On Ext^2 between Weyl modules for quantum GL_n

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In the study of highest weight categories, the class of Weyl modules $\Delta(\lambda)$ and their duals $\nabla(\lambda)$ are of central interest; this is for example motivated by the problem of finding the characters of the simple modules. Weyl modules form the building blocks for the category $\mathcal{F}(\Delta)$, whose objects have a filtration $0 = M_0 \leq M_1 \leq \cdots \leq M_{i-1} \leq M_i = M$ with quotients isomorphic to $\Delta(\lambda)$ for various λ. Knowing $\mathrm{Ext}^r(\Delta(\lambda), \Delta(\mu))$ is essential for the understanding of this category.

In [3, 13], we determined Ext^1 for Weyl modules of $\mathrm{SL}(2, k)$ and q-$\mathrm{GL}(2, k)$ over an infinite field k of characteristic $p > 0$. Here we are able to extend these results to determine Ext^2 for Weyl modules in both these cases (see (4.6)). Moreover, this also gives Ext^2 between any pair of Weyl modules $\Delta(\lambda)$, $\Delta(\mu)$ for q-$\mathrm{GL}(n, k)$ (where $n \geq 2$) such that both λ and μ have at most two rows or two columns, or where they differ by some multiple of a simple root (see Section 7).

Consider (for simplicity) polynomial representations of degree d for $\mathrm{GL}(n, k)$. A partition of d which has at most two rows is uniquely determined by the difference in the row lengths, which we use as a label for the partition. In this case our main result is

Theorem *Suppose λ and μ are partitions of d with at most two rows. We label these by the difference in row lengths as above. Let $\lambda = pn + i < \mu = pm + r$ lie in the same block with $0 \leq i \leq p-2$ and $j = p - 2 - i$.*

(i) If $r = i$ and $m - n$ is even then

$$\mathrm{Ext}^2(\Delta(\lambda), \Delta(\mu)) \cong \begin{cases} k & \text{if } m - n = 2 \\ \mathrm{Ext}^1(\Delta(0), \Delta(m-n-2)) & \text{if } m - n > 2. \end{cases}$$

(ii) If $r = j$ and $m - n$ is odd then

$$\mathrm{Ext}^2(\Delta(\lambda), \Delta(\mu)) \cong \begin{cases} 0 & \text{if } m - n = 1 \\ k & \text{if } m - n = 3 \\ \mathrm{Ext}^2(\Delta(n), \Delta(m-1)) & \text{if } m - n \neq 2p^a + 1 \text{ and } m - n > 3 \\ k^2 & \text{if } m - n = 2p^a + 1 > 3 \text{ and } n \not\equiv -1 \pmod{p^a} \\ k & \text{if } m - n = 2p^a + 1 > 3 \text{ and } n \equiv -1 \pmod{p^a}. \end{cases}$$

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(iii) $\operatorname{Ext}^2(\Delta(pn+p-1), \Delta(pm+p-1)) \cong \operatorname{Ext}^2(\Delta(n), \Delta(m))$.

To prove this, we exploit the representation theory of the first Frobenius kernel, via the Lyndon–Hochschild–Serre spectral sequence. Most of the proof consists of a refinement of the methods in [3, 13]. However, the case $p = 2$ cannot be solved in this way, and for this we use a new filtration of certain tilting modules.

It is a basic general question to characterise Δ-finite quasi-hereditary algebras; that is those for which $\mathcal{F}(\Delta)$ has only finitely many indecomposable objects. The main motivation for this paper is to answer this question for the class of (q-)Schur algebras. By [3, 13] the Ext^1-spaces are only one-dimensional and hence do not give information in this direction. However, in certain cases (as indicated in the theorem above) the Ext^2-spaces are two-dimensional — and we show that this implies the existence of infinitely many non-isomorphic indecomposable modules in $\mathcal{F}(\Delta)$. Combining this with our main result, we obtain the following sufficient condition for $S_q(n, d)$ to be Δ-infinite. (For simplicity we shall consider the classical Schur algebra in odd characteristic; the general case is given in Section 6.)

Corollary *Suppose that $p > 2$. If $d \geq 2p^2 + p - 2$, then for all $n \geq 2$ the algebra $S(n, d)$ is Δ-infinite.*

Let $\mathcal{H} = \mathcal{H}_q(d)$ be the Hecke algebra corresponding to Σ_d over k. For each partition λ of d there is a Specht module S^λ of $\mathcal{H}$. Denote by $\mathcal{F}(\mathrm{Sp}_{\leq n})$ the full subcategory of $\operatorname{Mod}(\mathcal{H})$ consisting of those modules filtered by Specht modules S^λ for partitions λ of at most n parts. As a consequence of the last corollary, we obtain a sufficient condition for $\mathcal{F}(\mathrm{Sp}_{\leq n})$ to be of infinite type (see (7.7)).

1 Preliminaries

In this section we briefly survey the main results and conventions that will be needed later. We consider the quantum general linear group q-$\mathrm{GL}(2, k)$ defined by Dipper and Donkin [5], over an infinite field k of characteristic $p > 0$. If q is not a root of unity, then by [10, 4(8)] the corresponding module category is semisimple, so we will always assume that q is either 1 or a primitive lth root of unity. In either case, we shall denote our group by G.

We largely follow the notation and conventions of [3]. This will be our basic reference in the quantum setting; the corresponding classical results can be found in [13]. In both cases there is a Borel subgroup B in G, and we can define for each dominant weight λ the corresponding induced module $\nabla(\lambda)$. Then the Weyl modules $\Delta(\lambda)$ are defined as the duals of appropriate induced modules as in the classical case (see [10, Section 4]). Since this duality fixes the simple modules, we have $\mathrm{Ext}^s(\Delta(\lambda), \Delta(\mu)) \cong \mathrm{Ext}^s(\nabla(\mu), \nabla(\lambda))$ for all $s \geq 0$. We shall also need to consider the tilting module $T(\lambda)$, the unique indecomposable module of highest weight λ in $\mathcal{F}(\Delta)$ whose dual is also in $\mathcal{F}(\Delta)$ (see [7, 8]).

There is a Frobenius morphism $F : G \longrightarrow \mathrm{GL}(2, k)$ and a corresponding Frobenius kernel G_1. The definition of these, along with some of the basic representation theory of G_1, can be found in [11, Chapter 3]. There is a related factor group $\bar{G} \cong \mathrm{GL}(2, k)$ which defines G_1 (see [10, Remark after Corollary 1.4]).

Our main tool will be the Lyndon–Hochschild–Serre spectral sequence, which will allow us to exploit the representation theory of G_1. More precisely, if V is a G-module, then we have a spectral sequence with E_2-page given by $E_2^{rs} = H^r(\bar{G}, H^s(G_1, V))$, and converging to $H^*(G, V)$ (see [10, Proposition 1.6]).

We next recall some notation from [3]. Let $\lambda = (\lambda_1, \lambda_2) = (\mu + \delta, \delta)$ with $0 \leq \mu \leq l - 2$, and note that henceforth we shall reserve the symbol λ for elements of this form. Define $\bar{\mu}$ by $\mu + \bar{\mu} = l - 2$. Then we set

$$\tilde{\lambda} = (\lambda_2 - 1, \lambda_1 + 1 - l) = \bar{\mu}\rho + (\mu - l + 1 + \delta)\varpi,$$

where $\varpi = (1, 1)$ and $\rho = (1, 0)$. Note that $\tilde{\tilde{\lambda}} = \lambda - l\varpi$. Our results will concern Hom and Ext between $\Delta(\lambda + ln\rho + t\varpi)$ and either $\Delta(\tilde{\lambda} + lm\rho)$ or $\Delta(\lambda + lm\rho)$. The integer t will always be chosen so that the degrees of each module agree; in the former case we shall assume that $2t = l(m - n - 1)$, and in the latter that $2t = l(m - n)$, and we shall set $t = lu$.

Consideration of these cases will be sufficient by the description of the blocks of G in [3, Theorem 2.1], along with the fact that for all $r \geq 0$, integers a and dominant weights θ and χ we have

$$\mathrm{Ext}^r_G(\Delta(\theta + a\varpi), \Delta(\chi + a\varpi)) \cong \mathrm{Ext}^r_G(\Delta(\theta), \Delta(\chi)).$$

This follows from the isomorphism $\Delta(\theta + a\varpi) \cong \Delta(\theta) \otimes q\text{-}\mathrm{det}^a$, where q-det is the quantum analogue of the determinant module (see [5, 2.1.8]). As noted in [3] we may also assume that $\theta < \chi$, as otherwise $\mathrm{Ext}^i(\Delta(\theta), \Delta(\chi)) = 0$ for $i \geq 0$. This is clear for $i = 0$, and follows for $i > 0$ as in the classical case [1, 3.2 Corollary].

Finally, we should note certain conventions that we shall follow during the course of this paper. So as to provide a uniform proof in both the classical and quantum cases, we shall adopt the convention that when $q = 1$, we shall work in the appropriate classical setting. Thus in this case G_1 will just be the usual Frobenius kernel, F the usual Frobenius morphism etc. Consequently we shall set $l = p$ when considering the classical versions of our results. The one remaining complication involves proof by induction. In the classical setting this will be as usual, but for the quantum version we shall often assume that the corresponding classical result is already known instead.

2 Two short exact sequences

In this section we shall consider two short exact sequences of G-modules from [3], quantum analogues of corresponding sequences due to Xanthopoulos [20]. These were the main tool used in [3, 13] and shall also play a central role in what follows. As our first application of these we conclude this section by determining the Hom-spaces between pairs of Weyl modules.

Recall from [3, Proposition 3.4] that for $n \geq 0$ there exists an exact sequence of G-modules

$$0 \to \Delta(\tilde{\lambda} + l(n+1)\rho) \to \Delta(n\rho)^F \otimes Q(\lambda) \to \Delta(\lambda + ln\rho) \to 0, \tag{1}$$

where $Q(\lambda) = T(\tilde{\lambda} + l\rho)$, the indecomposable tilting module of highest weight $\tilde{\lambda} + l\rho$. As noted in the proof of [3, Corollary 3.5], this is also the injective hull of $L(\lambda)$ as a G_1-module. If $n > 0$, we also have [3, Proposition 3.3] the exact sequence

$$0 \to \Delta((n-1)\rho)^F \otimes \Delta(\tilde{\lambda} + l\varpi) \to \Delta(\lambda + ln\rho) \to \Delta(n\rho)^F \otimes \Delta(\lambda) \to 0. \tag{2}$$

These two sequences will allow us to proceed in many cases by induction on the size of weights. When using the second sequence, the following result will be useful.

Proposition 2.1 *For $n \geq 0$, $k \geq 1$ and $\tau \in \{\lambda, \tilde{\lambda}\}$ we have*

$$\mathrm{Ext}^k_G(\Delta(\lambda + ln\rho + t\varpi), \Delta(m\rho)^F \otimes Q(\tau))$$

$$\cong \begin{cases} \mathrm{Ext}^k_{\mathrm{GL}_2}(\Delta(n\rho + v\varpi), \Delta(m\rho)) & \begin{cases} \text{if } m-n \text{ even and } \tau = \lambda \\ \text{if } m-n \text{ even, } \tau = \tilde{\lambda} \text{ and } \mu = \bar{\mu} \end{cases} \\ \mathrm{Ext}^k_{\mathrm{GL}_2}(\Delta((n-1)\rho + w\varpi), \Delta(m\rho)) & \begin{cases} \text{if } m-n \text{ odd, } \tau = \tilde{\lambda} \text{ and } n > 0 \\ \text{if } m-n \text{ odd, } \tau = \lambda, \mu = \bar{\mu} \text{ and } n > 0 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

where $v = \frac{1}{2}(m-n)$ and $w = \frac{1}{2}(m-n+1)$.

Proof: Let $V = \Delta(m\rho)^F \otimes Q(\tau)) \otimes \Delta^*(\lambda + ln\rho + t\varpi)$. We have

$$H^k(G_1, V) = \Delta(m\rho)^F \otimes \mathrm{Ext}^k_{G_1}(\Delta(\lambda + ln\rho + t\varpi), Q(\tau)).$$

As $Q(\tau)$ is injective as a G_1-module, this implies that $H^k(G_1, V) = 0$ for $k \geq 1$. Now from the Lyndon–Hochschild–Serre spectral sequence we obtain an exact sequence

$$0 \to H^k(\bar{G}, V^{G_1}) \to H^k(G, V) \to H^k(G_1, V) = 0$$

and hence $H^k(\bar{G}, V^{G_1}) \cong H^k(G, V)$. Next we calculate V^{G_1}. Clearly $V^{G_1} = \Delta(m\rho)^F \otimes W^{G_1}$, where $W = Q(\tau) \otimes \Delta^*(\lambda + ln\rho + t\varpi)$. First suppose that $n > 0$. Then we have a short exact sequence

$$0 \to \Delta((n-1)\rho)^F \otimes \Delta(\tilde{\lambda} + (l+t)\varpi) \to \Delta(\lambda + ln\rho + t\varpi) \to \Delta(n\rho)^F \otimes \Delta(\lambda + t\varpi) \to 0.$$

Applying $\mathrm{Hom}_{G_1}(-, Q(\tau))$ to this we obtain the exact sequence

$$0 \to \mathrm{Hom}_{G_1}(\Delta(\lambda + t\varpi), Q(\tau)) \otimes \Delta^*(n\rho)^F \to W^{G_1}$$

$$\to \mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda} + (l+t)\varpi), Q(\tau)) \otimes \Delta^*((n-1)\rho)^F \to 0.$$

Arguing as in the proof of [3, Lemmas 4.2 and 4.3], we see that

$$W^{G_1} \cong \begin{cases} \Delta^*(n\rho + v\varpi)^F & \begin{cases} \text{if } m-n \text{ even and } \tau = \lambda \\ \text{if } m-n \text{ even, } \tau = \tilde{\lambda} \text{ and } \mu = \bar{\mu} \end{cases} \\ \Delta^*((n-1)\rho + w\varpi)^F & \begin{cases} \text{if } m-n \text{ odd and } \tau = \tilde{\lambda} \\ \text{if } m-n \text{ odd, } \tau = \lambda \text{ and } \mu = \bar{\mu} \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

When $n = 0$ set $W = Q(\tau) \otimes \Delta^*(\lambda + t\varpi)$. Then $W^{G_1} = \Delta^*(v\varpi)$ if $m - n$ is even and either $\tau = \lambda$ or $\tau = \tilde{\lambda}$ and $\mu = \bar{\mu}$ (and zero otherwise).

Now with $Y^F = W^{G_1}$ we obtain

$$H^k(G, V) \cong H^k(\bar{G}, \Delta(m\rho)^F \otimes Y^F) \cong H^k(\mathrm{GL}_2, \Delta(m\rho) \otimes Y)$$

which gives the result.

Our first main application of the above sequences is to determine the Hom-spaces between Weyl modules. By the remarks in the last section, it is enough to consider the cases covered by the following proposition, as all other Hom-spaces will be zero. Most of the work for this has already been carried out in [3], using the sequences above, and it just remains to collect these various results together.

Proposition 2.2 *For $\tau \in \{\lambda, \tilde{\lambda}\}$ we have*
(i) If $m-n>0$ then

$$\begin{aligned} &\mathrm{Hom}_G(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tau + lm\rho)) \\ \cong &\begin{cases} \mathrm{Hom}_{\mathrm{GL}_2}(\Delta(n\rho + v\varpi), \Delta((m-1)\rho)) & \begin{cases} \text{if } m-n \text{ odd and } \tau = \tilde{\lambda} \\ \text{if } m-n \text{ odd and } \mu = \bar{\mu} \end{cases} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $v = \frac{1}{2}(m-n-1)$.
(ii) If $m = 2s$ then

$$\mathrm{Hom}_G(\Delta(s\varpi), \Delta(m\rho)) \cong \begin{cases} k & \text{if } m = 2(lp^a - 1) \text{ or } 0 \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Suppose that $n = l - 1 + lN$ and $m = l - 1 + lM$. Then we have

$$\mathrm{Hom}_G(\Delta(n\rho + lu\varpi), \Delta(m\rho)) \cong \mathrm{Hom}_{\mathrm{GL}_2}(\Delta(N\rho + u\varpi), \Delta(M\rho)).$$

Proof: First consider (i). Denoting the given weights in each case by θ and χ, we have

$$\mathrm{Hom}_G(\Delta(\theta), \Delta(\chi)) \cong [(\Delta(\chi) \otimes \Delta^*(\theta))^{G_1}]^G.$$

By [3, Lemmas 4.8–9] we see that if $m-n$ even, or $\tau = \lambda$ and $\mu \neq \bar{\mu}$ then this is zero. Otherwise

$$\begin{aligned} [(\Delta(\chi) \otimes \Delta^*(\theta))^{G_1}]^G &\cong [(q\text{-det}^{-v} \otimes \Delta((m-1)\rho) \otimes \Delta^*(n\rho))^F]^G \\ &\cong \mathrm{Hom}_{\mathrm{GL}_2}(\Delta(n\rho + v\varpi), \Delta((m-1)\rho)), \end{aligned}$$

as required. For the last part, we note that $\Delta(n\rho + lu\varpi) \cong \Delta(N\rho + u\varpi)^F \otimes \Delta((l-1)\rho)$ by [3, Proposition 3.1(ii)], and similarly for $\Delta(m\rho)$. So we have

$$\mathrm{Hom}_G(\Delta(n\rho + lu\varpi), \Delta(m\rho)) \cong (\mathrm{Hom}_{G_1}(\Delta((l-1)\rho), \Delta((l-1)\rho)) \otimes \Delta(M\rho)^F \otimes \Delta^*(N\rho + u\varpi)^F)^G$$

which implies the result. Finally, (ii) is just [3, Lemma 2.3].

We shall give a closed form for this result in Section 5.

3 The spectral sequence

Our main tool in the following sections will be the Lyndon–Hochschild–Serre spectral sequence, which will allow us to deduce results about the cohomology of G from that of the

Frobenius kernel. In this section we review the construction of this sequence, and conclude with a pair of lemmas that will allow us to calculate the desired Ext-spaces.

Consider the G-module $V = \Delta(\chi) \otimes \Delta^*(\theta)$. By [10, Proposition 1.6] we have a spectral sequence converging to $H^*(G, V)$ with E_2-page given by

$$E^{rs} = H^r(\bar{G}, H^s(G_1, V)) \cong H^r(\mathrm{GL}_2, W_s)$$

where $W_s^F = \mathrm{Ext}^s_{G_1}(\Delta(\theta), \Delta(\chi))$. This gives rise to the five term exact sequence

$$0 \to H^1(\mathrm{GL}_2, W_0) \to H^1(G, V) \to H^0(\mathrm{GL}_2, W_1) \to H^2(\mathrm{GL}_2, W_0) \to H^2(G, V).$$

In certain circumstances this sequence can be extended. In particular we have

Lemma 3.1 *(i) If $E_2^{rs} = 0$ for $s = 0, 2$ when $r \leq 2$ then we have*

$$H^1(G, V) \cong H^0(G, W_1) \qquad \text{and} \qquad H^2(G, V) \cong H^1(G, W_1).$$

(ii) If $H^i(\mathrm{GL}_2, W_1) = 0$ for $i = 1, 2$, then we can extend the above sequence to

$$\cdots \to H^0(\mathrm{GL}_2, W_1) \to H^2(\mathrm{GL}_2, W_0) \to H^2(G, V) \to W_2^{\mathrm{GL}_2} \to H^3(\mathrm{GL}_2, W_0) \to H^3(G, V).$$

Proof: (i) The first part follows from the five term sequence above. For the second part, consider the terms E_2^{rs} with $r + s = 2$. The only term which can be non-zero is E_2^{11}, and this is in the kernel of the map $d_2 : E_2^{11} \longrightarrow E_2^{30} = 0$.

(ii) Clearly all maps d_2 are zero, and so the terms in the E_2-page coincide with those in the E_3-page. Consider the line $r + s = 2$ on the E_3-page. The only possible non-trivial map starting or ending on this line is $d_3 : E_3^{02} \longrightarrow E_3^{30}$, and E^{30} is in the kernel of all maps and does not intersect with any other image. Hence the result follows.

To apply this result, we need to calculate the W_i. Note that in the following lemma (and throughout this paper) we adopt the notation and conventions from Section 1.

Lemma 3.2 *Let $\theta = \lambda + ln\rho + t\varpi < \chi = \tau + lm\rho$, where $\tau = \lambda$ or $\tilde{\lambda}$ and $m \geq n \geq 0$. Set $W_s^F = \mathrm{Ext}^s_{G_1}(\Delta(\theta), \Delta(\chi))$. Then*
(i) if $s = 0$ then

$$W_s \cong \begin{cases} \Delta((m-1)\rho) \otimes \Delta^*(n\rho + v\varpi) & \begin{cases} \text{if } m-n \text{ is odd and } \tau = \tilde{\lambda} \\ \text{if } m-n \text{ is odd, } \tau = \lambda \text{ and } \mu = \bar{\mu} \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

(ii) if $0 < s < m-n$ then

$$W_s \cong \begin{cases} \Delta((m-n-s-1)\rho - v\varpi) & \begin{cases} \text{if } m-n \text{ even, } s \text{ odd and } \tau = \lambda \\ \text{if } m-n \text{ odd, } s \text{ even and } \tau = \tilde{\lambda} \\ \text{if } m-n \text{ even, } s \text{ odd, } \tau = \tilde{\lambda} \text{ and } \mu = \bar{\mu} \\ \text{if } m-n \text{ odd, } s \text{ even, } \tau = \lambda \text{ and } \mu = \bar{\mu} \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

(iii) if $s = m-n$ then

$$W_s \cong \begin{cases} \Delta(0) & \begin{cases} \text{if } m-n \text{ odd and } \tau = \tilde{\lambda} \\ \text{if } m-n \text{ even and } \tau = \lambda \\ \text{if } m-n \text{ even, } \tau = \tilde{\lambda} \text{ and } \mu = \bar{\mu} \\ \text{if } m-n \text{ odd, } \tau = \lambda \text{ and } \mu = \bar{\mu} \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

where $v = \frac{1}{2}(m-n-s-1)$.

Proof: (i) is just [3, Lemmas 4.8–9]. For the remaining cases, let $1 \leq s \leq m-n$. Just as in [3, Lemma 4.5] we have

$$\mathrm{Ext}^s_{G_1}(\Delta(\theta), \Delta(\chi)) \cong \mathrm{Ext}^s_{G_1}(\Omega^{-n}\Delta(\theta), \Omega^{-n}\Delta(\chi)) \cong \mathrm{Ext}^1_{G_1}(\Omega^{-n}\Delta(\theta), \Omega^{-(n+s-1)}\Delta(\chi)).$$

By [3, Lemma 4.1 and the preceding remark] this equals

$$\begin{cases} \mathrm{Ext}^1_{G_1}(\Delta(\lambda + (2u+n)\frac{l}{2}\varpi), \Delta(\tilde{\tau} + xl\rho + (n+s)\frac{l}{2}\varpi)) & \text{if } n \text{ and } s \text{ even} \\ \mathrm{Ext}^1_{G_1}(\Delta(\lambda + (2u+n)\frac{l}{2}\varpi), \Delta(\tau + xl\rho + (n+s-1)\frac{l}{2}\varpi)) & \text{if } n \text{ even and } s \text{ odd} \\ \mathrm{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + (2u+n+1)\frac{l}{2}\varpi), \Delta(\tau + xl\rho + (n+s-1)\frac{l}{2}\varpi)) & \text{if } n \text{ odd and } s \text{ even} \\ \mathrm{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + (2u+n+1)\frac{l}{2}\varpi), \Delta(\tilde{\tau} + xl\rho + (n+s)\frac{l}{2}\varpi)) & \text{if } n \text{ and } s \text{ odd} \end{cases}$$

where $x = m-n-s+1$, and the result now follows from [3, Lemmas 4.6–7].

4 Ext^2 calculations

In this section we shall determine Ext^2 between all possible pairs of Weyl modules. We begin by considering certain special pairs of weights, starting with those that are 'close together'.

Lemma 4.1 *We have for $n \geq 0$ that*

$$\mathrm{Ext}^r_G(\Delta(\lambda + ln\rho), \Delta(\tilde{\lambda} + l(n+1)\rho)) \cong \begin{cases} k & \text{if } r = 1 \\ 0 & \text{if } r > 1 \end{cases}$$

Proof: Apply $\mathrm{Hom}_G(\Delta(\lambda+ln\rho),-)$ to (1) and note that by (2.1), we have

$$\mathrm{Ext}^r_G(\Delta(\lambda+ln\rho),\Delta(n\rho)^F\otimes Q(\lambda))\cong \mathrm{Ext}^r_{\mathrm{GL}_2}(\Delta(n\rho),\Delta(n\rho))=0$$

for $r\geq 1$. If $r=1$ then the result follows from [3, Lemma 5.3], while for $r\geq 2$ we get

$$\mathrm{Ext}^r_G(\Delta(\lambda+ln\rho),\Delta(\tilde{\lambda}+l(n+1)\rho))\cong \mathrm{Ext}^{r-1}_G(\Delta(\lambda+ln\rho),\Delta(\lambda+ln\rho))=0$$

as required.

Lemma 4.2 *We have for* $n\geq 0$

$$\mathrm{Ext}^r_G(\Delta(\lambda+ln\rho+l\varpi),\Delta(\lambda+l(n+2)\rho))\cong \begin{cases} k & \text{if } r=1,2\\ 0 & \text{if } r>2\end{cases}$$

Proof: The case $r=1$ follows from [3, Lemma 5.4], so we assume that $r\geq 2$. First, we exploit the spectral sequence using (3.2). For $m-n=2$ we have $W_1=W_2=\Delta(0)$ and $W_0=0$. But $H^i(\mathrm{GL}_2,\Delta(0))=0$ for $i=1,2$. So we apply (3.1)(ii) and get

$$\cdots\to H^2(\mathrm{GL}_2,W_0)\to H^2(G,V)\to k\to H^3(\mathrm{GL}_2,W_0)$$

which gives the Ext^2_G result.

Now we use the long exact sequence. Apply $\mathrm{Hom}_G(\Delta(\lambda+ln\rho+l\varpi),-)$ to the sequence

$$0\to\Delta(\lambda+l(n+2)\rho)\to\Delta((n+1)\rho+\varpi)^F\otimes Q(\tilde{\lambda})\to\Delta(\tilde{\lambda}+l(n+1)\rho+l\varpi)\to 0.$$

By (2.1), we have

$$\mathrm{Ext}^r_G(\Delta(\lambda+ln\rho+l\varpi),\Delta((n+1)\rho+\varpi)^F\otimes Q(\tilde{\lambda}))\cong \mathrm{Ext}^r_{\mathrm{GL}_2}(\Delta((n-1)\rho+\varpi),\Delta((n+1)\rho))$$

(or zero if $n=0$) for $r\geq 1$. This is zero unless $n-1=pN+(p-2)$ and $n+1=p(N+1)$. In this case (4.1) gives this is zero for $r\geq 2$. Returning to the long exact sequence, we deduce that

$$\mathrm{Ext}^r_G(\Delta(\lambda+ln\rho+l\varpi),\Delta(\lambda+l(n+2)\rho))\cong \mathrm{Ext}^{r-1}_G(\Delta(\lambda+ln\rho),\Delta(\tilde{\lambda}+l(n+1)\rho))=0$$

for $r\geq 3$, by (4.1).

The last small case that we consider is

Lemma 4.3 *We have for $n \geq 0$ that*

$$\mathrm{Ext}^r_G(\Delta(\lambda + ln\rho + l\varpi), \Delta(\tilde{\lambda} + l(n+3)\rho)) \cong \begin{cases} k & \text{if } r = 2, 3 \\ 0 & \text{if } r > 3 \end{cases}$$

Proof: We first exploit the spectral sequence. By (3.2) we have $W_1 = 0$, $W_0 = \Delta((n+2)\rho) \otimes \Delta^*(n\rho + \varpi)$, and $W_2 = \Delta(0)$, so we can apply (3.1)(ii) to obtain

$$0 \to \mathrm{Ext}^2_{\mathrm{GL}_2}(\Delta(n\rho + \varpi), \Delta((n+2)\rho))$$

$$\to \mathrm{Ext}^2_G(\Delta(\lambda + ln\rho + l\varpi), \Delta(\tilde{\lambda} + l(n+3)\rho)) \to k \to \mathrm{Ext}^3_{\mathrm{GL}_2}(\Delta(n\rho + \varpi), \Delta((n+2)\rho)).$$

The first and last terms are zero, as either $n\rho + \varpi$ and $(n+2)\rho$ are in different blocks, or we can apply (4.1). This gives the case $r = 2$.

For $r \geq 3$ we use the long exact sequence. Apply $\mathrm{Hom}_G(\Delta(\lambda + ln\rho + l\varpi), -)$ to the exact sequence

$$0 \to \Delta(\tilde{\lambda} + l(n+3)\rho) \to \Delta((n+2)\rho)^F \otimes Q(\lambda) \to \Delta(\lambda + l(n+2)\rho) \to 0.$$

Consider terms of the form $\mathrm{Ext}^r_G(\Delta(\lambda + ln\rho + l\varpi), \Delta((n+2)\rho)^F \otimes Q(\lambda))$. By (2.1) this is isomorphic to $\mathrm{Ext}^r_{\mathrm{GL}_2}(\Delta(n\rho+\varpi), \Delta((n+2)\rho))$, and this is zero for $r \geq 2$ as in the last lemma. So we deduce that for $r \geq 3$,

$$\mathrm{Ext}^r_G(\Delta(\lambda + ln\rho + l\varpi), \Delta(\tilde{\lambda} + l(n+3)\rho)) \cong \mathrm{Ext}^{r-1}_G(\Delta(\lambda + ln\rho + l\varpi), \Delta(\lambda + l(n+2)\rho)),$$

and the result now follows from (4.2).

Lemma 4.4 *Assume that $n \geq 0$. Then we have*

$$\mathrm{Ext}^2_G(\Delta((l-1+ln)\rho + lu\varpi), \Delta((l-1+lm)\rho)) \cong \mathrm{Ext}^2_{\mathrm{GL}_2}(\Delta(n\rho + u\varpi), \Delta(m\rho)).$$

Proof: We exploit the spectral sequence using (3.1). As $\Delta((l-1+ln)\rho+lu\varpi)$ is projective as a G_1-module, we have that $W_1 = W_2 = 0$. So by (3.1(ii)) we have $H^2(\mathrm{GL}_2, W_0) \cong H^2(G, V)$. The result now follows from [3, Lemma 4.10].

Lemma 4.5 *Assume that $n \geq 0$ and $a \geq 1$. Then*
(i) if $n \equiv -1 \pmod{lp^{a-1}}$ then $\mathrm{Ext}^s_G(\Delta(n\rho + lp^{a-1}\varpi), \Delta((n + 2lp^{a-1})\rho)) \cong 0$ for $s \geq 2$;
(ii) if $n \not\equiv -1 \pmod{lp^{a-1}}$ then $\mathrm{Ext}^2_G(\Delta(n\rho + lp^{a-1}\varpi), \Delta((n + 2lp^{a-1})\rho)) \cong k$.

Proof: First consider the case $n \equiv -1 \pmod{lp^{a-1}}$. Writing $n = l - 1 + Nl$, we have

$$\mathrm{Ext}^s_G(\Delta(n\rho + lp^{a-1}\varpi), \Delta((n + 2lp^{a-1})\rho)) \cong \mathrm{Ext}^s_{\mathrm{GL}_2}(\Delta(N\rho + p^{a-1}\varpi), \Delta((N + 2p^{a-1})\rho)).$$

When $a = 1$ this is zero as in the proof of (4.3), while for $a > 1$ the result follows by induction. For (ii), we proceed by induction on a. The case $a = 1$ follows from (4.2), so we assume that $a \geq 2$. Now if $n \equiv -1 \pmod{l}$ then writing n as above we have

$$\mathrm{Ext}^2_G(\Delta(n\rho + lp^{a-1}\varpi), \Delta((n + 2lp^{a-1})\rho)) \cong \mathrm{Ext}^2_{\mathrm{GL}_2}(\Delta(N\rho + p^{a-1}\varpi), \Delta((N + 2p^{a-1})\rho)).$$

Since $n \not\equiv -1 \pmod{lp^{a-1}}$, we have $N \not\equiv -1 \pmod{p^{a-1}}$, and we get the desired result by induction. Finally, if $n \not\equiv -1 \pmod{l}$ then write $n\rho = \lambda + lN\rho$. Then $(n + 2lp^{a-1})\rho = \lambda + l(N + 2p^{a-1})\rho = \lambda + lM\rho$. If $M - N = 2$ then the result follows from (4.2). If $M - N > 2$ then we apply (3.1). We have $W_q = 0$ for $0 \leq q < M - N$ if q is even. So by (3.1(i)) we have

$$\mathrm{Ext}^2_G(\Delta(n\rho + lp^{a-1}\varpi), \Delta((n + 2lp^{a-1})\rho)) \cong \mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta((p^{a-1} - 1)\varpi), \Delta((2p^{a-1} - 2)\rho)),$$

and the result now follows from [13, Theorem 3.6].

We are now in a position to prove our main result which will enable us to calculate Ext^2_G between Weyl modules. Note that by consideration of the blocks of G, the following result (in conjunction with (4.4)) includes all possible cases where this could be non-zero.

Theorem 4.6 *(i) Assume that $m - n$ is even, and let $v = \frac{1}{2}(m - n - 2)$. Then we have*

$$\mathrm{Ext}^2_G(\Delta(\lambda + ln\rho + t\varpi), \Delta(\lambda + lm\rho)) \cong \begin{cases} k & \text{if } m - n = 2 \\ \mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta(v\varpi), \Delta((m - n - 2)\rho)) & \text{if } m - n > 2. \end{cases}$$

(ii) Assume that $m - n$ is odd, and let $u = \frac{1}{2}(m - n - 1)$. Then we have

$$\mathrm{Ext}^2_G(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tilde{\lambda} + lm\rho))$$

$$\cong \begin{cases} 0 & \text{if } m - n = 1 \\ k & \text{if } m - n = 3 \\ \mathrm{Ext}^2_{\mathrm{GL}_2}(\Delta(n\rho + u\varpi), \Delta((m - 1)\rho)) & \text{if } m - n \neq 2p^a + 1 \text{ and } m - n > 3 \\ k & \text{if } m - n = 2p^a + 1 \text{ and } n \equiv -1 \pmod{p^a} \\ k^2 & \text{if } m - n = 2p^a + 1, n \not\equiv -1 \pmod{p^a}. \end{cases}$$

(iii) Suppose that $\mu = \bar{\mu}$. If $m - n$ is odd and $\tau = \lambda$, or $m - n$ is even and $\tau = \tilde{\lambda}$ then we have

$$\mathrm{Ext}^2_G(\Delta(\lambda + ln\rho + t\varpi), \Delta(\tau + lm\rho)) \cong \mathrm{Ext}^2_G(\Delta(\lambda + ln\rho + t'\varpi), \Delta(\tilde{\tau} + lm\rho))$$

where $t' = t - \frac{1}{2}l$.

Proof: Part (iii) follows immediately from the definition of $\tilde{\tau}$. For the remaining two parts, if $m-n \leq 3$ then the result follows from (4.1), (4.2) and (4.3). So we may assume that $m-n>3$. If $m-n$ is even, then the result follows from (3.1(i)) and (3.2). Now suppose that $m-n$ is odd. From (3.1(ii)) and (3.2) we have an exact sequence

$$0 \to \mathrm{Ext}^2_{\mathrm{GL}_2}(\Delta(n\rho+u\varpi), \Delta((m-1)\rho)) \to \mathrm{Ext}^2_G(\Delta(\lambda+ln\rho+t\varpi), \Delta(\tilde{\lambda}+lm\rho))$$

$$\to \mathrm{Hom}_{\mathrm{GL}_2}(\Delta((u-1)\varpi), \Delta((m-n-3)\rho)) \to \mathrm{Ext}^3_{\mathrm{GL}_2}(\Delta(n\rho+u\varpi), \Delta((m-1)\rho)). \qquad (3)$$

If $m-n \neq 2p^a+1$ then $\mathrm{Hom}_{\mathrm{GL}_2}(\Delta((u-1)\varpi), \Delta((m-n-3)\rho)) = 0$ by (2.2), and the first two terms are isomorphic as required.

We have now obtained everything except the cases in (ii) when $m-n = 2p^a+1 > 3$. If $n \equiv -1 \pmod{p^a}$ then the $\mathrm{Ext}^2_{\mathrm{GL}_2}$ and $\mathrm{Ext}^3_{\mathrm{GL}_2}$ terms of the exact sequence are zero by (4.5) and we get k, otherwise (as the first term is non-zero by (4.5)) we either get k or k^2. Before continuing with the proof we note

Lemma 4.7 *If $p>2$ then for all s and $b \geq 0$ we have*

$$\mathrm{Ext}^2_G(\Delta(s\rho+(lp^b-1)\varpi), \Delta((s+2(lp^b-1)\rho)) = 0.$$

Proof: We proceed by induction on b. If $s \not\equiv 0 \pmod{l}$ then s and $s+2(lp^b-1)$ are in different blocks. Now let $s=lN$, then $s+2(lp^b-1) = lM+l-2$ where $M = N+2p^b-1$. If $b=0$ then we are done by the first case in (ii) above. Otherwise, by the last case in the proof above, we have

$$\mathrm{Ext}^2_G(\Delta(s\rho+(lp^b-1)\varpi), \Delta((s+2(lp^b-1)\rho)) \cong \mathrm{Ext}^2_{\mathrm{GL}_2}(\Delta(N\rho+(p^b-1)\varpi), \Delta((N+2(p^b-1)\rho))$$

which is zero by the inductive hypothesis.

This lemma no longer holds when $p=2$, as when $l=1$ and $b=2$ we have $M=N+3$, and Ext^2 is then non-zero by (4.3). It is for this reason that the remainder of the proof of our main result — to which we now return — will consider the $p=2$ case separately. So we first assume that $p>2$.

Suppose that $m-n = 2p^a+1$. Then the third term in (3) is isomorphic to k by (2.2(ii)). By (4.5) we know that the first term in (3) is 0 if $n \equiv -1 \pmod{p^a}$ and isomorphic to k otherwise. So we shall be done if we can show that the last term in (3) is zero. Thus the following lemma will complete the proof for p odd.

Lemma 4.8 *If $p > 2$ and $a \geq 1$ then we have*

$$\mathrm{Ext}^3_G(\Delta(n\rho + lp^{a-1}\varpi), \Delta((n + 2lp^{a-1})\rho)) = 0.$$

Proof: We proceed by induction on a. Assume first that $a = 1$. If $n \not\equiv -1 \pmod{l}$, the result follows from (4.2), while in the case $n = l - 1 + lN$ we have

$$\mathrm{Ext}^3_G(\Delta(n\rho + l\varpi), \Delta((n + 2l)\rho) \cong \mathrm{Ext}^3_{\mathrm{GL}_2}(\Delta(N\rho + \varpi), \Delta((N + 2)\rho)).$$

This is zero either by (4.1), or because the weights lie in different blocks. So we may assume that $a \geq 2$. Suppose first that $n \equiv -1 \pmod{l}$. Write $n = l - 1 + lN$, and then

$$\mathrm{Ext}^3_G(\Delta(n\rho + lp^{a-1}\varpi), \Delta((n + 2lp^{a-1})\rho)) \cong \mathrm{Ext}^3_{\mathrm{GL}_2}(\Delta(N\rho + p^{a-1}\varpi), \Delta((N + 2p^{a-1})\rho))$$

and the result follows by induction.

Finally, suppose that $n \not\equiv -1 \pmod{l}$. Write $n\rho = \tau + lN\rho$, where $|\tau| \leq l - 2$, and then $(n + 2lp^{a-1})\rho = \tau + l(N + 2p^{a-1})\rho = \tau + lM\rho$. We have the exact sequence

$$0 \to \Delta(\tau + lM\rho) \to \Delta((M - 1)\rho)^F \otimes Q(\lambda) \to \Delta(\lambda + l(M - 1)\rho) \to 0$$

for some λ such that $\tilde{\lambda} = \tau$. Applying $\mathrm{Hom}_G(\Delta(n\rho + lp^{a-1}\varpi), -)$ to this we obtain

$$\cdots \to \mathrm{Ext}^2_G(\Delta(n\rho + lp^{a-1}\varpi), \Delta(\lambda + l(M - 1)\rho)) \to \mathrm{Ext}^3_G(\Delta(n\rho + lp^{a-1}\varpi), \Delta((n + 2lp^{a-1})\rho))$$
$$\to \mathrm{Ext}^3_{\mathrm{GL}_2}(\Delta((N - 1)\rho + p^{a-1}\varpi), \Delta((M - 1)\rho)) \to \cdots$$

by (2.1), where if $N = 0$ the last term is taken to be zero. By the third case of (4.6(ii)), the left-hand term in this sequence is isomorphic to $\mathrm{Ext}^2_{\mathrm{GL}_2}(\Delta(N\rho + (p^{a-1} - 1)\varpi), \Delta((M - 2)\rho))$, which is zero by the last lemma. Hence we are left with

$$0 \to \mathrm{Ext}^3_G(\Delta(n\rho + lp^{a-1}\varpi), \Delta((n + 2lp^{a-1})\rho)) \to \mathrm{Ext}^3_{\mathrm{GL}_2}(\Delta((N - 1)\rho + p^{a-1}\varpi), \Delta((M - 1)\rho)).$$

The last term is zero either by definition if $N = 0$ or by induction, and we are done.

It remains to consider the case when $p = 2$ and $m - n = 2p^a + 1$, with $n \not\equiv -1 \pmod{p^a}$. We shall show that in this case the Ext^2-space is 2-dimensional. The proof of this will take the rest of this section, and requires a new filtration of certain tilting modules, described below. We shall also require this filtration without restriction on p in Section 6, so in what follows p will be arbitrary.

Given a dominant weight τ, we define the module $X(\tau)$ via the short exact sequence

$$0 \to \Delta(\tau) \to T(\tau) \to X(\tau) \to 0. \tag{4}$$

Note that by [11, 2.1(13)], or [7, Proposition 3.1], $X(\tau)$ is filtered by $\Delta(\gamma)$'s with $\gamma < \tau$. Now let $\tau = \tilde{\lambda} + l\rho + l\alpha$ where $\alpha = r\rho$ with $r > 0$. Since $\mathrm{Ext}^i_G(-, T(\tau))$ vanishes on $\mathcal{F}(\Delta)$ for $i \geq 1$ (by [10, Section 4 (2)]), we have that for any dominant weight θ,

$$\mathrm{Ext}^2_G(\Delta(\theta), \Delta(\tau)) \cong \mathrm{Ext}^1_G(\Delta(\theta), X(\tau)). \tag{5}$$

Hence it is enough to show that the latter Ext-space is 2-dimensional. By the remarks after (3), it has dimension at most two. We begin by considering a new filtration of $X(\tau)$. By [11, 3.3(7)] we have

$$T(\tilde{\lambda} + l\rho) \otimes T(\alpha)^F \cong T(\tau)$$

and hence we obtain the short exact sequence

$$0 \to T(\tilde{\lambda} + l\rho) \otimes \Delta(\alpha)^F \to T(\tau) \to T(\tilde{\lambda} + l\rho) \otimes X(\alpha)^F \to 0.$$

As $Q(\lambda) \cong T(\tilde{\lambda} + l\rho)$, we obtain from (1) and (4) the sequence

$$0 \longrightarrow \Delta(\lambda + l\alpha) \longrightarrow X(\tau) \xrightarrow{\pi} T(\tilde{\lambda} + l\rho) \otimes X(\alpha)^F \longrightarrow 0. \tag{6}$$

We now claim that there exists a short exact sequence

$$0 \to \Delta(\lambda + l\alpha) \oplus \big(\nabla(\lambda) \otimes X(\alpha)^F\big) \to X(\tau) \to \nabla(\tilde{\lambda} + l\rho) \otimes X(\alpha)^F \to 0. \tag{7}$$

The proof of this is postponed until the end of this section; we first consider some of its consequences. For this we shall use

Lemma 4.9 *Suppose that l is odd, or $M \equiv N \pmod 2$. Then we have*

$$\mathrm{Hom}_G(\Delta(\lambda + lN\rho + t\varpi), \nabla(\tilde{\lambda} + l\rho) \otimes L(M\rho)^F) = 0.$$

Hence if X is any module such that all of its composition factors have highest weights $M\rho + w\varpi$ with $M \equiv N \pmod 2$ then $\mathrm{Hom}_G(\Delta(\lambda + lN\rho + t\varpi), \nabla(\tilde{\lambda} + l\rho) \otimes X^F) = 0$.

Proof: In the first part, any non-zero homomorphism must map the weight $\lambda + lN\rho + t\varpi$ to $\tilde{\lambda} + l\rho + lM\rho$, by the ordering of weights. By inspection this cannot occur under the above hypotheses. The second part is now immediate.

We now assume that $m - n = 2p^a + 1$, with $n \not\equiv -1 \pmod{p^a}$ (but p still arbitrary). Apply the functor $\mathrm{Hom}_G(\Delta(\lambda + ln\rho + t\varpi), -)$ to the exact sequence in (7) with $\alpha = (n+2p^a)\rho$. By the preceding lemma, the last homomorphism space is zero, and we obtain

$$\begin{aligned} 0 \to \mathrm{Ext}^1_G(\Delta(\lambda + ln\rho + t\varpi), \Delta(\lambda + l\alpha)) \oplus \mathrm{Ext}^1_G(\Delta(\lambda + ln\rho + t\varpi), \nabla(\lambda) \otimes X(\alpha)^F) \\ \to \mathrm{Ext}^1_G(\Delta(\lambda + ln\rho + t\varpi), X(\tau)). \end{aligned} \tag{8}$$

The first Ext-space is isomorphic to k by [3, Theorem 5.5], so it remains to show that the second space is non-zero. By the remarks after (5), this will complete the proof of (4.6) when $p = 2$. It will also be needed in the general case in Section 6.

We consider the exact sequence

$$0 \to \Delta((n-1)\rho)^F \otimes \Delta(\tilde{\lambda} + l\varpi + t\varpi) \to \Delta(\lambda + ln\rho + t\varpi) \to \Delta(n\rho)^F \otimes \Delta(\lambda + t\varpi) \to 0$$

coming from (2), and apply $\mathrm{Hom}_G(-, \nabla(\lambda) \otimes X(\alpha)^F)$. Consider $\mathrm{Hom}_G(\Delta((n-1)\rho)^F \otimes \Delta(\tilde{\lambda} + l\varpi + t\varpi), \nabla(\lambda) \otimes X(\alpha)^F)$. By Steinberg's tensor product theorem [11, 3.2(5)], the two sides have no common composition factors unless $\mu = \bar{\mu}$. But in this latter case, as $n - 1 \not\equiv n + 2p^a \pmod 2$, the two twisted modules lie in different blocks, and so the Hom-space is zero. Hence we have an injection

$$0 \to \mathrm{Ext}^1_G(\Delta(n\rho)^F \otimes \Delta(\lambda + t\varpi), \nabla(\lambda) \otimes X(\alpha)^F) \to \mathrm{Ext}^1_G(\Delta(\lambda + ln\rho + t\varpi), \nabla(\lambda) \otimes X(\alpha)^F).$$

From the five term exact sequence, we obtain an injection

$$0 \to \mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta(n\rho + u\varpi), X((n+2p^a)\rho)) \to \mathrm{Ext}^1_G(\Delta(n\rho)^F \otimes \Delta(\lambda + t\varpi), \nabla(\lambda) \otimes X(\alpha)^F)$$

where $lu = t$, and hence it is enough to show that the first of these spaces is non-zero. But applying the dimension shift arising from the defining sequence for X, and (4.5(ii)) we have

$$\mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta(n\rho + u\varpi), X((n+2p^a)\rho)) \cong \mathrm{Ext}^2_{\mathrm{GL}_2}(\Delta(n\rho + u\varpi), \Delta((n+2p^a)\rho)) \cong k$$

as required.

So it now just remains to verify that the sequence (7) exists, as claimed. For this we shall use the following general result.

Lemma 4.10 *Let L be a simple G-module, that remains simple on restriction to G_1. Then for any finite-dimensional G-module M we have an isomorphism of G-modules*

$$\mathrm{Hom}_{G_1}(L, M) \otimes L \cong (\mathrm{soc}_{G_1} M)_L$$

where $(\mathrm{soc}_{G_1} M)_L$ is the isotypic component of $\mathrm{soc}_{G_1} M$ of type L.

Proof: This follows just as in [17, I 6.15(2)], using that the G_1 fixed points of a G-module themselves have a G-module structure by [10, Proposition 1.5(i)].

With the above lemma we can now prove

Lemma 4.11 *Let $\alpha = r\rho$ and $\tau = \tilde{\lambda} + l\rho + l\alpha$, with $r > 0$. Then we have a short exact sequence of G-modules*

$$0 \to \Delta(\lambda + l\alpha) \oplus \big(\nabla(\lambda) \otimes X(\alpha)^F\big) \to X(\tau) \to \nabla(\tilde{\lambda} + l\rho) \otimes X(\alpha)^F \to 0.$$

Proof: Since $\nabla(\lambda) \leq T(\tilde{\lambda} + l\rho)$ with quotient $\nabla(\tilde{\lambda} + l\rho)$ we have an injection

$$\theta : \nabla(\lambda) \otimes X(\alpha)^F \to T(\tilde{\lambda} + l\rho) \otimes X(\alpha)^F$$

and so it is enough to show that this map factors through π in (6). In order to do this we shall consider the G_1-socle of $X(\tau)$ using the last lemma. By block considerations we must consider $\mathrm{Hom}_{G_1}(Y, X(\tau))$, where Y is either $\Delta(\lambda)$ or $\Delta(\tilde{\lambda})$. Since this is trivial as a G_1-module it is of the form $W(Y)^F$ for some GL_2-module $W(Y)$ by [10, Proposition 1.5(i)]. Applying $\mathrm{Hom}_{G_1}(Y, -)$ to (6) we get

$$\begin{aligned} 0 \to \mathrm{Hom}_{G_1}(Y, \Delta(\lambda + l\alpha)) \to W(Y)^F \to \mathrm{Hom}_{G_1}(Y, T(\tilde{\lambda} + l\rho) \otimes X(\alpha)^F) \\ \to \mathrm{Ext}^1_{G_1}(Y, \Delta(\lambda + l\alpha)) \to \mathrm{Ext}^1_{G_1}(Y, X(\tau)) \to 0 \end{aligned} \tag{9}$$

where the last term is zero as $T(\tilde{\lambda} + l\rho)$ is injective as a G_1-module.

Since $T(\tau)$ is injective as a G_1-module by [11, 3.3(2)], we have

$$\mathrm{Ext}^1_{G_1}(Y, X(\tau)) \cong \mathrm{Ext}^2_{G_1}(Y, \Delta(\tau)) \cong \mathrm{Ext}^1_{G_1}(Y, \Omega^{-1}\Delta(\tau)).$$

Now by [3, Lemma 4.1] we have $\Omega^{-1}\Delta(\tau) \cong \Delta(\lambda + l\alpha)$ and so the two $\mathrm{Ext}^1_{G_1}$-terms in (9) are isomorphic. Hence for each Y there is a short exact sequence

$$0 \to \mathrm{Hom}_{G_1}(Y, \Delta(\lambda + l\alpha)) \to W(Y)^F \to \mathrm{Hom}_{G_1}(Y, T(\tilde{\lambda} + l\rho) \otimes X(\alpha)^F) \to 0.$$

The G_1-socle of $T(\tilde{\lambda} + l\rho) \otimes X(\alpha)^F$ is just $\nabla(\lambda) \otimes X(\alpha)^F$, so using [3, Lemmas 4.2–3] we deduce that this sequence splits as one of the outer terms is zero. Hence by tensoring up (9) with Y and using the previous lemma, we deduce that the G_1-socle of $X(\tau)$ is the direct sum of the G_1-socles of the outer terms in (6).

Now the G_1-socle of $T(\tilde{\lambda} + l\rho) \otimes X(\alpha)^F$ is just $\nabla(\lambda) \otimes X(\alpha)^F$, which by the above is a direct summand of the G_1-socle of $X(\tau)$, and hence is a G-submodule of $X(\tau)$. But this cannot intersect $\Delta(\lambda + l\alpha)$ by the G_1-socle considerations above, and so the result now follows.

As noted after (8), this completes the proof of (4.6).

5 Closed forms

In this section we shall provide a closed form for the results on Hom- and Ext-spaces obtained so far. For simplicity we shall first just deal with the classical case. We wish to define certain sets which will play a similar role to the set $\Psi(r)$ in [13].

For any integer a with $0 \leq a \leq p-1$. we define $\hat{a}$ by $a+\hat{a} = p-1$. Also, if $r = \sum_{i=0}^{v} r_i p^i$, we let $r^s = \sum_{i>s} r_i p^{i-s-1}$. Note that for any r we have $r = r_0 + pr^0$. We now define

$$\Psi^0(r) = \left\{ \sum_{i=0}^{u-1} \hat{r}_i p^i : u \geq 0 \right\},$$

$$\Psi^1(r) = \left\{ \sum_{i=0}^{u-1} \hat{r}_i p^i + p^{u+a} : \hat{r}_u \neq 0,\ a \geq 1,\ u \geq 0 \right\} \bigcup \left\{ \sum_{i=0}^{u} \hat{r}_i p^i : \hat{r}_u \neq 0,\ u \geq 0 \right\},$$

$$\Psi^{2,1}(r) = \left\{ p^{u+a},\ \sum_{i=0}^{u} \hat{r}_i p^i + p^{u+a},\ \sum_{i=0}^{u-1} \hat{r}_i p^i + p^{u+a} + p^{u+b} : \hat{r}_u \neq 0,\ b > a \geq 1,\ u \geq 0 \right\},$$

$$\Psi^{2,2}(r) = \left\{ \sum_{i=0}^{u} \hat{r}_i p^i + p^{u+a+1} : \hat{r}_u \neq 0,\ r^u \not\equiv -1 \pmod{p^a},\ a \geq 1,\ u \geq 0 \right\}.$$

With these sets, we can now give a closed form for our previous results.

Theorem 5.1 *Let $\theta = (r+d, d)$ and $\chi = (s+d', d')$. Then we have*

$$\mathrm{Hom}_{\mathrm{GL}_2}(\Delta(\theta), \Delta(\chi)) \cong \begin{cases} k & \text{if } r+2d = s+2d' \text{ and } s = r+2e \text{ with } e \in \Psi^0(r) \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta(\theta), \Delta(\chi)) \cong \begin{cases} k & \text{if } r+2d = s+2d' \text{ and } s = r+2e \text{ with } e \in \Psi^1(r) \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{Ext}^2_{\mathrm{GL}_2}(\Delta(\theta), \Delta(\chi)) \cong \begin{cases} k^2 & \text{if } r+2d = s+2d' \text{ and } s = r+2e \text{ with } e \in \Psi^{2,2}(r) \\ k & \text{if } r+2d = s+2d' \text{ and } s = r+2e \text{ with } e \in \Psi^{2,1}(r) \backslash \Psi^{2,2}(r) \\ 0 & \text{otherwise.} \end{cases}$$

Proof: The Ext^1_G result is just [13, Theorem 3.6]. In each of the remaining cases, we shall rewrite the appropriate conditions from (2.2), (4.4) and (4.6) to describe the desired sets by a series of recurrence relations. It will then be straightforward to verify that the sets above are determined by the same relations, and hence obtain the result. Before beginning, we note that all these relations imply that for a non-zero result we must have the homogeneity condition in each case. So we assume that $r + 2d = s + 2d'$. As we show below, in each non-zero case we also have $s = 2e$, so we express our relations in terms of this variable e.

To begin we consider the Hom case. Let $\Phi^0(r)$ be the set of e such that the corresponding Hom-space is isomorphic to k. Then by (2.2) we have $e \in \Phi^0(r)$ if and only if one of the following conditions hold:

$$\begin{array}{ll} (1) & e_0 = \hat{r}_0 > 0 \text{ and } e^0 \in \Phi^0(r^0) \\ (2) & e = p^a - 1, r = 0 \text{ and } a \geq 0 \\ (3) & e_0 = \hat{r}_0 = 0 \text{ and } e^0 \in \Phi^0(r^0) \\ (4) & d = 0 \end{array}$$

where the last condition is obvious. Clearly we can combine (1) and (3) to obtain

$$(1') \quad e_0 = \hat{r}_0 \text{ and } e^0 \in \Phi^0(r^0).$$

Now it is straightforward to verify that (1′), (2) and (4) also generate the set

$$\left\{ \sum_{i=0}^{u-1} \hat{r}_i p^i : u \geq 0 \right\} \bigcup \left\{ \sum_{i=0}^{v} \hat{r}_i p^i + p^{a+v+1} - p^{v+1} : a \geq 0 \right\}$$

where $r = \sum_{i=0}^{v} r_i p^i$. But clearly this equals

$$\left\{ \sum_{i=0}^{u-1} \hat{r}_i p^i : u \geq 0 \right\} \bigcup \left\{ \sum_{i=0}^{w} \hat{r}_i p^i : w \geq v \right\} = \left\{ \sum_{i=0}^{u-1} \hat{r}_i p^i : u \geq 0 \right\}$$

and so $\Phi^0(r) = \Psi^0(r)$ as required. Next we consider the Ext^2 case for $p > 2$. We write $\Phi^{2,2}(r)$ for the set of e's giving rise to a two-dimensional Ext^2 case, and $\Phi^{2,1}(r)$ for the set of e's for which Ext^2 is non-zero. Now by (4.4) and (4.6), we have $e \in \Phi^{2,2}(r)$ if and only if one of the following conditions hold:

$$\begin{array}{ll} (5) & e_0 = \hat{r}_0 > 0, e^0 \in \Phi^{2,2}(r^0) \text{ and } e^0 \neq p^a \text{ for some } a > 0 \\ (6) & e_0 = \hat{r}_0 = 0 \text{ and } e^0 \in \Phi^{2,2}(r^0) \\ (7) & e_0 = \hat{r}_0 > 0, e^0 = p^a \text{ for some } a > 0 \text{ and } r^0 \not\equiv -1 \pmod{p^a}. \end{array}$$

Consider (5). If $e^0 = p^a$ for some $a > 0$ then the case is covered by (7) if $r^0 \not\equiv -1 \pmod{p^a}$. So to show that the last condition in (5) is redundant it is enough to show that if $r \equiv$

$-1 \pmod{p^a}$ then $p^a \notin \Phi^{2,2}(r)$. But this follows immediately from (4.5). Hence, we may replace (5) and (6) by the condition

$$(5') \quad e_0 = \hat{r}_0 \text{ and } e^0 \in \Phi^{2,2}(r^0).$$

It is now straightforward to verify that conditions (5′) and (7) generate $\Psi^{2,2}(r)$ as required.

Finally, we consider the remaining case. For this it will be easier to calculate when Ext^2 is non-zero. It is enough to show that $\Phi^{2,1}(r) = \Psi^{2,1}(r)$. Now by (4.4) and (4.6), we have $e \in \Phi^{2,1}(r)$ if and only if one of the following conditions hold:

$$\begin{array}{rl}
(8) & e_0 = \hat{r}_0 > 0, e^0 \in \Phi^{2,1}(r^0) \text{ and } e^0 \neq p^a \text{ for some } a > 0 \\
(9) & e_0 = \hat{r}_0 = 0 \text{ and } e^0 \in \Phi^{2,1}(r^0) \\
(10) & e_0 = \hat{r}_0 > 0 \text{ and } e^0 = p^a \text{ for some } a > 0 \\
(11) & e_0 = \hat{r}_0 > 0 \text{ and } e^0 = p^0 \\
(12) & e_0 = 0, \hat{r}_0 > 0 \text{ and } e^0 = 1 \\
(13) & e_0 = 0, \hat{r}_0 > 0 \text{ and } e^0 \in \{p^{u+1}, p^u + p^{u+a} : u \geq 0, a \geq 1\}.
\end{array}$$

Now (8–11) simplify to give:

$$\begin{array}{rl}
(8') & e_0 = \hat{r}_0 \text{ and } e^0 \in \Phi^{2,1}(r^0) \\
(10') & e_0 = \hat{r}_0 > 0 \text{ and } e^0 = p^a \text{ for some } a \geq 0,
\end{array}$$

while (12) and (13) become

$$(12') \quad \hat{r}_0 > 0 \text{ and } e \in \{p^a, p^a + p^b : b > a \geq 1\}.$$

As before, it is now routine to verify that (8′), (10′) and (12′) also generate $\Psi^{2,1}(r)$ as required, which completes the proof.

A similar result holds in the quantum case, with appropriate modifications as in [3, Theorem 5.5], and is left to the reader.

6 Δ-infinite q-Schur algebras

In this section we shall consider representations of the q-Schur algebra of Dipper and James. The definition of this, along with a review of its basic properties, can be found in [18] (or [14] in the classical case).

Our previous results have shown that there exist 2-dimensional Ext^2-spaces between certain Weyl modules. By dimension shifting, this gives rise to certain modules M and N in $\mathcal{F}(\Delta)$ for an appropriate q-Schur algebra, such that $\mathrm{Ext}^1_G(M, N)$ is 2-dimensional. We shall show that this implies that the category $\mathcal{F}(\Delta)$ contains infinitely many indecomposable modules. For this we will use a general result concerning modules for an arbitrary finite dimensional quasi-hereditary k-algebra A, with respect to $(\Lambda, \leq)$, for which we need the following lemma.

Lemma 6.1 *Suppose that $P = P(\theta)$ is an indecomposable projective A-module. If $\sigma : P \to P$ is a homomorphism, then σ maps $\Omega(\Delta(\theta))$ into itself.*

Proof: First assume that σ is an isomorphism. We have the exact sequence

$$0 \to U(\theta) \to P \to \Delta(\theta) \to 0$$

where $U(\theta)$ is defined to be the sum of the images of all homomorphisms $\psi : P(\chi) \to P$ with $\chi \not\leq \theta$ (see for example [7]). We identify $U(\theta)$ with $\Omega(\Delta(\theta))$. Now consider $x \in U(\theta)$. Without loss of generality, we may assume that $x = \psi(w)$ where $\psi : P(\chi) \to P$ and $\chi \not\leq \theta$. As $\sigma^{-1} : P \to P$ is onto, there exists a homomorphism $\eta : P(\chi) \to P$ such that $\psi = \sigma^{-1}\eta$. Now $\eta(w) \in U(\theta)$, so $x \in \sigma^{-1}(U(\theta))$ and hence $\sigma(U(\theta)) \subseteq U(\theta)$.

If σ is not an isomorphism, then it is nilpotent. Hence $1 + \sigma$ is an isomorphism, and by the above takes $U(\theta)$ into $U(\theta)$. Thus for any $x \in U(\theta)$ we have $x + \sigma(x) \in U(\theta)$, and hence $\sigma(x) \in U(\theta)$ as required.

Remark 6.2 *The above argument also implies that every homomorphism from $\Omega(\Delta(\theta))$ to P maps into $\Omega(\Delta(\theta))$.*

We wish to study modules $\Delta(\theta)$ and indecomposable projective modules X such that $\mathrm{Ext}^1_A(\Delta(\theta), X)$ is two-dimensional. Let $0 \to \Omega(\Delta(\theta)) \to P \to \Delta(\theta) \to 0$ be a projective cover of $\Delta(\theta)$. We regard $\mathrm{Ext}^1_A(\Delta(\theta), X)$ as $\mathrm{Hom}_A(\Omega(\Delta(\theta)), X)/\mathrm{Im}(\mathrm{Hom}_A(P, X))$. Thus it is a module both for $\mathrm{End}_A(X)$ and for $\mathrm{End}_A(\Omega(\Delta(\theta)))$.

Proposition 6.3 *Let X be an indecomposable projective A-module such that $\mathrm{Ext}^1_A(\Delta(\theta), X)$ is at least two-dimensional for some $\theta \in \Lambda$. Assume that $\mathrm{Ext}^1_A(\Delta(\theta), X)$ is semi-simple as a module for $\mathrm{End}_A(X)$ and also for $\mathrm{End}_A(\Omega(\Delta(\theta)))$. Then there is a family of modules*

$$0 \to X \to E_a \to \Delta(\theta) \to 0$$

with $a \in k \backslash \{0\}$ such that $E_a \not\cong E_b$ for $a \neq b$.

Proof: Take two fixed maps ϕ_1 and ϕ_2 from $\Omega = \Omega(\Delta(\theta)) \to X$ which are linearly independent in $\mathrm{Ext}^1_A(\Delta(\theta), X)$. For $0 \neq a \in k$ we define $f_a = \phi_1 + a\phi_2$ and let E_a be the push-out of f_a along the projective cover. We want to show that if $E_a \cong E_b$ then $a = b$. By definition, $E_a \cong (X \oplus P)/I_a$ where $I_a = \{(f_a w, w) : w \in \Omega\}$. So we have an exact sequence

$$0 \to I_a \to X \oplus P \to E_a \to 0.$$

Suppose that $E_a \cong E_b$. Then, as X is projective, the given isomorphism lifts to a homomorphism $\psi : X \oplus P \to X \oplus P$ which takes I_a into I_b.

For $p \in P$, we write $\psi(0, p) = (\rho(p), \sigma(p))$. Now σ is an isomorphism of P, as $L(\theta)$ occurs in the top of both E_a and E_b. Similarly, for $x \in X$ we write $\psi(x, 0) = (\delta(x), \eta(x))$. Since ψ maps I_a into I_b, there is an endomorphism ϵ of Ω such that

$$\psi(f_a w, w) = (f_b \epsilon(w), \epsilon(w))$$

for all $w \in \Omega(\Delta(\theta))$. Hence we deduce that for $w \in \Omega(\Delta(\theta))$,

$$\rho(w) + \delta f_a(w) = f_b \epsilon(w), \quad \text{and} \quad \sigma(w) + \eta f_a(w) = \epsilon(w).$$

Thus $f_b \epsilon - \delta f_a = \rho|_\Omega \in \mathrm{Im}(\mathrm{Hom}_A(P, X))$. We wish to show that there is a k-linear combination of f_a and f_b in $\mathrm{Im}(\mathrm{Hom}_A(P, X))$. There are two cases to consider.

First suppose that $\mathrm{Hom}_A(X, \Delta(\theta)) = 0$. Then η maps into $\Omega(\Delta(\theta))$, and hence $f_b \eta$ is an endomorphism of X. Substituting for ϵ, we obtain (on $\Omega\Delta(\theta)$)

$$f_b \sigma - (\delta - f_b \eta) f_a = \rho. \tag{10}$$

Now the endomorphism ring of X is local, so $\delta - f_b \eta = d \cdot 1 + \nu$ where ν is nilpotent. Similarly $\sigma = c \cdot 1 + \gamma$ where γ is nilpotent, and $c \neq 0$. By (6.1), we have that γ maps $\Omega(\Delta(\theta))$ into itself. Since $\mathrm{Ext}^1_A(\Delta(\theta), X)$ is semi-simple as a module for $\mathrm{End}_A(X)$ and for $\mathrm{End}_A(\Omega(\Delta(\theta))$ the left hand side of (10) equals $cf_b - df_a$, modulo maps which factor through P. As $c \neq 0$ it follows that f_a and f_b are linearly dependent in $\mathrm{Ext}^1_A(\Delta(\theta), X)$, and hence that $a = b$.

Next suppose that $\mathrm{Hom}_A(X, \Delta(\theta)) \neq 0$. Then X is not a direct summand of $\Omega(\Delta(\theta))$, since $\mathrm{Hom}_A(\Omega(\Delta(\theta)), \Delta(\theta)) = 0$ for a quasi-hereditary algebra. Note that ηf_a maps into $\Omega(\Delta(\theta))$ by (6.2). We show that in this case ηf_a must be nilpotent.

If ηf_a is not nilpotent, then $f_a(\eta f_a)\eta : X \to X$ is not nilpotent and hence must be an isomorphism. Then it follows that X is a summand of $\Omega(\Delta(\theta))$, a contradiction. Now we get as before that $\epsilon = c \cdot 1 + \nu$ with ν nilpotent and $c \neq 0$, and similarly for δ. Then, again as before, f_a and f_b are linearly dependent modulo the image of $\mathrm{Hom}_A(P, X)$, implying that $a = b$. This completes the proof of (6.3).

Suppose now that $\theta = \lambda + ln\rho + t\varpi$ and $\chi = \tilde{\lambda} + lm\rho$ with $m - n = 2p^a + 1$ and $n \not\equiv -1 \pmod{p^a}$. Then by (4.6) we have $\mathrm{Ext}^2_G(\Delta(\theta), \Delta(\chi)) = k^2$. As remarked in Section 4, this implies that

$$\mathrm{Ext}^1_G(\Delta(\theta), X(\chi)) = k^2$$

where $X(\chi)$ is defined as in (4). Setting $X = X(\chi) \otimes q\text{-}\det^{-1}$ and $A = S_q(2, |\chi| - 2)$ we have

$$\mathrm{Ext}^1_A(\Delta(\theta - \varpi), X) \cong \mathrm{Ext}^1_G(\Delta(\theta - \varpi), X) = k^2.$$

Thus, if we can show that X and $\mathrm{Ext}^1_A(\Delta(\theta - \varpi), X)$ satisfy the conditions in (6.3), this will imply that $S_q(2, |\chi| - 2)$ is Δ-infinite. We first show that X is a projective indecomposable module for A, using the following property of tilting modules.

Lemma 6.4 *The tilting module $T(\chi)$ is an indecomposable injective and projective module for $S_q(2, |\chi|)$.*

Proof: Consider first the classical case. By [8, Example 1], $T(\chi)$ has simple socle, which we denote by $L(s\chi)$. So it is enough to show that

$$(T(\chi) : \Delta(\tau)) = [\Delta(\tau) : L(s\chi)]$$

for all $\tau \leq \chi$. But this follows from [16, 5.9 Satz] and [8, Proposition 2.1 and Example 2]. Now consider the quantum case. As the results in [8] used above all generalise to the quantum setting (using [11, Sections 3.3 and 3.4] and [2, Lemmas 3.2 and 5.1]), it is enough to show that an appropriate analogue of [16, 5.9 Satz] holds. But using (1), (2) and [11, 3.3(7)] we obtain that this holds for q-$\mathrm{GL}(2, k)$ from the corresponding classical case.

Corollary 6.5 *The module X is a projective indecomposable module for A.*

Proof: Let $\pi = \Lambda^+(2, |\chi|)\backslash\{\chi\}$. This is a saturated subset of weights, so we have the associated generalised q-Schur algebra $S(\pi)$ and truncation functor O_π as in [11, Section 4.2]. Clearly

$$O_\pi(S_q(2, |\chi| - 2) \otimes q\text{-}\det) = S_q(2, |\chi| - 2) \otimes q\text{-}\det$$

has a natural algebra structure induced from $S_q(2, |\chi|-2)$, and the same dimension as $S(\pi)$ by the dimension arguments in [10, Section 4]. Thus $S(\pi) \cong S_q(2, |\chi|-2) \otimes q$-det.

As noted in [11, Section 2.1], we have an indempotent $e = \xi_{(r)} \in S = S_q(2, |\chi|)$ corresponding to the weight (r). Now Se is an indecomposable projective module, isomorphic to $\Delta(\chi)$. More generally, if S is an algebra containing an idempotent e, we consider S/SeS-modules as S-modules which are annihilated by the ideal SeS. For any projective module P we have that P/SeP is an S/SeS-module, and is projective as such since

$$\mathrm{Hom}_{S/SeS}(P/SeP, -) \cong \mathrm{Hom}_S(P, -)|_{\mathrm{Mod}(S/SeS)}$$

is an exact functor.

Now take S and e as above. Setting $P = T(\chi)$ we have $SeP = \Delta(\chi)$, and hence $X(\chi)$ is projective as an S/SeS-module. But $S/SeS \cong S(\pi)$, and so we are done.

We use the exact sequence

$$0 \to \Delta(\lambda + l\alpha) \oplus \left(\nabla(\lambda) \otimes X(\alpha)^F\right) \to X(\chi) \xrightarrow{\pi} \nabla(\tilde{\lambda} + l\rho) \otimes X(\alpha)^F \to 0 \tag{11}$$

from (7), where $\alpha = (n + 2p^a)\rho$. By (4.6) and the calculations after (4.9) this induces an isomorphism

$$\mathrm{Ext}^1_G(\Delta(\theta), X(\chi)) \cong \mathrm{Ext}^1_G(\Delta(\theta), \Delta(\lambda + l\alpha)) \oplus \mathrm{Ext}^1_G(\Delta(\theta), \nabla(\lambda) \otimes X(\alpha)^F). \tag{12}$$

Tensoring both sides with q-det^{-1} we see that $\mathrm{Ext}^1_A(\Delta(\theta - \varpi), X)$ is the direct sum of two one-dimensional modules for $\mathrm{End}_A(\Omega\Delta(\theta - \varpi))$ as required. So it just remains to show that it is also semi-simple as a module for $\mathrm{End}_A(X)$. Indeed, it is enough to show that (12) is semi-simple as a module for $\mathrm{End}_G(X(\chi))$, as the result again follows by tensoring with q-det^{-1}.

Recall from earlier in this section our identification of $\mathrm{Ext}^1_A(\Delta(\theta), Y)$ with the space $\mathrm{Hom}_A(\Omega(\Delta(\theta)), Y)/\mathrm{Im}(\mathrm{Hom}_A(P, Y))$. We fix homomorphisms

$$\phi_1 : \Omega(\Delta(\theta)) \to \Delta(\lambda + l\alpha) \quad \text{and} \quad \phi_2 : \Omega(\Delta(\theta)) \to \nabla(\lambda) \otimes X(\alpha)^F$$

whose push-outs are non-split exact sequences. We can now show

Lemma 6.6 *The* $\mathrm{End}_G(X(\chi))$*-module* $\mathrm{Ext}^1_G(\Delta(\theta), X(\chi))$ *is semi-simple.*

Proof: Let γ be an endomorphism of $X(\chi)$. As $\lambda+l\alpha$ is the highest weight of X, and occurs with multiplicity one, we have $\gamma(\Delta(\lambda+l\alpha)) \subseteq \Delta(\lambda+l\alpha)$. Now ϕ_1 maps into $\Delta(\lambda+l\alpha)$, and hence so too does $\gamma\phi_1$. Thus we have $\gamma\phi_1 = c\phi_1$ for some $c \in k$ (modulo the image of $\mathrm{Hom}_A(P, X(\chi))$). Thus it is enough to show that $\gamma\phi_2$ maps into $\nabla(\lambda) \otimes X(\alpha)^F$ (modulo maps factoring through P).

By the first part we have that γ induces an endomorphism $\bar{\gamma}$ of $X(\chi)/\Delta(\lambda+l\alpha)$, which by (6) is isomorphic to $T(\tilde{\lambda}+l\rho)\otimes X(\alpha)^F$. As $\nabla(\lambda)$ is a submodule of $T(\tilde{\lambda}+l\rho)$, the composition of this map with that induced by π from (11), gives a homomorphism from $\nabla(\lambda) \otimes X(\alpha)^F$ to $\nabla(\tilde{\lambda}+l\rho) \otimes X(\alpha)^F$. Writing X' for $X(\alpha)$ we have

$$\mathrm{Hom}_G(\nabla(\lambda) \otimes X'^F, \nabla(\tilde{\lambda}+l\rho) \otimes X'^F) \cong (\mathrm{Hom}_{G_1}(\nabla(\lambda), \nabla(\tilde{\lambda}+l\rho)) \otimes X'^F \otimes (X'^F)^*)^G$$

and this is zero by arguments as in [3, Lemmas 4.2–3]. Thus γ maps $\nabla(\lambda) \otimes X(\alpha)^F$ into $\ker\pi = \Delta(\lambda+l\alpha) \oplus (\nabla(\lambda) \otimes X(\alpha)^F)$. Hence we will be done if we can show that

$$\mathrm{Hom}_G(\nabla(\lambda) \otimes X(\alpha)^F, \Delta(\lambda+l\alpha)) = 0.$$

Apply $\mathrm{Hom}_G(\nabla(\lambda) \otimes X(\alpha)^F, -)$ to the short exact sequence

$$0 \to \Delta(\alpha-\rho)^F \otimes \Delta(\tilde{\lambda}+l\varpi) \to \Delta(\lambda+l\alpha) \to \Delta(\alpha)^F \otimes \Delta(\lambda) \to 0 \tag{13}$$

from (2). This gives an exact sequence

$$0 \to \mathrm{Hom}_G(\nabla(\lambda) \otimes X(\alpha)^F, \Delta(\alpha-\rho)^F \otimes \Delta(\tilde{\lambda}+l\varpi)) \to$$

$$\to \mathrm{Hom}_G(\nabla(\lambda) \otimes X(\alpha)^F, \Delta(\lambda+l\alpha)) \to \mathrm{Hom}_G(\nabla(\lambda) \otimes X(\alpha)^F, \Delta(\alpha)^F \otimes \Delta(\lambda)).$$

Let $f \in \mathrm{Hom}_G(\nabla(\lambda) \otimes X(\alpha)^F, \Delta(\lambda+l\alpha))$, and suppose that it is non-zero. By [19, 3.9, Theorem], the socle of $\Delta(\lambda+l\alpha)$ is simple, and hence must be contained in $\mathrm{Im}(f)$. By (13) the socle of $\Delta(\lambda+l\alpha)$ is also contained in $\mathrm{soc}(\Delta(\alpha-\rho)^F \otimes \Delta(\tilde{\lambda}+l\varpi))$ and hence $\nabla(\lambda) \otimes X(\alpha)^F$ and $\mathrm{soc}(\Delta(\alpha-\rho)^F \otimes \Delta(\tilde{\lambda}+l\varpi))$ must share a common composition factor.

By Steinberg's tensor product theorem [11, 3.2(5)] all composition factors of $\nabla(\lambda) \otimes X(\alpha)^F$ are of the form $\nabla(\lambda) \otimes L(\beta)^F$. However, setting $\eta = \alpha-\rho$, we have

$$\mathrm{Hom}_G(\nabla(\lambda)\otimes L(\beta)^F, \Delta(\eta)^F\otimes\Delta(\tilde{\lambda}+l\varpi)) \cong (\mathrm{Hom}_{G_1}(\nabla(\lambda), \Delta(\tilde{\lambda}+l\varpi))\otimes(L(\beta)^F)^*\otimes\Delta(\eta)^F))^G$$

and this is zero by arguments as in [3, Lemma 4.3]. Thus $\nabla(\lambda) \otimes X(\alpha)^F$ and $\mathrm{soc}(\Delta(\alpha-\rho)^F \otimes \Delta(\tilde{\lambda}+l\varpi))$ have no common composition factors, which gives the desired contradiction and completes the proof.

Combining the results of this section we obtain

Corollary 6.7 *Assume that θ and χ satisfy the conditions given before (6.4). Then the category $\mathcal{F}(\Delta)$ for $S_q(2, |\chi| - 2)$ contains infinitely many non-isomorphic indecomposable modules.*

We next show that, if $S_q(2, d)$ is Δ-infinite, then so is $S_q(N, d)$ for all $N \geq 2$. In order to prove this, we shall briefly consider once again the more general setting of quasi-hereditary algebras.

Suppose that A is a quasi-hereditary algebra with respect to the partially ordered set $(\Lambda, \leq)$ labelling the simple modules. Let Γ be a coideal in Λ — that is, that $\Lambda \backslash \Gamma$ is saturated. Consider a set of orthogonal primitive idempotents $\{e_\lambda : \lambda \in \Gamma\}$ in A, such that Ae_λ is indecomposable projective, with simple quotient $L(\lambda)$. Then, for $e = \sum_{\lambda \in \Gamma} e_\lambda$, the algebra $A_\Gamma = eAe$ is also quasi-hereditary, with weight poset Γ (see [12, 1.6 Lemma] for details). We wish to consider the functor induced by the map $X \longmapsto eX$ from A-mod to eAe-mod. In [12, 1.6 Lemma], it was shown that for $\lambda \in \Lambda$, we either have $e\Delta(\lambda) = 0$ if $\lambda \notin \Gamma$, or $e\Delta(\lambda)$ is the Weyl module for A_Γ of highest weight λ otherwise. We shall require the following slightly stronger result.

Theorem 6.8 *The above functor induces an equivalence of categories*

$$\mathcal{F}_\Gamma(\Delta) \cong \mathcal{F}_{A_\Gamma}(\Delta)$$

where $\mathcal{F}_\Gamma(\Delta)$ is the full subcategory of $\mathcal{F}_A(\Delta)$ with objects those modules all of whose Δ-quotients have highest weights in Γ.

Proof: This is [4, Theorem 1.2], an easy consequence of [7, Theorem 2].

Clearly $\mathcal{F}_B(\Delta)$ is equivalent to $\mathcal{F}_C(\Delta)$ for any pair of Morita equivalent algebras B and C. In conjunction with the last result this gives

Corollary 6.9 *If $S_q(n, d)$ is Δ-infinite, then so is $S_q(N, d)$ for any $N \geq n$.*

Proof: Take $\Gamma = \{\lambda \in \Lambda^+(N, d) : \lambda \text{ has at most } n \text{ parts}\}$. Then the complement of Γ in $\Lambda^+(N, d)$ is saturated. Taking $\bar{e} \in S_q(N, d)$ to be the idempotent as in [6, remarks after Corollary 8.9], we have that $\bar{e}S_q(N, d)\bar{e}$ is isomorphic to $S_q(n, d)$. Moreover, as $\bar{e}L(\lambda)$ is non-zero if and only if $\lambda \in \Gamma$ by [6], we also have that $\bar{e}S_q(N, d)\bar{e}$ is Morita equivalent to $S_q(N, d)_\Gamma$. The result now follows from the preceding proposition.

Let F^r denote the composition of the Frobenius morphism of Section 1 with the $(r-1)$th power of the classical Frobenius morphism, as in [2, Section 3]. Then we have

Lemma 6.10 *The functor* $\Phi : \mathrm{mod}\ \mathrm{GL}(n,k) \to \mathrm{mod}\ G$ *given by*

$$\Phi(V) = (q\text{-det})^a \otimes \mathrm{St}_r \otimes V^{F^r} \quad \text{and} \quad \Phi(\phi) = 1 \otimes 1 \otimes \phi$$

induces an equivalence of categories between appropriate blocks of the corresponding (q-)Schur algebras.

Proof: This is an easy generalisation of [9, Section 4, Theorem]. That Φ gives rise to a map between blocks has been shown in [2, Proposition 5.4]. For the equivalence of categories, the arguments in [9, Section 1(3) and Section 4, Theorem] also hold here, once we have proved an analogue of [17, 10.4 Proposition]. But the proof given there also holds in this setting, using (4.10).

To conclude this section, we combine (4.6), (6.7) and (6.9) to obtain

Corollary 6.11 *For* $n \geq 2$ *we have*
(i) $S_q(n,d)$ *is* Δ*-infinite if (a)* $l > 2$ *and* $d \geq 2lp + l - 2$*, or*
(b) $l = 2$ *and* $d \geq \begin{cases} 4p & \text{if } d \text{ even} \\ 4p^2 + 2p - 3 & \text{if } d \text{ odd.} \end{cases}$
(ii) $S(n,d)$ *is* Δ*-infinite if (a)* $p > 2$ *and* $d \geq 2p^2 + p - 2$*, or*
(b) $p = 2$ *and* $d \geq \begin{cases} 8 & \text{if } d \text{ even} \\ 17 & \text{if } d \text{ odd.} \end{cases}$

Proof: We prove part (ii), the quantum case is similar. First, we reduce to the case $n = 2$ using (6.9). For (ii)(a), it is enough to show, by (6.7), that for $d \in \{2p^2 + p, 2p^2 + p + 1\}$, there exist a pair of weights θ and χ such that $\mathrm{Ext}^2_G(\Delta(\theta), \Delta(\chi)) = k^2$, as the result then follows for $d \geq 2p^2 + p$ by tensoring up with an appropriate power of the determinant representation. But in these cases we can take $\theta = (p-2)\rho + (p^2+1)\varpi$ and $\chi = (2p^2+p)\rho$, and $\theta = (p-3)\rho + (p^2+2)\varpi$ and $\chi = (2p^2+p+1)\rho$, respectively.

We next consider (ii)(b). If $d = 2e$ with $e \geq 5$ take $\theta = e\varpi$ and $\chi = 10\rho + (e-5)\varpi$. Then we have $\mathrm{Ext}^2_G(\Delta(\theta), \Delta(\chi)) = k^2$ by (5.1), and the result follows from (6.7). Finally, suppose $d = 2e+1$, with $e \geq 8$. If $e = 8$ the result follows from the even case above and the classical analogues of (6.10) and [2, Lemma 5.1]. We obtain the result for $e > 8$ by tensoring up with an appropriate power of the determinant representation.

7 Further consequences

In this section we shall extend our results on Ext between Weyl modules to $G = q\text{-GL}(n,k)$, for certain special families of partitions. We shall also consider a related question concerning modules with a Specht filtration for Hecke algebras. We begin by recalling the following result from [11].

Theorem 7.1 *Let Σ be a subset of the simple roots of G, and G_Σ be the corresponding Levi subgroup (see [11, Section 4.2]). If λ and μ are partitions such that $\lambda - \mu \in \mathbb{Z}\Sigma$, then for all $i \geq 0$ we have*

$$\operatorname{Ext}^i_G(\Delta(\lambda), \Delta(\mu)) \cong \operatorname{Ext}^i_{G_\Sigma}(\Delta_\Sigma(\lambda), \Delta_\Sigma(\mu)).$$

When $|\Sigma| = 1$ and $i \leq 2$ the right-hand side can be obtained from (5.1) by identifying G_Σ with q-$GL(2,k)$.

Proof: See [11, 4.2(17)] (or [13, Section 4] in the classical case).

Assume that A is a quasi-hereditary algebra, with respect to the poset $(\Lambda, \leq)$ ordering the simple modules. Let T be a full tilting module for A, and $A' = \operatorname{End}_A(T)$ be the Ringel dual of A (which is unique up to Morita equivalence). Now A' is also quasi-hereditary, but with respect to $(\Lambda, \leq^{\text{op}})$. Further, the functor $\operatorname{Hom}_A(T, -)$ induces an equivalence $\mathcal{F}_A(\nabla) \to \mathcal{F}_{A'}(\Delta)$ which takes $\nabla(\theta)$ to $\Delta(\theta)$. From this we obtain

Proposition 7.2 *For θ,$\chi \in \Lambda$ and $s \geq 1$ we have*

$$\operatorname{Ext}^s_A(\nabla(\theta), \nabla(\chi)) \cong \operatorname{Ext}^s_{A'}(\Delta(\theta), \Delta(\chi)).$$

Proof: See for example [11, Proposition A4.8].

Taking $A = S_q(n,d)$ we have

Theorem 7.3 *Suppose θ and χ are partitions of d, and that $i \leq 2$.*
(i) If θ and χ have two rows then

$$\operatorname{Ext}^i_G(\Delta(\theta), \Delta(\chi)) \cong \operatorname{Ext}^i_{q-\mathrm{GL}_2(2,k)}(\Delta(\theta), \Delta(\chi)).$$

(ii) If θ and χ have two columns and at most n parts then

$$\operatorname{Ext}^i_G(\Delta(\theta), \Delta(\chi)) \cong \operatorname{Ext}^i_{q-\mathrm{GL}_2(2,k)}(\Delta(\chi'), \Delta(\theta')).$$

In each case the right-hand side is given by (5.1).

Proof: For polynomial representations of degree d we have $\mathrm{Ext}^s_G(-,-) = \mathrm{Ext}^s_A(-,-)$. Let Γ be the coideal of $\Lambda^+(n,d)$ such that $eAe \cong S_q(2,d)$ as in the proof of (6.9). By [11, Proposition A3.13] (or a small refinement of (6.8)) we have

$$\mathrm{Ext}^s_A(\Delta(\theta),\Delta(\chi)) \cong \mathrm{Ext}^s_{eAe}(e\Delta(\theta), e\Delta(\chi)).$$

But $e\Delta(\theta) \cong \Delta_{eAe}(\theta)$ for $\theta \in \Gamma$, and Ext^s_{eAe} is the same as $\mathrm{Ext}^s_{q-\mathrm{GL}_2(2,k)}$ for polynomial modules of degree d. This proves (i). Now (ii) follows from (i) and [11, Proposition 4.1.5(iii)] (as A is Morita equivalent to its own Ringel dual when $d \leq n$).

We have the following application to Specht modules for Hecke algebras. Details can be found in [11, Section 4.7] (or [12] in the classical case). For fixed $d > 2$, let $\mathcal{H} = \mathcal{H}_q(d)$ be the Hecke algebra corresponding to the symmetric group Σ_d over k, and set $A = S_q(2,d)$. Taking E to be the natural module for q-$\mathrm{GL}(2,k)$, there is an action of $\mathcal{H}$ on $E^{\otimes d}$ such that $\mathrm{End}_{\mathcal{H}}(E^{\otimes d})$ is isomorphic to A. Moreover, the natural map $\mathcal{H} \to \mathrm{End}_A(E^{\otimes d})$ is surjective, and hence $\mathrm{End}_A(E^{\otimes d}) \cong \mathcal{H}/I$ where I is the kernel of the above action.

Corollary 7.4 *Let θ and χ be partitions of d with at most two parts. Then unless d is even and either $l = 1$ and $p = 2$, or $l = 2$, we have*

$$\mathrm{Ext}^2_{\mathcal{H}/I}(S^\chi, S^\theta) \cong \mathrm{Ext}^2_A(\Delta(\theta),\Delta(\chi))$$

and the right-hand side is given by (5.1).

Proof: Let $T = E^{\otimes d}$, which is a tilting module for A. The functor $\mathrm{Hom}_A(T,-)$ takes $\nabla(\theta)$ to the Specht module S^θ, if θ is a partition of d with at most two parts. Under the given hypotheses, T is a full tilting module for A, and hence $A' = \mathcal{H}/I$ is a Ringel dual of A. Further, $\Delta_{A'}(\theta)$ is identified with the Specht module S^θ. The result now follows from (7.2).

Note that the hypotheses of the last result cannot be omitted. For example, take $p = 2$ and $l = 1$; then $\mathcal{H}/I$ is isomorphic to $k\Sigma_2$. In this case we have $S^{(2)} \cong k$, and $\mathrm{Ext}^s_{\Sigma_2}(k,k) \neq 0$ for all $s \geq 1$. However, for a quasi-hereditary algebra $\mathrm{Ext}^s_A(\Delta(\theta),\Delta(\theta)) = 0$ for all $s \geq 1$ and $\theta \in \Lambda$.

It is natural to ask whether Ext^s between Specht modules is the same for $\mathcal{H}/I$ and $\mathcal{H}$. If $s = 1$ and $l > 2$ then this follows from [11, 4.7(5)]. In the case $s = 2$ we have

Proposition 7.5 *Let $R = \mathcal{H}/I$, and θ and χ be partitions of d with at most two parts. Then there is a canonical inclusion*

$$\mathrm{Ext}^2_R(S^\theta, S^\chi) \longrightarrow \mathrm{Ext}^2_{\mathcal{H}}(S^\theta, S^\chi).$$

Proof: Suppose that A is any ring with ideal J, and let $B = A/J$. If W is a B-module such that $\mathrm{Ext}^1_A(B, W) = 0$, then it is easily verified that $\mathrm{Ext}^1_B(-, W)$ agrees with $\mathrm{Ext}^1_A(-, W)$ on $\mathrm{Mod}(B)$. Further, this condition is satisfied if $\mathrm{Hom}_A(J, W) = 0$. Thus to show that

$$\mathrm{Ext}^1_R(-, S^\chi) = \mathrm{Ext}^1_{\mathcal{H}}(-, S^\chi) \tag{14}$$

on $\mathrm{Mod}(R)$, it is enough to show that $\mathrm{Hom}_{\mathcal{H}}(I, S^\chi) = 0$. For this we use the inclusion $\mathrm{Hom}_{\mathcal{H}}(I, S^\chi) \to \mathrm{Hom}_{\mathcal{H}}(I, M^\chi)$, where M^χ is the q-permutation module corresponding to χ. This module is filtered by S^γ where $\gamma \notin \Lambda^+(2, d)$. By [6, 8.7 Corollary] (or [15, Theorem 13.13] in the classical case), we have $\mathrm{Hom}_{\mathcal{H}}(S^\gamma, M^\chi) = 0$, and hence (14) follows.

Let P_R be a projective cover of S^θ as an R-module, with corresponding kernel Ω_R. Applying $\mathrm{Hom}_{\mathcal{H}}(-, S^\chi)$ to

$$0 \to \Omega_R \to P_R \to S^\theta \to 0$$

we get an exact sequence

$$\cdots \to \mathrm{Ext}^1_{\mathcal{H}}(P_R, S^\chi) \to \mathrm{Ext}^1_{\mathcal{H}}(\Omega_R, S^\chi) \to \mathrm{Ext}^2_{\mathcal{H}}(S^\theta, S^\chi) \to \mathrm{Ext}^2_{\mathcal{H}}(P_R, S^\chi).$$

The first term is zero by (14). Again by (14), we have $\mathrm{Ext}^1_{\mathcal{H}}(\Omega_R, S^\chi) \cong \mathrm{Ext}^1_R(\Omega_R, S^\chi)$ and this equals $\mathrm{Ext}^2_R(S^\theta, S^\chi)$ by dimension shift. This completes the proof of the proposition.

Remark 7.6 *This inclusion need not be an isomorphism. For example, take $p = d = 3$ in the classical case. If $\theta = (2, 1)$ then S^θ is projective as an R-module, and hence $\mathrm{Ext}^2_R(S^\theta, -) = 0$. On the other hand, $\mathrm{Ext}^2_{\mathcal{H}}(S^\theta, S^\chi) \cong \mathrm{Ext}^1_{\mathcal{H}}(\Omega(S^\theta), S^\chi)$, and $\Omega(S^\theta) \cong D^\theta$. Taking $\chi = (3)$ we have $S^\chi \cong k \cong \mathrm{rad}(S^\theta)$, and hence $\mathrm{Ext}^1_{\mathcal{H}}(\Omega(S^\theta), S^\chi) \neq 0$.*

Finally, we use our results on Δ-infinite Schur algebras to show that certain categories of modules with a Specht filtration have infinite representation type. Define $\mathcal{F}_d(\mathrm{Sp}_{\leq n})$ to be the full subcategory of $\mathrm{Mod}(\mathcal{H})$ whose objects are all modules with filtrations by Specht modules S^θ for partitions θ of at most n parts.

Corollary 7.7 *The category $\mathcal{F}_d(\mathrm{Sp}_{\leq n})$ is of infinite type for all $n \geq 2$ in each of the following cases:*
(i) $l > 2$ and $d \geq 2lp + l - 2$.
(ii) $l = 2$, d odd, and $d \geq 4p^2 + 2p - 3$.
(iii) $l = 1$, $p > 2$, and $d \geq 2p^2 + p - 2$.
(iv) $l = 1$, $p = 2$, d odd, and $d \geq 17$.

Proof: Let $A = S_q(2, d)$, and $\mathcal{H}/I$ be as above. As in (7.4), we have that $\mathcal{H}/I$ is isomorphic to a Ringel dual of A, and hence that $\mathcal{F}_{\mathcal{H}/I}(\mathrm{Sp}_{\leq 2})$ is equivalent to $\mathcal{F}_A(\nabla)$ by (7.2). This is dual to $\mathcal{F}_A(\Delta)$, and hence is of infinite type in the cases listed by (6.11). The result now follows as $\mathcal{F}_{\mathcal{H}/I}(\mathrm{Sp}_{\leq 2})$ is a full subcategory of $\mathcal{F}_{\mathcal{H}}(\mathrm{Sp}_{\leq n})$.

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