hat{r}_{a1} - \hat{r}_{a2} \cdots\cdots \hat{r}_{an} = 0$

because of the treatments in the last row.

2.7.3 $T_{i.(..)} = n(\hat{\mu} + \hat{p}_i)$

2.7.4 $T_{.j(..)} = n(\hat{\mu} + \hat{s}_j) - \hat{r}_{aj}$ since no carry over effects from the treatments applied at the last period are available.

2.7.5 $T_{..(k.)} = n(\hat{\mu} + \hat{t}_k) + \hat{r}_{k1} + \hat{r}_{k2} + \cdots\cdots + \hat{r}_{kn}$

where k1 is the treatment applied before treatment k in column 1.

k2 is the treatment applied before treatment k in column 2.

and so on.

2.7.6 $T_{..(.l)} = (n-1)(\hat{\mu} + \hat{r}_l) - \hat{p}_1 - \hat{s}_{bl} + \hat{t}_{k1} + \hat{t}_{k2} \cdots + \hat{t}_{kn}$

S_{bl} = Subjects receiving lth treatment in the last row.

Note: (a) Summation only over rows 2.....n hence $\sum p_i = -p_1$

(b) When a treatment is given at the last period of the experiment then there is no carry over effect available.

(c) k_1 is the treatment following treatment k in column 1

k_2 " " " " k " " 2

and so on.

Now we can express in matrix form all the expressions from 2.7.3 to 2.7.6 and proceed with the solution of normal equations.

Let $\underline{\beta}$ be an (n x n) matrix indicatig which treatments precede and follow each other. For example in design 2.4.1

$\underline{\beta}$ = (Following Treatments × Previous Treatments)

	1	2	3	4
1	0	0	0	3
2	3	0	0	0
3	0	3	0	0
4	0	0	3	0

and in design 2.4.2

$\underline{\beta}$ = (following Treatments × Previous Treatments)

	1	2	3	4
1	0	1	1	1
2	1	0	1	1
3	1	1	0	1
4	1	1	1	0

$\underline{\beta}$ changes with the changing patterns of Latin-square designs. Thus treatment 2 is preceded three times by treatment 1 in 2.4.1 and only once in 2.4.2. Similarly we can read other values of the Betas.

Writing $\bar{Y} = \frac{1}{n^2} T_{..}$ so that $\hat{\mu} = \bar{Y}$

The least square equations may now be written as:

(I) $n\,\hat{\underline{p}} = \underline{P} - n\,\bar{Y}\,\underline{h}$ from 2.7.3

Since
$$\begin{bmatrix} \hat{r}_{a1} \\ \hat{r}_{a2} \\ \vdots \\ \hat{r}_{an} \end{bmatrix} = \underline{\alpha}\ \underline{\hat{r}} \qquad \text{we have}$$

(II) $n\,\underline{\hat{s}} - \underline{\alpha}\ \underline{\hat{r}} = \underline{S} - n\,\bar{Y}\,\underline{h}$ from 2.7.4

(III) $n\,\underline{\hat{t}} + \underline{\beta}\ \underline{\hat{r}} = \underline{T} - n\,\bar{Y}\,\underline{h}$ from 2.7.5

(IV) $(n-1)\,\underline{\hat{r}} - \underline{h}\,\underline{e}_1'\,\underline{\hat{p}} - \underline{\alpha}'\,\underline{\hat{s}} + \underline{\beta}'\,\underline{\hat{t}}$

$= \underline{R} - (n-1)\,\bar{Y}\,\underline{h}$ from 2.7.6

Here
$$\underline{S} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{bmatrix} \qquad \underline{h} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \qquad \underline{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \text{ jth}$$

Writing
$$\underline{X} = \begin{bmatrix} \underline{P} - n\,\bar{Y}\,\underline{h} \\ \underline{S} - n\,\bar{Y}\,\underline{h} \\ \underline{T} - n\,\bar{Y}\,\underline{h} \\ \underline{R} - (n-1)\,\bar{Y}\,\underline{h} \end{bmatrix} \qquad \underline{\hat{\theta}} = \begin{bmatrix} \underline{\hat{p}} \\ \underline{\hat{s}} \\ \underline{\hat{t}} \\ \underline{\hat{r}} \end{bmatrix}$$

From I, II, III and IV we get

$$\underline{M} = \begin{bmatrix} n\,\underline{I} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & n\,\underline{I} & \underline{0} & -\underline{\alpha} \\ \underline{0} & \underline{0} & n\,\underline{I} & \underline{\beta} \\ -\underline{h}\,\underline{e}_1' & -\underline{\alpha}' & \underline{\beta}' & (n-1)\,\underline{I} \end{bmatrix}$$

The least square equations may now be written as:

$$\begin{bmatrix} n\underline{I} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & n\underline{I} & \underline{0} & -\underline{\alpha} \\ \underline{0} & \underline{0} & n\underline{I} & \underline{\beta} \\ -\underline{h}\,\underline{e}_1' & -\underline{\alpha}' & \underline{\beta}' & (n-1)\,\underline{I} \end{bmatrix} \begin{bmatrix} \underline{\hat{p}} \\ \underline{\hat{s}} \\ \underline{\hat{t}} \\ \underline{\hat{r}} \end{bmatrix} = \begin{bmatrix} \underline{P} - n\bar{Y}\,\underline{h} \\ \underline{S} - n\bar{Y}\,\underline{h} \\ \underline{T} - n\bar{Y}\,\underline{h} \\ \underline{R} - (n-1)\,\bar{Y}\,\underline{h} \end{bmatrix}$$

where

$\underline{I}$ is the (n x n) identity matrix

$$\underline{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \ldots\ldots\ldots 0 \\ 0 & 1 & 0 & 0 & 0 \ldots\ldots\ldots 0 \\ 0 & 0 & 1 & 0 & 0 \ldots\ldots\ldots 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 \ldots\ldots\ldots 1 \end{bmatrix}$$

$\underline{e}_1$ is an (n x 1) vector with the first element equal to 1 and other elements equal to zero.

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

$\underline{h}$ is an (n x 1) vector with all the elements equal to 1.

$$\underline{h} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$$

$\underline{\alpha}$ is a permutation matrix depending on the treatments in the last row of the Latin-square.

Solving the equations $\underline{M}\,\underline{\hat{\theta}} = \underline{X}$ we get $\underline{\hat{\theta}} = \underline{M}^{-1}\,\underline{X}$

2.7.7 Inversion of Matrix M.

$$\underline{M}^{-1} = \frac{\text{Adjoint } M}{|M|} = \begin{bmatrix} \frac{m_{11}}{|M|} & \frac{m_{21}}{|M|} & \cdots\cdots & \frac{m_{n1}}{|M|} \\ \frac{m_{12}}{|M|} & \frac{m_{22}}{|M|} & \cdots\cdots & \frac{m_{n2}}{|M|} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{m_{1n}}{|M|} & \frac{m_{2n}}{|M|} & \cdots\cdots & \frac{m_{nn}}{|M|} \end{bmatrix}$$

where m_{ij} are cofactors of the elements of the ith row of $\underline{M}$ and are the elements of the ith column of adjoint M.

Hence we get

$$\underline{M}^{-1} = \frac{1}{n} \begin{bmatrix} \underline{I} & \underline{0} & \underline{0} & \underline{0} \\ \underline{\alpha}\,\underline{V}^{-1}\underline{h}\,\underline{e}_1' & \underline{I} + \underline{\alpha}\,\underline{V}^{-1}\underline{\alpha}' & -\underline{\alpha}\,\underline{V}^{-1}\underline{\beta}' & n\underline{\alpha}\,\underline{V}^{-1} \\ -\underline{\beta}\,\underline{V}^{-1}\underline{h}\,\underline{e}_1' & -\underline{\beta}\,\underline{V}^{-1}\underline{\alpha}' & \underline{I} + \underline{\beta}\,\underline{V}^{-1}\underline{\beta}' & -n\,\underline{\beta}\,\underline{V}^{-1} \\ n\,\underline{V}^{-1}\underline{h}\,\underline{e}_1' & n\,\underline{V}^{-1}\underline{\alpha}' & -n\,\underline{V}^{-1}\underline{\beta}' & n^2\,\underline{V}^{-1} \end{bmatrix}$$

where

$$\underline{V} = (n^2 - n)\,\underline{I}^2 - \underline{\beta}'\underline{\beta} - \underline{\alpha}'\underline{\alpha}$$

since $\underline{\alpha}'\underline{\alpha} = \underline{I}$

$$\underline{V} = (\,n^2 - n - 1\,)\,\underline{I} - \underline{\beta}'\underline{\beta}$$

The equation for parameter estimates is solved by substituting the values of the inverted matrix given above.

2.7.8 Parameter Estimates.

$\hat{\underline{\theta}} = \underline{M}^{-1} \underline{X}$ can now be expressed as:

$$\begin{bmatrix} \hat{\underline{p}} \\ \hat{\underline{s}} \\ \hat{\underline{t}} \\ \hat{\underline{r}} \end{bmatrix} = \frac{1}{n} \begin{bmatrix} \underline{I} & \underline{0} & \underline{0} & \underline{0} \\ \underline{\alpha}\,\underline{V}^{-1}\underline{h}\,\underline{e}_1' & \underline{I}+\underline{\alpha}\,\underline{V}^{-1}\underline{\alpha}' & -\underline{\alpha}\,\underline{V}^{-1}\underline{\beta}' & n\underline{\alpha}\,\underline{V}^{-1} \\ -\underline{\beta}\,\underline{V}^{-1}\underline{h}\,\underline{e}_1' & -\underline{\beta}\,\underline{V}^{-1}\underline{\alpha}' & \underline{I}+\underline{\beta}\,\underline{V}^{-1}\underline{\beta}' & -n\underline{\beta}\,\underline{V}^{-1} \\ n\underline{V}^{-1}\underline{h}\,\underline{e}_1' & n\underline{V}^{-1}\underline{\alpha}' & -n\underline{V}^{-1}\underline{\beta}' & n^2\underline{V}^{-1} \end{bmatrix} \begin{bmatrix} \underline{P} - n\bar{Y}\underline{h} \\ \underline{S} - n\bar{Y}\underline{h} \\ \underline{T} - n\bar{Y}\underline{h} \\ \underline{R} - (n-1)\bar{Y}\underline{h} \end{bmatrix}$$

From the above expression we obtain the parameter estimates.

For design 2.4.1 the estimates will be as follows:

$$\hat{\underline{p}} = \frac{1}{n}\left(\underline{P} - n\,\bar{Y}\,\underline{h} \right) \quad \text{i.e.} \quad \hat{p}_1 = \frac{P_1}{4} - \frac{G}{16}$$

$$\hat{p}_2 = \frac{P_2}{4} - \frac{G}{16}$$

and so on for other values of period effects. $G = \frac{\text{Grand Total}}{16}$

$$\hat{\underline{s}} = \frac{1}{n}\left(\frac{3}{2}\underline{S} + \frac{n}{2}\underline{R} - \frac{3}{2}\underline{T} + \frac{1}{2}\underline{P} - \frac{n^2}{2}\bar{Y}\,\underline{h} \right)$$

i.e. $\hat{s}_1 = \frac{3}{8} S_1 + \frac{1}{2} R_4 - \frac{1}{8} G$

............and so on.

$$\hat{\underline{t}} = \frac{1}{n}\left(\frac{11}{2}\underline{T} - \frac{3n}{2}\underline{R} - \frac{3}{2}\underline{P} - \frac{3}{2}\underline{S} + \frac{3n^2 - 8n}{2}\bar{Y}\,\underline{h} \right)$$

i.e. $\hat{t}_1 = \frac{11}{8} T_1 - \frac{3}{2} R_1 - \frac{3}{8} P_1 - \frac{3}{8} S_1 + \frac{1}{8} G$

............and so on for $\hat{t}_2$ to $\hat{t}_4$.

$$\hat{\underline{r}} = \frac{1}{n}\left(\frac{n^2}{2}\underline{R} - \frac{3n}{2}\underline{T} + \frac{n}{2}\underline{P} + \frac{n}{2}\underline{S} - n^2\,\bar{Y}\,\underline{h} \right)$$

i.e. $\hat{r}_1 = 2R_1 - \frac{3}{2} T_2 + \frac{1}{2} P_1 + \frac{1}{2} S_2 - \frac{1}{4} G$

.............and similarly other values of residual estimates.

For design 2.4.2

$$\underline{V} = 11\,\underline{I} - \begin{bmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{bmatrix}$$

or

$$\underline{V} = \begin{bmatrix} 8 & -2 & -2 & -2 \\ -2 & 8 & -2 & -2 \\ -2 & -2 & 8 & -2 \\ -2 & -2 & -2 & 8 \end{bmatrix}$$

and taking the inverse we get

$$\underline{V}^{-1} = \frac{1}{2000}\begin{bmatrix} 400 & 200 & 200 & 200 \\ 200 & 400 & 200 & 200 \\ 200 & 200 & 400 & 200 \\ 200 & 200 & 200 & 400 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.2 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.2 \end{bmatrix}$$

Hence the parameter estimates for this design are as follows:

$$\hat{\underline{p}} = \frac{1}{n}(\ \underline{P} - n\ \bar{Y}\ \underline{h}\ )$$

$$\hat{\underline{s}} = \frac{1}{n}\Big\{\underline{\alpha}\ \underline{V}^{-1}\underline{h}\ \underline{e}_1'\ \underline{P} + (\underline{I} + \underline{V}^{-1}\underline{\alpha}')\ \underline{S} - \underline{\alpha}\ \underline{V}^{-1}\underline{\beta}'\ \underline{T} + n\underline{\alpha}\ \underline{V}^{-1}\underline{R}$$

$$-n\bar{Y}\underline{h}(\ \underline{\alpha}\ \underline{V}^{-1}\underline{h}\ \underline{e}_1' + \underline{I} + \underline{V}^{-1} - \underline{\alpha}\ \underline{V}^{-1}\underline{\beta}' + n\underline{\alpha}\ \underline{V}^{-1})$$

$$+n\ \underline{V}^{-1}\underline{\alpha}\ \bar{Y}\ \underline{h}\Big\}$$

which gives

$$\hat{s}_1 = 0.3S_1 + 0.025S_2 + 0.025S_3 + 0.025S_4 - 0.1T_1 - 0.1T_2$$

$$-0.1T_3 - 0.075T_4 + 0.1R_1 + 0.1R_2 + 0.1R_3 + 0.2R_4$$

$$+ 0.125P_1 - 0.125G$$

Similarly we calculate $\hat{s}_2$, $\hat{s}_3$ and $\hat{s}_4$.

$$\hat{\underline{t}} = \frac{1}{n}\Big\{ -\underline{\beta}\, \underline{V}^{-1}\underline{h}\, \underline{e}_1'\, \underline{P} - \underline{\beta}\, \underline{V}^{-1}\, \underline{\alpha}'\, \underline{S} + (\underline{I} + \underline{\beta}\, \underline{V}^{-1}\, \underline{\beta}')\underline{T}$$

$$-n\,\underline{\beta}\, \underline{V}^{-1}\, \underline{R} - n\bar{Y}\underline{h}(-\underline{\beta}\, \underline{V}^{-1}\underline{h}\, \underline{e}_1' - \underline{\beta}\, \underline{V}^{-1}\underline{\alpha}' + \underline{I}$$

$$+ \underline{\beta}\, \underline{V}^{-1}\underline{\beta}' - n\,\underline{\beta}\, \underline{V}^{-1}) + n\underline{\beta}\, \underline{V}^{-1}\bar{Y}\, \underline{h} \Big\}$$

$$\therefore\ \hat{t}_1 = 0.55T_1 + 0.275T_2 + 0.275T_3 + 0.275T_4 - 0.3R_1 - 0.4R_2$$

$$-0.4R_3 - 0.4R_4 - 0.1S_1 - 0.1S_2 - 0.1S_3 - 0.075S_4 - 0.375P_1 + 0.125G$$

.... and similarly for $\hat{t}_2$, $\hat{t}_3$ and $\hat{t}_4$.

$$\hat{\underline{r}} = \frac{1}{n}\Big\{ n\, \underline{V}^{-1}\underline{h}\, \underline{e}_1'\, \underline{P} + n\, \underline{V}^{-1}\underline{\alpha}'\, \underline{S} - n\, \underline{V}^{-1}\underline{\beta}'\, \underline{T} + n^2\, \underline{V}^{-1}\, \underline{R}$$

$$- n\, \bar{Y}\, \underline{h}(n\, \underline{V}^{-1}\underline{h}\, \underline{e}_1' + n\, \underline{V}^{-1}\underline{\alpha}' - n\, \underline{V}^{-1}\underline{\beta}' + n^2\underline{V}^{-1})$$

$$+ n^2\, \underline{V}^{-1}\, \bar{Y}\, \underline{h} \Big\}$$

gives

$$\hat{r}_1 = 0.8R_1 + 0.4R_2 + 0.4R_3 + 0.4R_4 - 0.3T_1 - 0.4T_2 - 0.4T_3$$

$$- 0.4T_4 + 0.1S_1 + 0.1S_2 + 0.1S_3 + 0.2S_4 + 0.5P_1 - 0.25G$$

....

.... similarly for other values of the residual effects. Further details of calculations of parameter estimates will appear in the following chapters.

2.8 Alternative method for the estimation of Parameter Effects.

The period effect estimates can be obtained directly and independently from their set of normal equations. For the subject, treatment and residual effects the normal equations can be expressed in matrix form as follows:

$$n\,\hat{\underline{s}} - \underline{\alpha}\,\hat{\underline{r}} = \underline{S} - n\,\hat{\mu}\,\underline{h} \quad \text{........(Subject Effects)}$$

$$n\,\hat{\underline{t}} + \underline{\beta}\,\hat{\underline{r}} = \underline{T} - n\,\hat{\mu}\,\underline{h} \quad \text{........(Treatment Effects)}$$

$$(n-1)\,\hat{\underline{r}} + \underline{\beta}'\,\hat{\underline{t}} - \underline{\alpha}'\,\hat{\underline{s}} = \underline{R} - n\,\hat{\mu}\,\underline{h} + \frac{P_1}{n}\,\underline{h}$$

........(Residual Effects)

where $\hat{\mu} = \frac{G}{n^2}$

$\hat{\underline{s}}$, $\hat{\underline{t}}$, $\hat{\underline{r}}$, $\underline{S}$, $\underline{T}$, $\underline{R}$, $\underline{\alpha}$ and $\underline{\beta}$ as defined earlier.

P_1 is the total of observations in the first row.

The above three equations can now be written as:

$$\begin{bmatrix} n\,\underline{I} & \underline{0} & -\underline{\alpha} \\ \underline{0} & n\,\underline{I} & \underline{\beta} \\ -\underline{\alpha}' & \underline{\beta}' & (n-1)\,\underline{I} \end{bmatrix} \begin{bmatrix} \hat{\underline{s}} \\ \hat{\underline{t}} \\ \hat{\underline{r}} \end{bmatrix} = \begin{bmatrix} \underline{S} - n\,\hat{\mu}\,\underline{h} \\ \underline{T} - n\,\hat{\mu}\,\underline{h} \\ \underline{R} - n\,\hat{\mu}\,\underline{h} + \frac{P_1}{n}\,\underline{h} \end{bmatrix}$$

giving us

$$\begin{bmatrix} \hat{\underline{s}} \\ \hat{\underline{t}} \\ \hat{\underline{r}} \end{bmatrix} = \begin{bmatrix} n\,\underline{I} & \underline{0} & -\underline{\alpha} \\ \underline{0} & n\,\underline{I} & \underline{\beta} \\ -\underline{\alpha}' & \underline{\beta}' & (n-1)\,\underline{I} \end{bmatrix}^{-1} \begin{bmatrix} \underline{S} - n\,\hat{\mu}\,\underline{h} \\ \underline{T} - n\,\hat{\mu}\,\underline{h} \\ \underline{R} - n\,\hat{\mu}\,\underline{h} + \frac{P_1}{n}\,\underline{h} \end{bmatrix}$$

Now

$$\begin{bmatrix} n\,\underline{I} & \underline{0} & -\underline{\mathcal{L}} \\ \underline{0} & n\,\underline{I} & \underline{\beta} \\ -\underline{\mathcal{L}}' & \underline{\beta}' & (n-1)\underline{I} \end{bmatrix}^{-1} = \frac{1}{n}\begin{bmatrix} n(n-1)\,\underline{V}^{-1} - \underline{\beta}'\,\underline{V}^{-1}\underline{\beta} & -\underline{\beta}'\,\underline{V}^{-1}\underline{\mathcal{L}} & n\,\underline{\mathcal{L}}\,\underline{V}^{-1} \\ -\underline{\mathcal{L}}'\underline{V}^{-1}\underline{\beta} & n(n-1)\,\underline{V}^{-1} - \underline{\mathcal{L}}'\underline{V}^{-1}\underline{\mathcal{L}} & -n\,\underline{\beta}\,\underline{V}^{-1} \\ n\,\underline{V}^{-1}\underline{\mathcal{L}}' & -n\,\underline{V}^{-1}\underline{\beta}' & n^2\,\underline{V}^{-1} \end{bmatrix}$$

Hence we get

$$\hat{\theta} = \begin{bmatrix} \hat{\underline{s}} \\ \hat{\underline{t}} \\ \hat{\underline{r}} \end{bmatrix} = \frac{1}{n}\begin{bmatrix} n(n-1)\,\underline{V}^{-1} - \underline{\beta}'\,\underline{V}^{-1}\underline{\beta} & -\underline{\beta}'\,\underline{V}^{-1}\underline{\mathcal{L}} & n\,\underline{\mathcal{L}}\,\underline{V}^{-1} \\ -\underline{\mathcal{L}}'\underline{V}^{-1}\underline{\beta} & n(n-1)\,\underline{V}^{-1} - \underline{\mathcal{L}}'\underline{V}^{-1}\underline{\mathcal{L}} & -n\,\underline{\beta}\,\underline{V}^{-1} \\ n\,\underline{V}^{-1}\underline{\mathcal{L}}' & -n\,\underline{V}^{-1}\underline{\beta}' & n^2\,\underline{V}^{-1} \end{bmatrix}\begin{bmatrix} \underline{S} - n\,\hat{\mu}\,\underline{h} \\ \underline{T} - n\,\hat{\mu}\,\underline{h} \\ \underline{R} - n\,\hat{\mu}\underline{h} + \frac{P}{n}1\,\underline{h} \end{bmatrix}$$

Since this is the most simplified form of matrix approach towards the estimation of subject, treatment and residual effects, it can be followed for any further calculations of parameter estimates.

2.9 <u>Computer Program for solving Normal Equations</u>.

In design 2.4.1 each treatment has been preceded by the same treatment on all occasions. The normal equations for the estimation of parameters are quite simple in this case and can be solved simultaneously without much difficulty. For more complicated forms of Latin-square designs, involving carry-over effects, it is preferable to put the normal equations in matrix form and use matrix theory to find the estimates of the parameters. Matrix methods offer a straightforward means of solving simultaneous equations which can be easily adapted for use with calculating machines and can also be programmed for computers.

As an example the normal equations 2.5.2, 2.5.3 and 2.5.4 of design 2.4.1 can be put in matrix form as given below:

2.9.1 Matix form of the normal equations.

$$\begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & -1 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \\ \hat{t}_4 \\ \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ \hat{s}_4 \end{bmatrix} = \begin{bmatrix} T_1 - \frac{G}{4} \\ T_2 - \frac{G}{4} \\ T_3 - \frac{G}{4} \\ T_3 - \frac{G}{4} \\ R_1 - \frac{G}{4} + \frac{1}{4} P_1 \\ R_2 - \frac{G}{4} + \frac{1}{4} P_1 \\ R_3 - \frac{G}{4} + \frac{1}{4} P_1 \\ R_4 - \frac{G}{4} + \frac{1}{4} P_1 \\ S_1 - \frac{G}{4} \\ S_2 - \frac{G}{4} \\ S_3 - \frac{G}{4} \\ S_4 - \frac{G}{4} \end{bmatrix}$$

Let $\underline{X}$ be the information matrix representing the coefficients of the parameters to be estimated, $\hat{\underline{\theta}}$ the column vector of estimates of parameters representing the treatment effects, treatment residual effects and subject effects and $\underline{Y}$ the column vector of totals of observations on treatments, treatments with residual effects and subjects along with other factors brought to the right hand side of the normal equations while setting them in matrix form.

Period effects have been left out of the matrix since they can be determined directly from their normal equations without any complication and thus the matrix size is kept to a minimum.

$$\underline{X}\,\hat{\underline{\theta}} = \underline{Y}$$

$$\therefore \quad \hat{\underline{\theta}} = \underline{X}^{-1}\,\underline{Y}$$

The inversion of the matrices have been performed by computer using the Fortran language with the programme as in Appendix I.

2.9.2 The inverted matrix for design 2.4.1 can be seen on the next page.

1.37500	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	-1.50000	-0.37500	0.00000	0.00000	0.00000
0.00000	1.37500	0.00000	0.00000	-1.50000	0.00000	0.00000	0.00000	0.00000	-0.37500	0.00000	0.00000
0.00000	0.00000	1.37500	0.00000	0.00000	-1.50000	0.00000	0.00000	0.00000	0.00000	-0.37500	0.00000
0.00000	0.00000	0.00000	1.37500	0.00000	0.00000	-1.50000	0.00000	0.00000	0.00000	0.00000	-0.37500
0.00000	-1.50000	0.00000	0.00000	2.00000	0.00000	0.00000	0.00000	0.00000	0.50000	0.00000	0.00000
0.00000	0.00000	-1.50000	0.00000	0.00000	2.00000	0.00000	0.00000	0.00000	0.00000	0.50000	0.00000
0.00000	0.00000	0.00000	-1.50000	0.00000	0.00000	2.00000	0.00000	0.00000	0.00000	0.00000	0.50000
-1.50000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	2.00000	0.50000	0.00000	0.00000	0.00000
-0.37500	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.50000	0.37500	0.00000	0.00000	0.00000
0.00000	-0.37500	0.00000	0.00000	0.50000	0.00000	0.00000	0.00000	0.00000	0.37500	0.00000	0.00000
0.00000	0.00000	-0.37500	0.00000	0.00000	0.50000	0.00000	0.00000	0.00000	0.00000	0.37500	0.00000
0.00000	0.00000	0.00000	-0.37500	0.00000	0.00000	0.50000	0.00000	0.00000	0.00000	0.00000	0.37500

2.9.2

2.9.3 The matrices for the normal equations such as 2.9.1 can be derived directly from the design pattern without writing down any normal equations. The general form of such matrices may be expressed as:

$$\underline{X} = \begin{bmatrix} n\,\underline{I} & \underline{\beta} & \underline{0} \\ \underline{\beta}' & (n-1)\,\underline{I} & -\underline{\alpha}' \\ \underline{0} & -\underline{\alpha} & n\,\underline{I} \end{bmatrix}$$

$$\underline{\hat{\theta}} = \begin{bmatrix} \underline{\hat{t}} \\ \underline{\hat{r}} \\ \underline{\hat{s}} \end{bmatrix} \qquad \underline{Y} = \begin{bmatrix} \underline{T} - n\,\hat{\mu}\,\underline{h} \\ \underline{R} - n\,\hat{\mu}\,\underline{h} + \frac{P}{n}1\,\underline{h} \\ \underline{S} - n\,\hat{\mu}\,\underline{h} \end{bmatrix}$$

Taking for example design 2.4.1 with reference to 2.9.1 sub-matrices of $\underline{X}$ can be written as below.

$$n\,\underline{I} = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad (n-1)\,\underline{I} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$n\,\underline{I}$ is a diagonal matrix with each diagonal element equal to n, the number of treatments in a Latin-square. $(n-1)\,\underline{I}$ is also a diagonal matrix having each diagonal element equal to (n-1), the number of times each treatment has produced residual effects present in the design.

$$\underline{\beta} = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \qquad \underline{\alpha} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The elements in $\underline{\beta}$ represent the number of times treatments precede and follow each other. $\underline{\alpha}$ is a matrix based on the last row of the design as already explained under 2.7 and $\underline{0}$ is of order (n x n).

2.10 Parameter Estimates through Inverted Matrix.

With reference to 2.9.2 we can now write down the estimates of the parameters of design 4.2.1 as follows:-

2.10.1 Subject effects.

$$\hat{s}_1 = 0.375\ S_1 - 0.375\ T_1 + 0.5\ R_4 + 0.125\ P_1 - 0.125\ G$$

$$\hat{s}_2 = 0.375\ S_2 - 0.375\ T_2 + 0.5\ R_1 + 0.125\ P_1 - 0.125\ G$$

$$\hat{s}_3 = 0.375\ S_3 - 0.375\ T_3 + 0.5\ R_2 + 0.125\ P_1 - 0.125\ G$$

$$\hat{s}_4 = 0.375\ S_4 - 0.375\ T_4 + 0.5\ R_3 + 0.125\ P_1 - 0.125\ G$$

2.10.2 Treatment direct effects.

$$\hat{t}_1 = 1.375\ T_1 - 1.5\ R_4 - 0.375\ S_1 - 0.375\ P_1 + 0.125\ G$$

$$\hat{t}_2 = 1.375\ T_2 - 1.5\ R_1 - 0.375\ S_2 - 0.375\ P_1 + 0.125\ G$$

$$\hat{t}_3 = 1.375\ T_3 - 1.5\ R_2 - 0.375\ S_3 - 0.375\ P_1 + 0.125\ G$$

$$\hat{t}_4 = 1.375\ T_4 - 1.5\ R_3 - 0.375\ S_4 - 0.375\ P_1 + 0.125\ G$$

2.10.3 Treatment residual effects.

$$\hat{r}_1 = 2\ R_1 - 1.5\ T_2 + 0.5\ S_2 + 0.5\ P_1 - 0.25\ G$$

$$\hat{r}_2 = 2\ R_2 - 1.5\ T_3 + 0.5\ S_3 + 0.5\ P_1 - 0.25\ G$$

$$\hat{r}_3 = 2\ R_3 - 1.5\ T_4 + 0.5\ S_4 + 0.5\ P_1 - 0.25\ G$$

$$\hat{r}_4 = 2\ R_4 - 1.5\ T_1 + 0.5\ S_1 + 0.5\ P_1 - 0.25\ G$$

Period effects are obtained directly from the set of normal equations 2.5.1 and are given under 2.6.1.

2.11 Variance-covariance Matrix.

The parameter estimates are given by:

$$\underline{\hat{\theta}} = \underline{M}^{-1}\ \underline{X}$$

Then $\text{Cov}(\ \underline{\hat{\theta}}\ \underline{\hat{\theta}}'\ ) = \underline{M}^{-1}\ \text{Cov}(\ \underline{X}\ \underline{X}'\ )(\underline{M}^{-1})'$

From 2.7.8 we know that

$$\underline{X} = \begin{bmatrix} \underline{P} - n\,\bar{Y}\,\underline{h} \\ \underline{S} - n\,\bar{Y}\,\underline{h} \\ \underline{T} - n\,\bar{Y}\,\underline{h} \\ \underline{R} - (n-1)\bar{Y}\underline{h} \end{bmatrix}$$

Now writing

$$\underline{P} - n\,\bar{Y}\,\underline{h} = (\ \underline{I} - \tfrac{1}{n}\,\underline{h}\,\underline{h}'\ )\ \underline{P} = \underline{K}\ \underline{P}$$

$$\underline{S} - n\,\bar{Y}\,\underline{h} = (\ \underline{I} - \tfrac{1}{n}\,\underline{h}\,\underline{h}'\ )\ \underline{S} = \underline{K}\ \underline{S}$$

$$\underline{T} - n\,\bar{Y}\,\underline{h} = (\ \underline{I} - \tfrac{1}{n}\,\underline{h}\,\underline{h}'\ )\ \underline{T} = \underline{K}\ \underline{T}$$

$$\underline{R} - (n-1)\bar{Y}\,\underline{h} = (\ \underline{I} - \frac{n-1}{n^2}\,\underline{h}\,\underline{h}'\ )\ \underline{R} - \frac{n-1}{n^2}\,\underline{h}\,\underline{e}_1'\underline{P}$$

$$= (\underline{K} + \frac{1}{n^2}\underline{h}\,\underline{h}')\ \underline{R} - \frac{n-1}{n^2}\,\underline{h}\,\underline{e}_1'\ \underline{P}$$

where $\underline{K} = \underline{I} - \frac{1}{n}\,\underline{h}\,\underline{h}'$

Therefore $\underline{X}$ can now be expressed as follows:

$$\underline{X} = \begin{bmatrix} \underline{K} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{K} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K} & \underline{0} \\ -\frac{n-1}{n^2}\underline{h}\underline{e}_1' & \underline{0} & \underline{0} & \underline{K} + \frac{1}{n^2}\,\underline{h}\,\underline{h}' \end{bmatrix} \begin{bmatrix} \underline{P} \\ \underline{S} \\ \underline{T} \\ \underline{R} \end{bmatrix} = \underline{J}\ \underline{X}_o$$

$$\underline{J} = \begin{bmatrix} \underline{K} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{K} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K} & \underline{0} \\ -\frac{n-1}{n^2}\underline{h}\underline{e}_1' & \underline{0} & \underline{0} & \underline{K} + \frac{1}{n^2}\underline{h}\ \underline{h} \end{bmatrix}$$

$$\underline{X}_o = \begin{bmatrix} \underline{P} \\ \underline{S} \\ \underline{T} \\ \underline{R} \end{bmatrix} \qquad \underline{Y} = \begin{bmatrix} \underline{Y}_1 \\ \underline{Y}_2 \\ \vdots \\ \underline{Y}_n \end{bmatrix}$$

is $(n^2 \times 1)$ vector, where $\underline{Y}_j$ is the jth column.

$$\underline{P} = \sum_j Y_{ij}$$

$$\underline{S} = \left[\underline{e}_j \, \underline{h}' \, Y_{ij} \right]$$

$$\underline{T} = \sum_j \underline{\tau}_j' \, Y_{ij} \quad \text{where} \quad \underline{\tau}_j \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} = \begin{bmatrix} \text{Treatment in row 1 of column j} \\ \text{" \quad " \; " 2 " \quad " \quad "} \\ \cdots\cdots \\ \cdots\cdots \\ \text{" \quad " \; " n " \quad " \quad "} \end{bmatrix}$$

$$\underline{R} = \sum_j \underline{\rho}_j' \, Y_{ij} \quad \text{where} \quad \underline{\rho}_j \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} = \begin{bmatrix} \text{Residual in row 1 of column j} \\ \text{" \quad " \; " 2 " \quad " \quad "} \\ \cdots\cdots \\ \cdots\cdots \\ \text{" \quad " \; " n " \quad " \quad "} \end{bmatrix}$$

$$\text{or} \quad \underline{\rho}_j \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix} = \left[\begin{array}{c|c} 0 & 0 \\ \hline \underline{I}_{n-1} & 0 \end{array} \right] \underline{\tau}_j \begin{bmatrix} 1 \\ 2 \\ \vdots \\ n \end{bmatrix}$$

$$\therefore \quad \underline{\rho}_j = \underline{G} \, \underline{\tau}_j$$

$$\text{where} \quad \underline{G} = \begin{bmatrix} 0 & \cdots & 0 \\ & & \vdots \\ \underline{I}_{n-1} & & 0 \end{bmatrix}$$

$$\underline{G}' = \begin{bmatrix} 0 & & \underline{I}_{n-1} \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$$

$$\sum_j \underline{\tau}_j' = \underline{h} \, \underline{h}' \quad \text{and} \quad \sum_j \underline{\rho}_j' = \sum_j \underline{\tau}_j' \, \underline{G} = \underline{h} \, \underline{h}' \underline{G} = \underline{h}(\underline{h}' - \underline{e}_1')$$

$$\sum_j \underline{\tau}_j' \, \underline{\tau}_j = n \, \underline{I} \quad \text{and} \quad \sum_j \underline{\rho}_j' \, \underline{\rho}_j = (n-1) \, \underline{I}$$

The symbols and expressions from the last page can be explained by taking one of the designs as an example, say for 2.4.1, we have the design matrix as below:

$$
\begin{bmatrix}
1&0&0&0 & 1&0&0&0 & 1&0&0&0 & 1&0&0&0 \\
0&1&0&0 & 0&1&0&0 & 0&1&0&0 & 0&1&0&0 \\
0&0&1&0 & 0&0&1&0 & 0&0&1&0 & 0&0&1&0 \\
0&0&0&1 & 0&0&0&1 & 0&0&0&1 & 0&0&0&1 \\
1&1&1&1 & 0&0&0&0 & 0&0&0&0 & 0&0&0&0 \\
0&0&0&0 & 1&1&1&1 & 0&0&0&0 & 0&0&0&0 \\
0&0&0&0 & 0&0&0&0 & 1&1&1&1 & 0&0&0&0 \\
0&0&0&0 & 0&0&0&0 & 0&0&0&0 & 1&1&1&1 \\
1&0&0&0 & 0&0&0&1 & 0&0&1&0 & 0&1&0&0 \\
0&1&0&0 & 1&0&0&0 & 0&0&0&1 & 0&0&1&0 \\
0&0&1&0 & 0&1&0&0 & 1&0&0&0 & 0&0&0&1 \\
0&0&0&1 & 0&0&1&0 & 0&1&0&0 & 1&0&0&0 \\
0&1&0&0 & 0&0&0&0 & 0&0&0&1 & 0&0&1&0 \\
0&0&1&0 & 0&1&0&0 & 0&0&0&0 & 0&0&0&1 \\
0&0&0&1 & 0&0&1&0 & 0&1&0&0 & 0&0&0&0 \\
0&0&0&0 & 0&0&0&1 & 0&0&1&0 & 0&1&0&0
\end{bmatrix}
\begin{bmatrix}
Y_{11} \\ Y_{21} \\ Y_{31} \\ Y_{41} \\ Y_{12} \\ Y_{22} \\ Y_{32} \\ Y_{42} \\ Y_{13} \\ Y_{23} \\ Y_{33} \\ Y_{43} \\ Y_{14} \\ Y_{24} \\ Y_{34} \\ Y_{44}
\end{bmatrix}
$$

Now we can write down $\underline{X}_o$ in terms of new values and we get

$$
\underline{X}_o = \begin{bmatrix}
\underline{I} & \underline{I} & \underline{I} \cdots\cdots\cdots & \underline{I} \\
\underline{e}_1\,\underline{h}' & \underline{e}_2\,\underline{h}' & \underline{e}_3\,\underline{h}' \cdots\cdots\cdots & \underline{e}_n\,\underline{h}' \\
\underline{\tau}_1' & \underline{\tau}_2' & \underline{\tau}_3' \cdots\cdots\cdots & \underline{\tau}_n' \\
\underline{\rho}_1' & \underline{\rho}_2' & \underline{\rho}_3' \cdots\cdots\cdots & \underline{\rho}_n'
\end{bmatrix} \underline{Y}_{(n^2 \times 1)}
$$

We may write for convenience $\underline{X}_o = \underline{A}\,\underline{Y}$

where

$$
\underline{A} = \begin{bmatrix}
\underline{I} & \underline{I} & \underline{I} \cdots\cdots\cdots & \underline{I} \\
\underline{e}_1\,\underline{h}' & \underline{e}_2\,\underline{h}' & \underline{e}_3\,\underline{h}' \cdots\cdots\cdots & \underline{e}_n\,\underline{h}' \\
\underline{\tau}_1' & \underline{\tau}_2' & \underline{\tau}_3' \cdots\cdots\cdots & \underline{\tau}_n' \\
\underline{\rho}_1' & \underline{\rho}_2' & \underline{\rho}_3' \cdots\cdots\cdots & \underline{\rho}_n'
\end{bmatrix}
$$

and

$$\underline{A}' = \begin{bmatrix} \underline{I} & \underline{h}\,\underline{e}_1' & \underline{\tau}_1 & \underline{\rho}_1 \\ \underline{I} & \underline{h}\,\underline{e}_2' & \underline{\tau}_2 & \underline{\rho}_2 \\ \underline{I} & \underline{h}\,\underline{e}_3' & \underline{\tau}_3 & \underline{\rho}_3 \\ \vdots & \vdots & \vdots & \vdots \\ \underline{I} & \underline{h}\,\underline{e}_n' & \underline{\tau}_n & \underline{\rho}_n \end{bmatrix}$$

$$\underline{A}\,\underline{A}' = \begin{bmatrix} n\,\underline{I} & \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' & (\underline{h} - \underline{e}_1)\,\underline{h}' \\ \underline{h}\,\underline{h}' & n\,\underline{I} & \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' - \underline{\alpha} \\ \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' & n\,\underline{I} & \underline{\beta} \\ \underline{h}(\underline{h}'-\underline{e}_1') & \underline{h}\,\underline{h}' - \underline{\alpha}' & \underline{\beta}' & (n-1)\,\underline{I} \end{bmatrix}$$

$$\text{Cov}(\underline{X}_o\,\underline{X}_o') = \underline{A}\,\text{Cov}(\underline{Y}\,\underline{Y}')\,\underline{A}'$$

$$\text{Cov}(\underline{Y}\,\underline{Y}') = \sigma^2\,\underline{I}_n$$

Hence

$$\text{Cov}(\underline{X}_o\,\underline{X}_o') = \sigma^2\,\underline{A}\,\underline{A}' =$$

$$\sigma^2 \begin{bmatrix} n\,\underline{I} & \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' & (\underline{h} - \underline{e}_1)\,\underline{h}' \\ \underline{h}\,\underline{h}' & n\,\underline{I} & \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' - \underline{\alpha} \\ \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' & n\,\underline{I} & \underline{\beta} \\ \underline{h}(\underline{h}'-\underline{e}_1') & \underline{h}\,\underline{h}' - \underline{\alpha}' & \underline{\beta}' & (n-1)\,\underline{I} \end{bmatrix}$$

But $\text{Cov}(\underline{X}\,\underline{X}') = \underline{J}\,\text{Cov}(\underline{X}_o\,\underline{X}_o')\,\underline{J}'$

Therefore

$$\frac{1}{\sigma^2} \underline{J} \, \text{Cov}(\underline{X}_o \underline{X}_o') =$$

$$\begin{bmatrix} \underline{K} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{K} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K} & \underline{0} \\ -\frac{n-1}{n^2}\underline{h}\,\underline{e}_1' & \underline{0} & \underline{0} & \underline{K}+\frac{1}{n^2}\underline{h}\,\underline{h}' \end{bmatrix} \begin{bmatrix} n\,\underline{I} & \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' & (\underline{h}-\underline{e}_1)\,\underline{h}' \\ \underline{h}\,\underline{h}' & n\,\underline{I} & \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' - \underline{\alpha} \\ \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' & n\,\underline{I} & \underline{\beta} \\ \underline{h}(\underline{h}'-\underline{e}_1') & \underline{h}\,\underline{h}'-\underline{\alpha}' & \underline{\beta}' & (n-1)\,\underline{I} \end{bmatrix}$$

$$= \begin{bmatrix} n\,\underline{K} & \underline{0} & \underline{0} & -\underline{K}\,\underline{e}_1\,\underline{h}' \\ \underline{0} & n\,\underline{K} & \underline{0} & -\underline{K}\,\underline{\alpha} \\ \underline{0} & \underline{0} & n\,\underline{K} & \underline{K}\,\underline{\beta} \\ \underline{h}(\underline{h}'-\underline{e}_1') & -\underline{K}\,\underline{\alpha}' & \underline{K}\,\underline{\beta}' & (n-1)(\underline{K} + \frac{1}{n^2}\underline{h}\,\underline{h}') \end{bmatrix}$$

$$\because \; \underline{K}\,\underline{h} = (\underline{I} - \frac{1}{n}\underline{h}\,\underline{h}')\,\underline{h} = \underline{0}$$

$$\underline{h}'\underline{K} = \underline{0} \quad \text{and} \quad \underline{K}\,\underline{h}\,\underline{h}' = \underline{0}$$

$$\frac{1}{\sigma^2} \underline{J} \, \text{Cov}(\underline{X}_o \underline{X}_o') \, \underline{J}' =$$

$$\begin{bmatrix} n\,\underline{K} & \underline{0} & \underline{0} & -\underline{K}\,\underline{e}_1\,\underline{h}' \\ \underline{0} & n\,\underline{K} & \underline{0} & -\underline{K}\,\underline{\alpha} \\ \underline{0} & \underline{0} & n\,\underline{K} & \underline{K}\,\underline{\beta} \\ \underline{h}(\underline{h}'-\underline{e}_1') & -\underline{K}\,\underline{\alpha}' & \underline{K}\,\underline{\beta}' & (n-1)(\underline{K}+\frac{1}{n^2}\underline{h}\,\underline{h}') \end{bmatrix} \begin{bmatrix} \underline{K} & \underline{0} & \underline{0} & -\frac{n-1}{n^2}\underline{e}_1\,\underline{h}' \\ \underline{0} & \underline{K} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{K} + \frac{1}{n^2}\underline{h}\,\underline{h}' \end{bmatrix}$$

Note: $\underline{K}\,\underline{K} = \underline{K}$

Hence we get

$$\frac{1}{\sigma^2}\text{Cov}(\underline{X}\ \underline{X}') = n\begin{bmatrix} \underline{K} & \underline{0} & \underline{0} & -\frac{1}{n}\underline{K}\ \underline{e}_1\ \underline{h}' \\ \underline{0} & \underline{K} & \underline{0} & -\frac{1}{n}\underline{K}\ \mathcal{L}\ \underline{K} \\ \underline{0} & \underline{0} & \underline{K} & \frac{1}{n}\underline{K}\ \beta\ \underline{K} \\ -\frac{1}{n}\underline{h}\ \underline{e}_1'\underline{K} & -\frac{1}{n}\underline{K}\ \mathcal{L}'\ \underline{K} & \frac{1}{n}\underline{K}\ \beta'\ \underline{K} & \frac{n-1}{n}(\underline{K} + \frac{1}{n^3}\underline{h}\ \underline{h}') \end{bmatrix}$$

$$\text{Cov}(\ \hat{\underline{\theta}}\ \hat{\underline{\theta}}'\ ) = \underline{M}^{-1}\ \text{Cov}(\ \underline{X}\ \underline{X}'\ )(\ \underline{M}^{-1}\ )'$$

Now $\frac{1}{\sigma^2}\ \underline{M}^{-1}\ \text{Cov}(\ \underline{X}\ \underline{X}'\ ) =$

$$\begin{bmatrix} \underline{I} & \underline{0} & \underline{0} & \underline{0} \\ \mathcal{L}\ \underline{V}^{-1}\underline{h}\ \underline{e}_1' & \underline{I}+\mathcal{L}\ \underline{V}^{-1}\mathcal{L}' & -\mathcal{L}\ \underline{V}^{-1}\ \beta' & n\mathcal{L}\ \underline{V}^{-1} \\ -\beta\ \underline{V}^{-1}\underline{h}\ \underline{e}_1' & -\beta\ \underline{V}^{-1}\mathcal{L}' & \underline{I}+\beta\ \underline{V}^{-1}\beta' & -n\beta\ \underline{V}^{-1} \\ n\ \underline{V}^{-1}\ \underline{h}\ \underline{e}_1' & n\ \underline{V}^{-1}\ \mathcal{L}' & -n\ \underline{V}^{-1}\ \beta' & n^2\ \underline{V}^{-1} \end{bmatrix} \begin{bmatrix} \underline{K} & \underline{0} & \underline{0} & -\frac{1}{n}\underline{K}\underline{e}_1\underline{h}' \\ \underline{0} & \underline{K} & \underline{0} & -\frac{1}{n}\underline{K}\mathcal{L}\underline{K} \\ \underline{0} & \underline{0} & \underline{K} & \frac{1}{n}\underline{K}\beta\underline{K} \\ -\frac{1}{n}\underline{K}\underline{h}\underline{e}_1' & -\frac{1}{n}\underline{K}\mathcal{L}'\underline{K} & \frac{1}{n}\underline{K}\beta'\underline{K} & \frac{n-1}{n}(\underline{K}+\frac{1}{n^3}\underline{h}\ \underline{h}) \end{bmatrix}$$

$$= \begin{bmatrix} \underline{K} & \underline{0} & \underline{0} & -\frac{1}{n}\ \underline{K}\ \underline{e}_1\ \underline{h}' \\ \underline{0} & \underline{K} & \underline{0} & \frac{n-1}{n^3}\ \mathcal{L}\ \underline{V}^{-1}\underline{h}\ \underline{h}' \\ \underline{0} & \underline{0} & \underline{K} & -\frac{n-1}{n^3}\ \beta\ \underline{V}^{-1}\underline{h}\ \underline{h}' \\ \underline{0} & \underline{0} & \underline{0} & \underline{K} + \frac{n(n-1)}{n^3}\ \underline{V}^{-1}\underline{h}\ \underline{h}' \end{bmatrix}$$

and multiplying the matrix obtained above by $(\ \underline{M}^{-1})'$ we get the Variance-covariance matrix of parameter estimates.

Therefore $\text{Cov}(\underline{\hat{\theta}}\,\underline{\hat{\theta}}) = \underline{M}^{-1}\,\text{Cov}(\underline{X}\,\underline{X}')(\underline{M}^{-1})' =$

$$\frac{\sigma^2}{n}\begin{bmatrix} \underline{K} & \underline{0} & \underline{0} & -\frac{1}{n}\underline{K}\,\underline{e}_1\,\underline{h}' \\ \underline{0} & \underline{K} & \underline{0} & \frac{n-1}{n^3}\underline{V}^{-1}\underline{h}\,\underline{h}' \\ \underline{0} & \underline{0} & \underline{K} & -\frac{n-1}{n^3}\underline{V}^{-1}\underline{h}\,\underline{h}' \\ \underline{0} & \underline{0} & \underline{0} & \underline{K}+\frac{n(n-1)}{n^3}\underline{V}^{-1}\underline{h}\underline{h}' \end{bmatrix}\begin{bmatrix} \underline{I} & \underline{e}_1\underline{h}'\,\underline{V}^{-1}\underline{\mathcal{L}}' & -\underline{e}_1\underline{h}'\,\underline{V}^{-1}\underline{\beta}' & n\underline{e}_1\underline{h}'\,\underline{V}^{-1} \\ \underline{0} & \underline{I}+\underline{\mathcal{L}}\,\underline{V}^{-1}\underline{\mathcal{L}}' & -\underline{\mathcal{L}}\,\underline{V}^{-1}\underline{\beta}' & n\,\underline{\mathcal{L}}\,\underline{V}^{-1} \\ \underline{0} & -\underline{\beta}\,\underline{V}^{-1}\underline{\mathcal{L}}' & \underline{I}+\underline{\beta}\,\underline{V}^{-1}\underline{\beta}' & -n\underline{\beta}\,\underline{V}^{-1} \\ \underline{0} & n\underline{V}^{-1}\underline{\mathcal{L}}' & -n\,\underline{V}^{-1}\underline{\beta}' & n^2\,\underline{V}^{-1} \end{bmatrix}$$

After simplification we get $\text{Cov}(\underline{\hat{\theta}}\,\underline{\hat{\theta}}') =$

$$\frac{\sigma^2}{n}\begin{bmatrix} \underline{K} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \begin{matrix}\underline{K}+\underline{K}\,\underline{V}^{-1} \\ +\frac{n-1}{n^2}\underline{V}^{-1}\underline{h}\,\underline{h}'\,\underline{V}^{-1}\end{matrix} & \begin{matrix}-\underline{K}\,\underline{\mathcal{L}}\,\underline{V}^{-1}\underline{\beta}' \\ -\frac{n-1}{n^2}\underline{V}^{-1}\underline{\mathcal{L}}\,\underline{h}\,\underline{h}'\underline{\beta}\,\underline{V}^{-1}\end{matrix} & \begin{matrix}n\underline{K}\,\underline{\mathcal{L}}\,\underline{V}^{-1} \\ +\frac{n-1}{n}\underline{V}^{-1}\underline{\mathcal{L}}\,\underline{h}\,\underline{h}'\,\underline{V}^{-1}\end{matrix} \\ \underline{0} & \begin{matrix}-\underline{\beta}\,\underline{V}^{-1}\,\underline{\mathcal{L}}'\underline{K}' \\ -\frac{n-1}{n^2}\underline{V}^{-1}\underline{\beta}\,\underline{h}\,\underline{h}'\,\underline{\mathcal{L}}\,\underline{V}^{-1}\end{matrix} & \begin{matrix}\underline{K}+\underline{\beta}\,\underline{K}\,\underline{V}^{-1}\underline{\beta}' \\ +\frac{n-1}{n^2}\underline{V}^{-1}\underline{\beta}\,\underline{h}\,\underline{h}'\underline{\beta}\,\underline{V}^{-1}\end{matrix} & \begin{matrix}-n\underline{K}\,\underline{\beta}\,\underline{V}^{-1} \\ -\frac{n-1}{n}\underline{V}^{-1}\underline{\beta}\,\underline{h}\,\underline{h}'\,\underline{V}^{-1}\end{matrix} \\ \underline{0} & \begin{matrix}n\,\underline{V}^{-1}\,\underline{\mathcal{L}}'\underline{K}' \\ +\frac{n-1}{n}\underline{V}^{-1}\underline{h}\,\underline{h}'\,\underline{\mathcal{L}}\,\underline{V}^{-1}\end{matrix} & \begin{matrix}-n\,\underline{V}^{-1}\,\underline{\beta}'\,\underline{K} \\ -\frac{n-1}{n}\underline{V}^{-1}\,\underline{h}\,\underline{h}'\underline{\beta}\,\underline{V}^{-1}\end{matrix} & \begin{matrix}n^2\,\underline{K}\,\underline{V}^{-1} \\ +(n-1)\,\underline{V}^{-1}\,\underline{h}\,\underline{h}'\,\underline{V}^{-1}\end{matrix} \end{bmatrix}$$

when

$$\underline{\hat{\theta}} = \begin{bmatrix} \underline{\hat{p}} \\ \underline{\hat{s}} \\ \underline{\hat{t}} \\ \underline{\hat{r}} \end{bmatrix}$$

2.12 Precision in Measurements.

The variance of the difference between parameter estimates provides a means for comparing the relative precision of two experiments. Similarly we can study how precisely a certain parameter is measured in relation to another parameter within an experiment, such as relative efficiency of measuring treatment estimates over residual estimates in a Latin-square.

The variance-covariance matrix obtained under 2.11 provides variance-covariance matrices for different parameter estimates from where the variance of the difference between parameter estimates can be calculated for knowing the precision in measurements. This method is a bit cumbersome as it involves a number of matrices, one of which is to be inverted before conducting arithmetical operations to obtain the final results.

Since the main interest of this study lies in choosing the most efficient Latin-square design which provides the best estimates for treatment as well as residual effects therefore we shall mostly concentrate on these parameters only.

From the variance-covariance matrices obtained for different parameters, the variance of the difference between two parameter estimates are calculated as:

$$V(\hat{t}_i - \hat{t}_u) = V(\hat{t}_i) + V(\hat{t}_u) - 2\,Cov(\hat{t}_i \hat{t}_u)$$

$$V(\hat{r}_i - \hat{r}_u) = V(\hat{r}_i) + V(\hat{r}_u) - 2\,Cov(\hat{r}_i \hat{r}_u)$$

$$V(\hat{s}_i - \hat{s}_u) = V(\hat{s}_i) + V(\hat{s}_u) - 2\,Cov(\hat{s}_i \hat{s}_u)$$

where $i = 1, 2, \ldots\ldots n$ and $u = 1, 2, \ldots\ldots\ldots n$.

2.12.1 Using the variance-covariance matrix derived under 2.11 we can now write down the variance-covariance matrices for period, subject, treatment and residual effects for the designs 2.4.1 to 2.4.4. We can also calculate the variances of the differences between two parameter estimates, useful for calculating the efficiency ratio and comparison, from these matrices.

Period effects:

Design 2.4.1

$$V\begin{bmatrix}\hat{p}_1\\ \hat{p}_2\\ \hat{p}_3\\ \hat{p}_4\end{bmatrix}=\begin{bmatrix}0.1875 & -0.0625 & -0.0625 & -0.0625\\ -0.0625 & 0.1875 & -0.0625 & -0.0625\\ -0.0625 & -0.0625 & 0.1875 & -0.0625\\ -0.0625 & -0.0625 & -0.0625 & 0.1875\end{bmatrix}\sigma^2$$

which gives $V(\hat{p}_1) = \dots\dots\dots = V(\hat{p}_4) = 0.1875\,\sigma^2$

$$V(\hat{p}_i - \hat{p}_j) = V(\hat{p}_i) + V(\hat{p}_j) - 2\,\mathrm{Cov}(\hat{p}_i\ \hat{p}_j)$$

$$V(\hat{p}_1 - \hat{p}_2) = \dots\dots\dots = V(\hat{p}_1 - \hat{p}_4) = 0.1875\,\sigma^2 + 0.1875\,\sigma^2 + 0.125\,\sigma^2$$
$$= 0.5\,\sigma^2$$

and for other 4x4 Latin-square designs variance-covariance matrices for the estimates of period effects are the same as above.

Subject effects:

Design 2.4.1

$$V\begin{bmatrix}\hat{s}_1\\ \hat{s}_2\\ \hat{s}_3\\ \hat{s}_4\end{bmatrix}=\begin{bmatrix}0.328 & -0.047 & -0.047 & -0.047\\ -0.047 & 0.328 & -0.047 & -0.047\\ -0.047 & -0.047 & 0.328 & -0.047\\ -0.047 & -0.047 & -0.047 & 0.328\end{bmatrix}\sigma^2$$

$$V(\hat{s}_1 - \hat{s}_2) = \ldots\ldots = V(\hat{s}_3 - \hat{s}_4) = 0.328\sigma^2 + 0.328\sigma^2 + 0.094\sigma^2$$

$$= 0.75\sigma^2$$

Design 2.4.2

$$V\begin{bmatrix}\hat{s}_1\\ \hat{s}_2\\ \hat{s}_3\\ \hat{s}_4\end{bmatrix} = \begin{bmatrix}0.302 & 0.027 & 0.027 & 0.027\\ 0.027 & 0.302 & 0.027 & 0.027\\ 0.027 & 0.027 & 0.302 & 0.027\\ 0.027 & 0.027 & 0.027 & 0.302\end{bmatrix}\sigma^2$$

$$V(\hat{s}_1 - \hat{s}_2) = \ldots\ldots = V(\hat{s}_3 - \hat{s}_4) = 0.55\sigma^2$$

Design 2.4.3

$$V\begin{bmatrix}\hat{s}_1\\ \hat{s}_2\\ \hat{s}_3\\ \hat{s}_4\end{bmatrix} = \begin{bmatrix}0.23 & -0.062 & -0.053 & -0.053\\ -0.062 & 0.23 & -0.053 & -0.053\\ -0.053 & -0.053 & 0.23 & -0.062\\ -0.053 & -0.053 & -0.062 & 0.23\end{bmatrix}\sigma^2$$

$$V(\hat{s}_1 - \hat{s}_2) = V(\hat{s}_3 - \hat{s}_4) = 0.584\sigma^2$$

$$V(\hat{s}_1 - \hat{s}_3) = V(\hat{s}_1 - \hat{s}_4) = V(\hat{s}_2 - \hat{s}_3) = V(\hat{s}_2 - \hat{s}_4) = 0.566\sigma^2$$

Design 2.4.4

$$V\begin{bmatrix}\hat{s}_1\\ \hat{s}_2\\ \hat{s}_3\\ \hat{s}_4\end{bmatrix} = \begin{bmatrix}0.243 & -0.082 & -0.032 & -0.082\\ -0.082 & 0.243 & -0.082 & -0.032\\ -0.032 & -0.082 & 0.243 & -0.082\\ -0.082 & -0.032 & -0.082 & 0.243\end{bmatrix}\sigma^2$$

$$V(\hat{s}_1 - \hat{s}_2) = V(\hat{s}_1 - \hat{s}_4) = V(\hat{s}_2 - \hat{s}_3) = V(\hat{s}_3 - \hat{s}_4) = 0.65\sigma^2$$

$$V(\hat{s}_1 - \hat{s}_3) = V(\hat{s}_2 - \hat{s}_4) = 0.55\sigma^2$$

Treatment effects:

Design 2.4.1

$$V\begin{bmatrix}\hat{t}_1\\ \hat{t}_2\\ \hat{t}_3\\ \hat{t}_4\end{bmatrix} = \begin{bmatrix}1.453 & 0.078 & 0.078 & 0.078\\ 0.078 & 1.453 & 0.078 & 0.078\\ 0.078 & 0.078 & 1.453 & 0.078\\ 0.078 & 0.078 & 0.078 & 1.453\end{bmatrix}\sigma^2$$

Design 2.4.2

$$V\begin{bmatrix}\hat{t}_1\\ \hat{t}_2\\ \hat{t}_3\\ \hat{t}_4\end{bmatrix} = \begin{bmatrix}0.628 & 0.353 & 0.353 & 0.353\\ 0.353 & 0.628 & 0.353 & 0.353\\ 0.353 & 0.353 & 0.628 & 0.353\\ 0.353 & 0.353 & 0.353 & 0.628\end{bmatrix}\sigma^2$$

Design 2.4.3

$$V\begin{bmatrix}\hat{t}_1\\ \hat{t}_2\\ \hat{t}_3\\ \hat{t}_4\end{bmatrix} = \begin{bmatrix}0.7195 & 0.262 & 0.353 & 0.353\\ 0.262 & 0.7195 & 0.353 & 0.353\\ 0.353 & 0.353 & 0.7195 & 0.353\\ 0.353 & 0.353 & 0.262 & 0.7195\end{bmatrix}\sigma^2$$

Design 2.4.4

$$V\begin{bmatrix}\hat{t}_1\\ \hat{t}_2\\ \hat{t}_3\\ \hat{t}_4\end{bmatrix} = \begin{bmatrix}0.903 & 0.078 & 0.628 & 0.078\\ 0.078 & 0.903 & 0.078 & 0.628\\ 0.628 & 0.078 & 0.903 & 0.078\\ 0.078 & 0.628 & 0.078 & 0.903\end{bmatrix}\sigma^2$$

Residual effects:

Design 2.4.1

$$V\begin{bmatrix}\hat{r}_1\\ \hat{r}_2\\ \hat{r}_3\\ \hat{r}_4\end{bmatrix}=\begin{bmatrix}1.688 & -0.313 & -0.313 & -0.313\\ -0.313 & 1.688 & -0.313 & -0.313\\ -0.313 & -0.313 & 1.688 & -0.313\\ -0.313 & -0.313 & -0.313 & 1.688\end{bmatrix}\sigma^2$$

Design 2.4.2

$$V\begin{bmatrix}\hat{r}_1\\ \hat{r}_2\\ \hat{r}_3\\ \hat{r}_4\end{bmatrix}=\begin{bmatrix}0.49 & 0.09 & 0.09 & 0.09\\ 0.09 & 0.49 & 0.09 & 0.09\\ 0.09 & 0.09 & 0.49 & 0.09\\ 0.09 & 0.09 & 0.09 & 0.49\end{bmatrix}\sigma^2$$

Design 2.4.3

$$V\begin{bmatrix}\hat{r}_1\\ \hat{r}_2\\ \hat{r}_3\\ \hat{r}_4\end{bmatrix}=\begin{bmatrix}0.555 & -0.11 & 0.02 & 0.02\\ -0.11 & 0.555 & 0.02 & 0.02\\ 0.02 & 0.02 & 0.555 & -0.11\\ 0.02 & 0.02 & -0.11 & 0.555\end{bmatrix}\sigma^2$$

Design 2.4.4

$$V\begin{bmatrix}\hat{r}_1\\ \hat{r}_2\\ \hat{r}_3\\ \hat{r}_4\end{bmatrix}=\begin{bmatrix}0.888 & -0.31 & 0.49 & -0.31\\ -0.31 & 0.888 & -0.31 & 0.49\\ 0.49 & -0.31 & 0.888 & -0.31\\ -0.31 & 0.49 & -0.31 & 0.888\end{bmatrix}\sigma^2$$

2.12.2 Variances of the differences between two treatment and two residual estimates.

Design 2.4.1

$$V(\hat{t}_1 - \hat{t}_2) = V(\hat{t}_1) + V(\hat{t}_2) - 2\,\mathrm{Cov}(\hat{t}_1\hat{t}_2)$$

$$= 1.453\,\sigma^2 + 1.453\,\sigma^2 - 2(0.078)\,\sigma^2$$

$$= 2.75\,\sigma^2$$

$$V(\hat{t}_2 - \hat{t}_3) = V(\hat{t}_2 - \hat{t}_4) = \ldots\ldots\ldots = V(\hat{t}_1 - \hat{t}_4) = 2.75\,\sigma^2$$

$$V(\hat{r}_1 - \hat{r}_2) = V(\hat{r}_1) + V(\hat{r}_2) - 2\,\mathrm{Cov}(\hat{r}_1\hat{r}_2)$$

$$= 1.688\,\sigma^2 + 1.688\,\sigma^2 - 2(-0.313)\,\sigma^2$$

$$= 4\,\sigma^2$$

$$V(\hat{r}_2 - \hat{r}_3) = \ldots\ldots\ldots\ldots = V(\hat{r}_1 - \hat{r}_4) = 4\,\sigma^2$$

Design 2.4.2

$$V(\hat{t}_1 - \hat{t}_2) = 0.628\,\sigma^2 + 0.628\,\sigma^2 - 2(0.353)\,\sigma^2$$

$$= 0.55\,\sigma^2$$

$$V(\hat{t}_1 - \hat{t}_3) = \ldots\ldots\ldots\ldots\ldots = V(\hat{t}_2 - \hat{t}_4) = 0.55\,\sigma^2$$

$$V(\hat{r}_1 - \hat{r}_2) = 0.49\,\sigma^2 + 0.49\,\sigma^2 - 2(0.09)\,\sigma^2$$

$$= 0.8\,\sigma^2$$

$$V(\hat{r}_1 - \hat{r}_3) = V(\hat{r}_2 - \hat{r}_4) = \ldots\ldots\ldots\ldots = 0.8\,\sigma^2$$

Design 2.4.3

$$V(\hat{t}_1 - \hat{t}_2) = 0.7195\,\sigma^2 + 0.7195\,\sigma^2 - 2(0.262)\,\sigma^2 = 0.915\,\sigma^2$$

$$V(\hat{t}_1 - \hat{t}_3) = 0.7195\,\sigma^2 + 0.7195\,\sigma^2 - 2(0.353)\,\sigma^2 = 0.733\,\sigma^2$$

$$V(\hat{t}_1 - \hat{t}_4) = V(\hat{t}_2 - \hat{t}_4) = \ldots\ldots\ldots\ldots\ldots = 0.733\,\sigma^2$$

$$V(\hat{t}_3-\hat{t}_4) = 0.915\sigma^2$$

$$V(\hat{r}_1-\hat{r}_2) = 0.555\sigma^2 + 0.555\sigma^2 - 2(-0.11)\sigma^2$$
$$= 1.33\sigma^2$$

$$V(\hat{r}_1-\hat{r}_3) = 0.555\sigma^2 + 0.555\sigma^2 - 2(0.02)\sigma^2$$
$$= 1.07\sigma^2$$

$$V(\hat{r}_1-\hat{r}_4) = V(\hat{r}_2-\hat{r}_4) = \ldots\ldots\ldots\ldots = 1.07\sigma^2$$

$$V(\hat{r}_3-\hat{r}_4) = 1.33\sigma^2$$

Design 2.4.4

$$V(\hat{t}_1-\hat{t}_2) = 0.903\sigma^2 + 0.903\sigma^2 - 2(0.078)\sigma^2$$
$$= 1.65\sigma^2$$

$$V(\hat{t}_1-\hat{t}_3) = 0.903\sigma^2 + 0.903\sigma^2 - 2(0.628)\sigma^2$$
$$= 0.55\sigma^2$$

$$V(\hat{t}_1-\hat{t}_4) = V(t_3-t_4) = \ldots\ldots\ldots = 1.65\sigma^2$$

$$V(\hat{t}_2-\hat{t}_4) = 0.55\sigma^2$$

$$V(\hat{r}_1-\hat{r}_2) = 0.888\sigma^2 + 0.888\sigma^2 - 2(-0.31)\sigma^2$$
$$= 2.4\sigma^2$$

$$V(\hat{r}_1-\hat{r}_3) = 0.888\sigma^2 + 0.888\sigma^2 - 2(0.49)\sigma^2$$
$$= 0.8\sigma^2$$

$$V(\hat{r}_1-\hat{r}_4) = V(t_3-t_4) = \ldots\ldots\ldots = 2.4\sigma^2$$

$$V(\hat{r}_2-\hat{r}_4) = 0.8\sigma^2$$

All the above results can be found more easily by using the theory of linear functions.

2.13 Variance of Linear Functions.

According to Cochran and Cox 1957 if

$$z = l_1y_1 + l_2y_2 + \dots\dots\dots + l_ny_n.$$

is a linear function such that $l_1 + l_2 + \dots\dots\dots + l_n = 0$ and $y_1, y_2, \dots\dots\dots\dots, y_n$ are individual observations, then the variance of any kind of this type of linear function can as a general rule be given as:

$$\sigma_z^2 = l_1^2\sigma^2 + l_2^2\sigma^2 + \dots\dots\dots\dots + l_n^2\sigma^2$$

Using the above rule the variance of the difference between any pair of parameter estimates can be obtained by first expressing such differences as linear functions of the parameter estimates. For example the linear function of the difference between two treatment effects can be written as

$$z = l_1\hat{t}_i + l_2\hat{t}_u = \hat{t}_i - \hat{t}_u \qquad \text{since } l_1 = 1 \text{ and } l_2 = -1 .$$

Now writing $\hat{t}_i - \hat{t}_u = aT_i + bT_u + \dots\dots\dots$

a and b being coefficients of ith and uth treatment totals, the variance of the difference

$$V(\hat{t}_i - \hat{t}_u) = (a - b)\sigma^2$$

Similarly the variance of the difference between residual effects can be found. This is a simpler and a quicker method of finding the variance of the differences between the estimates of the parameter effects. By making use of this rule we totally bypass the variance-covariance matrix of the parameter estimates and cut down a great deal of arithmetical calculations. Since the calculations made for the variances of the differences between parameter effects for designs 2.4.1 to 2.4.4 produce exactly the same answers we shall continue our calculations by the latter method.

The variance of the linear component can be found by using the same rule as stated on the previous page. Here in places of l_1, l_2 , ... l_n one could use the coefficients of orthogonal polynomials which will ensure the fulfilment of the necessary condition of orthogonality of the rule. The coefficients of the orthogonal polynomial can be readily and easily consulted from any standard statistical tables such as Fisher and Yates or Pearson and Hartley.

As an example the variance of the linear component for treatment effects for a 4x4 Latin-square design can be obtained by first expressing the four treatment estimates as a linear function

$$z = l_1\hat{t}_1 + l_2\hat{t}_2 + l_3\hat{t}_3 + l_4\hat{t}_4$$

and taking $l_1 = -3$, $l_2 = -1$, $l_3 = 1$, and $l_4 = 3$, we get

$$z = -3\hat{t}_1 - \hat{t}_2 + \hat{t}_3 + 3\hat{t}_4.$$

By substituting the values of the estimates in terms of treatment totals, etc., the above linear expression becomes

$$z = a_1T_1 + a_2T_2 + a_3T_3 + a_4T_4 + \ldots\ldots\ldots\ldots$$

where a_1, a_2, a_3, and a_4 are coefficients of treatment totals T_1 to T_4 respectively.

$$\sigma_z^2 = V(-3\hat{t}_1 - \hat{t}_2 + \hat{t}_3 + 3\hat{t}_4)$$
$$= (-3a_1 -a_2 + a_3 + 3a_4)\ \sigma^2$$

Following the above procedure we can find the variance of the linear component for treatment as well as residual effects for any size of the Latin-square under study for the purpose of comparing and choosing the most efficient design.

2.14 Efficiency Estimation.

The ratio of the reciprocals of the variances of the treatment differences and the residual differences measures the efficiency of the measurements of treatment differences over the measurement of residual differences.

To find the efficiency ratio for design 2.4.1 we calculate the variances of the differences between measurements by using the theory mentioned under 2.13.

2.14.1 Variance of the difference between two treatment direct effects:

$$V(\hat{t}_1-\hat{t}_2) = V\{ (1.375T_1 - 1.5R_4 - 0.375S_1 - 0.375P_1+0.125G) -(1.375T_2 - 1.5R_1 - 0.375S_2 - 0.375P_1 + 0.125G)\}$$

$$= (1.375 + 1.375) \sigma^2$$

$$= 2.75 \sigma^2$$

Similarly $V(\hat{t}_1 - \hat{t}_3) = V(\hat{t}_2 - \hat{t}_4) = \ldots\ldots = V(\hat{t}_3 - \hat{t}_4) = 2.75 \sigma^2$

It can be stated for this design that the variance of the difference between any two treatment effects is $2.75 \sigma^2$.

2.14.2 Variance of the difference between two residual effects:

$$V(\hat{r}_1 - \hat{r}_2) = V (2R_1 - 1.5T_2 + 0.5S_2 - 0.5P_1 - 0.25G) -(2R_2 - 1.5T_3 + 0.5S_3 + 0.5P_1 - 0.25G)$$

$$= (2 + 2) \sigma^2$$

$$= 4 \sigma^2$$

and $V(\hat{r}_1 - \hat{r}_3) = V(\hat{r}_2 - \hat{r}_3) = \ldots\ldots\ldots = V(\hat{r}_3 - \hat{r}_4) = 4 \sigma^2$

Variance of the difference between any two residual effects of this design is equal to $4 \sigma^2$.

∴ Efficiency Ratio (Treament / Residual) expressed as a percentage

$$= \frac{4\sigma^2}{2.75\sigma^2} \times 100$$

$$= 145\ \%.$$

The efficiency percentage shows that the treatment differences are measured 45 % more efficiently than the residual differences as far as this design is concerned.

It has been pointed out by Yates (1951), Lucas (1957) and Cochran (1957) that the residual effects are less accurately determined than the direct effects in the designs of this type. Approximately equal precision, in the estimation of direct and residual effects, can be attained by introducing an extra period to the design.

2.15 Variance of Linear Component.

There may be experiments where the treatments take the form of an ordered variable such as cows after 0, 1, 2 and 3 units of rations or patients after 2, 4, 6 and 8 hours of diets etc. There could also be cases where treatments consist of increasing amount of food or increasing amount of dose of medicine etc.

The variance of the linear component can, therefore, be investigated for both treatment as well as residual effects which can later be used for making comparison between different designs of the same order.

For design 2.4.1, a Latin-square of order 4x4, the coefficients of orthogonal polynomials (ξ_i), available from the Fisher and Yates tables, are given as:

Treatment applied	1	2	3	4
ξ_1	-3	-1	1	3
ξ_2	1	-1	-1	1
ξ_3	-1	3	-3	1

2.15.1 Variance of linear component of treatment direct effects.

$$\phi_t = -3\hat{t}_1 - \hat{t}_2 + \hat{t}_3 + 3\hat{t}_4 = -4.125\ T_1 - 1.375\ T_2 + 1.375\ T_3 + 4.125\ T_4 + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\begin{aligned} V(\phi_t) = V(-3\hat{t}_1 - \hat{t}_2 + \hat{t}_3 + 3\hat{t}_4) &= V(-3x{-}4.125T_1 - 1x{-}1.375T_2 + 1x1.375T_3 \\ &\quad + 3x4.125T_4 + \ldots\ldots\ldots\ldots) \\ &= V(12.375T_1 + 1.375T_2 + 1.375T_3 + 12.375T_4 + \ldots.) \\ &= 27.5\ \sigma^2. \end{aligned}$$

2.15.2 Variance of linear component of treatment residual effects.

$$\phi_r = -3\hat{r}_1 - \hat{r}_2 + \hat{r}_3 + 3\hat{r}_4 = -6R_1 - 2R_2 + 2R_3 + 6R_4 + \ldots\ldots\ldots\ldots$$

$$\begin{aligned} V(\phi_r) = V(-3\hat{r}_1 - \hat{r}_2 + \hat{r}_3 + 3\hat{r}_4) &= V(-3x{-}6R_1 - 1x{-}2R_2 + 1x2R_3 \\ &\quad + 3x6R_4 + \ldots\ldots\ldots\ldots) \\ &= V(18R_1 + 2R_2 + 2R_3 + 18R_4 + \ldots\ldots.) \\ &= 40\ \sigma^2. \end{aligned}$$

2.16 True Effects of Treatments.

In the study of treatment effects on subjects our main aim is to know the total effect of the treatments and in such cases the estimates of the residual effects can be used as a correction to arrive at the true effects of the treatments. For example, in a dairy farm, when a group of cows are tested with a number of feeds given on successive occasions, our main aim is to determine the particular feed which gives the best yield of milk. The total effect of treatment will, therefore, be the sum of the treatment direct effect and its residual effect $(\hat{t}_k + \hat{r}_l)$ where k=l.

2.16.1 Total effects of treatments for design 2.4.1 can be obtained from 2.10.2 and 2.10.3 as below.

$$\hat{t}_1 + \hat{r}_1 = 1.375T_1 - 1.5R_4 + 2R_1 - 1.5T_2 - 0.375S_1 + 0.125P_1 + 0.5S_2 - 0.125G.$$

$$\hat{t}_2 + \hat{r}_2 = 1.375T_2 - 1.5R_1 + 2R_2 - 1.5T_3 - 0.375S_2 + 0.125P_1 + 0.5S_3 - 0.125G.$$

$$\hat{t}_3 + \hat{r}_3 = 1.375T_3 - 1.5R_2 + 2R_3 - 1.5T_4 - 0.375S_3 + 0.125P_1 + 0.5S_4 - 0.125G.$$

$$\hat{t}_4 + \hat{r}_4 = 1.375T_4 - 1.5R_3 + 2R_4 - 1.5T_1 - 0.375S_4 + 0.125P_1 + 0.5S_1 - 0.125G.$$

2.16.2 Variance of the difference between permanent effects of two treatments of design 2.4.1 may be given as:

$$V\left\{(\hat{t}_1 + \hat{r}_1) - (\hat{t}_2 + \hat{r}_2)\right\} = V(1.375T_1 - 1.375T_2 + 2R_1 - 2R_2 + \ldots\ldots) = 6.75\,\sigma^2.$$

Chapter 3

Parameter Estimates when Residual Effects Persist for Longer Periods.

3.1 Residual Effects Persisting for Two Successive Periods.

Let us now assume that the residual effects of the treatments persist for two successive periods immediately after their application to the subjects and study the effect of this assumption on single Latin-square designs.

The terms and notations to be used in the following study follow from 2.7 and any new term or symbol used shall be explained as it appears.

The model under 2.3 is now changed to:

$$Y_{ij(kl)} = \mu + p_i + s_j + t_k + r_l + c_l + e_{ij(kl)}$$

r_l = First residual effect of lth treatment.

c_l = Second carry-over or residual effect of lth treatment.

$\Sigma \hat{r}_l = \Sigma \hat{c}_l = 0$ and all other assumptions of the model 2.3 are observed by the above model.

3.1.1 The normal equations for the estimation of period, subject, treatment, first residual and second residual effects are respectively:

$$\underline{P} = n \bar{Y} \underline{h} + n \hat{\underline{p}}$$

$$\underline{S} = n \bar{Y} \underline{h} + n \hat{\underline{s}} - \underline{\alpha}_1 \hat{\underline{r}} - (\underline{\alpha}_1 + \underline{\alpha}_2) \hat{\underline{c}}$$

$$\underline{T} = n \bar{Y} \underline{h} + n \hat{\underline{t}} + \underline{\beta}_1 \hat{\underline{r}} + \underline{\beta}_2 \hat{\underline{c}}$$

$$\underline{R} = (n-1) \bar{Y} \underline{h} + (n-1) \hat{\underline{r}} + \underline{\beta}_1' \hat{\underline{t}} + \underline{\beta}_3 \hat{\underline{c}} - \underline{h}\, \underline{e}_1' \hat{\underline{p}} - \underline{\alpha}_1' \hat{\underline{s}}$$

$$\underline{C} = (n-2) \bar{Y} \underline{h} + (n-2) \hat{\underline{c}} + \underline{\beta}_2' \hat{\underline{t}} + \underline{\beta}_3' \hat{\underline{r}} - \underline{h}\, \underline{e}_1' \hat{\underline{p}} - \underline{h}\, \underline{e}_2' \hat{\underline{p}} - (\underline{\alpha}_1' + \underline{\alpha}_2') \hat{\underline{s}}$$

$\underline{C}$ is the vector of totals for observations having second residual i.e. the element C_i is the total of observations receiving the second residual of the ith treatment.

$\underline{\beta}_1$ is an (n x n) matrix indicating which treatments precede and follow each other immediately.

$\underline{\beta}_2$ is an (n x n) matrix indicating which treatments precede and follow each other in the second positions.

$\underline{\beta}_3$ is an (n x n) matrix indicating which treatments precede and follow each other immediately while ignoring the last row of the design.

$\underline{\mathcal{L}}_1$ is a permutation (n x n) matrix of the treatments in the nth row of the design.

$\underline{\mathcal{L}}_2$ is a permutation (n x n) matrix of the treatments in the (n-1)th row of the design.

$\underline{\mathcal{L}}_1$ and $\underline{\mathcal{L}}_2$ are orthogonal matrices. Hence $\underline{\mathcal{L}}'_1 \underline{\mathcal{L}}_1 = \underline{\mathcal{L}}'_2 \underline{\mathcal{L}}_2 = \underline{I}$

3.1.2 Estimation of Parameter Effects.

Period effects being orthogonal with all other effects can be found directly as:

$$\underline{\hat{p}} = \frac{1}{n}(\underline{P} - n \bar{Y} \underline{h}) = \frac{1}{n} \underline{P} - \frac{G}{n^2} \underline{h}$$

For estimating the other effects their normal equations can be set out in the matrix form as follows:

$$\begin{bmatrix} \underline{\hat{s}} \\ \underline{\hat{t}} \\ \underline{\hat{r}} \\ \underline{\hat{c}} \end{bmatrix} = \begin{bmatrix} n\underline{I} & \underline{0} & -\underline{\mathcal{L}}_1 & -(\underline{\mathcal{L}}_1 + \underline{\mathcal{L}}_2) \\ \underline{0} & n\underline{I} & \underline{\beta}_1 & \underline{\beta}_2 \\ -\underline{\mathcal{L}}'_1 & \underline{\beta}'_1 & (n-1)\underline{I} & \underline{\beta}_3 \\ -(\underline{\mathcal{L}}'_1 + \underline{\mathcal{L}}'_2) & \underline{\beta}'_2 & \underline{\beta}'_3 & (n-2)\underline{I} \end{bmatrix}^{-1} \begin{bmatrix} \underline{S} - n\bar{Y}\underline{h} \\ \underline{T} - n\bar{Y}\underline{h} \\ \underline{R} - n\bar{Y}\underline{h} + \frac{P_1}{n}\underline{h} \\ \underline{C} - n\bar{Y}\underline{h} + \frac{P_1+P_2}{n}\underline{h} \end{bmatrix}$$

Let $\underline{\hat{\theta}} = \underline{M}^{-1} \underline{X}$, where $\underline{\hat{\theta}}$ is the column vector of parameter estimates, $\underline{M}$ is the normal equations matrix representing the coefficients of the parameters to be estimated and $\underline{X}$ is a column vector of the totals of observations and other known factors.

Therefore $\hat{\underline{\theta}}$ =

$$
\left[\begin{array}{llll}
\begin{array}{l} n(n-1)(n-2)\,\underline{W}^{-1} \\ -(n-2)\,\underline{\beta}_1\underline{W}^{-1}\underline{\beta}_1' \\ +\underline{\beta}_1'\underline{\beta}_2\underline{W}^{-1}\underline{\beta}_3' \\ +\underline{\beta}_1\underline{W}^{-1}\underline{\beta}_2'\underline{\beta}_3 \\ -(n-1)\,\underline{\beta}_2\underline{W}^{-1}\underline{\beta}_2' \\ -n\,\underline{\beta}_3\underline{W}^{-1}\underline{\beta}_3' \end{array} &
\begin{array}{l} -(n-2)\underline{\mathcal{L}}_1'\underline{W}^{-1}\underline{\beta}_1 \\ -(n-1)\,\underline{\mathcal{L}}_1'\underline{W}^{-1}\underline{\beta}_2 \\ -(n-1)\,\underline{\mathcal{L}}_2'\underline{W}^{-1}\underline{\beta}_2 \\ +\underline{\beta}_1\underline{\mathcal{L}}_1'\underline{W}^{-1}\underline{\beta}_3 \\ +\underline{\beta}_2\underline{\mathcal{L}}_2'\underline{W}^{-1}\underline{\beta}_3 \\ +\underline{\beta}_2\underline{\mathcal{L}}_1'\underline{W}^{-1}\underline{\beta}_3' \end{array} &
\begin{array}{l} n(n-2)\,\underline{\mathcal{L}}_1'\underline{W}^{-1} \\ +\underline{\beta}_1'\underline{\mathcal{L}}_1'\underline{W}^{-1}\underline{\beta}_2 \\ +\underline{\beta}_1'\underline{\mathcal{L}}_2'\underline{W}^{-1}\underline{\beta}_2 \\ -n\,\underline{\mathcal{L}}_1'\underline{W}^{-1}\underline{\beta}_3 \\ -n\,\underline{\mathcal{L}}_2'\underline{W}^{-1}\underline{\beta}_3 \\ -\underline{\beta}_2'\underline{\mathcal{L}}_1\underline{W}^{-1}\underline{\beta}_2 \end{array} &
\begin{array}{l} -n\underline{\mathcal{L}}_1'\underline{W}^{-1}\underline{\beta}_3 \\ +\underline{\beta}_1\underline{\mathcal{L}}_1'\underline{W}^{-1}\underline{\beta}_2' \\ -\underline{\beta}_1\underline{\mathcal{L}}_1'\underline{W}^{-1}\underline{\beta}_1' \\ -\underline{\beta}_1\underline{\mathcal{L}}_2'\underline{W}^{-1}\underline{\beta}_1' \\ +n(n-1)\,\underline{\mathcal{L}}_1'\underline{W}^{-1} \\ +n(n-1)\,\underline{\mathcal{L}}_2'\underline{W}^{-1} \end{array} \\
\\
\begin{array}{l} -(n-2)\,\underline{\beta}_1'\underline{W}^{-1}\underline{\mathcal{L}}_1 \\ -(n-1)\,\underline{\beta}_2'\underline{W}^{-1}\underline{\mathcal{L}}_1 \\ -(n-1)\,\underline{\beta}_2'\underline{W}^{-1}\underline{\mathcal{L}}_2 \\ +\underline{\beta}_3'\underline{W}^{-1}\underline{\mathcal{L}}_1\underline{\beta}_1' \\ +\underline{\beta}_3'\underline{W}^{-1}\underline{\mathcal{L}}_2\underline{\beta}_2' \\ +\underline{\beta}_3'\underline{W}^{-1}\underline{\mathcal{L}}_1\underline{\beta}_2' \end{array} &
\begin{array}{l} n(n-1)(n-2)\underline{W}^{-1} \\ -(3n-4)\,\underline{W}^{-1} \\ -n\,\underline{\beta}_3\underline{W}^{-1}\underline{\beta}_3' \\ -2(n-1)\,\underline{\mathcal{L}}_1\underline{W}^{-1}\underline{\mathcal{L}}_2' \\ +\underline{\mathcal{L}}_1'\underline{\beta}_3\underline{W}^{-1}\underline{\mathcal{L}}_2 \\ +\underline{\mathcal{L}}_1\underline{\beta}_3\underline{W}^{-1}\underline{\mathcal{L}}_2' \\ +\underline{\beta}_3'\underline{W}^{-1}+\underline{\beta}_3\underline{W}^{-1} \end{array} &
\begin{array}{l} -n(n-1)\,\underline{\beta}_1'\underline{W}^{-1} \\ +n\,\underline{\beta}_2'\underline{W}^{-1}\underline{\beta}_3 \\ -\underline{\beta}_2'\underline{W}^{-1}+2\underline{\beta}_1'\underline{W}^{-1} \\ -\underline{\mathcal{L}}_1'\underline{\beta}_2'\underline{W}^{-1}\underline{\mathcal{L}}_2 \\ +2\,\underline{\mathcal{L}}_1'\underline{\beta}_1'\underline{W}^{-1}\underline{\mathcal{L}}_2 \end{array} &
\begin{array}{l} n\,\underline{\beta}_1'\underline{W}^{-1}\underline{\beta}_3' \\ -n(n-1)\,\underline{\beta}_2'\underline{W}^{-1} \\ -\underline{\beta}_1'\underline{W}^{-1}+\underline{\beta}_2'\underline{W}^{-1} \\ -\underline{\mathcal{L}}_1\underline{W}^{-1}\underline{\beta}_1\underline{\mathcal{L}}_2' \end{array} \\
\\
\begin{array}{l} n(n-2)\,\underline{W}^{-1}\underline{\mathcal{L}}_1 \\ +\underline{\beta}_2'\underline{W}^{-1}\underline{\mathcal{L}}_1\underline{\beta}_1 \\ +\underline{\beta}_2'\underline{W}^{-1}\underline{\mathcal{L}}_2\underline{\beta}_1 \\ -n\,\underline{\beta}_3'\underline{W}^{-1}\underline{\mathcal{L}}_1 \\ -n\,\underline{\beta}_3'\underline{W}^{-1}\underline{\mathcal{L}}_2 \\ -\underline{\beta}_2'\underline{W}^{-1}\underline{\mathcal{L}}_1\underline{\beta}_2 \end{array} &
\begin{array}{l} -n(n-1)\,\underline{W}^{-1}\underline{\beta}_1 \\ +n\,\underline{\beta}_3'\underline{W}^{-1}\underline{\beta}_2 \\ -\underline{W}^{-1}\underline{\beta}_2+2\underline{W}^{-1}\underline{\beta}_1 \\ -\underline{\mathcal{L}}_2'\underline{W}^{-1}\underline{\beta}_2\underline{\mathcal{L}}_1 \\ +2\,\underline{\mathcal{L}}_2'\underline{W}^{-1}\underline{\beta}_1\underline{\mathcal{L}}_1 \end{array} &
\begin{array}{l} n^2(n-1)\,\underline{W}^{-1} \\ -2n\,\underline{W}^{-1} \\ -2\,\underline{\mathcal{L}}_1'\underline{W}^{-1}\underline{\mathcal{L}}_2' \\ -n\,\underline{\beta}_2'\underline{W}^{-1}\underline{\beta}_2 \end{array} &
\begin{array}{l} +n\,\underline{W}^{-1} \\ +n\,\underline{\mathcal{L}}_1\underline{W}^{-1}\underline{\mathcal{L}}_2' \\ +n\,\underline{\beta}_1\underline{W}^{-1}\underline{\beta}_2' \\ -n^2\underline{\beta}_3'\underline{W}^{-1} \end{array} \\
\\
\begin{array}{l} -n\,\underline{\beta}_3'\underline{W}^{-1}\underline{\mathcal{L}}_1 \\ +\underline{\beta}_2\underline{W}^{-1}\underline{\mathcal{L}}_1\underline{\beta}_1' \\ -\underline{\beta}_1\underline{W}^{-1}\underline{\mathcal{L}}_1\underline{\beta}_1' \\ -\underline{\beta}_1\underline{W}^{-1}\underline{\mathcal{L}}_2\underline{\beta}_1 \\ +n(n-1)\,\underline{W}^{-1}\underline{\mathcal{L}}_1 \\ +n(n-1)\,\underline{W}^{-1}\underline{\mathcal{L}}_2 \end{array} &
\begin{array}{l} n\,\underline{\beta}_3\underline{W}^{-1}\underline{\beta}_1 \\ -n(n-1)\,\underline{W}^{-1}\underline{\beta}_2 \\ -\underline{W}^{-1}\underline{\beta}_1+\underline{W}^{-1}\underline{\beta}_2 \\ -\underline{\mathcal{L}}_2\underline{\beta}_1\underline{W}^{-1}\underline{\mathcal{L}}_1' \end{array} &
\begin{array}{l} +n\,\underline{W}^{-1} \\ +n\,\underline{\mathcal{L}}_2\underline{W}^{-1}\underline{\mathcal{L}}_1' \\ +n\,\underline{\beta}_2\underline{W}^{-1}\underline{\beta}_1' \\ -n^2\underline{W}^{-1}\underline{\beta}_3 \end{array} &
\begin{array}{l} n^2(n-1)\,\underline{W}^{-1} \\ -n\,\underline{W}^{-1} \\ -n\,\underline{\beta}_1'\underline{W}^{-1}\underline{\beta}_1 \end{array}
\end{array}\right] \underline{X}
$$

where $$\underline{W} = (n^4 - 3n^3 - n^2 + 2n)\,\underline{I} + \left\{1 - n(n-1)\right\}\underline{\beta}_2'\underline{\beta}_2 + 2n\,\underline{\beta}_3 + 2\,\underline{\beta}_1 - n(n-1)\,\underline{\beta}_1'\underline{\beta}_1 - \underline{\beta}_1'\underline{\beta}_2 - \underline{\beta}_1\underline{\beta}_2' + n\,\underline{\beta}_1\underline{\beta}_2'\underline{\beta}_3 + n\,\underline{\beta}_1'\underline{\beta}_2\underline{\beta}_3' + \left\{n\,\underline{\beta}_3 + 2\underline{\beta}_1 - \underline{\beta}_1\underline{\beta}_2' + \underline{\beta}_3 - \underline{\beta}_1'\underline{\beta}_2 - 2n(n-1)\right\}\underline{\alpha}_1'\underline{\alpha}_2 .$$

3.1.3 As an example, the matrix form of the normal equations of design 25(9.3.1), when residual effects persist for two successive periods, can be given as:

$$\begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ \hat{s}_4 \\ \hat{s}_5 \\ \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \\ \hat{t}_4 \\ \hat{t}_5 \\ \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \\ \hat{r}_5 \\ \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \\ \hat{c}_5 \end{bmatrix} = \left[\begin{array}{ccccc ccccc ccccc ccccc}
5&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&-1&-1\\
0&5&0&0&0&0&0&0&0&0&-1&0&0&0&0&-1&0&-1&0&0\\
0&0&5&0&0&0&0&0&0&0&0&-1&0&0&0&0&-1&0&0&-1\\
0&0&0&5&0&0&0&0&0&0&0&0&-1&0&0&0&-1&-1&0&0\\
0&0&0&0&5&0&0&0&0&0&0&0&0&-1&0&-1&0&0&-1&0\\
0&0&0&0&0&5&0&0&0&0&0&1&1&2&0&0&0&2&1&0\\
0&0&0&0&0&0&5&0&0&0&1&0&1&0&2&2&0&0&0&1\\
0&0&0&0&0&0&0&5&0&0&0&2&0&1&1&1&0&0&0&2\\
0&0&0&0&0&0&0&0&5&0&1&0&2&0&1&0&3&0&0&0\\
0&0&0&0&0&0&0&0&0&5&2&1&0&1&0&0&0&1&2&0\\
0&-1&0&0&0&0&1&0&1&2&4&0&0&0&0&0&1&0&2&0\\
0&0&-1&0&0&1&0&2&0&1&0&4&0&0&0&1&0&1&0&1\\
0&0&0&-1&0&1&1&0&2&0&0&0&4&0&0&0&1&0&1&1\\
0&0&0&0&-1&2&0&1&0&1&0&0&0&4&0&0&0&2&0&1\\
-1&0&0&0&0&0&2&1&1&0&0&0&0&0&4&2&1&0&0&0\\
0&-1&0&0&-1&0&2&1&0&0&0&1&0&0&2&3&0&0&0&0\\
0&0&-1&-1&0&0&0&0&3&0&1&0&1&0&1&0&3&0&0&0\\
-1&0&0&-1&0&2&0&0&0&1&0&1&0&2&0&0&0&3&0&0\\
0&-1&0&0&-1&1&0&0&0&2&2&0&1&0&0&0&0&0&3&0\\
-1&0&-1&0&0&0&1&2&0&0&0&1&1&1&0&0&0&0&0&3
\end{array}\right]^{-1} \begin{bmatrix} S_1-0.2G \\ S_2-0.2G \\ S_3-0.2G \\ S_4-0.2G \\ S_5-0.2G \\ T_1-0.2G \\ T_2-0.2G \\ T_3-0.2G \\ T_4-0.2G \\ T_5-0.2G \\ R_1-0.2G+0.2P_1 \\ R_2-0.2G+0.2P_1 \\ R_3-0.2G+0.2P_1 \\ R_4-0.2G+0.2P_1 \\ R_5-0.2G+0.2P_1 \\ C_1-0.2G+0.2(P_1+P_2) \\ C_2-0.2G+0.2(P_1+P_2) \\ C_3-0.2G+0.2(P_1+P_2) \\ C_4-0.2G+0.2(P_1+P_2) \\ C_5-0.2G+0.2(P_1+P_2) \end{bmatrix}$$

The above matrix can be inverted by computer using the programme in Appendix I.

Period effects can be found directly as suggested under 3.1.2.

3.2 Residual Effects Persisting for Longer Periods.

The process developed under 3.1 can systematically be extended for extracting the residual effects of treatments, persisting for any number of periods, beyond their period of application to the subjects.

By some changes and rearrangements of the symbols, the system of writing the normal equations matrix, for subjects, treatments and residual effects for longer periods, can be expressed as:

$$
\begin{bmatrix} \hat{\underline{s}} \\ \hat{\underline{t}} \\ \hat{\underline{r}}\cdot \\ \hat{\underline{r}}\cdot\cdot \\ \hat{\underline{r}}\cdot\cdot\cdot \\ \hat{\underline{r}}\cdot\cdot\cdot\cdot \\ \vdots \end{bmatrix}
=
\begin{bmatrix}
n\underline{I} & \underline{0} & -\underline{\alpha}_1 & -\underline{\alpha}_1-\underline{\alpha}_2 & -\underline{\alpha}_1-\underline{\alpha}_2-\underline{\alpha}_3 & -\underline{\alpha}_1-\underline{\alpha}_2-\underline{\alpha}_3-\underline{\alpha}_4 & \cdots \\
\underline{0} & n\underline{I} & \underline{\beta}_1 & \underline{\beta}_2 & \underline{\beta}_3 & \underline{\beta}_4 & \cdots \\
-\underline{\alpha}_1' & \underline{\beta}_1' & (n-1)\underline{I} & \underline{\gamma}_1 & \underline{\delta}_1 & \underline{\delta}_2 & \cdots \\
-\underline{\alpha}_1'-\underline{\alpha}_2' & \underline{\beta}_2' & \underline{\gamma}_1' & (n-2)\underline{I} & \underline{\gamma}_2 & \underline{\delta}_1\cdot & \cdots \\
-\underline{\alpha}_1'-\underline{\alpha}_2'-\underline{\alpha}_3' & \underline{\beta}_3' & \underline{\delta}_1' & \underline{\gamma}_2' & (n-3)\underline{I} & \underline{\gamma}_3 & \cdots \\
-\underline{\alpha}_1'-\underline{\alpha}_2'-\underline{\alpha}_3'-\underline{\alpha}_4' & \underline{\beta}_4' & \underline{\delta}_2' & \underline{\delta}_1'\cdot & \underline{\gamma}_3' & (n-4)\underline{I} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}^{-1}
\begin{bmatrix}
\underline{S} - n\bar{Y}\underline{h} \\
\underline{T} - n\bar{Y}\underline{h} \\
\underline{R}\cdot - n\bar{Y}\underline{h} + \frac{1}{n}P_1\underline{h} \\
\underline{R}\cdot\cdot - n\bar{Y}\underline{h} + \frac{1}{n}(P_1 + P_2)\underline{h} \\
\underline{R}\cdot\cdot\cdot - n\bar{Y}\underline{h} + \frac{1}{n}(P_1 + P_2 + P_3)\underline{h} \\
\underline{R}\cdot\cdot\cdot\cdot - n\bar{Y}\underline{h} + \frac{1}{n}(P_1 + P_2 + P_3 + P_4)\underline{h} \\
\vdots \quad \vdots
\end{bmatrix}
$$

$\underline{\alpha}_1$ is a permutation matrix of the nth row of the Latin-square.

$\underline{\alpha}_2$ " " " " " " (n-1)th " " " .

$\underline{\alpha}_3$ " " " " " " (n-2)th " " " .

$\underline{\alpha}_4$ " " " " " " (n-3)th " " " .

- -
- -

$\underline{\alpha}_i$ " " " " " " (n-i+1)th " " " " .

$\underline{\beta}_1$ is a matrix indicating which treatments precede and follow each other immediately.

$\underline{\beta}_2$ is a matrix indicating which treatments precede and follow each other by one period.

$\underline{\beta}_3$ is a matrix indicating which treatments precede and follow each other by two periods.

$\underline{\beta}_4$ is a matrix indicating which treatments precede and follow each other by three periods.

- -
- -

$\underline{\beta}_i$ is a matrix indicating which treatments precede and follow each other by (i-1) periods.

$\underline{\gamma}_1$ is a matrix indicating which treatments precede and follow each other in the absence of the nth row.

$\underline{\gamma}_2$ is a matrix indicating which treatments precede and follow each other in the absence of the nth and (n-1)th rows.

$\underline{\gamma}_3$ is a matrix indicating which treatments precede and follow each other in the absence of the nth, (n-1)th and (n-2)the rows.

- -
- -

$\underline{\gamma}_i$ is a matrix indicating which treatments precede and follow each other in the absence of the (n-i+1)th to nth rows.

$\underline{\delta}_1$ is a matrix indicating which treatments precede and follow each other by one period in the absence of the nth row.

$\underline{\delta}_2$ is a matrix indicating which treatments precede and follow each other by two periods in the absence of the nth row.

- -
- -

$\underline{\delta}_i$ is a matrix indicating which treatments precede and follow each other by i periods in the absence of the nth row.

$\underline{\delta}_1.$ is a matrix indicating which treatments precede and follow each other by (1+1) periods in the absence of the nth and (n-1)th rows.

- -
- -

$\underline{\delta}_i.$ is a matrix indicating which treatments precede and follow each other by (i+1) periods in the absence of the nth and (n-1)th rows.

$\underline{\delta}_1..$ can be used to denote a matrix indicating which treatments precede and follow each other by three periods in the absence of the nth, (n-1)th and (n-2)th rows.

$\underline{\delta}_i..$ can be used to denote a matrix indicating which treatments precede and follow each other by (i+2) periods in the absence of the nth, (n-1)th and (n-2)th rows.

Following the above mentioned system the coefficient matrix can be written without any complication. The numerical analysis is then carried out, after inverting the coefficient matrix on a computer, as suggested earlier.

$\underline{R}.$ is a vector of the totals of observations containing the first residual effect of the treatments.

$\underline{R}..$ is a vector of the totals of observations containing the second residual effect of the treatments.

$\underline{R}......$(i dots) is a vector of the totals of observations containing the ith residual effect of the treatments.

$\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$, $\underline{\delta}$, $\underline{\delta}.$, and $\underline{\delta}..$ are all (n x n) matrices.

3.3 Residual Effect as a fraction of Direct Effect.

It is often convenient to assume that the residual effect of treatment t_i, in the following period, is ct_i, where c is a fraction ranging between 0 and and 1.

The normal equations for the parameter estimates then take the form:

$$\underline{P} = n\hat{\mu}\,\underline{h} + n\,\hat{\underline{p}}$$

$$\underline{S} = n\hat{\mu}\,\underline{h} + n\,\hat{\underline{s}} - c\,\underline{\alpha}\,\hat{\underline{t}}$$

$$\underline{T} = n\hat{\mu}\,\underline{h} + n\,\hat{\underline{t}} + c\,\underline{\beta}\,\hat{\underline{t}}$$

$$\underline{R} = n\hat{\mu}\,\underline{h} + (n-1)\,c\,\hat{\underline{t}} + \underline{\beta}'\,\hat{\underline{t}} - \underline{h}\,\underline{e}_1'\,\hat{\underline{p}} - \underline{\alpha}'\hat{\underline{s}}$$

Rearranging and simplifying the above equations we get:

$$n\,\hat{\underline{s}} - c\,\underline{\alpha}\,\hat{\underline{t}} = \underline{S} - n\,\bar{Y}\,\underline{h}$$

$$n\,\hat{\underline{t}} + c\,\underline{\beta}\,\hat{\underline{t}} = \underline{T} - n\,\bar{Y}\,\underline{h}$$

$$(n-1)\,c\,\hat{\underline{t}} + \underline{\beta}'\,\hat{\underline{t}} - \underline{\alpha}'\hat{\underline{s}} = \underline{R} - (n-1)\,\bar{Y}\,\underline{h} + \frac{P}{n}1\,\underline{h} - \bar{Y}$$

$$= \underline{R} - n\,\bar{Y}\,\underline{h} + \frac{P}{n}1\,\underline{h}$$

Period effects being orthogonal to the other effects can be found directly. $\hat{\underline{p}} = \underline{P} - n\,\bar{Y}\,\underline{h}$

Since there are now only two unknowns, $\underline{t}$ and $\underline{s}$, we further simplify by adding the last two equations in the above set.

$$\therefore \quad n\,\hat{\underline{s}} - c\,\underline{\alpha}\,\hat{\underline{t}} = \underline{S} - n\,\bar{Y}\,\underline{h}$$

$$\left[\left\{n + c(n-1)\right\}\underline{I} + \underline{\beta}' + c\,\underline{\beta}\right]\hat{\underline{t}} - \underline{\alpha}'\hat{\underline{s}} = \underline{T} + \underline{R} - 2n\,\bar{Y}\,\underline{h} + \frac{P}{n}1\,\underline{h}$$

The terms and symbols used in this section have already been defined under 2.7.

3.3.1

$$\begin{bmatrix} \hat{\underline{s}} \\ \hat{\underline{t}} \end{bmatrix} = \begin{bmatrix} n\,\underline{I} & -c\,\underline{\mathcal{L}} \\ -\underline{\mathcal{L}}' & \{n + c(n-1)\}\,\underline{I} + \underline{\beta}' + c\,\underline{\beta} \end{bmatrix}^{-1} \begin{bmatrix} \underline{S} - n\,\bar{Y}\,\underline{h} \\ \underline{T} + \underline{R} - 2n\,\bar{Y}\,\underline{h} + \frac{P}{n}1\,\underline{h} \end{bmatrix}$$

$$= \underline{D}^{-1} \begin{bmatrix} \{n + c(n-1)\}\,\underline{I} + \underline{\beta}' + c\,\underline{\beta} & c\,\underline{\mathcal{L}} \\ \underline{\mathcal{L}}' & n\,\underline{I} \end{bmatrix} \begin{bmatrix} \underline{S} - n\,\bar{Y}\,\underline{h} \\ \underline{T} + \underline{R} - 2n\,\bar{Y}\,\underline{h} + \frac{P}{n}1\,\underline{h} \end{bmatrix}$$

where $D = n\left[\{n + c(n-1)\}\,\underline{I} + \underline{\beta}' + c\,\underline{\beta}\right] - c\,\underline{\mathcal{L}}'\underline{\mathcal{L}}$

or $= (n^2 + c\,n^2 - c\,n - c)\,\underline{I} + \underline{\beta}' + c\,\underline{\beta}$

Therefore from 3.1.1 we get:

$$\hat{\underline{s}} = \underline{D}^{-1}\Big[\{n\underline{I} + c(n-1)\underline{I} + \underline{\beta}' + c\,\underline{\beta}\}\{\underline{S} - n\,\bar{Y}\,\underline{h}\} + c\,\underline{\mathcal{L}}\,(\underline{T} + \underline{R} - 2n\,\bar{Y}\,\underline{h} + \frac{P}{n}1\,\underline{h})\Big]$$

$$\hat{\underline{t}} = \underline{D}^{-1}\Big[\underline{\mathcal{L}}'(\underline{S} - n\,\bar{Y}\,\underline{h}) + n(\underline{T} + \underline{R} - 2n\,\bar{Y}\,\underline{h} + \frac{P}{n}1\,\underline{h})\Big]$$

and $\hat{\underline{r}} = c\,\hat{\underline{t}} = c\,\underline{D}^{-1}\Big[\underline{\mathcal{L}}'(\underline{S} - n\,\bar{Y}\,\underline{h}) + n(\underline{T} + \underline{R} - 2n\,\bar{Y}\,\underline{h} + \frac{P}{n}1\,\underline{h})\Big]$

3.3.2 If c, the residual fraction, varies from treatment to treatment, that is c_i is the residual fraction for t_i, then c is replaced by $\underline{c}$ when estimating the parameter effects. $\underline{c}$ is a diagonal matrix given by

$$\underline{c} = \begin{bmatrix} c_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & 0 & \cdots & 0 \\ \cdots & & & & & \cdots \\ 0 & 0 & 0 & 0 & \cdots & c_n \end{bmatrix}$$

The matrix form of the normal equations is

$$\begin{bmatrix} \underline{\hat{s}} \\ \underline{\hat{t}} \end{bmatrix} = \begin{bmatrix} n\,\underline{I} & -\underline{\mathcal{L}}\,\underline{c} \\ -\underline{\mathcal{L}}' & n\,\underline{I} + (n-1)\,\underline{c} + \underline{\beta}' + \underline{\beta}\,\underline{c} \end{bmatrix}^{-1} \begin{bmatrix} \underline{S} - n\,\bar{Y}\,\underline{h} \\ \underline{T} + \underline{R} - 2n\bar{Y}\underline{h} + \frac{P}{n}1\,\underline{h} \end{bmatrix}$$

and we get

$$\begin{bmatrix} \underline{\hat{s}} \\ \underline{\hat{t}} \end{bmatrix} = \underline{D}^{-1} \begin{bmatrix} n\,\underline{I} + (n-1)\,\underline{c} + \underline{\beta}' + \underline{\beta}\,\underline{c} & \underline{\mathcal{L}}\,\underline{c} \\ \underline{\mathcal{L}}' & n\,\underline{I} \end{bmatrix} \begin{bmatrix} \underline{S} - n\,\bar{Y}\,\underline{h} \\ \underline{T} + \underline{R} - 2n\bar{Y}\underline{h} + \frac{P}{n}1\,\underline{h} \end{bmatrix}$$

where $\underline{D} = n\left[\left\{n\,\underline{I} + (n-1)\,\underline{c} + \underline{\beta}' + \underline{\beta}\,\underline{c}\right\} - \underline{c}\,\underline{\mathcal{L}}'\underline{\mathcal{L}}\right.$

$$= n^2\underline{I} + (n^2 - n - 1)\,\underline{c} + n\underline{\beta}' + n\underline{\beta}\,\underline{c}$$

Therefore the parameter estimates can now be written as:

$$\underline{\hat{s}} = \underline{D}^{-1}\Big[\left\{n\,\underline{I} + (n-1)\,\underline{c} + \underline{\beta}' + \underline{\beta}\,\underline{c}\right\}(\underline{S} - n\,\bar{Y}\,\underline{h}) + \underline{\mathcal{L}}\,\underline{c}\;(\underline{T} + \underline{R} - 2n\,\bar{Y}\,\underline{h} + \tfrac{P}{n}1\,\underline{h})\Big]$$

$$\underline{\hat{t}} = \underline{D}^{-1}\left\{\underline{\mathcal{L}}'(\underline{S} - n\,\bar{Y}\,\underline{h}) + n\,(\underline{T} + \underline{R} - 2n\,\bar{Y}\,\underline{h} + \tfrac{P}{n}1\,\underline{h})\right\}$$

$$\underline{\hat{r}} = \underline{c}\,\underline{\hat{t}} = \underline{c}\,\underline{D}^{-1}\left\{\underline{\mathcal{L}}'(\underline{S} - n\,\bar{Y}\,\underline{h}) + n\,(\underline{T} + \underline{R} - 2n\,\bar{Y}\,\underline{h} + \tfrac{P}{n}1\,\underline{h})\right\}$$

Similarly it can be assumed that the residual fraction varies according to treatment as well as period of application. But for our present study we shall adhere to the usual assumption of the existance of an unknown quantity of a residual effect for a treatment.

Chapter 4

Subjects carrying Residual Effects in the Initial Period of the Application of Treatments.

4.1 Subjects Prepared for the Experiments.

In certain situations it may be convenient to treat the subjects in such a way that they carry residual effects of the treatments to be applied in the nth period, before the actual experiment is started. This may be done by simply treating the subjects, in period zero, with the treatments they are to receive in the nth period. Thus the residual effects which are lost at the end of the experiment are gained in period one.

Under this assumption the normal equations for the parameter estimates are:

$$\underline{P} = n\hat{\mu}\,\underline{h} + n\,\hat{\underline{p}}$$

$$\underline{S} = n\hat{\mu}\,\underline{h} + n\,\hat{\underline{s}}$$

$$\underline{T} = n\hat{\mu}\,\underline{h} + n\,\hat{\underline{t}} + \underline{\lambda}\,\hat{\underline{r}}$$

$$\underline{R} = n\hat{\mu}\,\underline{h} + n\,\hat{\underline{r}} + \underline{\lambda}'\,\hat{\underline{t}}$$

where $\underline{\lambda}$ is an (n x n) matrix indicating which treatments precede and follow each other including treatments in period zero.

Hence $\hat{\underline{p}} = \underline{P} - n\bar{Y}\,\underline{h}$

$$\hat{\underline{s}} = \underline{S} - n\bar{Y}\,\underline{h}$$

and

$$\begin{bmatrix} \hat{\underline{t}} \\ \hat{\underline{r}} \end{bmatrix} = \begin{bmatrix} n\,\underline{I} & \underline{\lambda} \\ \underline{\lambda}' & n\,\underline{I} \end{bmatrix}^{-1} \begin{bmatrix} \underline{T} - n\bar{Y}\,\underline{h} \\ \underline{R} - n\bar{Y}\,\underline{h} \end{bmatrix}$$

$$= \begin{bmatrix} n\,\underline{A}^{-1} & -\underline{\lambda}\,\underline{A}^{-1} \\ -\underline{A}^{-1}\underline{\lambda}' & n\,\underline{A}^{-1} \end{bmatrix} \begin{bmatrix} \underline{T} - n\bar{Y}\,\underline{h} \\ \underline{R} - n\bar{Y}\,\underline{h} \end{bmatrix}$$

Here $\underline{A} = n^2\,\underline{I} - \underline{\lambda}'\,\underline{\lambda}$

We obtain

$$\hat{\underline{t}} = n\,\underline{A}^{-1}\,(\,\underline{T} - n\,\bar{Y}\,\underline{h}\,) - \underline{\lambda}\,\underline{A}^{-1}(\,\underline{R} - n\,\bar{Y}\,\underline{h}\,)$$

$$\hat{\underline{r}} = n\,\underline{A}^{-1}\,(\,\underline{R} - n\,\bar{Y}\,\underline{h}\,) - \underline{A}^{-1}\,\underline{\lambda}'\,(\,\underline{T} - n\,\bar{Y}\,\underline{h}\,)$$

4.1.1 As an example taking design 2.4.2 to explain the numerical calculations of the parameter estimates ($\hat{\underline{p}}$, $\hat{\underline{s}}$, $\hat{\underline{t}}$ and $\hat{\underline{r}}$), assuming the presence of residual effects in period one, when prepared subjects have been used for the design.

The period and subject effects are calculated directly from their normal equations.

$\hat{\underline{p}} = 0.25\,\underline{P} - \bar{Y}\,\underline{h}$ giving

$$\hat{p}_1 = 0.25\,P_1 - 0.0625\,G$$
$$\hat{p}_2 = 0.25\,P_2 - 0.0625\,G$$
$$\hat{p}_3 = 0.25\,P_3 - 0.0625\,G$$
$$\hat{p}_4 = 0.25\,P_4 - 0.0625\,G$$

$\hat{\underline{s}} = 0.25\,\underline{S} - \bar{Y}\,\underline{h}$ giving

$$\hat{s}_1 = 0.25\,S_1 - 0.0625\,G$$
$$\hat{s}_2 = 0.25\,S_2 - 0.0625\,G$$
$$\hat{s}_3 = 0.25\,S_3 - 0.0625\,G$$
$$\hat{s}_4 = 0.25\,S_4 - 0.0625\,G$$

For calculating the treatment and residual effects we solve their normal equations simultaneously as follows:

$$\begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \\ \hat{t}_4 \\ \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 4 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 4 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 4 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 4 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 4 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 4 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}^{-1} \begin{bmatrix} T_1 - 0.25\,G \\ T_2 - 0.25\,G \\ T_3 - 0.25\,G \\ T_4 - 0.25\,G \\ R_1 - 0.25\,G \\ R_2 - 0.25\,G \\ R_3 - 0.25\,G \\ R_4 - 0.25\,G \end{bmatrix}$$

The above matrix can be inverted on a computer by using the programme in Appendix I.

4.2 Variance-covariance Matrix of Parameter Estimates.

Deriving the variance-covariance matrix of the treatment and residual estimates, when the presence of residual effects in period 1 is assumed.

$$\mathrm{Cov}(\,\underline{\hat{\theta}}\,\underline{\hat{\theta}}'\,) = \underline{M}^{-1}\,\mathrm{Cov}(\,\underline{X}\,\underline{X}'\,)\,(\,\underline{M}^{-1}\,)'$$

where

$$\underline{\hat{\theta}} = \begin{bmatrix} \underline{\hat{t}} \\ \underline{\hat{r}} \end{bmatrix} \qquad \underline{X} = \begin{bmatrix} \underline{T} - n\,\bar{Y}\,\underline{h} \\ \underline{R} - n\,\bar{Y}\,\underline{h} \end{bmatrix} = \begin{bmatrix} \underline{K} & \underline{O} \\ \underline{O} & \underline{K} \end{bmatrix} \begin{bmatrix} \underline{T} \\ \underline{R} \end{bmatrix}$$

$\underline{K} = \underline{I} - \frac{1}{n}\,\underline{h}\,\underline{h}'$ and $\underline{h}$ is a column vector of order n having each element equal to 1.

Let $\underline{X} = \underline{J}\,\underline{X}_o$ $\quad \underline{X}_o = \underline{A}\,\underline{Y}$ $\quad$ $\underline{A}$ is a design matrix.

$\underline{Y}$ is a matrix of observations.

$$\underline{X}_o = \begin{bmatrix} \underline{T} \\ \underline{R} \end{bmatrix}$$

$$\underline{A}\,\underline{A}' = \begin{bmatrix} n\,\underline{I} & \underline{\lambda} \\ \underline{\lambda}' & n\,\underline{I} \end{bmatrix}$$ is the matrix form of the normal equations.

$$\mathrm{Cov}(\,\underline{X}_o\,\underline{X}_o'\,) = \underline{A}\,\mathrm{Cov}(\,\underline{Y}\,\underline{Y}'\,)\,\underline{A}' = \underline{A}\,\underline{A}'\,\sigma^2$$

Therefore

$$\underline{J}\,\underline{A}\,\underline{A}' = \begin{bmatrix} \underline{K} & \underline{O} \\ \underline{O} & \underline{K} \end{bmatrix} \begin{bmatrix} n\,\underline{I} & \underline{\lambda} \\ \underline{\lambda}' & n\,\underline{I} \end{bmatrix} = \begin{bmatrix} n\,\underline{K} & \underline{K}\,\underline{\lambda} \\ \underline{\lambda}'\,\underline{K} & n\,\underline{K} \end{bmatrix}$$

$$\underline{J}\,\underline{A}\,\underline{A}'\,\underline{J}' = \underline{J}\,\mathrm{Cov}(\,\underline{X}_o\,\underline{X}_o'\,)\,\underline{J}' = \mathrm{Cov}\,(\,\underline{X}\,\underline{X}'\,)$$

$$= \sigma^2 \begin{bmatrix} n\,\underline{K} & \underline{K}\,\underline{\lambda} \\ \underline{\lambda}'\,\underline{K} & n\,\underline{K} \end{bmatrix} \begin{bmatrix} \underline{K} & \underline{O} \\ \underline{O} & \underline{K} \end{bmatrix} \qquad \underline{K}\,\underline{K} = \underline{K}, \quad \underline{K} = \underline{K}'$$

$$\text{Cov}(\underline{X}\,\underline{X}') = \sigma^2 \begin{bmatrix} n\,\underline{K} & \underline{K}\,\underline{\lambda} \\ \underline{\lambda}'\underline{K} & n\,\underline{K} \end{bmatrix}$$

$$\text{Cov}(\underline{\hat{\theta}}\,\underline{\hat{\theta}}') = \underline{M}^{-1}\,\text{Cov}(\underline{X}\,\underline{X}')\,(\underline{M}^{-1})'$$

$$\therefore\ \underline{M}^{-1}\text{Cov}(\underline{X}\,\underline{X}') = \sigma^2 \begin{bmatrix} n\,\underline{A}^{-1} & -\underline{\lambda}\,\underline{A}^{-1} \\ -\underline{A}^{-1}\,\underline{\lambda}' & n\,\underline{A}^{-1} \end{bmatrix} \begin{bmatrix} n\,\underline{K} & \underline{K}\,\underline{\lambda} \\ \underline{\lambda}'\underline{K} & n\,\underline{K} \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} \underline{K} & \underline{O} \\ \underline{O} & \underline{K} \end{bmatrix}$$

$$\underline{M}^{-1}\,\text{Cov}(\underline{X}\,\underline{X}')(\underline{M}^{-1})' = \sigma^2 \begin{bmatrix} \underline{K} & \underline{O} \\ \underline{O} & \underline{K} \end{bmatrix} \begin{bmatrix} n\,\underline{A}^{-1} & -\underline{\lambda}\,\underline{A}^{-1} \\ -\underline{A}^{-1}\,\underline{\lambda}' & n\,\underline{A}^{-1} \end{bmatrix}$$

$$\text{Cov}(\underline{\hat{\theta}}\,\underline{\hat{\theta}}') = \sigma^2 \begin{bmatrix} n\,\underline{K}\,\underline{A}^{-1} & -\underline{K}\,\underline{A}^{-1}\,\underline{\lambda} \\ -\underline{\lambda}'\underline{A}^{-1}\,\underline{K} & n\,\underline{K}\,\underline{A}^{-1} \end{bmatrix}$$

We obtain $V(\underline{\hat{t}}) = n\,\underline{K}\,\underline{A}^{-1}\,\sigma^2$ and $V(\underline{\hat{r}}) = n\,\underline{K}\,\underline{A}^{-1}\,\sigma^2$

It shows that the treatment and residual effects are measured with the same precision.

Chapter 5

Variance of the Linear Components of Parameter Estimates for 4x4 Latin-square Designs.

5.1 Other Patterns of 4 x 4 Latin-square Design.

In design 2.4.2 each treatment has been followed by every other treatment except by itself. Designs 2.4.3 and 2.4.4 have treatments preceded twice by one treatment and once by another treatment.

After obtaining the inverted matrices, as suggested under 2.9, we can easily calculate the parameter effects for the above mentioned designs.

5.1.1 Treatment direct and Treatment residual effects of Design 2.4.2.

$$\hat{t}_1 = 0.55T_1+0.275T_2+0.275T_3+0.275T_4-0.3R_1-0.4R_2-0.4R_3-0.4R_4$$
$$-0.1S_1-0.1S_2-0.1S_3-0.075S_4-0.375P_1+0.125G$$

$$\hat{t}_2 = 0.275T_1+0.55T_2+0.275T_3+0.275T_4-0.4R_1-0.3R_2-0.4R_3-0.4R_4$$
$$-0.1S_1-0.1S_2-0.075S_3-0.1S_4-0.375P_1+0.125G$$

$$\hat{t}_3 = 0.275T_1+0.275T_2+0.55T_3+0.275T_4-0.4R_1-0.4R_2-0.3R_3-0.4R_4$$
$$-0.1S_1-0.075S_2-0.1S_3-0.1S_4-0.375P_1+0.125G$$

$$\hat{t}_4 = 0.275T_1+0.275T_2+0.275T_3+0.55T_4-0.4R_1-0.4R_2-0.4R_3-0.3R_4$$
$$-0.075S_1-0.1S_2-0.1S_3-0.1S_4-0.375P_1+0.125G$$

$$\hat{r}_1 = 0.8R_1+0.4R_2+0.4R_3+0.4R_4-0.3T_1-0.4T_2-0.4T_3-0.4T_4$$
$$+0.1S_1+0.1S_2+0.1S_3+0.2S_4+0.5P_1-0.25G$$

$$\hat{r}_2 = 0.4R_1+0.8R_2+0.4R_3+0.4R_4-0.4T_1-0.3T_2-0.4T_3-0.4T_4$$
$$+0.1S_1+0.1S_2+0.2S_3+0.1S_4+0.5P_1-0.25G$$

$$\hat{r}_3 = 0.4R_1+0.4R_2+0.8R_3+0.4R_4-0.4T_1-0.4T_2-0.3T_3-0.4T_4$$
$$+0.1S_1+0.2S_2+0.1S_3+0.1S_4+0.5P_1-0.25G$$

$$\hat{r}_4 = 0.4R_1+0.4R_2+0.4R_3+0.8R_4-0.4T_1-0.4T_2-0.4T_3-0.3T_4$$
$$+0.2S_1+0.1S_2+0.1S_3+0.1S_4+0.5P_1-0.25G$$

5.1.2 Treatment direct and Treatment residual effects of Design 2.4.3

$\hat{t}_1 = 0.642T_1+0.183T_2+0.275T_3+0.275T_4-0.233R_1-0.567R_2-0.433R_3$
$-0.267R_4+$............other terms in S, P_1 and G.

$\hat{t}_2 = 0.183T_1+0.642T_2+0.275T_3+0.275T_4-0.567R_1-0.233R_2-0.267R_3$
$-0.433R_4+$..

$\hat{t}_3 = 0.275T_1+0.275T_2+0.642T_3+0.183T_4-0.267R_1-0.433R_2-0.233R_3$
$-0.567R_4+$..

$\hat{t}_3 = 0.275T_1+0.275T_2+0.183T_3+0.642T_4-0.433R_1-0.267R_2-0.567R_3$
$-0.233R_4+$..

$\hat{r}_1 = 0.933R_1+0.267R_2+0.4R_3+0.4R_4-0.233T_1-0.567T_2-0.267T_3$
$-0.433T_4+$..

$\hat{r}_2 = 0.267R_1+0.933R_2+0.4R_3+0.4R_4-0.567T_1-0.233T_2-0.433T_3$
$-0.267T_4+$..

$\hat{r}_3 = 0.4R_1+0.4R_2+0.933R_3+0.267R_4-0.433T_1-0.267T_2-0.233T_3$
$-0.567T_4+$..

$\hat{r}_4 = 0.4R_1+0.4R_2+0.267R_3+0.933R_4-0.267T_1-0.433T_2-0.567T_3$
$-0.233T_4+$..

5.1.3 Treatment direct and residual effects of Design 2.4.4

$\hat{t}_1 = 0.825T_1+0.55T_3-0.8R_2-0.7R_4-0.175S_1-0.2S_3-0.375P_1+0.125G$

$\hat{t}_2 = 0.825T_2+0.55T_4-0.8R_1-0.7R_3-0.175S_2-0.2S_4-0.375P_1+0.125G$

$\hat{t}_3 = 0.825T_3+0.55T_1-0.8R_4-0.7R_2-0.175S_3-0.2S_2-0.375P_1+0.125G$

$\hat{t}_4 = 0.825T_4+0.55T_2-0.8R_3-0.7R_1-0.175S_4-0.2S_2-0.375P_1+0.125$

$\hat{r}_1 = 1.2R_1+0.8R_3-0.8T_2-0.7T_4+0.2S_2+0.3S_4+0.5P_1-0.25G$

$\hat{r}_2 = 1.2R_2+0.8R_4-0.8T_1-0.7T_3+0.2S_1+0.3S_3+0.5P_1-0.25G$

$\hat{r}_3 = 1.2R_3+0.8R_1-0.8T_4-0.7T_2+0.2S_4+0.3S_2+0.5P_1-0.25G$

$\hat{r}_4 = 1.2R_4+0.8R_2-0.8T_3-0.7T_1+0.2S_3+0.3S_1+0.5P_1-0.25G$

5.2 Variance of the Difference between two Estimates.

Following the statistical theory of the variance of linear components, stated under 2.13, we calculate the variances of the differences between two treatment and two residual effects for the designs under consideration.

The variance of the differences calculated by this method produce exactly the same results as by the method mentioned under 2.12 with an extra advantage of simplicity and ease in calculations.

5.2.1 Variances of the differences between two treatment direct and treatment residual effects of Design 2.4.2.

$$\hat{t}_1-\hat{t}_2 = 0.55T_1+0.275T_2-0.275T_1-0.55T_2+\ldots\ldots\ldots\ldots$$

$$V(\hat{t}_1-\hat{t}_2) = V(\,0.275T_1 - 0.275T_2\,)$$

$$= 0.55\sigma^2$$

Similarly we find:

$$V(\hat{t}_1-\hat{t}_3) = V(\hat{t}_1-\hat{t}_4) = V(\hat{t}_2-\hat{t}_3) = V(\hat{t}_2-\hat{t}_4) = V(\hat{t}_3-\hat{t}_4) = 0.55\sigma^2$$

$$\hat{r}_1-\hat{r}_2 = 0.8R_1+0.4R_2-0.4R_1-0.8R_2+\ldots\ldots\ldots\ldots\ldots$$

$$V(\hat{r}_1-\hat{r}_2) = V(\,0.4R_1 - 0.4R_2\,)$$

$$= 0.8\sigma^2$$

and $V(\hat{r}_1-\hat{r}_3) = V(\hat{r}_1-\hat{r}_4) = V(\hat{r}_2-\hat{r}_3) = V(\hat{r}_2-\hat{r}_4) = V(\hat{r}_3-\hat{r}_4) = 0.8\sigma^2$

Average $V(\hat{t}_i-\hat{t}_j) = 0.55\sigma^2$

Average $V(\hat{r}_i-\hat{r}_j) = 0.8\sigma^2$

The variances calculated above indicate that, for design 2.4.2 (Williams' Design), the differences of all the pairs of treatments are measured with the same precision. Similarly we find that the differences of all the pairs of residuals are measured with the same precision, though less precisely than the treatment differences as their variance is higher than the variance of corresponding treatment differences.

5.2.2 Variances of the differences between two treatment direct and treatment residual effects of Design 2.4.3.

$V(\hat{t}_1-\hat{t}_2) = V(0.459T_1-0.459T_2) = 0.918\sigma^2 = V(\hat{t}_3-\hat{t}_4)$

$V(\hat{t}_1-\hat{t}_3) = V(0.367T_1-0.367T_3) = 0.734\sigma^2$

$V(\hat{t}_1-\hat{t}_4) = V(\hat{t}_2-\hat{t}_3) = V(\hat{t}_2-\hat{t}_4) = 0.737\sigma^2$

$V(\hat{r}_1-\hat{r}_2) = V(0.933R_1+0.267R_2-0.267R_1-0.933R_2) = 1.332\sigma^2$

$V(\hat{r}_3-\hat{r}_4) = 1.332\sigma^2$

$V(\hat{r}_1-\hat{r}_3) = V(\ 0.533R_1-0.533R_3) = 1.067\sigma^2$

$V(\hat{r}_1-\hat{r}_4) = V(\hat{r}_2-\hat{r}_3) = V(\hat{r}_2-\hat{r}_4) = 1.067\sigma^2$

Average $V(\hat{t}_i-\hat{t}_j) = 0.797\sigma^2$

Average $V(\hat{r}_i-\hat{r}_j) = 1.155\sigma^2$

This design, unlike designs 2.4.1 and 2.4.2, does not measure the differences of treatment effects or residual effects with the same precision.

5.2.3 Variances of the differences between two treatment direct and treatment residual effects of Design 2.4.4.

$V(\hat{t}_1-\hat{t}_2) = V(0.825T_1-0.825T_2) = 1.65\sigma^2$

$V(\hat{t}_1-\hat{t}_4) = V(\hat{t}_2-\hat{t}_3) = V(\hat{t}_3-\hat{t}_4) = 1.65\sigma^2$

$V(\hat{t}_1-\hat{t}_3) = V(0.825T_1+0.55T_3-0.55T_1-0.825T_3) = 0.55\sigma^2$

$V(\hat{t}_2-\hat{t}_4) = 0.55\sigma^2$

$V(\hat{r}_1-\hat{r}_2) = V(1.2R_1-1.2R_2) = 2.4\sigma^2$

$V(\hat{r}_1-\hat{r}_4) = V(\hat{r}_2-\hat{r}_3) = V(\hat{r}_3-\hat{r}_4) = 2.4\sigma^2$

$V(\hat{r}_1-\hat{r}_3) = V(1.2R_1+0.8R_3-1.2R_3-0.8R_1) = 0.8\sigma^2 = V(\hat{r}_2-\hat{r}_4)$

Average $V(\hat{t}_i-\hat{t}_j) = 1.283\sigma^2$

Average $V(\hat{r}_i-\hat{r}_j) = 1.867\sigma^2$

This design also does not measure the treatment differences or residual differences with the same precision.

5.3 Variance of Linear Components.

Using orthogonal polynomials, as in 2.15, we calculate the variance of linear component of treatment direct effects as well as treatment residual effects for the Latin-square designs 2.4.2 to 2.4.4.

5.3.1 Variance of Linear components of Design 2.4.2.

$$\phi_t = -3\hat{t}_1-\hat{t}_2+\hat{t}_3+3\hat{t}_4 = -0.825T_1-0.275T_2+0.275T_3+0.825T_4+ \cdots$$

$$V(\phi_t) = V(-3x-0.825T_1-1x-0.275T_2+1x0.275T_3+3x0.825T_4)$$
$$= V(2.475T_1+0.275T_2+0.275T_3+2.475T_4)$$
$$= 5.5\sigma^2$$

$$\phi_r = -3\hat{r}_1-\hat{r}_2+\hat{r}_3+3\hat{r}_4 = -1.2R_1-0.4R_2+0.4R_3+1.2R_4+\cdots$$

$$V(\phi_r) = V(-3x-1.2R_1-1x-0.4R_2+1x0.4R_3+3x1.2R_4)$$
$$= V(3.6R_1+0.4R_2+0.4R_3+3.6R_4)$$
$$= 8\sigma^2$$

5.3.2 Variance of Linear components of Design 2.4.3.

$$V(\phi_t) = V(-3x-1.008T_1-1x-0.092T_2+1x0.092T_3+3x1.008T_4)$$
$$= 6.232\,\sigma^2$$
$$V(\phi_r) = V(-3\hat{r}_1-\hat{r}_2+\hat{r}_3+3\hat{r}_4) = V(-3x1.467R_1-1x-0.133R_2+1x0.133R_3$$
$$+3x1.467R_4) = 9.068\sigma^2$$

5.3.3 Variance of Linear components of Design 2.4.4.

$$V(\phi_t) = V(-3\hat{t}_1-\hat{t}_2+\hat{t}_3+3\hat{t}_4) = V(-3x-1.925T_1-1x+0.825T_2+1x-0.825T_3$$
$$+3x1.925T_4) = 9.9\,\sigma^2$$

$$V(\phi_r) = V(-3\hat{r}_1-\hat{r}_2+\hat{r}_3+3\hat{r}_4) = V(8.4R_1-1.2R_2-1.2R_3+8.4R_4)$$
$$= 14.4\,\sigma^2$$

5.4 Variance of the Difference between Permanent Effects.

The variance of the difference between two permanent effects for designs 2.4.2 to 2.4.4 can be derived from the results obtained under 5.2.

5.4.1 Variance of the difference between two permanent effects of Design 2.4.2.

Let $\hat{t}_i + \hat{r}_i$ = Permanent effect of treatment i.

The variance of the difference between permanent effects of this design is the same for any pair of treatments.

$$V\{(\hat{t}_i + \hat{r}_i) - (\hat{t}_u + \hat{r}_u)\} = 1.35\sigma^2$$

5.4.2 Variance of the difference between two permanent effects of Design 2.4.3.

$$V\{(\hat{t}_1+\hat{r}_1) - (\hat{t}_2+\hat{r}_2)\} = V\{(\hat{t}_3+\hat{r}_3) - (\hat{t}_4+\hat{r}_4)\} = 2.25\sigma^2$$

$$V\{(\hat{t}_1+\hat{r}_1)-(\hat{t}_3+\hat{r}_3)\} = V\{(\hat{t}_1+\hat{r}_1)-(\hat{t}_4+\hat{r}_4)\} = V\{(\hat{t}_2+\hat{r}_2)-(\hat{t}_3+\hat{r}_3)\} =$$
$$V\{(\hat{t}_2+\hat{r}_2)-(\hat{t}_4+\hat{r}_4)\} = 1.801\sigma^2$$

5.4.3 Variance of the difference between two permanent effects of Design 2.4.4.

$$V\{(\hat{t}_1+\hat{r}_1)-(\hat{t}_2+\hat{r}_2)\} = V\{(\hat{t}_1+\hat{r}_1)-(\hat{t}_4+\hat{r}_4)\} = V\{(\hat{t}_2+\hat{r}_2)-(\hat{t}_3+\hat{r}_3)\} =$$
$$V\{(\hat{t}_3+\hat{r}_3)-(\hat{t}_4+\hat{r}_4)\} = 4.05\sigma^2$$

$$V\{(\hat{t}_1+\hat{r}_1) - (\hat{t}_3+\hat{r}_3)\} = V\{(\hat{t}_2+\hat{r}_2) - (\hat{t}_4+\hat{r}_4)\} = 1.35\sigma^2$$

Again we find that the differences between two permanent effects of designs 2.4.3 and 2.4.4 are not measured with the same precision for different pairs of treatments.

5.5 Efficiency Ratio.

The efficiency of measuring the treatment differences over residual differences or other such measures, can be calculated by taking the ratios of the reciprocals of their respective variances.

5.5.1 Efficiency of measuring Treatment Effects over Residual Effects of Design 2.4.2.

Efficiency (Differences) (All Pairs) $= \frac{0.8\sigma^2}{0.55\sigma^2} \times 100 = 145\ \%$

Efficiency (Linear Effects) $= \frac{8\sigma^2}{5.5\sigma^2} \times 100 = 145\ \%$

5.5.2 Efficiency of measuring Treatment Effects over Residual Effects of Design 2.4.3.

Efficiency (Difference) (Pairs, 1-2, 3-4) $= \frac{1.332\sigma^2}{0.918\sigma^2} \times 100 = 145\ \%$

Efficiency (Differences) (Pairs, 1-3, 1-4, 2-3, 2-4) $= \frac{1.067\sigma^2}{0.734\sigma^2} \times 100 = 145\ \%$

Efficiency (Difference) (Average) $= \frac{1.155\sigma^2}{0.797\sigma^2} \times 100 = 145\ \%$

Efficiency (Linear Effects) $= \frac{9.068\sigma^2}{6.232\sigma^2} \times 100 = 145\ \%$

5.5.3 Efficiency of measuring Treatment Effects over Residual Effects of Design 2.4.4

Efficiency (Differences) (Pairs: 1-2, 1-4, 2-3, 3-4) and 1-3, 2-4) $= \frac{2.4\sigma^2}{1.65\sigma^2} \times 100 = 145\ \% = \frac{0.8\sigma^2}{0.55\sigma^2}$

Efficiency (Difference) (Average) $= \frac{1.867\sigma^2}{1.283\sigma^2} \times 100 = 145\ \%$

Efficiency (Linear Effects) $= \frac{14.4\sigma^2}{9.9\sigma^2} \times 100 = 145\ \%$

Chapter 6

Variance-covariance Matrix and Efficiency of Measuring Treatment Effects over Residual Effects.

6.1 Variance-covariance Matrix of Parameter Estimates ignoring Period Effects.

As explained under 2.8 period effects can be calculated directly and independently of other parameter effects. Estimates of subject, treatment and residual effects are given by:

$$\underline{\hat{\theta}} = \begin{bmatrix} \underline{\hat{s}} \\ \underline{\hat{t}} \\ \underline{\hat{r}} \end{bmatrix} = \frac{1}{n} \begin{bmatrix} n(n-1)\,\underline{V}^{-1} - \underline{\beta}'\,\underline{V}^{-1}\underline{\beta} & -\underline{\beta}'\,\underline{V}^{-1}\,\underline{\mathcal{L}} & n\,\underline{\mathcal{L}}\,\underline{V}^{-1} \\ -\underline{\mathcal{L}}'\,\underline{V}^{-1}\,\underline{\beta} & n(n-1)\,\underline{V}^{-1} - \underline{\mathcal{L}}'\,\underline{V}^{-1}\,\underline{\mathcal{L}} & -n\,\underline{\beta}\,\underline{V}^{-1} \\ n\,\underline{V}^{-1}\,\underline{\mathcal{L}}' & -n\,\underline{V}^{-1}\,\underline{\beta}' & n^2\,\underline{V}^{-1} \end{bmatrix} \begin{bmatrix} \underline{S} - n\,\bar{Y}\,\underline{h} \\ \underline{T} - n\,\bar{Y}\,\underline{h} \\ \underline{R} - n\,\bar{Y}\,\underline{h} + \frac{P}{n}{}_1\,\underline{h} \end{bmatrix}$$

Let $\underline{\hat{\theta}} = \underline{M}^{-1}\,\underline{X}$

Therefore we can now consider the variance-covariance matrix of parameter estimates ignoring period effects.

$$\text{Cov}(\,\underline{\hat{\theta}}\,\underline{\hat{\theta}}'\,) = \underline{M}^{-1}\,\text{Cov}(\,\underline{X}\,\underline{X}'\,)(\,\underline{M}^{-1}\,)'$$

$$\underline{X} = \begin{bmatrix} \underline{S} - n\,\bar{Y}\,\underline{h} \\ \underline{T} - n\,\bar{Y}\,\underline{h} \\ \underline{R} - n\,\bar{Y}\,\underline{h} + \frac{P}{n}{}_1\,\underline{h} \end{bmatrix}$$

Writing

$$\underline{S} - n\,\bar{Y}\,\underline{h} = (\,\underline{I} - \tfrac{1}{n}\,\underline{h}\,\underline{h}'\,)\,\underline{S} = \underline{K}\,\underline{S}$$

$$\underline{T} - n\,\bar{Y}\,\underline{h} = (\,\underline{I} - \tfrac{1}{n}\,\underline{h}\,\underline{h}'\,)\,\underline{T} = \underline{K}\,\underline{T}$$

$$\underline{R} - n\,\bar{Y}\,\underline{h} = (\,\underline{I} - \tfrac{1}{n}\,\underline{h}\,\underline{h}'\,)\,\underline{R} - \tfrac{1}{n}\,\underline{h}\,\underline{e}_1'\underline{P} + \tfrac{1}{n}\,\underline{h}\,\underline{e}_1'\underline{P}$$
$$= \underline{K}\,\underline{R}$$

The symbols and the terms used above have already been defined under 2.7.

$$\therefore \ \underline{X} = \begin{bmatrix} \underline{O} & \underline{O} & \underline{O} & \underline{O} \\ \underline{O} & \underline{K} & \underline{O} & \underline{O} \\ \underline{O} & \underline{O} & \underline{K} & \underline{O} \\ \underline{O} & \underline{O} & \underline{O} & \underline{K} \end{bmatrix} \begin{bmatrix} \underline{P} \\ \underline{S} \\ \underline{T} \\ \underline{R} \end{bmatrix}$$

Let $\underline{X} = \underline{J}\,\underline{X}_o$

where

$$\underline{J} = \begin{bmatrix} \underline{O} & \underline{O} & \underline{O} & \underline{O} \\ \underline{O} & \underline{K} & \underline{O} & \underline{O} \\ \underline{O} & \underline{O} & \underline{K} & \underline{O} \\ \underline{O} & \underline{O} & \underline{O} & \underline{K} \end{bmatrix}$$

and

$$\underline{X}_o = \begin{bmatrix} \underline{P} \\ \underline{S} \\ \underline{T} \\ \underline{R} \end{bmatrix} = \begin{bmatrix} \underline{I} & \underline{I} \ \cdots\cdots\cdots & \underline{I} \\ \underline{e}_1\underline{h}' & \underline{e}_2\underline{h}' \cdots\cdots\cdots & \underline{e}_n\underline{h}' \\ \underline{\tau}_1' & \underline{\tau}_2' \cdots\cdots\cdots & \underline{\tau}_n' \\ \underline{\rho}_1' & \underline{\rho}_2' \cdots\cdots\cdots & \underline{\rho}_n' \end{bmatrix} \underline{Y}_{(n^2 x 1)}$$

or $\underline{X}_o = \underline{A}\,\underline{Y}$. $\underline{A}$ is the design matrix.

$$\underline{A}\,\underline{A}' = \begin{bmatrix} n\,\underline{I} & \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' & (\underline{h} - \underline{e}_1)\,\underline{h}' \\ \underline{h}'\,\underline{h} & n\,\underline{I} & \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' - \underline{\alpha} \\ \underline{h}'\,\underline{h} & \underline{h}'\,\underline{h} & n\,\underline{I} & \underline{\beta} \\ \underline{h}(\underline{h}' - \underline{e}_1') & \underline{h}'\,\underline{h} - \underline{\alpha}' & \underline{\beta}' & (n-1)\,\underline{I} \end{bmatrix}$$

$\text{Cov}(\underline{X}_o\,\underline{X}_o') = \underline{A}\,\text{Cov}(\underline{Y}\,\underline{Y}')\,\underline{A}'$ and $\text{Cov}(\underline{Y}\,\underline{Y}') = \sigma^2\,\underline{I}$

$\therefore \ \text{Cov}(\underline{X}_o\,\underline{X}_o') = \underline{A}\,\underline{A}'\,\sigma^2$

$$Cov(\underline{X}\,\underline{X}') = \underline{J}\,Cov(\underline{X}_o\,\underline{X}_o')\,\underline{J}' = \underline{J}\,\underline{A}\,\underline{A}'\,\underline{J}'\sigma^2$$

$$\underline{J}\,\underline{A}\,\underline{A}' = \begin{bmatrix} \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{K} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{K} \end{bmatrix} \begin{bmatrix} n\,\underline{I} & \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' & (\underline{h} - \underline{e}_1)\,\underline{h}' \\ \underline{h}'\,\underline{h} & n\,\underline{I} & \underline{h}\,\underline{h}' & \underline{h}\,\underline{h}' - \underline{\alpha} \\ \underline{h}'\,\underline{h} & \underline{h}'\,\underline{h} & n\,\underline{I} & \underline{\beta} \\ \underline{h}(\underline{h}'-\underline{e}_1') & \underline{h}'\,\underline{h} - \underline{\alpha}' & \underline{\beta}' & (n-1)\,\underline{I} \end{bmatrix}$$

$$= \begin{bmatrix} \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & n\underline{K} & \underline{0} & -\underline{K}\,\underline{\alpha} \\ \underline{0} & \underline{0} & n\underline{K} & \underline{K}\,\underline{\beta} \\ \underline{0} & -\underline{K}\,\underline{\alpha}' & \underline{K}\,\underline{\beta}' & (n-1)\,\underline{K} \end{bmatrix}$$

$$\underline{J}\,\underline{A}\,\underline{A}'\,\underline{J}'\sigma^2 = \underline{J}\,Cov(\underline{X}_o\,\underline{X}_o')\,\underline{J}' = Cov(\underline{X}\,\underline{X}')$$

$$= \sigma^2 \begin{bmatrix} \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & n\underline{K} & \underline{0} & -\underline{K}\,\underline{\alpha} \\ \underline{0} & \underline{0} & n\underline{K} & \underline{K}\,\underline{\beta} \\ \underline{0} & -\underline{K}\,\underline{\alpha}' & \underline{K}\,\underline{\beta}' & (n-1)\,\underline{K} \end{bmatrix} \begin{bmatrix} \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & \underline{K} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} & \underline{K} \end{bmatrix}$$

Hence

$$COv(\underline{X}\,\underline{X}') = \sigma^2 \begin{bmatrix} \underline{0} & \underline{0} & \underline{0} & \underline{0} \\ \underline{0} & n\underline{K} & \underline{0} & -\underline{K}\,\underline{\alpha} \\ \underline{0} & \underline{0} & n\underline{K} & \underline{K}\,\underline{\beta} \\ \underline{0} & -\underline{K}\,\underline{\alpha}' & \underline{K}\,\underline{\beta}' & (n-1)\,\underline{K} \end{bmatrix} \qquad \because \underline{K}\,\underline{h} = \underline{0}, \quad \underline{K}\,\underline{K} = \underline{K}$$

$$\mathrm{Cov}(\underline{\hat\theta}\,\underline{\hat\theta}') = \underline{M}^{-1}\,\mathrm{Cov}(\underline{X}\,\underline{X}')(\underline{M}^{-1})'$$

$$\underline{M}^{-1} = \frac{1}{n}\begin{bmatrix} n(n-1)\,\underline{V}^{-1}-\underline{\beta}'\underline{V}^{-1}\underline{\beta} & -\underline{\beta}'\underline{V}^{-1}\underline{\mathcal{L}} & n\,\underline{\mathcal{L}}\,\underline{V}^{-1} \\ -\underline{\mathcal{L}}'\underline{V}^{-1}\underline{\beta} & n(n-1)\,\underline{V}^{-1}-\underline{\mathcal{L}}'\underline{V}^{-1}\underline{\mathcal{L}} & -n\,\underline{\beta}\,\underline{V}^{-1} \\ n\,\underline{V}^{-1}\underline{\mathcal{L}}' & -n\,\underline{V}^{-1}\underline{\beta}' & n^2\,\underline{V}^{-1} \end{bmatrix}$$

from 2.8

Therefore $\underline{M}^{-1}\,\mathrm{Cov}(\underline{X}\,\underline{X}') =$

$$\frac{\sigma^2}{n}\begin{bmatrix} n(n-1)\,\underline{V}^{-1} -\underline{\beta}'\underline{V}^{-1}\underline{\beta} & -\underline{\beta}'\underline{V}^{-1}\underline{\mathcal{L}} & n\,\underline{\mathcal{L}}\,\underline{V}^{-1} \\ -\underline{\mathcal{L}}'\underline{V}^{-1}\underline{\beta} & n(n-1)\,\underline{V}^{-1} -\underline{\mathcal{L}}'\underline{V}^{-1}\underline{\mathcal{L}} & -n\,\underline{\beta}\,\underline{V}^{-1} \\ n\,\underline{V}^{-1}\underline{\mathcal{L}}' & -n\,\underline{V}^{-1}\underline{\beta}' & n^2\,\underline{V}^{-1} \end{bmatrix}\begin{bmatrix} n\,\underline{K} & \underline{0} & -\underline{K}\,\underline{\mathcal{L}} \\ \underline{0} & n\,\underline{K} & \underline{K}\,\underline{\beta} \\ -\underline{K}\,\underline{\mathcal{L}}' & \underline{K}\,\underline{\beta}' & (n-1)\underline{K} \end{bmatrix}$$

$$= \frac{\sigma^2}{n}\begin{bmatrix} n^2(n-1)\,\underline{K}\,\underline{V}^{-1} -n\,\underline{\mathcal{L}}'\underline{K}\,\underline{V}^{-1}\,\underline{\mathcal{L}} -n\,\underline{\beta}'\underline{K}\,\underline{V}^{-1}\,\underline{\beta} & \underline{0} & \underline{0} \\ \underline{0} & n^2(n-1)\,\underline{K}\,\underline{V}^{-1} -n\,\underline{\mathcal{L}}'\underline{K}\,\underline{V}^{-1}\,\underline{\mathcal{L}} -n\,\underline{\beta}\,\underline{K}\,\underline{V}^{-1}\,\underline{\beta} & \underline{0} \\ \underline{0} & \underline{0} & n^2(n-1)\,\underline{K}\,\underline{V}^{-1} -n\,\underline{\mathcal{L}}'\underline{K}\,\underline{V}^{-1}\,\underline{\mathcal{L}} -n\,\underline{\beta}'\underline{K}\,\underline{V}^{-1}\,\underline{\beta} \end{bmatrix}$$

After simplification we find that

$$\underline{M}^{-1}\,\mathrm{Cov}(\underline{X}\,\underline{X}') = \frac{\sigma^2}{n}\begin{bmatrix} n\,\underline{K}\,\underline{V}\,\underline{V}^{-1} & \underline{0} & \underline{0} \\ \underline{0} & n\,\underline{K}\,\underline{V}\,\underline{V}^{-1} & \underline{0} \\ \underline{0} & \underline{0} & n\,\underline{K}\,\underline{V}\,\underline{V}^{-1} \end{bmatrix} = \begin{bmatrix} \underline{K} & \underline{0} & \underline{0} \\ \underline{0} & \underline{K} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K} \end{bmatrix}\sigma^2$$

and $\underline{M}^{-1}\,\mathrm{Cov}(\,\underline{X}\,\underline{X}'\,)(\,\underline{M}^{-1})' =$

$$\frac{\sigma^2}{n}\begin{bmatrix} \underline{K} & \underline{0} & \underline{0} \\ \underline{0} & \underline{K} & \underline{0} \\ \underline{0} & \underline{0} & \underline{K} \end{bmatrix}\begin{bmatrix} n(n-1)\,\underline{V}^{-1} - \underline{\beta}'\,\underline{V}^{-1}\,\underline{\beta} & -\underline{\beta}'\,\underline{V}^{-1}\,\underline{\mathcal{L}} & n\,\underline{\mathcal{L}}\,\underline{V}^{-1} \\ -\underline{\mathcal{L}}'\,\underline{V}^{-1}\,\underline{\beta} & n(n-1)\,\underline{V}^{-1} - n\,\underline{\mathcal{L}}'\,\underline{V}^{-1}\,\underline{\mathcal{L}} & -n\,\underline{\beta}\,\underline{V}^{-1} \\ n\,\underline{V}^{-1}\,\underline{\mathcal{L}}' & -n\,\underline{V}^{-1}\,\underline{\beta}' & n^2\,\underline{V}^{-1} \end{bmatrix}$$

Hence

$$\text{Var-Cov}(\,\hat{\underline{\theta}}\,\hat{\underline{\theta}}'\,) =$$

$$\frac{\sigma^2}{n}\begin{bmatrix} n(n-1)\,\underline{K}\,\underline{V}^{-1} - \underline{\beta}'\,\underline{K}\,\underline{V}^{-1}\underline{\beta} & -\underline{\beta}'\,\underline{K}\,\underline{V}^{-1}\underline{\mathcal{L}} & n\underline{\mathcal{L}}\,\underline{K}\,\underline{V}^{-1} \\ -\underline{\mathcal{L}}'\,\underline{V}^{-1}\,\underline{K}\,\underline{\beta} & n(n-1)\,\underline{K}\,\underline{V}^{-1} - \underline{\mathcal{L}}'\,\underline{K}\,\underline{V}^{-1}\underline{\mathcal{L}} & -n\underline{\beta}\,\underline{K}\,\underline{V}^{-1} \\ n\,\underline{V}^{-1}\,\underline{K}\,\underline{\mathcal{L}}' & -n\,\underline{V}^{-1}\,\underline{K}\,\underline{\beta}' & n^2\,\underline{K}\,\underline{V}^{-1} \end{bmatrix}$$

6.2 Variance of Estimable Function.

The normal equation $\underline{X}'\underline{X}\,\hat{\underline{\theta}} = \underline{X}'\underline{Y}$ may, for simplicity, be denoted as: $\underline{M}\,\hat{\underline{\theta}} = \underline{S}$ where $\underline{M} = \underline{X}'\underline{X}$ and $\underline{S} = \underline{X}'\underline{Y}$

The essential property of normal equation is

$$D(\underline{S}) = \sigma^2 \underline{M}$$

Let $\underline{C}$ be the inverse of $\underline{M}$ such that

$$\hat{\underline{\theta}} = \underline{M}^{-1}\underline{S} = \underline{C}\,\underline{S}$$

$\underline{\phi}'\underline{\theta}$ is a linear contrast if $\underline{\phi} = \left[\xi_k\right]$ subject to the condition that $\sum_{k=1}^{n}\xi_k = 0$

Let $\underline{\phi}'\hat{\underline{\theta}}$ be the the least square estimator of the estimable function $\underline{\phi}'\underline{\theta}$ and $\underline{C}$ be any generalised inverse of $\underline{M}$.

Then $V(\underline{\phi}'\hat{\underline{\theta}}) = \sigma^2\,\underline{\phi}'\underline{C}\,\underline{\phi}$ which may be proved as:

$$
\begin{aligned}
V(\underline{\phi}'\hat{\underline{\theta}}) &= V(\underline{\phi}'\underline{C}\,\underline{S}) \\
&= \underline{\phi}'\underline{C}\,D(\underline{S})\,\underline{C}'\underline{\phi} \\
&= \sigma^2\,\underline{\phi}'\underline{C}\,\underline{M}\,\underline{C}'\underline{\phi} \qquad \because D(\underline{S}) = \sigma^2\underline{M} \\
&= \sigma^2\,\underline{\phi}'\underline{C}\,\underline{\phi}
\end{aligned}
$$

From 2.8

$$
\hat{\underline{\theta}} = \frac{1}{n}\begin{bmatrix} n(n-1)\,\underline{V}^{-1} - \underline{\beta}'\underline{V}^{-1}\underline{\beta} & -\underline{\beta}'\underline{V}^{-1}\underline{\alpha} & n\,\underline{\alpha}\,\underline{V}^{-1} \\ -\underline{\alpha}'\underline{V}^{-1}\underline{\beta} & n(n-1)\,\underline{V}^{-1} - \underline{\alpha}'\underline{V}^{-1}\underline{\alpha} & -n\,\underline{\beta}\,\underline{V}^{-1} \\ n\,\underline{V}^{-1}\underline{\alpha}' & -n\,\underline{V}^{-1}\underline{\beta}' & n^2\,\underline{V}^{-1} \end{bmatrix}\begin{bmatrix} \underline{S} - n\bar{Y}\underline{h} \\ \underline{T} - n\bar{Y}\underline{h} \\ \underline{R} - n\bar{Y}\underline{h} + \frac{P}{n}1\underline{h} \end{bmatrix}
$$

This is the reduced form of $\hat{\underline{\theta}}$ since our main interest lies in the treatment direct and treatment residual effects.

The estimable function can now be expressed as:

$$\underline{\phi}'\hat{\underline{\theta}} = \underline{\phi}'\frac{1}{n}\begin{bmatrix} n(n-1)\,\underline{V}^{-1} - \underline{\beta}'\underline{V}^{-1}\underline{\beta} & -\underline{\beta}'\underline{V}^{-1}\underline{\alpha} & n\,\underline{\alpha}\,\underline{V}^{-1} \\ -\underline{\alpha}'\underline{V}^{-1}\underline{\beta} & n(n-1)\,\underline{V}^{-1} - \underline{\alpha}'\underline{V}^{-1}\underline{\alpha} & -n\,\underline{\beta}\,\underline{V}^{-1} \\ n\,\underline{V}^{-1}\underline{\alpha}' & -n\,\underline{V}^{-1}\underline{\beta}' & n^2\,\underline{V}^{-1} \end{bmatrix}\begin{bmatrix} \underline{S} - n\bar{Y}\,\underline{h} \\ \underline{T} - n\bar{Y}\,\underline{h} \\ \underline{R} - n\bar{Y}\underline{h} + \frac{P}{n}1\underline{h} \end{bmatrix}$$

Therefore

2.6.1

$$V(\underline{\phi}'\hat{\underline{\theta}}) = \frac{\sigma^2}{n}\underline{\phi}'\begin{bmatrix} n(n-1)\,\underline{V}^{-1} - \underline{\beta}'\underline{V}^{-1}\underline{\beta} & -\underline{\beta}'\underline{V}^{-1}\underline{\alpha} & n\,\underline{\alpha}\,\underline{V}^{-1} \\ -\underline{\alpha}'\underline{V}^{-1}\underline{\beta} & n(n-1)\,\underline{V}^{-1} - \underline{\alpha}'\underline{V}^{-1}\underline{\alpha} & -n\,\underline{\beta}\,\underline{V}^{-1} \\ n\,\underline{V}^{-1}\underline{\alpha}' & -n\,\underline{V}^{-1}\underline{\beta}' & n^2\,\underline{V}^{-1} \end{bmatrix}\underline{\phi}$$

If

$$\underline{\phi}_{t(ij)} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ (s) \\ \\ \\ \\ (t) \\ \\ \\ \\ (r) \\ \\ \end{matrix}$$

where 1 corresponds to the ith treatment and -1 corresponds to the jth treatment.

then

$$\underline{\phi}'_{t(ij)}\,\hat{\underline{\theta}} = \hat{t}_i - \hat{t}_j$$

Note that $\underline{\phi}_{t(ij)}$ is reduced to $\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{bmatrix}$ with 1 in the ith position and -1 in the jth position.

Therefore

$$V(\underline{\phi}'_{t(ij)} \hat{\underline{t}}) = \frac{\sigma^2}{n} \underline{\phi}'_{t(ij)} \left[n(n-1)\,\underline{V}^{-1} - \underline{\alpha}'\underline{V}^{-1}\underline{\alpha} \right] \underline{\phi}_{t(ij)}$$

Similarly writing $\underline{\phi}$ for residual effects as:

$$\underline{\phi}_{r(ij)} = \begin{bmatrix} 0 \\ \vdots\ (s) \\ 0 \\ 0 \\ \vdots\ (t) \\ 0 \\ 0 \\ \cdot \\ 1 \\ \cdot\ (r) \\ -1 \\ \cdot \\ 0 \end{bmatrix}$$

where 1 corresponds to the ith residual

-1 corresponds to the jth residual.

We get

$$\underline{\phi}'_{r(ij)} \hat{\underline{\theta}} = \hat{r}_i - \hat{r}_j$$

and $\underline{\phi}_{r(ij)}$ is reduced to

$$\begin{bmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ -1 \\ \cdot \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \text{ith position} \\ \\ \text{jth position} \\ \\ \\ \end{matrix}$$

Hence $V(\underline{\phi}'_{r(ij)} \hat{\underline{r}}) = \frac{\sigma^2}{n} \underline{\phi}'_{r(ij)} \left[n^2\, \underline{V}^{-1} \right] \underline{\phi}_{r(ij)}$

Now 6.2.1 can be used to calculate the variance of any linear function of $\hat{\theta}_i$ provided the corresponding linear function in $\hat{\theta}_i$ is estimable.

6.3 Efficiency of measuring Treatment Effects over Residual Effects.

From the Variance-covariance matrix of the parameter estimates derived under 6.1 we get:

$$V(\hat{\underline{t}}) = \sigma^2 \left\{ (n-1)\, \underline{K}\, \underline{V}^{-1} - \frac{1}{n} \underline{\mathcal{L}}' \underline{K}\, \underline{V}^{-1} \underline{\mathcal{L}} \right\}$$

The diagonal elements of this matrix represent the variances of their respective treatments. Therefore the trace of this matrix provides us with the total variance of the set of treatment effects given as:

$$\text{Var(Treatments)} = \sigma^2\, \text{Tr} \left\{ (n-1)\, \underline{K}\, \underline{V}^{-1} - \frac{1}{n} \underline{\mathcal{L}}' \underline{K}\, \underline{V}^{-1} \underline{\mathcal{L}} \right\}$$

If $\underline{P}$ is an orthogonal matrix, $\text{Tr}(\underline{P}' \underline{A}\, \underline{P}) = \text{Tr}(\underline{P}' \underline{P}\, \underline{A})$

$$= \text{Tr}(\underline{A})$$

Hence $\text{Var(Treatments)} = (n-1)\,\sigma^2\, \text{Tr}(\underline{K}\, \underline{V}^{-1}) - \frac{\sigma^2}{n} \text{Tr}(\underline{K}\, \underline{V}^{-1})$

Since $\underline{\mathcal{L}}$ is orthogonal

$$= (n-1-\tfrac{1}{n})\, \sigma^2\, \text{Tr}(\underline{K}\, \underline{V}^{-1})$$

Again from the Variance-covariance matrix we get:

$V(\hat{\underline{r}}) = \sigma^2\, n\, \underline{K}\, \underline{V}^{-1}$ and the trace of this matrix provides us with the total variance of the set of residual effects in the design.

$$\text{Var(Residuals)} = n\, \sigma^2\, \text{Tr}(\underline{K}\, \underline{V}^{-1})$$

To find the efficiency of measuring the treatment effects over residual effects we take the inverse ratio of their respective variances.

$$\therefore \text{ Efficiency(Treatments/ Residuals)} = \frac{n\,\sigma^2\, \text{Tr}(\underline{K}\, \underline{V}^{-1})}{(n-1-\frac{1}{n})\,\sigma^2\, \text{Tr}(\underline{K}\, \underline{V}^{-1})}$$

$$= \frac{n^2}{n^2 - n - 1}$$

where n is the number of treatments or the size of the Latin-square design used for the experiment.

Alternatively

$$\text{Efficiency(Residuals/ Treatments)} = \frac{n^2 - n - 1}{n^2}$$

$$= 1 - \frac{n - 1}{n^2}$$

For example, when n = 4

$$\text{Effy(T / R)} = \frac{16}{11} = 1.45 \quad \text{or} \quad 145\ \%$$

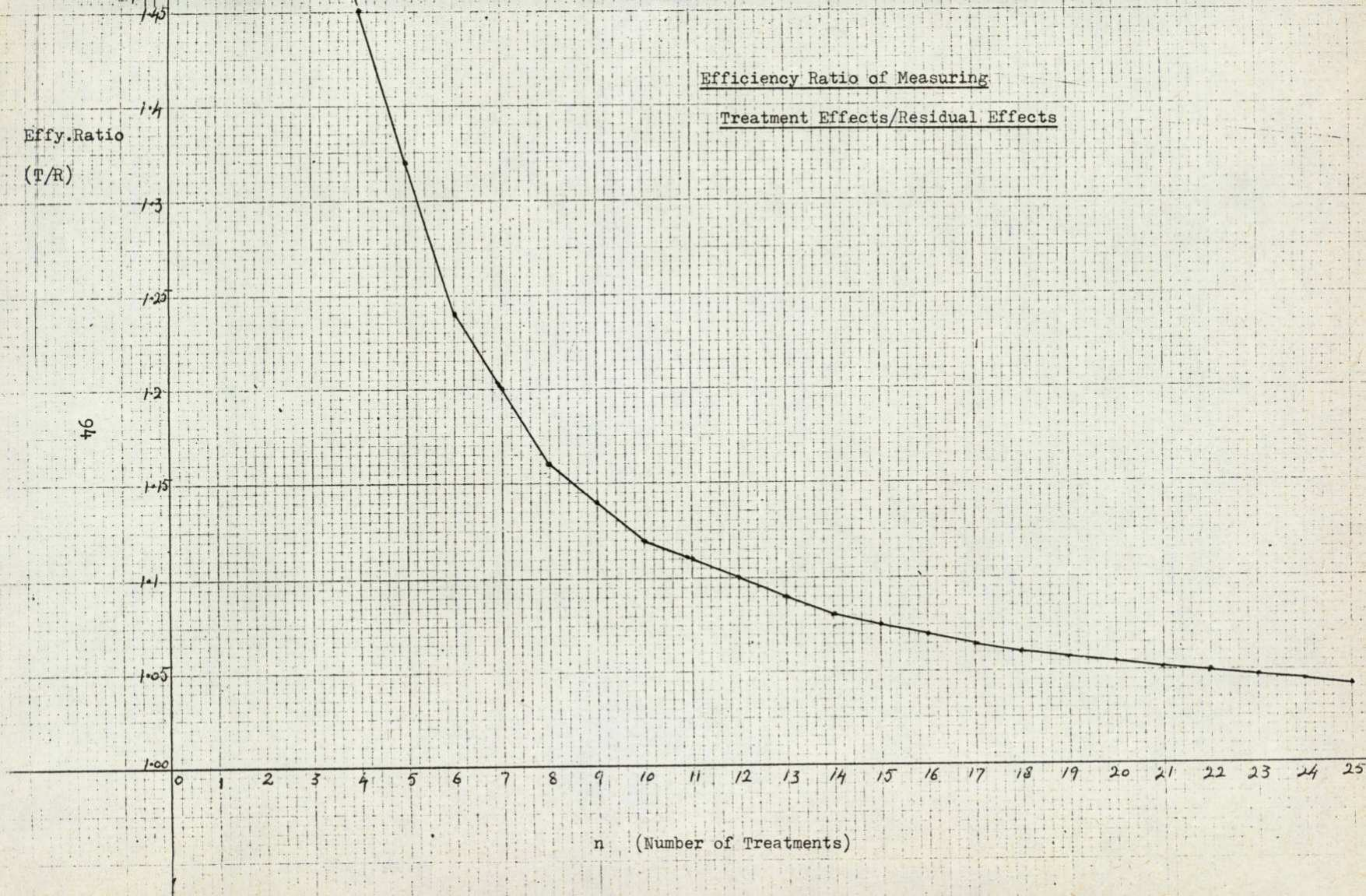
and $$\text{Effy(R / T)} = 0.6875 \quad \text{or } 69\ \%.$$

For n = 5

$$\text{Effy(T / R)} = \frac{25}{19} = 1.32 \quad \text{or} \quad 132\ \%$$

and $$\text{Effy(R / T)} = 0.76 \quad \text{or } 76\ \%.$$

As n increases the gap between the efficiencies of measuring treatment effects and residual effects closes as long as residual effects persist for one period only. This can be visualised from the graph on the next page.


Efficiency Ratio of Measuring
Treatment Effects/Residual Effects
Effy.Ratio
(T/R)
1·45
1·4
1·3
1·25
1·2
1·15
1·1
1·05
1·00
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25
n (Number of Treatments)

6.4 Efficiency Comparison.

For comparing different patterns of Latin-squares we calculate the relative efficiencies by taking the inverse ratio of the variances of the two designs or the two parameters as the case may be.

6.4.1 The table below shows the relative efficiencies calculated as stated above, for 4x4 Latin-square designs.

Design	2.4.1	2.4.2	2.4.3	2.4.4
Efficiency % (Treat./Res.)				
$V(\hat{r}_i-\hat{r}_u)/V(\hat{t}_i-\hat{t}_u)$	145	145	145	145
Treat./Treat.(2.4.2)				
$V(\hat{t}_1-\hat{t}_2)/V(\hat{t}_1-\hat{t}_2)$	20	100	60	33
$V(\hat{t}_2-\hat{t}_3)/V(\hat{t}_2-\hat{t}_3)$	20	100	75	33
$V(\hat{t}_1-\hat{t}_3)/V(\hat{t}_1-\hat{t}_3)$	20	100	75	100
$V(\hat{t}_3-\hat{t}_4)/V(\hat{t}_3-\hat{t}_4)$	20	100	60	33
Res./Res.(2.4.2)				
$V(\hat{r}_1-\hat{r}_2)/V(\hat{r}_1-\hat{r}_2)$	20	100	60	33
$V(\hat{r}_2-\hat{r}_3)/V(\hat{r}_2-\hat{r}_3)$	20	100	75	33
$V(\hat{r}_1-\hat{r}_3)/V(\hat{r}_1-\hat{r}_3)$	20	100	75	100
$V(\hat{r}_3-\hat{r}_4)/V(\hat{r}_3-\hat{r}_4)$	20	100	60	33

From the above table we find that the relative efficiency of measuring treatment as well as residual effects is the highest for design 2.4.2 and is the lowest for 2.4.1. Design 2.4.1 is the cyclic design where each treatment follows the same treatment at all occasions.

The comparison of the 4x4 Latin-square designs, with different patterns for carry-over or residual effects, can now be made with respect to variance of linear components and average variance of the differences between two estimates of treatment direct and treatment residual effects.

6.4.2

Design	Variance of Linear Components		Average Variance of Differences		Efficiency Ratio
	Treatment Direct Effects	Treatment Residual Effects	Treat. Direct Effects	Treat. Res. Effects	T.D.E / T.R.E
2.4.1	$27.5\sigma^2$	$40.0\sigma^2$	$2.75\sigma^2$	$4\sigma^2$	1.45
2.4.2	$5.5\sigma^2$	$8.0\sigma^2$	$0.55\sigma^2$	$0.8\sigma^2$	1.45
2.4.3	$6.232\sigma^2$	$9.064\sigma^2$	$0.797\sigma^2$	$1.155\sigma^2$	1.45
2.4.4	$9.9\sigma^2$	$14.4\sigma^2$	$1.283\sigma^2$	$1.867\sigma^2$	1.45

T.D.E = Treatment Direct Effect.

T.R.E = Treatment Residual Effect.

Efficiency ratio relates to both Variance of Linear Components and Average Variance of Differeneces.

To choose the most efficient design from the above table we look for the design with minimum variance of linear components and minimum average variance of differences for treatment as well as residual effects. Design 2.4.2 has the minimum variances and design 2.4.1 has the maximum variances, therefore they can be classified as the most efficient and the most inefficient designs respectively.

Chapter 7

Conclusions Derived from the Study of 4x4 Latin-square Designs.

7.1 Conclusions as to the study of 4x4 Latin-squares.

All the 4x4 Latin-square designs considered under 2.4, irrespective of their pattern, measure the difference between two treatment direct effects 45% more efficiently than the difference between two treatment residual effects, when it is assumed that the residual effects are present for one period only.

From table 6.4.2 we find that design 2.4.2 is the most efficient for measuring both treatment direct and treatment residual effects. In this design each treatment was preceded by another treatment only once.

The second best design is 2.4.3 and third in order of efficiency is 2.4.4. The general pattern of these designs is the same as in both the designs each treatment was preceded twice by one treatment and once by another treatment. Though there is similarity in the general pattern of these designs yet they do not measure the parameter estimates with the same precision.

Design 2.4.1 is the most inefficient design for measuring both treatment direct effects as well as treatment residual effects. Therefore, designs with each treatment followed by the same treatment throughtout constitute inefficient designs and should be avoided for practical use when the presence of residual effects is suspected.

From the last column of table 6.4.2 we notice that the variance of the linear component of residual effects

is 1.45 times more than the variance of the linear component of treatment effects and hence we find again that the treatment linear effects, for all 4x4 Latin-square designs, are measured 45% more efficiently than the residual linear effects; exactly the same degree of efficiency as the treatment differences over residual differences calculated under 2.14 and 5.5.
The loss in efficiency in measuring the residual effects is not purely due to the nature of the designs but also because of the fact that only $\frac{3}{4}$ of the number of observations have been used for extracting residual effects.

Another observation from table 6.4.2 is that the variance of the linear component of parameters for the most efficient design 2.4.2 is not widely different from that of 2.4.3 design. This may lead us to conclude that there could be designs other than balanced designs which measure treatment or residual effects equally efficiently and could be worthwhile considering from the point of view of practical convenience. Design 2.4.2 may be called a balanced design since each treatment in this design was followed by a different treatment on all occasions though not by itself on any occasion, but in design 2.4.3 there is no such balance.

Chapter 8

Survey of the Balanced Designs for Odd Number of Treatments.

8.1 Balanced Designs.

According to Williams (1949) Latin-square designs can be balanced for the residual effects, persisting for only one period, when n the number of treatments is even and to achieve balance when n the number of treatments is odd we require 2n replications. In other words, for an odd number of treatments, we need to conduct two Latin-squares, of the same size side by side.

Therefore to achieve balance, as suggested by Williams, for a 5x5 Latin-square design we require at least ten subjects and the experiment would involve twice as much work as expected from a single 5x5 Latin-square.

For example, when it is required to test five treatments over five periods we would require ten subjects and the design will take the form as below.

		Subjects									
		S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
	P_1	1	2	3	4	5	1	2	3	4	5
	P_2	3	4	5	1	2	2	3	4	5	1
Periods	P_3	2	3	4	5	1	4	5	1	2	3
	P_4	5	1	2	3	4	5	1	2	3	4
	P_5	4	5	1	2	3	3	4	5	1	2

In this design each treatment follows every other treatment twice.

The true balance is still incomplete because a treatment is never preceded by itself as suggested by Cochran (page 134), Lucas (1957) and Berenblut (1964).

In this design, where we are testing five treatments over five periods, the condition of ten subjects may increase the size of the experiment to an unmanageable size. Therefore, the advantage one is looking for in the balance may be lost in the magnitude of the experiment. Just five observations for five subjects at five time intervals should be sufficient for testing five treatments.

Lucas (1956) suggests introducing an extra period by repeating the last row of the design so that each treatment is preceded by itself as well and the true balance is completed. For example, a pair of 5x5 Latin-square designs balanced with an extra period, where each treatment is preceded by every other treatment and is also preceded by itself equal number of times, twice in this case, is given below.

		Subjects									
		S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
Periods	P_1	1	2	3	4	5	1	2	3	4	5
	P_2	3	4	5	1	2	2	3	4	5	1
	P_3	2	3	4	5	1	4	5	1	2	3
	P_4	5	1	2	3	4	5	1	2	3	4
	P_5	4	5	1	2	3	3	4	5	1	2
	P_6	4	5	1	2	3	3	4	5	1	2

Lucas (1956) produced an orthogonal set of four squares for five treatments to be used as a complete set for the experiment.

Lucas mentions that the advantages of extra-period designs over the regular Latin-square designs decrease as the number of treatments is increased. Extra-period designs are less efficient than the Latin-square designs if residual effects are assumed not to exist.

The point made by Cochran et al about Williams' designs and the solution produced by Lucas by introducing an extra period even does not provide ideal designs for experiments to be conducted, for example, with dairy-cattle, agricultural crops, hospital patients, factory machines etc, where time is an important factor. One may not be able to find unrestricted number of subjects and may fail to comply with Williams' condition of balance for odd number of treatments. This demonstrates the importance of single Latin-squares,especially when the number of treatments to be tested is odd.

Chapter 9

Parameter Estimates of 5x5 Latin-square Designs when Residual Effects Persist for one Period only.

9.1 Aim of studying 5x5 Latin-square Designs.

It is desired to investigate the 5x5 Latin-square designs which measure highly efficiently the treatment direct and residual effects, persisting only for one period, without putting any extra conditions on them. A single Latin-square design carries a great advantage over the combination of two or more Latin-squares operated side by side or by introducing an extra period for the obvious reasons mentioned earlier.

A 5x5 Latin-square is of special quality and importance. The number of treatments, subjects and periods required to conduct such an experiment are very reasonable. A 3x3 Latin-square is too small and a 7x7 Latin-square is too big to be of common use. For instance, the typical working week of five days adds to the importance of this design if, say, faced with a study of labour output over a number of days.

The aim of reducing the bulk of the design can be achieved by constructing balanced incomplete designs by special arrangement of the sequences as described by Patterson (1951, 1952).

Such designs, if used, would produce insufficient balance. To overcome this difficulty Patterson and Lucas (1959) suggest the use of an extra period in these designs. The analysis of such designs have produced bias in the estimation of variances though not of significant magnitude to be of practical importance.

9.2 5x5 Latin-square Design.

As already explained under 2.2 a single Latin-square is not suitable for estimating the residual effects as it will then fail to provide any degree of freedom for the error term for making tests of significance. Therefore the simplest single design for the odd number of treatments we can consider is a 5x5 Latin-square design.

To investigate the efficiency in measuring the treatment direct and residual effects, in the patterns of 5x5 Latin-squares, a sample of 31 designs of order 5 given by Fisher and Yates(1953) has been considered.

For each of the designs listed under 9.3.1 normal equations for periods, subjects, treatment direct effects and treatment residual effects were obtained.

The period effects are directly available from their individual normal equations.

The subject, treatment direct and treatment residual effects are interdependent, therefore the normal equations of subjects, treatments and residuals are to be solved simultaneously to obtain the respective effects. This set of normal equations amounts to 15 equations and therefore it is not easy to solve these equations manually. Hence the normal equations are written in matrix form under 9.5.2.

The coefficient matrices are inverted by computer using an ICL scientific subroutine in Fortran. For the programme see Appendix I.

9.3 5x5 Latin-square Designs given by Fisher and Yates(1953).

9.3.1

1.

1	2	3	4	5
2	1	5	3	4
3	4	1	5	2
4	5	2	1	3
5	3	4	2	1

2.

1	2	3	4	5
2	1	4	5	3
3	5	2	1	4
4	3	5	2	1
5	4	1	3	2

3.

1	2	3	4	5
2	1	5	3	4
3	5	4	1	2
4	3	2	5	1
5	4	1	2	3

4.

1	2	3	4	5
2	1	4	5	3
3	4	5	1	2
4	5	2	3	1
5	3	1	2	4

5.

1	2	3	4	5
2	3	4	5	1
3	5	2	1	4
4	1	5	2	3
5	4	1	3	2

6.

1	2	3	4	5
2	3	4	5	1
3	5	1	2	4
4	1	5	3	2
5	4	2	1	3

7.

1	2	3	4	5
2	3	5	1	4
3	4	2	5	1
4	5	1	3	2
5	1	4	2	3

8.

1	2	3	4	5
2	3	5	1	4
3	1	4	5	2
4	5	2	3	1
5	4	1	2	3

9.

1	2	3	4	5
2	3	4	5	1
3	1	5	2	4
4	5	1	3	2
5	4	2	1	3

10.

1	2	3	4	5
2	3	1	5	4
3	5	4	1	2
4	1	5	2	3
5	4	2	3	1

11.

1	2	3	4	5
2	3	1	5	4
3	4	5	2	1
4	5	2	1	3
5	1	4	3	2

12.

1	2	3	4	5
2	4	5	3	1
3	1	2	5	4
4	5	1	2	3
5	3	4	1	2

13.

1	2	3	4	5
2	4	1	5	3
3	5	4	2	1
4	3	5	1	2
5	1	2	3	4

14.

1	2	3	4	5
2	4	5	3	1
3	1	4	5	2
4	5	1	2	3
5	3	2	1	4

15.

1	2	3	4	5
2	4	1	5	3
3	5	4	2	1
4	1	5	3	2
5	3	2	1	4

16.

1	2	3	4	5
2	4	5	1	3
3	5	4	2	1
4	3	1	5	2
5	1	2	3	4

17.

1	2	3	4	5
2	4	1	5	3
3	5	2	1	4
4	1	5	3	2
5	3	4	2	1

18.

1	2	3	4	5
2	4	5	1	3
3	5	1	2	4
4	3	2	5	1
5	1	4	3	2

19.

1	2	3	4	5
2	5	4	1	3
3	1	2	5	4
4	3	5	2	1
5	4	1	3	2

20.

1	2	3	4	5
2	5	4	1	3
3	1	5	2	4
4	3	1	5	2
5	4	2	3	1

21.

1	2	3	4	5
2	5	1	3	4
3	4	5	2	1
4	1	2	5	3
5	3	4	1	2

22.

1	2	3	4	5
2	5	4	3	1
3	1	5	2	4
4	3	1	5	2
5	4	2	1	3

23.

1	2	3	4	5
2	5	4	3	1
3	4	5	1	2
4	1	2	5	3
5	3	1	2	4

24.

1	2	3	4	5
2	5	1	3	4
3	4	2	5	1
4	3	5	1	2
5	1	4	2	3

25.

1	2	3	4	5
2	5	4	1	3
3	4	1	5	2
4	3	5	2	1
5	1	2	3	4

26.

1	2	3	4	5
2	3	5	1	4
3	5	4	2	1
4	1	2	5	3
5	4	1	3	2

27.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

28.

1	2	3	4	5
2	4	5	3	1
3	5	2	1	4
4	3	1	5	2
5	1	4	2	3

29.

1	2	3	4	5
2	4	1	5	3
3	1	5	2	4
4	5	2	3	1
5	3	4	1	2

30.

1	2	3	4	5
2	5	4	1	3
3	4	2	5	1
4	1	5	3	2
5	3	1	2	4

31.

1	2	3	4	5
2	5	1	3	4
3	1	4	5	2
4	3	5	2	1
5	4	2	1	3

This sample of designs covers all possible types of patterns for 5x5 Latin-squares.

9.4 Analysis of 5x5 Latin-square Design.

The design 9.3.1(1) has been chosen to illustrate the derivation of the normal equations for the estimation of treatment, residual, subject and period effects of a 5x5 Latin-square design with the extra assumption of residual effects persisting for one period only.

		Subjects				
		S_1	S_2	S_3	S_4	S_5
	P_1	1	2	3	4	5
	P_2	2	1	5	3	4
Periods	P_3	3	4	1	5	2
	P_4	4	5	2	1	3
	P_5	5	3	4	2	1

Normal equations.

9.4.1 Period effects.

$P_1=5\hat{\mu}+5\hat{p}_1$

$P_2=5\hat{\mu}+5\hat{p}_2$

$P_3=5\hat{\mu}+5\hat{p}_3$

$P_4=5\hat{\mu}+5\hat{p}_4$

$P_5=5\hat{\mu}+5\hat{p}_5$

$\hat{\mu}$ =(Grand total)÷25 $=\frac{G}{25}$ $=0.04G$

9.4.2 subject effects.

$$S_1 = 5\,\hat{\mu} + 5\hat{s}_1 - \hat{r}_5$$

$$S_2 = 5\,\hat{\mu} + 5\hat{s}_2 - \hat{r}_3$$

$$S_3 = 5\,\hat{\mu} + 5\hat{s}_3 - \hat{r}_4$$

$$S_4 = 5\,\hat{\mu} + 5\hat{s}_4 - \hat{r}_2$$

$$S_5 = 5\,\hat{\mu} + 5\hat{s}_5 - \hat{r}_1$$

9.4.3 Treatment effects.

$$T_1 = 5\,\hat{\mu} + 5\hat{t}_1 + \hat{r}_2 + \hat{r}_3 + 2\hat{r}_5$$

$$T_2 = 5\,\hat{\mu} + 5\hat{t}_2 + 3\hat{r}_1 + \hat{r}_4$$

$$T_3 = 5\,\hat{\mu} + 5\hat{t}_3 + 2\hat{r}_2 + \hat{r}_4 + \hat{r}_5$$

$$T_4 = 5\,\hat{\mu} + 5\hat{t}_4 + \hat{r}_1 + \hat{r}_2 + \hat{r}_3 + \hat{r}_5$$

$$T_5 = 5\,\hat{\mu} + 5\hat{t}_5 + 2\hat{r}_3 + 2\hat{r}_4$$

9.4.4 Residual effects.

$$R_1 = 4\,\hat{\mu} + 4\hat{r}_1 + 3\hat{t}_2 + \hat{t}_4 - \hat{s}_5 - \hat{p}_1$$

$$R_2 = 4\,\hat{\mu} + 4\hat{r}_2 + \hat{t}_1 + 2\hat{t}_3 + \hat{t}_4 - \hat{s}_4 - p_1$$

$$R_3 = 4\,\hat{\mu} + 4\hat{r}_3 + \hat{t}_1 + \hat{t}_4 + 2\hat{t}_5 - \hat{s}_2 - \hat{p}_1$$

$$R_4 = 4\,\hat{\mu} + 4\hat{r}_4 + \hat{t}_2 + \hat{t}_3 + 2\hat{t}_5 - \hat{s}_3 - \hat{p}_1$$

$$R_5 = 4\,\hat{\mu} + 4\hat{r}_5 + 2\hat{t}_1 + \hat{t}_3 + \hat{t}_4 - \hat{s}_1 - \hat{p}_1$$

Capital letters represent the total response and small letters represent individual effects.

9.5 <u>Parameter estimates of the 5x5 Latin-square Design.</u>

Period effects can be estimated directly by solving the normal equations for period effects mentioned under 9.4.1. To estimate the treatment, residual and subject effects we must solve the normal equations 9.4.2, 9.4.3 and 9.4.4 simultaneously. After some simplification the fifteen normal equations of the parameters can be put in matrix form and then using matrix theory we get the parameter estimates.

9.5.1 Estimates of period effects.

$$\hat{p}_1 = 0.2\ P_1 - 0.4\ G$$
$$\hat{p}_2 = 0.2\ P_2 - 0.4\ G$$
$$\hat{p}_3 = 0.2\ P_3 - 0.4\ G$$
$$\hat{p}_4 = 0.2\ P_4 - 0.4\ G$$
$$\hat{p}_5 = 0.2\ P_5 - 0.4\ G$$

9.5.2 Matrix form of the normal equations.

$$\begin{bmatrix}
5 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 2 & 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 1 & 2 & 0 & 0 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & -1 & 0 & 0 \\
2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{bmatrix}
\begin{bmatrix}
\hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \\ \hat{t}_4 \\ \hat{t}_5 \\ \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \\ \hat{r}_5 \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ \hat{s}_4 \\ \hat{s}_5
\end{bmatrix}
=
\begin{bmatrix}
T_1-0.2G \\ T_2-0.2G \\ T_3-0.2G \\ T_4-0.2G \\ T_5-0.2G \\ R_1-0.2G+0.2P_1 \\ R_2-0.2G+0.2P_1 \\ R_3-0.2G+0.2P_1 \\ R_4-0.2G+0.2P_1 \\ R_5-0.2G+0.2P_1 \\ S_1-0.2G \\ S_2-0.2G \\ S_3-0.2G \\ S_4-0.2G \\ S_5-0.2G
\end{bmatrix}$$

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0.28431	0.20287	-0.36616	-0.07323	-0.05686	-0.04057	-0.06122	-0.03478
0.14492	0.55333	0.16521	0.20834	0.19487	-0.49166	-0.17992	-0.19553	-0.29165	-0.17457	-0.03491	-0.03911	-0.05833	-0.03598	-0.09833
0.23624	0.16521	0.45693	0.21341	0.19487	-0.18659	-0.35882	-0.22089	-0.26629	-0.30074	-0.06015	-0.04418	-0.05326	-0.07176	-0.03732
0.22609	0.20834	0.21341	0.42396	0.19487	-0.27604	-0.28339	-0.27363	-0.21355	-0.28672	-0.05734	-0.05473	-0.04271	-0.05668	-0.05521
0.19487	0.19487	0.19487	0.19487	0.48718	-0.20513	-0.20513	-0.35897	-0.35897	-0.20513	-0.04103	-0.07179	-0.07179	-0.04103	-0.04103
-0.17391	-0.49166	-0.18659	-0.27604	-0.20513	0.72396	0.21661	0.22637	0.28645	0.21328	0.04266	0.04527	0.05729	0.04332	0.14479
-0.30603	-0.17992	-0.35882	-0.28339	-0.20513	0.21661	0.60713	0.26309	0.24973	0.33010	0.06602	0.05262	0.04995	0.12143	0.04332
-0.28431	-0.19553	-0.22089	-0.27363	-0.35897	0.22637	0.26309	0.59892	0.29852	0.27977	0.05595	0.11978	0.05970	0.05262	0.04527
-0.20287	-0.29165	-0.26629	-0.21355	-0.35897	0.28645	0.24973	0.29852	0.59892	0.23305	0.04661	0.05970	0.11978	0.04995	0.05729
-0.36616	-0.17457	-0.30074	-0.28672	-0.20513	0.21328	0.33010	0.27977	0.23305	0.61047	0.12209	0.05595	0.04661	0.06602	0.04266
-0.07323	-0.03491	-0.06015	-0.05734	-0.04103	0.04266	0.06602	0.05595	0.04661	0.12209	0.22442	0.01119	0.00932	0.01320	0.00853
-0.05686	-0.03911	-0.04418	-0.05473	-0.07179	0.04527	0.05262	0.11978	0.05970	0.05595	0.01119	0.22396	0.01194	0.01052	0.00905
-0.04057	-0.05833	-0.05326	-0.04271	-0.07179	0.05729	0.04995	0.05970	0.11978	0.04661	0.00932	0.01194	0.22396	0.00999	0.01146
-0.06122	-0.03598	-0.07176	-0.05668	-0.04103	0.04332	0.12143	0.05262	0.04995	0.06602	0.01320	0.01052	0.00999	0.22429	0.00866
-0.03478	-0.09833	-0.03732	-0.05521	-0.04103	0.14479	0.04332	0.04527	0.05729	0.04266	0.00853	0.00905	0.01146	0.00866	0.22696

In matrix notation

$$\underline{X}\,\hat{\underline{\theta}} = \underline{Y}$$

and $$\hat{\underline{\theta}} = \underline{X}^{-1}\,\underline{Y}$$

where $\underline{X}$ is 15x15 matrix of the coefficients of the parameters to be estimated.

$\underline{Y}$ is the column vector of totals of observations plus factors G and P.

$\hat{\underline{\theta}}$ is the column vector of parameters to be estimated.

9.5.3 Inverted matrix i.e. $\underline{X}^{-1}$ at page 113.

9.5.4 Estimates of treatment effects.

$$\hat{t}_1 = 0.464T_1+0.145T_2+0.236T_3+0.226T_4+0.195T_5-0.174R_1-0.306R_2 -0.284R_3-0.203R_4-0.366R_5-0.073S_1-0.057S_2-0.041S_3 -0.061S_4-0.035S_5+0.067G -0.267P_1$$

$$\hat{t}_2 = 0.145T_1+0.553T_2+0.165T_3+0.208T_4+0.195T_5-0.492R_1-0.18R_2 -0.2R_3-0.292R_4-0.175R_5-0.035S_1-0.039S_2-0.058S_3 -0.036S_4-0.098S_5+0.067G -0.267P_1$$

$$\hat{t}_3 = 0.236T_1+0.165T_2+0.457T_3+0.213T_4+0.195T_5-0.187R_1-0.359R_2 -0.221R_3-0.266R_4-0.301R_5-0.06S_1-0.044S_2-0.053S_3 -0.072S_4-0.037S_5+0.067G -0.267P_1$$

$$\hat{t}_4 = 0.226T_1+0.208T_2+0.213T_3+0.424T_4+0.195T_5-0.276R_1-0.283R_2 -0.274R_3-0.214R_4-0.287R_5-0.057S_1-0.055S_2-0.043S_3 -0.057S_4-0.55S_5+0.067G -0.267P_1$$

$$\hat{t}_5 = 0.195T_1+0.195T_2+0.195T_3+0.195T_4+0.487T_5-0.205R_1$$
$$-0.205R_2-0.359R_3-0.359R_4-0.205R_5-0.041S_1-0.072S_2$$
$$-0.072S_3-0.041S_4-0.041S_5+0.067G-0.267P_1$$

9.5.5 Estimates of residual effects.

$$\hat{r}_1=0.724R_1+0.217R_2+0.226R_3+0.286R_4+0.213R_5-0.174T_1$$
$$-0.492T_2-0.187T_3-0.276T_4-0.205T_5+0.043S_1+0.045S_2$$
$$-0.057S_3+0.043S_4+0.145S_5-0.133G+0.333P_1$$

$$\hat{r}_2=0.217R_1+0.607R_2+0.263R_3+0.25R_4+0.33R_5-0.306T_1-0.18T_2$$
$$-0.359T_3-0.283T_4-0.205T_5+0.066S_1+0.053S_2+0.05S_3$$
$$+0.121S_4+0.043S_5-0.133G+0.333P_1$$

$$\hat{r}_3=0.226R_1+0.263R_2+0.599R_3+0.299R_4+0.28R_5-0.284T_1$$
$$-0.196T_2-0.221T_3-0.274T_4-0.359T_5+0.056S_1+0.12S_2$$
$$+0.06S_3+0.052S_4+0.045S_5-0.133G+0.333P_1$$

$$\hat{r}_4=0.286R_1+0.25R_2+0.3R_3+0.599R_4+0.233R_5-0.203T_1-0.292T_2$$
$$-0.266T_3-0.214T_4-0.359T_5+0.047S_1+0.06S_2+0.12S_3$$
$$+0.05S_4+0.057S_5-0.133G+0.333P_1$$

$$\hat{r}_5=0.213R_1+0.33R_2+0.28R_3+0.233R_4+0.61R_5-0.366T_1-0.175T_2$$
$$-0.301T_3-0.287T_4-0.205T_5+0.122S_1+0.056S_2$$
$$+0.047S_3+0.066S_4+0.043S_5-0.133G+0.333P_1$$

9.5.6 Subject effects.

$$\hat{s}_1 = 0.224S_1+0.011S_2+0.009S_3+0.013S_4+0.008S_5-0.073T_1$$
$$-0.035T_2-0.06T_3-0.057T_4-0.041T_5+0.043R_1+0.066R_2$$
$$+0.056R_3+0.047R_4+0.122R_5-0.067G +0.067P_1$$

$$\hat{s}_2=0.011S_1+0.224S_2+0.012S_3+0.011S_4+0.009S_5-0.057T_1$$
$$-0.039T_2-0.044T_3-0.055T_4-0.072T_5+0.045R_1+0.053R_2$$
$$+0.12R_3+0.06R_4+0.056R_5-0.067G +0.067P_1$$

$$\hat{s}_3=0.009S_1+0.012S_2+0.224S_3+0.01S_4+0.225S_5-0.041T_1$$
$$-0.0.058T_2-0.053T_3-0.043T_4-0.072T_5+0.573R_1$$
$$+0.05R_2+0.06R_3+0.12R_4+0.047R_5-0.067G +0.067P_1$$

$$\hat{s}_4=0.013S_1+0.011S_2+0.01S_3+0.224S_4+0.009S_5-0.061T_1-0.036T_2$$
$$-0.072T_3-0.057T_4-0.041T_5+0.043R_1+0.121R_2+0.053R_3$$
$$+0.05R_4+0.066R_5-0.067G +0.067P_1$$

$$\hat{s}_5=0.009S_1+0.009S_2+0.011S_3+0.009S_4+0.229S_5-0.035T_1$$
$$-0.098T_2-0.037T_3-0.055T_4-0.041T_5+0.145R_1+0.043R_2$$
$$+0.045R_3+0.057R_4+0.043R_5-0.067G +0.067P_1$$

9.6 <u>Efficiency factor.</u>

9.6.1 Variance of the difference between two treatment effects.

$$V(\hat{t}_1-\hat{t}_2) = V(0.464T_1+0.145T_2-0.145T_1-0.553T_2+\ldots\ldots)$$
$$= 0.727\sigma^2$$

9.6.2 Variance of the difference between two residual effects.

$$V(\hat{r}_1-\hat{r}_2) = V(0.724R_1+0.217R_2\ldots\ldots-0.217R_1-0.607R_2\ldots\ldots)$$
$$=0.897\,\sigma^2$$

$V(\hat{t}_3-\hat{t}_5)=V(0.457T_3+0.195T_5....-0.195T_3-0.487T_5....)$

$=0.554\ \sigma^2$

$V(\hat{t}_1-\hat{t}_3)=0.449\ \sigma^2$ $V(\hat{t}_2-\hat{t}_4)=0.561\ \sigma^2$

$V(\hat{t}_1-\hat{t}_4)=0.436\sigma^2$ $V(\hat{t}_2-\hat{t}_5)=0.650\sigma^2$

$V(\hat{r}_3-\hat{r}_5)=V(0.599R_3+0.28R_5.....-0.28R_3-61R_5......)$

$=0.649\ \sigma^2$

$V(\hat{r}_1-\hat{r}_3)=0.871\ \sigma^2$ $V(\hat{r}_2-\hat{r}_4)=0.706\ \sigma^2$

$V(\hat{r}_1-\hat{r}_4)=0.751\ \sigma^2$ $V(\hat{r}_2-\hat{r}_5)=0.557\ \sigma^2$

The efficiency ratios of the measurements of the difference between two treatment effects over the difference between two residual effects for 1&2, 1&3, 1&4, 2&4, 2&5, 3&5, etc. can be stated in percentages as 123%, 194%, 172%, 126%, 86%, 117%... ....respectively.

This is due to the fact that the residual effects are based on four observations only while the treatment effects are based on five observations. There is no uniformity in the treatments following other treatments, hence variable efficiency factors for measurements of differences. The efficiecy percentages vary from 86% to 194% in the above calculations, which shows that this pattern of Latin-square does not measure all the parameter effects equally efficiently.

9.7 Variance of Linear Components of 5x5 Latin-squares.

Investigating the variance of linear components of treatment direct and treatment residual effects.

Coefficients of orthogonal polynomials for linearity for five variables are given by ξ_1.

Treatment/ Residual	1	2	3	4	5
ξ_1	-2	-1	0	1	2

9.7.1 Variance of linear component of treatment direct effects.

$$-2\hat{t}_1 - \hat{t}_2 + 0 + \hat{t}_4 + \hat{t}_5 = -0.459T_1 - 0.245T_2 - 0.034T_3 + 0.154T_4 + 0.584T_5 + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\therefore V(-2\hat{t}_1 - \hat{t}_2 + \hat{t}_4 + 2\hat{t}_5) = V(-2x-0.459T_1 - 1x-0.245T_2 + 1x0.154T_4 + 2x0.584T_5 + \ldots\ldots\ldots\ldots)$$

$$= V(0.918T_1 + 0.245T_2 + 0.154T_4 + 1.168T_5 \ldots)$$

$$= 2.485\ \sigma^2$$

9.7.2 Variance of linear component of residual effects.

$$-2\hat{r}_1 - \hat{r}_2 + 0 + \hat{r}_4 + 2\hat{r}_5 = -0.953R_1 - 0.131R_2 + 0.144R_3 + 0.244R_4 + 0.697R_5 + \ldots\ldots\ldots\ldots$$

$$V(-2\hat{r}_1 - \hat{r}_2 + \hat{r}_4 + 2\hat{r}_5) = V(-2x-0.953R_1 - 1x-0.131R_2 + 1x0.244R_4 + 2x0.697R_5 + \ldots\ldots\ldots)$$

$$= V(1.906R_1 + 0.131R_2 + 0.244R_4 + 1.394R_5 \ldots.)$$

$$= 3.675\ \sigma^2$$

Chapter 10

Comparison of the Variances of Linear Components of the Treatment and Residual Effects for 5x5 Latin-squares.

10.1 Comparison of the Variance of Linear Components of the Parameters.

Following the same procedure of writing up the normal equations for all the parameters as in 9.4 and working out the variances of the linear components for treatment direct and residual effects, we can make the comparison between the designs both from treatment and residual point of view. Taking all the thirty one 5x5 Latin-square designs given under 9.3.1 and consulting their inverted matrices, the variances of the linear components of the parameters are:

	Variance of linear component	
Design Number.	Treatment effects.	Residual effects.
1	$2.485\ \sigma^2$	$3.675\ \sigma^2$
2	$3.543\ \sigma^2$	$4.072\ \sigma^2$
3	$2.918\ \sigma^2$	$3.392\ \sigma^2$
4	$8.072\ \sigma^2$	$4.661\ \sigma^2$
5	$7.414\ \sigma^2$	$9.961\ \sigma^2$
6	$2.771\ \sigma^2$	$5.346\ \sigma^2$
7	$2.342\ \sigma^2$	$2.893\ \sigma^2$
8	$2.844\ \sigma^2$	$3.796\ \sigma^2$
9	$3.099\ \sigma^2$	$4.661\ \sigma^2$
10	$2.750\ \sigma^2$	$4.897\ \sigma^2$
11	$2.562\ \sigma^2$	$3.563\ \sigma^2$
12	$2.815\ \sigma^2$	$3.414\ \sigma^2$
13	$7.553\ \sigma^2$	$4.676\ \sigma^2$
14	$2.703\ \sigma^2$	$3.375\ \sigma^2$

Design Number.	Variance of linear component	
	Treatment effects.	Residual effects.
15	$7.840\ \sigma^2$	$4.799\ \sigma^2$
16	$3.550\ \sigma^2$	$9.945\ \sigma^2$
17	$2.788\ \sigma^2$	$3.274\ \sigma^2$
18	$3.652\ \sigma^2$	$10.320\ \sigma^2$
19	$4.063\ \sigma^2$	$3.645\ \sigma^2$
20	$3.722\ \sigma^2$	$3.625\ \sigma^2$
21	$2.602\ \sigma^2$	$3.705\ \sigma^2$
22	$7.567\ \sigma^2$	$9.759\ \sigma^2$
23	$2.195\ \sigma^2$	$3.092\ \sigma^2$
24	$3.543\ \sigma^2$	$10.829\ \sigma^2$
25	$2.280\ \sigma^2$	$3.008\ \sigma^2$
26	$2.316\ \sigma^2$	$3.056\ \sigma^2$
27	$12.670\ \sigma^2$	$16.670\ \sigma^2$
28	$2.476\ \sigma^2$	$3.266\ \sigma^2$
29	$2.194\ \sigma^2$	$2.886\ \sigma^2$
30	$2.476\ \sigma^2$	$3.266\ \sigma^2$
31	$2.316\ \sigma^2$	$3.056\ \sigma^2$

10.1.1 Inferences from the study of the variances of the Linear Components.

Design 9.3.1(27) is the most inefficient design for estimating both treatment and residual effects as it has the highest variance of the linear components. This design has the character of a treatment preceded by the same treatment on all occasions, in other words it is a design of cyclic pattern. Therefore, to investigate the parameters of a 5x5 Latin-square, when the treatments are supposed to have carry-over effects, a design of this pattern should be avoided at all costs.

Design 9.3.1(29) gives minimum variance of linear components for both treatment and residual effects and therefore can be considered as the most efficient design, out of the sample of 31 designs given under 9.3.1. In this Latin-square each treatment has been preceded by two other treatments exactly twice.

Other designs such as 9.3.1(7, 12, 21, 23, 25, 26, 28, 30 and 31) can also be considered as efficient since their variances of linear components are not widely different from that of the most efficient design. It is hard to decide, at this stage, on the pattern of these designs since they do not follow any set pattern as in the case of the most efficient design or the most inefficient design. The patterns of these and other less efficient designs have treatments preceded by other treatments once and three times or a mixture of once, twice, three times and four times. Therefore, a further study would be needed to accept or reject these designs as efficient or inefficient.

Chapter 11

The Most Efficient Design.

11.1 Design with Minimum Variance of the Linear Components.

The most efficient design we selected on the basis of minimum variance of the linear components for both treatment direct and treatment residual effects;

		Subjects				
		S_1	S_2	S_3	S_4	S_5
	P_1	1	2	3	4	5
	P_2	2	4	1	5	3
Periods	P_3	3	1	5	2	4
	P_4	4	5	2	3	1
	P_5	5	3	4	1	2

can now be randomised by rows and this gives us 120 new designs. It is unnecessary to randomise by columns since it does not change the influence of residual effects in any way. The residual effects occur along periods and the periods are placed along rows, hence the rows have been randomised. Subjects once assigned columns are not changed.

The randomised designs appear on the next few pages.

11.1.1 The list of 120 new designs formed by randomising the most efficient design from our original sample of 31 designs. Design 1 in this list is the parent design.

1. 1 2 3 4 5
2 4 1 5 3
3 1 5 2 4
4 5 2 3 1
5 3 4 1 2

2. 1 2 3 4 5
2 4 1 5 3
3 1 5 2 4
5 3 4 1 2
4 5 2 3 1

3. 1 2 3 4 5
2 4 1 5 3
5 3 4 1 2
3 1 5 2 4
4 5 2 3 1

4. 1 2 3 4 5
5 3 4 1 2
2 4 1 5 3
3 1 5 2 4
4 5 2 3 1

5. 5 3 4 1 2
1 2 3 4 5
2 4 1 5 3
3 1 5 2 4
4 5 2 3 1

6. 5 3 4 1 2
1 2 3 4 5
2 4 1 5 3
4 5 2 3 1
3 1 5 2 4

7. 5 3 4 1 2
1 2 3 4 5
4 5 2 3 1
2 4 1 5 3
3 1 5 2 4

8. 5 3 4 1 2
4 5 2 3 1
1 2 3 4 5
2 4 1 5 3
3 1 5 2 4

9. 4 5 2 3 1
5 3 4 1 2
1 2 3 4 5
2 4 1 5 3
3 1 5 2 4

10. 4 5 2 3 1
5 3 4 1 2
1 2 3 4 5
3 1 5 2 4
2 4 1 5 3

11. 4 5 2 3 1
5 3 4 1 2
3 1 5 2 4
1 2 3 4 5
2 4 1 5 3

12. 4 5 2 3 1
3 1 5 2 4
5 3 4 1 2
1 2 3 4 5
2 4 1 5 3

13. 3 1 5 2 4
4 5 2 3 1
5 3 4 1 2
1 2 3 4 5
2 4 1 5 3

14. 3 1 5 2 4
4 5 2 3 1
5 3 4 1 2
2 4 1 5 3
1 2 3 4 5

15. 3 1 5 2 4
4 5 2 3 1
2 4 1 5 3
5 3 4 1 2
1 2 3 4 5

16. 3 1 5 2 4
2 4 1 5 3
4 5 2 3 1
5 3 4 1 2
1 2 3 4 5

17. 2 4 1 5 3
3 1 5 2 4
4 5 2 3 1
5 3 4 1 2
1 2 3 4 5

18. 2 4 1 5 3
3 1 5 2 4
4 5 2 3 1
1 2 3 4 5
5 3 4 1 2

19. 2 4 1 5 3
3 1 5 2 4
1 2 3 4 5
4 5 2 3 1
5 3 4 1 2

20. 2 4 1 5 3
1 2 3 4 5
3 1 5 2 4
4 5 2 3 1
5 3 4 1 2

21. 1 2 3 4 5
2 4 1 5 3
5 3 4 1 2
4 5 2 3 1
3 1 5 2 4

22. 1 2 3 4 5
2 4 1 5 3
4 5 2 3 1
5 3 4 1 2
3 1 5 2 4

23. 1 2 3 4 5
2 4 1 5 3
4 5 2 3 1
3 1 5 2 4
5 3 4 1 2

24. 1 2 3 4 5
5 3 4 1 2
4 5 2 3 1
3 1 5 2 4
2 4 1 5 3

25. 5 3 4 1 2
1 2 3 4 5
4 5 2 3 1
3 1 5 2 4
2 4 1 5 3

26. 5 3 4 1 2
1 2 3 4 5
3 1 5 2 4
4 5 2 3 1
2 4 1 5 3

27. 5 3 4 1 2
1 2 3 4 5
3 1 5 2 4
2 4 1 5 3
4 5 2 3 1

28. 5 3 4 1 2
4 5 2 3 1
3 1 5 2 4
2 4 1 5 3
1 2 3 4 5

29. 4 5 2 3 1
5 3 4 1 2
3 1 5 2 4
2 4 1 5 3
1 2 3 4 5

30. 4 5 2 3 1
5 3 4 1 2
2 4 1 5 3
3 1 5 2 4
1 2 3 4 5

31. 2 4 1 5 3
1 2 3 4 5
3 1 5 2 4
5 3 4 1 2
4 5 2 3 1

32. 2 4 1 5 3
3 1 5 2 4
1 2 3 4 5
5 3 4 1 2
4 5 2 3 1

33. 2 4 1 5 3
4 5 2 3 1
1 2 3 4 5
5 3 4 1 2
3 1 5 2 4

34. 2 4 1 5 3
1 2 3 4 5
4 5 2 3 1
5 3 4 1 2
3 1 5 2 4

35. 2 4 1 5 3
3 1 5 2 4
4 5 2 3 1
5 3 4 1 2
1 2 3 4 5

36. 2 4 1 5 3
4 5 2 3 1
3 1 5 2 4
5 3 4 1 2
1 2 3 4 5

37. 3 1 5 2 4
1 2 3 4 5
2 4 1 5 3
5 3 4 1 2
4 5 2 3 1

38. 3 1 5 2 4
1 2 3 4 5
4 5 2 3 1
5 3 4 1 2
2 4 1 5 3

39. 3 1 5 2 4
4 5 2 3 1
2 4 1 5 3
5 3 4 1 2
1 2 3 4 5

40. 3 1 5 2 4
2 4 1 5 3
4 5 2 3 1
5 3 4 1 2
1 2 3 4 5

41. 3 1 5 2 4
4 5 2 3 1
1 2 3 4 5
5 3 4 1 2
2 4 1 5 3

42. 3 1 5 2 4
2 4 1 5 3
1 2 3 4 5
5 3 4 1 2
4 5 2 3 1

43. 4 5 2 3 1
1 2 3 4 5
2 4 1 5 3
5 3 4 1 2
3 1 5 2 4

44. 4 5 2 3 1
1 2 3 4 5
3 1 5 2 4
5 3 4 1 2
2 4 1 5 3

45. 4 5 2 3 1
3 1 5 2 4
1 2 3 4 5
5 3 4 1 2
2 4 1 5 3

46. 4 5 2 3 1
3 1 5 2 4
2 4 1 5 3
5 3 4 1 2
1 2 3 4 5

47. 4 5 2 3 1
2 4 1 5 3
1 2 3 4 5
5 3 4 1 2
3 1 5 2 4

48. 4 5 2 3 1
2 4 1 5 3
3 1 5 2 4
5 3 4 1 2
1 2 3 4 5

49. 1 2 3 4 5
2 4 1 5 3
5 3 4 1 2
3 1 5 2 4
4 5 2 3 1

50. 1 2 3 4 5
3 1 5 2 4
5 3 4 1 2
2 4 1 5 3
4 5 2 3 1

51. 1 2 3 4 5
2 4 1 5 3
5 3 4 1 2
4 5 2 3 1
3 1 5 2 4

52.
1 2 3 4 5
4 5 2 3 1
5 3 4 1 2
3 1 5 2 4
2 4 1 5 3

53.
1 2 3 4 5
4 5 2 3 1
5 3 4 1 2
2 4 1 5 3
3 1 5 2 4

54.
1 2 3 4 5
3 1 5 2 4
5 3 4 1 2
4 5 2 3 1
2 4 1 5 3

55.
2 4 1 5 3
1 2 3 4 5
5 3 4 1 2
3 1 5 2 4
4 5 2 3 1

56.
2 4 1 5 3
3 1 5 2 4
5 3 4 1 2
1 2 3 4 5
4 5 2 3 1

57.
2 4 1 5 3
4 5 2 3 1
5 3 4 1 2
1 2 3 4 5
3 1 5 2 4

58.
2 4 1 5 3
1 2 3 4 5
5 3 4 1 2
4 5 2 3 1
3 1 5 2 4

59.
2 4 1 5 3
3 1 5 2 4
5 3 4 1 2
4 5 2 3 1
1 2 3 4 5

60.
2 4 1 5 3
4 5 2 3 1
5 3 4 1 2
3 1 5 2 4
1 2 3 4 5

61.
3 1 5 2 4
1 2 3 4 5
5 3 4 1 2
2 4 1 5 3
4 5 2 3 1

62.
3 1 5 2 4
1 2 3 4 5
5 3 4 1 2
4 5 2 3 1
2 4 1 5 3

63.
3 1 5 2 4
4 5 2 3 1
5 3 4 1 2
2 4 1 5 3
1 2 3 4 5

64.
3 1 5 2 4
2 4 1 5 3
5 3 4 1 2
4 5 2 3 1
1 2 3 4 5

65.
3 1 5 2 4
4 5 2 3 1
5 3 4 1 2
1 2 3 4 5
2 4 1 5 3

66.
3 1 5 2 4
2 4 1 5 3
5 3 4 1 2
1 2 3 4 5
4 5 2 3 1

67.
4 5 2 3 1
1 2 3 4 5
5 3 4 1 2
2 4 1 5 3
3 1 5 2 4

68.
4 5 2 3 1
1 2 3 4 5
5 3 4 1 2
3 1 5 2 4
2 4 1 5 3

69.
4 5 2 3 1
3 1 5 2 4
5 3 4 1 2
1 2 3 4 5
2 4 1 5 3

70.
4 5 2 3 1
3 1 5 2 4
5 3 4 1 2
2 4 1 5 3
1 2 3 4 5

71.
4 5 2 3 1
2 4 1 5 3
5 3 4 1 2
1 2 3 4 5
3 1 5 2 4

72.
4 5 2 3 1
2 4 1 5 3
5 3 4 1 2
3 1 5 2 4
1 2 3 4 5

73.
1 2 3 4 5
5 3 4 1 2
2 4 1 5 3
3 1 5 2 4
4 5 2 3 1

74.
1 2 3 4 5
5 3 4 1 2
3 1 5 2 4
2 4 1 5 3
4 5 2 3 1

75.
1 2 3 4 5
5 3 4 1 2
2 4 1 5 3
4 5 2 3 1
3 1 5 2 4

76.
1 2 3 4 5
5 3 4 1 2
4 5 2 3 1
3 1 5 2 4
2 4 1 5 3

77.
1 2 3 4 5
5 3 4 1 2
4 5 2 3 1
2 4 1 5 3
3 1 5 2 4

78.
1 2 3 4 5
5 3 4 1 2
3 1 5 2 4
4 5 2 3 1
2 4 1 5 3

79. 2 4 1 5 3
5 3 4 1 2
1 2 3 4 5
3 1 5 2 4
4 5 2 3 1

80. 2 4 1 5 3
5 3 4 1 2
3 1 5 2 4
1 2 3 4 5
4 5 2 3 1

81. 2 4 1 5 3
5 3 4 1 2
4 5 2 3 1
1 2 3 4 5
3 1 5 2 4

82. 2 4 1 5 3
5 3 4 1 2
1 2 3 4 5
4 5 2 3 1
3 1 5 2 4

83. 2 4 1 5 3
5 3 4 1 2
3 1 5 2 4
4 5 2 3 1
1 2 3 4 5

84. 2 4 1 5 3
5 3 4 1 2
4 5 2 3 1
3 1 5 2 4
1 2 3 4 5

85. 3 1 5 2 4
5 3 4 1 2
1 2 3 4 5
2 4 1 5 3
4 5 2 3 1

86. 3 1 5 2 4
5 3 4 1 2
1 2 3 4 5
4 5 2 3 1
2 4 1 5 3

87. 3 1 5 2 4
5 3 4 1 2
4 5 2 3 1
2 4 1 5 3
1 2 3 4 5

88. 3 1 5 2 4
5 3 4 1 2
2 4 1 5 3
4 5 2 3 1
1 2 3 4 5

89. 3 1 5 2 4
5 3 4 1 2
4 5 2 3 1
1 2 3 4 5
2 4 1 5 3

90. 3 1 5 2 4
5 3 4 1 2
2 4 1 5 3
1 2 3 4 5
4 5 2 3 1

91. 4 5 2 3 1
5 3 4 1 2
1 2 3 4 5
2 4 1 5 3
3 1 5 2 4

92. 4 5 2 3 1
5 3 4 1 2
1 2 3 4 5
3 1 5 2 4
2 4 1 5 3

93. 4 5 2 3 1
5 3 4 1 2
3 1 5 2 4
1 2 3 4 5
2 4 1 5 3

94. 4 5 2 3 1
5 3 4 1 2
3 1 5 2 4
2 4 1 5 3
1 2 3 4 5

95. 4 5 2 3 1
5 3 4 1 2
2 4 1 5 3
1 2 3 4 5
3 1 5 2 4

96. 4 5 2 3 1
5 3 4 1 2
2 4 1 5 3
3 1 5 2 4
1 2 3 4 5

97. 5 3 4 1 2
1 2 3 4 5
2 4 1 5 3
3 1 5 2 4
4 5 2 3 1

98. 5 3 4 1 2
1 2 3 4 5
3 1 5 2 4
2 4 1 5 3
4 5 2 3 1

99. 5 3 4 1 2
1 2 3 4 5
2 4 1 5 3
4 5 2 3 1
3 1 5 2 4

100. 5 3 4 1 2
1 2 3 4 5
4 5 2 3 1
3 1 5 2 4
2 4 1 5 3

101. 5 3 4 1 2
1 2 3 4 5
4 5 2 3 1
2 4 1 5 3
3 1 5 2 4

102. 5 3 4 1 2
1 2 3 4 5
3 1 5 2 4
4 5 2 3 1
2 4 1 5 3

103. 5 3 4 1 2
2 4 1 5 3
1 2 3 4 5
3 1 5 2 4
4 5 2 3 1

104. 5 3 4 1 2
2 4 1 5 3
3 1 5 2 4
1 2 3 4 5
4 5 2 3 1

105. 5 3 4 1 2
2 4 1 5 3
4 5 2 3 1
1 2 3 4 5
3 1 5 2 4

106.
5 3 4 1 2
2 4 1 5 3
1 2 3 4 5
4 5 2 3 1
3 1 5 2 4

107.
5 3 4 1 2
2 4 1 5 3
3 1 5 2 4
4 5 2 3 1
1 2 3 4 5

108.
5 3 4 1 2
2 4 1 5 3
4 5 2 3 1
3 1 5 2 4
1 2 3 4 5

109.
5 3 4 1 2
3 1 5 2 4
1 2 3 4 5
2 4 1 5 3
4 5 2 3 1

110.
5 3 4 1 2
3 1 5 2 4
1 2 3 4 5
4 5 2 3 1
2 4 1 5 3

111.
5 3 4 1 2
3 1 5 2 4
4 5 2 3 1
2 4 1 5 3
1 2 3 4 5

112.
5 3 4 1 2
3 1 5 2 4
2 4 1 5 3
4 5 2 3 1
1 2 3 4 5

113.
5 3 4 1 2
3 1 5 2 4
4 5 2 3 1
1 2 3 4 5
2 4 1 5 3

114.
5 3 4 1 2
3 1 5 2 4
2 4 1 5 3
1 2 3 4 5
4 5 2 3 1

115.
5 3 4 1 2
4 5 2 3 1
1 2 3 4 5
2 4 1 5 3
3 1 5 2 4

116.
5 3 4 1 2
4 5 2 3 1
1 2 3 4 5
3 1 5 2 4
2 4 1 5 3

117.
5 3 4 1 2
4 5 2 3 1
3 1 5 2 4
1 2 3 4 5
2 4 1 5 3

118.
5 3 4 1 2
4 5 2 3 1
3 1 5 2 4
2 4 1 5 3
1 2 3 4 5

119.
5 3 4 1 2
4 5 2 3 1
2 4 1 5 3
1 2 3 4 5
3 1 5 2 4

120.
5 3 4 1 2
4 5 2 3 1
2 4 1 5 3
3 1 5 2 4
1 2 3 4 5

11.2 Study of 120 Latin-square Designs.

The study of 120 Latin-square designs of order 5x5 listed under 11.1.1 shows that there are 20 designs where each treatment is followed by two other treatments exactly twice. The serial numbers of these designs are 1, 10, 15, 16, 21, 26, 33, 40, 52, 59, 61, 71, 77, 80, 86, 96, 99, 103, 113 and 118.

11.3 Matrix form of the Normal Equations of 5x5 Latin-square Designs.

The matrix form of the normal equations relating to subject, treatment and residual effects can be shown by taking 11.1.1(10), one of the 20 designs mentioned under 11.2.

$$
\begin{bmatrix}
5&0&0&0&0&0&2&0&0&2&0&0&0&0&0\\
0&5&0&0&0&0&0&2&2&0&0&0&0&0&0\\
0&0&5&0&0&2&0&0&2&0&0&0&0&0&0\\
0&0&0&5&0&2&0&0&0&2&0&0&0&0&0\\
0&0&0&0&5&0&2&2&0&0&0&0&0&0&0\\
0&0&2&2&0&4&0&0&0&0&0&0&0&-1&0\\
2&0&0&0&2&0&4&0&0&0&0&0&0&0&-1\\
0&2&0&2&0&0&0&4&0&0&0&-1&0&0&0\\
0&2&2&0&0&0&0&0&4&0&0&0&-1&0&0\\
2&0&0&2&0&0&0&0&0&4&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&5&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&5&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&5&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&5&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&5
\end{bmatrix}
\begin{bmatrix}
\hat{t}_1\\ \hat{t}_2\\ \hat{t}_3\\ \hat{t}_4\\ \hat{t}_5\\ \hat{r}_1\\ \hat{r}_2\\ \hat{r}_3\\ \hat{r}_4\\ \hat{r}_5\\ \hat{s}_1\\ \hat{s}_2\\ \hat{s}_3\\ \hat{s}_4\\ \hat{s}_5
\end{bmatrix}
=
\begin{bmatrix}
T_1-0.2G\\ T_2-0.2G\\ T_3-0.2G\\ T_4-0.2G\\ T_5-0.2G\\ R_1-0.2G+0.2P_1\\ R_2-0.2G+0.2P_1\\ R_3-0.2G+0.2P_1\\ R_4-0.2G+0.2P_1\\ R_5-0.2G+0.2P_1\\ S_1-0.2G\\ S_2-0.2G\\ S_3-0.2G\\ S_4-0.2G\\ S_5-0.2G
\end{bmatrix}
$$

11.3.1 The parameter estimates are obtained as follows.

$$\begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \\ \hat{t}_4 \\ \hat{t}_5 \\ \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \\ \hat{r}_5 \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ \hat{s}_4 \\ \hat{s}_5 \end{bmatrix} = \underline{X}^{-1} \begin{bmatrix} T_1-0.2G \\ T_2-0.2G \\ T_3-0.2G \\ T_4-0.2G \\ T_5-0.2G \\ R_1-0.2G+0.2P_1 \\ R_2-0.2G+0.2P_1 \\ R_3-0.2G+0.2P_1 \\ R_4-0.2G+0.2P_1 \\ R_5-0.2G+0.2P_1 \\ S_1-0.2G \\ S_2-0.2G \\ S_3-0.2G \\ S_4-0.2G \\ S_5-0.2G \end{bmatrix}$$

11.3.2 $\underline{X}^{-1}$ on page 132. .....Type I.

11.3.3 Inverted matrix of type II. Page 133.

We notice that the inverted matrices for the designs mentioned in 11.2 give only two types of matrices, one of type 11.3.2 and the other of type 11.3.3.

[illegible]	[illegible]	[illegible]	[illegible]	0.23803	-0.19687	0.19687	-0.39821	-0.39821	-0.14318	-0.07964	-0.02864	-0.03937	-0.07964	-0.03937
0.13602	0.51857	0.23803	0.13602	0.23803	-0.39821	-0.19687	-0.14318	-0.19687	-0.39821	-0.03937	-0.07964	-0.03937	-0.02864	-0.07964
0.13602	0.23803	0.51857	0.23803	0.13602	-0.19687	-0.39821	-0.19687	-0.14318	-0.39821	-0.02864	-0.07964	-0.07964	-0.03937	-0.03937
0.23803	0.13602	0.23803	0.51857	0.13602	-0.14318	-0.39821	-0.39821	-0.19687	-0.19687	-0.03937	-0.03937	-0.07964	-0.07964	-0.02864
0.23803	0.23803	0.13602	0.13602	0.51857	-0.39821	-0.14318	-0.19687	-0.39821	-0.19687	-0.07964	-0.03937	-0.02864	-0.03937	-0.07964
-0.19687	-0.39821	-0.19687	-0.14318	-0.39821	0.68233	0.17897	0.17897	0.31320	0.31320	0.06264	0.06264	-0.03579	-0.03579	-0.13647
-0.19687	-0.19687	-0.39821	-0.39821	-0.14318	0.17897	0.68233	0.31320	0.17897	0.31320	0.03579	0.06264	0.13647	0.06264	0.03579
-0.39821	-0.14318	-0.19687	-0.39821	-0.19687	0.17897	0.31320	0.68233	0.31320	0.17897	0.06264	0.03579	0.06264	0.13647	0.03579
-0.39821	-0.19687	-0.14318	-0.19687	-0.39821	0.31320	0.17897	0.31320	0.68233	0.17897	0.13647	0.03579	0.03579	0.06264	0.06264
-0.14318	-0.39821	-0.39821	-0.19687	-0.19687	0.31320	0.31320	0.17897	0.17897	0.68233	0.03579	0.13647	0.06264	0.03579	0.06264
-0.07964	-0.03937	-0.02864	-0.03937	-0.07964	0.06264	0.03579	0.06264	0.13647	0.03579	0.22729	0.00716	0.00716	0.01253	0.01253
-0.02864	-0.07964	-0.07964	-0.03937	-0.03937	0.06264	0.06264	0.03579	0.03579	0.13647	0.00716	0.22729	0.01253	0.00716	0.01253
-0.03937	-0.03937	-0.07964	-0.07964	-0.02864	0.03579	0.13647	0.06264	0.03579	0.06264	0.00716	0.01253	0.22729	0.01253	0.00716
-0.07964	-0.02864	-0.03937	-0.07964	-0.03937	0.03579	0.06264	0.13647	0.06264	0.03579	0.01253	0.00716	0.01253	0.22729	0.00716
-0.03937	-0.07964	-0.03937	-0.02864	-0.07964	0.13647	0.03579	0.03579	0.06264	0.06264	0.01253	0.01253	0.00716	0.00716	0.22729

[illegible]	[illegible]	[illegible]	[illegible]	0.13602	-0.19687	-0.14318	-0.39821	-0.19687	-0.39821	-0.02864	-0.03937	-0.03937	-0.07964	-0.07964
0.23803	0.51857	0.13602	0.23803	0.13602	-0.39821	-0.19687	-0.39821	-0.14318	-0.19687	-0.03937	-0.02864	-0.07964	-0.03937	-0.07964
0.23803	0.13602	0.51857	0.13602	0.23803	-0.14318	-0.19687	-0.19687	-0.39821	-0.39821	-0.03937	-0.07964	-0.02864	-0.07964	-0.03937
0.13602	0.23803	0.13602	0.51857	0.23803	-0.39821	-0.39821	-0.19687	-0.19687	-0.14318	-0.07964	-0.03937	-0.07964	-0.02864	-0.03937
0.13602	0.13602	0.23803	0.23803	0.51857	-0.19687	-0.39821	-0.14318	-0.39821	-0.19687	-0.07964	-0.07964	-0.03937	-0.03937	-0.02864
-0.19687	-0.39821	-0.14318	-0.39821	-0.19687	0.68233	0.31320	0.31320	0.17897	0.17897	0.06264	0.03579	0.13647	0.03579	0.06264
-0.14318	-0.19687	-0.19687	-0.39821	-0.39821	0.31320	0.68233	0.17897	0.31320	0.17897	0.13647	0.06264	0.06264	0.03579	0.03579
-0.39821	-0.39821	-0.19687	-0.19687	-0.14318	0.31320	0.17897	0.68233	0.17897	0.31320	0.03579	0.03579	0.06264	0.06264	0.13647
-0.19687	-0.14318	-0.39821	-0.19687	-0.39821	0.17897	0.31320	0.17897	0.68233	0.31320	0.06264	0.13647	0.03579	0.06264	0.03579
-0.39821	-0.19687	-0.39821	-0.14318	-0.19687	0.17897	0.17897	0.31320	0.31320	0.68233	0.03579	0.06264	0.03579	0.13647	0.06264
-0.02864	-0.03937	-0.03937	-0.07964	-0.07964	0.06264	0.13647	0.03579	0.06264	0.03579	0.22729	0.01253	0.01253	0.00716	0.00716
-0.03937	-0.02864	-0.07964	-0.03937	-0.07964	0.03579	0.06264	0.03579	0.13647	0.06264	0.01253	0.22729	0.00716	0.01253	0.00716
-0.03937	-0.07964	-0.02864	-0.07964	-0.03937	0.13647	0.06264	0.06264	0.03579	0.03579	0.01253	0.00716	0.22729	0.00716	0.01253
-0.07964	-0.03937	-0.07964	-0.02864	-0.03937	0.03579	0.03579	0.06264	0.06264	0.13647	0.00716	0.01253	0.00716	0.22729	0.01253
-0.07964	-0.07964	-0.03937	-0.03937	-0.02864	0.06264	0.03579	0.13647	0.03579	0.06264	0.00716	0.00716	0.01253	0.01253	0.22729

11.3.4 Efficiency in the measurement of treatment differences over residual differences.

Type I.

$V(\hat{t}_1-\hat{t}_2) = V(0.519T_1+0.136T_2-0.136T_1-0.519T_2+\ldots\ldots\ldots\ldots)$

$= 0.766\ \sigma^2 = V(\hat{t}_4-\hat{t}_5) = V(\hat{t}_2-\hat{t}_4) = \ldots\ldots\ldots\ldots$

$V(\hat{t}_2-\hat{t}_3) = V(0.519T_2+0.238T_3-0.238T_2-0.519T_3+\ldots\ldots\ldots\ldots)$

$= 0.562\ \sigma^2 = V(\hat{t}_3-\hat{t}_4) = V(\hat{t}_1-\hat{t}_3) = \ldots\ldots\ldots\ldots$

Average of the variances of the differences of the treatment effects: Av. $V(\hat{t}_i-\hat{t}_j) = 0.664\ \sigma^2$

$V(\hat{r}_1-\hat{r}_2) = V(0.682R_1+0.179R_2-0.179R_1-0.682R_2+\ldots\ldots\ldots\ldots)$

$= 1.006\ \sigma^2 = V(\hat{r}_4-\hat{r}_5) = V(\hat{r}_2-\hat{r}_4) = \ldots\ldots\ldots\ldots$

$V(\hat{r}_2-\hat{r}_3) = V(0.682R_2+0.313R_3-0.313R_2-0.682R_3+\ldots\ldots\ldots\ldots)$

$= 0.738\ \sigma^2 = V(\hat{r}_3-\hat{r}_4) = V(\hat{r}_1-\hat{r}_3) = \ldots\ldots\ldots\ldots$

Average of the variances of the differences of the treatment effects: Av. $V(\hat{r}_i-\hat{r}_j) = 0.872\ \sigma^2$

Efficiency of estimating differences (Treatment/Residual) $= \dfrac{1.006\ \sigma^2}{0.766\ \sigma^2} \times 100 = 132\ \%$

$= \dfrac{0.738\ \sigma^2}{0.562\ \sigma^2} \times 100 = \dfrac{0.872\ \sigma^2}{0.664\ \sigma^2} \times 100 = 132\ \%$

Type II.

$V(\hat{t}_1-\hat{t}_2) = 0.562\ \sigma^2 = V(\hat{t}_1-\hat{t}_3) = \ldots\ldots\ldots = V(\hat{t}_2-\hat{t}_4) = 0.562\ \sigma^2$

$V(\hat{t}_2-\hat{t}_3) = 0.766\ \sigma^2 = V(\hat{t}_3-\hat{t}_4) = \ldots\ldots\ldots = V(\hat{t}_1-\hat{t}_5) = 0.766\ \sigma^2$

Av. $V(\hat{t}_i-\hat{t}_j) = 0.664\ \sigma^2$

$V(\hat{r}_1-\hat{r}_2) = 0.738\ \sigma^2 = V(\hat{r}_1-\hat{r}_3) = \ldots\ldots\ldots = V(\hat{r}_2-\hat{r}_4) = 0.738\ \sigma^2$

$V(\hat{r}_2-\hat{r}_3) = 1.006\ \sigma^2 = V(\hat{r}_3-\hat{r}_4) = \ldots\ldots\ldots = V(\hat{r}_1-\hat{r}_5) = 1.006\ \sigma^2$

Av. $V(\hat{r}_i-\hat{r}_j) = 0.872\ \sigma^2$

$$\text{Effy. (Treatment/Residual)} = \frac{0.738\ \sigma^2}{0.562\ \sigma^2} \times 100 = 132\ \%$$

$$= \frac{1.006\ \sigma^2}{0.766\ \sigma^2} \times 100 = \frac{0.872\ \sigma^2}{0.664\ \sigma^2} \times 100 = 132\ \%$$

For design 9.3.1(27)

$$V(\hat{t}_1-\hat{t}_2) = V(1.267T_1 - 1.267T_2 + \ldots\ldots\ldots\ldots)$$

$$= 2.534\ \sigma^2 = V(\hat{t}_2-\hat{t}_3) = V(\hat{t}_4-\hat{t}_5) = \ldots\ldots\ldots = 2.534\ \sigma^2$$

$$V(\hat{r}_1-\hat{r}_2) = V(1.667R_1 - 1.667R_2 + \ldots\ldots\ldots\ldots)$$

$$= 3.334\ \sigma^2 = V(\hat{r}_2-\hat{r}_3) = V(\hat{r}_4-\hat{r}_5) = \ldots\ldots\ldots = 3.334\ \sigma^2$$

The differences between two direct effects are measured 32 % more efficiently than the differences between two residual effects for all the combinations. It is also true for design 9.3.1(27) since (3.334/2.534)x100 = 132 %.

All these designs are symmetrical since in type I and II designs the residual effect in each treatment has been drawn twice from two of the other treatments and in design 9.3.1(27) the residual effect in each treatment has been drawn from the same treatment on each of the four occasions.

Efficiency factors for non-symmetrical designs have been discussed under 9.6 where it is shown that, for each design, they vary from pair to pair depending upon their design pattern.

The computer inversion of the twenty matrices mentioned under11.2 produce two types of inversions, type I and typeII, as shown under11.3.2 and 11.3.3 respectively.

11.3.5 Linear component of variance for treatment direct effects.

Type I.

$$V(-2\hat{t}_1-\hat{t}_2+0+\hat{t}_4+2\hat{t}_5)=V(-2x-0.459T_1-1x-0.179T_2+1x0.179T_4$$
$$+2x0.459T_5+........)$$
$$=V(0.918T_1+0.179T_2+0.179T_4+0.918T_5....)$$
$$=2.194\ \sigma^2$$

Type II.

$$V(-2\hat{t}_1-\hat{t}_2+0+\hat{t}_4+2\hat{t}_5)=V(-2x-0.868T_1-1x-0.485T_2+1x0.485T_4$$
$$+2x0.868T_5......)$$
$$=V(1.736T_1+0.97T_2+0.97T_4+1.736T_5....)$$
$$=4.442\ \sigma^2$$

11.3.6 Linear component of variance for treatment residual effects.

Type I.

$$V(-2\hat{r}_1-\hat{r}_2+0+\hat{r}_4+\hat{r}_5)=V(-2x-0.604R_1-1x-0.235R_2+0.235x1R_4$$
$$+2x0.604R_5......)$$
$$=V(1.208R_1+0.235R_2+0.235R_4+1.208R_5....)$$
$$=2.886\ \sigma^2$$

Type II.

$$V(-2\hat{r}_1-\hat{r}_2+0+\hat{r}_4+2\hat{r}_5) = V(-2x-1.14R_1-1x-0.637R_2+1x0.637R_4$$
$$+2x1.14R_5+........)$$
$$= V(2.28R_1+0.637R_2+0.637R_4+2.28R_5+.......)$$
$$= 5.834\ \sigma^2$$

There are ten type I designs giving minimum variance of linear components for treatment as well as residual effects and hence can be taken as the most efficient designs. There are also ten type II designs giving a variance of linear components for treatment and residual effects which is twice as much as type I designs.

We are now in a position to separate out the ten most efficient designs (type I) out of the set of twenty Latin-square designs of order 5x5 and prepare a list for ready reference. The ten designs (type II) can be classified as inefficient ones along with other designs mentioned under 10.1.1. A separate list for these designs has also been prepared in order to avoid the confusion arising from the pattern of the designs, i.e. a treatment preceded by two other treatments exactly twice, which is the same in both the types.

11.4 Designs with Minimum Variance of Linear Components.

The set of 5x5 Latin-square designs with minimum variance of linear components for treatment direct as well as treatment residual effects, given below, are considered to be the most efficient designs, under the assumption of carry-over effects persisting for one period only. Treatments in all these designs have been preceded twice by other two treatments.

For practical purposes one of the following designs should be chosen when it is desired to measure the linear effects of the treatment direct or treatment residual effects most efficiently.

1.	1 2 3 4 5	2.	3 1 5 2 4	3.	2 4 1 5 3	4.	5 3 4 1 2
	2 4 1 5 3		1 2 3 4 5		4 5 2 3 1		3 1 5 2 4
	3 1 5 2 4		5 3 4 1 2		1 2 3 4 5		4 5 2 3 1
	4 5 2 3 1		2 4 1 5 3		5 3 4 1 2		1 2 3 4 5
	5 3 4 1 2		4 5 2 3 1		3 1 5 2 4		2 4 1 5 3
5.	4 5 2 3 1	6.	5 3 4 1 2	7.	4 5 2 3 1	8.	3 1 5 2 4
	5 3 4 1 2		4 5 2 3 1		2 4 1 5 3		5 3 4 1 2
	2 4 1 5 3		3 1 5 2 4		5 3 4 1 2		1 2 3 4 5
	3 1 5 2 4		2 4 1 5 3		1 2 3 4 5		4 5 2 3 1
	1 2 3 4 5		1 2 3 4 5		3 1 5 2 4		2 4 1 5 3
9.	2 4 1 5 3	10.	1 2 3 4 5				
	1 2 3 4 5		3 1 5 2 4				
	4 5 2 3 1		2 4 1 5 3				
	3 1 5 2 4		5 3 4 1 2				
	5 3 4 1 2		4 5 2 3 1				

These designs should be used in their original form and no re-arrangement of rows to periods should be made as this will upset the balance of the design and may turn into an inefficient design. In fact there are only two distinct designs in the above table and the others are their column permutations. Designs 1 to 5 when written in their reverse order of treatments in each column produce designs 6 to 10 respectively. Originally these designs were chosen from the designs obtained by permuting rows as given under 11.1.

11.5 Designs with high Variance of Linear Components.

The set of 5x5 Latin-square designs, given below, have the same general pattern of each treatment preceded by two other treatments twice but give twice as much variance of linear components for the parameter estimates as for the most efficient designs listed under 11.4.

These are inefficient designs for the estimation of treatment direct or treatment residual effects when carry-over effects persist for one period only. Therefore these designs must not be used without any further investigation as there are many more designs much more efficient than these. These designs have been listed to avoid confusion about 11.4 designs.

1.

3	1	5	2	4
2	4	1	5	3
4	5	2	3	1
5	3	4	1	2
1	2	3	4	5

2.

5	3	4	1	2
2	4	1	5	3
1	2	3	4	5
3	1	5	2	4
4	5	2	3	1

3.

3	1	5	2	4
4	5	2	3	1
2	4	1	5	3
1	2	3	4	5
5	3	4	1	2

4.

4	5	2	3	1
1	2	3	4	5
3	1	5	2	4
5	3	4	1	2
2	4	1	5	3

5.

1	2	3	4	5
4	5	2	3	1
5	3	4	1	2
3	1	5	2	4
2	4	1	5	3

6.

1	2	3	4	5
5	3	4	1	2
4	5	2	3	1
2	4	1	5	3
3	1	5	2	4

7.

4	5	2	3	1
3	1	5	2	4
1	2	3	4	5
2	4	1	5	3
5	3	4	1	2

8.

5	3	4	1	2
1	2	3	4	5
2	4	1	5	3
4	5	2	3	1
3	1	5	2	4

9.

2	4	1	5	3
5	3	4	1	2
3	1	5	2	4
1	2	3	4	5
4	5	2	3	1

10.

2	4	1	5	3
3	1	5	2	4
5	3	4	1	2
4	5	2	3	1
1	2	3	4	5

The most efficient designs grouped under 11.4 have a set pattern where each design has the same treatment appearing all along one of the diagonals and other treatments appear symmetrically parallel to the other diagonal which consists of all the five treatments. It is noticed that the top part of the design (above the diagonal made up of all the treatments) is the image of the bottom part. Another characteristic of these designs which has already been mentioned in the previous chapters is that each treatment has been preceded twice by other two treatments. For 5x5 Latin-squares we can have only ten such designs.

The inefficient designs grouped under 11.5 follow the same general pattern as explained before, yet lack the symmetry in the position of the treatments. Both of their diagonals are made up of all the five different treatments and in no way show any parallel occurrence or any image of one part to the other as observed for designs called most efficient.

As a general guide these designs having each treatment preceded twice by other two treatments have been listed separately as efficient and inefficient. There are only ten good designs we have identified so far, a limited number which may not satisfy the general need. Therefore we must look through other patterns which are nearly as efficient as 11.4 designs and see if a general pattern of the most efficient designs can be established.

Efficient designs of other patterns have been mentioned under 10.1.1, further study of which may reveal some interesting results.

11.6 Other Patterns.

For further investigation of designs 7, 12, 21, 23, 25, 26, 28, 30, and 31 as suggested under 10.1.1 the simplest approach would be to study their relative precision in measuring the linear effect of treatments as well as residuals along with the precision in measuring the differences between the estimates of related parameter effects.

A design or pattern of designs producing optimum precision for linear effect as well as differences of estimates may be taken as the most efficient pattern of designs for parameter estimates when residual effects are assumed to exist.

The most efficient design minimises average variance for the difference between two treatment as well as residual estimates. In other words

$$\text{Av. } V(\hat{t}_i - \hat{t}_j) = \text{Minimum.}$$

and

$$\text{Av. } V(\hat{r}_i - \hat{r}_j) = \text{Minimum.}$$

Secondly, the most efficient design has the smallest range for the variances of the differences between parameter estimates. Therefore in this case we are also looking for;

$$R_{V(\hat{t}_i - \hat{t}_j)} = \text{Minimum}$$

and

$$R_{V(\hat{r}_i - \hat{r}_j)} = \text{Minimum.} \qquad R = \text{Range}$$

Tables 11.6.1 and 11.6.2 have been constructed to compare the averages of the variances of the differences between estimates and their ranges.

11.6.1

Design	7	12*	21*	23	25*	26*	28*	29	30*	31*
$V(\hat{t}_1-\hat{t}_2)$	$0.585\sigma^2$	$0.525\sigma^2$	$0.477\sigma^2$	$0.521\sigma^2$	$0.564\sigma^2$	$0.509\sigma^2$	$0.509\sigma^2$	$0.766\sigma^2$	$0.477\sigma^2$	$0.509\sigma^2$
$V(\hat{t}_1-\hat{t}_3)$	$0.54\sigma^2$	$0.424\sigma^2$	$0.525\sigma^2$	$0.478\sigma^2$	$0.45\sigma^2$	$0.477\sigma^2$	$0.477\sigma^2$	$0.766\sigma^2$	$0.509\sigma^2$	$0.509\sigma^2$
$V(\hat{t}_1-\hat{t}_4)$	$0.585\sigma^2$	$0.538\sigma^2$	$0.477\sigma^2$	$0.529\sigma^2$	$0.538\sigma^2$	$0.509\sigma^2$	$0.477\sigma^2$	$0.562\sigma^2$	$0.477\sigma^2$	$0.477\sigma^2$
$V(\hat{t}_1-\hat{t}_5)$	$0.48\sigma^2$	$0.564\sigma^2$	$0.525\sigma^2$	$0.478\sigma^2$	$0.477\sigma^2$	$0.477\sigma^2$	$0.509\sigma^2$	$0.562\sigma^2$	$0.509\sigma^2$	$0.477\sigma^2$
$V(\hat{t}_2-\hat{t}_3)$	$0.719\sigma^2$	$0.525\sigma^2$	$0.538\sigma^2$	$0.567\sigma^2$	$0.538\sigma^2$	$0.509\sigma^2$	$0.477\sigma^2$	$0.562\sigma^2$	$0.509\sigma^2$	$0.477\sigma^2$
$V(\hat{t}_2-\hat{t}_4)$	$0.422\sigma^2$	$0.477\sigma^2$	$0.45\sigma^2$	$0.666\sigma^2$	$0.424\sigma^2$	$0.477\sigma^2$	$0.509\sigma^2$	$0.766\sigma^2$	$0.509\sigma^2$	$0.477\sigma^2$
$V(\hat{t}_2-\hat{t}_5)$	$0.489\sigma^2$	$0.477\sigma^2$	$0.564\sigma^2$	$0.567\sigma^2$	$0.525\sigma^2$	$0.477\sigma^2$	$0.477\sigma^2$	$0.562\sigma^2$	$0.509\sigma^2$	$0.509\sigma^2$
$V(\hat{t}_3-\hat{t}_4)$	$0.719\sigma^2$	$0.564\sigma^2$	$0.564\sigma^2$	$0.767\sigma^2$	$0.564\sigma^2$	$0.477\sigma^2$	$0.509\sigma^2$	$0.562\sigma^2$	$0.477\sigma^2$	$0.509\sigma^2$
$V(\hat{t}_3-\hat{t}_5)$	$0.732\sigma^2$	$0.538\sigma^2$	$0.424\sigma^2$	$0.422\sigma^2$	$0.477\sigma^2$	$0.509\sigma^2$	$0.509\sigma^2$	$0.766\sigma^2$	$0.477\sigma^2$	$0.477\sigma^2$
$V(\hat{t}_4-\hat{t}_5)$	$0.489\sigma^2$	$0.448\sigma^2$	$0.538\sigma^2$	$0.767\sigma^2$	$0.525\sigma^2$	$0.509\sigma^2$	$0.477\sigma^2$	$0.766\sigma^2$	$0.509\sigma^2$	$0.509\sigma^2$
Average	$0.576\sigma^2$	$0.508\sigma^2$	$0.508\sigma^2$	$0.576\sigma^2$	$0.508\sigma^2$	$0.493\sigma^2$	$0.493\sigma^2$	$0.664\sigma^2$	$0.493\sigma^2$	$0.493\sigma^2$
Range	$0.31\sigma^2$	$0.14\sigma^2$	$0.14\sigma^2$	$0.345\sigma^2$	$0.14\sigma^2$	$0.032\sigma^2$	$0.032\sigma^2$	$0.204\sigma^2$	$0.032\sigma^2$	$0.032\sigma^2$

11.6.2

Design	7	12*	21*	23	25*	26*	28*	29	30*	31*
$V(\hat{r}_1-\hat{r}_2)$	$1.008\sigma^2$	$0.709\sigma^2$	$0.59\sigma^2$	$0.644\sigma^2$	$0.69\sigma^2$	$0.67\sigma^2$	$0.67\sigma^2$	$1.006\sigma^2$	$0.628\sigma^2$	$0.67\sigma^2$
$V(\hat{r}_1-\hat{r}_3)$	$0.556\sigma^2$	$0.556\sigma^2$	$0.709\sigma^2$	$0.964\sigma^2$	$0.629\sigma^2$	$0.628\sigma^2$	$0.628\sigma^2$	$1.006\sigma^2$	$0.67\sigma^2$	$0.67\sigma^2$
$V(\hat{r}_1-\hat{r}_4)$	$0.747\sigma^2$	$0.741\sigma^2$	$0.629\sigma^2$	$0.644\sigma^2$	$0.69\sigma^2$	$0.67\sigma^2$	$0.628\sigma^2$	$0.738\sigma^2$	$0.628\sigma^2$	$0.628\sigma^2$
$V(\hat{r}_1-\hat{r}_5)$	$0.63\sigma^2$	$0.69\sigma^2$	$0.741\sigma^2$	$0.634\sigma^2$	$0.629\sigma^2$	$0.628\sigma^2$	$0.67\sigma^2$	$0.738\sigma^2$	$0.67\sigma^2$	$0.628\sigma^2$
$V(\hat{r}_2-\hat{r}_3)$	$1.008\sigma^2$	$0.741\sigma^2$	$0.741\sigma^2$	$0.946\sigma^2$	$0.741\sigma^2$	$0.67\sigma^2$	$0.628\sigma^2$	$0.738\sigma^2$	$0.67\sigma^2$	$0.628\sigma^2$
$V(\hat{r}_2-\hat{r}_4)$	$0.877\sigma^2$	$0.592\sigma^2$	$0.629\sigma^2$	$0.556\sigma^2$	$0.556\sigma^2$	$0.628\sigma^2$	$0.67\sigma^2$	$1.006\sigma^2$	$0.628\sigma^2$	$0.628\sigma^2$
$V(\hat{r}_2-\hat{r}_5)$	$0.696\sigma^2$	$0.629\sigma^2$	$0.709\sigma^2$	$0.771\sigma^2$	$0.708\sigma^2$	$0.628\sigma^2$	$0.628\sigma^2$	$0.738\sigma^2$	$0.67\sigma^2$	$0.67\sigma^2$
$V(\hat{r}_3-\hat{r}_4)$	$0.747\sigma^2$	$0.709\sigma^2$	$0.69\sigma^2$	$0.946\sigma^2$	$0.708\sigma^2$	$0.628\sigma^2$	$0.67\sigma^2$	$0.738\sigma^2$	$0.628\sigma^2$	$0.67\sigma^2$
$V(\hat{r}_3-\hat{r}_5)$	$0.63\sigma^2$	$0.69\sigma^2$	$0.556\sigma^2$	$0.71\sigma^2$	$0.592\sigma^2$	$0.67\sigma^2$	$0.67\sigma^2$	$1.006\sigma^2$	$0.628\sigma^2$	$0.628\sigma^2$
$V(\hat{r}_4-\hat{r}_5)$	$0.687\sigma^2$	$0.629\sigma^2$	$0.69\sigma^2$	$0.771\sigma^2$	$0.741\sigma^2$	$0.67\sigma^2$	$0.628\sigma^2$	$1.006\sigma^2$	$0.67\sigma^2$	$0.67\sigma^2$
Average	$0.759\sigma^2$	$0.669\sigma^2$	$0.669\sigma^2$	$0.759\sigma^2$	$0.668\sigma^2$	$0.649\sigma^2$	$0.649\sigma^2$	$0.872\sigma^2$	$0.649\sigma^2$	$0.649\sigma^2$
Range	$0.452\sigma^2$	$0.185\sigma^2$	$0.185\sigma^2$	$0.215\sigma^2$	$0.185\sigma^2$	$0.042\sigma^2$	$0.042\sigma^2$	$0.268\sigma^2$	$0.042\sigma^2$	$0.042\sigma^2$

Taking account of the variances of the differences along with the variances of the linear components we find that though the variances of linear components of designs 7, 23 and 29 do not differ widely from the variances of linear components of designs 12, 21, 25, 26, 28, 30 and 31, yet we notice from tables 11.6.1 and 11.6.2 that they lack uniformity in the magnitude of variances of the differences between different pairs of treatment as well as residual estimates.

Now looking at the averages of the variances of the differences and ranges of the variances of the differences we find that designs 7, 23 and 29 carry higher values than other designs in the tables 11.6.1 and 11.6.2. Therefore we conclude that designs 7, 23 and 29 measure the differences less precisely than other designs mentioned in the tables, for both treatment and residual effects.

Similarly we find that designs 9.3.1(3, 8, 11, 14 and 17) carry smaller variances of linear components, yet have higher values for their averages and ranges of the variances of the differences and therefore can be classified, along with designs 7, 23 and 29, as designs of less precision.

Considering the average or range of the variances of the differences between two estimates, designs 12, 21, 25, 26, 28, 30 and 31 can be grouped into two separate groups:

Group A: Designs 12, 21 and 25.

Group B: Designs 26, 28, 30 and 31.

Group B designs are slightly better than designs of group A, though they may not be so when considering the variance of the linear component.

Finding that a wide variation exists in the precision of measurements of the differences between treatment as well as residual effects for designs 7, 23 and 29, we can exclude these designs from our list of most efficient designs.

The slight difference between the precision of group A and and group B designs is due to the fact that group B designs have treatments distributed symmetrically around a diagonal made up of whole set of treatments. In group A designs the symmetry in the distribution of treatments does not exist. Therefore group B designs have the advantage of having slightly better precision in the estimation of parameter differences than the designs in group A.

Since the general pattern of the designs of both the groups is the same and there is no significant difference in the overall precision of parameter estimates, we can safely classify all the designs of these two groups, as the most efficient designs and the pattern of these designs shall be taken as a pattern of the most efficient design. In these designs each treatment has been followed twice by one treatment and once by other two treatments.

These designs can be used in addition to the designs grouped under 11.4 which are the most efficient designs based on the test of minimum variance of linear component. 11.4 designs are of less value, when the interest lies in the differences of the estimates, than these new designs.

11.4 designs are limited in number and designs of type 12, 21, 25, etc can be generated to a good number which will be discussed in the later chapters. Further tests will be made to confirm the above results regarding the pattern of designs giving optimum precision in the estimation of parameter effects.

11.7 Numerical Study of the Most Efficient Design.

Taking design 9.3.1(25), one of the most efficient designs, for numerical illustrations.

It is a 5x5 Latin-square and its design pattern is given below.

11.7.1

Subjects

Periods		S_1	S_2	S_3	S_4	S_5
	P_1	1	2	3	4	5
	P_2	2	5	4	1	3
	P_3	3	4	1	5	2
	P_4	4	3	5	2	1
	P_5	5	1	2	3	4

Treatments are assigned to the digits in the table.

The observations of the above table can be set out as follows:

11.7.2

Subjects

Periods		S_1	S_2	S_3	S_4	S_5	Totals
	P_1	Y_{11}	Y_{12}	Y_{13}	Y_{14}	Y_{15}	$Y_{1.}$
	P_2	Y_{21}	Y_{22}	Y_{23}	Y_{24}	Y_{25}	$Y_{2.}$
	P_3	Y_{31}	Y_{32}	Y_{33}	Y_{34}	Y_{35}	$Y_{3.}$
	P_4	Y_{41}	Y_{42}	Y_{43}	Y_{44}	Y_{45}	$Y_{4.}$
	P_5	Y_{51}	Y_{52}	Y_{53}	Y_{54}	Y_{55}	$Y_{5.}$
Totals		$Y_{.1}$	$Y_{.2}$	$Y_{.3}$	$Y_{.4}$	$Y_{.5}$	$Y_{..}$

Y_{ij} denotes the observation in the ith row and the jth column.

$Y_{.j}$ denotes the total for jth subject but S_j has been used instead.

$Y_{i.}$ " " " " ith period " P_i " " " " .

11.7.3 The design matrix of 11.7.1 can be written as:

$$
\begin{bmatrix}
1\,0\,0\,0\,0 & 1\,0\,0\,0\,0 & 1\,0\,0\,0\,0 & 1\,0\,0\,0\,0 & 1\,0\,0\,0\,0 \\
0\,1\,0\,0\,0 & 0\,1\,0\,0\,0 & 0\,1\,0\,0\,0 & 0\,1\,0\,0\,0 & 0\,1\,0\,0\,0 \\
0\,0\,1\,0\,0 & 0\,0\,1\,0\,0 & 0\,0\,1\,0\,0 & 0\,0\,1\,0\,0 & 0\,0\,1\,0\,0 \\
0\,0\,0\,1\,0 & 0\,0\,0\,1\,0 & 0\,0\,0\,1\,0 & 0\,0\,0\,1\,0 & 0\,0\,0\,1\,0 \\
0\,0\,0\,0\,1 & 0\,0\,0\,0\,1 & 0\,0\,0\,0\,1 & 0\,0\,0\,0\,1 & 0\,0\,0\,0\,1 \\
1\,1\,1\,1\,1 & 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0 & 1\,1\,1\,1\,1 & 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & 1\,1\,1\,1\,1 & 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & 1\,1\,1\,1\,1 & 0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & 0\,0\,0\,0\,0 & 1\,1\,1\,1\,1 \\
1\,0\,0\,0\,0 & 0\,0\,0\,0\,1 & 0\,0\,1\,0\,0 & 0\,1\,0\,0\,0 & 0\,0\,0\,1\,0 \\
0\,1\,0\,0\,0 & 1\,0\,0\,0\,0 & 0\,0\,0\,0\,1 & 0\,0\,0\,1\,0 & 0\,0\,1\,0\,0 \\
0\,0\,1\,0\,0 & 0\,0\,0\,1\,0 & 1\,0\,0\,0\,0 & 0\,0\,0\,0\,1 & 0\,1\,0\,0\,0 \\
0\,0\,0\,1\,0 & 0\,0\,1\,0\,0 & 0\,1\,0\,0\,0 & 1\,0\,0\,0\,0 & 0\,0\,0\,0\,1 \\
0\,0\,0\,0\,1 & 0\,1\,0\,0\,0 & 0\,0\,0\,1\,0 & 0\,0\,1\,0\,0 & 1\,0\,0\,0\,0 \\
0\,1\,0\,0\,0 & 0\,0\,0\,0\,0 & 0\,0\,0\,1\,0 & 0\,0\,1\,0\,0 & 0\,0\,0\,0\,1 \\
0\,0\,1\,0\,0 & 0\,1\,0\,0\,0 & 0\,0\,0\,0\,0 & 0\,0\,0\,0\,1 & 0\,0\,0\,1\,0 \\
0\,0\,0\,1\,0 & 0\,0\,0\,0\,1 & 0\,1\,0\,0\,0 & 0\,0\,0\,0\,0 & 0\,1\,0\,0\,0 \\
0\,0\,0\,0\,1 & 0\,0\,0\,1\,0 & 0\,0\,1\,0\,0 & 0\,1\,0\,0\,0 & 0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0 & 0\,0\,1\,0\,0 & 0\,0\,0\,0\,1 & 0\,0\,0\,1\,0 & 0\,1\,0\,0\,0
\end{bmatrix}
\begin{bmatrix}
Y_{11} \\ Y_{21} \\ Y_{31} \\ Y_{41} \\ Y_{51} \\ Y_{12} \\ Y_{22} \\ Y_{32} \\ Y_{42} \\ Y_{52} \\ Y_{13} \\ Y_{23} \\ Y_{33} \\ Y_{43} \\ Y_{53} \\ Y_{14} \\ Y_{24} \\ Y_{34} \\ Y_{44} \\ Y_{54} \\ Y_{15} \\ Y_{25} \\ Y_{35} \\ Y_{45} \\ Y_{55}
\end{bmatrix}
$$

The left hand side matrix is the design matrix and on the right hand side is the column vector of observations.

11.7.4 The coefficient matrix of the normal equations of parameter estimates of the design under study along with column vectors of totals and parameter estimates can now be written as:

$$
\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{bmatrix}
=
\begin{bmatrix}
5 & 5&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
5 & 0&5&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
5 & 0&0&5&0&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
5 & 0&0&0&5&0 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
5 & 0&0&0&0&5 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&0 \\
5 & 0&0&0&0&0 & 5&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&-1 \\
5 & 0&0&0&0&0 & 0&5&0&0&0 & 0&0&0&0&0 & -1&0&0&0&0 \\
5 & 0&0&0&0&0 & 0&0&5&0&0 & 0&0&0&0&0 & 0&-1&0&0&0 \\
5 & 0&0&0&0&0 & 0&0&0&5&0 & 0&0&0&0&0 & 0&0&-1&0&0 \\
5 & 0&0&0&0&0 & 0&0&0&0&5 & 0&0&0&0&0 & 0&0&0&-1&0 \\
5 & 0&0&0&0&0 & 0&0&0&0&0 & 5&0&0&0&0 & 0&1&1&2&0 \\
5 & 0&0&0&0&0 & 0&0&0&0&0 & 0&5&0&0&0 & 1&0&1&0&2 \\
5 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&5&0&0 & 0&2&0&1&1 \\
5 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&5&0 & 1&0&2&0&1 \\
5 & 0&0&0&0&0 & 0&0&0&0&0 & 0&0&0&0&5 & 2&1&0&1&0 \\
4 & 0&0&0&0&0 & 0&-1&0&0&0 & 0&1&0&1&2 & 4&0&0&0&0 \\
4 & 0&0&0&0&0 & 0&0&-1&0&0 & 1&0&2&0&1 & 0&4&0&0&0 \\
4 & 0&0&0&0&0 & 0&0&0&-1&0 & 1&1&0&2&0 & 0&0&4&0&0 \\
4 & 0&0&0&0&0 & 0&0&0&0&-1 & 2&0&1&0&1 & 0&0&0&4&0 \\
4 & 0&0&0&0&0 & -1&0&0&0&0 & 0&2&1&1&0 & 0&0&0&0&4
\end{bmatrix}
\begin{bmatrix} \hat{\mu} \\ \hat{p}_1 \\ \hat{p}_2 \\ \hat{p}_3 \\ \hat{p}_4 \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ \hat{s}_4 \\ \hat{s}_5 \\ \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \\ \hat{t}_4 \\ \hat{t}_5 \\ \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \\ \hat{r}_5 \end{bmatrix}
$$

T_i is the total of observations on treatment i and R_u is the total of observations containing the residual effect of the uth treatment.

For this design

$$\underline{\alpha} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\underline{\beta} = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 2 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Normal equations for period effects are simple, therefore the estimates of period effects can be derived directly from these equations. We get

$$\hat{p}_1 = 0.2\ P_1 - 0.04\ G$$

$$\hat{p}_2 = 0.2\ P_2 - 0.04\ G$$

$$\hat{p}_3 = 0.2\ P_3 - 0.04\ G$$

$$\hat{p}_4 = 0.2\ P_4 - 0.04\ G$$

$$\hat{p}_5 = 0.2\ P_5 - 0.04\ G$$

11.7.5 Normal equations for period effects have already been used for estimating period effects. The coefficient matrix is therefore rearranged to a convenient form for matrix inversion and estimation of other parameter effects.

$$
\begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \hat{t}_3 \\ \hat{t}_4 \\ \hat{t}_5 \\ \hat{r}_1 \\ \hat{r}_2 \\ \hat{r}_3 \\ \hat{r}_4 \\ \hat{r}_5 \\ \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \\ \hat{s}_4 \\ \hat{s}_5 \end{bmatrix}
=
\begin{bmatrix}
5&0&0&0&0&0&1&1&2&0&0&0&0&0&0 \\
0&5&0&0&0&1&0&1&0&2&0&0&0&0&0 \\
0&0&5&0&0&0&2&0&1&1&0&0&0&0&0 \\
0&0&0&5&0&1&0&2&0&1&0&0&0&0&0 \\
0&0&0&0&5&2&1&0&1&0&0&0&0&0&0 \\
0&1&0&1&2&4&0&0&0&0&0&-1&0&0&0 \\
1&0&2&0&1&0&4&0&0&0&0&0&-1&0&0 \\
1&1&0&2&0&0&0&4&0&0&0&0&0&-1&0 \\
2&0&1&0&1&0&0&0&4&0&0&0&0&0&-1 \\
0&2&1&1&0&0&0&0&0&4&-1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&-1&5&0&0&0&0 \\
0&0&0&0&0&-1&0&0&0&0&0&5&0&0&0 \\
0&0&0&0&0&0&-1&0&0&0&0&0&5&0&0 \\
0&0&0&0&0&0&0&-1&0&0&0&0&0&5&0 \\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&5
\end{bmatrix}^{-1}
\begin{bmatrix}
T_1 - 0.2\,G \\ T_2 - 0.2\,G \\ T_3 - 0.2\,G \\ T_4 - 0.2\,G \\ T_5 - 0.2\,G \\
R_1 - 0.2\,G + 0.2\,P_1 \\ R_2 - 0.2\,G + 0.2\,P_1 \\ R_3 - 0.2\,G + 0.2\,P_1 \\ R_4 - 0.2\,G + 0.2\,P_1 \\ R_5 - 0.2\,G + 0.2\,P_1 \\
S_1 - 0.2\,G \\ S_2 - 0.2\,G \\ S_3 - 0.2\,G \\ S_4 - 0.2\,G \\ S_5 - 0.2\,G
\end{bmatrix}
$$

Here $G = Y_{..}$ = Grand total of the observations.

The matrix has been inverted by computer using the programme given in appendix I.

The computer readings have been recorded to the nearest six decimal places but for calculations of results only three decimal places have been considered.

```
0.45563  0.17626  0.23137  0.18872  0.21469 -0.20904
-0.05964 -0.05312 -0.07144

0.17626  0.46014  0.18872  0.24834  0.19322 -0.28814 -0.19655 -0.29818 -0.19328 -0.35719 -0.07144 -0.05763
-0.03931 -0.05964 -0.03866

0.23137  0.18872  0.45563  0.17626  0.21469 -0.20904 -0.35719 -0.20332 -0.29818 -0.26561 -0.05312 -0.04181
-0.07144 -0.04066 -0.05964

0.18872  0.24834  0.17626  0.46014  0.19322 -0.28814 -0.19328 -0.35719 -0.19655 -0.29818 -0.05964 -0.05763
-0.03866 -0.07144 -0.03931

0.21469  0.19322  0.21469  0.19322  0.45085 -0.33898 -0.28814 -0.20904 -0.28814 -0.20904 -0.04181 -0.06780
-0.05763 -0.04181 -0.05763

-0.20904 -0.28814 -0.20904 -0.28814 -0.33898  0.59322  0.25424  0.28249  0.25424  0.28249  0.05650  0.11864
0.05085  0.05650  0.05085

-0.29818 -0.19655 -0.35719 -0.19328 -0.28814  0.25424  0.60545  0.23192  0.32676  0.24831  0.04966  0.05085
0.12109  0.04638  0.06535

-0.26561 -0.29818 -0.20332 -0.35719 -0.20904  0.28249  0.23192  0.59952  0.24831  0.30444  0.06089  0.05650
0.04638  0.11990  0.04966

-0.35719 -0.19328 -0.29818 -0.19655 -0.28814  0.25424  0.32676  0.24831  0.60545  0.23192  0.04638  0.05085
0.06535  0.04966  0.12109

-0.20332 -0.35719 -0.26561 -0.29818 -0.20904  0.28249  0.24831  0.30444  0.23192  0.59952  0.11990  0.05650
0.04966  0.06089  0.04638

-0.04066 -0.07144 -0.05312 -0.05964 -0.04181  0.05650  0.04966  0.06089  0.04638  0.11990  0.22398  0.01130
0.00993  0.01218  0.00928

-0.04181 -0.05763 -0.04181 -0.05763 -0.06780  0.11864  0.05085  0.05650  0.05085  0.05650  0.01130  0.22373
0.01017  0.01130  0.01017

-0.05964 -0.03931 -0.07144 -0.03866 -0.05763  0.05085  0.12109  0.04638  0.06535  0.04966  0.00993  0.01017
0.22422  0.00928  0.01307

-0.05312 -0.05964 -0.04066 -0.07144 -0.04181  0.05650  0.04638  0.11990  0.04966  0.06089  0.01218  0.01130
0.00928  0.22398  0.00993

-0.07144 -0.03866 -0.05964 -0.03931 -0.05763  0.05085  0.06535  0.04966  0.12109  0.04638  0.00928  0.01017
0.01307  0.00993  0.22422
```

11.7.7 Using inverted matrix 11.7.6 we calculate parameter effects as follows:

$$\hat{t}_1 = 0.456T_1+0.176T_2+0.231T_3+0.189T_4+0.215T_5-0.210R_1\ 0.298R_2 -0.266R_3-0.357R_4-0.203R_5-0.407S_1-0.042S_2-0.06S_3-0.053S_4 -0.071S_5+0.067G\ -0.267P_1$$

$$\hat{t}_2 = 0.176T_1+0.46T_2+0.189T_3+0.248T_4+0.193T_5-0.288R_1-0.197R_2 -0.298R_3-0.193R_4-0.357R_5-0.071S_1-0.058S_2-0.039S_3-0.06S_4 -0.039S_5+0.067G\ -0.267P_1$$

$$\hat{t}_3 = 0.231T_1+0.189T_2+0.456T_3+0.176T_4+0.215T_5-0.209R_1-0.357R_2 -0.203R_3-0.298R_4-0.266R_5-0.053S_1-0.042S_2-0.071S_3-0.041S_4 -0.06S_5+0.067G\ -0.267P_1$$

$$\hat{t}_4 = 0.189T_1+0.248T_2+0.176T_3+\ 0.46T_4+0.193T_5-0.288R_1-0.193R_2 -0.357R_3-0.197R_4-0.298R_5-0.06S_1-0.058S_2-0.039S_3 -0.071S_4-0.039S_5+0.067G\ -0.267P_1$$

$$\hat{t}_5 = 0.215T_1+0.193T_2+0.215T_3+0.193T_4+0.451T_5-0.339R_1-0.288R_2 -0.209R_3-0.288R_4-0.209R_5-0.416S_1-0.068S_2-0.058S_3-0.042S_4 -0.058S_5+0.067G\ -0.267P_1$$

$$\hat{r}_1 = -0.209T_1-0.288T_2-0.209T_3-0.288T_4-0.339T_5+0.593R_1+0.254R_2 +0.282R_3+0.254R_4+0.282R_5+0.0566S_1+0.119S_2+0.051S_3+0.057S_4 +0.051S_5-0.133G\ +0.333P_1$$

$$\hat{r}_2 = -0.298T_1-0.197T_2-0.357T_3-0.193T_4-0.288T_5+0.254R_1+0.605R_2 +0.232R_3+0.327R_4+0.248R_5+0.05S_1+0.05S_2+0.121S_3+0.046S_4 +0.065S_5-0.133G\ +0.333P_1$$

$$\hat{r}_3 = -0.266T_1-0.298T_2-0.203T_3-0.357T_4-0.209T_5+0.282R_1+0.232R_2 +0.6R_3+0.248R_4+0.304R_5+0.061S_1+0.057S_2+0.046S_3+0.12S_4 +0.05S_5-0.133G\ +0.333P_1$$

$$\hat{r}_4 = -0.357T_1-0.193T_2-0.298T_3-0.197T_4-0.288T_5+0.254R_1+0.327R_2$$
$$+0.248R_3+0.605R_4+0.232R_5+0.046S_1+0.051S_2+0.065S_3+0.05S_4$$
$$+0.121S_5-0.133G+0.333P_1$$

$$\hat{r}_5 = -0.203T_1-0.357T_2-0.267T_3-0.298T_4-0.209T_5+0.282R_1+0.248R_2$$
$$+0.304R_3+0.232R_4+0.6R_5+0.12S_1+0.057S_2+0.05S_3+0.061S_4$$
$$+0.046S_5-0.133G+0.333P_1$$

$$\hat{s}_1 = -0.041T_1-0.071T_2-0.053T_3-0.06T_4-0.042T_5+0.057R_1+0.05R_2$$
$$+0.061R_3+0.046R_4+0.12R_5+0.224S_1+0.011S_2+0.01S_3+0.012S_4$$
$$+0.009S_5-0.067G+0.067P_1$$

$$\hat{s}_2 = -0.042T_1-0.058T_2-0.042T_3-0.058T_4-0.068T_5+0.119R_1+0.051R_2$$
$$+0.057R_3+0.051R_4+0.057R_5+0.011S_1+0.224S_2+0.01S_3+0.011S_4$$
$$+0.01S_5-0.067G+0.067P_1$$

$$\hat{s}_3 = -0.06T_1-0.039T_2-0.071T_3-0.039T_4-0.058T_5+0.051R_1+0.121R_2$$
$$+0.046R_3+0.065R_4+0.05R_5+0.009S_1+0.01S_2+0.224S_3+0.009S_4$$
$$+0.013S_5-0.067G+0.067P_1$$

$$\hat{s}_4 = -0.053T_1-0.06T_2-0.041T_3-0.071T_4-0.042T_5+0.057R_1-0.046R_2$$
$$+0.12R_3+0.05R_4+0.061R_5+0.012S_1+0.011S_2+0.009S_3+0.224S_4$$
$$+0.224S_5-0.067G+0.067P_1$$

$$\hat{s}_5 = -0.071T_1-0.039T_2-0.06T_3-0.039T_4-0.058T_5+0.051R_1+0.065R_2$$
$$+0.05R_3+0.121R_4+0.046R_5+0.009S_1+0.01S_2+0.013S_3+0.01S_4$$
$$+0.224S_5-0.067G+0.067P_1$$

11.7.8 Variance of the difference between two parameter estimates.

$V(\hat{t}_1 - \hat{t}_2) = V(0.28T_1 - 0.284T_2) = 0.564\sigma^2$

$V(\hat{t}_1 - \hat{t}_3) = V(0.225T_1 - 0.225T_3) = 0.45\ \sigma^2$

$V(\hat{t}_1 - \hat{t}_4) = V(0.267T_1 - 0.271T_4) = 0.538\ \sigma^2$

$V(\hat{t}_1 - \hat{t}_5) = V(0.241T_1 - 0.236T_5) = 0.477\ \sigma^2$

$V(\hat{t}_2 - \hat{t}_3) = V(0.271T_2 - 0.267T_3) = 0.538\ \sigma^2$

$V(\hat{t}_2 - \hat{t}_4) = V(0.212T_2 - 0.212T_4) = 0.422\ \sigma^2$

$V(\hat{t}_2 - \hat{t}_5) = V(0.267T_2 - 0.258T_5) = 0.525\ \sigma^2$

$V(\hat{t}_3 - \hat{t}_4) = V(0.28T_3 - 0.284T_4) = 0.564\ \sigma^2$

$V(\hat{t}_3 - \hat{t}_5) = V(0.241T_3 - 0.236T_5) = 0.477\ \sigma^2$

$V(\hat{t}_4 - \hat{t}_5) = V(0.267T_4 - 0.258T_5) = 0.525\ \sigma^2$

$V(\hat{r}_1 - \hat{r}_2) = V(0.339R_1 - 0.351R_2) = 0.69\ \sigma^2$

$V(\hat{r}_1 - \hat{r}_3) = V(0.311R_1 - 0.318R_3) = 0.629\ \sigma^2$

$V(\hat{r}_1 - \hat{r}_4) = V(0.339R_1 - 0.351R_4) = 0.69\ \sigma^2$

$V(\hat{r}_1 - \hat{r}_5) = V(0.311R_1 - 0.318R_5) = 0.629\ \sigma^2$

$V(\hat{r}_2 - \hat{r}_3) = V(0.373R_2 - 0.368R_3) = 0.741\ \sigma^2$

$V(\hat{r}_2 - \hat{r}_4) = V(0.278R_2 - 0.278R_4) = 0.556\ \sigma^2$

$V(\hat{r}_2 - \hat{r}_5) = V(0.357R_2 - 0.352R_5) = 0.709\ \sigma^2$

$V(\hat{r}_3 - \hat{r}_4) = V(0.352R_3 - 0.357R_4) = 0.709\ \sigma^2$

$V(\hat{r}_3 - \hat{r}_5) = V(0.296R_3 - 0.296R_5) = 0.592\ \sigma^2$

$V(\hat{r}_4 - \hat{r}_5) = V(0.373R_4 - 0.368R_5) = 0.741\ \sigma^2$

11.7.9 Efficiency of measuring treatment differences over residual differences.

(i) Effy. $(\hat{t}_1 - \hat{t}_2) / (\hat{r}_1 - \hat{r}_2)$ = 0.69 / 0.564 = 1.223 = 122 %

(ii) Effy. $(\hat{t}_1 - \hat{t}_3) / (\hat{r}_1 - \hat{r}_3)$ = 0.629/ 0.45 = 1.398 = 140 %

(iii) Effy. $(\hat{t}_1 - \hat{t}_4) / (\hat{r}_1 - \hat{r}_4)$ = 0.69 / 0.538 = 1.283 = 128 %

(iv) Effy. $(\hat{t}_1 - \hat{t}_5) / (\hat{r}_1 - \hat{r}_5)$ = 0.629/ 0.477 = 1.319 = 132 %

(v) Effy. $(\hat{t}_2 - \hat{t}_3) / (\hat{r}_2 - \hat{r}_3)$ = 0.741/ 0.538 = 1.378 = 138 %

(vi) Effy. $(\hat{t}_2 - \hat{t}_4) / (\hat{r}_2 - \hat{r}_4)$ = 0.556/ 0.422 = 1.318 = 132 %

(vii) Effy. $(\hat{t}_2 - \hat{t}_5) / (\hat{r}_2 - \hat{r}_5)$ = 0.709/ 0.525 = 1.35 = 135 %

(viii) Effy. $(\hat{t}_3 - \hat{t}_4) / (\hat{r}_3 - \hat{r}_4)$ = 0.709/ 0.564 = 1.257 = 126 %

(ix) Effy. $(\hat{t}_3 - \hat{t}_5) / (\hat{r}_3 - \hat{r}_5)$ = 0.592/ 0.477 = 1.241 = 124 %

(x) Effy. $(\hat{t}_4 - \hat{t}_5) / (\hat{r}_4 - \hat{r}_5)$ = 0.741/ 0.525 = 1.411 = 141 %

The overall efficiecy of measuring treatment effects over residual effects is

$$= \frac{n^2}{n^2 - n - 1} = \frac{25}{19} = 1.316 = 132\ \%$$

The average of the above ten efficiency factors is equal to 132 %, as expected, since these ten efficiency factors relate to all the possible pairs of treatment as well as residual effects in this design.

The efficiency percentages vary from 122 % to 141 %, which is a smaller range than any other pattern of designs of Latin-squares of order 5x5. This means that this pattern of designs provides better uniformity in measuring parameter effects than any other pattern.

11.7.10 Variance of linear component of treatment effects, using coefficients of orthogonal polynomials for linearity.

$$V(-2\hat{t}_1-\hat{t}_2+0\hat{t}_3+\hat{t}_4+2\hat{t}_5) = V(-2x-0.469T_1-1x-0.178T_2 + 0 + 1x0.22T_4 +2x0.472T_5)$$

$$= (0.938 +0.178 +0.22 +0.944)\sigma^2$$

$$= 2.28\sigma^2$$

Variance of linear component of residual effects.

$$V(-2\hat{r}_1-\hat{r}_2+0\hat{r}_3+\hat{r}_4+2\hat{r}_5) = V(-2x-0.622R_1-1x-0.29R_2+ 0 + 1x0.234R_4+2x0.62R_5)$$

$$= (1.244 +0.29 +0.234 +1.24)\sigma^2$$

$$= 3.008\sigma^2$$

Efficiency ratio of measuring linear component of treatment effects over residual effects is

$$= 3.008 / 2.28$$

$$= 1.319$$

$$= 132 \%$$

The variances of linear components for both treatment and residual effects are smaller than any other design pattern of Latin-squares of the same order.

11.7.11 Permanent effects of treatments are calculated as:

$$\hat{t}_1+\hat{r}_1 = 0.247T_1-0.112T_2+0.022T_3-0.099T_4-0.124T_5+0.383R_1 -0.044R_2+0.016R_3-0.103R_4+0.079R_5-0.351S_1+0.077S_2 -0.009S_3+0.004S_4-0.02S_5-0.066G +0.066P_1$$

$$\hat{t}_2+\hat{r}_2 = -0.122T_1+0.263T_2-0.168T_3+0.055T_4-0.095T_5-0.034R_1 +0.408R_2-0.066R_3+0.134R_4-0.109R_5-0.021S_1-0.008S_2 +0.082S_3-0.014S_4+0.026S_5-0.066G +0.066P_1$$

$$\hat{t}_3+\hat{r}_3 = -0.035T_1-0.109T_2+0.253T_3-0.181T_4+0.006T_5+0.073R_1 -0.125R_2+0.397R_3-0.05R_4+0.038R_5+0.008S_1+0.015S_2 -0.025S_3+0.079S_4-0.01S_5-0.066G +0.066P_1$$

$$\hat{t}_4+\hat{r}_4 = -0.168T_1+0.055T_2-0.122T_3+0.263T_4-0.095T_5-0.034R_1 +0.134R_2-0.109R_3+0.408R_4-0.066R_5-0.014S_1-0.007S_2 +0.026S_3-0.021S_4+0.082S_5-0.066G +0.066P_1$$

$$\hat{t}_5+\hat{r}_5 = 0.012T_1-0.164T_2-0.052T_3-0.105T_4+0.242T_5-0.057R_1 -0.04R_2+0.095R_3-0.056R_4+0.391R_5-0.296S_1-0.011S_2 -0.008S_3+0.019S_4-0.012S_5-0.066G +0.066P_1$$

11.7.12 Variance of the difference between two permanent effects of treatments.

$$V\left\{(\hat{t}_i+\hat{r}_i)-(\hat{t}_u+\hat{r}_u)\right\} = V\left\{(\hat{t}_i-\hat{t}_u)+(\hat{r}_i-\hat{r}_u)\right\}$$

Therefore from 11.7.8 we get

$$V\left\{(\hat{t}_1+\hat{r}_1)-(\hat{t}_2+\hat{r}_2)\right\} = (0.564+0.69)\sigma^2 = 1.254\,\sigma^2$$

$$V\left\{(\hat{t}_1+\hat{r}_1)-(\hat{t}_3+\hat{r}_3)\right\} = (0.45+0.629)\sigma^2 = 1.079\,\sigma^2$$

$$V\left\{(\hat{t}_1+\hat{r}_1)-(\hat{t}_4+\hat{r}_4)\right\} = (0.538+0.69)\sigma^2 = 1.228\,\sigma^2$$

$$V\left\{(\hat{t}_1+\hat{r}_1)-(\hat{t}_5+\hat{r}_5)\right\} = (0.477+0.629)\sigma^2 = 1.106\,\sigma^2$$

$$V\left\{(\hat{t}_2+\hat{r}_2)-(\hat{t}_3+\hat{r}_3)\right\} = (0.538 + 0.741)\sigma^2 = 1.279\sigma^2$$

$$V\left\{(\hat{t}_2+\hat{r}_2)-(\hat{t}_4+\hat{r}_4)\right\} = (0.422 + 0.556)\sigma^2 = 0.978\sigma^2$$

$$V\left\{(\hat{t}_3+\hat{r}_3)-(\hat{t}_5+\hat{r}_5)\right\} = (0.525 + 0.709)\sigma^2 = 1.234\sigma^2$$

$$V\left\{(\hat{t}_3+\hat{r}_3)-(\hat{t}_5+\hat{r}_5)\right\} = (0.565 + 0.709)\sigma^2 = 1.273\sigma^2$$

$$V\left\{(\hat{t}_4+\hat{r}_4)-(\hat{t}_5+\hat{r}_5)\right\} = (0.477 + 0.592)\sigma^2 = 1.069\sigma^2$$

Average of the variances of the differences between two treatment effects. Av. $V(\hat{t}_i-\hat{t}_j) = 0.508\sigma^2$

Average of the variances of the differences between two residual effects. Av. $V(\hat{r}_i-\hat{r}_j) = 0.668\sigma^2$

Average of the variances of the differences between two permanent effects of treatments = $1.176\sigma^2$.

$i, j = 1, 2, 3, 4$ and 5

Chapter 12

D-optimality Test and Structure of Beta ($\underline{\beta}$) for the Most Efficient Design.

12.1 D-optimality.

T.J.Mitchell (1973) and V.V.Fedorov (1972) suggest that for looking for the most efficient design in a series of designs one should look for the design with D-optimality which means that the determinant $\left| \underline{X}'\underline{X} \right|$ should be maximum or equivalently the generalised variance determinant $\left| (\underline{X}'\underline{X})^{-1} \sigma^2 \right|$ of the estimated coefficients should be minimum.

By maximising $\left| \underline{X}'\underline{X} \right|$, we in fact are minimising the generalised variance of the set of all parameters involved in the model.

Mitchell (1973) further suggests that we may minimise the generalised variance of the parameters of interest only rather than all the parameters introduced in the design. This can be achieved by partitioning the generalised variance matrix as follows.

$$(\underline{X}'\underline{X})^{-1} = \begin{bmatrix} \underline{X}_1'\underline{X}_1 & \underline{X}_1'\underline{X}_2 \\ \underline{X}_2'\underline{X}_1 & \underline{X}_2'\underline{X}_2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{bmatrix} \quad \text{and} \quad \left| \underline{A}_{11} \right| = \frac{\left| \underline{X}_2'\underline{X}_2 \right|}{\left| \underline{X}'\underline{X} \right|}$$

$\underline{X}$ is the matrix of independent variables at which a response is measured from the model. $\underline{X}'\underline{X}$ is the (m x m) matrix of normal equations where m is the number of parameters to be estimated.

In the partitioned form $\underline{X}$ has been written as $[\underline{X}_1 \ \underline{X}_2]$

where $\underline{X}_1$ is associated with the variables of the parameters of interest.

$\underline{X}_2$ is associated with the variables of the parameters of less interest or nuisance parameters.

Therefore, the design measuring the parameters of interest, most efficiently, can now be obtained simply by the minimisation of $|\underline{A}_{11}|$ only.

$|\underline{A}_{11}| = \dfrac{|\underline{X}_2' \underline{X}_2|}{|\underline{X}' \underline{X}|}$ means that maximisation of $|\underline{X}' \underline{X}|$

is in fact minimisation of the generalised variance of the parameters of interest. Moreover, for calculation purposes it is better to base conclusions on the basis of maximisation of $|\underline{X}' \underline{X}|$ rather than minimisation of $|(\underline{X}' \underline{X})^{-1} \sigma^2|$ or $|\underline{A}_{11}|$ etc., as this will cut down lot of calculations.

For the linear model of 2.3

$$Y_{ij(kl)} = \mu + p_i + s_j + t_k + r_l + e_{ij(kl)}$$

where all possible combinations of treatments and blocks have been applied, Mitchell (1973) suggests that D-optimal design i.e. $|\underline{X}' \underline{X}|$ maximum, is a sufficient condition for efficiency.

The efficiency of an experiment depends upon its magnitude of dispersion. The bigger the value of dispersion the smaller will be the degree of efficiency and vice versa. Our aim is to find a design or pattern of designs which gives us smaller dispersion.

12.2 Determinant values of $(\underline{X}'\underline{X})$ Matrices and Dispersion Matrices.

Design Number	$\lvert \underline{X}'\underline{X} \rvert$	$\lvert (\underline{X}'\underline{X})^{-1} \rvert$	
1	0.365 E9	0.274 E-8	
2	0.680 E8	0.147 E-7	
3	0.402 E9	0.249 E-8	
4	0.680 E8	0.147 E-7	
5	0.680 E8	0.147 E-7	
6	0.104 E9	0.959 E-8	
7	0.389 E9	0.257 E-8	
8	0.365 E9	0.274 E-8	
9	0.680 E8	0.147 E-7	
10	0.765 E8	0.131 E-7	
11	0.365 E9	0.274 E-8	
12	0.506 E9	0.198 E-8	*
13	0.680 E8	0.147 E-7	
14	0.365 E9	0.274 E-8	
15	0.765 E8	0.131 E-7	
16	0.680 E8	0.147 E-7	
17	0.365 E9	0.274 E-8	
18	0.765 E8	0.131 E-7	
19	0.104 E9	0.959 E-8	
20	0.765 E8	0.131 E-7	

Design Number	$\lvert \underline{X}'\underline{X} \rvert$	$\lvert (\underline{X}'\underline{X})^{-1} \rvert$	
21	0.506 E9	0.198 E-8	*
22	0.680 E8	0.147 E-7	
23	0.347 E9	0.289 E-8	
24	0.680 E8	0.147 E-7	
25	0.506 E9	0.198 E-8	*
26	0.536 E9	0.187 E-8	*
27	0.759 E6	0.132 E-5	
28	0.536 E9	0.187 E-8	*
29	0.208 E9	0.480 E-8	
30	0.536 E9	0.187 E-8	*
31	0.536 E9	0.187 E-8	*

* indicates maximum determinant of $\underline{X}'\underline{X}$ and minimum determinant of $(\underline{X}'\underline{X})^{-1}$.

As explained on pages 143 and 144 we again notice from the above list that the numerical value of group A designs (12, 21 and 25) slightly differ from group B designs(26, 28, 30 and 31). The difference arises due to the fact that group B designs are symmetrical designs and group A designs are non-symmetrical. This difference is quite small and does not introduce sufficient error to reduce materially the efficiency of the designs in group A as compared with group B designs. Therefore we shall consider all these designs of equal importance and the difference in the numerical values for $\lvert \underline{X}'\underline{X} \rvert$ and $\lvert (\underline{X}'\underline{X})^{-1} \rvert$ shall be ignored for any discussion.

From the list under 12.2 we find that designs 12, 21, 25, 26, 28, 30 and 31 have maximum determinant values of their $\underline{X}'\underline{X}$ matrices and minimum values of their dispersion matrices. Therefore, by the D-optimality test, we can classify these designs as the most efficient ones. In all of these designs each treatment was preceded twice by one treatment, once by two other treatments and was never preceded by itself or the fourth treatment.

For example, for design 12 the lay-out of the design pattern is as:

		Preceding Treatment				
		1 (A)	2 (B)	3 (C)	4 (D)	5 (E)
Following Treatment	1 (A)	0	2	0	1	1
	2 (B)	1	0	1	0	2
	3 (C)	0	1	0	2	1
	4 (D)	2	1	1	0	0
	5 (E)	1	0	2	1	0

There is a similar pattern for the other designs, i.e. 21, 25, 26, 28, 30 and 31. The only change in the patterns is the position of the numbers 0, 1 and 2.

The numbers 0, 1 and 2 in the above table indicate the number of times a treatment has been preceded by another treatment in the design.

12.3 Test based on Parameters of interest only.

Now considering the reduced form of the matrices obtained by partitioning the design matrices as suggested under 12.1. Since we are mainly interested in the study of treatment direct and treatment residual effects we may use only that part of the design matrix which is associated with these parameters only.

The matrix $(\underline{X}'\underline{X})$ may be partitioned as follows:

$$(\underline{X}'\underline{X}) = \begin{bmatrix} \underline{X}_1'\underline{X}_1 & \underline{X}_1'\underline{X}_2 \\ \underline{X}_2'\underline{X}_1 & \underline{X}_2'\underline{X}_2 \end{bmatrix}$$

where $(\underline{X}_1'\underline{X}_1)$ can be taken as associated with the parameters of interest i.e. treatment direct and treatment residual effects. For example the partitions for design 9.3.1 (1) can be shown as below.

$$\text{12.3.1} \quad \left[\begin{array}{cccccccccc|ccccc}
5 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 2 & 1 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 1 & 2 & 0 & 0 & 4 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & -1 & 0 & 0 \\
2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5
\end{array}\right]$$

12.3.2 The partitioned matrix of normal equations, based on the parameters of interest and other parameters, can be expressed as:

$$\underline{X}'\underline{X} = \left[\begin{array}{cc|c} n\,\underline{I} & \underline{\beta} & \underline{0} \\ \underline{\beta}' & (n-1)\,\underline{I} & -\underline{\alpha}' \\ \hline \underline{0} & -\underline{\alpha} & n\,\underline{I} \end{array}\right]$$

Therefore for determining the efficiency of the designs under consideration we need to apply the D-optimality test only to the top left hand corner of the above partitioned matrix.

$$\underline{X}_1'\,\underline{X}_1 = \begin{bmatrix} n\,\underline{I} & \underline{\beta} \\ \underline{\beta}' & (n-1)\,\underline{I} \end{bmatrix}$$

Under the D-optimality test this matrix ($\underline{X}_1'\,\underline{X}_1$) should have maximum determinant or equivalently minimum determinant for its inverse ($\underline{X}_1'\,\underline{X}_1$)$^{-1}$ for the most efficient design.

$$(\underline{X}_1'\,\underline{X}_1)^{-1} = \begin{bmatrix} (n-1)\,\underline{U}^{-1} & -\underline{\beta}'\,\underline{U}^{-1} \\ -\underline{\beta}\,\underline{U}^{-1} & n\,\underline{U}^{-1} \end{bmatrix}$$

where $\underline{U} = n(n-1)\,\underline{I} - \underline{\beta}'\underline{\beta}$

Here we notice that $\underline{\beta}$ is the most important unit in the formation of a design. Therefore if we can control $\underline{\beta}$ we can produce the most efficient design of any order. Parameters of interest in this case are the treatment and residual effects.

12.3.3 Now looking at the determinants of the matrices considered under 12.3.2.

We know that

$$\begin{vmatrix} \underline{A} & \underline{C} \\ \underline{B} & \underline{D} \end{vmatrix} = |\underline{A}|\,|\underline{D} - \underline{B}\,\underline{A}^{-1}\,\underline{C}|$$

Therefore

$$|\underline{X}_1'\,\underline{X}_1| = \begin{vmatrix} n\,\underline{I} & \underline{\beta} \\ \underline{\beta}' & (n-1)\,\underline{I} \end{vmatrix}$$

$$= |n\,\underline{I}|\,|(n-1)\,\underline{I} - \underline{\beta}'(n\,\underline{I})^{-1}\underline{\beta}|$$

$$= |n(n-1)\,\underline{I} - \underline{\beta}'\underline{\beta}|$$

According to Graybill(1961) if $\underline{A}_1$ and $\underline{A}_2$ are symmetric and $\underline{A}_2$ is positive definite and if $(\underline{A}_1 - \underline{A}_2)$ is positive semidefinite (or positive definite) then $|\underline{A}_1| \geqslant |\underline{A}_2|$, implying that $|\underline{A}_1| - |\underline{A}_2| \geqslant 0$.

($\underline{X}_1'\,\underline{X}_1$) and ($\underline{\beta}'\underline{\beta}$) are positive definite as every principal minor of these matrices is positive. We get the determinant

$$|\underline{X}_1'\,\underline{X}_1| = |n(n-1)\,\underline{I}| - |\underline{\beta}'\underline{\beta}| > 0$$

Hence to maximise $|\underline{X}_1'\,\underline{X}_1| = |n(n-1)\,\underline{I}| - |\underline{\beta}'\underline{\beta}|$ we minimise $|\underline{\beta}'\underline{\beta}|$.

Keeping to the assumptions regarding $\underline{\beta}$, as $|\underline{\beta}'\underline{\beta}| \rightarrow$ minimum

$|\underline{X}_1'\,\underline{X}_1| \rightarrow$ maximum.

Expressing the determinant of $\underline{X}_1'\,\underline{X}_1$ as:

$$|\underline{X}_1'\,\underline{X}_1| = |n\,\underline{I}|\,|(n-1)\,\underline{I}| - |\underline{\beta}'|\,|\underline{\beta}|$$

we notice that to maximise $|\underline{X}_1'\,\underline{X}_1|$ we simply minimise $|\underline{\beta}|$.

Alternatively, the determinant of $(\underline{X}_1'\,\underline{X}_1)^{-1}$ is minimum for the most efficient design, therefore, we may also study the effect of $|\underline{\beta}|$ on $|(\underline{X}_1'\,\underline{X}_1)^{-1}|$.

$$\left|(\underline{X}_1' \underline{X}_1)^{-1}\right| = \begin{vmatrix} (n-1)\,\underline{U}^{-1} & -\underline{\beta}\,\underline{U}^{-1} \\ -\underline{U}^{-1}\,\underline{\beta}' & n\,\underline{U}^{-1} \end{vmatrix}$$

$$= \left|(n-1)\,\underline{U}^{-1}\right| \left|n\,\underline{U}^{-1} - (n-1)^{-1}\,\underline{\beta}'\,\underline{U}\,\underline{\beta}\right|$$

$$= \left|\underline{U}^{-1}\right|$$

$$= \frac{1}{\left|n(n-1)\,\underline{I} - \underline{\beta}'\underline{\beta}\right|}$$

In this case maximum denominator ($|n(n-1)\,\underline{I} - \underline{\beta}'\underline{\beta}|$) produces minimum $|(\underline{X}_1' \underline{X}_1)^{-1}|$.

As $|n(n-1)\,\underline{I} - \underline{\beta}'\underline{\beta}| \rightarrow$ maximum, $|(\underline{X}_1' \underline{X}_1)^{-1}| \rightarrow$ minimum. Following the earlier argument about determinant we find that as $|\underline{\beta}'\underline{\beta}| \rightarrow$ minimum $|n(n-1)\,\underline{I} - \underline{\beta}'\underline{\beta}| \rightarrow$ maximum and hence $|(\underline{X}_1' \underline{X}_1)^{-1}| \rightarrow$ minimum.

12.3.4 The results under 12.3.3 have been obtained from submatrices of parameters of interest only. The determinants of these matrices produce the same results as the determinants of the main matrices $\underline{X}'\underline{X}$ and $(\underline{X}'\underline{X})^{-1}$ as shown below.

$$\left|\underline{X}'\underline{X}\right| = \begin{vmatrix} n\,\underline{I} & \underline{\beta} & \underline{0} \\ \underline{\beta}' & (n-1)\,\underline{I} & -\underline{\alpha}' \\ \underline{0} & -\underline{\alpha} & n\,\underline{I} \end{vmatrix}$$

$$= \left|n\,\underline{I}\right| \begin{vmatrix} (n-1)\,\underline{I} & -\underline{\alpha}' \\ -\underline{\alpha} & n\,\underline{I} \end{vmatrix} - \left|\underline{\beta}\right| \begin{vmatrix} \underline{\beta}' & -\underline{\alpha}' \\ \underline{0} & n\,\underline{I} \end{vmatrix} + \left|\underline{0}\right|$$

$$= \left|n\,\underline{I}\right| \left|n(n-1)\,\underline{I} - \underline{\alpha}'\underline{\alpha}\right| - \left|\underline{\beta}\right| \left|n\,\underline{\beta}'\right|$$

$$= \left|n^2(n-1)\,\underline{I} - \underline{\alpha}'\underline{\alpha}\right| - \left|n\,\underline{\beta}'\underline{\beta}\right|$$

$$= \left|n^2(n-1)\,\underline{I} - n\,\underline{I}\right| - \left|n\,\underline{\beta}'\underline{\beta}\right| \qquad \because\ \underline{\alpha}'\underline{\alpha} = \underline{I}$$

and $\left|\underline{B}\,\underline{A}\right| = \left|\underline{A}\right|\left|\underline{B}\right|$

$$= \left|n(n^2 - n - 1)\,\underline{I}\right| - \left|n\,\underline{\beta}'\underline{\beta}\right|$$

Here again as $\left|\underline{\beta}'\underline{\beta}\right| \longrightarrow 0$ $\quad \left|\underline{X}'\underline{X}\right| \longrightarrow$ maximum.

$$\left|(\underline{X}'\underline{X})^{-1}\right| = \frac{1}{\left|\underline{X}'\underline{X}\right|} = \frac{1}{\left|n(n^2 - n - 1)\,\underline{I} - n\,\underline{\beta}'\underline{\beta}\right|}$$

For minimum $\left|(\underline{X}'\underline{X})^{-1}\right|$ we should have minimum $\underline{\beta}$ since it is the only variable quantity. Once n, the size of the design, has been decided, it cannot be changed.

12.3.5 The left hand corner of the partitioned matrix 12.3.1 is the matrix associated with treatment direct and treatment residual effects only. Likewise, reducing the size of the other designs and using the computer program in Appendix II, we get their determinant values as under.

Design Number	$\lvert \underline{X}_1' \underline{X}_1 \rvert$	$\lvert (\underline{X}_1' \underline{X}_1)^{-1} \rvert$
1	0.207 E6	0.483 E-5
2	0.487 E5	0.205 E-4
3	0.207 E6	0.483 E-5
4	0.487 E5	0.205 E-4
5	0.487 E5	0.205 E-4
6	0.717 E5	0.139 E-4
7	0.198 E6	0.505 E-5
8	0.207 E6	0.483 E-5
9	0.487 E5	0.205 E-4
10	0.538 E5	0.186 E-4
11	0.207 E6	0.483 E-5
12	0.280 E6	0.358 E-5 *
13	0.487 E5	0.205 E-4
14	0.207 E6	0.483 E-5
15	0.538 E5	0.186 E-4
16	0.487 E5	0.205 E-4
17	0.207 E6	0.483 E-5
18	0.683 E5	0.146 E-4
19	0.717 E5	0.139 E-4
20	0.538 E5	0.186 E-4

Design Number	$\left| \underline{X}_1' \underline{X}_1 \right|$	$\left| (\underline{X}_1' \underline{X}_1)^{-1} \right|$	
21	0.280 E6	0.358 E-5	*
22	0.487 E5	0.205 E-4	
23	0.198 E6	0.505 E-5	
24	0.487 E5	0.205 E-4	
25	0.280 E6	0.358 E-5	*
26	0.294 E6	0.340 E-5	*
27	0.102 E4	0.977 E-3	
28	0.294 E6	0.340 E-5	*
29	0.124 E6	0.807 E-5	
30	0.294 E6	0.340 E-5	*
31	0.294 E6	0.340 E-5	*

* Most efficient design.

Studying the determinant values given above and applying the D-optimality test we find that designs 12, 21, 25, 26, 28, 30 and 31 are the most efficient designs, which are the same as found under 12.2 and 11.6. Hence it confirms the efficiency of the designs and also the fact that the matrices restricted to parameters of interest only produce the same results as the design and dispersion matrices based on all the information.

12.4 Determinants of $X'X$ for 4x4 Latin-squares.

Determinants of $X'X$ and dispersion matrices of 4x4 Latin-square designs considered under 2.4 are:

Design Number	$\lvert X'X \rvert$	$\lvert (X'X)^{-1} \rvert$
1	0.410 E4	0.244 E-3
2	0.512 E6	0.195 E-5
3	0.184 E6	0.543 E-5
4	0.102 E6	0.977 E-5

Determinants of the above mentioned matrices when reduced to parameters of interest only i.e. treatment direct and treatment residual effects only.

Design Number	$\lvert X_1' X_1 \rvert$	$\lvert (X_1' X_1)^{-1} \rvert$
1	0.810 E2	0.123 E-1
2	0.399 E4	0.250 E-3
3	0.162 E4	0.618 E-3
4	0.109 E4	0.918 E-3

From the above two tables of determinants we notice that design 2 has maximum value for its determinant $\lvert X'X \rvert$ and minimum value for its dispersion determinant $\lvert (X'X)^{-1} \rvert$ which is the necessary condition of the D-optimality test for the most efficient design. Hence we find that design 2.4.2, classified as the most efficient pattern of 4x4 Latin-square designs, under the previous tests, is also classified as the most efficient design under the D-optimality test.

The above tables also show that design 1 is the most inefficient since it has minimum value of its determinant $\lvert X'X \rvert$ and maximum value of its dispersion determinant which is in line with our findings under the previous tests.

The study of 4x4 Latin-square designs leads us to the conclusion that the structure of the most efficient design is based on the fact that each treatment is followed by every other treatment once only, which may be stated that each treatment is preceded by every other treatment, apart from itself, once only. This conclusion is based on the condition of minimum variance of linear components of parameters and the D-optimality test.

Following the same basis of study as for the 4x4 Latin-square designs we found that the most efficient designs among 5x5 Latin-squares are those where each treatment was preceded twice by any other two treatments. Further study of the designs of this pattern disclosed that there are twenty such designs of order 5x5. Ten of these designs (11.4) are the most efficient and the other ten (11.5) are 50 % less efficient as based on the condition of minimum variance of linear components. Therefore, we cannot adopt this pattern as a general pattern for the purpose of describing a most efficient design from among the 5x5 Latin-square designs available for measuring the treatment direct and treatment residual.effects.

This then leads us to the application of D-optimality test (12.1) and brings us to the conclusion that for the 5x5 Latin-square designs, the most efficient has the pattern of each treatment preceded twice by one of the treatments and once by other two tratments (12.2.1). Again this brings us much closer to our conclusion about 4x4 Latin-square (2.4.2) as the most efficient design for measuring treatment direct as well as treatment residual effects.

From the set of 5x5 Latin-square designs there is no design where each treatment is preceded by every other treatment. Next to this pattern is the pattern where a treatment is preceded twice by one treatment and once by other two treatments and this is now accepted as a general pattern for the most efficient design of Latin-square of order 5x5. The average of the variances of the differences between two parameter effects of the most efficient design is smaller than the designs of other patterns as can be seen under 11.6.1 and 11.6.2.

The most inefficient design is one where each treatment was preceded by the same treatment on all occasions and has been classified as such by the methods of variance of linear components, average variance of the differences and D-optimality.

The determinant values of designs 11.4 and 11.5 are the same and therefore, D-optimality test does not make any distinction between the two sets of designs.

12.5 Beta ($\underline{\beta}$) Structure for the Most Efficient Designs.

Let

$$\underline{\beta} = \begin{bmatrix} b_{11} & b_{12} & b_{13}\cdots\cdots\cdots b_{1n} \\ b_{21} & b_{22} & b_{23}\cdots\cdots\cdots b_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots\cdots\cdots \\ b_{n1} & b_{n2} & b_{n3}\cdots\cdots\cdots b_{nn} \end{bmatrix}$$

The elements b_{ij} indicate the number of times treatments follow and precede each other.

12.5.1 When n the number of treatments is even

$b_{ij} = 0$ for $i=j$ diagonal elements are 0.

and $b_{ij} = 1$ for $i \neq j$ off-diagonal elements are 1.

such that $\sum_{i=1}^{n} b_{ij} = \sum_{j=1}^{n} b_{ij} = n-1$

12.5.2 When n the number of treatments is odd

$b_{ij} = 0$ for $i=j$ i.e. $b_{11} = b_{22} = \cdots\cdots b_{nn} = 0$

and b_{ij} ($i \neq j$) = 0 once in each row and once in each column.

= 2 " " " " " " " " "

= 1 for the rest of n-3 places in each row and each column.

such that $\sum_{i=1}^{n} b_{ij} = \sum_{j=1}^{n} b_{ij} = n-1$

For example for Latin-square designs of order 5 x 5 and 6 x 6 Beta structures are respectively:

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

For further study of Beta structure writing

$$\beta' \beta = \begin{bmatrix} c_{11} & c_{12} & c_{13} \cdots\cdots\cdots c_{1n} \\ c_{21} & c_{22} & c_{23} \cdots\cdots\cdots c_{2n} \\ \cdots & \cdots & \cdots\cdots\cdots \\ \cdots & \cdots & \cdots\cdots\cdots \\ c_{n1} & c_{n2} & c_{n3} \cdots\cdots\cdots c_{nn} \end{bmatrix}$$

Since $\beta = \left[b_{ij} \right]$ and $\beta' = \left[b'_{ij} \right]$

hence $$c_{ij} = \sum_{k=1}^{n} b_{ik} . b'_{kj}$$

12.5.3 For n odd

$$c_{ij} = n + 1 \quad \text{when } i = j$$

and the trace of $\beta' \beta = n(n + 1)$. The trace of other designs classified as inefficient is always greater than n(n + 1).

12.5.4 For n even

$$\underset{(i=j)}{c_{ij}} = n - 1$$ i.e. all diagonal elements are equal to n-1.

The trace of $\beta' \beta = n(n - 1)$ and this is greater than n(n - 1) for all other designs known as inefficient.

As an example for Betas mentioned under 12.5.2 the $\beta' \beta$ diagonals and traces are as follows.

n = 5

$$\begin{bmatrix} 6 & 2 & 3 & 2 & 3 \\ 2 & 6 & 2 & 3 & 3 \\ 3 & 2 & 6 & 3 & 2 \\ 2 & 3 & 3 & 6 & 2 \\ 3 & 3 & 2 & 2 & 6 \end{bmatrix}$$

Trace = 5 x 6 = 30

n = 6

$$\begin{bmatrix} 5 & 4 & 4 & 4 & 4 & 4 \\ 4 & 5 & 4 & 4 & 4 & 4 \\ 4 & 4 & 5 & 4 & 4 & 4 \\ 4 & 4 & 4 & 5 & 4 & 4 \\ 4 & 4 & 4 & 4 & 5 & 4 \\ 4 & 4 & 4 & 4 & 4 & 5 \end{bmatrix}$$

Trace = 6 x 5 = 30

12.6 Design Matrix.

Each row of the design matrix $\underline{X}$ represents a combination of factors at which a response is measured and the expected response is assumed to have a linear function of the factor variables.

Rao (1973) explains that if $\underline{X}$ is the design matrix then the dispersion matrix of least square estimator $\underline{\hat{\theta}}$ is $(\underline{X}'\Sigma^{-1}\underline{X})^{-1}$. We choose $\underline{X}$ to make $(\underline{X}'\Sigma^{-1}\underline{X})^{-1}$ as small as possible. Therefore by choosing the factor values in an unlimited range the values of $(\underline{X}'\Sigma^{-1}\underline{X})^{-1}$ can be made as small as required. For example by choosing $\underline{X}$ as $a\underline{X}$ the dispersion matrix becomes $a^{-2}(\underline{X}'\Sigma^{-1}\underline{X})^{-1}$ tending to zero as a tends to infinity. We limit the ranges to small regions of interest only within which we have measurements of the response function.

$$\text{Taking } \underline{X}_i'\Sigma^{-1}\underline{X}_i = C_i^2 \quad i= 1,2,\ldots\ldots\ldots n.$$

where $\underline{X}_i$ is the column vector of $\underline{X}$ representing the n levels of the ith factor used in an experiment. We choose the combinations of the factor values which lead to $\underline{\hat{\theta}}$ estimators with the least variance.

According to Rao (1973) let $\underline{X}$ be a design matrix and $\hat{\theta}_i$ be the least square estimator of θ_i then under the above conditions on $\underline{X}$,

(i) $V(\hat{\theta}_i) \geqslant \frac{1}{C_i^2}$ and the minimum is attained when $\underline{X}_i'\Sigma^{-1}\underline{X}_j = 0$ for $j = 1,2,\ldots i-1, i+1,\ldots\ldots n$.

(ii) The optimum choice of combinations (rows of $\underline{X}$) is when $\underline{X}_i'\underline{X}_j = 0$, that is, the columns of $\underline{X}$ are orthogonal.

12.6.1 Considering the submatrix under 12.3.2

$$\underline{X}_1' \underline{X}_1 = \begin{bmatrix} n\,\underline{I} & \underline{\beta} \\ \underline{\beta}' & (n-1)\,\underline{I} \end{bmatrix}$$

Under condition (i) of 12.6 minimum variance is attained when $\underline{\beta} = \underline{0}$, that is, $b_{ij} = 0$ for $i \neq j$ in $\underline{\beta}$ stated under 12.5. This is contrary to the basic assumption of this study. Hence a Latin-square design having $\underline{\beta} = \underline{0}$ does not in any way exist. Following the assumption of residual effects persisting for one period only, condition (i) can be modified accordingly by stating that:

(iii) $V(\hat{\theta}_i) \geqslant \frac{1}{C_i^2}$ and the minimum is attained

when $\underline{X}_i' \underline{\Sigma}^{-1} \underline{X}_j = 1$ for $j = 1,2,......i-1, i+1,......n.$

According to this condition (iii) $b_{ij} = 1$ when $i \neq j$. It is to be noted that for this study of residual effects the maximum value of b_{ij} when $i \neq j$ is $n - 1$.

Observing the modified condition (iii) we can choose $\underline{\beta}$ which determines the most efficient Latin-square design when n the number of treatments is even. For example,

$$\underline{\beta} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ 2n' \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \cdots\cdots & 2n' \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & \cdots\cdots & 1 \\ 1 & 0 & 1 & \cdots\cdots & 1 \\ 1 & 1 & 0 & \cdots\cdots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & & 0 \end{bmatrix} \end{array}$$

$2n'$ = Even number of treatments.

12.6.2 As already explained, Latin-square designs for odd number of treatments, having $\underline{\beta}$ with elements $b_{ij} = 1$ when $i \neq j$, do not exist. To construct $\underline{\beta}$ for n odd we have to further modify condition (ii) as:

(iv) $V(\hat{\theta}_i) \geqslant \frac{1}{C_i^2}$ and the minimum is attained

if $\underline{X}_i' \Sigma^{-1} \underline{X}_j$ = 0 for one position only.

= 1 for n-3 positions.

= 2 for one position only.

when j = 1, 2, ... i-1, i+1, ... n.

Therefore under this condition (iv) b_{ij} set of values for each i and each j when $i \neq j$ is a combination of 0, 1, and 2 such that $\sum_{i=1}^{n} b_{ij} = \sum_{j=1}^{n} b_{ij} = n-1$. 0 and 2 appearing only once in each row and each column. $b_{ij} = 0$ for $i = j$.

For example, when n = 7

Preceding Treatment

Following Treatment $\underline{\beta}$ =

	A	B	C	D	E	F	G
A	0	1	2	1	1	0	1
B	1	0	1	2	1	1	0
C	0	1	0	1	2	1	1
D	1	0	1	0	1	2	1
E	1	1	0	1	0	1	2
F	2	1	1	0	1	0	1
G	1	2	1	1	0	1	0

Higher order $\underline{\beta}$ can be constructed by extending the rows with digits in relation to the diagonal pattern.

12.7 Choice of $\underline{\beta}$.

To study the character of $\underline{\beta}'\underline{\beta}$ for the most efficient design we observe the following restrictions.

(i) Elements of $\underline{\beta}$ are made up by the number of times a treatment follows another treatment in a Latin-square.

(ii) Combination of row and column elements is the same for all rows and columns of $\underline{\beta}$.

(iii) Rows and columns are the permutations of numbers under restriction (i).

(iv) Sum of each row and column elements = n-1.

(v) An element in $\underline{\beta}$ varies from 0 to n-1.

(vi) Diagonal elements of $\underline{\beta}$ are zero as a treatment does not follow itself.

(vii) The most efficient design has $\underline{\beta}'\underline{\beta}$ minimum as proved under 12.3.3.

12.7.1 Let $\underline{\beta} = \left[b_{ij} \right]$ and $\underline{\beta}'\underline{\beta} = \left[c_{ij} \right] = \left[b_{ik}\ b_{kj} \right]$

For simplicity let $b_{1j} = z_j$ for j = 1, 2,...... n. $z_1 = 0$

$$c_{ii} = \sum b_{ij}\, b_{ji} \quad \text{and } c_{11} = z_1^2 + z_2^2 + \ldots\ldots + z_n^2 = \sum_{j=1}^{n} z_j^2$$

$$\text{Trace}(\underline{\beta}'\underline{\beta}) = \sum_{i=1}^{n} c_{ii}$$

If $z_i = n-1$ then $z_j = 0$ for j = 1, 2, ...i-1, i+1, ... n.

$$\text{Therefore } \sum_{j=1}^{n} z_j^2 = 0 + 0 + \ldots\ldots\ldots + (n-1)^2 + 0 + \ldots\ldots + 0.$$

$$= n(n-2) + 1 \qquad \text{------------ I}$$

The maximum value z_j can take is (n-1) according to restriction (v) above.

If $z_i = n-2$, $z_k = 1$ and $z_j = 0$ for $j \neq i$ and k

$$\text{Then } \sum_{j=1}^{n} z_j^2 = (n-2)^2 + 1 + 0 + \ldots\ldots\ldots + 0$$

$$= n(n-4) + 5 \qquad \text{------------II}$$

We find that II < I.

Again if $z_i = n - 3$, $z_k = 2$ and $z_j = 0$ for $j \neq i,k$.

Or $z_k = 1$, $z_l = 1$ and $z_j = 0$ otherwise.

$$\text{Then} \quad \sum_{j=1}^{n} z_j^2 = (n - 3)^2 + 2^2 + 0 + \dots\dots\dots\dots + 0$$

$$= n^2 - 6n + 13 \quad = n(n - 6) + 13 \quad \text{------}I'$$

$$\text{or} \quad = (n - 3)^2 + 1 + 1 + 0 + \dots\dots\dots + 0$$

$$= n(n - 6) + 11 \quad \text{--------}II'. \quad \text{Hence } II' < I'$$

Similarly if we go on splitting z_i into smaller and smaller components we get closer and closer to the smallest possible value of c_{11} or $\sum_{j=1}^{n} z_j^2$.

12.7.1 In general let:

(a) $z_i = (n - 1 - x)$, $z_k = x$, $z_j = 0$ for $j \neq k$

(b) or $z_i = (n-1-x)$, $z_k = (x-1)$, $z_l = 1$, $z_j = 0$ for $j \neq k,l$.

....

....

(c) or $z_i = (n-1-x), z_k = 2, z_l = 1, \dots\dots (x-2)$ terms of $z_j = 1$

(d) or $z_i = (n-1-x), z_k = 1 \dots\dots\dots x$ terms of $z_j = 1$

otherwise $z_j = 0$ for all the above cases.

$$\text{Therefore} \quad \sum_{j=1}^{n} z_j^2 = (n-1-x)^2 + x^2 + 0 + \dots\dots\dots\dots + 0.$$

(a′) $= n(n-2x-2) + (2x^2 + 2x + 1)$

(b′) or $= n(n-2x-2) + (2x^2 + 3)$

....

....

(c′) or $= n(n-2x-2) + (x^2 + 3x + 3)$

(d′) or $= n(n-2x-2) + (x^2 + 3x + 1)$.

Similarly we get other terms of c_{ii} for $i = 2,3,\dots\dots\dots n.$, following conditions (ii) and (iii).

(c) and (d) derived above are suitable for Latin-square designs when n is odd and even respectively.

Here x denotes the number of elements in a row or a column in $\underline{\beta}$ having value equal to 1 and $0 < x < (n-1)$.

12.7.1(d) is due to maximum value of x and 12.7.1(c) has second maximum value of x.

12.7.2 $\quad Tr(\underline{\beta}'\underline{\beta}) = \sum_{i=1}^{n} c_{ii}$

12.7.1(c') and (d') are respectively diagonal elements of $\underline{\beta}'\underline{\beta}$ for n odd and even.

$\underline{\beta}'\underline{\beta}$ is symmetric since it is a square matrix and $c_{ij} = c_{ji}$.

According to Scheffe (1961) and Rao (1973) when $\underline{A}$ and $\underline{B}$ are symmetric and $\underline{B}$ is positive definite and if $\underline{A} - \underline{B}$ is positive semidefinite or positive definite, then:

(i) Trace $\underline{A} \geqslant$ Trace $\underline{B}$

and (ii) $|\underline{A}| \geqslant |\underline{B}|$

According to Graybill(1961) a necessary and sufficient condition that the matrix $\underline{A}$ be positive definite, where

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \cdots\cdots\cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots\cdots\cdots a_{2n} \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots \\ a_{n1} & a_{n2} \cdots\cdots\cdots\cdots a_{nn} \end{bmatrix}$$

is that the following inequalities hold:

$$a_{11} > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \cdots\cdots\cdots\cdots \begin{vmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \cdots\cdots & \cdots\cdots\cdots \\ \cdots\cdots & \cdots\cdots\cdots \\ a_{n1} & a_{n2} \cdots\cdots a_{nn} \end{vmatrix} > 0$$

For positive semidefinite we change the above inequalities to $\geqslant$.

For the most efficient design, when n the number of treatments is even, the trace of $\underline{\beta}'\underline{\beta}$ is expressed as

12.7.3 $\mathrm{Tr}(\underline{\beta}'\underline{\beta}) = n^2(n - 2x - 2) + n(x^2 + 3x + 1)$

For n odd the most efficient design has

12.7.4 $\mathrm{Tr}(\underline{\beta}'\underline{\beta}) = n^2(n - 2x - 2) + n(x^2 + 3x + 1)$

By decreasing the value of x we increase $\mathrm{Tr}(\underline{\beta}'\underline{\beta})$ and hence increase determinant of $\underline{\beta}'\underline{\beta}$. The traces 12.7.3 and 12.7.4 are minimum for Latin-square designs of order n, subject to n being even and odd respectively.

According to (i) and (ii) under 12.7.2 minimum trace implies minimum determinant and the minimum $|\underline{\beta}'\underline{\beta}|$ indicates maximum $|\underline{X}'\underline{X}|$ or minimum $|(\underline{X}'\underline{X})^{-1}|$, a necessary condition for the most efficient design.

Therefore we conclude that for studying the residual effects in Latin-square designs, persisting for one period only we choose such a Latin-square which produces $\underline{\beta}$ so that 12.7.1(c) is satisfied when n is odd and 12.7.1(d) is satisfied when n is even. Numerical examples of such Beta structures have already been cited under 12.5.

It can be stated that $\underline{\beta}$ is non-singular and has rank n. $\underline{\beta}'\underline{\beta}$ is positive definite as every principal minor of it is positive and its rank is n.

Chapter 13

Construction of the Most Efficient Latin-square Design for the Estimation of Residual Effects.

13.1 Derivation of Designs.

Williams (1948) has given methods of deriving the balanced designs for even and odd number of treatments. For an even number of treatments there are single designs which according to Williams are balanced for the estimation of residual effects but for an odd number of treatments a single Latin-square design is not possible.

For even number of treatments the initial rows of balanced or most efficient Latin-squares can be derived from:

0 1 n-1 2 n-2 3 n-3 ....n/2

and the successive differences between rows can be given by

1 n-2 3 n-4 5 n-6 ....

13.1.1 For n=4 a balanced or in our language a most efficient Latin-square can be written as:

0	1	3	2	initial row.
1	2	0	3	initial row + 1
3	0	2	1	second row +(2=n-2)
2	3	1	0	third row +3

Other designs of this pattern can be obtained by simply adding an arbitrary digit to each of the design digits representing treatments. For example by adding 1 we get another most efficient design

1	2	0	3
2	3	1	0
0	1	3	2
3	0	2	1

and so on.

The columns can be arranged at random without altering the character of the design and this in fact is the same thing as adding an arbitrary number to the design digits.

Similarly when n=6 the initial row is:

0 1 5 2 4 3

or 0 5 2 1 4 3 by random arrangement.

and so on.

For most efficient designs, when n is even, it is unnecessary to devise any structure of the initial row. Any order, random or not random, of design digits will form the initial row. It is important to know the successive differences without which it is difficult to lay the pattern of the most efficient design.

13.1.2 n Even

Number of treatments(n)	Successive differences
4	1 2 3.
6	1 4 3 2 5.
8	1 6 3 4 5 2 7.
10	1 8 3 6 5 4 7 2 9.
12	1 10 3 8 5 6 7 4 9 2 11.

13.2 Designs for Odd Number of Treatments.

When we are dealing with an odd number of treatments the pattern of the most efficient design can be obtained by first writing the initial row either as given under 13.1 or in the natural order of digits or simply by putting the design digits in a random order, since the assumption is that the residual effects are placed along columns, and then adding the appropriate successive differences to the rows.

13.2.1 When n=5 the most efficient design can be written as:

(a)	0	1	4	2	3	initial row.			
	1	2	0	3	4	"	"	+ 1	Successive difference.
	3	4	2	0	1	2nd	"	+ 2	" "
	2	3	1	4	0	3rd	"	+ 4	" "
	4	0	3	1	2	4th	"	+ 2	" "

Or

(b)	0	1	2	3	4	initial row.			
	1	2	3	4	0	"	"	+ 1	Successive difference.
	3	4	0	1	2	2nd	"	+ 2	" "
	2	3	4	0	1	3rd	"	+ 4	" "
	4	0	1	2	3	4th	"	+ 2	" "

Both (a) and (b) are equally good provided we stick to the assumption about residual effects arising along columns. If by any chance it is required to estimate residual effects supposed to be arising along rows then the above two designs or any other design based on the addition of an arbitrary number to these design digits would be rendered as the most inefficient designs. Change of assumption regarding the existance of residual effects would change the pattern of the design based on the initial assumption.

13.2.2 For n=7 the Latin-square may be derived as:

									Successive difference.
0	1	6	2	5	3	4	initial row.		
1	2	0	3	6	4	5	"	"	+ 1
4	5	3	6	2	0	1	2nd	"	+ 3
2	3	1	4	0	5	6	3rd	"	+ 5
6	0	5	1	4	2	3	4th	"	+ 4
5	6	4	0	3	1	2	5th	"	+ 6
3	4	2	5	1	6	0	6th	"	+ 5

Other designs of this nature can be obtained either by adding an arbitrary number or using other series of successive differences given in the table 13.3.

13.2.3 9x9 Latin-square can also be shown as:

											S.D.
0	1	8	2	7	3	6	4	5	initial row.		
1	2	0	3	8	4	7	5	6	"	"	+ 1
5	6	4	7	3	8	2	0	1	2nd	"	+ 4
3	4	2	5	1	6	0	7	8	3rd	"	+ 7
8	0	7	1	6	2	5	3	4	4th	"	+ 5
7	8	6	0	5	1	4	2	3	5th	"	+ 8
4	5	3	6	2	7	1	8	0	6th	"	+ 6
6	7	5	8	4	0	3	1	2	7th	"	+ 2
2	3	1	4	0	5	8	6	7	8th	"	+ 5

Similarly designs of higher order of odd values of n can be derived by an initial row and series of successive differences.

13.3. Successive Differences.

n Odd

Number of Treatments (n)	Successive Differences for Successive Rows
5	(i) d, d, 2d, 4d.
	(ii) 4d, 2d, d, d.
	(iii) d, 2d, 4d, 2d.
	(iv) 2d, 4d, 2d, d.
7	(i) d, 2d, 3d, 6d, 4d, 2d.
	(ii) d, 2d, 3d, 5d, d, 4d.
	(iii) d, 3d, 5d, 4d, 6d, 5d.
	(iv) d, 3d, 2d, 4d, 6d, 3d.
	(v) d, 4d, 4d, 2d, 6d, 3d.
	(vi) d, 5d, 3d, 2d, 6d, 2d.
	(vii) d, 5d, 4d, 6d, 3d, 6d.
	(viii) d, 5d, 4d, 2d, 6d, 5d.
	(ix) 5d, 6d, 4d, 5d, 3d, d.
	(x) 3d, 6d, 4d, 2d, 3d, d.
	(xi) 3d, 6d, 2d, 4d, 4d, d.
	(xii) 2d, 6d, 2d, 3d, 5d, d.
	(xiii) 6d, 3d, 6d, 4d, 5d, d.
	(xiv) 5d, 6d, 2d, 4d, 5d, d.
	....
	....

Number of Treatments (n)	Successive Differences for Successive Rows
9	(i) d, 3d, 4d, 6d, 2d, 5d, 8d, 4d.
	(ii) 4d, 8d, 5d, 2d, 6d, 4d, 3d, d.
	(iii) d, 4d, 7d, 5d, 8d, 6d, 2d, 5d.
	(iv) 5d, 2d, 6d, 8d, 5d, 7d, 4d, d.
	(v) d, 4d, 6d, 2d, 3d, 5d, 5d, 7d.
	(vi) 7d, 5d, 5d, 3d, 2d, 6d, 4d, d.
	(vii) 2d, 3d, 5d, 6d, 8d, 2d, 4d, d.
	(viii) d, 4d, 2d, 8d, 6d, 5d, 3d, 2d.
	(ix) d, 2d, 3d, 5d, 2d, 4d, 8d, 7d.
	(x) 7d, 8d, 4d, 2d, 5d, 3d, 2d, d.
	....
11	(i) d, 2d, 4d, 6d, 8d, 10d, 7d, 3d, 9d, 9d..
	(ii) 9d, 9d, 3d, 7d, 10d, 8d, 6d, 4d, 2d, d.
	(iii) 2d, 4d, 6d, 8d, 10d, 7d, d, 9d, 4d, 3d.
	(iv) 3d, 4d, 9d, d, 7d, 10d, 8d, 6d, 4d, 2d.
	....

d can take any values from 1 to n-1. $1 \leq d \leq (n-1)$

Series of successive differences can be generated such that no more than two differences of the same magnitude should occur in the same series and when used for successive rows must produce a Latin-square.

13.4. Most Efficient Designs of Order (5 x 5).

Using the initial row (1, 2, 3, 4, 5) and the successive differences given under 13.3., for d carrying values from 1 to 4, we get the following set of the most efficient designs of order (5 x 5).

1.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
5	1	2	3	4
4	5	1	2	3

2.

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
4	5	1	2	3
2	3	4	5	1

3.

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
3	4	5	1	2
5	1	2	3	4

4.

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
2	3	4	5	1
3	4	5	1	2

5.

1	2	3	4	5
5	1	2	3	4
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3

6.

1	2	3	4	5
4	5	1	2	3
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1

7.

1	2	3	4	5
3	4	5	1	2
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4

8.

1	2	3	4	5
2	3	4	5	1
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2

9.

1	2	3	4	5
2	3	4	5	1
4	5	1	2	3
3	4	5	1	2
5	1	2	3	4

10.

1	2	3	4	5
3	4	5	1	2
2	3	4	5	1
5	1	2	3	4
4	5	1	2	3

11.

1	2	3	4	5
4	5	1	2	3
5	1	2	3	4
2	3	4	5	1
3	4	5	1	2

12.

1	2	3	4	5
5	1	2	3	4
3	4	5	1	2
4	5	1	2	3
2	3	4	5	1

13.

1	2	3	4	5
3	4	5	1	2
2	3	4	5	1
4	5	1	2	3
5	1	2	3	4

14.

1	2	3	4	5
5	1	2	3	4
3	4	5	1	2
2	3	4	5	1
4	5	1	2	3

15.

1	2	3	4	5
2	3	4	5	1
4	5	1	2	3
5	1	2	3	4
3	4	5	1	2

16.

1	2	3	4	5
4	5	1	2	3
5	1	2	3	4
3	4	5	1	2
2	3	4	5	1

By permuting the columns we get 120 new most efficient designs for each of the above 16 designs and hence a set of 1,920 most efficient designs of order (5 x 5) can be written down.

To serve as an example and for practical use a table of the most efficient Latin-squares, appearing under 13.5., has been constructed by simply taking five permutations of the columns for each of the 16 designs. This table also includes the most efficient Latin-square designs of order (7 x 7) derived by using the initial row (1, 2, 3, 4, 5, 6, 7) and successive differences 13.3.(7i) for d ranging from 1 to 6.

13.5. Latin-squares Table.

The (5 x 5) Latin-squares

1.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
5	1	2	3	4
4	5	1	2	3

2.

2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
1	2	3	4	5
5	1	2	3	4

3.

3	4	5	1	2
4	5	1	2	3
5	1	2	3	4
2	3	4	5	1
1	2	3	4	5

4.

4	5	1	2	3
5	1	2	3	4
1	2	3	4	5
3	4	5	1	2
2	3	4	5	1

5.

5	4	3	1	2
1	2	3	4	5
2	3	4	5	1
4	5	1	2	3
3	4	5	1	2

6.

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
4	5	1	2	3
2	3	4	5	1

7.

2	3	4	5	1
4	5	1	2	3
1	2	3	4	5
5	1	2	3	4
3	4	5	1	2

8.

3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
1	2	3	4	5
4	5	1	2	3

9.

4	5	1	2	3
1	2	3	4	5
3	4	5	1	2
2	3	4	5	1
5	1	4	2	3

10.

5	1	2	3	4
2	3	4	5	1
4	5	1	2	3
3	4	5	1	2
1	2	3	4	5

11.

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
3	4	5	1	2
5	1	2	3	4

12.

2	3	4	5	1
5	1	2	3	4
3	4	5	1	2
4	5	1	2	3
1	2	3	4	5

13.

3	4	5	1	2
1	2	3	4	5
4	5	1	2	3
5	1	2	3	4
2	3	4	5	1

14.

4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
1	2	3	4	5
3	4	5	1	2

15.

5	1	2	3	4
3	4	5	1	2
1	2	3	4	5
2	3	4	5	1
4	5	1	2	3

16.

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
2	3	4	5	1
3	4	5	1	2

17.

2	3	4	5	1
1	2	3	4	5
5	1	2	3	4
3	4	5	1	2
4	5	1	2	3

18.

3	4	5	1	2
2	3	4	5	1
1	2	3	4	5
4	5	1	2	3
5	1	2	3	4

19. 4 5 1 2 3
3 4 5 1 2
2 3 4 5 1
5 1 2 3 4
1 2 3 4 5

20. 5 1 2 3 4
4 5 1 2 3
3 4 5 1 2
1 2 3 4 5
2 3 4 5 1

21. 1 2 3 4 5
5 1 2 3 4
2 3 4 5 1
3 4 5 1 2
4 5 1 2 3

22. 2 3 4 5 1
1 2 3 4 5
3 4 5 1 2
4 5 1 2 3
5 1 2 3 4

23. 3 4 5 1 2
2 3 4 5 1
4 5 1 2 3
5 1 2 3 4
1 2 3 4 5

24. 4 5 1 2 3
3 4 5 1 2
5 1 2 3 4
1 2 3 4 5
2 3 4 5 1

25. 5 1 2 3 4
4 5 1 2 3
1 2 3 4 5
2 3 4 5 1
3 4 5 1 2

26. 1 2 3 4 5
4 5 1 2 3
3 4 5 1 2
5 1 2 3 4
2 3 4 5 1

27. 2 3 4 5 1
5 1 2 3 4
4 5 1 2 3
1 2 3 4 5
3 4 5 1 2

28. 3 4 5 1 2
1 2 3 4 5
5 1 2 3 4
2 3 4 5 1
4 5 1 2 3

29. 4 5 1 2 3
2 3 4 5 1
1 2 3 4 5
3 4 5 1 2
5 1 2 3 4

30. 5 1 2 3 4
3 4 5 1 2
2 3 4 5 1
4 5 1 2 3
1 2 3 4 5

31. 1 2 3 4 5
3 4 5 1 2
4 5 1 2 3
2 3 4 5 1
5 1 2 3 4

32. 2 3 4 5 1
4 5 1 2 3
5 1 2 3 4
3 4 5 1 2
1 2 3 4 5

33. 3 4 5 1 2
5 1 2 3 4
1 2 3 4 5
4 5 1 2 3
2 3 4 5 1

34. 4 5 1 2 3
1 2 3 4 5
2 3 4 5 1
5 1 2 3 4
3 4 5 1 2

35. 5 1 2 3 4
2 3 4 5 1
3 4 5 1 2
1 2 3 4 5
4 5 1 2 3

36. 1 2 3 4 5
2 3 4 5 1
5 1 2 3 4
4 5 1 2 3
3 4 5 1 2

37.

2	3	4	5	1
3	4	5	1	2
1	2	3	4	5
5	1	2	3	4
4	5	1	2	3

38.

3	4	5	1	2
4	5	1	2	3
2	3	4	5	1
1	2	3	4	5
5	1	2	3	4

39.

4	5	1	2	3
5	1	2	3	4
3	4	5	1	2
2	3	4	5	1
1	2	3	4	5

40.

5	1	2	3	4
1	2	3	4	5
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

41.

1	2	3	4	5
2	3	4	5	1
4	5	1	2	3
3	4	5	1	2
5	1	2	3	4

42.

2	3	4	5	1
3	4	5	1	2
5	1	2	3	4
4	5	1	2	3
1	2	3	4	5

43.

3	4	5	1	2
4	5	1	2	3
1	2	3	4	5
5	1	2	3	4
2	3	4	5	1

44.

4	5	1	2	3
5	1	2	3	4
2	3	4	5	1
1	2	3	4	5
3	4	5	1	2

45.

5	1	2	3	4
1	2	3	4	5
3	4	5	1	2
2	3	4	5	1
4	5	1	2	3

46.

1	2	3	4	5
3	4	5	1	2
2	3	4	5	1
5	1	2	3	4
4	5	1	2	3

47.

2	3	4	5	1
4	5	1	2	3
3	4	5	1	2
1	2	3	4	5
5	1	2	3	4

48.

3	4	5	1	2
5	1	2	3	4
4	5	1	2	3
2	3	4	5	1
1	2	3	4	5

49.

4	5	1	2	3
1	2	3	4	5
5	1	2	3	4
3	4	5	1	2
2	3	4	5	1

50.

5	1	2	3	4
2	3	4	5	1
1	2	3	4	5
4	5	1	2	3
3	4	5	1	2

51.

1	2	3	4	5
4	5	1	2	3
5	1	2	3	4
2	3	4	5	1
3	4	5	1	2

52.

2	3	4	5	1
5	1	2	3	4
1	2	3	4	5
3	4	5	1	2
4	5	1	2	3

53.

3	4	5	1	2
1	2	3	4	5
2	3	4	5	1
4	5	1	2	3
5	1	2	3	4

54.

4	5	1	2	3
2	3	4	5	1
3	4	5	1	2
5	1	2	3	4
1	2	3	4	5

55.	5	1	2	3	4	56.	1	2	3	4	5	57.	2	3	4	5	1
	3	4	5	1	2		5	1	2	3	4		1	2	3	4	5
	4	5	1	2	3		3	4	5	1	2		4	5	1	2	3
	1	2	3	4	5		4	5	1	2	3		5	1	2	3	4
	2	3	4	5	1		2	3	4	5	1		3	4	5	1	2

58.	3	4	5	1	2	59.	4	5	1	2	3	60.	5	1	2	3	4
	2	3	4	5	1		3	4	5	1	2		4	5	1	2	3
	5	1	2	3	4		1	2	3	4	5		2	3	4	5	1
	1	2	3	4	5		2	3	4	5	1		3	4	5	1	2
	4	5	1	2	3		5	1	2	3	4		1	2	3	4	5

61.	1	2	3	4	5	62.	2	3	4	5	1	63.	3	4	5	1	2
	3	4	5	1	2		4	5	1	2	3		5	1	2	3	4
	2	3	4	5	1		3	4	5	1	2		3	4	5	1	2
	4	5	1	2	3		5	1	2	3	4		1	2	3	4	5
	5	1	2	3	4		1	2	3	4	5		2	3	4	5	1

64.	4	5	1	2	3	65.	5	1	2	3	4	66.	1	2	3	4	5
	1	2	3	4	5		2	3	4	5	1		5	1	2	3	4
	4	5	1	2	3		5	1	2	3	4		3	4	5	1	2
	2	3	4	5	1		3	4	5	1	2		2	3	4	5	1
	3	4	5	1	2		4	5	1	2	3		4	5	1	2	3

67.	2	3	4	5	1	68.	3	4	5	1	2	69.	4	5	1	2	3
	1	2	3	4	5		2	3	4	5	1		3	4	5	1	2
	4	5	1	2	3		5	1	2	3	4		1	2	3	4	5
	3	4	5	1	2		4	5	1	2	3		5	1	2	3	4
	5	1	2	3	4		1	2	3	4	5		2	3	4	5	1

70.	5	1	2	3	4	71.	1	2	3	4	5	72.	2	3	4	5	1
	4	5	1	2	3		2	3	4	5	1		3	4	5	1	2
	2	3	4	5	1		4	5	1	2	3		5	1	2	3	4
	1	2	3	4	5		5	1	2	3	4		1	2	3	4	5
	3	4	5	1	2		3	4	5	1	2		4	5	1	2	3

73.
3 4 5 1 2
4 5 1 2 3
1 2 3 4 5
2 3 4 5 1
5 1 2 3 4

74.
4 5 1 2 3
5 1 2 3 4
2 3 4 5 1
3 4 5 1 2
1 2 3 4 5

75.
5 1 2 3 4
1 2 3 4 5
3 4 5 1 2
4 5 1 2 3
2 3 4 5 1

76.
1 2 3 4 5
4 5 1 2 3
5 1 2 3 4
3 4 5 1 2
2 3 4 5 1

77.
2 3 4 5 1
5 1 2 3 4
1 2 3 4 5
4 5 1 2 3
3 4 5 1 2

78.
3 4 5 1 2
1 2 3 4 5
2 3 4 5 1
5 1 2 3 4
4 5 1 2 3

79.
4 5 1 2 3
2 3 4 5 1
3 4 5 1 2
1 2 3 4 5
5 1 2 3 4

80.
5 1 2 3 4
3 4 5 1 2
4 5 1 2 3
2 3 4 5 1
1 2 3 4 5

The (7 x 7) Latin-squares

1.
1 2 3 4 5 6 7
2 3 4 5 6 7 1
4 5 6 7 1 2 3
7 1 2 3 4 5 6
6 7 1 2 3 4 5
3 4 5 6 7 1 2
5 6 7 1 2 3 4

2.
1 2 3 4 5 6 7
3 4 5 6 7 1 2
7 1 2 3 4 5 6
6 7 1 2 3 4 5
4 5 6 7 1 2 3
5 6 7 1 2 3 4
2 3 4 5 6 7 1

3.
1 2 3 4 5 6 7
4 5 6 7 1 2 3
3 4 5 6 7 1 2
5 6 7 1 2 3 4
2 3 4 5 6 7 1
7 1 2 3 4 5 6
6 7 1 2 3 4 5

4.
1 2 3 4 5 6 7
5 6 7 1 2 3 4
6 7 1 2 3 4 5
4 5 6 7 1 2 3
7 1 2 3 4 5 6
2 3 4 5 6 7 1
3 4 5 6 7 1 2

5.
1 2 3 4 5 6 7
6 7 1 2 3 4 5
2 3 4 5 6 7 1
3 4 5 6 7 1 2
5 6 7 1 2 3 4
4 5 6 7 1 2 3
7 1 2 3 4 5 6

6.
1 2 3 4 5 6 7
7 1 2 3 4 5 6
5 6 7 1 2 3 4
2 3 4 5 6 7 1
3 4 5 6 7 1 2
6 7 1 2 3 4 5
4 5 6 7 1 2 3

Balanced designs for an odd number of treatments given by Williams (1948), Extra-period change-over designs produced by Patterson and Lucas (1959), Tied-double-change-over designs suggested by Federer and Atkinson (1964) and Symmetrical balanced designs developed by Berenblut (1968) are less likely to be required in practice because too many periods and subjects are required to apply the treatments. The construction and analysis of these designs is more complex than single Latin-squares. Williams (1948) and Ferris (1957) have shown that one Latin-square is sufficient to estimate treatment direct and treatment residual effects when n the number of treatments is even and the pattern of these designs has already been elaborated under 13.1.1.

The construction of Latin-square designs for estimating residual and direct effects, when n the number of treatments is odd and can be used as single designs, has been demonstrated under 11.2. The rows are to be used for successive periods of application of treatments, while the columns are to represent the subjects under treatments.

The Latin-square designs suggested under 13.2 are suitable for residual effects coming from immediately preceding treatments only.

Chapter 14

The Analysis of Variance.

14.1 Analysis of variance table for single Latin-squares where the treatments are assumed to produce residual effects.

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Periods	n-1	$\sum \hat{p}_i P_i$	$\frac{\sum \hat{p}_i P_i}{n-1}$	-
Subjects (Residuals ignored)	n-1	$\frac{1}{n}\sum S_j^2$-C.F.	$\frac{\frac{1}{n}\sum S_j^2\text{-C.F.}}{n-1}$	-
Treatment direct effects (Unadjusted)	n-1	$\frac{1}{n}\sum T_k^2$-C.F.	$\frac{\frac{1}{n}\sum T_k^2\text{-C.F.}}{n-1}$	-
Treatment residual effects (Adjusted)	n-1	*****	$\frac{*****}{n-1}$	-
Error	(n-1)(n-3)	By Difference	EMS	
Total	n^2-1	TSS		

$TSS = Y_{ij}^2 - \frac{G^2}{n^2}$ $\quad Y_{ij}$ is the value of the observation in the ith row and jth column.

G is the grand total of observed values.

P, S, T, and R are the totals for each period, subject, treatment and residual respectively.

$\hat{p}$, $\hat{s}$, $\hat{t}$, and $\hat{r}$ are individual effects of periods, subjects, treatments and residuals respectively.

F = (M.S.) ÷ (EMS)

Residual effects are entangled with the treatment direct and subject effects, therefore it is necessary to adjust their sum of squares to add up crrectly to the total sum of squares.

***** Residual Sum of Squares (adjusted for subjects and treatments) $= \Sigma \hat{r}_i R_i + \Sigma \hat{s}_j S_j - SSS + \Sigma \hat{t}_k T_k - TSS$

where SSS is the subject sum of squares and TSS is the treatment sum of squares.

i,j,k,l=1,2..........n.

As a check:

Direct effects(unadjusted) + Residual effects(adjusted)

= Direct effects(adjusted) + Residual effects(unadjusted).

Similarly for subjects and residuals.

$EMS = TSS - \Sigma \hat{p}_i P_i - \Sigma \hat{s}_j S_j - \Sigma \hat{t}_k T_k - \Sigma \hat{r}_l R_l$

Chapter 15

Conclusions as to the Study of Single Latin-square Designs in Change-over Experiments.

15.1 Conclusion.

Though this study concentrates on 4x4 and 5x5 Latin-square designs with residual effects existing for one period only, the methods developed for the estimation of parameter effects can be extended to Latin-squares of higher order with residual effects persisting for any number of periods.

The investigation of the Latin-square designs reveals that the most efficient design for an even number of treatments has a form where each treatment is preceded by a different treatment on different occasions. $\underline{\beta}$ for such designs is a matrix with diagonal elements equal to zero and off-diagonal elements equal to 1.

For an odd number of treatments the pattern of the most efficient design takes a form where each treatment is preceded twice by one treatment, not at all by another treatment and once by the remaining n-3 treatments (n being the number of treatments). Due to the basic property of the Latin-squares no treatment is followed by itself. $\underline{\beta}$ for these designs is a matrix with the diagonal elements equal to zero and off-diagonal elements equal to 0 in one position, 1 in n-3 positions and 2 in one position in each row and each column. The dimension of $\underline{\beta}$ is n x n.

The optimum or the most efficient design satisfies most of the following conditions simultaneously.

(i) Minimum average variance of the differences between two parameter estimates.

(ii) Minimum variance of the linear components of the parameter estimates.

(iii) Minimum generalised variance $\left| (\underline{X}' \underline{X})^{-1} \sigma^2 \right|$.

(iv) Maximum determinant of $\underline{X}' \underline{X}$.

The Latin-square designs where each treatment is preceded by the same treatment on all occasions, known as cyclic designs, are the most inefficient designs for measuring both the residual and the treatment effects for all purposes.

The efficiency ratio of measuring the treatment effects over residual effects existing for one period only has been found to be equal to $\frac{n^2}{n^2-n-1}$, which is dependent on n, the size of the design. The ratio is equal to 1 when the existence of the residual effects in period one is also included in the basic assumption about residual effects in an experiment.

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```
      MASTER JUDE
C
      DIMENSION A(576),W(576),B(576),C(576)
C
C
C------------MATRICES OF VARIOUS DIMENSIONS ARE TO BE INVERTED
C
C---------------THIS IS DONE USING AN ICL SCIENTIFIC SUBROUTINE, NAMELY,
C----------------------FPMGEIN
C
C------------THE ORIGINAL MATRIX IS TO BE PRINTED AS IS THE INVERTED
C----------------MATRIX
C
C
C----------READ IN THE NUMBER OF MATRICES TO BE INVERTED AND WRITE THE
C-------------ANSWER
      READ(1,100)NUM
100   FORMAT(10)
      WRITE(2,200)NUM
200   FORMAT(1H0,38HTHE NUMBER OF MATRICES TO BE INVERTED=,I2)
C
C
C
C----------FOR EACH MATRIX READ IN AND WRITE THE MATRIX NUMBER AS A
C--------------HEADING
C----------ALSO READ IN AND WRITE THE DIMENSION OF EACH MATRIX
C----------ALSO READ AND PRINT THE MATRICES BEFORE INVERSION
      DO 30 L=1,NUM
      WRITE (2,300)L
300   FORMAT(1H1,/,50X,7HMATRIX ,I2,/,50X,9H=========)
C
```


continue

```
0042              READ(1,400)N,K
0043        400   FORMAT(2I0)
0044        C
0045              WRITE(2,500)N,K
0046        500   FORMAT(1H0,31HTHE MATRIX TO BE INVERTED IS A ,1H(,I2,1H,,I2,1H))
0047        C
0048        C
0049              JJ=N*K
0050              WRITE(2,700)
0051        700   FORMAT(1H0,/,10X,23HMATRIX BEFORE INVERSION,/,10X,25H-------------
0052             1------------,)
WARNING 193 REDUNDANT COMMA IGNORED AT ABOUT COLUMN 18, LINE 0052

0053              DO 20 J=1,N
0054              READ(1,600)(A(I),I=J,JJ,N)
0055        600   FORMAT(1000F0.0)
COMMENT 178 IS THIS LARGE A REPEAT COUNT INTENDED AT ABOUT COLUMN 16, LINE 0055

0056        20    WRITE(2,800)(A(I),I=J,JJ,N)
0057        800   FORMAT(1H ,24F5.1)
0058              WRITE(2,900)
0059        900   FORMAT(1H0)
0060        C
0061        C
0062        C---------WRITE OUT THE HEADING FOR THE INVERTED MATRIX
0063        C
0064        C
0065              WRITE(2,1000)
0066        1000  FORMAT(1H0,/,10X,15HINVERTED MATRIX,/,10X,17H-----------------)
0067        C
0068        C
0069        C----        SET N,E AND A BEFORE CALLING ROUTINE
0070        C                AND CALL SUBROUTINE FPMGEIN
```


continue

```
C
      E=1E-5
      WRITE(2,800)
C
      CALL FPMGEIN(N,E,A(1),W(1),DET,IRANK,NRR)
C
C---------CHECK FOR SINGULAR MATRIX
      IF(NRR.EQ.128)WRITE(2,1100)
1100  FORMAT(1H ,55H*.N.B. THIS MATRIX IS SINGULAR,IT CAN NOT BE INVERTE
     2D *)
      IF(NRR.NE.128)GO TO 1
      GO TO 30
C
C
C---------WRITE OUT INVERTED MATRIX
C
1     DO 40J=1,N
40    WRITE(2,1200)(A(I),I=J,JJ,N)
1200  FORMAT(1H0,12F9.5/1X,12F9.5)
30    CONTINUE
      STOP
      END

END OF SEGMENT, LENGTH   218, NAME  JUDE        - WARNINGS
```

```
      MASTER JUDE
       DIMENSION A(576),W(576),B(576),REINT(576)
C
C
C
C-----------MATRICES OF VARIOUS DIMENSIONS ARE TO BE INVERTED
C
C--------------THIS IS DONE USING AN ICL SCIENTIFIC SUBROUTINE, NAMELY,
C-------------------FPMGEIN
C
C-----------THE ORIGINAL MATRIX IS TO BE PRINTED AS IS THE INVERTED
C--------------MATRIX
C
C
C---------READ IN THE NUMBER OF MATRICES TO BE INVERTED AND WRITE THE
C------------ANSWER
      READ(1,100)NUM
100   FORMAT(I0)
      WRITE(2,200)NUM
200   FORMAT(1H0,38HTHE NUMBER OF MATRICES TO BE INVERTED=,I2)
C
C
C
C---------FOR EACH MATRIX READ IN AND WRITE THE MATRIX NUMBER AS A
C------------HEADING
C---------ALSO READ IN AND WRITE THE DIMENSION OF EACH MATRIX
C---------ALSO READ AND PRINT THE MATRICES BEFORE INVERSION
      DO 30 L=1,NUM
      WRITE (2,300)L
300     FORMAT(1H1,/,50X,7HMATRIX ,I2,/,50X,9H=========)
```


continue

```
0041        C
0042              READ(1,400)N,K
0043        400   FORMAT(2I0)
0044        C
0045              WRITE(2,500)N,K
0046        500   FORMAT(1H0,31HTHE MATRIX TO BE INVERTED IS A ,1H(,I2,1H,,I2,1H))
0047        C
0048        C
0049              JJ=N*K
0050              WRITE(2,700)
0051        700   FORMAT(1H0,/,10X,23HMATRIX BEFORE INVERSION,/,10X,25H------------
0052             1------------,)
0053        REDUNDANT COMMA IGNORED AT ABOUT COLUMN 18, LINE 0052
WARNING 193 REDUNDANT COMMA IGNORED AT ABOUT COLUMN 18, LINE 0052

WARNING  24 SUPERFLUOUS CHARACTER(S) IGNORED AT ABOUT COLUMN 19, LINE 0052

0054              DO 20 J=1,N
0055                READ(1,600)(A(I),I=J,JJ,N)
0056        600   FORMAT(1000F0.0)
0057        IS THIS LARGE A REPEAT COUNT INTENDED AT ABOUT COLUMN 16, LINE 0055
COMMENT 178 IS THIS LARGE A REPEAT COUNT INTENDED AT ABOUT COLUMN 16, LINE 0056

WARNING  24 SUPERFLUOUS CHARACTER(S) IGNORED AT ABOUT COLUMN 21, LINE 0056

0058        20    WRITE(2,800)(A(I),I=J,JJ,N)
0059        800   FORMAT(1H ,24F5.1)
0060              WRITE(2,900)
0061        900   FORMAT(1H0)
0062        C
0063              DO 50 I =1,JJ
0064              B(I) = A(I)
0065        50    CONTINUE
```


continue

```
      CALL F4DET(B,N,JJ,D,ID,REINT,IT)
      IF(IT.EQ.0)GO TO 2
      WRITE(2,1300)
1300  FORMAT(1H ,91H* N.B. THIS MATRIX IS SINGULAR,ITS DETERMINANT CAN N
     1OT BE CALCULATED NOR CAN IT BE INVERTED)
      GO TO 30
2     DETB = D * 2.0 ** ID
      WRITE(2,1400)DETB
1400  FORMAT(1H ,37HDETERMINANT OF ORIGINAL MATRIX IS - :,F11.5,/,38(1H=
     1))
C
C---------WRITE OUT THE HEADING FOR THE INVERTED MATRIX
C
C
      WRITE(2,1000)
1000  FORMAT(1H0,/,10X,15HINVERTED MATRIX,/,10X,17H-----------------)
C
C
C----      SET N,E AND A BEFORE CALLING ROUTINE
C                 AND CALL SUBROUTINE FPMGEIN
C
      E=1E-5
      WRITE(2,800)
C
      CALL FPMGEIN(N,E,A(1),W(1),DET,IRANK,NRR)
C
C---------CHECK FOR SINGULAR MATRIX
      IF(NRR.EQ.128)WRITE(2,1100)
1100  FORMAT(1H ,55H*.N.B. THIS MATRIX IS SINGULAR.IT CAN NOT BE INVERTE
     2D *)
```


continue

```
            IF(NRR.NE.128)GO TO 1
            GO TO 30
C
C
C---------WRITE OUT INVERTED MATRIX
C
1           DO 40J=1,N
40          WRITE(2,1200)(A(I),I=J,JJ,N)
1200        FORMAT(1H0,12F9.5/1X,12F9.5)
            CALL F4DET(A,N,JJ,D,ID,REINT,IT)
            IF(IT.EQ.0)GO TO 3
            WRITE(2,1500)
1500        FORMAT(1H ,68H* N.B. THIS MATRIX IS SINGULAR.ITS DETERMINANT CAN N
           1OT BE CALCULATED)
            GO TO 30
3           DETB = D * 2.0 ** ID
            WRITE(2,1600)DETB
1600        FORMAT(1H0,37HDETERMINANT OF INVERTED MATRIX IS - :,F11.5,/,38(1H=
           1))
30          CONTINUE
            STOP
            END

END OF SEGMENT, LENGTH   332, NAME  JUDE        - WARNINGS
```