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AN ANALYSIS OF COMBINED NATURAL AND FORCED CONVECTION
AND OTHER PROBLEMS IN INTERNAL LAMINAR FLOW

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A thesis presented for the degree of
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ABSTRACT

Internal developing laminar flow is described by the full partial differential equations in two-dimensional cylindrical co-ordinates. Making assumptions only about property variations, these are expressed as finite-difference equations, which are treated by a computer program in Fortran. This involves a marching procedure and Gaussian elimination. A feature of the approach is that the energy balance at any axial position is guaranteed by using it as a constituent equation, rather than as a check.

The predictive ability of this treatment has been demonstrated firstly in adiabatic flow. Then it has been rigorously tested in a number of forced and combined convection situations against both analytical and experimental data. These cover a range of thermal (constant heat flux and constant temperature) and fluid (undeveloped, partially - or fully - developed entry flow) boundary conditions; they involve working fluids which include air, water, and oils. In general, agreement is consistently good.

In combined convection, the phenomenon of transition from laminar flow is studied, for the differing cases of natural convection assisting or opposing the forced flow. The various processes accompanying transition are identified, and comparison made with published data.

The complete nature of the equations used means it should be possible to use the model as a check on more approximate analyses. This has been carried out successfully in an instance involving heat transfer to transformer oil.

The problem of viscous dissipation demonstrates the flexibility of this approach, in its ability to treat the complete relevant terms in the energy equation. A study of this problem shows that viscous dissipation is usually a minor effect, tending to retard heat transfer.

Flexibility is also apparent in the application of the treatment to a concentric annulus. Initial results are for adiabatic developing flow.

Care has been taken to maintain the accuracy of the results for the wide range of situations reported. To this end, a number of improvements have been made in the original program.

Relative importance of heat-transfer phenomena

In this study, most applications are within the range of entry effects which themselves increase heat transfer.

Relative to such increase, the combined convection effect depends on the fluid. With air the effect is not large, but should not be neglected. With water the increase can be substantial, but with oils it is small. These comments apply only to vertical flows, the effect being of no significance in the horizontal cases considered.

With regard to property variations, the most important is that of viscosity. With air the effect of this is very small, with water small, but with oils substantial. Further property variation within the energy equation has little effect on heat transfer.

In conclusion, the results imply that the allowances made for density and viscosity variation in this work are sound ones.

NOMENCLATURE

A	Wall area
C_{01}	$C_{04} = C_4 - C_5 T_o'^2 + C_6 T_o'^2$
C_{02}	$C_{05} = 2T_o'(C_6/C_5) - 1$
	----- * {Viscosity variation parameters, defined in text. -----
C_{03}	$C_{06} = \frac{C_6}{C_5^2 G}$
C_p	Specific heat of fluid at constant pressure.
G	$= R^3 g / \nu_A^2$ dimensionless.
Gz	Graetz number $= \gamma \times Pe / Z$
g	Acceleration due to gravity.
k	Thermal conductivity of fluid.
M	Number of radial divisions.
m,n	Radial and axial positions.
P	Dimensionless pressure $= (p - p_o) / \rho_m U_o^2$
Pe	Peclet number $= Re \times Pr$
Pr	Prandtl number $= \mu_A (C_p / k)_m$
p	Pressure.
Q	Wall heat flow rate.
R	Tube radius.
Re	Reynolds number based on radius $= U_z R \rho / \mu_A$
r'	Radial co-ordinate.
r	Dimensionless radial co-ordinate $= r' / R$
S	Natural Convection parameter, defined in text.
T'	Temperature.
T	Dimensionless temperature $= GC_5 (T' - T_o')$
U	Dimensionless axial velocity $= U_z / U_o$
V	Dimensionless radial velocity $= U_r / U_o$
U_r, U_z	Radial and axial velocities.

- z' Axial co-ordinate.
- Z Dimensionless axial co-ordinate = z'/R .
- μ, ν Dynamic, kinematic viscosities.
- ρ Density.

Subscripts

- A Property reference point, equations (5) and (6).
- L Centre line.
- m Mean bulk value (bulk value, midway between entry and exit).
- R Wall condition.
- T Transition point.
- r, z Radial and axial directions.
- O Entrance condition.

Finite Difference Parameters

HFLUX Dimensionless wall heat flux $\left\{ = \frac{Q}{A} \cdot \frac{G \cdot C_p R}{M k} \right\}$

QDISS Dimensionless viscous dissipation parameter $\left\{ = \frac{G C_p U_o^2}{C_p Re} \right\}$

Alternatives for constant density

T $\left\{ = \frac{G}{T_o} (T' - T_o') \right\}$

HFLUX $\left\{ = \frac{Q \cdot G R}{A T_o' M k} \right\}$

QDISS $\left\{ = \frac{G U_o^2}{T_o' C_p Re} \right\}$

ADDITIONAL

- | | | |
|------------|---|---------------------|
| f_{av} | Average (pressure drop) friction factor | } defined on p.55 |
| f_p | Local (pressure drop) friction factor | |
| f_s | Local (shear stress) friction factor | |
| Nu | Local Nusselt number | defined on pp.53,54 |
| T_1, T_2 | Velocity profile parameters defined on pp.101,106 | |

PART I

INTRODUCTION

1. GENERAL INTRODUCTION

Heat transfer by convection between a solid surface and a fluid has two modes, 'natural' and 'forced'. The force which causes fluid movement is due respectively to density differences (Jakob (1) pp 12-13) or to external means (e.g. a pump).

Three convection regimes, therefore, can be identified: natural, forced and combined natural and forced convection. The first two regimes have covered most cases of interest in the past. However, combined convection has become more important in recent years (e.g. in emergency cooling of nuclear reactors, and in cooling of electrical transformers) and research is needed in this field (Métais and Eckert (2)).

Combined convection is of considerable analytical and experimental interest, because the relevant parameters include all those which affect natural and forced convection independently. The more important parameters are:

- (a) the condition of the convection. This may vary from 'pure' natural (forced effects absent) to 'pure' forced (natural effects negligible, though always present).
- (b) the direction of each convective mode. The forced flow may be at any angle to the free (usually vertical) convection, horizontal and vertical flows being of most interest.
- (c) the geometry. The flow may be internal or external, with a variety of boundary shapes.
- (d) the flow condition, which may be laminar, transitional or turbulent.

A comparative investigation of the main parameters does not exist, and would aid basic understanding of the combined convection process. The intention of this treatment is to deal respectively with:

- (a) forced and combined convection,
- (b) any orientation (forced convection), vertical flows
(combined convection),
- (c) an internal cylindrical co-ordinate flow-field,
- and (d) laminar flow.

2. LITERATURE SURVEY

2.1 Introduction

The body of literature on combined convection has grown increasingly in recent years, and covers both analytical and experimental aspects. Despite, or perhaps, because of, the broadness of the subject, there are few references of general interest and it is necessary to synthesise the remainder.

2.2 Boundaries of Regime

One preliminary question that arises is that of defining possible boundaries between the three regimes mentioned. Several attempts have been made, viz: Ostrach (3), by dimensional deduction from the momentum equation, Sparrow, Eichhorn and Gregg (4) by similarity boundary layer analysis, and Sparrow and Gregg (5), and Eshghy (6), by analysis of convection from a vertical flat plate. The main practical use of these criteria is to show when existing correlations for natural and forced convection regimes cannot be used without much error (see also (2)).

2.3 Dimensional Analysis

If this problem is tackled by dimensional analysis (e.g. Schlichting (7) p. 254), it is found that the Nusselt Number is a function of the Reynolds, Grashof and Prandtl Numbers, and a geometrical dimensionless group (e.g. for a pipe, the length-to-diameter ratio). The Grashof and Reynolds Numbers are basically independent as representatives of natural and forced convection respectively, but it has been shown by Brown and Marco (8) p. 111, and partly modified by Jakob (1) pp 492-493, that for convection from a vertical flat plate, the square root of the Grashof Number is equivalent to the Reynolds Number. Jackson, Harrison and Boteler (9) developed this idea by using the heat transfer coefficient from a forced convection relationship in macroscopic free convection energy and momentum balances.

The two flow effects are vectorially combined, the correlation giving reasonable comparison with experimental data. This is further treated by Jackson, Spurlock and Purdy (10).

2.4 Theoretical Analysis

A full analytical treatment demands simultaneous solutions of the partial differential equations for conservation of mass (continuity), momentum (Navier-Stokes equations) and energy. These are stated and discussed in Chapter 5.

The complex nature of the equations has resulted in various simplifying assumptions (e.g. Martinelli and Boelter (11) p. 23), made at first more to obtain a solution, than on justifiable theoretical grounds. As time has passed, the magnitude of the assumptions has decreased. The present availability of highspeed digital computers and the use of finite-difference techniques has reduced the necessity for simplified equations, and approximations can be viewed more objectively.

Most of the combined convection problems are for vertical geometries - normally a tube or flat plate. In these cases the forced and natural convections are in line, thus permitting a two-dimensional model.

2.5 Fully-developed Flow

The major common simplifying assumption has been that a fully-developed flow. This postulates a region far from the entrance where there is no variation in velocity or temperature patterns. This implies an essentially one-dimensional treatment.

The first analytical solution was formulated by Martinelli and Boelter (11) in 1942. The simplified equations were:

$$0 = -\frac{\partial p}{\partial z} \pm \rho g + \mu \left(\frac{\partial^2 v_z}{\partial r^2} \right) \quad \text{c.f. equn (2)}^*$$

$$\text{and} \quad v_z \frac{\partial T}{\partial z} = \frac{k}{\rho c_p} \left(\frac{\partial^2 T}{\partial r^2} \right) \quad \text{c.f. equn (4)}$$

where $\frac{\partial p}{\partial z}$ and wall temperature, are constant.

* In this chapter, dimensional variables are unprimed.

Basically, the first equation was solved for U_z as a power series in $(R - r)$. The value of $\frac{\partial U_z}{\partial r}$ at the wall was substituted into the energy equation, giving a temperature solution. Hence the wall heat transfer coefficient was calculated, resulting in a Nusselt Number as a fairly complex function of the Graetz, Grashof and Prandtl Numbers. The analysis was virtually limited to the wall, so questions of flow field and instability could not be touched on. More recently, Pigford (12) allowed for temperature variation of viscosity, using the same model.

A large amount of analysis was carried out by Ostroumov (13), (14), (14)A, dealing with fully-developed velocity and temperature profiles for constant wall heat flux. Hallman (15) summarised and extended the above, in one point correcting Ostroumov (13). The equations considered

were:
$$0 = -\frac{\partial p}{\partial z} \pm \rho g + \mu \left(\frac{1}{r} \cdot \frac{\partial U_z}{\partial r} + \frac{\partial^2 U_z}{\partial r^2} \right) \quad \text{c.f. equn (2)}$$

and
$$U_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left(\frac{\partial^2 T}{\partial z^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \right) + \frac{q_w}{\rho C_p} \quad \text{c.f. equn (4)}$$

Velocity and temperature profiles were solved as Bessel functions, and Nusselt Numbers calculated.

Hanratty, Rosen and Kabel (16) considered the same equations (omitting last two terms of energy equation) for constant wall heat flux. Bessel functions were again used, and at certain values of (Grashof/Reynolds Number) ratio results were correlated with experimental evidence of transition to a form of turbulence.

Morton (17) solved the equations:

$$0 = -\frac{\partial p}{\partial z} \pm \rho g + \mu \left(\frac{\partial^2 U_z}{\partial z^2} + \frac{1}{r} \cdot \frac{\partial U_z}{\partial r} + \frac{\partial^2 U_z}{\partial r^2} \right)$$

$$U_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left(\frac{\partial^2 T}{\partial z^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \right)$$

Upward and downward flows were treated, and annular (i.e. partially reversed) flows predicted, being considered probably unstable. Velocity and temperature profiles were presented.

Brown and Grassman (18) and (19) also obtained fully-developed velocity and temperature profiles for constant wall heat flux, and considered mass transfer in addition. Madijski (20), using Bessel functions, obtained local and average Nusselt Numbers, mainly for liquid flows.

2.6 Other Vertical Geometries

A not insubstantial amount of work deals with vertical geometries other than cylindrical co-ordinate internal flows. The significance here depends on the basic assumptions.

Han (22), Tao (23) and Agrawal (24) treated the problem of a rectangular channel with constant heat flux, where the fluid had internal heat generation. Different methods of solution were used.

Parallel-plate analyses have been attempted, by Ostrach (21), with constant heat flux, and Rao and Morris (25) where the plates were adiabatic/constant heat flux. Sitharamarao and Barrow (27) applied a single vertical plate (of constant temperature) analysis to this case, using a similarity solution.

Regular polygonal ducts have been treated by Iqbal et al (26), as an instance of varied geometry.

External flows have been attempted, by Acrivos (28) (vertical flat plate as particular instance of an inclined plate) and Hatton and Woolley (29) (vertical flat plate).

Finally Zeldin and Schmidt (36) mention instances of the effect of a vertical magnetic field between parallel plates (their references 19 - 21).

With the exception of (27) and (29) the fully-developed concept is used above, and its validity will now be examined.

2.7 Entrance Flow Analyses

The assumption of fully-established flow has never been really justified. This is admitted by Hallman (60) and evident from experimental data of Hanratty, Rosen and Kabel (16). It is criticised by Chato and Lawrence (32) because modest temperature changes can cause significant Rayleigh Number changes. Relevant references are discussed below.

Sparrow and Gregg (5) and Eshghy (6) analysed flows between vertical parallel plates for the entrance region, it being claimed that the results were applicable to pipes. The velocity and temperature fields were not investigated, however, the results being presented as regime boundary criteria. The similar form of momentum and energy equations as used above was evident, and Sparrow, Eichhorn and Gregg (4), and Brindley (30) have considered similarity solutions. The geometrics were a wedge and contoured wall (Sparrow et al), and a heated vertical wall (Brindley).

Parallel development of two-dimensional treatments occurred in forced convection (e.g. Siegel, Sparrow and Hallman (48) and natural convection (e.g. Bodoia and Osterle (31)).

There now exist several treatments of a developing model, by Rosen and Hanratty (34), Lawrence and Chato (33), and Marner and McMillan (38), the equations considered being:

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0 \quad (\text{Continuity})$$

$$\rho \left(v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial p}{\partial z} - \rho g + \mu \left(\frac{1}{r} \cdot \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial r^2} \right) + \left(\frac{\partial \mu}{\partial r} \cdot \frac{\partial v_z}{\partial r} \right) \quad (\text{Axial Momentum})$$

$$v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left(\frac{1}{r} \cdot \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \right) \quad (\text{Energy})$$

Rosen and Hanratty obtained approximate solutions by assuming temperature and velocity profiles with axially-dependent coefficients, and substituting these into the differential equations. Distances for transition to turbulence were predicted adequately where this occurred far from the entrance. However, as the authors stated, velocity and temperature fields, and hence heat transfer results, were not sufficiently accurate.

Lawrence and Chato treated the equations numerically and velocity and temperature profiles were calculated at increasing distances from entry. Centre-line velocities compared well with experiment. Only upward flow with constant wall heat flux was considered, and the necessity of a developing model was demonstrated.

The fluid treated in both cases above was water and the common conclusion was that temperature dependence of density (in buoyancy term) and viscosity were both necessary.

Although a later paper, Marner and McMillan assumed constant viscosity ($\bar{\mu}$ in equations zero), and compared mean Nusselt Numbers with the original Martinelli-Boelter correlation. Developing velocity and temperature profiles (for unity Prandtl Number) were also presented. The wall was at constant temperature.

Also relevant is the paper by Scheele and Hanratty (35) which presents analytical fully-developed profiles of Hallman (15), Hanratty et al (16), and Morton (17) together with results of comprehensive dye-injection experiments. Although the premise of fully-developed flow is accepted, this was not found completely possible in practice.

Under what conditions can flow fields be said to be genuinely developed according to these data?

Firstly, if the wall is at constant temperature, the velocity profile initially becomes distorted (c.f. (38)), and then recovers as the fluid temperature approaches that of the wall.

Finally, the whole field becomes isothermal and hence adiabatic. A developing model should predict the tube length this takes, whereas the one-dimensional treatment will not be valid except at the tube boundaries.

Alternatively, if the wall is at constant heat flux the density and viscosity (in particular) are continuously changing, and hence affecting the forces in the momentum equation, distorting the velocity profile. A developing model is necessary to describe this. It is, however, possible to conceive of property variations being such that the momentum equation remains balanced in its terms.

However, a one-dimensional solution which varied, say, with Rayleigh Number might give reliable point values, if one could relate these to tube length. On balance, it is difficult to justify objectively a simpler model from choice.

An additional point is the possibility of flow reversal and transition to turbulence, e.g. (33) and (35). A laminar flow developing analysis should shed light on this problem.

Finally, there exists another analysis (Takhar (39) and (40)) using the same equations as Marner and McMillan, for upward heated (39) and cooled (40) flow. Results are consistent in principle with other data, although no comparisons are made.

2.8 Elliptic Equations

The above logic, however, may be taken further. The full partial differential equations are elliptic, but the boundary layer assumptions render them parabolic (Bayley et al (37) pp 166-172). Lawrence and Chato mention that discrepancies in their analytical/experimental comparison near tube entry, may be due to this common set of assumptions. In fact, elliptic treatments for adiabatic developing flow have been introduced, and will be discussed in Chapter 8.

To progress to the elliptic equation model appears to be the next logical step, and Zeldin and Schmidt (36) have attempted this. Again, developing flow profiles were calculated, and a comparison made with experimental results for air. The constant viscosity limitation was presumably taken to be valid for a gas. The solution was for constant wall temperature, in terms of potential functions, and used a fully developed boundary condition at axial length ∞ . The particular comparison did not appear consistent with full development at exit.

2.9 The Internal Cylindrical Flow-field

While most of the literature surveyed is for tubes, the internal cylindrical flow-field covers other cases as well. The basic equations are identical, but the geometry differs according to the boundary conditions: for example, a tube is formed by (i) a wall and (ii) an axis.

Two references are relevant to sector tubes, viz Lu (41) and Tao (42). However, this problem, while satisfying the definition, does not have axial symmetry, which the present model requires.

Carter and Gill (43) and Gill et al (44) treated the case of the porous wall. This was obtained by making the radial velocity finite instead of zero at the wall. The model used in (43) was a fully-developed one.

The analytical work of Sherwin and Wallis, (45) - (47b) covered the case of a vertical annulus, formed by (i) an inner wall, (ii) an outer wall, as boundary conditions. The analytical models progressed from a one- to a two-dimensional treatment, and in (47b) the equations are the constant viscosity form used by the entrance flow analysts. Water only was considered.

This brief consideration shows that a theoretical model which permits boundary condition changes could be of considerable use.

2.10 Variable Property Effects

With such a broad subject as combined convection, it is not surprising to find relevant literature under a different subject, and it is possible to conceive of the topic as a special case of convection with variable physical properties, namely, variable density, since this gives rise to natural convection. Once again, there is a logical progression in the analytical models attempted.

Siegel, Sparrow and Hallman (48), Ulrichson and Schmitz (49), and Yang (50), deal with a developing model, the two former assuming constant properties, the latter with varying viscosity. These, therefore, are forced convection analyses.

Others are true combined convection analyses. Bradley and Entwistle (51), and Koppel and Smith (52) both attempted a fully-developed model with all properties varying, the former for air and the latter for Carbon Dioxide. Worsøe-Schmidt and Leppert (53) used a developing model with all properties varying (applied to air), and even considered viscous dissipation effects. The boundary layer assumptions only are made, thus making this work one of the most general analyses.

2.11 Experimental Work - Vertical

Turning to experimental data, the variables of major importance may be summarised as below.

- (a) The geometrical arrangement. Firstly, this determines the direction of the forced convection, and hence the angle made with the (vertical) free convection. The geometry may be internal or external, in addition. These cases are all vertical.
- (b) The forced-flow and heating or cooling rates. These jointly determine the relative magnitudes of the forced and free convection effects, and the flow character, whether laminar, transitional or turbulent.

(c) The boundary conditions. These may be thought of in terms of the flow conditions at entry, and the wall thermal condition — constant heat flux or temperature.

(d) The fluid. Significant differences may be expected between the behaviour of liquids and gases, because of the importance of the density and viscosity variation with temperature.

The bulk of vertical data is for circular pipes, viz: Watzinger and Johnson (54), Martinelli et al (55), Clark and Rohsenow (56), Eckert et al (57), (58), Hanratty, Rosen and Kabel (16), Jackson, Harrison and Boteler (9), Scheele, Rosen and Hanratty (59), Hallman (60), Kemeny and Somers (61), Scheele and Hanratty (35), and Lawrence and Chato (33). Guerrieri and Hanna (62) gave data for parallel plates and Altman and Staub (63) for rectangular ducts. Some free convection effects were noted by Poppendiek (64) and Gross and Van Ness (65) in laminar forced convection work. Data for natural circulation loops for Freon - 12 (Holman and Boggs (66)) and water (Van Putte (67)) indicated the inadequacy of an equivalent purely forced convection correlation.

For a developing study, length-to-diameter ratios are an important geometrical feature, and these covered a wide range, from 5 (58) and 15 (9) to 762 (35). Actual lengths ranged from 9.4 inches (56) to 50ft. (35), and the bulk of data was for tube diameters of 1 inch or less.

Given a vertical geometry four clear cases of combined convection arise, two where free convection assists the forced, and two of opposition. The former occurs with heated upward or cooled downward flow, the latter with opposite effects. All four cases have been treated, but not in the same body of work - except for the visual results of Hanratty et al (16).

The flow character was usually laminar or transitional. The comprehensive dye injection work of Hanratty, Kabel, Rosen and Scheele (4 references) give useful data on transition. Unusual effects (e.g.

(56) and (63) have been attributed to transition phenomena.

Boundary conditions covered a wide range. Calming lengths varied from zero to as long as the heat transfer section itself. Wall heating was normally either constant temperature (e.g. steam), or constant heat flux (e.g. electrical resistance heating).

Fluids tested have been mainly the liquids water and types of oil (55). The only gas tested has been air.

The analysis of results has usually been in terms of dimensionless groups. The earliest correlations were empirical (Colburn (68) and Sieder and Tate (69)) and Martinelli and Boelter's correlation has also been used.

Most of the data is undoubtedly useful for some performance purposes. There is, however, only limited data for the flow fields, viz: temperature profiles (Wetjen (70), Brown (19), Lawrence and Chato (33), and Zeldin and Schmidt (36)) and velocity profiles (36). For more fundamental understanding of the combined convection process, leading to reliable predictions, more data are required.

2.12 Non-vertical Geometries

If the geometry is other than vertical, the flow will be asymmetrical in a circular tube. This leads to a 3-dimensional requirement for a developing model.

Most non-vertical cases are horizontal, where the free convection is at 90° to the forced. Analytical references for tubes may be found in a general heat transfer review (Eckert et al (71)) also Morton (72), and Shannon and Depew (75). Experimental data are also available - Jackson, Spurlock, and Purdy (10), McComas and Eckert (73), Shannon and Depew (74), and Depew and August (76). An overall correlation is presented in the latter paper, for shorter tubes.

Data also exists for horizontal rectangular tubes (Altman and Staub (63) and Cheng and Hwang (77)), forced laminar flow over a

heated horizontal plate, Mori (78), and convection from spheres in air, Yuge (79).

2.13 Discussion

The literature net has been cast fairly wide, partly because of the lack of a general review. That of Metais (80) is not fully comprehensive or recent enough.

Despite the large number of sources, only the more recent analytical references, with a small number of experimental ones, help towards a fundamental understanding of combined convection effects.

Only where the analytical model becomes two-dimensional do results become of general value; in addition, much data is for arbitrary Prandtl Numbers rather than real fluids, which makes comparisons awkward. Similarly, since (as will be seen) the entry temperature is a necessary variable, work which omits this consideration is difficult to assimilate. Sufficient evidence exists, however, to suggest that a solution of the full elliptic equations could give better results at (i) extreme entrance and (ii) the transition regions.

Turning to property variation, any solution claiming generality should include the temperature variation of viscosity. Density variation is normally allowed only in the buoyancy term (of the axial momentum equation); this condition is sometimes termed 'quasi-incompressible'. Usually other properties are assumed constant, because of the modest temperature changes frequently associated with combined convection, unlike specific 'property variation' which involves large temperature variations. For our purposes, three classes of fluids may be specified:

- (i) Gases - low viscosity, but temperature variation in opposite sense to liquids. Example: air.
- (ii) Low viscosity liquids. Example: water.
- (iii) High viscosity liquids. Example: oils.

As mentioned, a cylindrical co-ordinate treatment which allows

boundary changes can be made applicable to internal flows other than the standard tube. Also, when the buoyancy term is omitted, the solution becomes a forced convection one; this relates the model to a further body of data.

Finally, a comment on experimental data. One or two references (namely Scheele and Hanratty (55) and Kemeny and Somers (61)) give sufficient overall information to permit an analytical prediction of the heat transfer results obtained. Apart from these, there are few sources which give temperature and velocity profile data (e.g. Lawrence Chato (33)). In toto, there does seem sufficient evidence to provide comprehensive checks for a unified analysis.

3. PHILOSOPHY OF APPROACH

3.1 Purposes of the Analysis

The main purposes of this analysis may be stated as follows:-

- (a) to give a unified treatment of laminar combined convection,
- (b) as a consequence of (a), to be able to deal with a comprehensive range of problems,

and (c) to be as accurate as possible for each such problem.

The literature survey has shown that the first purpose has not been achieved to date. The nearest approach to it is in the work of Scheele, Hanratty, Rosen and Kabel ((16), (34), (35), (59)) but there the concentration is mainly on transition studies. Other analyses are usually restricted to (i) the limited conditions, and (ii) the working fluid, of the accompanying experimental work. For example, Lawrence and Chato do not test their transition conclusions against the work of Scheele et al.

The second aim is comprehensiveness in predictive ability. Zeldin and Schmidt's model, for example, will probably be inappropriate for (i) liquids, because it assumes a constant viscosity, and (ii) transition studies, because it requires an outlet boundary condition.

Finally, accuracy in each problem is an important aim, especially since reliability in one application is not a guarantee of reliability in another. This implies as full an analytical treatment as possible, which, in its turn, opens up the possibility of achieving better reconciliation between theory and experiment. In one or two instances (e.g. ⁽⁶¹⁾ Kemeny and Somers), experimental results are not accompanied by corresponding theory, which gives particular scope for the analysis.

3.2 Features of the Analysis

3.2.1 Generality of Equations

The above reasons require the theory to be as general as possible, and apart from property variations, the full set of partial differential

equations are used in two-dimensional cylindrical co-ordinates. The only strictly comparable model is that of Zeldin and Schmidt (36).

3.2.2 Variables used

There is a choice here. Velocities and pressures may be used as variables (as Lawrence and Chato (33)), or stream function and velocity potential (as Zeldin and Schmidt (36)). In addition the dimensions may be untransformed, or transformed ((33) or (36)). At the time, only the former (33) was published; as it seemed quite successful, this work used the same concepts. In retrospect, this expression of the variables has permitted all the changes needed for the various applications.

3.2.3 Elliptic equations

The full equations are elliptic, by virtue of the double derivative terms in the axial direction, that is, the axial diffusion of heat and momentum. If these terms are omitted, the equations become parabolic; the boundary layer assumptions, on the grounds of 'order of magnitude' of terms ((37), pp 166-172) result in not only the above terms being ignored, but also the radial momentum equation.

A further consideration is most important, however, involving the modes of solution of finite-difference equations. (Morse and Feshbach (81), pp 696-706). For parabolic equations with an open boundary a unique stable solution is possible in the positive direction, i.e. a marching procedure is permitted from inlet to outlet of a tube. For elliptic equations with an open boundary the solution is unstable, a closed boundary being required, i.e. giving an iterative solution.

There is a conceptual problem, therefore. Let us say we wish to study transition phenomena. Rosen and Hanratty (34) admit that for shorter tubes 'the boundary-layer assumptions are probably poor'. If we then use the elliptic treatment of Zeldin and Schmidt (36), however, we find near transition states that 'no numerical solutions could be

obtained.....in the sense that the iteration algorithm could not be forced to converge'. To adhere strictly to the above criteria, the only answer would be to use the maximum possible parabolic equations, that is, to omit only the axial diffusion terms. In fact, to anticipate the results, it was found quite acceptable to use the complete equations and a marching procedure. This implies that the equations are only weakly elliptic within the bounds of this study.*

A final point is that the entry boundary condition is a Cauchy one ((81) p. 706) demanding 'slope' as well as 'value'. This point is discussed in Chapters 8 (momentum) and 12 (heat).

* At one stage, the axial diffusion terms were omitted, no difference being found in the results.

3.2.4 Properties

Ignoring the energy equation, the two properties of interest are viscosity and density.

Viscosity variation with temperature is particularly important for liquids, and is allowed for in full. In fact, the program accommodates any function of temperature which may be differentiated.

Density variation with temperature is allowed for, as a quadratic function, only in the buoyancy term of the axial momentum equation. This limitation is sometimes termed 'quasi-incompressible' and was originally suggested by Boussinesq.

The energy equation properties are assumed constant in the 'Standard' Program, but allowed to vary linearly with temperature in the 'Additional Property Variation' Program. This latter was specifically to compare predictions with Allen's results (Chapter 10), but it could then also be used as a check on other results.

In the 'Viscous Dissipation' extension to the Standard Program, viscosity is allowed to vary fully.

3.3 Strategy of the Analysis

3.3.1 Introduction

The strategy of the analysis follows a logical progression. A study of adiabatic flow basically tests the continuity and momentum equations, and then forced convection tests the energy equation, both against a substantial body of data. The investigation can then proceed to combined convection via the buoyancy effects.

Two other recent bodies of work follow a similar path. The integral profile analysis of Martin results in an adiabatic developing flow study (Fargie and Martin (82)), a forced convection study (Martin and Fargie (83)), and a viscous dissipation analysis (Martin (84)). The above work of Zeldin and Schmidt (36) is preceded by an adiabatic flow analysis. (Schmidt and Zeldin (85)).

3.3.2 Forced Convection

Even with forced convection, the range of parameters is wide, and covers the following:

(a) fluid boundary condition. The entry velocity profile may be:

- (1) Uniform (rounded entry)
- (2) Partially developed before heating starts
- (3) Fully developed or parabolic

In addition, experimental work may involve (4) A sharp entry, which is difficult to treat analytically.

(b) thermal boundary condition at the wall. This may be:

- (1) Constant heat flux
- (2) Constant temperature

(c) fluid. Interpreting types of fluid in terms of viscosity variation, we have three common alternatives:

- (1) Air (approximates to constant property)
- (2) Water (recent data for water-glycerol mixtures)
- (3) Oils

Authors and References	Experimental (E)	Fluid Tested	Fluid boundary condition (see Section 3.3.2)	Thermal boundary condition Constant wall:	Comments
	Analytical (A)	Property Variation			
Butterworth and Hazell <u>(86)</u> , <u>(87)</u>	E	Glycerol-Water	(2), (3), (4)	Heat flux,	
	A	Nil	(1), (2), (3)	Heat flux	
Martin and Fargie <u>(83)</u> , <u>(88)</u>	E	Oil	(4)	Heat flux	
	A	Viscosity	(1)	Heat flux	
Zeldin and Schmidt <u>(36)</u>	E	Air	(1)	Temperature	
	A	Nil	(1), (3)	Temperature	
Test <u>(89)</u>	E	Oil	(3)	Temperature	Properties of oil used not published (Appendix 11)
	A	Viscosity	(3)	Temperature	

TABLE 1 FORCED CONVECTION DATA

Table 1 summarises the experimental data available, which are accompanied by corresponding analyses. No single analysis, to the author's knowledge, has covered the above range of variables.

3.3.3 Combined Convection

The variables include those relevant for forced convection, together with the vertical combined convection parameter which may be:

- (1) Upward heated flow ('natural' aiding 'forced')
- (2) Downward " " (" opposing ")
- (3) Upward cooled flow (" " ")
- (4) Downward " " (" aiding ")

Table 2 (p. 24) summarises data for which comparisons may be made. The key to further abbreviations is:

U	Upward flow	D	Downward flow
+	Heated	-	Cooled
Q	Constant heat flux	T	Constant temperature

Table 2 shows that sufficient evidence exists to be able to test the analytical model and its predictive ability over a range of fluids and boundary conditions, and for transition phenomena over the full range of combined convection parameter.

3.3.4 Other Problems

Two problems are selected as other applications of the internal cylindrical flow-field concept, namely, viscous dissipation and the annulus.

For viscous dissipation the aim is to investigate the effect in:

- (a) adiabatic flows,
- and (b) forced convection, as it has been postulated (83), (86) as an aiding effect in heat transfer.

For the annulus, which has combined convection significance (45) - (47b), the aim is only a preliminary one, namely to test the flow-field solution.

3.3.5 Other elliptic treatments

Because the full equations are used, it is apposite to summarise the other elliptic treatments, which are compared in the relevant chapters, for:

- (a) Adiabatic developing flow - Friedmann et al (91), Vrentas et al (92), and Schmidt and Zeldin (85).
- (b) Forced convection - Hennecke (93), Singh (94), and Jones (95).
- (c) Combined convection - Zeldin and Schmidt (36).

Authors and References	Experimental (E)	Fluid Tested	Combined Convection Parameter (see 3.3.3)	Fluid b.c. (see 3.3.2)	Wall b.c. (see 3.3.3)	Comments on experimental data
	Analytical (A)	Property Variation (Other than ρ)				
Lawrence and Chato (33)	E	Water	U+	(1)	Q	Velocity profiles
	A	Viscosity				
Scheele and Hanratty (35)	E	Water	U+	(3)	Q	Wall temperatures
	A	Viscosity				
Allen (90) and Appendix 8	E	Oil	U+	(3)	Q	Wall temperatures, Pressure drops
	A	All				
Kemeny and Somers (61)	E	Oil	U+	(4)	Q	Wall temperatures
Zeldin and Schmidt (36)	E	Air	U+	(1)	T	Velocities, Temperatures
	A	Nil	U+	(1), (3)		
Marnier and McMillan (38)	A	Nil	U+	(3)	T	
Scheele et al (16), (34), (35), (59)	E	Water	U+, U- D+, D-	(3)	T, Q	Transition phenomena
	A	Viscosity				

TABLE 2 COMBINED CONVECTION DATA

4. AUTHOR'S PUBLISHED WORK

4.1 Adiabatic Developing Flow (Appendix 3)

This work comprises three main parts:

- (a) comparison of developing axial velocities with existing data,
- (b) study of effect of altering radial and axial step sizes,
- (c) comparison of pressure results with some existing data.

The comparison (a) is very reasonable except with Nikuradse's data close to entry. Experimental details are lacking in the literature, but The City University Library was able to locate Prof. Nikuradse, who confirmed (96) (translation - Appendix 4) that such details were not available.

Step size alterations (b) are found not to be very significant, indicating that for adiabatic flow quite reasonable results are possible with a small number of radial positions.

The comparison (c) is also reasonable.

Although the elliptic/parabolic issue is not raised explicitly, in the written discussion the issue of initial concavity in the axial velocity profile implies it. Also the lower bound to the initial axial step size, because of stability troubles, may well be related to the inlet boundary condition.

Lockett discusses the effect of entry configuration in experiment, and this is found to be quite significant.

Because of additional data, these results are re-assessed in Chapter 8.

4.2 Forced convection and Viscous dissipation (Preliminary)(Appendix 5)

This Appendix (5A and 5B) comprises respectively written discussion to references (83) and (84) of Martin.

In 5A a solution for the high temperature forced convection run in (83) is mentioned. The necessity is recognised for an incremental energy balance in the context of rapid oil viscosity change with

temperature. Results based on this lead to a reasonable velocity and fair heat transfer comparison with experiment. Also, qualitative conclusions are drawn about viscous dissipation effects in adiabatic flow. Although these were obtained by the author before the introduction of the incremental energy balance, they indicate that such effects are very small.

In 5B fresh calculations are reported, with a better viscosity variation with temperature, and other refinements. The forced convection run above agrees with measured wall temperatures to within 3 - 4K, and viscous dissipation effects are calculated, which show that Nusselt numbers are reduced. Martin, in (84), presents analytical predictions of viscous dissipation effects in adiabatic flow, which are directly comparable with measurements of Polak (97) and Hersey and Zimmer (98). Further, in 5B, the author presents his own predictions as a comparison with the above. The agreement is not unreasonable.

4.3 Forced convection and Viscous dissipation (Final)(Appendix 6)

This paper reports heat transfer and pressure drop data for:

- (a) Viscous dissipation effects in adiabatic flow,
- (b) Forced convection, with and without viscous dissipation, and in so doing gives final calculations which are an improvement on those reported in Appendix 5.

The first set of data compares with predictions of Martin (84) and measurements of Polak (97) and Hersey and Zimmer (98). The comparison is very reasonable. Axial and radial velocity variations are also studied. A finding of Martin is confirmed, namely that viscosity variation tends to reduce viscous dissipation effects, this running counter to Polak's hypothesis of a thermal instability situation.

The forced convection data is a specific investigation into experimental results of Martin and Fargie (83) and Butterworth and

Hazell (86). In both these references analytical predictions underestimated the heat transfer substantially, and the common conclusion was that this could be attributed to viscous dissipation. However, the author's predictions ignoring viscous dissipation, showed sufficiently adequate agreement with experiment to conclude that it was unnecessary to postulate an additional effect. It was also found, by calculations of axial velocities, that the sharp entry effects on the flow-field were not insignificant. This was interesting confirmation of Lockett's discussion (Appendix 3). Further calculations incorporating viscous dissipation, showed it to be a minor effect, which, by increasing wall temperatures more than bulk mean values, tended to reduce Nusselt numbers. In fact, Iqbal et al (99) had already calculated such reductions, but comparison with his results could not be made because of ambiguities in the definition of his Eckert number (Appendix 8).

Finally, it should be mentioned that in adiabatic flows, to obtain a solution free from initial 'negative' temperatures, it was necessary to reduce the number of radial positions. This may be connected with the similar lower limit on the axial step size.

4.4 Combined convection with oil and water (Appendix 9)

(co-authors, P.H.G. Allen and O. Szpiro, Electrical Engineering Department, Imperial College).

This manuscript, to be submitted for publication, reports results of a collaborative study, which may be divided into two main parts.

(1) Comparison with data in the literature

The data in question are all for upward heated convection with constant wall heat flux, which relates to part (2) below. The data comprise:

- (i) Lawrence and Chato (33). Developing centre-line velocities

for water with a uniform entry velocity. (Experiment and analysis).

(ii) Scheele and Hanratty (35). Wall temperatures for water, with a parabolic entry profile. (Experiment and analysis).

(iii) Kemeny and Somers (61). Wall temperatures for an oil with a sharp entry. (Experiment only).

In general, data are compared with:

(a) Predictions of a simplified analytical model of Allen (101), which allows for complete property variation with temperature, and

(b) Predictions of Collins, allowing for additional property variation, so as to be comparable with (a).

(2) Comparison with Allen and Szpiro's measurements

Measurements of wall temperature, for a transformer oil with a parabolic entry profile, are compared with predictions of (a) and (b) above. Also, overall pressure drop was measured.

Both comparisons, (1) and (2) above, are in general very reasonable as far as this analysis is concerned. The data, incidentally, cover all fluid entry boundary conditions.

4.5 Comparative combined convection study (Appendix 10)

This paper reports results of a comparative study, using air, of the main combined convection parameters. Concentrating attention on the flow-field development from a uniform velocity profile, the first cases are of upward and downward heated flow for constant wall heat flux and temperature.

With downward flow, reversed flow regions near the wall were predicted, which correspond to those stable regions found for water by Scheele and Hanratty (35). However, while predictions can only indicate an annulus of reversal, observations recorded randomly asymmetric regions, the latter being presumably more stable.

For cooling flow, it is shown that even over an identical but reversed temperature range, the flow-field changes are different from the corresponding combined heating cases.

Even for air, the effect of viscosity change is not insignificant, and this justifies the present approach which requires inlet temperature and property data (as a function of temperature) as input parameters.

4.6 Conclusion

The range of work reported, covers broadly:

- (a) adiabatic flow,
- (b) forced convection with constant heat flux,
- (c) combined convection with constant heat flux,
- (d) effect of viscous dissipation,
- (e) comparative study of parameters.

In addition the following City University Research Memoranda have been written:

- (a) "Mixed Free and Forced Convection - A Survey" ML 15 (1969)
- (b) "Combined Convection in Vertical Tubes - A Computer Program in Fortran IV" ML 25 (1970)
- (c) "Combined Convection - further computer developments and a Program for Annuli" ML 28 (1971)

The scope of these is covered within this Thesis, but parts of the following internal publication are used as Appendices 1 and 12.

- (d) "The basic equations for Convective heat transfer in cylindrical co-ordinates, with Computer solutions using Fortran" Post Experience Course Notes TCU (1974)

PART II

ANALYSIS AND COMPUTING

5. BASIC EQUATIONS

5.1 Partial differential equations

The basic equations are obtained for a standard two-dimensional field inside a tube. The cylindrical co-ordinates z' and r' represent axial distance from entry and radius from axis respectively. Circumferential effects are absent for axially symmetric flow, which is assumed to be steady and laminar.

Equations (1) - (4) are standard equations for

- (1) conservation of mass (continuity)
- (2) $\left. \begin{array}{l} \text{momentum} \end{array} \right\} \begin{array}{l} \text{axial (Navier-Stokes)} \\ \text{radial} \end{array}$
- (3) $\left. \begin{array}{l} \text{momentum} \end{array} \right\}$
- (4) conservation of energy

$$\begin{aligned} \frac{\partial v_r}{\partial r'} + \frac{v_r}{r'} + \frac{\partial v_z}{\partial z'} &= 0 \quad \dots (1) \\ \rho \left(v_r \frac{\partial v_z}{\partial r'} + v_z \frac{\partial v_z}{\partial z'} \right) &= -\frac{\partial p}{\partial z'} - S\rho g + \mu \left(\frac{\partial^2 v_z}{\partial z'^2} + \frac{1}{r'} \frac{\partial v_z}{\partial r'} + \frac{\partial^2 v_z}{\partial r'^2} \right) \\ &\quad + 2 \frac{\partial \mu}{\partial z'} \cdot \frac{\partial v_z}{\partial z'} + \frac{\partial \mu}{\partial r'} \left(\frac{\partial v_r}{\partial z'} + \frac{\partial v_z}{\partial r'} \right) \quad \dots (2) \\ \rho \left(v_r \frac{\partial v_r}{\partial r'} + v_z \frac{\partial v_r}{\partial z'} \right) &= -\frac{\partial p}{\partial r'} + \mu \left(\frac{\partial^2 v_r}{\partial r'^2} + \frac{1}{r'} \frac{\partial v_r}{\partial r'} - \frac{v_r}{r'^2} + \frac{\partial^2 v_r}{\partial z'^2} \right) \\ &\quad + 2 \frac{\partial \mu}{\partial r'} \cdot \frac{\partial v_r}{\partial r'} + \frac{\partial \mu}{\partial z'} \left(\frac{\partial v_r}{\partial z'} + \frac{\partial v_z}{\partial r'} \right) \quad \dots (3) \\ v_r \frac{\partial T'}{\partial r'} + v_z \frac{\partial T'}{\partial z'} &= \frac{k}{\rho C_p} \left(\frac{\partial^2 T'}{\partial z'^2} + \frac{1}{r'} \frac{\partial T'}{\partial r'} + \frac{\partial^2 T'}{\partial r'^2} \right) + \frac{\mu}{\rho C_p} \phi \quad \dots (4) \end{aligned}$$

For a derivation of these equations from first principles, see Appendix 1. Although they are classical forms no complete cylindrical-co-ordinate treatment is available in the literature. Of more significance is that the derivation includes the effect of viscous dissipation.

Property relationships are as follows: viscosity variation is allowed for in full, and the treatment allows any expression which may be differentiated w.r.t. temperature.

Hence, for example, the term $\frac{\partial \mu}{\partial r'}$, in equation (2) becomes

$$\frac{\partial \mu}{\partial r'} = \frac{d\mu}{dT'} \cdot \frac{\partial T'}{\partial r'}$$

Allowing up to three input constants,

$$\mu = \mu_A \cdot f_1(C_1, C_2, C_3, T') \quad \cdot \quad \cdot \quad \cdot \quad (5)$$

C_p and k are assumed constant, and ρ constant, except in the buoyancy term of equation (2). This term gives natural convection effects in vertical flows, where $s = +1$, upward flow, or -1 , downward flow. These effects are assumed absent in forced convection of any orientation, and in adiabatic flow ($s = 0$).

The allowed density variation is quadratic:

$$\rho = \rho_A(C_4 - C_5 T' + C_6 T'^2) \quad (6)$$

Finally, ϕ , the viscous dissipation function, is given by:

$$\phi = 2 \left\{ \left(\frac{\partial v_r}{\partial r'} \right)^2 + \left(\frac{v_r}{r'} \right)^2 + \left(\frac{\partial v_z}{\partial z'} \right)^2 \right\} + \left\{ \frac{\partial v_z}{\partial r'} + \frac{\partial v_r}{\partial z'} \right\}^2 \quad (7)$$

In this numerical solution, the integral continuity equation is used, namely:

$$\pi R^2 U_0 = \int_0^R 2\pi r' \cdot v_z \cdot dr' \quad \cdot \quad \cdot \quad \cdot \quad (8)$$

In addition, and it is somewhat of an innovation, the integral energy equation is used as part of the analysis, rather than an energy balance check. By so using it, the energy balance is guaranteed, and the wall thermal boundary condition need not be explicitly specified. The situation is analogous to the use of equation (8), and

is discussed further in Chapter 6.5.

For a constant wall temperature, the statement is discussed in Chapter 12, but allowing for constant wall heat flux, and viscous dissipation, the equation is:

$$\pi R^2 \rho U_0 C_p (\Delta T')_m = \frac{Q}{A} \cdot (2\pi R) \Delta z' + \int_0^R \mu \phi 2\pi r' \cdot dr' \cdot \Delta z' \quad (9)$$

where $(\Delta T')_m$ is the increase in mixed-mean temperature over a length $\Delta z'$.

The boundary conditions are as follows:

Entry condition:

$$v_z = U_0, \quad v_r = 0, \quad p = p_0, \quad T' = T'_0$$

Tube axis:

$$v_r = 0, \quad \frac{\partial v_z}{\partial r'} = 0, \quad \frac{\partial T'}{\partial r'} = 0$$

Wall flow condition:

$$v_z = 0, \quad v_r = 0$$

Wall thermal condition, either

(a) Constant heat flux:

$$\frac{Q}{A} = k \left(\frac{\partial T'}{\partial r'} \right)_R = \text{const}$$

or (b) constant temperature:

$$T' = T'_R = \text{const}$$

$\frac{\partial p}{\partial r'} = 0 \rightarrow \text{see P 210 Card. 565}$

The wall thermal condition becomes redundant, however, when specified implicitly in equation (9).

5.2 Non-dimensional equations

The above equations may be made non-dimensional, using the substitutions defined in the Nomenclature.

Equations (1) - (9) become equations (10) - (18) respectively:

$$\frac{\partial V}{\partial r} + \frac{V}{r} + \frac{\partial U}{\partial z} = 0 \quad (10)$$

$$\begin{aligned} V \frac{\partial U}{\partial r} + U \frac{\partial U}{\partial z} = & -\frac{\partial P}{\partial z} + \frac{1}{Re} \left[T_a \left(\frac{\partial^2 U}{\partial z^2} + \frac{1}{r} \cdot \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial r^2} \right) \right. \\ & \left. + T_b \left\{ 2 \frac{\partial T}{\partial z} \cdot \frac{\partial U}{\partial z} + \frac{\partial T}{\partial r} \left(\frac{\partial V}{\partial z} + \frac{\partial U}{\partial r} \right) \right\} \right] \pm \frac{1}{Re^2} [G \cdot C_{04} + T(C_{05} + C_{06} T)] \frac{\rho}{\rho_A} \end{aligned} \quad (11)$$

$$V \frac{\partial V}{\partial r} + U \frac{\partial V}{\partial z} = -\frac{\partial P}{\partial r} + \frac{1}{Re} \left[T_a \left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial V}{\partial r} - \frac{V}{r^2} + \frac{\partial^2 V}{\partial z^2} \right) + T_\beta \left\{ 2 \frac{\partial T}{\partial r} \cdot \frac{\partial V}{\partial r} + \frac{\partial T}{\partial z} \left(\frac{\partial V}{\partial z} + \frac{\partial U}{\partial r} \right) \right\} \right] \quad (12)$$

$$V \frac{\partial T}{\partial r} + U \frac{\partial T}{\partial z} = \frac{1}{Pr \cdot Re} \left(\frac{\partial^2 T}{\partial z^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \right) + \phi_{ND} \quad (13)$$

$$\mu = \mu_A f_2(C_{01}, C_{02}, C_{03}, T) = \mu_A \cdot T_a \quad (14)$$

$$\rho = \rho_A \left(C_{04} + \frac{C_{05} T}{G} + \frac{C_{06} T^2}{G} \right) \quad (15)$$

$$\phi_{ND} = Q_{DISS} \times T_a \times \left\{ 2 \left[\left(\frac{\partial V}{\partial r} \right)^2 + \left(\frac{V}{r} \right)^2 + \left(\frac{\partial U}{\partial z} \right)^2 \right] + \left(\frac{\partial V}{\partial z} + \frac{\partial U}{\partial r} \right)^2 \right\} \quad (16)$$

$$\frac{1}{2} = \int_0^1 U \cdot r \cdot dr \quad (17)$$

$$\frac{1}{2}(\Delta T)_m = \frac{HFLUX}{Re \cdot Pr} \cdot M \cdot \Delta z + \int_0^1 \phi_{ND} \cdot r \cdot dr \cdot \Delta z \quad (18)$$

In addition, T_β represents $\frac{d\mu}{dT}$ and is defined by:

$$T_\beta = \frac{d}{dT}(T_a) \quad (19)$$

The boundary conditions become

- (1) $U = 1, V = 0, P = P_0, T = T_0;$
- (2) $V = 0, \frac{\partial U}{\partial r} = 0, \frac{\partial T}{\partial r} = 0;$
- (3) $U = 0, V = 0;$
- (4) (a) $\frac{Q}{A} = k \left(\frac{\partial T}{\partial r} \right)_1 \left(\frac{v_A^2}{R^4 C_{5g}} \right)$ (b) $T = T_R$

5.3 Scope of treatment

5.3.1 Property Limitations

The restricted density variation is quite reasonable for liquids, and for gases with modest temperature changes. This is consistent with assuming k and C_p constant.

5.3.2 Geometry

The meaning of s in the buoyancy term of equation (2) has already been discussed.

As a further application, $s = \sin\alpha$ is valid for inclined tubes, where α is the angle made with the horizontal by the z axis. To be valid, there must be no buoyancy effects at right angles to the tube, which can only be considered reasonable for small diameters. The problem is not considered in this thesis.

5.3.3 Boundary Conditions

The fluid entry boundary condition (sub-section 3.3.2) has been specified for:

- (1) A uniform velocity profile.

Alternatives are:

- (2) Partial adiabatic development; this is a programming problem (section 7.7 and Appendix 2).
- (3) Fully-developed velocity profile.

The boundary condition is stated as:

$$\frac{V_z}{U_0} = 2 \left[1 - \left(\frac{r'}{R} \right)^2 \right] \quad (\text{dimensional})$$

where U_0 = mean entry velocity

$$\text{and } U = 2(1 - r^2) \quad (\text{non-dimensional})$$

Also, the problem of wall porosity can be defined by introducing finite V_r at the wall and re-stating equations (8) and (9) accordingly. Although this is a problem not considered here, the programming alterations necessary are rather minor. See Carter and Gill (43).

5.3.4 Laminar and Turbulent Flow

Laminar flow analyses have engineering significance under certain conditions. Apart from high-viscosity oils used in transformer cooling, Sherwin (46) mentions heat transfer where:

- (1) At least one fluid in a heat exchanger is in laminar flow,
- (2) Design flow rates vary widely down to laminar flow,
- (3) Process is automatic and laminar flow represents an emergency or breakdown condition.

Turbulent flow analyses are much more complex, and present treatments involve using the laminar equations (where time average velocities are understood) with eddy and thermal diffusivities re-defining the viscosity, and possibly thermal conductivity. For such re-definition a turbulence model is necessary, and current work is exemplified by Launder and Spalding (102) and Bankston and McEligot (103). An internal M.Sc. project (Ross 104) incorporated some turbulence models into the present program. The attempt was not without success, but demonstrated that the radial grid spacing needed improvement.

6. FINITE-DIFFERENCE EQUATIONS

6.1 Introduction

The finite-difference grid is a rectangular axial (suffix 'n') - radial (suffix 'm') one. The dimensionless radius r is re-expressed as $r = \frac{m-1}{M}$, for $m = 1$ (at axis)..... $M+1$ (at wall). This gives $r = 0,1$ at axis and wall.

A set of linear equations for the unknowns T, U, V, P may be written for the unknown axial level ' $n+1$ '. Where the product of two unknowns occurs, linearity is achieved by putting one unknown at its value at the previous step ($X_{n+1} \rightarrow X_n$) an approximation which should not be too serious. In practice, (section 7.1) a check solution is made, which effectively removes the approximation.

6.2 Continuity equation

Backward differences are used in both directions. Terms from equation (10) become:

$$\frac{(\partial V)}{(\partial r)_{n+1,m}} = \frac{V_{n+1,m} - V_{n+1,m-1}}{\frac{1}{M}}$$

$$\frac{(V)}{(r)_{n+1,m}} = \frac{V_{n+1,m}}{\frac{(m-1)}{M}}$$

$$\frac{(\partial U)}{(\partial z)_{n+1,m}} = \frac{U_{n+1,m} + U_{n+1,m-1} - U_{n,m} - U_{n,m-1}}{2\Delta z}$$

The last term is an average backward difference, with backward radial positions (after Lawrence (105)).

Substituting the above into equation (10) gives the set of finite-difference equations:

$$\left[\frac{2Mm\Delta z}{m-1} \right] V_{n+1,m} - [2\Delta z M] V_{n+1,m-1} + [1] U_{n+1,m} + [1] U_{n+1,m-1} = [U_{n,m} + U_{n,m-1}] \quad (19)$$

where quantities in square brackets are known coefficients of the variables $V_{n+1,m}$ etc.

6.3 Alternatives

Some alternative finite-difference schemes may be noted in the literature. For example, Sherwin and Wallis (47a) use a simpler expression for $\left(\frac{\partial U}{\partial z}\right)$. They also state that a central difference form of $\left(\frac{\partial V}{\partial r}\right)$ will only give a solution for an odd number of radial spacings.

If each $(n + 1)$ level unknown variable is expressed as a function of (n) and $(n - 1)$ level known variables, then the unknown variables may be solved directly, and the scheme is called explicit. Any other scheme is termed implicit. Worsøe-Schmidt and Leppert (53) use a central/backward difference scheme in the radial/axial directions, but also define an axial position $n + (\sigma \Delta z)$, where σ is a weighting parameter, giving an explicit scheme when zero. Stability and convergence criteria led to the choice of $\sigma = \frac{3}{4}$.

6.4 Other equations

For the other equations, mainly after Lawrence, and similar to Sherwin and Wallis, straightforward central/backward differences were used radially/axially. The $(n - 1)$ level terms only appear in the approximations for $\left(\frac{\partial^2 U}{\partial z^2}\right), \left(\frac{\partial^2 V}{\partial z^2}\right), \left(\frac{\partial^2 T}{\partial z^2}\right)$. The parabolic equations do not have the double derivatives, so the $(n - 1)$ terms are not needed. The approximations are:

Radial

$$\left(\frac{\partial U}{\partial r}\right)_{n+1,m} = \frac{U_{n+1,m+1} - U_{n+1,m-1}}{\left(\frac{2}{M}\right)} \quad \left(\frac{\partial V}{\partial r}\right)_{n+1,m} = \frac{V_{n+1,m+1} - V_{n+1,m-1}}{\left(\frac{2}{M}\right)}$$

$$\left(\frac{\partial T}{\partial r}\right)_{n+1,m} = \frac{T_{n+1,m+1} - T_{n+1,m-1}}{\left(\frac{2}{M}\right)} \quad \left(\frac{\partial P}{\partial r}\right)_{n+1,m} = \frac{P_{n+1,m+1} - P_{n+1,m-1}}{\left(\frac{2}{M}\right)}$$

$$\left(\frac{\partial^2 U}{\partial r^2}\right)_{n+1,m} = \frac{U_{n+1,m+1} - 2U_{n+1,m} + U_{n+1,m-1}}{\left(\frac{1}{M^2}\right)} \quad \left(\text{Similarly for } \frac{\partial^2 V}{\partial r^2}, \frac{\partial^2 T}{\partial r^2}\right)$$

Axial

$$\left(\frac{\partial U}{\partial z}\right)_{n+1,m} = \frac{U_{n+1,m} - U_{n,m}}{\Delta z} \quad \left(\text{Similarly for } \frac{\partial V}{\partial z}, \frac{\partial T}{\partial z}, \frac{\partial P}{\partial z}\right)$$

$$\left(\frac{\partial^2 U}{\partial z^2}\right)_{n+1,m} = \frac{U_{n+1,m} - 2U_{n,m} + U_{n-1,m}}{\Delta z^2} \quad \left(\text{Similarly for } \frac{\partial^2 T}{\partial z^2}, \frac{\partial^2 V}{\partial z^2}\right)$$

Using the above approximations, equations (11) - (13) become equations

(20) - (22) respectively.

$$\begin{aligned} & \left[\frac{M^2}{2Re} \left\{ T_A \frac{(2m-1)}{m-1} + \frac{T_B}{2} (T_{n+1,m+1} - T_{n+1,m-1}) \right\} \right. \\ & \quad - \frac{M}{2} V_{n,m} \left. \right] U_{n+1,m+1} + \left[\frac{1}{Re} \left\{ T_A \left(\frac{1}{\Delta z^2} - 2M^2 \right) \right. \right. \\ & \quad \left. \left. + \frac{2T_B}{\Delta z^2} (T_{n+1,m} - T_{n,m}) \right\} - \frac{U_{n,m}}{\Delta z} \right] U_{n+1,m} + \left[\frac{M^2}{2Re} \right. \\ & \quad \times \left\{ T_A \frac{(2m-3)}{(m-1)} - \frac{T_B}{2} (T_{n+1,m+1} - T_{n+1,m-1}) \right\} \\ & \quad \left. + \frac{M}{2} V_{n,m} \right] U_{n+1,m-1} - \frac{P_{n+1,m}}{[\Delta z]} \\ & = \left[\pm \frac{1}{Re^2} \left\{ G \cdot C_{04} + T_{n+1,m} (C_{05} + C_{06} T_{n+1,m}) \right\} \cdot \frac{\rho}{\rho_A} \right. \\ & \quad - \frac{P_{n,m}}{\Delta z} - \frac{U_{n,m}^2}{\Delta z} + \frac{2U_{n,m}}{\Delta z^2 \cdot Re} \{ T_A + T_B (T_{n+1,m} - T_{n,m}) \} \\ & \quad \left. - \frac{U_{n-1,m}}{\Delta z^2 \cdot Re} T_A - \frac{MT_B}{2Re \Delta z} (T_{n+1,m+1} - T_{n+1,m-1}) \right. \\ & \quad \left. \times (V_{n,m} - V_{n-1,m}) \right] \end{aligned} \quad (20)$$

$$\begin{aligned} & \left[\frac{M^2}{2Re} \left\{ \frac{T_A(2m-1)}{(m-1)} + T_B (T_{n+1,m+1} - T_{n+1,m-1}) \right\} - \frac{M}{2} V_{n,m} \right] \\ & \quad \times V_{n+1,m+1} + \left[\frac{1}{Re} \left\{ T_A \left(\frac{1}{\Delta z^2} - \frac{M^2}{(m-1)^2} (2m^2 - 4m + 3) \right) \right. \right. \\ & \quad \left. \left. + \frac{T_B}{\Delta z^2} (T_{n+1,m} - T_{n,m}) \right\} - \frac{U_{n,m}}{\Delta z} \right] V_{n+1,m} \\ & \quad + \left[\frac{M^2}{2Re} \left\{ T_A \frac{(2m-3)}{(m-1)} - T_B (T_{n+1,m+1} - T_{n+1,m-1}) \right\} \right. \\ & \quad \left. + \frac{M}{2} V_{n,m} \right] V_{n+1,m-1} + \left[\frac{M}{12} \right] P_{n+1,m+2} \\ & \quad - \left[\frac{2M}{3} \right] P_{n+1,m+1} + \left[\frac{2M}{3} \right] P_{n+1,m-1} - \left[\frac{M}{12} \right] P_{n+1,m-2} \\ & = \left[-\frac{U_{n,m} \cdot V_{n,m}}{\Delta z} + \frac{2V_{n,m} \cdot T_A}{Re \cdot \Delta z^2} + \frac{V_{n-1,m}}{\Delta z^2 \cdot Re} \left\{ \frac{T_B}{2} (T_{n+1,m} \right. \right. \\ & \quad \left. \left. - T_{n,m}) - T_A \right\} - \frac{MT_B}{2Re \Delta z} (T_{n+1,m} - T_{n,m}) \right. \\ & \quad \left. \times (U_{n,m+1} - U_{n,m-1}) \right] \end{aligned} \quad (21)$$

$$\begin{aligned} & \left[\frac{M}{2} V_{n,m} + \frac{M^2(1-2m)}{2Pr \cdot Re(m-1)} \right] T_{n+1,m+1} + \left[\frac{U_{n,m}}{\Delta z} \right. \\ & \quad \left. + \frac{1}{Pr \cdot Re} \left(2M^2 - \frac{1}{\Delta z^2} \right) \right] T_{n+1,m} + \left[\frac{M^2(3-2m)}{2Pr \cdot Re(m-1)} \right. \\ & \quad \left. - \frac{M}{2} V_{n,m} \right] T_{n+1,m-1} \\ & = \left[\left\{ U_{n,m} - \frac{2}{Pr \cdot Re \Delta z} \right\} \frac{T_{n,m}}{\Delta z} + \frac{T_{n-1,m}}{\Delta z^2 \cdot Pr \cdot Re} \right] + \phi \text{FD} \end{aligned} \quad (22)$$

(An early attempt to eliminate instability effects was to put

$$\left(\frac{\partial P}{\partial r}\right)_m = \frac{8P_{m+1} - 8P_{m-1} - P_{m+2} + P_{m-2}}{\left(\frac{12}{M}\right)}$$

The attempt was unsuccessful, but the expression was left in.)

TA and TB, the viscosity variation parameters, are given by:

$$\left. \begin{array}{l} T_\alpha = TA \\ T_\beta = TB \end{array} \right\} \text{ when } T = T_{n+1,m}$$

In the energy equation (22), ϕ_{FD} , the finite-difference viscous dissipation parameter, has the effect of a (known) addition to the r.h.s. It is given by:

$$\phi_{FD} = QDISS \times TA \left[2 \left\{ (V_{n+1,m} - V_{n+1,m-1})^2 M^2 + \left(\frac{MV_{n+1,m}}{m-1} \right)^2 + \left(\frac{U_{n+1,m} - U_{n,m}}{\Delta z} \right)^2 \right\} + \left\{ \left(\frac{V_{n+1,m} - V_{n,m}}{\Delta z} \right) + \left(\frac{U_{n+1,m} - U_{n+1,m-1}}{\frac{1}{M}} \right) \right\}^2 \right] \quad (23)$$

This term can include the unknowns $V_{n+1,m}$ etc. because (Section 7.1) the set of equations (19) - (21) is solved separately from, and before, the set of equations (22).

6.5 Integral equations

Applying Simpson's Rule to equation (17) gives the finite-difference equivalent:

$$\frac{3M^2}{8} = \sum_{m=1}^{\frac{M}{2}} (2m - 1)U_{2m} + \sum_{m=1}^{\frac{M}{2} - 1} (m)U_{2m-1} \quad (24)$$

Here 'm' is not the radial position, but a variable. It should also be noted that Simpson's Rule proved quite adequate. A much more exact form (closed 6 - interval Newton-Coates Rule, Ref. (100)) gave the same flow-field results.

The equivalent of equation (18), term for term, is:

$$\sum_{m=2}^M (m-1) U_{n+1,m} T_{n+1,m} - \sum_{m=2}^M (m-1) U_{n,m} T_{n,m} \\ = \frac{\text{HFLUX}}{\text{Re.Pr}} M^3 \Delta Z + \Delta Z \cdot \sum_{m=2}^M \phi_{FD}(m-1) \quad (25)$$

6.6 Boundary conditions

The boundary conditions become:

- (1) $U_{1,m} = 1, V_{1,m} = 0, P_{1,m} = P_0, T_{1,m} = 0$
- (2) $U_{n,1} = U_{n,2}, V_{n,1} = 0, P_{n,1} = P_{n,2},$
 $T_{n,1} = T_{n,2}$
- (3) $U_{n,M+1} = 0, V_{n,M+1} = 0$
- (4) (a) $T_{n,M+1} = T_{n,M} + \frac{QR^4 C_{3g}}{Av_A^2 M k};$ (b) $T_{n,M+1} = T_R$

Additional entry conditions

The diffusion terms giving an 'n-1' level require the specifying of $U_{0,m}, V_{0,m}, T_{0,m}$. In the absence of other data, we use $U_{0,m} = U_{1,m}, V_{0,m} = V_{1,m}, T_{0,m} = T_{1,m}$. Assuming no axial diffusion at entry, this is not strictly correct, but the point is fully discussed in Chapter 8.

Additionally, for a fully-developed velocity profile, the inlet axial distribution is given by:

$$U_{2,m} = 2 \left[1 - \frac{(m-1)^2}{M^2} \right] \quad \text{for } m \text{ between } 2 \text{ and } M \quad (26)$$

Wall thermal condition

(4)(a)* is the redundant wall thermal condition for constant heat flux. It basically gives an estimated wall temperature resulting from the equivalent for $\left(\frac{\partial T}{\partial r}\right)$ given in Section 6.4, and as used by Lawrence. This has a first-order truncation error, but was subsequently re-placed by a second-order equivalent (as used by Butterworth and Hazeli (87), namely,

* i.e. boundary condition (4)(a) on this page

$$T_{n,wall} = (2 \text{ HFLUX} + 4T_{n,M} - T_{n,M-1})/3 \quad (27)$$

It is an important point, because the temperature gradient is highest at the wall. However, by using the energy balance equation, the difficulty is eliminated, because the single finite-difference equation involving $T_{n,M+1}$ is not then needed.

Interestingly, this has effectively been done in equation (19) for continuity. It will be noticed for the last field point, $m = M$, neither U_{M+1} nor V_{M+1} are employed. The zero U_{M+1} , however, is implicit in the integral continuity equation. *In the literature, the latter is commonly used, but to the author's knowledge, no comparable analysis has used the corresponding energy 'implicit boundary condition'. The attention given to overall energy balance in the literature, is surveyed in Chapter 16.

* For the porous wall case, $V_{M+1} \neq 0$, the wall flow would also be implicit in the re-stated equation.

7. COMPUTER PROGRAM
7.1 Method of Solution
7.1.1 Introduction

The basic method is as follows. Assuming a second-order effect of temperature on the flow-field, the latter is solved first, and the solution used to solve the energy equation. This assumes the U, V, P solution is reasonable. As a check, and in view of T_m depending on U values, the temperature solution so obtained is substituted into the UVP equations for a second solution, and in turn a second T solution. Differences have usually been slight, justifying the above assumptions. This also implies that temperature dependent viscosity and density appearing in the momentum equations, have their virtually correct values the 'second time round', rather than their values at the previous step.

In more detail, equations (19), (20), (21) (and (24)) are written for each radial position at the first axial step from entry, and the known coefficients and r.h.s. evaluated. This gives $3M-2$ unknowns (U, V, P for each radial position and P_{M+1}) with the same number of equations. The solution is by a standard Gaussian elimination technique suggested by McCormack and Salvadori (106). Equation (22) is similarly written and solved with U_{n+1} 's and V_{n+1} 's from the first solution. Here $M-1$ equations (including equation (25)) solve for the $M-1$ unknown T's.

The semi-iterative check is carried out in the same order. The U, V, P matrix used T_{n+1} 's from the temperature solution, and finally the T solution is re-obtained.

The marching is carried out by progression to the next axial step, and so on.

A major part of the programming is associated with calculating the coefficients ('A' on the l.h.s. and 'B' on the r.h.s.) for the T and

U, V, P matrices, and the structure of these is discussed. It should be noted that with the simple Gaussian elimination scheme, the A_{ii} values must all be non-zero.

7.1.2 Temperature Matrix

As the T matrix is somewhat easier to follow, it will be considered first.

Equation (22) may be written:

$$F_4 (T_{m-1}) + F_3 (T_m) + F_2 (T_{m+1}) = B$$

and equation (25) as:

$$\sum F_1 T_m = B$$

The additional subscript 'n+1' is understood, and F_1, F_2, F_3, F_4 are functions only of the radial position 'm'. Hence they are known, as coefficients A_{ij} , for a given m, as is B.

The resulting set of equations is given in Table 3.

	Value of m
$(F_4 + F_3)T_2 + F_2T_3 + 0$	$= B_1$ 2
$F_4T_2 + F_3T_3 + F_2T_4$	$= B_2$ 3
$0 + F_4T_3 + F_3T_4 + F_2T_5$	$= B_3$ 4
	1
	1
	1
$F_4T_{M-2} + F_3T_{M-1} + F_2T_M$	$= B_{M-2}$ M-1
$F_1T_2 + F_1T_3 + F_1T_4 + F_1T_5 + \dots + F_1T_{M-2} + F_1T_{M-1} + F_1T_M$	$= B_{M-1}$ M

Table 3. Energy Equations

The axis boundary condition $T_1 = T_2$ results in the coefficient of T_2 , in the first equation, being $(F_4 + F_3)$ instead of F_3 .

7.1.3 U, V, P Matrix

In principle, this set is similar to that for T, except that there are $3M-2$ equations, giving a UVP matrix of left hand side coefficients about 9 times larger than that for T.

With the constraint of non-zero A_{ii} 's, and summarising equations (19), (20), (21) and (24) as below, the matrix is formed as in Table 3.

$$F_5^V m + F_6^V m-1 + F_7^U m + F_8^U m-1 = B_I$$

$$F_9^U m+1 + F_{10}^U m + F_{11}^U m-1 + F_{12}^P m = B_{II}$$

$$F_{13}^V m+1 + F_{14}^V m + F_{15}^V m-1 + F_{16}^P m+1 + F_{17}^P m-1 \\ + F_{18}^P m+2 + F_{19}^P m-2 = B_{III}$$

$$\sum_{m=1}^{\frac{M}{2}} F_{20}^U m + \sum_{m=1}^{\frac{M}{2}-1} F_{21}^U m+1 = B_{IV}$$

Again, Table 4 should be a reasonably self-evident description of the equations.

At the axis, a previously unspecified equivalent is

$$\frac{(\partial P)}{(\partial r)_{m=2}} = \frac{-P_{n+1,m+2} + 4P_{n+1,m+1} - 3P_{n+1,m}}{\frac{(2)}{(M)}}$$

Hence, it may be noted that a major part of the programming involves specifying the 'A' Matrix, and this consists of:

- (i) Loop statements, where A's and their matrix positions vary regularly.
- (ii) Single statements for the A's for the boundary conditions.

EQU. NO. \ VARIABLE VALUE ROW		P_2	P_3	P_4	-----	P_M	P_{M+1}	V_2	V_3	---
		1	2	3	-----	M-1	M	M+1	M+2	---
(20)	2 1	F_{12}								
	3 2		F_{12}							
	4 3			F_{12}						

(21)	M M-1					F_{12}				

	M M					$F'_{17}(P_{M-1})$	F'_{16}			

(19)	2 M+1	F'_A	F'_B	F'_C				$(F_{15}^0 + F_4)$	F_{13}	
	3 M+2	$(F_{17} + F_{19})$		F_{16}	$F_{18}(P_5)$			F_{15}	F_{14}	$F_{13}(V_1)$

	M-1 2M-2					F_{16}	F_{18}			
(24)	-----									
	M 2M-1									
	2 2M							$(F_6^0 + F_5)$		
	3 2M+1							F_6	F_5	
(24)	-----									
	M-1 3M-3									
(24)	-----									
	3M-2									

← 'A' - COEFFICIENT MATRIX →

NOTES: ----- Indicates all unspecified ranges

----- Indicates limits of fields of equation types

F'_{17} and F'_{16} refer to original form of $\frac{\partial P}{\partial r}$ necessary at boundary

F'_A , F'_B , F'_C refer to special form of $\frac{\partial P}{\partial r}$ at inner boundary

F_{15}^0 , F_6^0 go to zero because V_1 is put to zero.

TABLE 4 - U, V, P Equations. (ALSO ON PAGE 47).

REFER TO PAGE 45 FOR DISCUSSION

$-V_{M-1}$	V_M	U_2	U_3	U_4	---	U_{M-1}	U_M		
$-2M-2$	$2M-1$	$2M$	$2M+1$	$2M+2$	---	$3M-3$	$3M-2$		
		$(F_{11}+F_{10})$	F_9					P_2	$B_{II}(row=1)$
		F_{11}	F_{10}	F_9				P_3	$B_{II}(2)$
			F_{11}	F_{10}	$F_9(U_5)$			P_4	$B_{II}(3)$
						F_{11}	F_{10}	P_M	$(B_{II}(M-1) - (F_9 \cdot U_{M+1})^{10})$
F_{15}	F_{14}							P_{M+1}	$(B_{III}(M) - (F_{13} \cdot U_{M+1})^{10})$
								V_2	$B_{III}(M+1)$
								V_3	$B_{III}(M+2)$
F_{14}	F_{13}							V_{M-1}	$B_{III}(2M-2)$
F_6	F_5					F_8	F_7	V_M	$B_{II}(2M-1)$
		(F_8+F_7)						U_2	$B_{II}(2M)$
		F_8	F_7					U_3	$B_{II}(2M+1)$
						$F_8(U_{M-2})$	F_7	U_{M-1}	$B_{II}(3M-3)$
								U_M	$B_{II}(3M-2)$
		F_{20}	F_{21}	F_{20}		F_{21}	F_{20}		
cont	'A' - COEFFICIENT MATRIX							↑ Solution Vector ↑ column Vector (r.h.s. b)	

NOTES: U_{M+1} , V_{M+1} are both zero.

Matrix notation $A(I,J)$ indicates Ith row

Jth column

TABLE 4 - U,V,P Equations. (ALSO ON PAGE 46).

REFER TO PAGE 45 FOR DISCUSSION

7.1.4 Axial increment changes

Normally, with finite-difference analyses, when axial changes become smaller, the axial increment (ΔZ here) is increased. With the known levels including 'n - 1' as well as 'n', this increase may only be a doubling of ΔZ . Figure 1 explains this. Normally after a solution the old values at 'n' and 'n + 1' become new values at 'n - 1' and 'n' respectively. ((i) Normal). For doubling ΔZ (as in (ii)) old 'n + 1' and 'n - 1' become new 'n' and 'n - 1' respectively, old 'n' data becoming redundant.

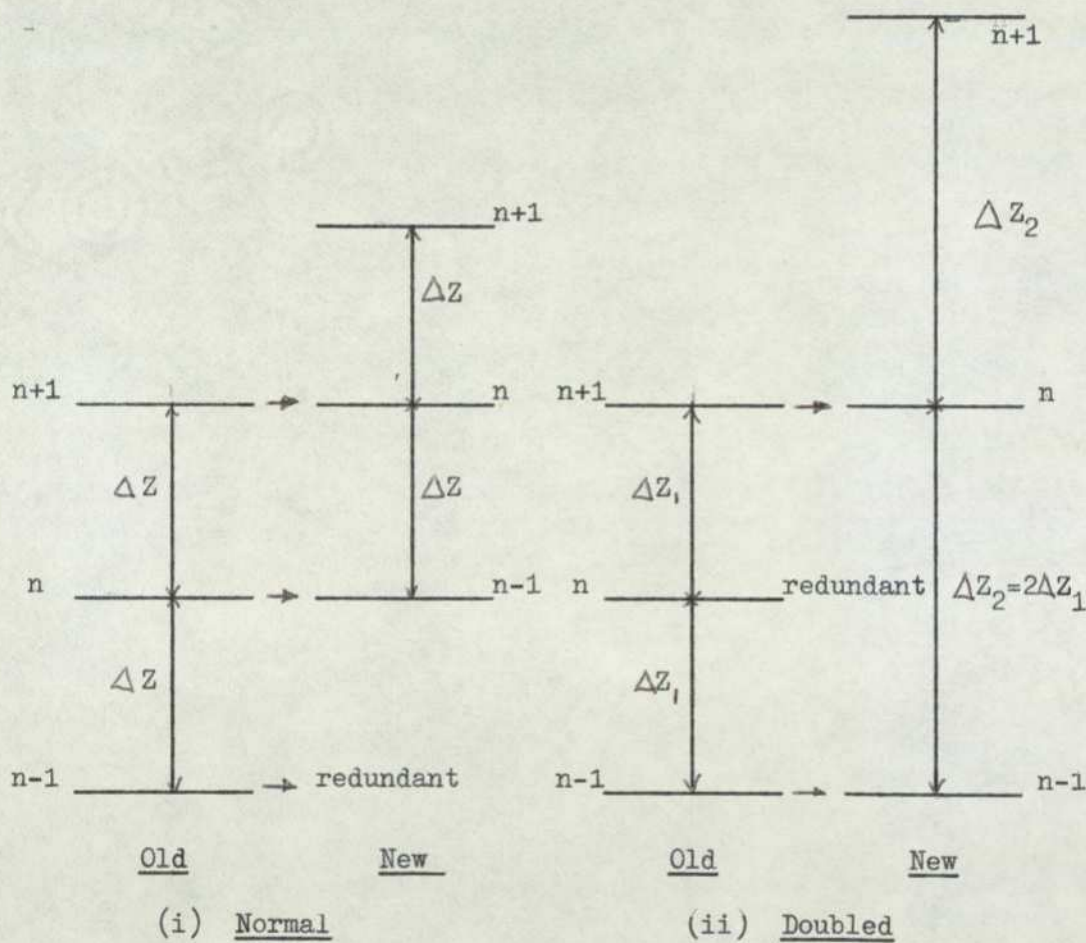


FIGURE 1.

7.1.5 Practical details

The program was written in Fortran IV on 64 character punched cards, which were ideal for making improvements. Input data are also on cards, output data using the line printer. When initially written the program used a number of independent Subroutines which were tested before the program was finally run as a whole. Eventually, most Subroutines were eliminated.

At first, the program was run on The City University ICT 1900 Computer, taking about 50 minutes for two runs of 40 axial and 30 radial positions. Then, using the University of London CDC 6600, this time was reduced to 1 - 1½ minutes. Finally, the ULCC CDC 7600 was used and with the program as in this Thesis, the time taken for a single run with 80 radial and 60 axial positions was 325 seconds of core time; however, the core-sharing procedure of this computer makes the comparison difficult.

7.1.6 Conclusion

The essential features of this approach may be summarised as:

- (i) Use of velocities (instead of stream function, and velocity potential).
- (ii) Direct solution for each axial position by Gaussian elimination, with a check solution (instead of full iteration).
- (iii) Marching-type solution for the unknown field (instead of iteration).

7.2 Input Data

A total of 7 or 8 cards is necessary for each run. All input parameters are explained below.

Card 1 (Integers)

ILAST Total number of axial positions.

Card 6 (Real number)

QDISS Viscous dissipation parameter. (For no effect, use
 0.0).

Card 7 (Real number)

ZBARX Axial length for partial adiabatic development (Card
 only used for ISUNU = 2).

Card 8 (Real number)

TWALL Wall temperature, for IFTOQ = 1
or HFLUX Wall heat flux, for IFTOQ = 2

7.3 Output Data

For each step, the following data are given:

ITALLY Step number, zero at entry, increased by 1 for each
 step, terminating as ILAST.

DELTZ Current value of ΔZ .

ZBAR Axial distance Z, zero at entry, increased by ΔZ after
 each step.

TMEAN Bulk mean temperature T_m .

TWALL Wall temperature for IFTOQ = 2.

NUSSLT Nusselt number.

TBAR Temperature obtained from heat input considerations,
 in this analysis co-incident with TMEAN.

Also Axial velocity profile (U)
 Radial velocity profile (V)
 Temperature profile (T)
 Pressure profile (P)

At the end of each run

ISEC Time taken for the run.

7.4 Heat transfer Evaluations

7.4.1 TMEAN and TBAR (Fluid mean temperatures)

TMEAN is the bulk mean, or mixing-cup, temperature at an axial

position, defined by:

$$T'_{\text{mean}} = \frac{\int_0^R \rho u_z T' (2\pi r') dr'}{\int_0^R \rho u_z (2\pi r') dr'}$$

the dimensionless equivalent is

$$T_{\text{mean}} = \frac{\int_0^1 \rho U T r dr}{\int_0^1 \rho U r dr}$$

and the finite-difference equivalent is

$$T_{\text{MEAN}} = \frac{\sum_{m=2}^M (m-1) U_{n+1,m} \cdot T_{n+1,m} \cdot \rho}{\sum_{m=2}^M (m-1) U_{n+1,m} \cdot \rho}$$

In equation (25), constant density is assumed, but in the program proper the above refinement is made on the grounds that near the wall contributions to TMEAN are significant, and it is there that density changes due to temperature are greatest. ρ is given the quadratic temperature allowance.

TBAR, originally a check temperature obtained by an energy balance, but here made to coincide with TMEAN, is defined for constant heat flux by:

$$\left(\frac{Q}{A}\right) \cdot \pi \cdot (2R) \cdot \Delta z' = (\pi R^2) U_o C_{pm} \rho_m (\overline{T}_{n+1}' - \overline{T}_n') \quad \text{Neglecting viscous dissipation}$$

Non-dimensionally this gives:

$$\overline{T}_{n+1} - \overline{T}_n = \frac{2 \cdot \text{HFLUX} \cdot M \cdot \Delta Z}{Re \cdot Pr} \cdot \frac{\rho_m}{\rho_A} \quad (28)$$

(again, allowing a bulk density correction, where equation (25) assumed constant density).

The final step is to make TBAR coincident with TMEAN to obtain the finite-difference equation (25).

7.4.2 TWALL (Wall temperatures)

Wall temperatures need only be calculated for the constant heat flux case. Using the constrained energy balance TWALL does not enter the T matrix, but is calculated subsequently as follows.

Dimensionally, the relationship

$$\frac{Q}{A} = k \left(\frac{\partial T}{\partial r} \right)_R = \text{const.} \quad (\text{Fourier's Law})$$

becomes the dimensionless

$$\text{HFLUX} = \frac{1}{M} \left(\frac{\partial T}{\partial r} \right)_1$$

Using a second-order (truncation error) equivalent, TWALL is given by equation (27).

Since this procedure is essentially a curve-fitting exercise to the solved temperature field, as checks third and fourth order equivalents were also incorporated into the program. Using expressions from Hausner (Ref. (107) p.433), the 3rd and 4th order equations for TWALL are respectively:

$$\text{TWALL} = [(6 \times \text{HFLUX}) + 18T_M - 9T_{M-1} + 2T_{M-2}]/11 \quad (27)A$$

$$\text{TWALL} = [(12 \times \text{HFLUX}) + 48T_M - 36T_{M-1} + 16T_{M-2} - 3T_{M-3}]/25 \quad (27)B$$

It may be objected that since the finite-difference equivalents within the field are second order (discussed in (53)), it would be inappropriate to use these higher order relationships. However, they are extrapolations where the temperature gradient is highest. Also, as emphasised in Appendix 6, for constant heat flux the Nusselt number is a function of wall temperature, so TWALL is a significant parameter.

7.4.3 Nusselt number

This is a local Nusselt number based on the wall-to-mean temperature difference.

Hence:

$$\text{Nu} = \frac{2(Q) R}{k(A)(T_R' - T_m')}$$

becomes:

$$\text{Nu} = \frac{2 \times \text{HFLUX} \times M}{(\text{TWALL} - \text{TMEAN})} \quad (29)$$

7.5 Constant wall temperature boundary condition

This will be discussed in Chapter 12. However, at this stage two alternatives may be discerned.

A) Use of the explicit wall temperature

If equation (22) is written for the outermost fluid station $m = M$, the unknown $T_{n,M+1}$ is the known input T_{WALL} , so the temperature matrix may be solved.

Using, for example, equation (27)B, the wall heat flux $HFLUX_T$ is given by

$$HFLUX_T = \frac{1}{12} [-25T_{WALL} + 48T_M - 36T_{M-1} + 16T_{M-2} - 3T_{M-3}]$$

and

$$Nu = \frac{2 \times HFLUX_T \times M}{(T_{WALL} - T_{MEAN})} \quad (29)A$$

Equation (28) then becomes a check energy balance, where $HFLUX$ is replaced by $HFLUX_T$, and T_{BAR} by T_{MEAN} .

B) Use of the implicit boundary condition

However, the logic may be reversed. The T_{WALL} version of equation (28) may be used (instead of (22) for $m = M$) in the temperature matrix. In this case, both sides of equation (28) include the unknown temperatures, but it will be appreciated that it is feasible to re-express the equation as

$$\sum F_1 \cdot T_m = B, \text{ as in Subsection 7.1.2.}$$

Nu is again given by equation (29)A.

7.6 Friction factor evaluations

Local pressure drop and friction factor are important performance data. The local friction factor may be defined either in terms of pressure drop (e.g. Martin and Fargie (83)) or wall shear stress (e.g. Worsøe-Schmidt and Leppert (53)). The two coincide only when density is constant, and momentum changes zero.

7.6.1 Pressure drop

The local friction factor, f_p , is defined dimensionally, non-dimensionally, and in finite-difference terms respectively, as:

$$\begin{aligned} f_p &= \frac{R}{\rho_m U_o^2} \cdot \frac{\partial p}{\partial z'} \\ f_p &= \frac{\partial P}{\partial z} \\ f_p &= \frac{P_{n+1} - P_n}{\Delta Z} \end{aligned} \quad (30)$$

the corresponding average friction factor, f_{av} , is

$$f_{av} = \frac{P_{n+1} - P_o}{Z} \quad (31)$$

As Allen (108) has pointed out, when density changes occur, they contribute to pressure differences. On elimination of these, a radial pressure distribution will result. This means that the non-dimensional P in equations (30) and (31) should be replaced by a weighted mean pressure over the cross-section. This is defined non-dimensionally by:

$$\bar{P} = 2 \int_0^1 P \cdot r \cdot dr. \quad (32)$$

7.6.2 Wall shear stress

The local friction factor, f_s , is defined dimensionally and non-dimensionally respectively as:

$$\begin{aligned} f_s &= \frac{\mu}{\frac{1}{2} \rho U_o^2} \left(\frac{\partial v_z}{\partial r'} \right)_R \\ f_s &= \frac{2}{Re_o} \cdot (T_\alpha)_1 \cdot \left(\frac{\partial U}{\partial r} \right)_1 \end{aligned} \quad (33)$$

$(T_\alpha)_1$, is the viscosity temperature dependence term at the wall (equation (14)). $\left(\frac{\partial U}{\partial r}\right)_1$ is obtained by a similar formula to $\left(\frac{\partial T}{\partial r}\right)_1$,

except that here $U_1 = 0.0$. Hence, the third and fourth-order equivalents respectively, are:

$$\left(\frac{\partial U}{\partial r}\right)_1 = \frac{1}{6} (U_M - 9U_{M-1} + 2U_{M-2}) \quad (34)$$

$$\left(\frac{\partial U}{\partial r}\right)_1 = \frac{1}{12} (48U_M - 36U_{M-1} + 16U_{M-2} - 3U_{M-3}) \quad (34)A$$

7.7 Structure

The flow pattern is given as Section 7.8. Subroutines are as follows:

MAT This solves the set of equations for U, V, P and then T, by the Gaussian elimination procedure.

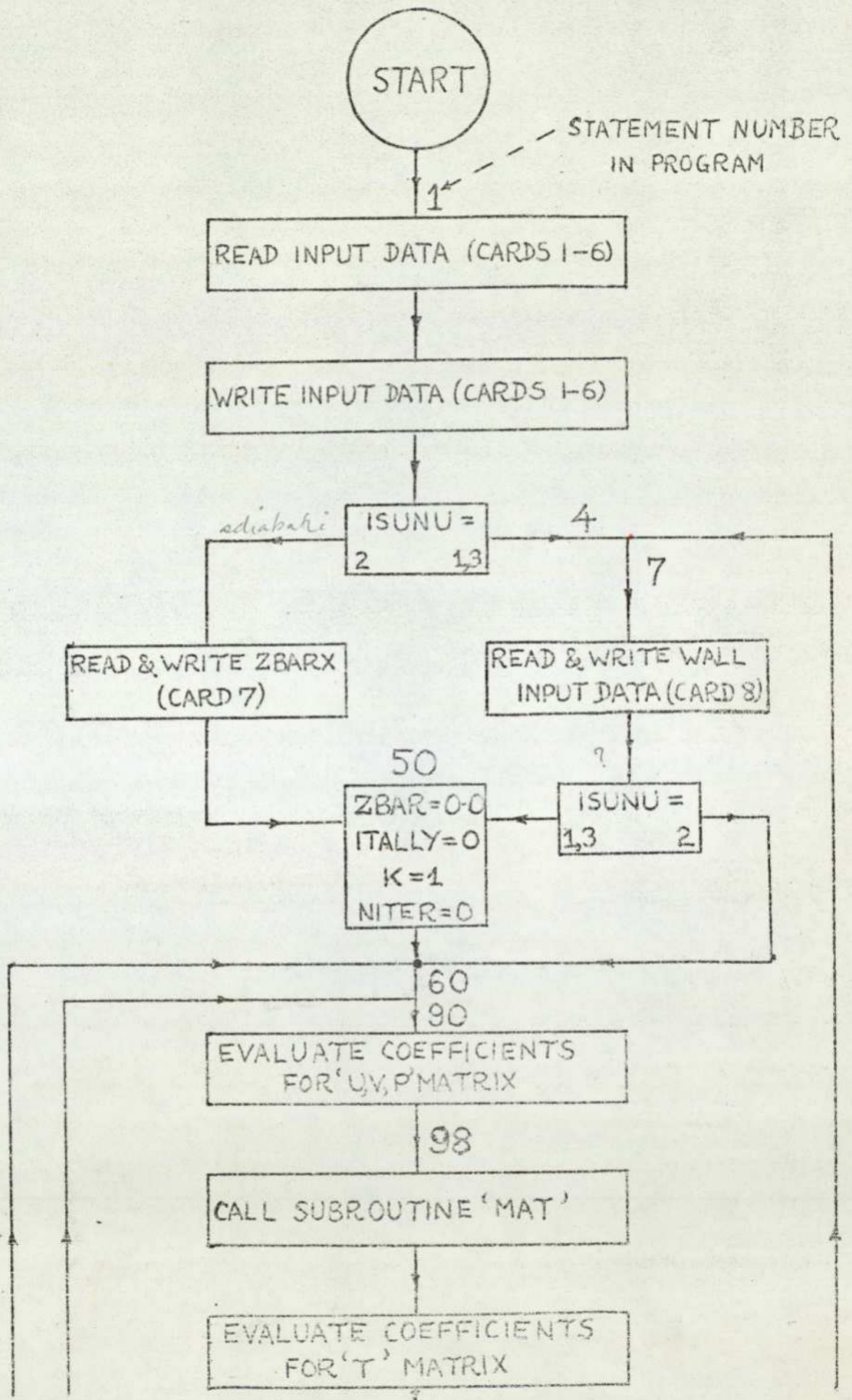
ITIME This calculates the run time

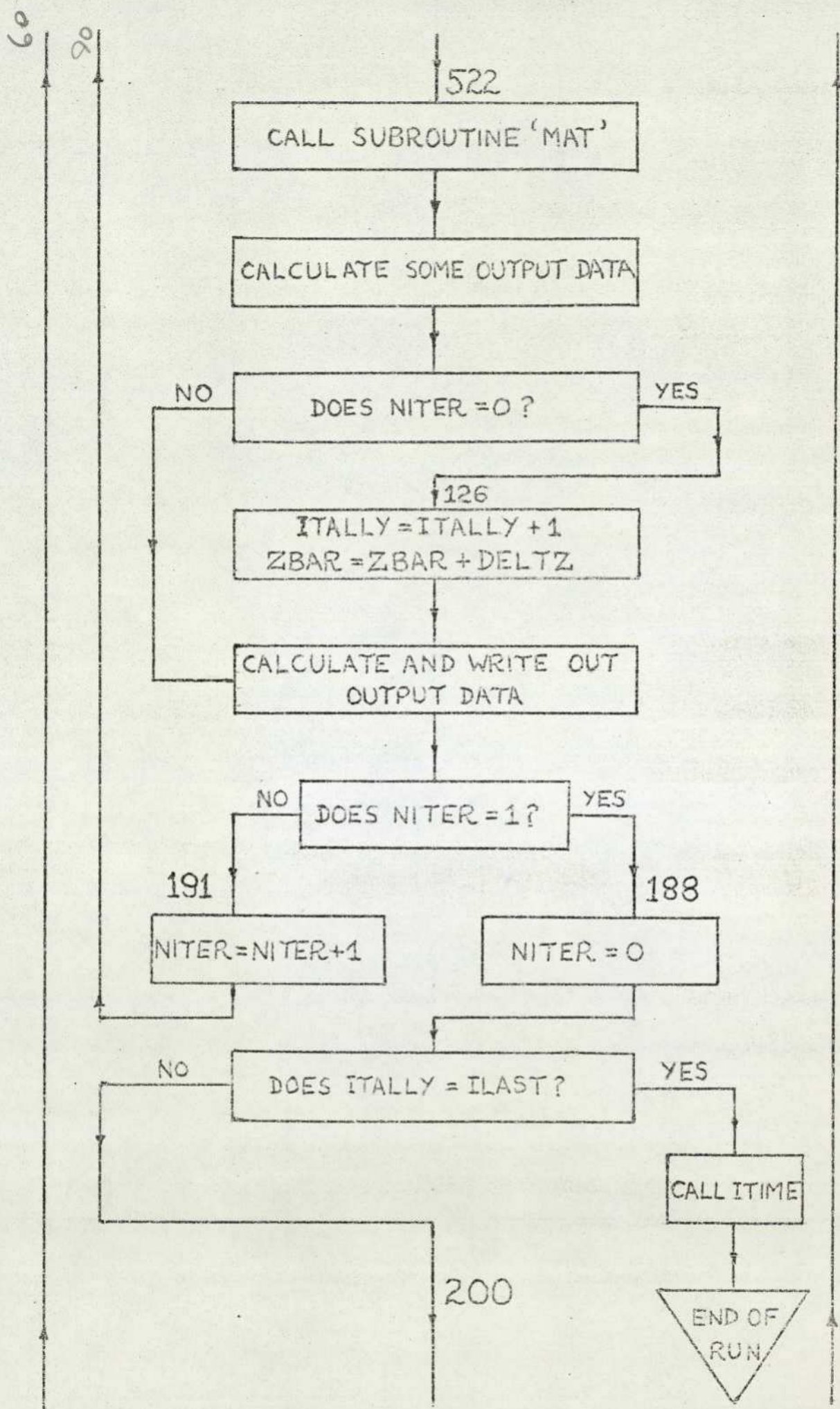
The following looped variables are used:

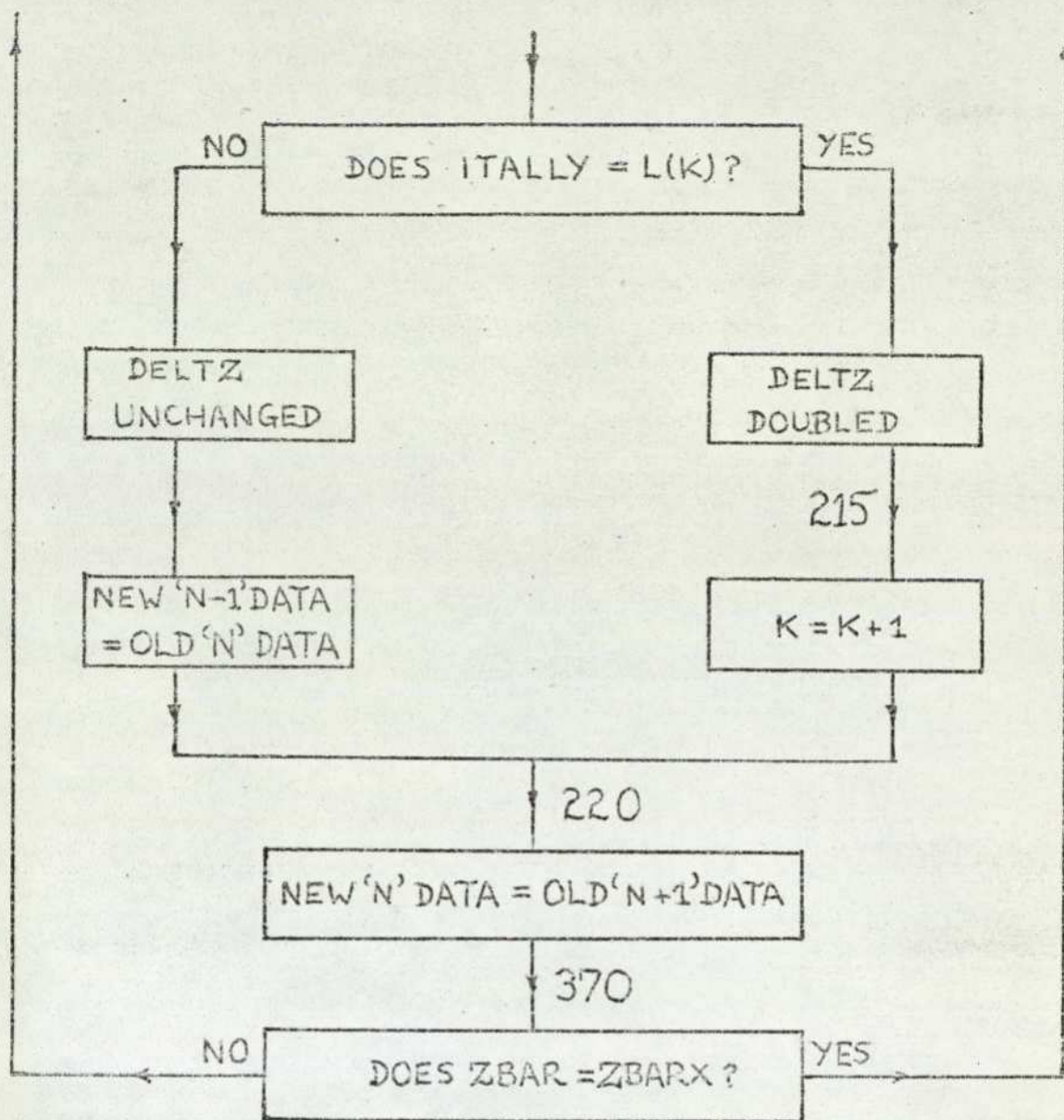
ITALLY When ITALLY reaches one of the specified values of L(K), DELTZ is doubled. When ITALLY reaches ILAST a run is complete.

K This is the step size doubling parameter. K is initially 1, then increased by 1 each time the step size is doubled. This enables ITALLY to be compared with the current L(K).

NITER This loop variable controls the semi-iteration check on the solution of U, V, P and T. It starts at 0, and is increased by 1 after solving the T matrix. On reaching 2 it is made 0 again.







P A R T I I I

R E S U L T S

8. ADIABATIC FLOW

8.1 Published paper

As Chapter 4 explains, a large proportion of the work reported by this Thesis has already been published. Although it is not necessary, therefore, to repeat the material, which is incorporated in the relevant Appendices, a logical order of presentation is maintained, with appropriate reference made to the published work in question.

Adiabatic flow results are reported in Appendix 3, but some additional discussion is needed to complete consideration of the topic.

8.2 Data of Zeldin and Schmidt (36)

Zeldin and Schmidt report developing velocity profiles for the case of forced convection ($Re = 254.75$) where the fluid and wall temperatures differed by less than $0.5K$, i.e. effectively adiabatic flow.

Table 5 compares Zeldin and Schmidt's data with predictions of Collins from (i) Appendix 3 ($M=30$) and (ii) a recent study with $M=80$.

$\frac{Z}{Re}$	Zeldin and Schmidt		Collins	
	Experiment	Analysis	$M=30$	$M=80$
0.01127	1.365	1.34	1.36	1.35
0.043	1.614	1.625	1.64	1.635
0.116	1.883	1.883	1.885	1.885
0.4915	2.00	2.00	1.995	1.999

TABLE 5 - DEVELOPING CENTRE-LINE
VELOCITY COMPARISON

The Table shows that the overall agreement with measurement is about the same for Collins as for Zeldin and Schmidt. Also, the refinement to 80 radial positions affects results only marginally.

8.3 Data of Fargie and Martin (82)

Fargie and Martin reported centre-line velocities significantly lower than predictions which originally were considered to be due to

viscous dissipation effects. However, since the entry used was an abrupt contraction, the results may not be strictly comparable with theory. However, Figure 3 of Appendix 6 does include a relevant comparison between Collins's and Fargie and Martin's adiabatic centre-line velocities.

8.4 Development lengths for low Reynolds numbers

It is only at lower Reynolds numbers that the elliptic terms representing diffusion (of vorticity (92)) are not insignificant. Then the solution becomes somewhat Re-dependent. Table 6 compares predictions of development length, of Friedmann et al (91), Vrentas et al (92) and Collins.

Re	Friedmann et al	Vrentas et al	Collins
250	0.224	-	0.2435
200	0.224	-	-
150	0.224	-	-
125	-	0.214	0.241
100	0.226	-	-
75	0.226	0.192	0.245
50	0.226	-	0.249
25	-	0.188	-
20	0.244	-	-
10	0.270	-	-
5	0.352	-	-
0.5	-	1.32	-

TABLE 6 - DEVELOPMENT* LENGTH COMPARISON

There is no real consistency in these results, and probably, (c.f. Hornbeck (109)) the reason is the difference in the inlet boundary conditions, which are summarised below.

Vrentas et al solved the field from $z' = -\infty$ to $+\infty$, with $V=0$ and $U=1$ at $z' = -\infty$, and at $z'=0$ the condition $V=U=0$ at the wall.

* i.e. 99% of fully-developed centre-line velocity

Friedmann et al solved the field from $z'=0$ to $+\infty$. A uniform axial profile was assumed at $z'=0$, but the vorticity over the boundary had to be modified to give a stable solution.

Schmidt and Zeldin also solved the $0 \rightarrow \infty$ field, assuming a uniform inlet profile without upstream vorticity diffusion, but did allow a radial pressure profile there.

Thus the three alternative elliptic solutions have different boundary conditions from each other and from Collins, whose conditions are closest to those of Schmidt and Zeldin.

8.5 Friction factor results

Figures 2 - 4 compare the three friction factors obtained, with predictions in the literature. These predictions involve a variety of analytical procedures, making the comparison a comprehensive one. Higher and lower Re results of Collins give some idea of Re-dependent effects, and Figure 3 includes a direct comparison for $Re = 250$ with results of Schmidt and Zeldin. The maximum difference is under 7%.

In general, the comparison is good, and gives confidence in the predictive ability for pressure drop and friction effects.

9. CONSTANT WALL HEAT FLUX - FORCED CONVECTION

9.1 Introduction

This chapter discusses results for forced convection with constant wall heat flux. The standard case of constant properties may be compared with data in the literature, and this extends the check on the program's reliability from the solution of the continuity and momentum equations (i.e. adiabatic flow) to the solution of the energy equation.

Further, variable-viscosity forced convection predictions may be compared with experimental and analytical data for water/glycerol mixtures (Butterworth and Hazell (86)) and oil (Martin and Fargie (83)). A large proportion of this further study has been published (Appendix 6).

9.2 Constant properties

9.2.1 Parabolic entry velocity

The simplest case is that of fully-developed flow, and here the solution of the UVP matrix remains static at a parabolic axial velocity profile, zero radial velocity, and constant friction factors. It is only the T matrix solution which develops to an eventual steady state. Further, because of the energy balance, the fluid bulk mean temperature varies linearly with distance, and so the essential heat transfer parameter, the Nusselt number, is a function of the developing wall temperature only.

Table 7 compares results of Collins with recent numerical boundary-layer analyses of Worsøpe-Schmidt and Leppert (53), Bankston and McEligot (103), and Butterworth and Hazell (87), together with analytical results of Leveque (111) and Siegel et al (48), and Kays (numerical) quoted in (87). The general comparison is good; Collins's highest Pe results should be most consistent with the boundary layer assumptions (i.e. negligible axial conduction).

Nu

$\left(\frac{Z}{Pe}\right)$	Kays	Butterworth & Hazell	Leveque(L) and Siegel et al	Bankston & McEligot	Worsøpe- Schmidt & Leppert	COLLINS		Pe =
						1000	12500	25000
5×10^{-5}		56.7	56.5 _(L)			-	60.39	56.89
1×10^{-4}		44.4	44.9 _(L)			-	45.37	44.13
5×10^{-4}		25.7	26.2 _(L)			-	25.42	25.31
0.001		20.2	20.8 _(L)			-	20.05	20.01
0.002			15.84	15.80	15.81	16.41	15.85	15.84
0.005	13.8	11.8				11.80	11.68	11.70
0.01	10.4	9.40	9.295	9.29	9.295	9.36	9.33	9.32
0.02	8.06	7.65	7.49	7.50	7.49	7.53	7.52	-
0.04	6.56	6.25	6.15	6.15	6.15	6.18	-	-
0.06	5.76	5.62	5.54	5.55		5.57	-	-
0.10	5.11	5.02	4.97	4.98		4.99	-	-
0.16	4.74	4.66	4.62	4.63		4.64	-	-
0.2	4.59	4.55	4.51	4.52		4.53	-	-
0.4		4.40	4.375	4.38		4.38	-	-
0.8		4.39	4.36	4.36		4.36	-	-

TABLE 7 - NUSSELT NUMBER COMPARISON - FORCED CONVECTION WITH
CONSTANT HEAT FLUX AND FULLY-DEVELOPED ENTRY FLOW

The range of (Z/Pe) for the three sets of Collins's results is limited for low Pe by the minimum initial value of ΔZ , and for high Pe by the length of run required. However, the common part of the range shows that the downstream results may be regarded as uniform.

Figure 5 compares the fully-developed temperature profile with graphs of Bradley and Entwistle (51) and Worsøe-Schmidt and Leppert (53). The comparison is very good.

9.2.2 Uniform entry velocity

For a uniform entry velocity, Nu depends on Pr , as shown by Figure 6, which compares results with predictions of Martin and Fargie (83) and Butterworth and Hazell (87). It is intriguing that upstream differences between the latter two cause two different Pr curves almost to coincide. However, Collins's results for $Re = 2500$ (comparable with boundary-layer assumptions) lie within the range of the others, for each Pr . The results also show an Re -dependent effect, although this is not mentioned by the other analysts. Later, some specific Re results are compared with data of Butterworth and Hazell.

Figure 7 is a similar comparison for $Pr = 0.7$ (i.e. air) with predictions of Manohar (110), Ulrichson and Schmitz (49) and measurements of Heaton et al (112) (quoted in (49)). Again, Collins's results are for Re of 100 and 2500, with, exceptionally, 10,000 to extend the range of GZ . Agreement is very reasonable, predictions of Collins and Ulrichson and Schmitz almost coinciding, and being above measured values of Heaton et al by 10% or less. The Re -dependent effect near entry is evident in present predictions.

Finally, Table 8 summarises the effect on Nu of the alternative inlet boundary conditions, for air. Agreement is very reasonable except at lowest (Z/Pe) but this is consistent with the differences of Figures 6 and 7.

$\left(\frac{Z}{Pe}\right)$	Bankston & McEligot	Ulrichson & Schmitz	Collins
0.02	1.192	1.22	1.43
0.04	1.124	1.125	1.17
0.10	1.0545	1.065	1.060
0.20	1.018	1.054	1.022

TABLE 8 - HEAT TRANSFER COMPARISON FOR DIFFERENT BOUNDARY CONDITIONS (PR = 0.7, RE = 100)

$$\left(\frac{\text{VALUES OF } \frac{Nu(\text{UNIFORM})}{Nu(\text{PARABOLIC})}} \right)$$

9.2.3 Axial conduction at entry

The inlet thermal boundary condition for this work assumed zero axial conduction. Hennecke (93), for the case of a parabolic velocity profile, concluded that this is reasonable for Pe (based on diameter) above 10. Even for air this only demands an Re_R above 14. A final comment, to be discussed more fully with constant wall temperatures, is that any boundary layer analysis must ignore entry conduction.

9.2.4 Partial adiabatic development

Figure 8 compares results for two values of unheated entry length, with those of Butterworth and Hazell (87). Also shown are the limiting values of 0 (uniform entry velocity) and ∞ (parabolic entry velocity). In general, results of Collins lie below those of B. and H., the difference reducing with distance.

9.3 Water/glycerol Mixtures

9.3.1 Introduction

This study relates to heat transfer measurements and predictions made by Butterworth and Hazell ((86) and (87)). A comparison with their Runs 35 and 36 is already reported in Figure 8 of Appendix 6.* Run 35 had measured values of Nu substantially higher than

* It has no significance, but should be mentioned, that the present definition of GZ differs from that of Appendix 6.

prediction, and this was interpreted by Butterworth and Hazell as being due to (1) viscosity variation with temperature and (2) viscous dissipation. Collins found, however, that (1) was a minor effect and (2) caused a slight reduction. As a check, a comparison was made with run 36, closest in conditions to run 35, and good agreement was reached. Since runs 35 and 36 were for a nearly sharp entry, it was decided to clarify matters by studying some more runs, as summarised in Table 9. Run 49 gives a further sharp entry check, run 47 a typical partial adiabatic development problem, and runs 52 and 55 the common parabolic entry profile case. Run 52 with high $\Delta \bar{T}$ and hence heat flux should clarify the effect of viscosity variation, especially in comparison with run 55 of low $\Delta \bar{T}$.

B. and H. run No.	Adiabatic Z	$\Delta \bar{T}$ °C	Re_o	Pr_o
35	2.5	0.81	85.75	553.2
36	2.5	0.71	271.4	262.6
49	2.5	0.99	715.8	73.14
47	69.375	0.89	757.7	71.25
52	∞	6.72	92.0	77.05
55	∞	0.48	1024.7	73.90

TABLE 9 - BUTTERWORTH AND HAZELL PARAMETERS

9.3.2 Viscosity variation with temperature

Deduced from data tabulated by Miner and Dalton (113) the temperature function used was:

$$\frac{\mu}{\mu_A} = \left[\frac{T' + C_1}{T'_A + C_1} \right]^{-C_2} \quad (35)$$

Besides water/glycerol mixtures, water itself and some oils satisfy this function, which is referred to as defining

VISCOSITY TYPE A

Using the alternative definition of dimensionless temperature T , this becomes:

$$\frac{\mu}{\mu_A} = T_\alpha = \left[\frac{C_{01} \cdot T}{G} + C_{02} \right]^{-C_{03}} \quad (36)$$

$$\text{and } \frac{\partial(\frac{\mu}{\mu_A})}{\partial T(\frac{\mu}{\mu_A})} = T_\beta = \frac{-C_{01} \cdot C_{03}}{G} \left[\frac{C_{01} \cdot T}{G} + C_{02} \right]^{-(C_{03} + 1.0)} \quad (37)$$

$$\text{where } C_{01} = \frac{T'_0}{T'_A + C_1} \quad (38)A$$

$$C_{02} = \frac{T'_0 + C_1}{T'_A + C_1} \quad (38)B$$

$$\text{and } C_{03} = C_2 \quad (38)C$$

9.3.3 Discussion of results

Results for constant and varying viscosity are plotted in Figs. 9 - 11 for the three values of Z , and are compared with measured and predicted Nu of Butterworth and Hazell.

Fig. 9 shows that Collins's constant viscosity prediction this time lies consistently below that of B. and H., the difference reducing from about 15% to 5%. This agreement is less good than for runs 35 and 36, but taking the three together, still permits a conclusion of reasonable agreement. The effect of viscosity variation in Fig. 9 is to increase heat transfer marginally with distance.

Measurements in Fig. 9, however, virtually agree with B. and H.'s prediction, and therefore lie above the present predictions. Again, comparing the latter with measurement for the three runs, there is no consistent trend with either Re , Pr or Pe numbers, which may point to experimental difficulties.

For adiabatic $Z = 69.375$ (Fig. 10 and run 47) there is no longer the uncertainty of the sharpness of entry. The constant property predictions agree to within 10 - 5%, this difference being reduced

somewhat by viscosity variation. However, measurements are again high, and this time diverge with increasing distance.

Runs 52 and 55 shown on Fig. 11 show virtual agreement for predictions, with the usual increase for viscosity variation. The larger temperature rise for run 52 results in a downstream increase in Nu of about 6%. However, the key feature is the substantially and consistently higher level of measured Nu. Apart from experimental uncertainty, it is difficult to postulate a reason for this. There is no doubt about the inlet fluid boundary condition, and the effect of viscosity variation would be least upstream where temperature rises are less. Yet the differences are largest upstream, and on the whole reduce gradually with distance.

As a general conclusion, this further study is one of the least satisfying in terms of agreement of measurement with the present analysis. However, if the comparison in Fig. 11 be viewed as quantifying experimental variation, (this being confirmed by agreement of the predictions) then the agreement for the other runs becomes quite reasonable.

9.4 Oil

9.4.1 Introduction

Martin and Fargie (83) reported fluid flow and heat transfer for M600, an oil of high Pr number. They compared measurement with analytical predictions allowing for viscosity variation, for two runs at the upper and lower bounds of experiment. In both cases, measured Nu values were above analytical ones.

Appendix 6 also reports Collins's predictions for the high temperature extreme of experiment (Martin and Fargie parameter

$\frac{\dot{q}''_D}{kT_0} = 60.3$), which gives good agreement with measurement for (1) developing centre-line velocity (Fig. 3 of Appendix 6), (2) Nu number

(Fig. 6, *ibid*) and (3) friction factor (Fig. 5 *ibid*). However, as with the glycerol/water results above, it was felt desirable to make a comparative study for the low temperature run $\frac{\dot{q}''D}{kT_0} = 11.3$

9.4.2 Viscosity variation with temperature

From a graph supplied by Martin (88), the temperature function used was:

$$\frac{\mu}{\mu_A} = \left(\frac{T'}{T_A'} \right)^{-C_1} \quad (39)$$

Besides M600, the liquids studied by Martin (84) can be expressed by this relationship, which is referred to as defining

VISCOSITY TYPE B

In reality, this is a simpler case of TYPE A.

Using, again, the alternative definition of T, this becomes:

$$\frac{\mu}{\mu_A} = T_\alpha = \left(\frac{T}{G} + 1 \right)^{-C_{01}} \quad (40)$$

$$\text{and} \quad \frac{\partial}{\partial T} \left(\frac{\mu}{\mu_A} \right) = T_\beta = -\frac{C_{01}}{G} \left(\frac{T}{G} + 1 \right)^{-C_{02}} \quad (41)$$

$$\text{where} \quad C_{01} = C_1 \quad (42)A$$

$$\text{and} \quad C_{02} = C_1 + 1.0 \quad (42)B$$

9.4.3 Discussion of results

Results of the present analysis (allowing for viscosity variation but ignoring viscous dissipation) are compared with Martin and Fargie's measurements and variable viscosity analysis in Figs. 12 - 14.

Predicted velocities in Fig. 12 are initially below, then above experiment, the downstream values being 1.895 and 1.835 respectively. This is at least partly attributable to the sharp entry used. However, while U_c for both runs (Fig. 7 Ref. (83)) ends below the standard adiabatic curve, initially the 11.3 parameter run is above that curve

while the 60.3 parameter run is always below, except at the first point: It may be that the sharp entry causes a relative acceleration initially, followed by a steady deceleration, this being Re-dependent and possibly associated with separation and re-attachment. Despite this entry difficulty, the present analysis agrees fairly well with experiment.

Fig. 13 shows poorer agreement with measurements than in Fig. 5 of Appendix 6. For the first two points Martin and Fargie's prediction is closer to experiment than the present one, but thereafter the reverse is true. For example, at the last station Collins and Martin and Fargie underestimate the pressure drop by 24% and 28% respectively. For the other run (60.3 parameter) agreement with both predictions is much better. In the light of the sharp-entry discussion above, possibly part of the discrepancy could be attributed to that.

On the other hand, the Nu comparison in Fig. 14 is good, and has some similarity with the equivalent Fig. 6 of Appendix 6. In both instances the second and third station values are higher than experiment, but this difference rapidly decreases downstream.

Overall, while the comparison for $\frac{\dot{q}''D}{kT_0} = 11.3$ is not quite as good as for 60.3, it is still convincing, and specifically confirms the conclusion that heat transfer may be adequately predicted with the present variable viscosity model.

10. CONSTANT WALL HEAT FLUX - COMBINED CONVECTION

10.1 Additional Property Variation

10.1.1 Introduction

Data in the literature for this problem are all relevant to the results of Allen (90) and predictions of Collins are essentially & Finn's contained in Appendix 9. While Allen's analytical model (101) is approximate in the sense of the equations used, it is comprehensive in respect of the allowed property variations. To make results of this program more comparable with those of Allen and Szpiro, the model was adapted to accommodate additional property variation within the energy equation. Specifically, the thermal conductivity k , thermal diffusivity $\alpha (= \frac{k}{\rho C_p})$, and the product (ρC_p) were allowed to vary linearly with temperature. These assumptions are consistent with the oil in question, and apply to other fluids in so far as changes with temperature are very modest.

10.1.2 Changes in analysis

Equation (4) becomes, when the viscous dissipation term is ignored:

$$v_r \frac{\partial T'}{\partial r'} + v_z \frac{\partial T'}{\partial z'} = \alpha \left(\frac{\partial^2 T'}{\partial z'^2} + \frac{1}{r'} \cdot \frac{\partial T'}{\partial r'} + \frac{\partial^2 T'}{\partial r'^2} \right) + \frac{1}{\rho C_p} \left(\frac{\partial T'}{\partial r'} \cdot \frac{\partial k}{\partial r'} + \frac{\partial T'}{\partial z'} \cdot \frac{\partial k}{\partial z'} \right) \quad (43)$$

Using the dimensionless relationships:

$$\alpha = \alpha_0 (1 + C_{10}T) \quad (44)$$

$$k = k_0 (1 + C_{11}T) \quad (45)$$

$$\rho C_p = (\rho C_p)_0 (1 + C_{12}T) \quad (46)$$

and taking, for example:

$$\frac{\partial k}{\partial r} = \frac{\partial k}{\partial T} \cdot \frac{\partial T}{\partial r}$$

equation (13) becomes:

$$\begin{aligned} V \frac{\partial T}{\partial r} + U \frac{\partial T}{\partial Z} = \frac{(1 + C_{10} T)}{\text{Pr}_o \cdot \text{Re}} \cdot \frac{\rho_m}{\rho_o} \left(\frac{\partial^2 T}{\partial Z^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \right) \\ + \frac{C_{11} \rho_m}{\text{Pr}_o \text{Re}} \left(\frac{\partial^2 T}{\partial r^2} + \frac{\partial^2 T}{\partial Z^2} \right) \end{aligned} \quad (47)$$

In the above:

$$\text{Pr}_o = \mu_A \left(\frac{C_p}{k} \right)_o, \text{ and HFLUX is based on } k_o \text{ rather than } k_m.$$

The mixed mean temperature is given by:

$$\begin{aligned} T'_{\text{mean}} &= \frac{\rho_o C_{p_o} \int_0^R u_{Z,T'} r' (1 + C'_{12} T') dr'}{\rho_o C_{p_o} \int_0^R u_{Z,T'} (1 + C'_{12} T') dr'} \\ T_{\text{mean}} &= \frac{\int_0^1 U \cdot T \cdot r \cdot (1 + C_{12} T) dr}{\int_0^1 U \cdot r \cdot (1 + C_{12} T) dr} \end{aligned}$$

The fluid mean temperature given by heat input becomes:

$$\left(\frac{Q}{A} \right) \pi (2R) \Delta Z' = \pi R^2 U_o C_{p_o} \rho_o (1 + C'_{12} \bar{T}_{n+1}') (\bar{T}_{n+1}' - \bar{T}_n')$$

and

$$\bar{T}_{n+1} - \bar{T}_n = \frac{2\text{HFLUX} \cdot M \cdot \Delta Z}{\text{Re} \cdot \text{Pr}} \frac{\rho_m}{\rho_o} (1 + C_{12} \bar{T}_{n+1}) \quad (48)$$

For consistency and convenience, the ratio $\frac{\rho_m}{\rho_o}$ in equations (47)

and (48) replaces that used in the standard program, namely $\frac{\rho_m}{\rho_A}$, by

simply making $T'_A = T'_O$ for density.

Finally, equation (22) becomes:

$$\begin{aligned} \left[\frac{M}{2V_{n,m}} + \frac{M^2(1-2m)(1+C_{10}T_{n,m})}{2 \cdot \text{Pr} \cdot \text{Re}(m-1)} - \frac{M}{2} \frac{C_{11}(T_{n,m} - T_{n,m-1})}{\text{Pr} \cdot \text{Re}(1+C_{12}T_{n,m})} \right] T_{n+1,m+1} + \\ \left[\frac{U_{n,m}}{\Delta Z} + \frac{(1+C_{10}T_{n,m})(2M^2 - \frac{1}{\Delta Z^2})}{\text{Pr} \cdot \text{Re}} - \frac{C_{11}(T_{n,m} - T_{n,m-1})}{\text{Pr} \cdot \text{Re}(1+C_{12}T_{n,m}) \Delta Z} \right] T_{n+1,m} + \end{aligned}$$

$$\begin{aligned}
& \left[\frac{M^2(3-2m)(1+C_{10}T_{n,m})}{2Pr.Re(m-1)} - \frac{M}{2V_{n,m}} + \frac{M^2}{2} \frac{C_{11}(T_{n,m}-T_{n,m-1})}{Pr.Re(1+C_{12}T_{n,m})} \right] T_{n+1,m-1} \\
& = \left[\frac{(T)_{n,m} \cdot U_{n,m}}{\Delta Z} - \frac{1}{Pr.Re \Delta Z^2} \left\{ \frac{(C_{11}(T)_{n,m}(T_{n,m}-T_{n-1,m}) + (1+C_{10}T_{n,m}) \times \right. \right. \\
& \quad \left. \left. (2T_{n,m} - T_{n-1,m}) \right\} \right] \quad (49)
\end{aligned}$$

Where all T's in square brackets, because they represent property values, are updated from n,n-1 values to n+1,n values respectively, when the second temperature solution is made at a given step. This remark does not apply to $(T)_{n,m}$ in B brackets.

The program is given as Appendix 7. Explanatory comment cards are omitted, but effects may be deduced from a comparison with the original standard program, Appendix 2.

The three additional property parameters are required as CTEN, CTEN1, and CTEN2 (from equations (44) to (46) respectively) on Card 5 (Section 7.2).

10.2 Paper submitted for Publication

The essence of the data reported in this Chapter is given as Appendix 9, and the remarks below discuss the logical relevance. The viscosity relationship for each liquid is of Type A, and all data are for upward heated flow.

10.3 Results for water

10.3.1 Lawrence and Chato's data (33) (105)

Input data for one run of this program was defined after some difficulty, and output data in the form of developing velocity profiles were checked against predictions and limited measurements given in (33). The comparison was good, and the effect of additional property variation negligible.

10.3.2 Scheele and Hanratty's data (35)

Input data for two runs of this program could be defined in terms of dimensionless parameters. Output data for wall temperatures

were compared with (i) measurement and (ii) results from an approximate theoretical analysis, which did not give good results close to entry. Predictions of this program can be reasonably said to resolve this problem.

Additional property variation was not incorporated here, because (a) the accuracy of the input data was not felt to justify it, and (b) the effect in 10.3.1 was so small.

10.3.3 Conclusion

Although information in the literature is somewhat limited, that of Scheele and Hanratty complements that of Lawrence and Chato, thus confirming the reliability of this program in predictions for this liquid.

10.4 Results for oils

10.4.1 Allen's data (90)

Measurements and predictions of wall temperatures and overall pressure drops were available, together with predicted outlet centre-line velocities. Three runs (covering the range of parameters) were studied for comparative purposes. Although Appendix 9 uses this program as a check for Allen's approximate analysis, the overall agreement may fairly be viewed as a reverse check also. With the fully-developed entry velocity profile, approximations in theory are felt to be more justifiable.

10.4.2 Kemeny and Somers's data (61)

It was only possible to deduce input data for one run of this program. Results could be compared with measured wall temperatures up to and after transition from laminar flow. Despite the sharp entry used in experiment, predicted temperatures up to transition compared very reasonably, and axial velocity predictions showed that transition would occur in approximately the correct region.

10.4.3 Conclusion

For oils, the effect of allowing for additional property variation is not necessarily negligible, but depends on the magnitude and sign of the property changes with temperature. Taken together, data for oils provide a further convincing check on this program's results.

11. VISCOUS DISSIPATION

11.1 Introduction

The material for this Chapter is given in Appendix 6. The fluid viscosity is of Type B, except for the water-glycerol mixtures of Type A.

11.2 Adiabatic flow

Measurements of wall and mean temperatures were available and compared with predictions of Martin (84). The results of this study lie generally within the range of that comparison.

11.3 Forced convection (constant wall heat flux)

Here it is shown that viscous dissipation has a small retarding effect, as expressed by the Nusselt number. Additional data of Test (89), for forced convection with constant wall temperature, and Iqbal et al (99) for combined convection, could not be incorporated into the study. This was because of a certain lack of data essential for input purposes, apparent from Appendices 11 and 8 respectively. However, the mentioned smallness of the effect is quite consistent with their results.

12. CONSTANT WALL TEMPERATURE - FORCED CONVECTION

12.1 Introduction

12.1.1 Constant Properties

This chapter reports (entirely unpublished) results for forced convection with constant wall temperature. The standard case of constant properties relates heat transfer results to a number of references. Unfortunately, an intention to complement this with a variable viscosity study (Test (89)) was frustrated by a lack of property data (Appendix 11). However, the constant properties study has some interesting features.

12.1.2 Alternative treatments

Firstly, as shown in Section 7.5, there are the alternative treatments of the wall thermal b.c. statement, viz. (1) the use of the explicit finite-difference energy equation at the wall, or (2) the use of the implicit energy balance equation.

12.1.3 Significance of T'_m

In Appendix 6 (Section 5.2) the point is made that Nu for constant heat flux is basically a function of wall temperature.

For constant wall temperature T'_R , and constant k , $Nu \propto \frac{q_T}{T'_R - T'_m}$

where q_T is the wall heat flux rate, and for a given axial length is a function of T'_m . Hence Nu is a function of T'_m . While it is difficult to deduce changes in T'_m from values of (local) Nu ,* fortunately Jones (95) and Singh (94) tabulate predictions of T'_m . These form a very useful check, therefore.

12.1.4 Discussion of literature

Besides Jones and Singh, Zeldin and Schmidt (36) provide an elliptic type solution. Their Nu values are criticised by Marner

* Conversely, Test (114) found his experimental values of Nu unsatisfactory, because they were based on changes in T'_m , between adjacent axial positions, of a few tenths of a degree.

(115) as being too high. Marner usefully pinpoints the difference in entry thermal b.c. as a contributory factor, but does not note the effect of this on T_m' . This issue will be dealt with fully.

In addition, various relevant boundary layer type analyses are comparable, and for the sake of clarity are listed with the above in Table 10, together with the comparisons made with the results of this work.

Authors and References	Type of solution	Fluid entry Profile	Comments	Comparison with this work
Jones (95)	Elliptic	Developed	} Mixed mean } temperat- } ures given	Table 11
Singh (94)	Elliptic	Developed		Table 12 Table 13 Table 14
Zeldin and Schmidt (36)	Elliptic	(1) Developed (2) Uniform		(1) Table 15 (2) Figure 15
Worsøe-Schmidt & Leppert (53)	Parabolic	Developed		} Table 16 }
Butterworth & Hazell (87)	Parabolic	Developed		
Sellars et al (111)	Parabolic	Developed		Table 15
Ulrichson & Schmitz (49)	Parabolic	Uniform	} Pr = 0.7 }	Figure 16
Manohar (110)	Parabolic	Uniform		

TABLE 10 - COMPARISON WITH LITERATURE
(CONSTANT WALL TEMPERATURE)

12.2 Parabolic Entry Velocity

12.2.1 Comparison with Jones and Singh

Predictions of Nu of Jones and Singh, are compared with those of Collins for (1) Explicit wall thermal b.c., and (2) Implicit energy balance statement. Tables 11, 12 and 13 are for Pe = 50, 100 and 1000, respectively. These give a good range of conditions, from where axial conduction could be significant, to where it should not be.

Table 14 effectively compares mixed mean temperature for the above values of Pe . In this case, Collins's predictions used the implicit statement for the wall thermal b.c.

$\frac{Z}{Pe}$	Jones	Collins (1)	Collins (2)
0.025	5.657	6.858	6.288
0.05	4.657	5.005	4.957
0.10	4.011	4.141	4.129
0.15	3.795	3.862	3.855
0.20	3.712	3.747	3.742
0.25	3.679	3.701	3.696
0.50	3.657	3.655	3.652
1.00	3.657	3.637	3.632

TABLE 11 - Nu COMPARISON ($PE = 50$)

$\frac{Z}{Pe}$	Jones	Singh	Collins (1)	Collins (2)
0.010	7.508	8.079	9.722	9.690
0.025	5.624	5.976	6.034	6.000
0.05	4.644	4.759	4.803	4.795
0.10	4.006	4.034	4.068	4.065
0.15	3.793	3.803	3.835	3.834
0.20	3.710	3.715	3.735	3.734
0.25	3.678	3.680	3.695	3.694
0.50	3.657	3.66	3.655	3.653
1.00	3.657	3.66	3.636	3.635

TABLE 12 - Nu COMPARISON ($PE = 100$)

$\frac{Z}{Pe}$	Jones	Singh	Collins (1)	Collins (2)
0.010	7.466	8.077	7.590	7.589
0.025	5.613	5.975	5.677	5.677
0.05	4.640	4.758	4.683	4.683
0.10	4.005	4.033	4.037	4.037
0.15	3.792	3.803	3.819	3.819
0.20	3.710	3.715	3.727	3.727
0.25	3.678	3.680	3.690	3.690
0.50	3.657	3.66	3.654	3.654
1.00	3.657	3.66	-	-

TABLE 13 - Nu COMPARISON (PE = 1000)

$\frac{Z}{Pe}$	Pe = 50		Pe = 100			Pe = 1000		
	Jones	Collins (2)	Jones	Singh	Collins (2)	Jones	Singh	Collins (2)
0.010	0.892	-	0.893	0.91	0.902	0.893	0.90	0.894
0.025	0.812	0.823	0.812	0.82	0.818	0.812	0.82	0.813
0.05	0.716	0.726	0.716	0.73	0.721	0.716	0.73	0.717
0.10	0.579	0.587	0.579	0.59	0.582	0.579	0.59	0.581
0.15	0.477	0.485	0.477	0.48	0.481	0.477	0.48	0.480
0.20	0.395	0.402	0.395	0.40	0.401	0.395	0.40	0.400
0.25	0.329	0.337	0.329	0.33	0.335	0.329	0.33	0.333
0.50	0.132	0.143	0.132	0.134	0.141	0.132	0.134	0.136
1.00	0.021	0.028	0.021	0.02	0.028	0.021	0.02	-

TABLE 14 - COMPARISON OF VALUES OF $(1 - \frac{T'_m}{T'_R})$

A number of points arises from Tables 11 - 14. Firstly, there is the comparison between results of Jones and Singh. Whereas Singh assumed a uniform entry temperature, hence zero entry T'_m , Jones solved for the region downstream of entry, with a zero wall temperature there. This gives a positive entry T'_m , which explains why Jones's temperatures from Table 14 are higher than Singh's, despite his Nu values being lower (Tables 12 and 13). Also, although the Nu values

for both Jones and Singh vary with Pe at the same (Z/Pe) value (Tables 11 - 13), the mean temperatures from Table 14 do not. This suggests that T'_m is possibly a more fundamental description of heat transfer than Nu in this context.

Turning to Collins's results (ignoring differences between (1) and (2)) it would be expected that they agree with Singh since the entry thermal b.c. is the same, but that Jones's solution would be the preferable standard of comparison. Collins's Nu 's are higher and lower than those of Singh in Tables 12 and 13 respectively, but the comparison of Table 14 is more illuminating. At $Pe = 1,000$, where the entry b.c. statement is probably less important, Collins's results virtually coincide with Jones's, this agreement being less good at lower Pe . However, at $Pe = 100$, the predictions lie about midway between Jones and Singh. Given that a change in statement for thermal entry is not wanted on the grounds of complexity being avoided, this is very creditable agreement.

12.2.2 Comparison of alternative solutions

The final issue is the comparison between Collins (1) and (2), ignoring the other solutions. Although initially the values of Nu differ somewhat, they rapidly converge downstream (Tables 11 - 13). Even these differences reflect only marginally different solutions of the temperature profile, and in Table 14, for example, at $Pe = 100$, solution (1) gives 0.8945 and 0.813 for the first two positions tabulated.

In finite-difference terms the agreement between the two solutions is remarkable. Within the author's knowledge, while other numerical solutions mention energy balance checks for the constant heat flux case (see Chapter 16), none do for constant wall temperature. These results imply a virtually exact energy balance for using alternative solution (1). Alternative solution (2) is a quite new way of

dealing with this wall b.c., as it was with constant heat flux. The fact that it can be used as successfully as the ~~ex~~PLICIT statement is quite interesting.

This leaves the actual choice of solution. A detailed study of the alternatives' results showed that the implicit statement method (2) was very marginally closer to Jones's results than that of (1). For this reason, and also to retain the implicit statement method through the Thesis, method (2) was chosen, but it is conceded that theoretically solution (1) may be more correct.

Perhaps one or two reservations should also be stated. The wall heat flux was assessed on the temperature profile at the end of each axial step, but near the entrance the flux may in fact vary slightly over a ΔZ . In fact, an attempt was made to use the average of values at start and end of ΔZ , but this led to oscillation with the semi-iteration, and had to be omitted. In any case, the approximation is not severe. Another difficulty, found with both solutions, is that the progression of T'_m towards T'_R for high Z/Pe (Table 14) is somewhat slow, and this is reflected in correspondingly low Nu values. Expressed as a percentage of the wall temperature, the maximum difference in Table 14 is 0.7%. (Collins compared with Jones).

12.2.3 Comparison with Zeldin and Schmidt

Marner (115) tabulates Nu of Zeldin and Schmidt, for $Pe = 126.25$, and compares this with results of Sellars et al (111) Table 15 further compares them with results of this analysis.

$\frac{Z}{Pe}$	Sellars et al	Zeldin & Schmidt	Collins
0.00905	7.713	8.01	9.357
0.01282	6.893	7.076	7.766
0.02198	5.831	5.919	6.217
0.05128	4.610	4.641	4.730
0.09524	4.037	4.059	4.101
0.2052	3.704	3.722	3.716
0.4616	3.656	3.659	3.655

TABLE 15 - Nu COMPARISON ($Pe = 126.25$)

Marner criticised Zeldin and Schmidt's results as being too high, but Collins's are even higher. At this Pe value the entry b.c. is more important, and Collins's results (Table 14) tend to under-predict heat transfer relative to Jones. If, therefore, we accept Jones's solution as using a preferable entry b.c., then the results of Table 15 indicate underprediction increasing from right to left, because all assume zero entry T'_m .

Comparing Table 15 (Sellars et al) with Table 12 (Jones), at roughly the same Pe , it can be seen that the Nu values are approximately the same, thus causing Marner to conclude the former to be a reliable solution in this context. As has been seen, however, consideration of T'_m leads to an opposite conclusion.

12.2.4 Comparison with boundary-layer solutions

A comparison with one boundary-layer solution, viz Sellars et al, has been made. Table 16 is a further comparison, using Collins's results for $Pe = 1000$, which should be high enough.

$\frac{Z}{Pe}$	Collins	W-Schmidt & Leppert	Butterworth & Hazell
0.010	7.589	7.47	7.53
0.025	5.677	5.55	5.70
0.05	4.683	4.65	4.55
0.10	4.037	4.00	3.95
0.15	3.819	3.80	3.73
0.20	3.727	3.70	3.70
0.25	3.690	3.67	3.67
0.50	3.654	3.66	3.65
1.00	-	3.66	3.65

TABLE 16 - Nu COMPARISON WITH PARABOLIC EQUATION SOLUTIONS

The agreement is within about 2% maximum, which is very acceptable. Strictly, the Sellars et al results should appear here rather than in Table 15, and these differ from Collins (Table 16) by perhaps less than 2%, allowing for differences in $\frac{Z}{Pe}$.

12.3 Uniform Entry Velocity

12.3.1 Comparison with Zeldin and Schmidt

Figure 15 compares results for a specific Re and Pr with those of Zeldin and Schmidt. Apart from extreme entry, agreement is very reasonable, the curve of Collins lying first above, then slightly below the other. Comparing this graph with Table 15 (same Pe, but parabolic entry profile) it may be noted that the point at which the predictions cross is closer to entry for this case.

A further comparison with the (parabolic-equation) solutions of Manohar and Ulrichson and Schmitz, for Pr = 0.7, is given in Figure 16. The two predictions of Collins are for (1) probably ignorable and (2) possibly significant axial conduction, for Pe = 1750 and 175 respectively. The former, except at extreme entry, almost agrees with Manohar and is within about 7% of Ulrichson and Schmitz's results.

Except at entry, the differences are so slight, that it is felt inappropriate to suggest reasons for them.

12.4 Conclusion

The constant properties problem has revealed some interesting aspects. In conclusion, two runs reported in Chapter 14 are for constant wall temperature forced convection with variable viscosity. They are, however, of little interest in this context.

13. CONSTANT WALL TEMPERATURE - COMBINED CONVECTION

13.1 Introduction

13.1.1 Sources

This Chapter reports unpublished results for combined convection with constant wall temperature, and in several ways it is an extension of Chapter 12. There are two main sources, viz. Zeldin and Schmidt (36), giving an elliptic equation analysis and measurement, and Marner and McMillan (38) with a boundary-layer analysis. Table 17 summarises the comparative studies which can be made.

Problem	Data	Source	Comparison
1. Parabolic entry profile (Pe=126.25)	Nu U_{L}	(Z & S(36) (M (115) Z & S	Table 18 Table 19*
2. Parabolic entry profile (Pr = 1)	Velocity profiles Temperature profiles Nu	M & M(38)	(Table 20 (Fig. 17 Fig. 18 Fig. 19
3. Uniform entry profile (Pe=126.25)	U_{L} Nu	Z & S	Fig. 20 Fig. 21
4. Uniform entry profile(Experiment)	U_{L} Velocity profiles Temperature profiles Mixed mean temps.	Z & S	(Fig. 22 (Table 22* Fig. 23 Fig. 24 Table 23*

* Also compares alternative solutions for wall thermal b.c.

TABLE 17 - KEY TO COMPARATIVE STUDIES

13.1.2 Problems

Two specific problems may also be discussed. As with forced convection, Marner (115) criticises the level of Zeldin and Schmidt's Nu values. This time, he compares them with his own predictions.

Secondly, Zeldin and Schmidt were unable to predict their measured mixed mean temperatures, and concluded that there had been

preheating of the air through the plastic nozzle used.

These problems are discussed from Tables 18 and 23 respectively.

13.1.3 Property variations

(i) DENSITY

The simpler linear relationship is used by both Zeldin & Schmidt and Marner & McMillan, viz:

$$\rho = \rho_0 (1 - (T' - T'_0)) \quad (50)$$

thus for the dimensionless T,

$$\rho = \rho_0 (1 - T) \quad (51)$$

$$\text{i.e. } C_{04} = 1.0, C_{05} = -1.0, C_{06} = 0.0 \quad (52)$$

Except for Problem 4, the above is used for comparative reasons.

(ii) VISCOSITY

Again, the assumption of constant viscosity is used by both (36) and (38). For the application in question (air) this is perhaps not unreasonable. For Problem 4, however, a quadratic relationship is more than adequate:

$$\mu = \mu_A (C_1 + C_2 T' - C_3 T'^2) \quad (53)$$

and this function serves to define

VISCOSITY TYPE C.

Using dimensionless T:

$$\frac{\mu}{\mu_A} = T_\alpha = (C_{01} + C_{02} T + C_{03} T^2) \quad (54)$$

$$\text{and } \frac{\partial}{\partial T} \left(\frac{\mu}{\mu_A} \right) = T_\beta = (C_{02} + 2C_{03} T) \quad (55)$$

$$\text{where } C_{01} = C_1 + C_2 T'_0 - C_3 T'^2_0 \quad (56)A$$

$$C_{02} = \left(\frac{1}{GC_5} \right) (C_2 - 2C_3 T'_0) \quad (56)B$$

$$C_{O3} = - \frac{C_3}{G \cdot C_5^2} \quad (56)c$$

13.2 Comparison with theory (parabolic entry profile)

Problem 1 Calculated values of Nu are compared in Table 18. Zeldin and Schmidt's data are taken from (115) which refers to Zeldin (116). These values are not completely consistent with Fig. 9 of (36)^{*}, but, on balance, are used in preference to them.

Z/Pe	Z & S	Marner	Collins
0.00905	8.724	8.347	9.664
0.01282	7.789	7.525	8.195
0.02198	6.648	6.479	6.726
0.05128	5.451	5.345	5.324
0.09524	4.989	4.910	4.766
0.2052	4.634	4.688	4.213

TABLE 18 - Nu COMPARISON (PARABOLIC ENTRY PROFILE)

Up to Z/Pe = 0.022 the comparison is as for Table 15, and there it was concluded that the higher Nu's of Zeldin and Schmidt and Collins did not reflect overestimated heat transfer. The lower values of Collins, for higher Z/Pe, are eventually similarly followed by Zeldin and Schmidt. Perhaps higher initial Nu's do result in lower long-distance values.

Comparison is made of centre-line velocity in Table 19, which includes representative values for alternative solution (1).

* At Z & S's Z/Pe = 0.011, Marner (115) states Nu = 6.65, whereas Fig. 9 (36) gives 7.6.

Z/P_e	Z & S	Collins (2)	Collins (1)
0.008	1.96	1.965	
0.014	1.934	1.936	
0.020	1.896	1.90588	1.90598
0.040	1.78	1.792	
0.080	1.52	1.534	
0.140	1.182	1.210	
0.190	1.12	1.11126	1.11149
0.198	1.12	1.10625	1.10645
0.40	1.48	1.425	
0.60	1.733	1.69590	1.69574
0.80	1.854	1.839	
1.40	1.972	1.975	
1.80	1.980	1.99235	1.99233
2.00	1.984	1.996	

Minimum U_{ξ} (Z & S)
Minimum U_{ξ} (COLLINS)

TABLE 19 - U_{ξ} COMPARISON (PROBLEM 1)

Agreement is good, the maximum difference being 0.055 at $Z/P_e = 0.40$. Most other values lie well within this. The predicted position of 'maximum profile distortion' (to use Marner's expression) is only 4% different, and the minimum U_{ξ} is very close. A further aspect is that Collins predicts a somewhat more rapid return to the parabolic profile, as the fluid reaches the wall temperature, giving isothermal flow. Zeldin and Schmidt's outlet fluid b.c. was a parabolic profile at $Z = \infty$.

Considering that the concepts used in the two analyses are quite different, the agreement is all the more pleasing.

It is interesting that this agreement is better than that for Nu , implying that the fluid flow equations are not affected too drastically by the energy equation.

Finally, comparing Collins (1) and (2), the differences may be noted as less than 0.1%.

Problem 2 This specifically compares with the run reported by Marner and McMillan (38), and is essentially the same as Problem 1. Table 20

(like Table 19) gives centre-line velocity predictions, comparing them with a few values of (38). Although Collins's solution was not carried to a final parabolic profile, it completely covered the range of (38).

Z/P_e	M & M	Collins	
0.01		1.871	
0.02	1.712	1.721	
0.04		1.362	
0.06		0.928	
0.08		0.466	
0.102	0.064	0.167	Minimum U_{ξ} (M & M)
0.11		0.138	Minimum U_{ξ} (COLLINS)
0.15		0.291	
0.20		0.579	
0.30		0.015	
0.40		1.314	
0.50	1.560	1.524	

TABLE 20 - U_{ξ} COMPARISON (PROBLEM 2)

Some of the aspects of Table 20 are the same as Table 19, namely the higher predictions of Collins and the position of maximum profile distortion relative to Marner and McMillan. On the whole, with a third analysis using boundary-layer equations, Tables 19 and 20 indicate quite reasonable agreement.

Other results for this problem are shown in Figs. 17 - 19. When shown as profiles (Fig. 17) the velocity predictions appear even closer. The first pair of profiles is almost co-incident. The pair for Marner and McMillan's minimum U_{ξ} is closer away from the centre-line, and the position and magnitude of maximum U are very close. Finally, the third pair shows quite good agreement. In Fig. 18 the maximum difference in temperature (expressed as a drop from the wall level) is about 4% at maximum profile distortion. Otherwise near entry and end of the run, agreement is very good.

However, T_m' , the mixed mean temperature is a function both of axial velocity and temperature, and since Nu is a non-linear function of T_m' , the generally good agreement of Figs. 17 and 18 is not reflected in the evidently much more sensitive Nu of Fig. 19.

Marner and McMillan's values are above Collins's by 5% increasing to 77%, which is surprising in view of the much closer agreement of Table 18 for a similar problem, the difference being generally of opposite sign. However, while Collins does not confirm the increase in Nu at maximum profile distortion, there is a marked flattening of the curve there, followed in both cases by a substantial drop.

13.3 Comparison with theory (uniform entry profile)

Problem 3 Results of this study are summarised and compared with those of Zeldin and Schmidt in Figs. 20 and 21. Values of developing U_e agree very well, to within 0.05 (Fig. 20), and the position and magnitude of maximum distortion are again very close. The Nu predictions (Fig. 21) do however differ, but Collins's high level at entry and low long-distance values have already been noted. Apart from this, Zeldin and Schmidt's prediction over Gz from, say 150 to 20 is about 10% higher than that of Collins. It is disappointing that a comparison of T_m' was not possible, as this could well have shown better agreement, in view of the similar effects on the developing velocity.

13.4 Comparison with experiment (Zeldin and Schmidt)

Problem 4 This problem represents a well-designed combined convection experiment, the results being compared with theory and fully discussed in (36). Because of this, a number of calculations were made by Collins, as summarised in Table 21.

13.4.1 Introduction

Calculation	Wall thermal b.c.	Viscosity	Density
A	Energy balance	Constant	Quadratic
B, B*	Energy balance	Constant	Linear
C	Explicit(constant T_R)	Constant	Quadratic
D	Energy balance	Quadratic	Quadratic

TABLE 21 - COMPARATIVE RUNS OF COLLINS (PROBLEM 4)

This gives useful scope for further comparison of the effects of wall thermal b.c., and property variations. Run B is closest to Zeldin and Schmidt's model, but Run D is the fullest solution, and is therefore used in the comparative graphs of Figs. 22 - 24. Tables 22 and 23 give full comparisons for the important variables at the experimental stations.

Z	Zeldin & Schmidt		Collins			
	Experiment	Theory	A	B	C	D
2.869	1.20	1.17	1.210	1.219	1.207	1.216
10.93	0.89	0.89	0.9616	0.997	0.9587	0.976
29.54	1.077	1.30	1.0721	1.097	1.0732	1.091
125.08	1.886	1.96	1.914	1.917	1.914	1.918
204.48	1.95	1.99	1.987	1.986	1.987	1.987

TABLE 22 - COMPARISON OF U_z (PROBLEM 4)

Z	Zeldin & Schmidt		Collins			
	Experiment	Theory	A	B	C	D
2.869	0.60	0.722	0.760	0.761	0.758	0.760
10.93	0.42	0.443	0.540	0.542	0.5385	0.541
29.54	0.2125	0.161	0.288	0.293	0.287	0.293
125.08	0.025	0.015	0.056	0.051	0.056	0.055
204.48	0.020	0.005	0.039	0.0385	0.039	0.039

TABLE 23 - COMPARISON OF $\frac{T_R - T_M}{T_R}$ (PROBLEM 4)

13.4.2. Initial comparison (Run B)

From Table 22 it is evident that Run B (except at $Z = 10.93$) is closer to experiment, and sometimes substantially so, than Zeldin and Schmidt's predictions. Not only so, but whereas the experiment was designed to give full development at exit, and failed (but not severely) to do so with an exit U_e of 1.95, Zeldin and Schmidt's theory with its closed boundary had to assume this. Although Collins's value is only about 10% closer to experiment at exit, the marching procedures permitted open boundary is advantageous.

The corresponding comparison of Table 23 has a number of aspects. Quantitatively, Run B's results may appear rather disappointing. However, qualitatively Collins's results confirm the conclusion of (36) that there must have been unintentional preheating of the entry fluid. This is because at $Z = 2.869$ the measured T_m was already 40% of T_R whereas both predictions give less than 30%. Also, whereas the theory again assumes exit isothermal flow (exit T_m only 0.5% different from T_R), this was not quite so in experiment (2% different) or in Collins's prediction (4% different). Quantitatively, as a %age of T_R , Collins's overprediction reduces from 16.1 (1st point) through 12.2 (2nd) to 8.0 (3rd). On balance, it is regarded as unwise to draw any conclusions further than this when the inlet boundary condition is suspect.

13.4.3. Revised entry temperature (Run B^{*})

However, in view of the stress laid upon T'_m in the context of a constant wall temperature, it was felt necessary to attempt to resolve the uncertainty, by making a predictive run B^{*}. Anticipating discussion of Fig. 24 the wall-to-entry temperature difference was reduced by 10%. This assumes, for the sake of simplicity, that the entry air temperature was uniform at about 2°F above the nominal value, all other parameters being taken as the same as Run B. Output data are

⊗ At $Z = \infty$

compared with experiment in Table 23A, on the understanding that the predictive validity is only illustrative. To avoid confusion $(T_R' - T_m')$ is given explicitly.

Z	U_c		$(T_R' - T_m')(^{\circ}F)$	
	Z & S	Run B*	Z & S	Run B*
				(B)
2.869	1.20	1.237	13.35	15.20 (16.90)
10.93	0.89	1.067	9.34	10.86 (12.05)
29.54	1.077	1.161	4.73	5.94 (6.525)
125.08	1.886	1.921	0.56	1.03 (1.13)
204.48	1.95	1.987	0.45	0.69 (0.86)

TABLE 23A - COMPARISON OF EXPERIMENT AND RUN B*

The temperature predictions are improved relative to Run B, whose values are also tabulated. They still differ from experiment, however, possibly suggesting that the preheating was greater than $2^{\circ}F$ near the wall. This would be a quite logical situation. The velocity predictions are now somewhat further from experiment, because the decreased temperature difference has reduced the natural convection effect. This again could possibly be corrected for by allowing for the effect of preheating on the inlet velocity profile. However, this would begin to be changing the inlet b.c. to give the results of experiment.

On the whole, Run B* confirms the probable presence of preheating on the temperature profiles.

13.4.4 Predictive Runs A, C and D

By comparing Run A, then Run D, with Run B in Tables 22 and 23, the effect of allowing for better density and then viscosity variation with temperature may be seen to be not negligible. The nett effect is

to bring U_c , but not T_m , closer to experiment.

The alternative solution, Run C, using the explicit wall boundary condition is comparable with Run A, and in fact, slightly closer to experiment. A closer study will be made in sub-section 13.4.6, but perhaps the more important feature is the closeness of the agreement.

13.4.5 Detailed comparison (Run D)

Figs. 22 - 24 give a detailed velocity and temperature comparison between Collins (Run D) and theory and experiment for Problem 4.

Fig. 22 shows that Collins's prediction gives better agreement with experiment than that of Zeldin and Schmidt, except at the second station. Fig. 23 demonstrates that even this latter difference is substantially less than the difference between Zeldin and Schmidt's prediction and experiment at $Z = 29.54$.

Fig. 24 gives four comparisons of which three positions coincide with Fig. 23. Collins's prediction is somewhat less close to experiment at the first position, is rather poor at the second (whereas the other is very close), and is about as good at the other positions

In detail, the first position being so close to entry, the two predictions virtually agree that preheating must have occurred, and this certainly by 2°F in the centre region of the pipe, and probably more towards the wall. If this is at all accepted, then the other temperature profiles of Collins become at least qualitatively consistent, and this is justified by the results of Run B^{*}.

Finally, whereas Collins's solution is above experiment in Fig. 22, and below in Fig. 24, Zeldin and Schmidt's is low (in both Figs.) at the first station, virtually coincides at the second, and then 'crosses over' to give high predictions thereafter. Possibly this could be viewed as the effect of applying the alternative procedures of marching and iteration.

It is felt that on the whole, the predictive ability of this solution is at least as good as that of Zeldin and Schmidt, without the difficulty, in other applications, of having to specify the outlet boundary conditions.

13.5 Study of wall thermal boundary condition

Z (ΔZ)	RUN A		RUN C			
	TMEAN	Δ (TMEAN)	TBAR	Δ (TBAR)	TMEAN	Δ (TMEAN)
0.625 (0.625)	3220.22	-	1786.31	-	3343.17	-
1.25 (0.625)	4384.23	1164.01	2796.31	1010.00	4466.63	1123.46
1.875 (0.625)	5229.54	845.31	3592.27	795.96	5298.71	832.08
2.50 (0.625)	5945.53	715.99	4290.96	698.69	6006.85	708.14
3.125 (0.625)	6591.09	645.56	4921.29	630.33	6647.30	640.45
10.00 (0.625)	11635.72	365.02	9897.49	361.20	11669.82	363.89
50.0 (2.5)	21840.67	552.72	20166.85	568.63	21846.25	551.32
125.0 (10.0)	24905.87	138.95	23329.33	143.81	24906.34	138.78
215.0 (10.0)	25351.42	15.53	23790.33	16.07	25351.44	15.52

TABLE 24 - THERMAL BOUNDARY CONDITION COMPARISON
(PROBLEM 4)

Table 24 compares the alternative solutions for the two expressions for wall thermal b.c. Dimensionless temperatures are tabulated (where the wall temperature difference of 22.3°F has a value of 26372.8). Run A (Problem 4) gives a TMEAN which coincides with TBAR for the energy balance method. Run C, using the explicit wall temperature method, calculates TBAR as a check. These are all tabulated for the first five (and then three typical subsequent) positions. For each

position, two sets of comparisons may be made, that for Z itself (upper), and the ΔZ or stepwise change (lower).

Run C, the conventional solution, shows that a bad TBAR at the first position affects the whole solution, although decreasingly so with Z. If it were used as an energy balance check, the solution's accuracy could be assessed as disappointing. However, $\Delta(TBAR)$ becomes increasingly close to $\Delta(TMEAN)$, to within $1\frac{1}{2}\%$ at $Z = 2.50$ and under 1% at $Z = 10.0$, but then the %age difference increases again. Assessed on this basis, the accuracy could be regarded as reasonable.

It has been mentioned that, for parabolic entry profiles, the alternative solutions give virtually identical results. Even for the 'more difficult' uniform entry, Run A is close to Run C. By $Z = 3.125$, both the Z TMEAN and the ΔZ change are within 1% of Run C, and are closer thereafter. This agreement justifies both solutions. However, Run A has the slight advantage of the energy balance being exact, although it is admitted that the wall heat transfer rate, based on the temperature curves, has an element of uncertainty in it.

14. TRANSITION FROM LAMINAR FLOW

14.1 Introduction

14.1.1 Stability of flows

The problem of transition from laminar flow to some form of turbulence is a major topic associated with the theory of instability of flows. For combined convection one would expect the instability features of both forced convection (e.g. Pai (117)) and natural convection (e.g. Gebhart (118)) to be relevant.

Lin stated (Pai (117) p. 314) that for inviscid fluids, the essential feature of unstable flow is that the velocity profile has a point of inflexion, when a self-excited disturbance exists. Pai, however, modified this by postulating (ibid P.315) that the viscosity is a substantial destabilizing influence, for viscous fluids with at least one solid boundary.

For natural convection, Gebhart (118) concluded that instability has the same initial mechanism as for forced convection. He pointed out that the term 'disturbance' in heated flow implies temperature effects as well as velocity. Finally, he considered the Prandtl number to be an important parameter.

The identity of instability with points of inflexion in the velocity profile was confirmed experimentally by Lawrence and Chato (33) and Scheele and Hanratty (35) by observing that transition always took place after the development of points of inflexion. Incidentally, both sets of data were for natural assisting forced convection, for water.

14.1.2 Transition

Whereas instability has theoretical significance, transition from laminar flow is more empirical. It is associated with the growth of small disturbances of velocity (and, as above, temperature), after an unstable condition has been reached. Thus the initial studies of

Scheele et al (59) involved observation of dye filaments injected in water. This identifies transition with a visible change in velocity. Later, Scheele and Hanratty (35) used temperature measurement as a transition criterion, with a minimum fluctuation of 0.1°C . Lawrence and Chato (33) also measured temperature fluctuations, defining turbulence as occurring with a minimum change of 0.03°C . They specifically rejected the dye filament method as being too insensitive.

14.1.3 Correlation of transition

The correlation of transition is of some interest. In the work of Scheele, Hanratty, Rosen and Kabel (16), (34), (35) and (59) the empirical correlation was in terms of the axial distance to transition from entry and the Grashof/Reynolds Number ratio. In the context of this work, the Grashof Number is proportional to the dimensionless heat flux, and equivalent to the dimensionless wall temperature, for the b.c.'s of constant wall heat flux and temperature respectively. No attempt, however, was made to relate this to the velocity profile at transition, possibly because the corresponding theoretical work was somewhat approximate.

Lawrence and Chato (33) quoted instability theory as specifying that a characteristic parameter identifying the velocity profile is a function of entrance Reynolds Number. The transition parameter they obtained by trial and error was:

$$T_1 = \frac{U_P + U_c}{(1 - r_p)} (\mu_R) \quad (57)$$

where U_P = dimensionless velocity at point of inflexion

U_c = " velocity at centre-line

r_p = dimensionless radius of point of inflexion

and μ_R is in (lb/ft hr) units.

This usefully combined the velocity profile statement of Lin, and the viscosity effect mentioned by Pai. After a point of inflexion appears in the velocity profile, T_1 varies with distance (increasing

and then decreasing Fig. 10 (33)) until it reaches the value given for its particular Re at entry, when transition occurs.

Two studies would be useful, therefore. The first is to check whether T_1 versus Re_0 can correlate Scheele et al's results, and the second is to obtain a similar correlation for the case of constant wall temperature.

14.1.4 Natural opposing forced convection

The above discussion is for natural convection assisting the forced flow, i.e. for upward heated (or downward cooled) flow. The opposite case occurs for upward cooled or downward heated flows, and Scheele et al give the only available data. They concluded (35) that instability is connected with flow separation at the wall and that transition is sudden and occurs shortly after such separation. They also observed transition to an asymmetric flow, which included a region of steady reversed flow over a portion of the wall, which was tentatively associated with points of inflexion in the velocity profile.

Another study would be useful, namely to investigate the flow as far as experimental transition, and to correlate if possible, some kind of transition parameter.

14.2 Comparative Study (air)

This published study (Appendix 10) investigates the effect of different parameters on flow of air in combined convection, and has been discussed in Section 4.5. In a general way, it shows similarity with results of Scheele et al for water, and in particular indicates the possibility of reversed stable flow. Because of the two-dimensional model this appears as an outer annulus.

14.3 Predictions for Water (Data of Scheele, Hanratty, Rosen and Kabel)

14.3.1 Introduction

These data, comprising Refs. (16), (34), (35) and (59), include

results of both experiment and analysis.

Experimental work involves transition studies for:

- (i) Upward heated flow - constant heat flux (Q/UP)^{*}
- (ii) Upward heated flow - constant temperature (T/UP)
- (iii) Downward heated flow - constant heat flux ($Q/DOWN$)
- (iv) Upward cooled flow - constant temperature ($-T/UP$)

Case (i) also includes wall temperature results for two runs, and Case (iv) includes the study of transition to stable reversed flow.

Two theoretical analyses are reported. The first assumed fully-developed flow for constant heat flux, and is presented in (16), and velocity profiles given in (35) and (39). In view of this not being comparable with any experiment^{**} it was felt to be of minor interest in the present context. A second approximate model for developing flow with constant temperature is presented in (34) and compared with experiment in (59). Some centre-line velocity curves (34) for (constant density/variable viscosity) and the converse, form useful comparisons with the present analysis, but it is admitted in (34) that accuracy is probably not good for short tube lengths.

Table 25 summarises the details of the present study. Runs 8 and 9 are not reported here as they concern wall temperature measurements.

In this study, the viscosity relationship TYPE A was used, and the linear density relationship involving β was assumed (equation 50)) partly because of lack of data.

^{*} Refers to Table 25, p. 104.

^{**} Except for wall temperatures discussed in Appendix 9.

Run No.	Case	Ref.	Source within Ref.	Theory ?	Re _o	Gr _Q * or Gr _T	Transition Z	Comments
1	Q/UP	(59)	Fig. 6		100.0	6315	157.0	
2	"	"	"		237.5	35000	"	
3	"	"	"		97.5	4500	228.6	
4	"	"	"		290.0	25500	"	
5	"	(35)	Fig. 9		256.8	10470	610.0	
6	"	"	"		486.0	30200	"	
7	"	"	"		1207.8	62750	1220.0	
8	"	"	Fig. 6	✓	339.0	10230	-	)Reported in)Appendix 9
9	"	"	"	✓	1132.0	50000	-	
10	Q/DOWN	(59)	Fig. 8		50.0	3152	228.6	
11	"	"	"		194.1	12155	"	
12	"	"	"		301.0	18820	"	
13	T/UP	"	)Fig.10 &)Table 1	✓	280.0	17545	155.2	
14	"	"		✓	116.0	4240	"	
15	"	"		✓	166.0	7335	"	
16	"	"		✓	122.0	6975	82.2	
17	-T/UP	"	)Fig.13 &)Table 2	✓	248.0	-18400	146.2	
18	"	"		✓	91.0	-10360	18.2	
19	"	"		✓	654.0	-48047	143.4 ^φ	
20	"	"		✓	296.0	-22670	65.6 ^φ	
21	"	"		✓	120.0	-8540	31.4 ^φ	
22	T/UP	(34)	Fig. 5	✓	100.0	10000	-	
23	"	"	Fig.12	✓	"	2700	-	
24	-T/UP	"	Fig. 5	✓	"	-3000	-	
25	"	"	Fig.12	✓	"	-300	-	
26	Q/UP	(59)	Fig. 6		175.0	17450	157.0	)Additional)to Runs 1-7)
27	"	"	"		220.0	15000	228.6	
28	"	(35)	Fig. 9		400.0	20000	610.0	
29	"	"	"		1446.0	72300	1524.0	

TABLE 25 - SUMMARY OF TRANSITION STUDY

* Gr_Q and Gr_T are Grashof Number definitions used in (59)

For constant heat flux, Gr_Q = HFLUX x M = HFLUX x 80.

For constant temperature, Gr_T = TWALL. In both cases C₅ = β.

^φ Transition to asymmetric partially reversed flow.

14.3.2 Comparison with theory

A) Constant viscosity, variable density

Runs 22 and 24 correspond to the constant viscosity, variable density runs for upward heated and cooled flows shown in Fig. 5 of (34). The comparison is plotted in Fig. 25, which shows very reasonable agreement. For heated flow the position of maximum distortion is virtually the same, its magnitude differing by less than 0.1. For cooled flow, while the magnitude of maximum distortion agrees well, its position is not so good. From $Z/Pe = 0.1$, the predicted U_{ϕ} differs by well under 0.1 for both flows. This confirms the comment in (59) that the theory is 'least satisfactory close to entry'.

B) Variable viscosity, constant density

Rosen and Hanratty (34) assumed that inverse viscosity varied linearly with temperature. This gives:

$$\frac{\mu}{\mu_0} = 1 + b(T' - T'_0) \quad (58)$$

where b is a constant.

This forms a special case of viscosity TYPE A, where for

$$T'_A = T'_0,$$

$$C_{01} = bT'_0 \quad C_{02} = 1.0 \quad C_{03} = 1.0 \quad (59)$$

Runs 23 and 25 correspond to the heated and cooled runs of Fig. 12 of (34) and the comparison is shown in Fig. 26. Bearing in mind that the vertical scale has been doubled, agreement is seen to be not as good, although the comparisons are quite similar to Fig. 25. This could be because whereas the natural convection term is the same for both analyses, Collins's viscosity temperature terms are complete. However, while the position of maximum distortion differs by about $\Delta Z/Pe = 0.05$, the magnitude agrees to within 0.1 (heated) and 0.05 (cooled). Again, after $Z/Pe = 0.1$, the U_{ϕ} predictions differ by less than 0.1.

Commenting on both Figs., agreement for an admittedly approximate analysis is possibly better than might be expected. However, since the application is to predict axial positions for certain velocity conditions (Tables 1 and 2 of (59)) this agreement will not necessarily be as good.

Finally, it should be noted that Case A is strictly relevant to the combined convection section 13.2, and Case B to Chapter 12.

14.3.3 Upward heated flow (constant heat flux)

As mentioned in sub-section 14.1.3 the Lawrence and Chato Transition Parameter T_1 can be checked against present predictions using Scheele et al's transition lengths. Fig. 27 shows that agreement with Lawrence and Chato's curve is poor. However, Fig. 28 shows that an alternative correlation is possible. If the wall viscosity factor is removed, a modified transition parameter T_2^* is obtained, defined by:

$$T_2 = \frac{U_P + U_q}{(1-r_p)} \quad (60)$$

Fig. 28 shows that this correlates well against (Re_o/Gr_Q) , the agreement being checked with some additional runs 26 - 29. Unfortunately, it is not possible to calculate T_2 for any of Lawrence and Chato's results, but for Collins's comparative prediction (Appendix 9) T_2 has the approximate value 9. Similarly, a value of about 19 is relevant for the Kemeny and Somers's run of Appendix 9. Both these are not inconsistent with the present predictions.

Although T_2 does not include the viscosity (previously mentioned as a stability factor) the effect of heating as such is represented by Gr_Q . It is not altogether surprising that the correlation is in terms of the ratio (Re_o/Gr_Q) as the inverse is consistently used as a parameter in (35), for example.

Lawrence and Chato postulated that by calculating their T_1 it is possible to predict the transition point for a given Re_o . However,

* but see qualification on p.112.

T_1 increases and then decreases with Z , the latter because μ_R continuously decreases as T_R' increases. Calculations show that T_2 is too insensitive for this purpose (without μ_R), and the transition length can only be correlated from the empirical results.

Figs. 29 and 30 are alternative presentations for (Z_T/Re_o) in terms of (Gr_Q/Re_o) and Gr_Q respectively. Three specific points are available from Lawrence and Chato, the others representing typical stations on the smooth curves of Fig. 6 (59) and Fig. 9 (35). Fig. 29, while having to omit factors like Pr due to lack of data, is essentially the correlation of Lawrence and Chato (Fig. 8 (33)). Its scatter is larger, but so is the range of Z_T . In fact the highest Z_T values give more or less the same curve. Fig. 30 is perhaps a clearer description of the phenomena. For all Z_T below about 300 a single curve is sufficient, and for higher Z_T clearly separate curves result.

Figs. 29 and 30 suggest that at low Z_T the standard dimensionless form (Z_T/Re_o) is a function of Gr_Q only,* but at higher Z_T this changes to become eventually a function of (Gr_Q/Re_o) only. This may imply property effects (c.f. Fig. 8 (33)) and unpresented graphs for more limited data, suggest somewhat closer correlation using outlet property values. However, this would defeat the intention of predicting Z_T from the inlet conditions.

Summarising, the transition parameter T_2 correlates well, but is not able to predict transition. However, for Z_T below about 300 a single curve from Fig. 30 may be used. Alternatively, all the data could be plotted as Gr_Q versus Re_o for different Z_T , in the manner of Fig. 6 of (59). This would provide a simple prediction of Z_T for water, but is omitted here as it would merely re-present the data of (59) and (35).

* reversing the axes of the graphs

14.3.4 Upward heated flow (constant temperature)

RUN NO.	EXPERIMENTAL INSTABILITY	AXIAL DISTANCES (Z) FOR:			
		FLAT PROFILE PREDICTION		MINIMUM U_L PREDICTION	
		(59)	COLLINS	(59)	COLLINS
13	155.2	43.0	55.0	167.8	160.0
14	155.2	38.4	40.0	98.8	70.0
15	155.2	42.8	45.0	134.2	100.0
16	82.2	20.0	27.5	75.2	70.0

TABLE 26 - COMPARISON FOR UPWARD HEATED FLOW
(CONSTANT WALL TEMPERATURE)

Runs 13 - 16 cover the range of conditions in Fig. 10 of (59). Table 26 shows that qualitatively, current predictions confirm the experimental observation (and prediction) of (59) that instability always occurred downstream of the flat profile location. Some of the compared distances vary considerably, however, but this is not surprising in view of the approximate nature of the analysis for (59). In fact, five of the eight comparisons agree to within $\Delta Z = 8$.

Fig. 31 shows that the modified transition parameter correlates acceptably for the case of constant wall temperature, where Gr_T replaces Gr_Q . T_2 , therefore, can be regarded (within the range of data) as a unifying parameter for natural assisting forced convection.*

14.3.5 Downward heated flow (constant heat flux)

Runs 10 - 12 cover the data range of Fig. 8 of (59), but an immediate reservation is that this is inconsistent with Fig. 7, which used a Gr , defined in our notation as:

$$Gr^* = T_R - T_\infty \quad (61)$$

The same values are given in Figs. 7 and 8 of (59) for Gr^* and Gr_Q respectively, and Table 27 shows that probably (and unfortunately) Fig. 8 is incorrect. The values of Gr^*/Re_0 at transition Z of about

* but see qualification on p.112.

230 vary widely, and Fig. 32 shows that Run 10 probably implies a much shorter transition distance. Again, except by trial and error, the true Gr_Q 's are not readily found.

RUN	Re_o	Z	Gr^*	Gr^*/Re_o
10	50.0	42.5	2623.2	52.5
		230.0	5951.7	119.0
11	194.1	150.0	9170.0	47.25
		230.0	10870	56.00
12	301.0	210.0	13692	45.5
		230.0	14302	47.5

TABLE 27 - RUNS 10-12 - VALUES OF Gr^*/Re_o AT ONSET OF REVERSED FLOW AND AT $Z = 230$

Data for $Z = 230$ are therefore ignored, but the predictions for flow reversal at the wall should still be valid. Fig. 2 of (59) using a fully-developed model, predicted onset of wall reversed flow at $Gr^*/Re_o = 49.2$. Fig. 32 shows that for these runs the assumption of a fully-developed state is not good, but the value itself compares to within 10% of those given in Table 27. The profiles themselves are compared in Fig. 33. As might be expected, with the additional constraint of zero gradient at the wall, the profiles are close, but at least the solution is confirmed. The empirical transition Gr^*/Re_o from (59) is 63, and Fig. 34 compares the predicted profile for this condition with the corresponding profile for Run 10 at the same value of Gr^*/Re_o . Considering the curve of (59) is a steady-state prediction, the comparison is very good, suggesting that solutions so obtained are valid, even though the distances are not predictable from a solution independent of Z .

14.3.6 Upward cooled flow (constant wall temperature)

With constant wall temperature and a parabolic entry profile, the fluid flow is affected by property changes due to temperature

effects. However as the fluid bulk gradually reaches the temperature of the wall, i.e. becomes isothermal, the flow field correspondingly begins to return to its entry parabolic profile. Scheele et al (59) observed that above a Gr_T/Re of 74 'an asymmetric region of reverse flow occurred.' At Re greater than about 260 this was rapidly followed by a break-up of the flow (instability) but at lower Re a certain range of asymmetric flow remained stable. Their data (Fig. 13 and Table 2 of (59)) suggest three regimes for comparison.

- (i) $Re < 260$ Observation of instability (Our runs 17, 18)
- (ii) $Re > 260$ Observation of asymmetry (Our runs 19, 20)
- (iii) $Re < 260$ Observation of asymmetry (Our run 21)

Runs 17 - 21 were taken only as far as their predicted locations of zero wall velocity gradient (z.w.v.g.). It must be conceded that whereas theory thereafter predicts an annular reversed flow region, the observed asymmetric flow had a randomly located area of reversed flow near the wall. Past the instability location, of course, any prediction is notional. Table 28 compares data ~~work~~ of Collins with those of (59), and Figs. 35 and 36 give corresponding U_d predictions.

RUN NO.	Data of (59)			Collins			
	Exptl. Instability	Exptl. Asymmetry	Predicted z.w.v.g.	Predicted z.w.v.g.	U_d for column 2	U_d for column 3	U_d at z.w.v.g.
17	Z 146.2	Z	Z 56.8	Z 6.25	3.741		2.192
18	18.2		14.8	1.25	4.421		2.145
19		45.6	143.4	17.5		2.590	2.207
20		22.8	65.6	6.25		2.690	2.163
21		13.6	31.4	3.75		2.902	2.231

TABLE 28 - COMPARATIVE RESULTS FOR RUNS 17 - 21

The only direct comparison is for location of z.w.v.g., and Collins's values are roughly an order of magnitude less. This difference, while severe, is consistent with Figs. 25 and 26, which show

Collins's predictions for cooled flow suggest more rapid development. However, the important point is that present predictions of z.w.v.g. are upstream of the experimental asymmetry location, rather than downstream for all predictions of Table 2 of (59). Scheele et al suggest that the former may be the correct flow situation. Another comparison is that of U_L at z.w.v.g. (59) predicts a uniform 3.0, whereas Table 28 gives values around 2.2. A final qualitative comparison is given by Run 21 for asymmetry without instability. While the mathematical correctness of marching into a negative velocity field may be queried, Run 21 (Fig. 36) does predict a limited negative flow region, consistent with the corresponding diagrams (second or third) in Fig. 11 of (59) which show eventual return to a symmetric state.

Finally, Figs. 37 and 38 give the predicted velocity profiles at instability and asymmetry respectively. Fig. 37 shows dissimilarity although with more data it might be possible to deduce a transition parameter. Fig. 38, however, suggests a uniform profile for inception of asymmetric flow ($U_L = 2.72 \pm 6.7\%$), although a trend with Re is apparent. On the whole, firm conclusions are not possible.

14.4 Conclusion

A fairly comprehensive study of transition and instability has been made, based on the observations of Scheele, Hanratty and Rosen. For natural assisting forced convection it has been possible to correlate the velocity profiles by simplifying Lawrence and Chato's transition parameter. This correlation is satisfactory for upward heated flow with both constant wall heat flux and temperature. For natural opposing forced convection, results are insufficient, partly because of an inconsistency in presented data for downward cooled flow in (59).

The approximate analysis of Rosen and Hanratty (34) compares reasonably in terms of U_c but not so well with distance predictions. In particular, for upward cooled flow, Collins predicts that reversed flow at the wall precedes, rather than follows, the observed start of asymmetric flow, a more logical situation. A limited comparison (for downward cooled flow) with the fully-developed theory of Hanratty et al (16) suggests that their predictions have some validity, even though the appropriate axial distance is not predicted.

In general, the fuller analytical model of the present work serves to identify a number of aspects of the overall problem.

Qualification regarding T_2

The parameter T_1 obtained by Lawrence and Chato (33) was claimed to be a 'functional relationship' able to predict the axial position of transition for a given entrance Reynolds number. It is suspect both in being dimensional and in that the profile shape parameter has no logical connection with transition. Fig. 27 demonstrates its inability to correlate present results. T_2 suffers from the same lack of connection. Also, while it is uniquely related to (Grashof Number/Reynolds Number), T_2 cannot be associated with transition, since it hardly varies with axial distance as the flow approaches transition (see para. 1, p. 107). For these reasons T_2 should not be regarded as a transition parameter, and any such indication in this chapter should be ignored.

15. THE ANNULUS

15.1 Introduction

In the context of this thesis, the intention here is to demonstrate the flexibility of the analysis. The annulus has the same outer boundary as the tube, but possesses an inner wall in place of the centre-line. In principle, therefore, it should be possible to change the effect of the inner boundary within the program.

Appendix 12 contains the essence of this approach, which is not repeated here.

15.2 Analysis

15.3 Program

15.4 Nomenclature

These sections correspond to Chapters 11, 12 and 13 of Appendix 12 respectively. For the purposes of this thesis, equation:

(5) ANN	should read	(8) ANN
A(14) ANN	" "	(17)ANN
(19)AANN	" "	(24)ANN

15.5 Results

Initial results for radius ratios of 0.5 and 0.25 are compared in Tables 29 and 30 with data of Heaton et al (121), where U_{MAX} and r_{MAX} are the magnitude and position of maximum axial velocity.

The Tables show very good agreement for a radius ratio of 0.50, and less agreement for a ratio of 0.25. The analysis used 30 radial positions, but no attempt was made to smooth the results to give a maximum velocity away from a solution point. This may account for some of the discrepancy in Table 30. Otherwise, the somewhat less rapid development may correspond to a comparison (for a ratio of 0.05) of (121) with Sparrow and Lin (122).

Information for annuli is relatively sparse. Sparrow and Lin's results for a ratio of 0.8 are confirmed by (a) experiment (123), and

Z	Heaton et al		Collins	
	U_{MAX}	r_{MAX}	U_{MAX}	r_{MAX}
3.371	1.3849	0.7393	1.3884	0.733
6.088	1.4545	0.7373	1.4527	0.733
8.590	1.4825	0.7364	1.4794	0.733
10.46	1.4932	0.7360	1.4901	0.733
13.18	1.5012	0.7358	1.4985	0.733
15.30	1.5040	0.7357	1.5022	0.733
19.68	1.5061	0.7356	1.5058	0.733
∞	1.5078	0.7355	1.5077	0.733

TABLE 29. ANNULUS RESULTS. COMPARISON FOR RADIUS RATIO OF 0.5.

Z	Heaton et al		Collins	
	U_{MAX}	r_{MAX}	U_{MAX}	r_{MAX}
1.287	1.3015	0.5994	1.2358	0.607
3.380	1.4104	0.5914	1.3758	0.575
6.250	1.4673	0.5868	1.4493	0.575
8.900	1.4919	0.5847	1.4835	0.575
13.30	1.5116	0.5830	1.5091	0.575
16.71	1.5189	0.5824	1.5183	0.575
21.87	1.5243	0.5819	1.5246	0.575
∞	1.5287	0.5815	1.5283	0.575

TABLE 30. ANNULUS RESULTS. COMPARISON FOR RADIUS RATIO OF 0.25.

(b) results of Spalding-Patankar method (p. 291 Ref. (37)). The published results are self-consistent, therefore.

Preliminary checks on convergence indicate sufficient accuracy (within about 1% for velocity), with a lower limit on initial ΔZ of about 0.375.

15.6 Conclusion

This comparison is encouraging, and confirms the reliability of the present approach.

16. ASPECTS OF ACCURACY

16.1 Introduction

Several aspects of accuracy are important in an analysis of this nature, and some have already been discussed.

Firstly, there is the degree of completeness of the partial differential equations used. For example, Worsøe-Schmidt and Leppert (53) estimate in retrospect the magnitude of the terms they neglect.

Then there are the problems of how closely:

(i) the finite-difference equations approximate to their originals (the discretization error - G.D. Smith (119) p. 56),

and (ii) the numerical solution approximates to the finite-difference solution (rounding-off or stability error (119) pp 57, 58).

Besides the partial differential equations there are the overall (integral) continuity, energy and momentum equations. As already discussed, the energy equation has particular significance in this work.

If the above is regarded as internal accuracy, external accuracy implies comparison with available empirical and analytical data.

Questions of a more philosophical kind, but of no less importance, can arise here. For instance, Test in discussion (120) criticises Yang's lack of experimental data in (50). Yang, however, points out that agreement of an approximate analysis with experiment is not necessarily conclusive, because a more exact analysis might show divergence from the experiment. Later, Test in his turn is criticised by Witte and Cox (114) for using results of analysis in preference to experiment.

All these aspects are dealt with below.

16.2 Nature of partial differential equations

Apart from property considerations the equations are complete. While such completeness may not be essential for many applications it

removes a source of uncertainty which Worsøe-Schmidt and Leppert (53) painstakingly resolve in retrospect. Even then, their fluid entry condition was the 'easiest' case of a parabolic profile.

On the other hand, the complete elliptic-type equations demand additional boundary conditions, and it is an interesting conclusion that a marching procedure is acceptable, as discussed in Chapter 13.

With reference to property changes, the complete viscosity variation again removes uncertainty. The allowance for density variation in the axial momentum equation is standard except in specific property variation or compressible flow studies. In the present work, bulk density changes are not more than a few per cent. The studies of Appendix 9 confirm that additional property effects are either small or negligible.

16.3 Finite-difference approximations

The main point is that present discretization (or truncation) errors are all second-order. This point is discussed at some length by Worsøe-Schmidt and Leppert (53). Due to a stability requirement (see section 6.3) their errors also included first-order elements. As already noted, present wall calculations involve higher-order errors (e.g. sub-section 7.4.2.).

16.4 Convergence and Stability

16.4.1 Definitions

The crux of accuracy is in these two concepts, explained below (after Smith (119) pp 55 - 57).

Let x_i be the exact solution of a set of partial differential equations with (our) independent variables r and Z . Let X_i be the exact solution of the finite-difference approximations. Then the solution is CONVERGENT if X_i tends to x_i for Δr and ΔZ tending to zero. Convergence, therefore, expresses the closeness of the finite-difference equations to their originals.

The second problem is to achieve the exact solution X_i , and errors here are due to computing calculation procedures. A set of finite-difference equations is STABLE if the cumulative effect of rounding-off errors is negligible.

16.4.2 Convergence

Smith comments((119) p. 58) that the dominant error in a stable and convergent process is due to discretization. In principle convergence may be investigated analytically, but Prof. Price^{*} advised that the effort would be prohibitive. Further, (confirmed by Dr. Jones^{*} in the elliptic/parabolic context) the general validity of the results obtained could be viewed as reflecting reasonable mathematical justification.

What is usually done is to examine the effect of reducing (for this problem) ΔZ and Δr on results for U, V, P, T profiles and Nu and friction factors (see Churchill's comment and author's reply in (115)).

One particular problem in this connection is the minimum initial ΔZ value of about 0.625. However, this did not appear significant. Appendix 3 includes such a study for the adiabatic flow field, and a similar study has been made for forced convection with constant properties and constant heat flux. Both cases of I. parabolic and II. uniform fluid entry have been treated, and typical results are given in Tables 31 - 34, for $Re = 100.0$ and $Pr = 0.7$.

^{*} Mathematics Department, The City University.

M=80	70	60	50	40	30	20
Z = 0.625						
11.13	11.13	11.12	11.10	11.07	11.01	10.88
.07998	.07998	.07997	.07996	.07994	.07990	.07980
.08	.08	.08	.08	.08	.08	.08001
Z = 5.0						
5.418	5.417	5.416	5.414	5.411	5.404	5.385
.08	.08	.08	.08001	.08001	.08002	.08004
.08	.08	.08	.08	.08	.08001	.08002
Z = 25.0						
4.391	4.391	4.390	4.388	4.386	4.380	4.364
.08001	.08001	.08001	.08002	.08002	.08004	.08010
.08	.08	.08001	.08001	.08001	.08002	.08005
Z = 100.0						
4.359	4.357	4.355	4.351	4.344	4.329	4.286
.08001	.08001	.08001	.08002	.08002	.08004	.08010
.08	.08	.08	.08001	.08001	.08002	.08005

TABLE 31 - EFFECT OF Δr WITH $\Delta Z = 0.625$

CONVERGENCE CASE I. PARABOLIC PROFILE. VALUES OF $\begin{pmatrix} \text{Nu} \\ \text{fp} \\ \text{fs} \end{pmatrix}$

TABLE 32 - EFFECT OF INITIAL ΔZ WITH M = 80

$\Delta Z=0.625$	0.75	1.00	$\Delta Z=0.625$	0.75	1.00
Z = 3.0			Z = 10.0		
	6.237	6.306	4.728		4.752
	.08	.08	.08		.08
	.08	.08	.08		.08
Z = 3.75			Z = 15.0		
5.828	5.854		4.514	4.516	4.520
.08	.08		.08001	.08001	.08001
.08	.08		.08	.08	.08
Z = 5.0			Z = 30.0		
5.418		5.473	4.375	4.375	4.376
.08		.08	.08001	.08001	.08001
.08		.08	.08	.08	.08
Z = 7.5			Z = 120.0		
4.965	4.976		4.358	4.358	4.358
.08	.08		.08001	.08001	.08001
.08	.08		.08	.08	.08

M=80	60	40	30	20
Z = 0.625				
23.36	23.07	22.49	21.91	20.83
.3831	.3837	.3854	.3876	.3935
.2709	.2706	.2699	.2686	.2646
Z = 5.0				
5.894	5.893	5.891	5.886	5.871
.1040	.1039	.1038	.1036	.1029
.90240	.09235	.09225	.09214	.09190
Z = 25.0				
4.415	4.414	4.410	4.404	4.389
.08115	.08116	.08117	.08118	.08123
.08050	.08050	.08050	.08050	.08051
Z = 100.0				
4.359	4.355	4.344	4.329	4.286
.08001	.08001	.08003	.08004	.08010
.08	.08001	.08001	.08002	.08005

TABLE 33 - EFFECT OF Δr WITH $\Delta Z = 0.625$

CONVERGENCE CASE II. UNIFORM PROFILE. VALUES OF $\begin{pmatrix} \text{Nu} \\ \text{fp} \\ \text{fs} \end{pmatrix}$

TABLE 34 - EFFECT OF INITIAL ΔZ WITH M=80

$\Delta Z=0.625$	0.75	1.00	$\Delta Z=0.625$	0.75	1.00
Z = 3.0			Z = 10.0		
	7.306	7.636	4.906		4.943
	.1544	.1848	.08992		.09055
	.1034	.1079	.08459		.08471
Z = 3.75			Z = 15.0		
6.529	6.615		4.602	4.605	4.609
.1131	.1346		.08471	.08473	.084
.09760	.09736		.08213	.08212	.08208
Z = 5.0			Z = 30.0		
5.894		6.039	4.388	4.388	4.389
.1040		.1169	.08059	.08059	.08057
.09240		.09312	.08026	.08025	.08025
Z = 7.5			Z = 120.0		
5.239	5.259		4.358	4.358	4.358
.09528	.09623		.08001	.08001	.08001
.08723	.08726		.08	.08	.08

Table 31 gives salient parameters for the first ($z = 0.625$) and typical subsequent steps. This shows very good convergence (less than 0.1% difference between $M=80$ and 70). Table 31 may also be used to give an estimate of the likely error incurred when lower values of M are used e.g. to save computer time. Even for $M=20$, the maximum error is only somewhat in excess of 2%.

Table 32 gives uninterpolated comparisons only, and these reveal somewhat larger differences, although they are virtually eliminated by $Z = 15.0$. Evidently it would be preferable to check with a still smaller initial ΔZ , though in absolute terms, differences are still small (0.5% in Nu at $Z = 3.75$). Incidentally, and not appearing in Table 32, a run with $\Delta Z = 0.3125$ failed to solve, showing the same characteristic as the uniform entry restriction on ΔZ .

Table 33 reveals less good convergence than Table 31, though the 1% $M = 80/60$ difference in Nu at $Z = 0.625$ reduces to 0.3% at the second step (not shown). Comparing Tables 31 and 33, by $Z = 100.0$ the two solutions virtually coincide. A more significant comment is that Tables 31 and 33 imply that less analytical 'liberties' can be taken with the uniform entry solution, a point already touched on.

Table 34 again shows less convergence than Table 33, especially with fp , although by $Z = 7.5$ all differences are below 1%. Initial ΔZ appears to be a more critical factor than Δr .

Taken as a whole, Tables 31 - 34 indicate that convergence errors are well under 1% for most of the solution field, and for a parabolic entry profile in particular are almost negligible.

16.4.3 Stability

All evidence points to the solution (involving the Gaussian elimination sub-routine) being stable.

A computation (suggested by Mr. Girling^{*}) compared the matrix

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coefficients with those obtained before back substitution, and this proved quite satisfactory. Further, using double-precision variables gave identical results. Finally, identical results were also obtained using a sophisticated elimination procedure, standard subroutine GELG, available from the University of London Computer Centre.

16.5 Energy balance

An independent check on convergence and stability is provided by the energy balance requirement. Suppose (i) that the equation is used as a check, and (ii) that the mixed mean temperature agrees exactly with that calculated from the heat input. Since the bulk mean temperature is a function of all U's and T's, such agreement implies an exact solution.

From this study, the following results have emerged, considering energy balance as a check:

- (i) The constant wall temperature case has, in general, a very good heat balance,
- (ii) The constant wall heat flux case with parabolic fluid entry has a fair heat balance,
- (iii) The constant wall heat flux case with uniform fluid entry has a poorer heat balance.

Therefore, without a heat balance requirement, accuracy reduces in the order (i), (ii), (iii), specifically near the duct entry.

It is not being claimed that the inclusion of the energy balance is a guarantee of very increased accuracy. For this model, however, it was necessary. But it seems arguable that for constant wall heat flux it represents a better statement of the wall boundary condition.

Logically, the momentum balance can also be used as a check. Although not so used in this analysis, exact agreement between pressure drop and shear stress friction factors occurs when the velocity profile is parabolic and developed (Table 31). This confirms the momentum balance in one instance.

Finally, Table 35 gives details of continuity and energy balances as used in the Literature.

REFERENCE	Q or T	CONTINUITY R or C	ENERGY R or C	COMMENTS (WITHIN REFERENCE)
Rosen and Hanratty (34)	T	R	R	Momentum equation used also.
Martin and Fargie (83)	Q	R	R	
Marner and McMillan (38)	T	R		
Zeldin and Schmidt (36)	T			
Manohar (110)	Q	R	C	1% error at $Z/Re=0.2$
	T	R		
Hennecke (93)	Q		C	1% general error
	T			
Bankston and McEligot (103)		R		1% overall error
Butterworth and Hazell (86)	Q	Langhaar solution		
Bradley and Entwistle (51)	Q			Continuity not fully satisfied
	T			
Worsøe-Schmidt and Leppert (53)	Q	R	C	Discrepancies in energy balance in first few steps for Q.
	T	R		
Lawrence and Chato (33)	Q	R	C	1% at end of tube
	T	R		

TABLE 35. CONTINUITY AND ENERGY BALANCES.

Q signifies Analysis for constant heat flux

T " " " temperature

R " Requirement

C " Check

The first two analyses (non finite-difference) use both integral continuity and energy equations. Perhaps it is surprising, therefore, that while a number of the other (mainly finite-difference) analyses

use the continuity balance as a requirement, none use the energy balance. It is fair, however, to note that checks are normal with constant heat flux, with 1% as a common criterion of error.

Thus the present analysis, with its constrained energy balance for both Q and T wall conditions, does represent an improvement.

16.6 Semi-iteration

To recapitulate, at a given axial distance, the continuity and momentum equations are solved (using temperatures from the previous step to obtain viscosity and density) to give all U's, V's and P's. The solution is then substituted into the energy equation to give T's. These T's are then used for a fresh solution for U, V, P and in turn a second solution for T.

This 'semi-iteration', therefore, is a check solution at a given Z, due to temperature-dependent properties affecting the momentum equations.

Semi-iteration is not needed for a constant properties solution, but is most necessary when viscosity and density are varying, i.e. for combined convection studies. The effect has already been stated to be small, but Table 36 represents a study for the Lawrence and Chato comparison, which had substantial velocity changes over a relatively short Z of about 75. The second case is a double iteration, giving a third solution for each Z.

Comparing the 2nd and final solutions for Case 2, f_s and Nu barely change, and U_e changes little (at most by just over 1%). Comparing the two final solutions shows virtual coincidence of f_s and Nu (about 0.1% maximum difference), but a slight divergence of U_e . At $Z = 70$ the error has become .0075 i.e. about 4% absolute, or under 1% relative to the entry velocity.

This justifies the semi-iterative procedure for all but the most extreme flow cases, and even then the errors are not unreasonable.

Z	CASE 1		CASE 2		
	1st	Final	1st	2nd	Final
0.625	1.17037	1.16669	1.17037	1.16669	1.16668
	.18850	.19327	.18850	.19327	.19333
	106.64	105.74	106.64	105.74	105.74
2.5	1.36483	1.35993	1.36487	1.35996	1.36011
	.06623	.06961	.06625	.06963	.06970
	22.808	22.683	22.814	22.688	22.693
10.0	1.50776	1.50589	1.50860	1.50673	1.50684
	.06960	.07036	.06955	.07032	.07032
	12.517	12.513	12.512	12.508	12.509
50.0	.66762	.65332	.67397	.65994	.66203
	.10716	.10839	.10708	.10833	.10837
	9.5394	9.5341	9.5248	9.5203	9.5228
70.0	.20673	.18956	.21117	.19426	.19703
	.11935	.12022	.11934	.12024	.12035
	9.7061	9.6927	9.6934	9.6806	9.6842

TABLE 36 - SEMI-ITERATION STUDY.

VALUES OF $\begin{cases} U \\ f_s \\ Nu \end{cases}$

Case 1 - Usual

Case 2 - Double iteration

16.7 Philosophy of Comparison

The preceding sections deal with aspects of internal accuracy. Here the question of attitude to external data is discussed. The 'classical' progression has been followed, through adiabatic results and constant property solutions to variable property problems. In general, the aim has been to try to reconcile results of this analysis with all comparable experimental and analytical data. In doing so there is not the same problem which frequently arises when results of experiment disagree with a (single) accompanying theory (as with Yang and Test - section 16.1).

Once the general reliability of this program had been demonstrated, a comparison (in such a case of disagreement) usually showed agreement with either:

(i) theory or

(ii) experiment

In the first instance, e.g. with Butterworth and Hazell's data, doubt was ultimately expressed about the measurements. However, this was only done reluctantly and with good cause, as with the heat transfer results for fully-developed entry velocity (see Fig. 11), where theory virtually co-incided.

In the second instance, the comparison probably indicated some approximation in the theory. For example, the forced convection experiments of Martin and Fargie were reasonably predicted. This possibly suggested some lack of accommodation in the corresponding theory of the substantial viscosity changes, just as initial comparison with this work highlighted the need for a better energy balance technique. Finally, even with an admittedly approximate theory, e.g. Rosen and Hanratty's analysis, results were not just dismissed, but accepted on their merits as far as possible.

16.8 Conclusion

While consistent care has been taken with accuracy and reliability, there is no real guarantee that accuracy for one problem automatically gives it for another. However, within the author's knowledge, the treatment gives good predictions in almost all contexts attempted.

17. GENERAL CONCLUSIONS

17.1 Introduction

The complete equations for internal laminar flow in two-dimensional cylindrical co-ordinates have been treated by a computer program solving the finite-difference equations. Complete variation of viscosity, and partial variation of density, with temperature are allowed for, and an alternative program accommodates some additional property variation.

The program in general has proved flexible and accurate, a particular feature being the use of the stepwise energy balance as a constituent equation rather than a check. A number of forced and combined convection cases have been treated, some of the results having already been published. The more detailed conclusions below relate to both published and unpublished data in a logical sequence.

17.2 Adiabatic flow

Developing velocity profiles show some divergence near entry with classic experiments of Nikuradse, but compare well with recent empirical and theoretical data. (Nikuradse confirmed that certain experimental details were not available). Pressure drops are somewhat higher than predicted by some boundary-layer analyses, but more consistent with the elliptic-type analysis of Schmidt and Zeldin. Development lengths for low Re are compared with those of Friedmann et al and Vrentas et al, but no consistency is apparent, this being attributed to differences in the inlet boundary condition used.

The effect upon accuracy is studied of reducing both radial and axial step sizes. Results for $M = 80$ differed little from those for $M = 30$. However, a lower limit of 0.625 exists for ΔZ below which program instability occurs.

17.3 Constant heat flux - forced convection

(i) Constant properties

Heat transfer results for fully-developed fluid entry compare well with a number of analyses. For uniform entry velocity, Nu varies with Pr (at constant Re) and results lie between those of Martin and Fargie and Butterworth and Hazell. Results for $Pr = 0.7$ compare very reasonably with data in the literature.

(ii) Water/glycerol mixtures

Calculations of Nu are compared with analytical and empirical data of Butterworth and Hazell. Constant property predictions agree to within 15%, and allowance for viscosity variation increased Nu by at most 6%. The rather inconsistent divergences of experiment are reluctantly attributed to measurement difficulties.

(iii) Oil

Predictions of Nu , centre-line velocity and friction factor give good agreement with ~~a~~ high temperature experimental data of Martin and Fargie, despite the latter being for a sharp entry. A similar comparison for a lower temperature run/^{is}nearly as good, except for friction factor.

17.4 Effect of viscous dissipation

(i) Adiabatic flow

Comparison of Nu and pressure drop is reasonable with Martin's predictions and good with experimental data of Hersey and Zimmer, and Polak. Polak's postulation of a hot slip zone is examined, but the concept of thermal instability is not confirmed.

(ii) Convection with constant heat flux

The effect of viscous dissipation is a small reduction in Nu , and this is consistent with the experimental data of Martin and Fargie, and Butterworth and Hazell in the previous section.

17.5 Constant heat flux - combined convection

(i) Comparison with an approximate analysis

The completeness of the present analysis has provided a basis for comparison with results of Allen's truncation method. Heat transfer and fluid flow parameters compare well for the experimental data below, and justify Allen's approach.

(ii) Water

Developing velocity profiles compare well with Lawrence and Chato's data, and wall temperature predictions with measurements of Scheele and Hanratty.

(iii) Oils

Predictions of wall temperature (up to transition) provide the first (and good) analytical comparison with a run of Kemeny and Somers. Comparison made with three runs of Allen is also good.

(iv) Additional property variation

For this section of the work additional property variation was accommodated within the program. The effect of this is found to be negligible or small, depending on both magnitude and sign of the property changes with temperature.

17.6 Constant wall temperature - forced convection

(i) Parabolic entry velocity

Nu values (for given Pe) lie between those obtained from the elliptic equation analyses of Jones and Singh. A comparison of mixed mean temperatures, however, show a pleasing agreement at high Pe with Jones. Comparison with boundary-layer analysis and the full equations treatment of Zeldin and Schmidt is reasonable, and a study of Marner's criticism of the latter's results shows that the problem could be resolved by a consideration of mixed mean temperatures.

(ii) Uniform entry velocity

Comparisons of Nu made with (a) Zeldin and Schmidt, and (b) Manohar, and Ulrichson and Schmitz, are very reasonable.

(iii) Alternative boundary condition treatments

Several comparisons are made for the methods using (1) the explicit wall thermal b.c., and (2) the implicit energy balance statement. Apart from extreme entry, agreement is very close. However, a reservation is that both methods appear to give rather low long-distance Nu values.

17.7 Constant wall temperature - combined convection

(i) Parabolic entry velocity

Comparison is good with the theoretical data of both Zeldin and Schmidt, and Marner and McMillan. In the latter case, however, good agreement of velocity and temperature profiles is not reflected in a comparison of Nu, and specifically, their prediction of a peak in Nu at maximum profile distortion is only partially confirmed.

(ii) Uniform entry velocity

Again, comparison is good with a theoretical result of Zeldin and Schmidt. A detailed comparison is made with an experimental run (and theory) of those authors, which show that Collins's results are closer to experiment for axial velocities, but not as close for temperatures. However, this is in the context of possible preheating of the fluid, which present calculations also suggest.

(iii) Alternative boundary condition treatments

Some comparisons are again made for the alternatives in 17.6 (iii) above. Agreement is very satisfying and, on balance, the 'exact energy balance' method is preferred in this context.

17.8 Transition and instability

A study, chiefly based on observations of Scheele et al, has

shed further light on these phenomena, partly because their accompanying analytical work was more approximate. For natural assisting forced convection, a simplified version of Lawrence and Chato's transition parameter is found to correlate the data. This is valid for upward heated flow, with both constant wall heat flux and temperature. For natural opposing forced convection, results are insufficient for correlation, partly because of discrepancies in presented data.

Comparison with the approximate analysis of Rosen and Hanratty is reasonable for centre-line velocity, but not as good for distances. However, Collins's predictions of these are probably more consistent with flow reversal observations. A limited comparison with the fully-developed theory of Hanratty et al suggests that their predictions have some validity even in a developing flow field.

17.9 Annulus

Initial adiabatic results are compared with those of Heaton et al, and felt to demonstrate the flexibility of the overall treatment in its adaptation to annular flows.

17.10 Accuracy

The various aspects of accuracy are felt to be satisfactory. Convergence in general is good, and stability very good. The semi-iteration procedure works well. The energy balance requirement is shown to be a useful aid to accuracy, not fully utilised in some current analyses.

17.11 Conclusion

The analysis presented here has been applied to a number of laminar flow problems involving a variety of working fluids and boundary conditions. In general it gives consistently good results, in some cases resolving differences between theory and experiment. The applications have shown it to be flexible and of general validity and reliability.

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$(f_p \times Re_R)$
 10^2

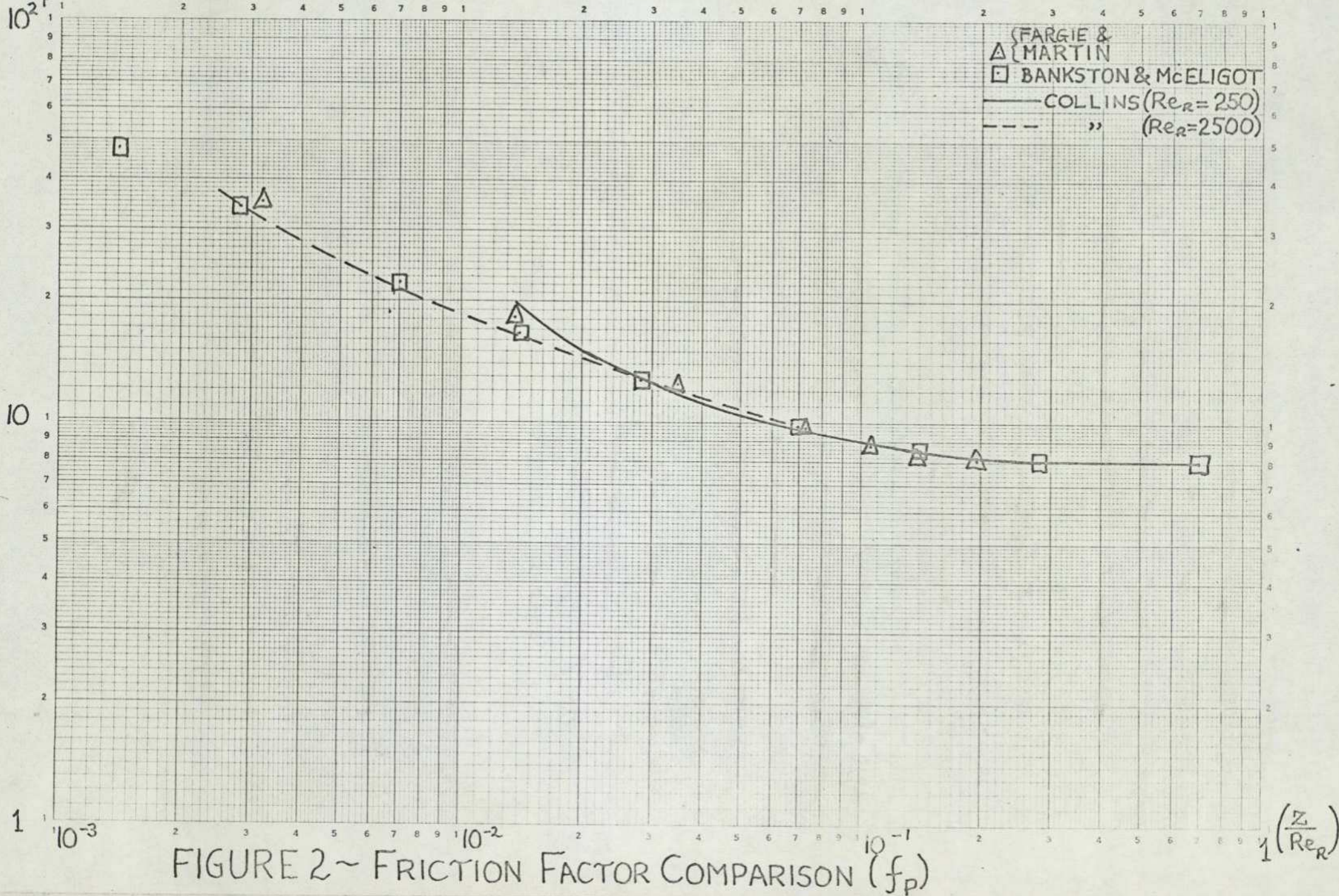


FIGURE 2~ FRICTION FACTOR COMPARISON (f_p)

$$(f_{av} \times Re_R) \times 10^2$$

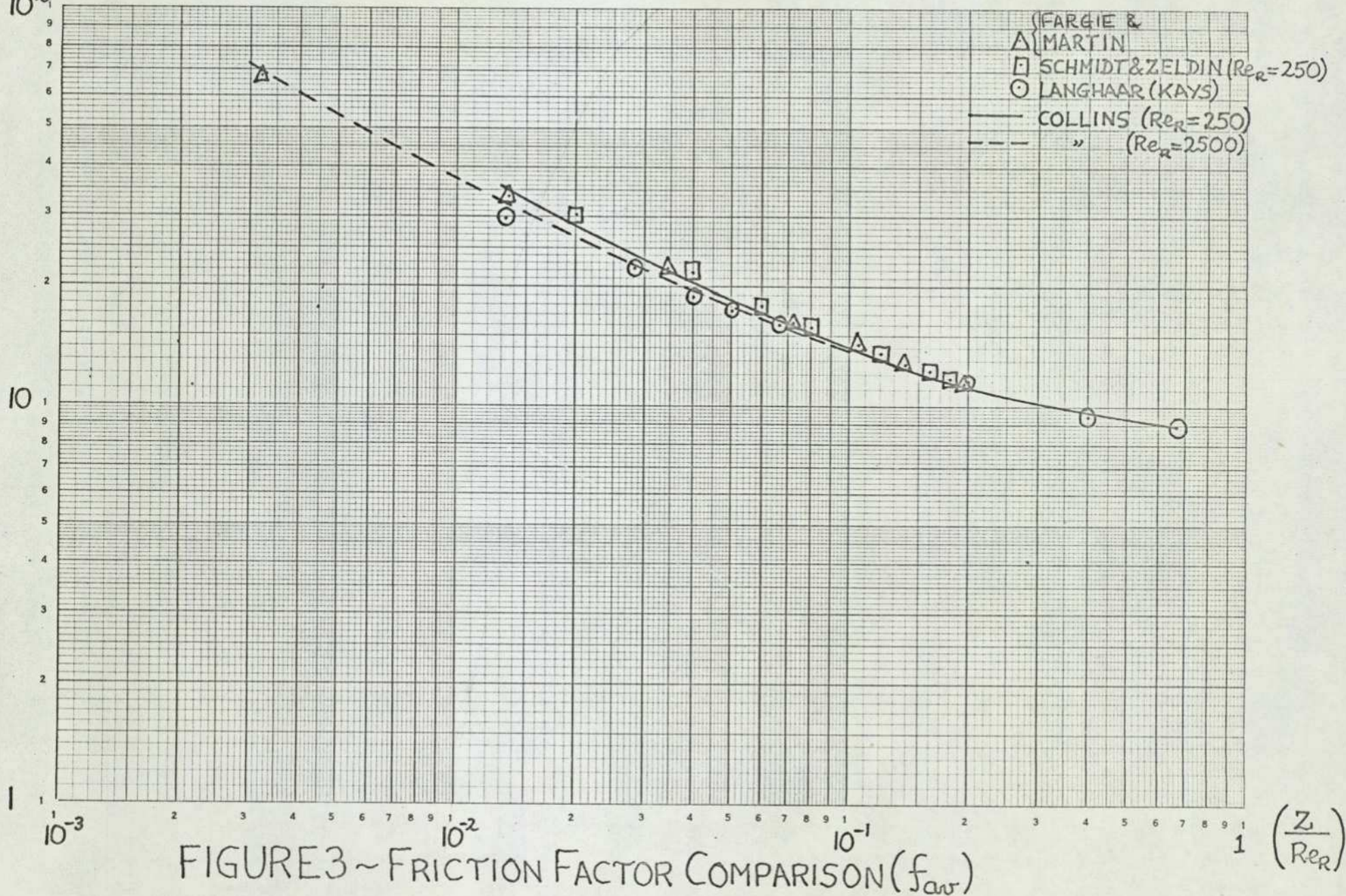
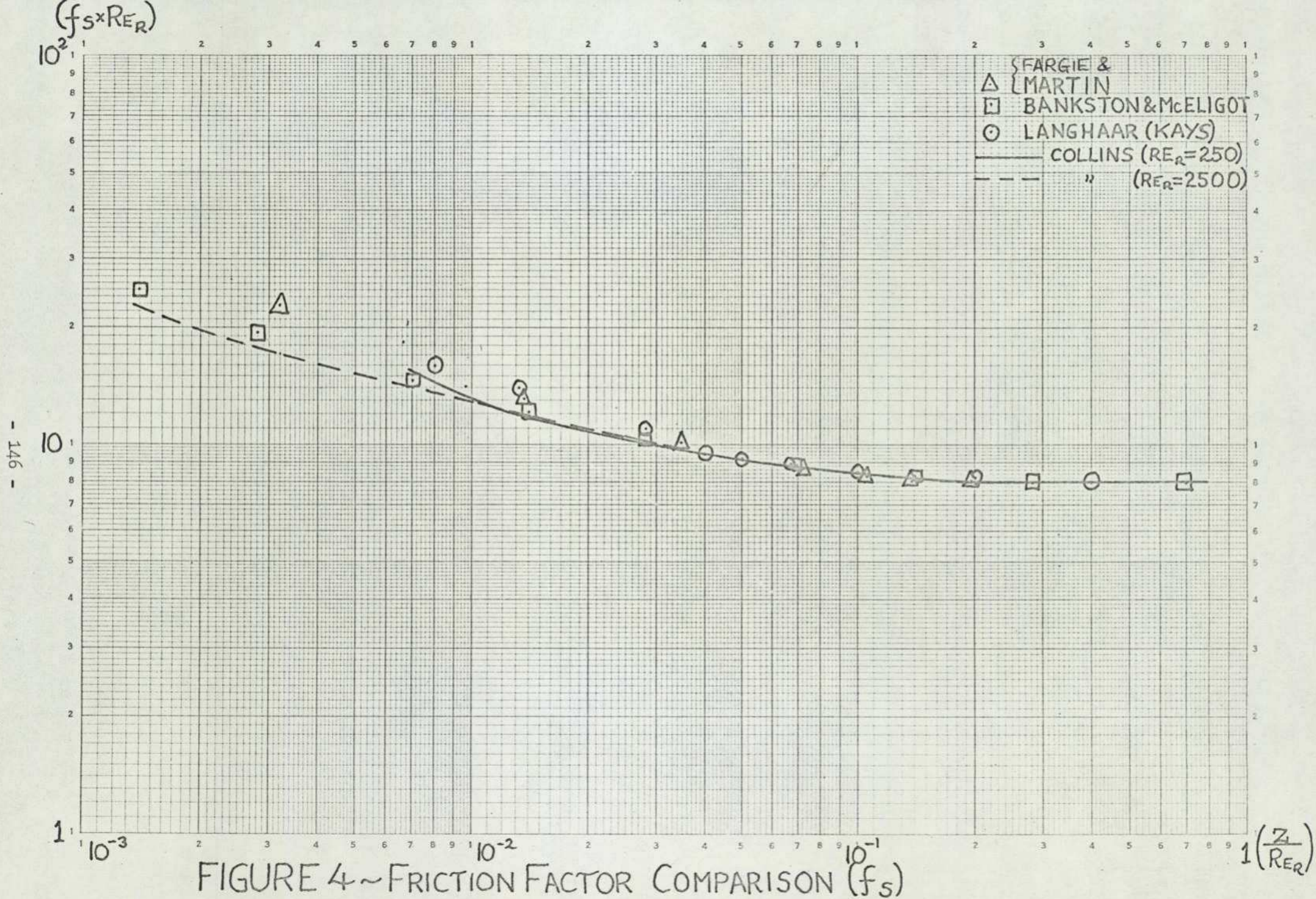


FIGURE 3 ~ FRICTION FACTOR COMPARISON (f_{av})

$$\left(\frac{Z}{Re_R} \right)$$



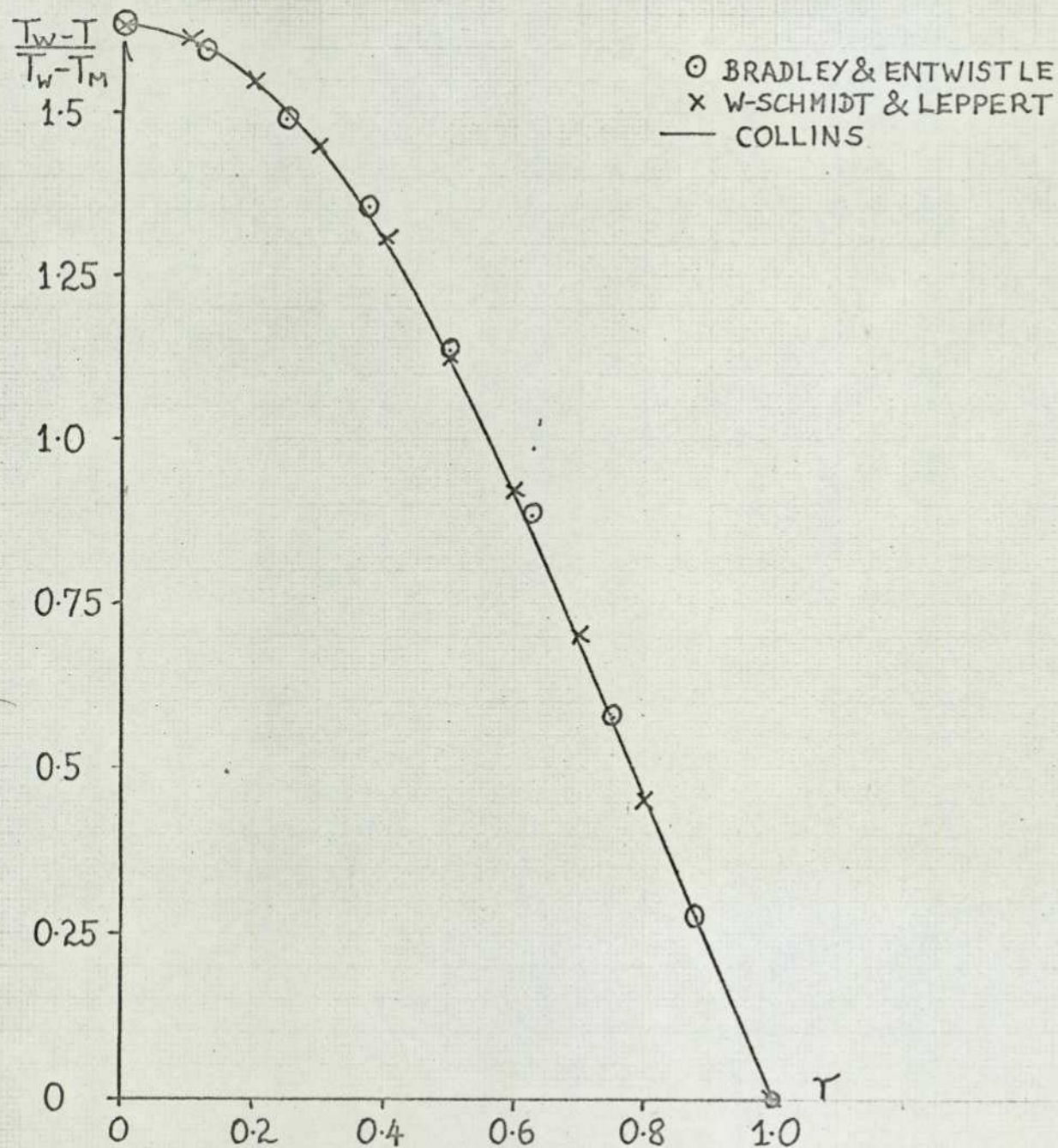


FIGURE 5
FULLY-DEVELOPED
FORCED CONVECTION
TEMPERATURE PROFILE
(CONSTANT PROPERTIES
AND CONSTANT HEAT FLUX)

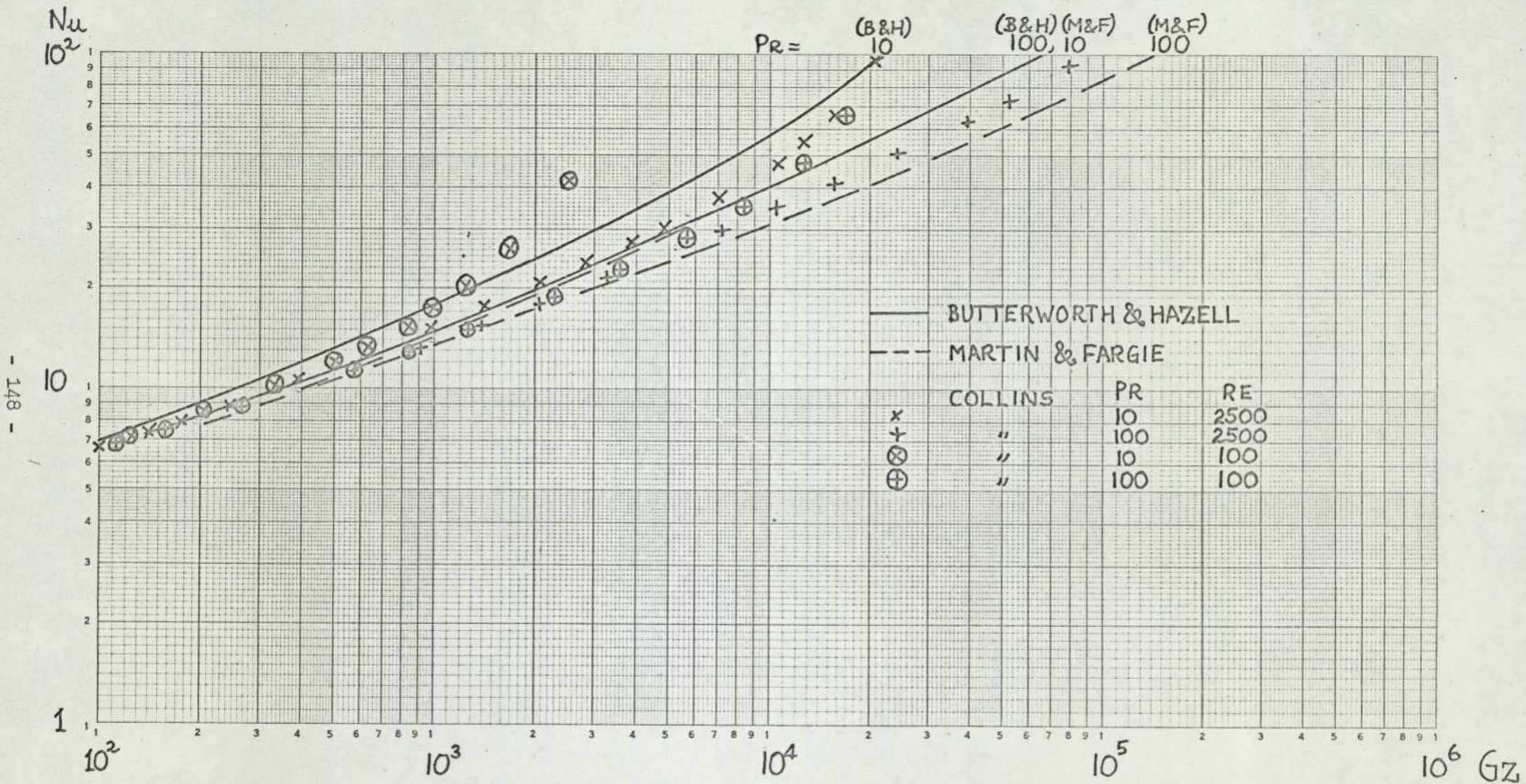


FIGURE 6 ~ EFFECT OF Pr ON HEAT TRANSFER WITH UNIFORM ENTRY VELOCITY ($Pr=10, 100$)

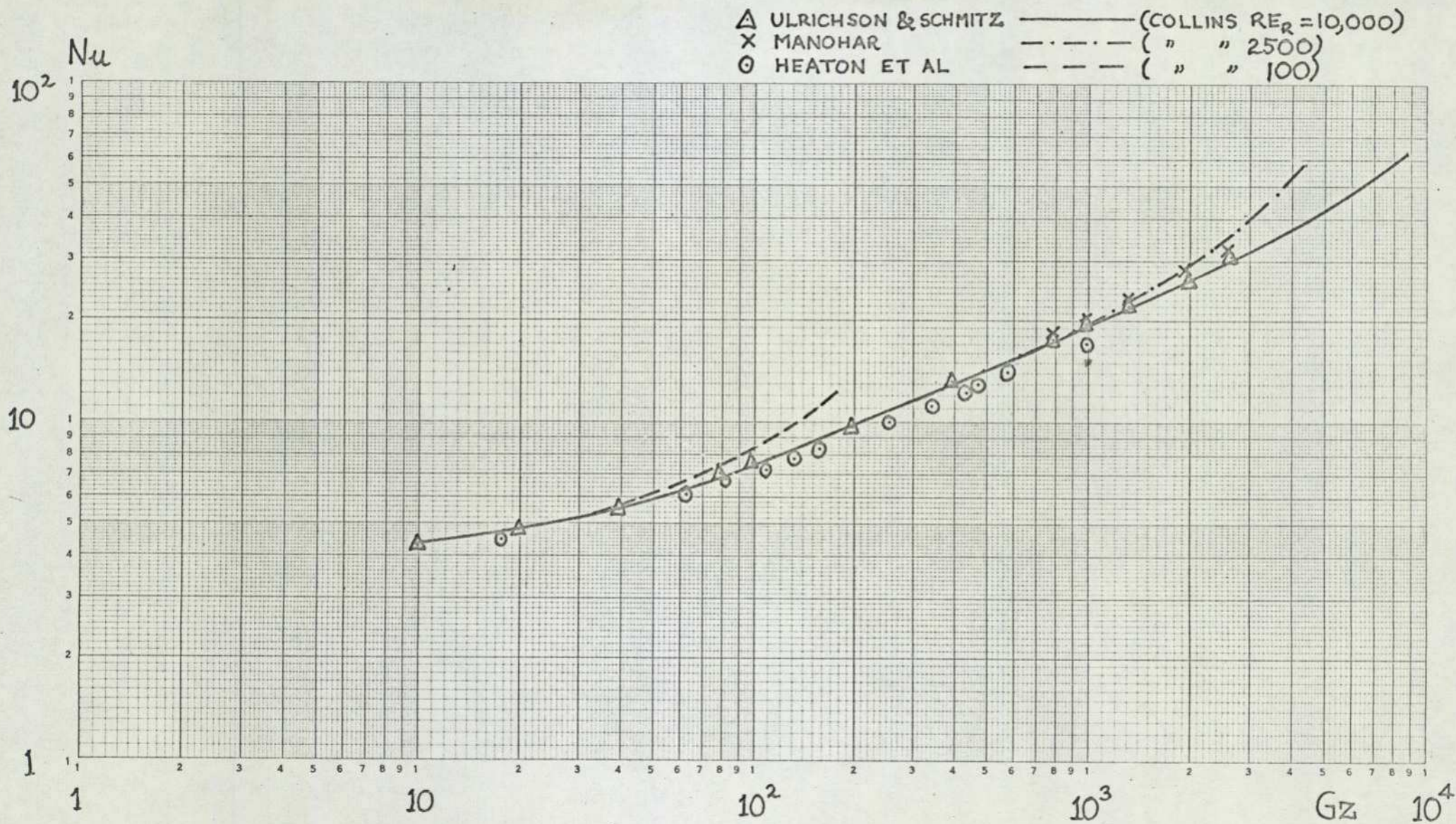


FIGURE 7~EFFECT OF Pr ON HEAT TRANSFER WITH UNIFORM ENTRY VELOCITY ($Pr=0.7$)

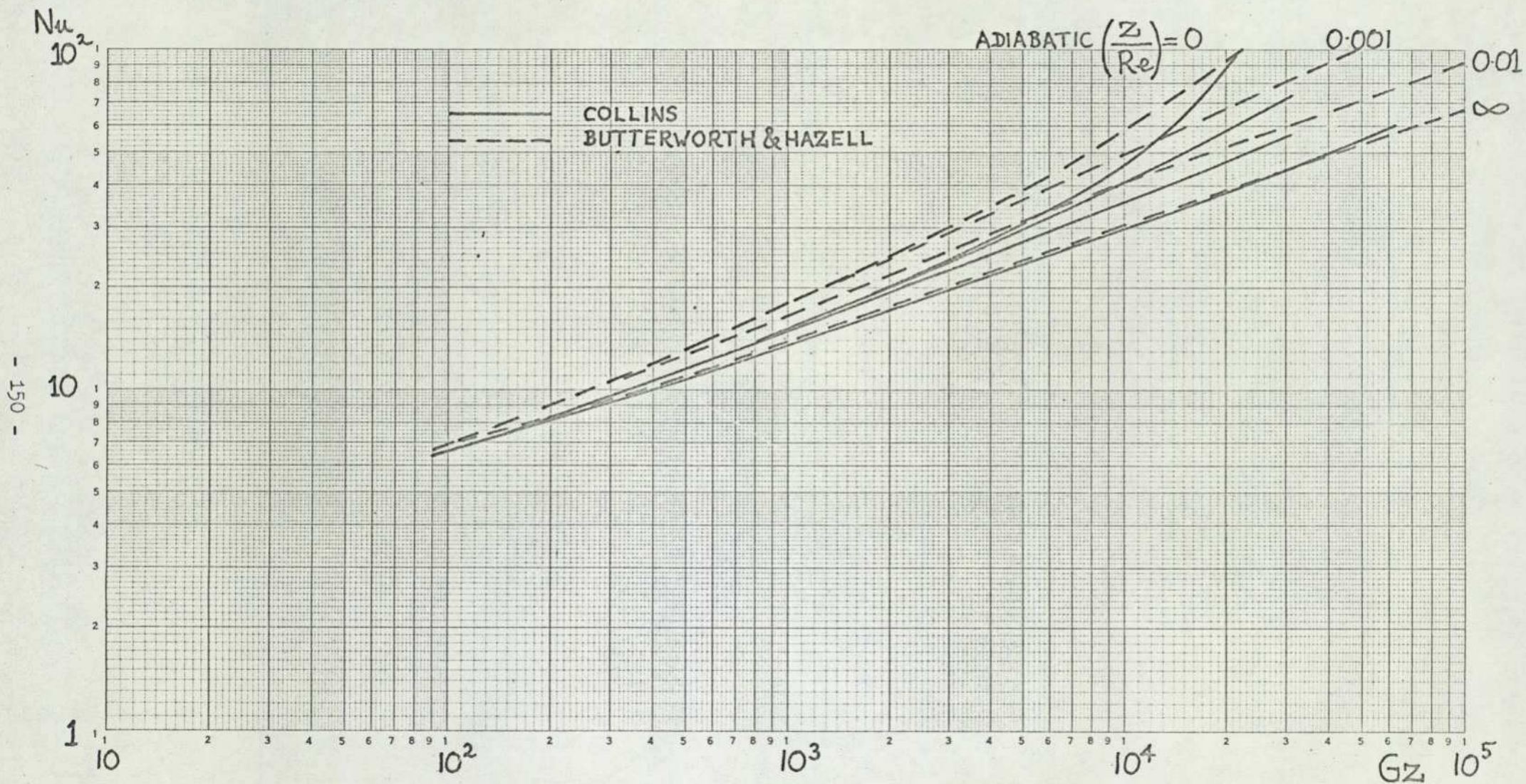


FIGURE 8~EFFECT OF ADIABATIC DEVELOPMENT ON HEAT TRANSFER ($Pr=10$)

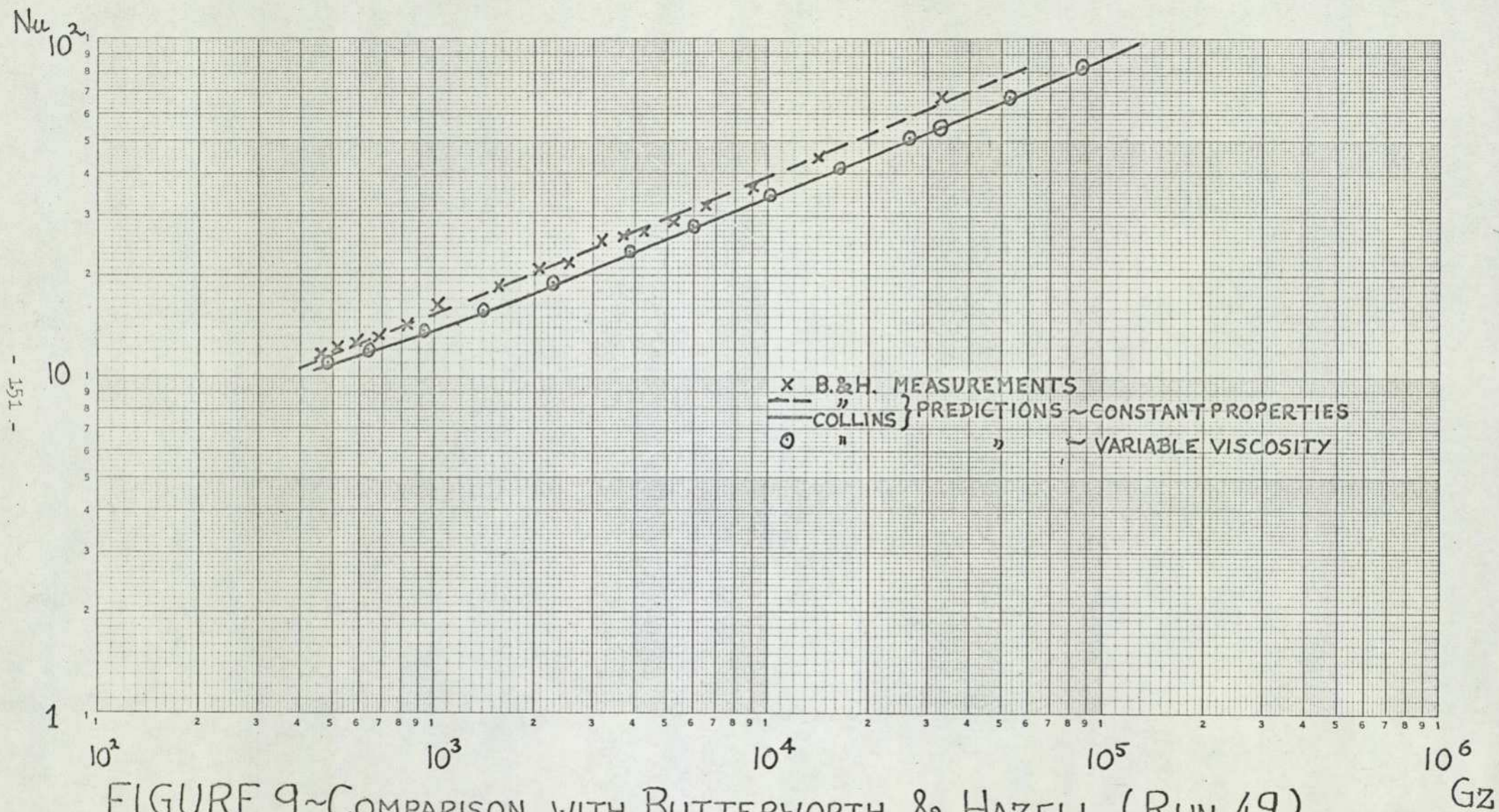


FIGURE 9~COMPARISON WITH BUTTERWORTH & HAZELL (RUN 49)

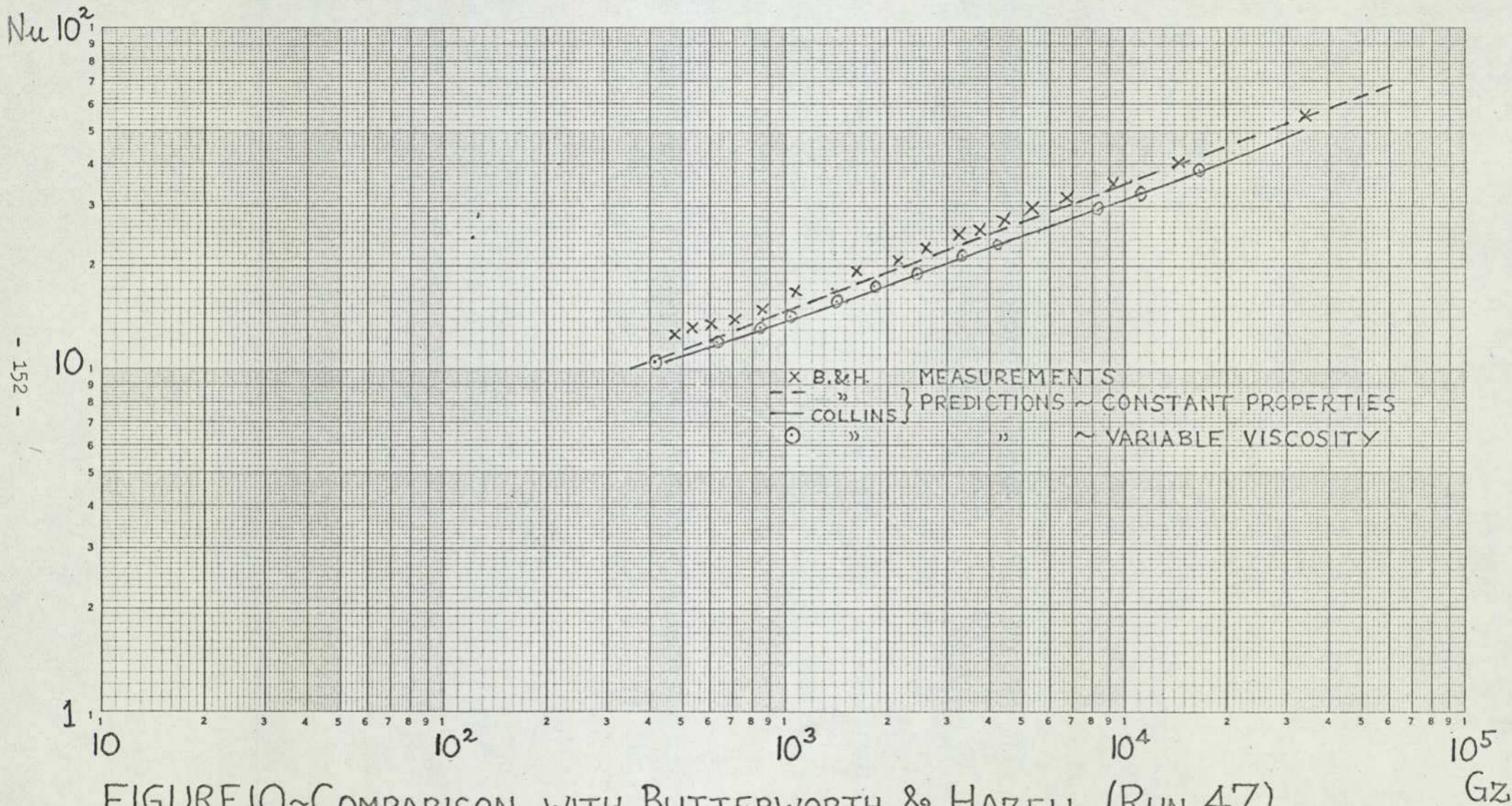


FIGURE 10~COMPARISON WITH BUTTERWORTH & HAZELL (RUN 47).

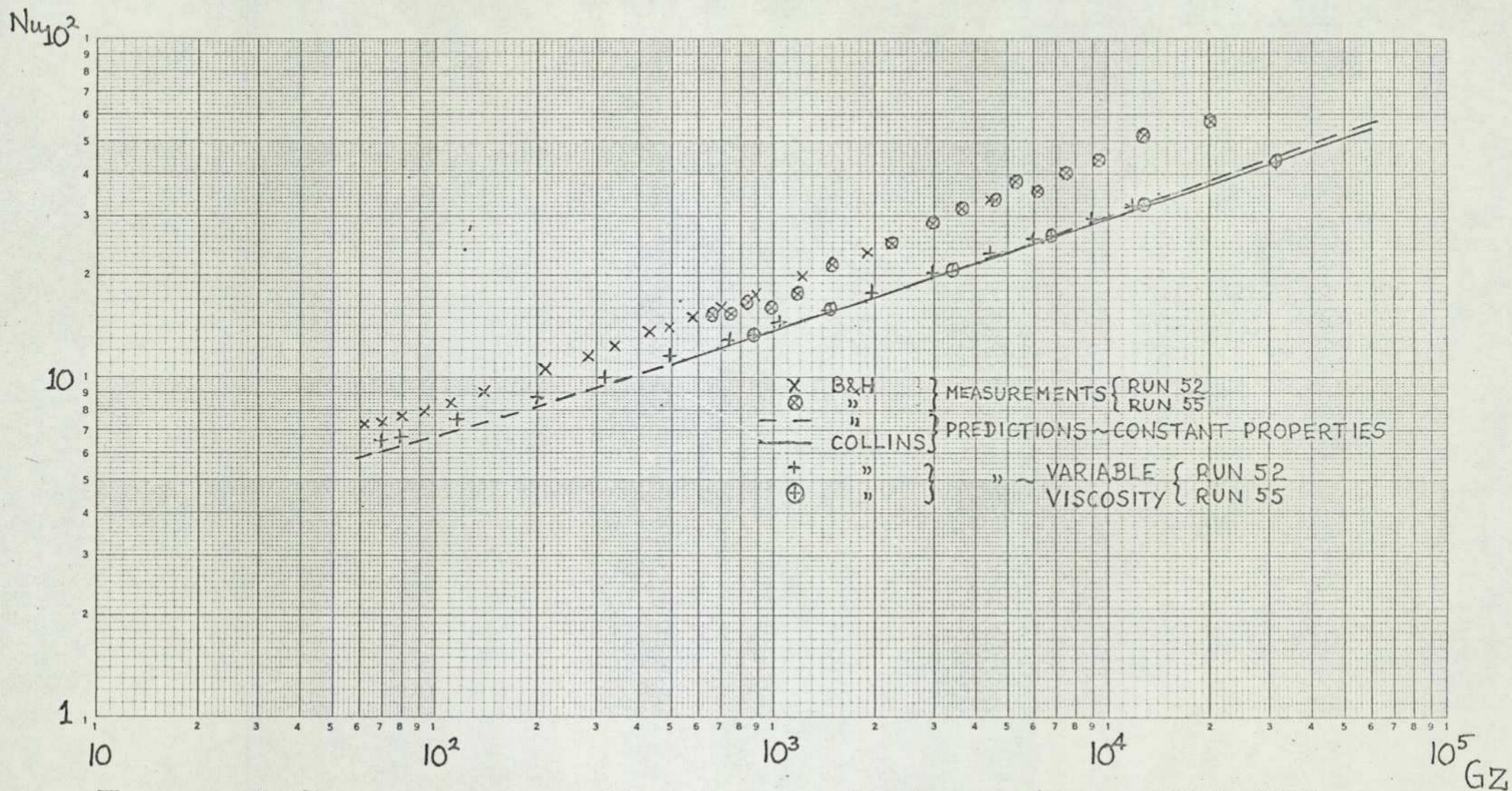


FIGURE 11. COMPARISON WITH BUTTERWORTH & HAZELL (RUNS 52 & 55)

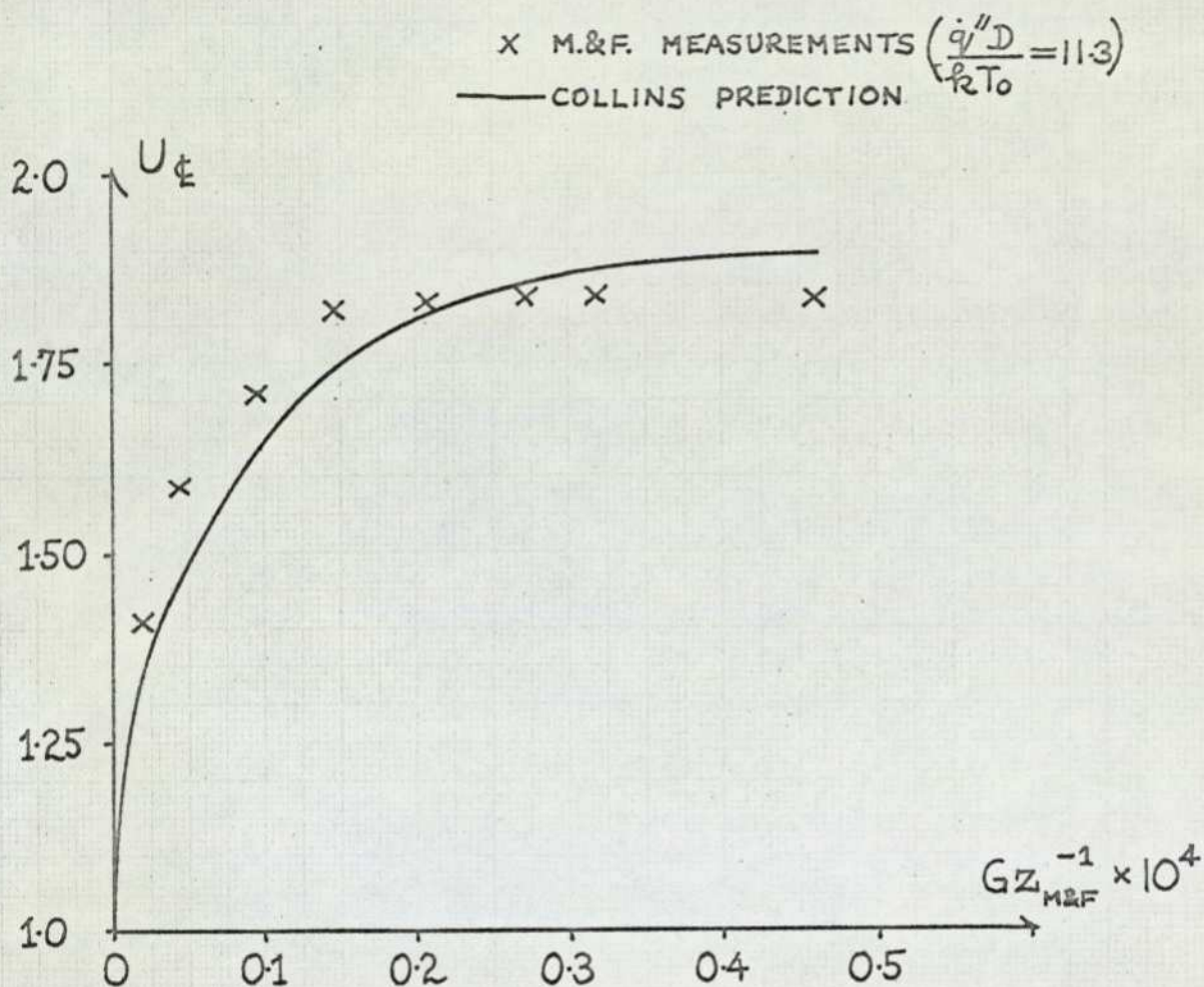


FIGURE 12 ~ COMPARISON OF CENTRE~LINE VELOCITIES
 (MARTIN & FARGIE)
 $[GZ_{M\&F} = 4 \times Pe/Z]$

$f_{AV} \times Pe$
 10^5
 10^4

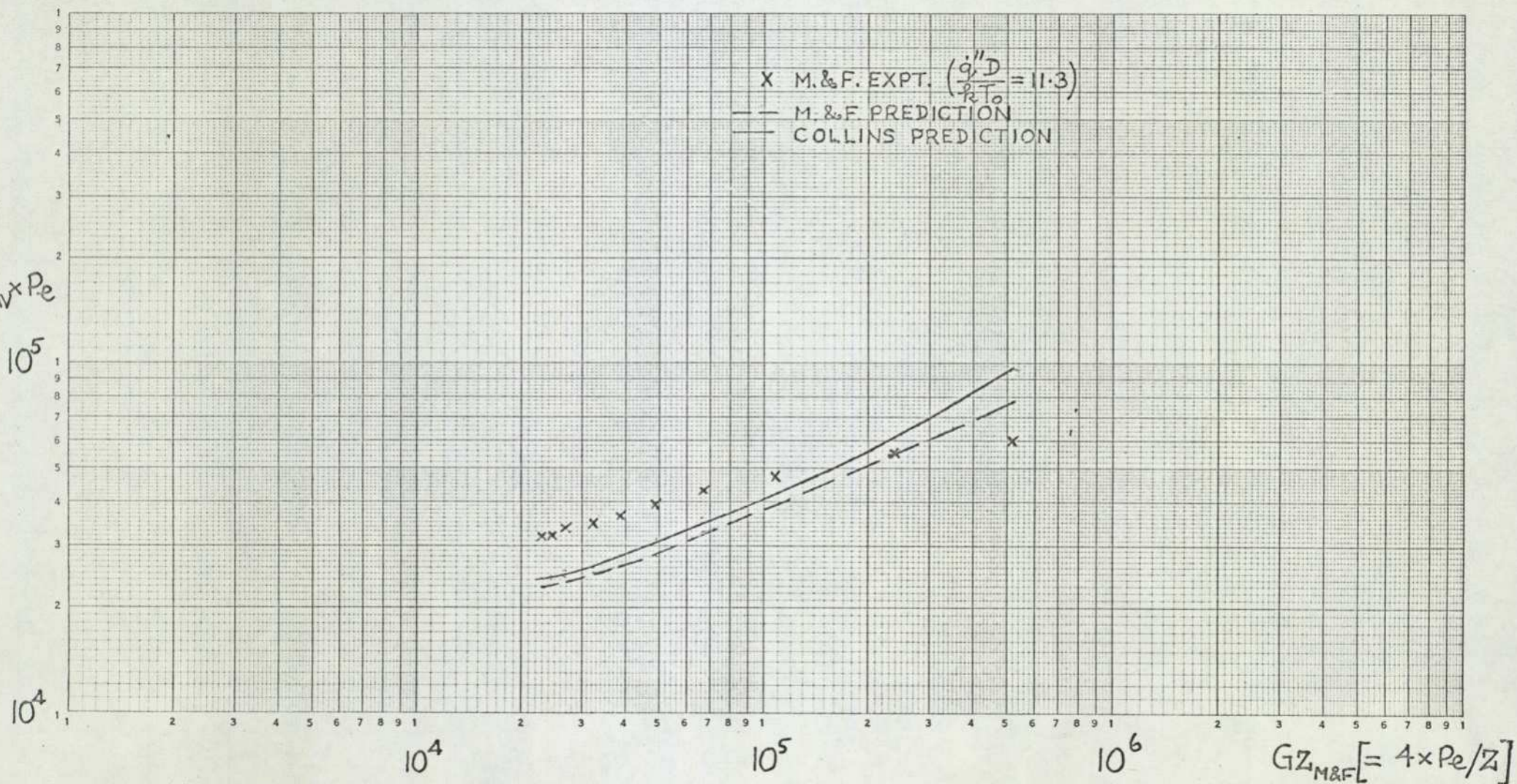


FIGURE 13~COMPARISON OF FRICTION FACTOR.(MARTIN & FARGIE)

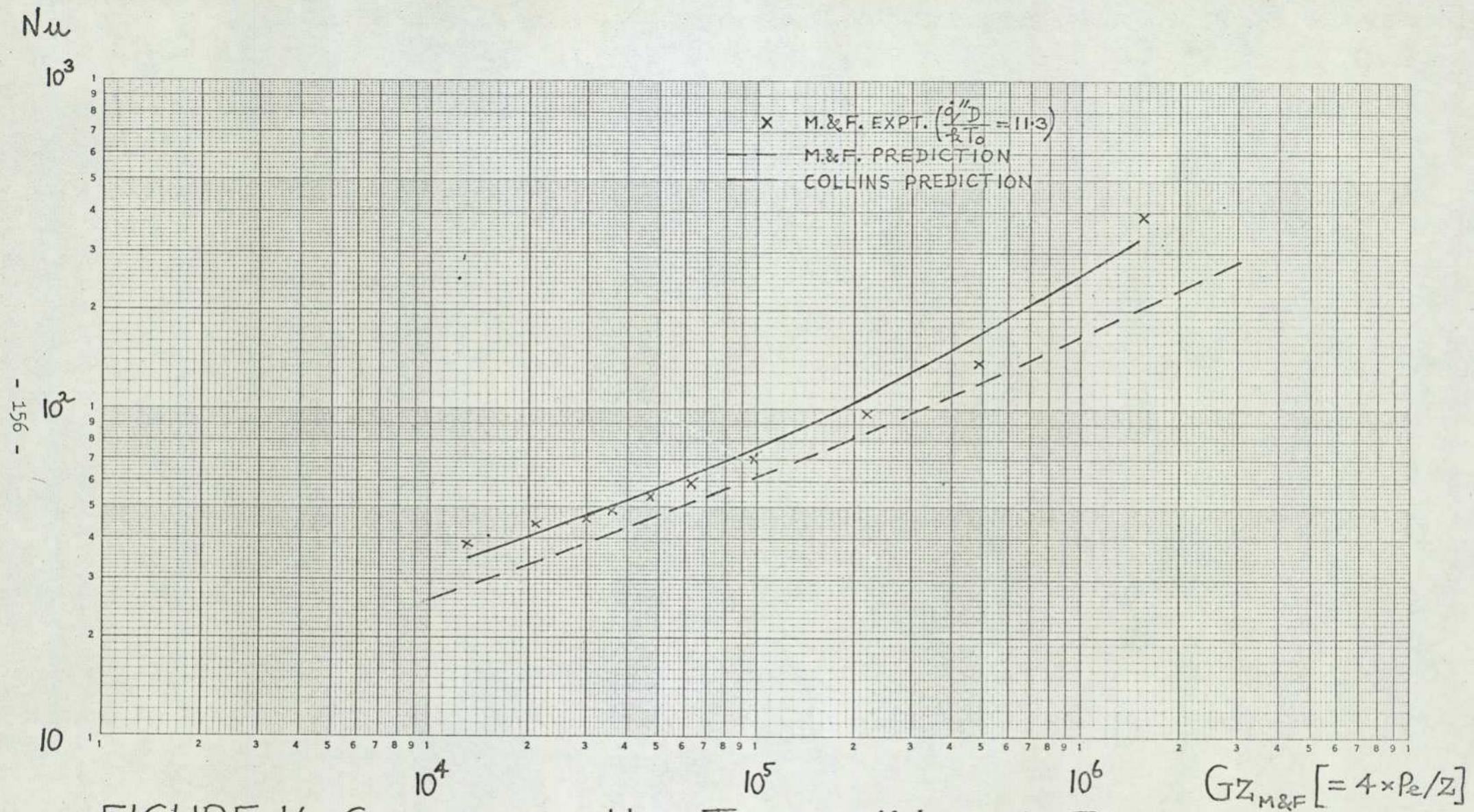


FIGURE 14.~COMPARISON OF HEAT TRANSFER (MARTIN & FARGIE)

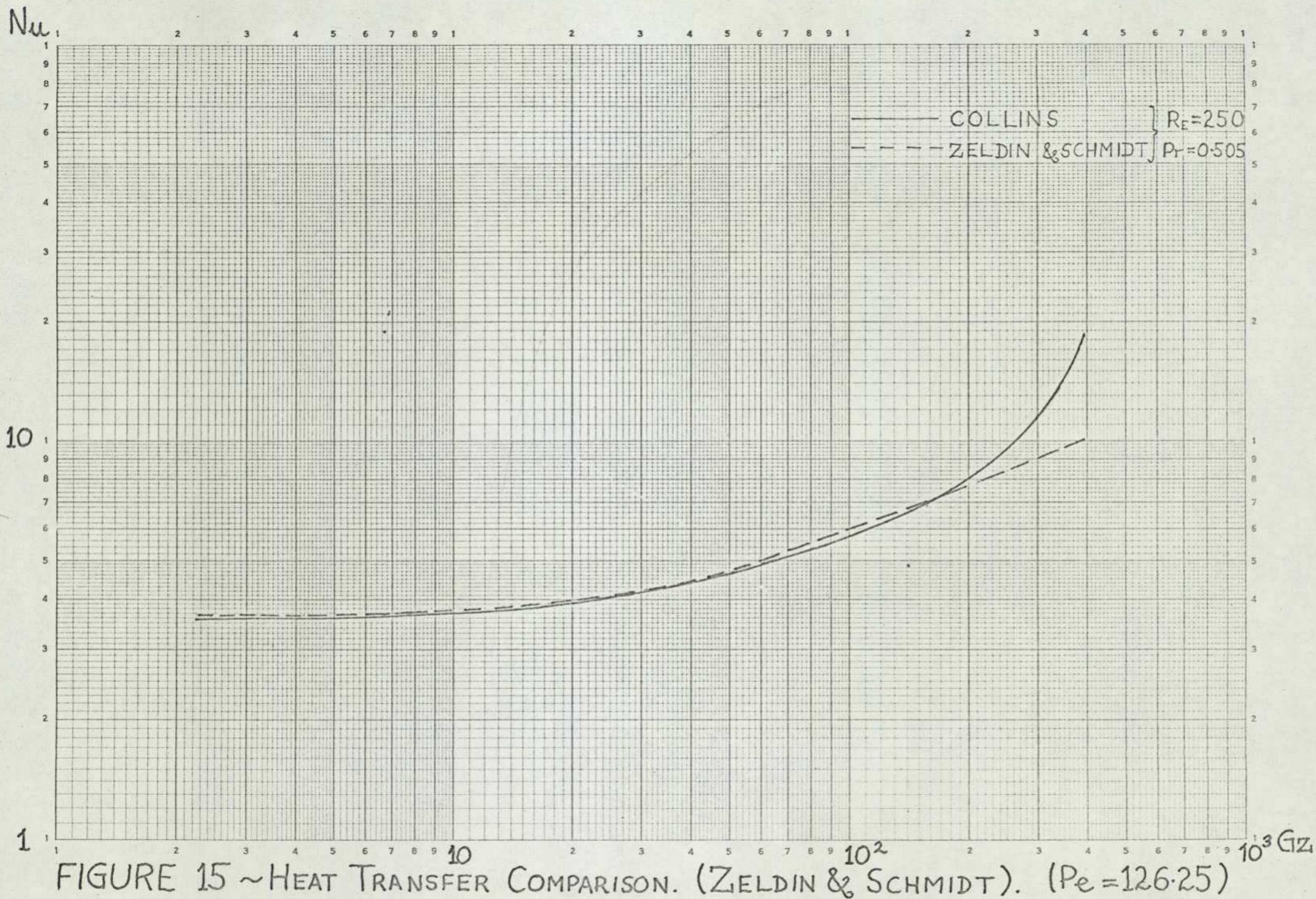


FIGURE 15 ~ HEAT TRANSFER COMPARISON. (ZELDIN & SCHMIDT). ($Pe = 126.25$)

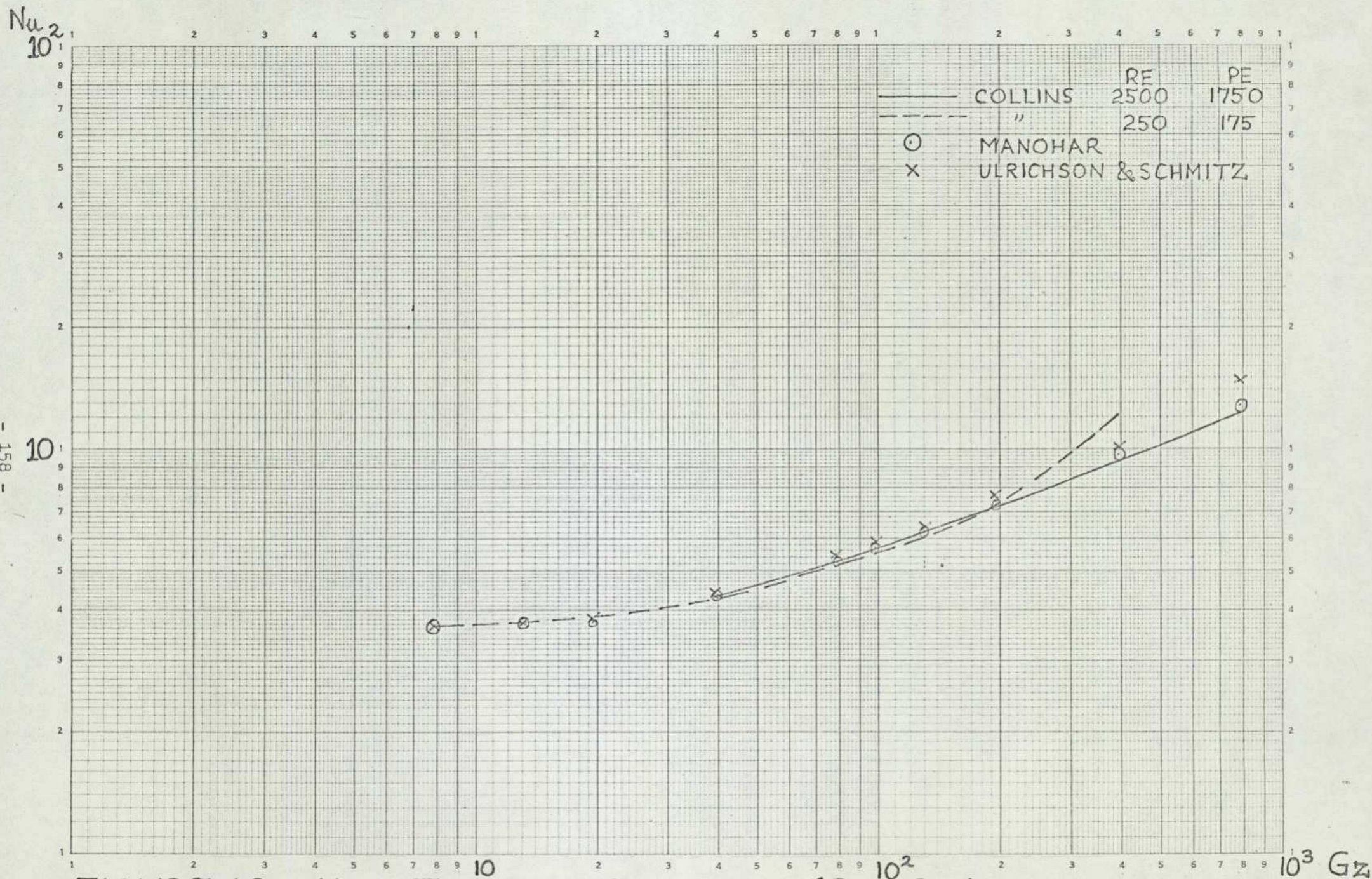


FIGURE 16 ~ HEAT TRANSFER COMPARISON. ($Pr = 0.7$)

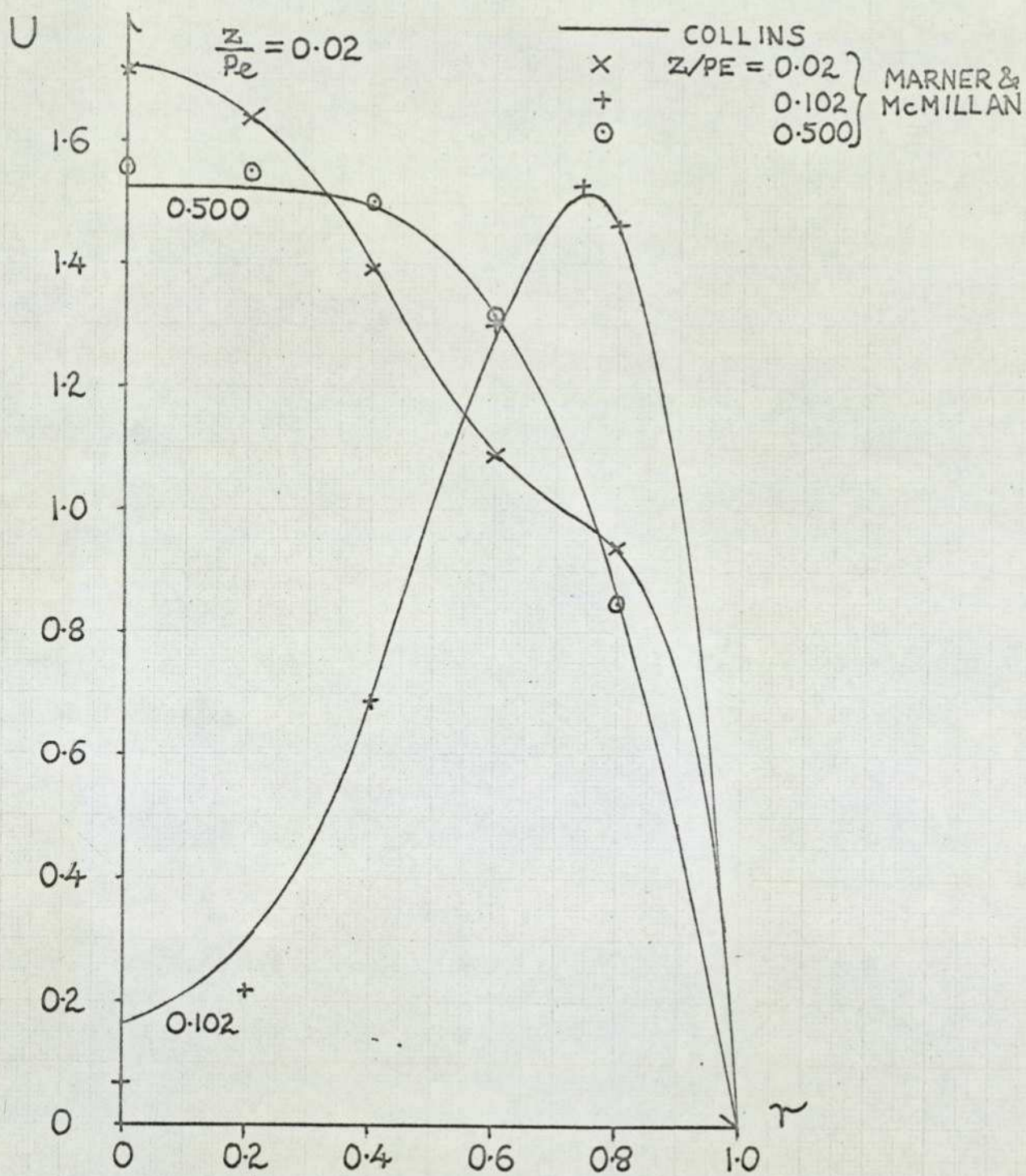


FIGURE 17 ~ VELOCITY PROFILES COMPARISON
(MARNER & McMILLAN)

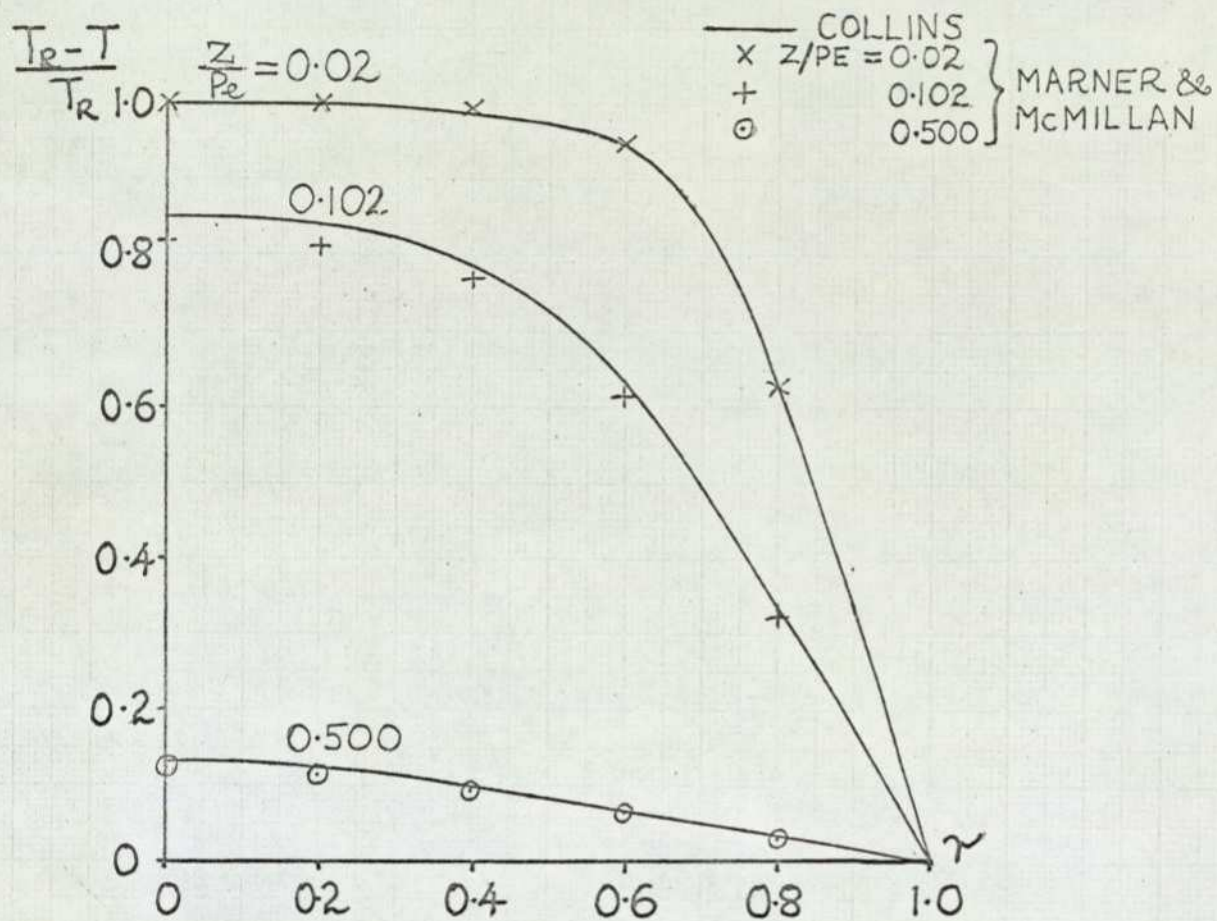


FIGURE 18 ~ TEMPERATURE PROFILES COMPARISON
(MARNER & McMILLAN)

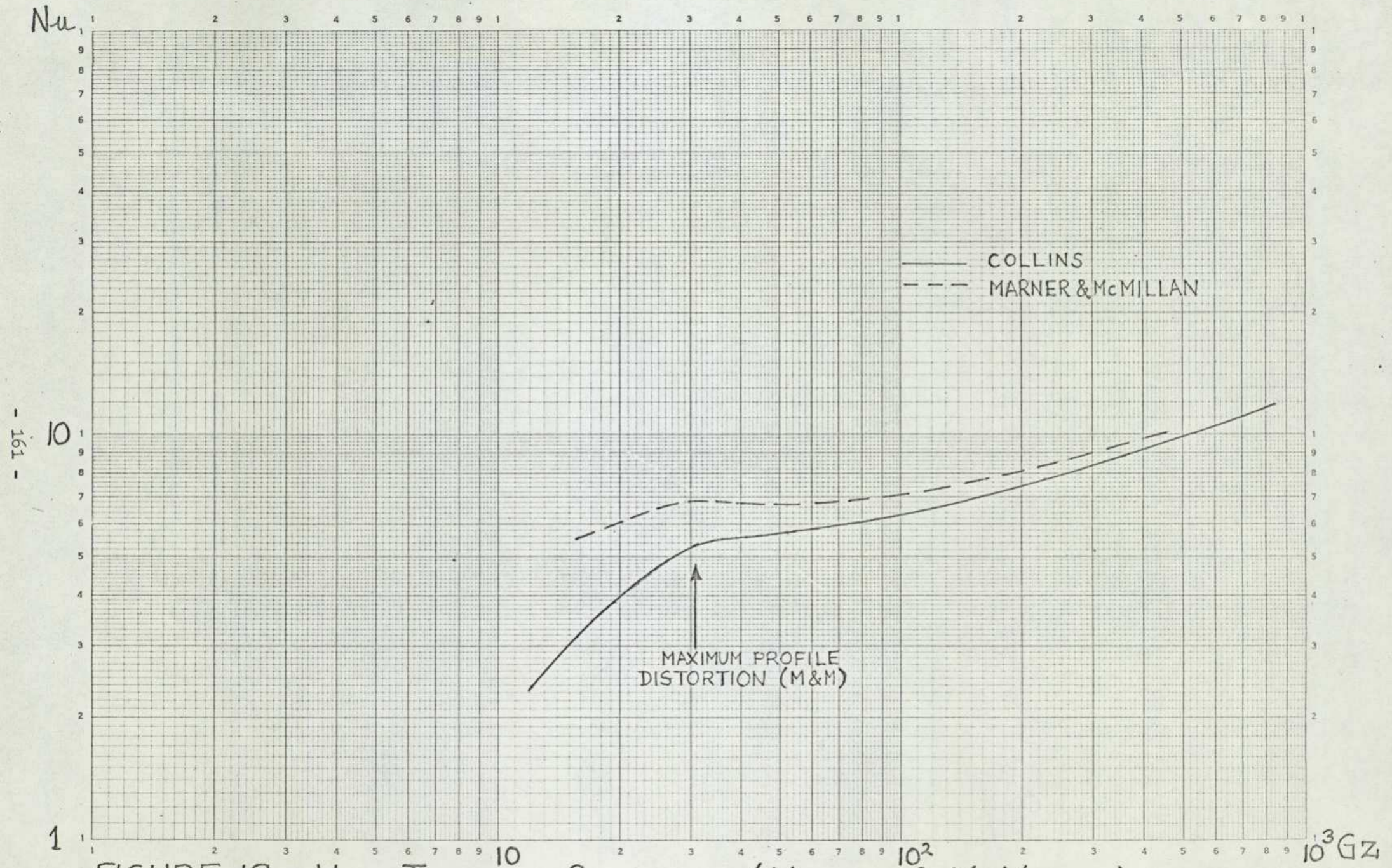


FIGURE 19 ~ HEAT TRANSFER COMPARISON (MARNER & McMILLAN).

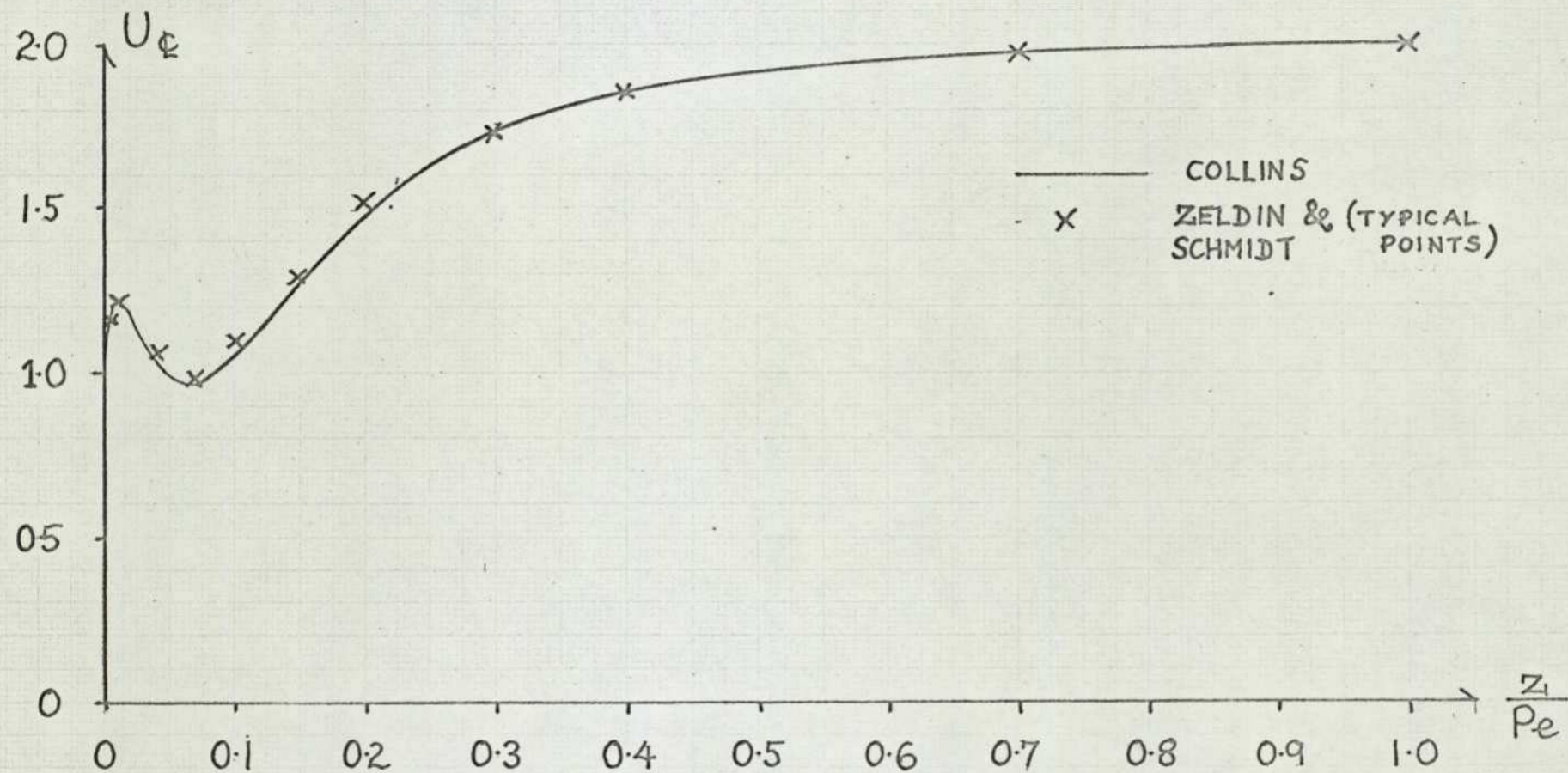
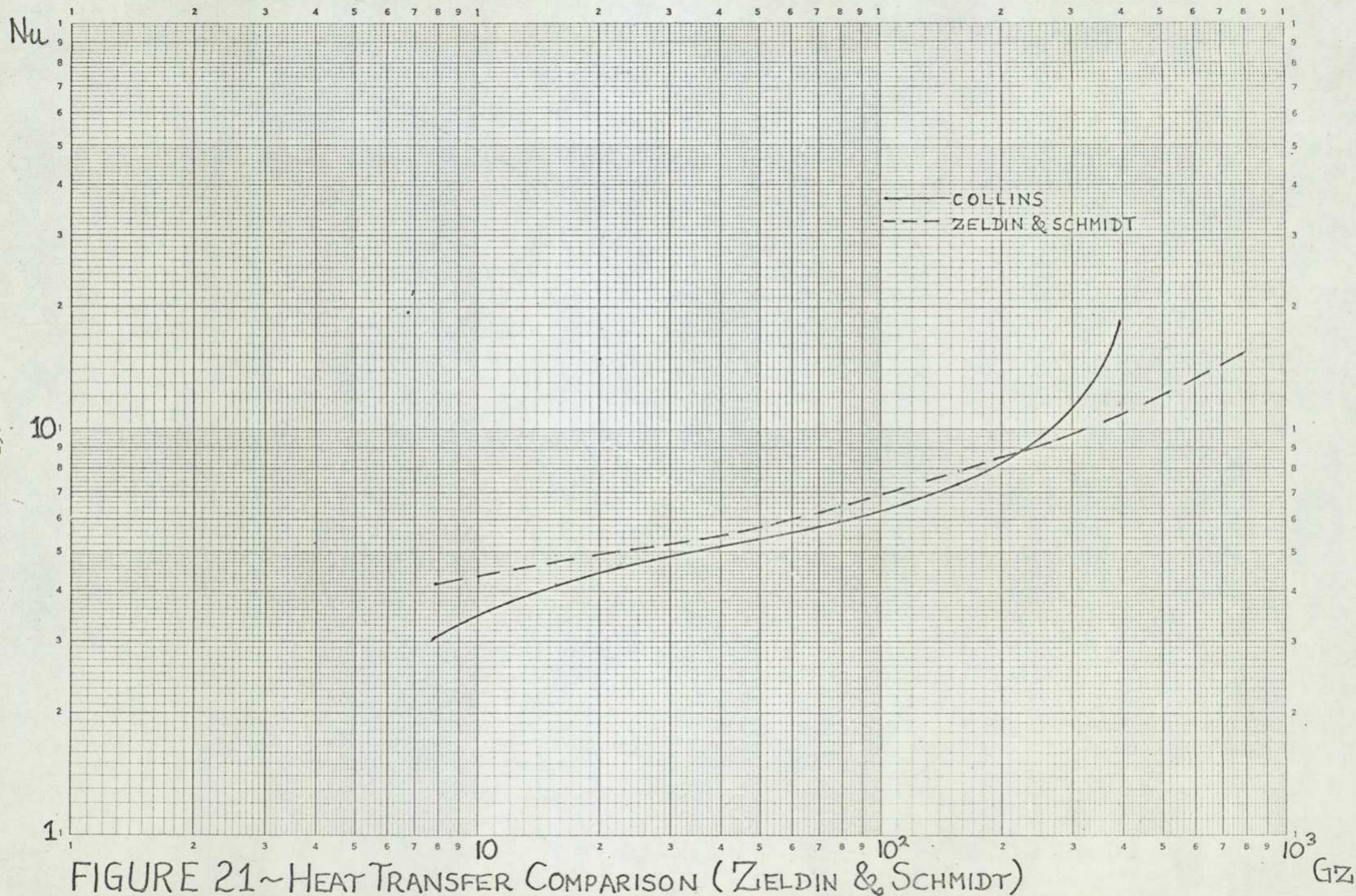
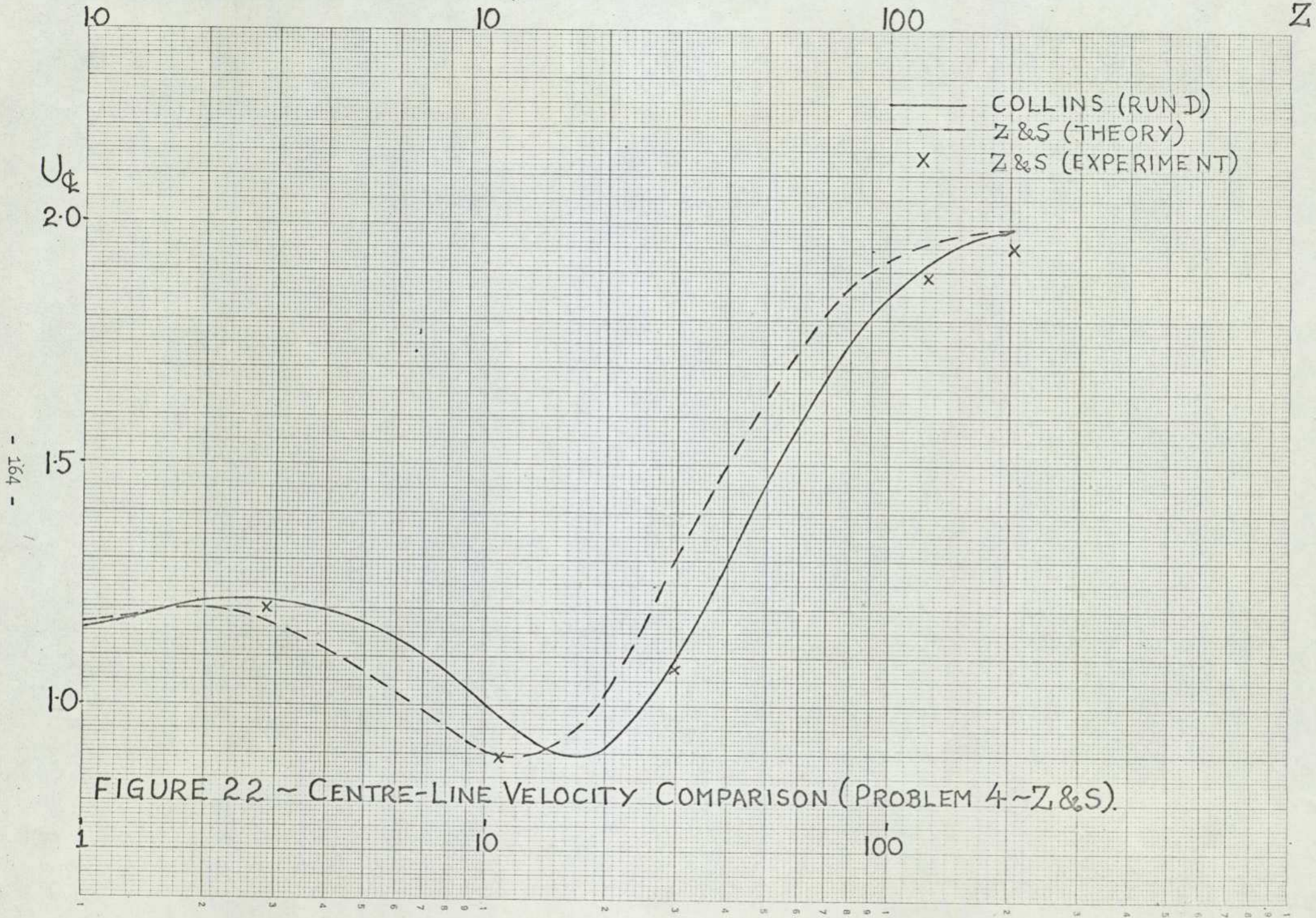


FIGURE 20 ~ CENTRE-LINE VELOCITY COMPARISON (ZELDIN & SCHMIDT)





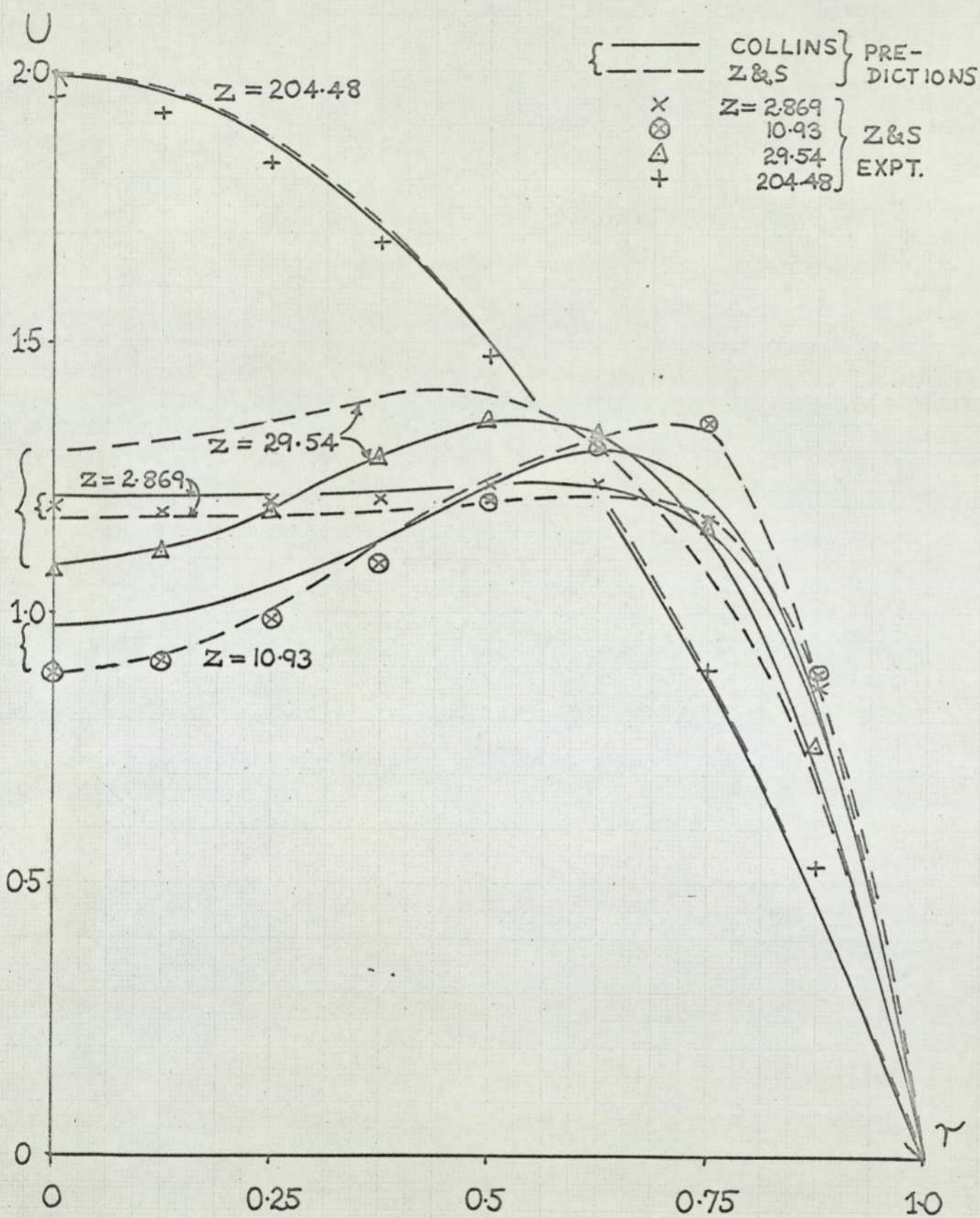


FIGURE 23 ~ VELOCITY PROFILE COMPARISON (PROBLEM 4 ~ Z&S)

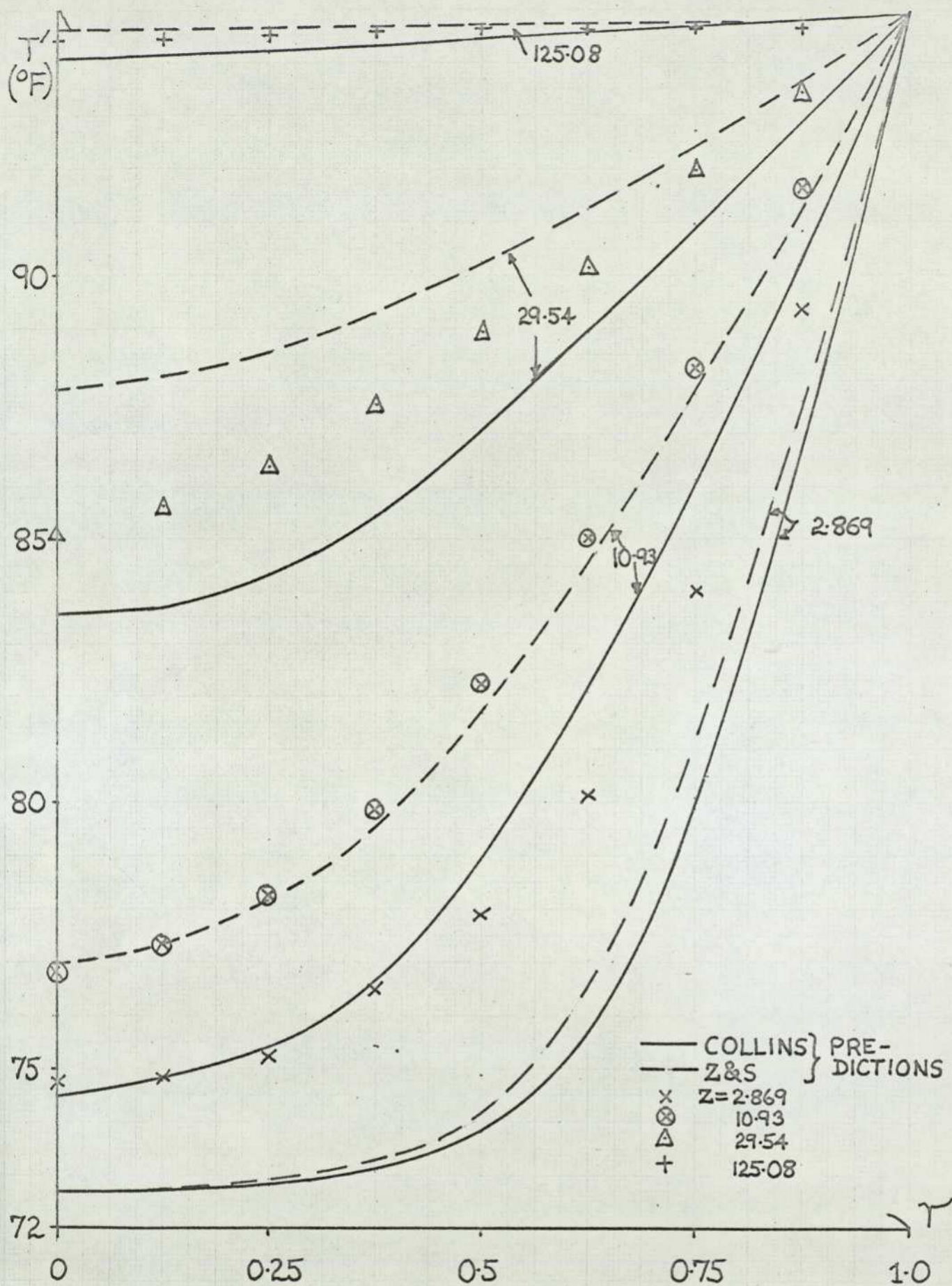


FIGURE 24 - TEMPERATURE PROFILE COMPARISON (PROBLEM 4-Z&S)

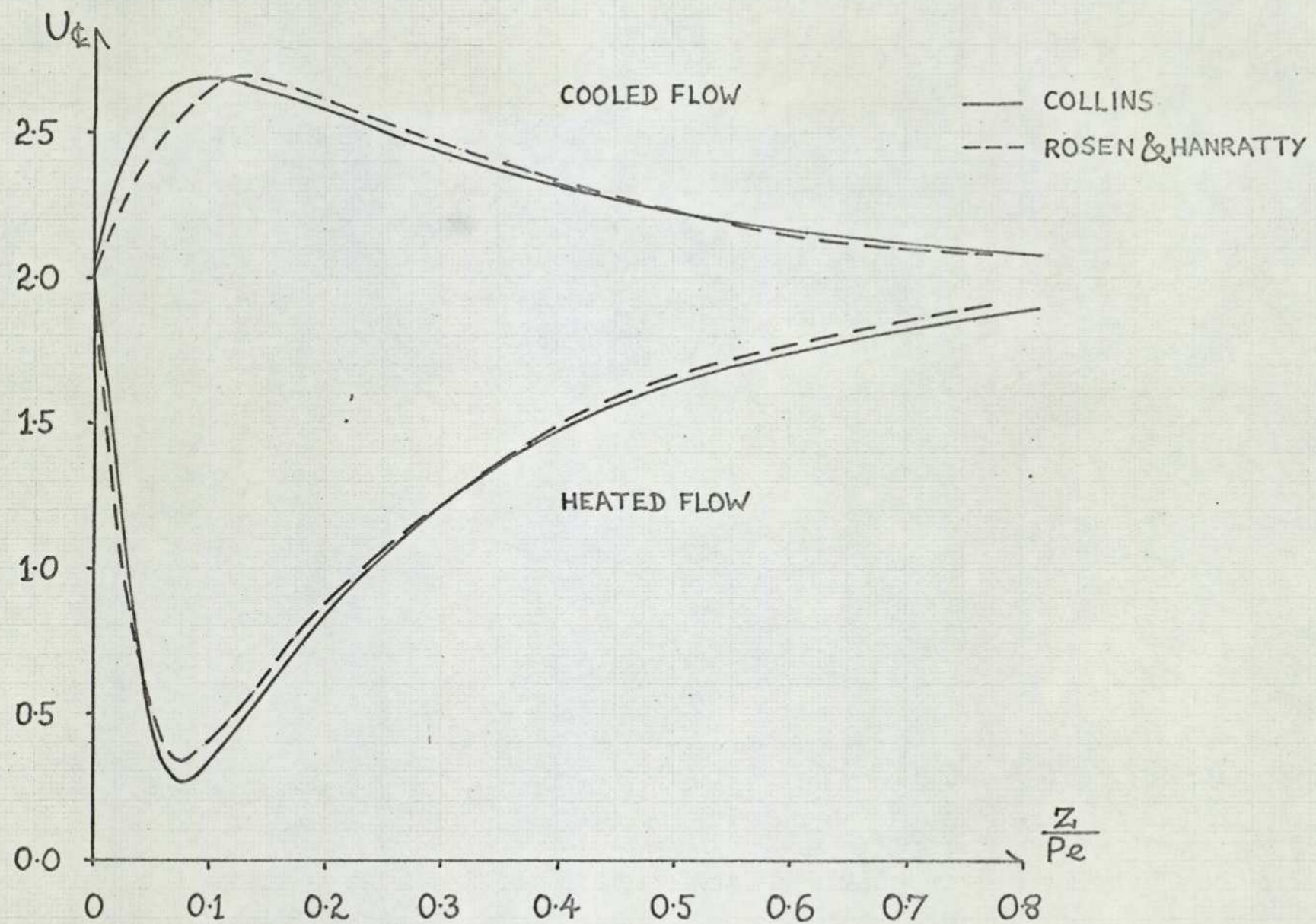


FIGURE 25 ~ VARIABLE-DENSITY VELOCITY COMPARISON (ROSEN & HANRATTY)

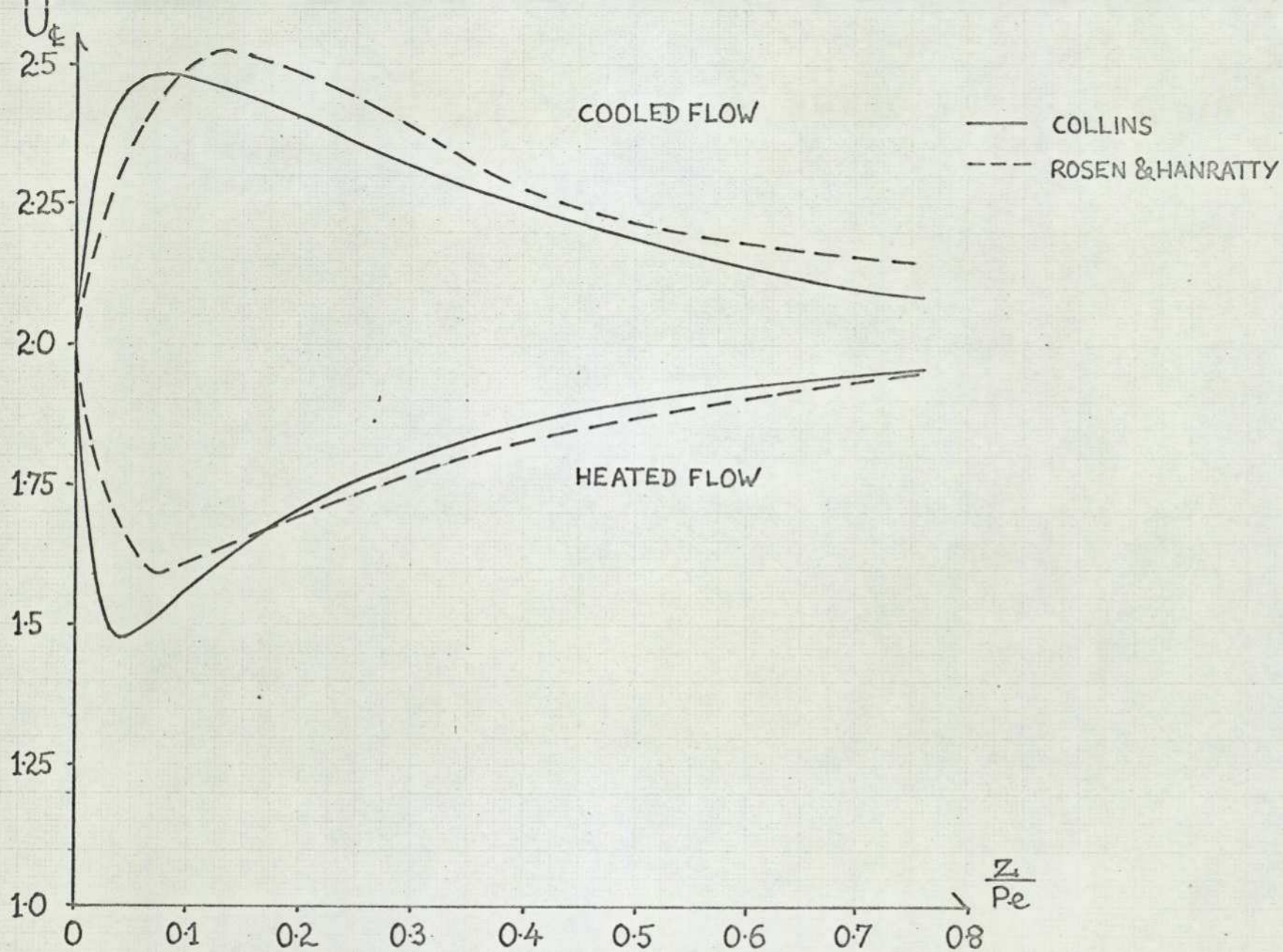


FIGURE 26 ~ VARIABLE-VISCOSITY VELOCITY COMPARISON (ROSEN & HANRATTY)

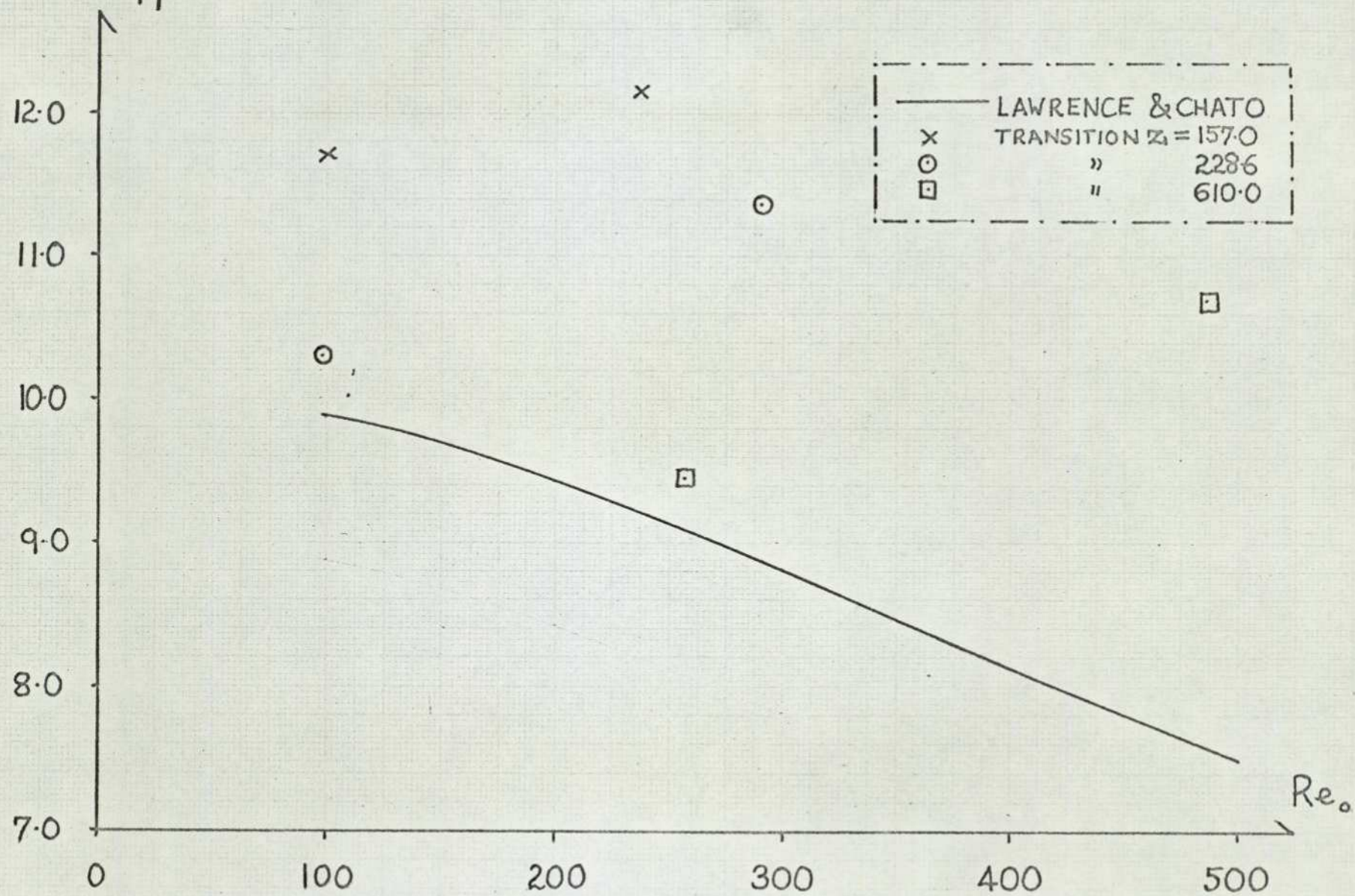


FIGURE 27 ~ LAWRENCE & CHATO TRANSITION PARAMETER STUDY

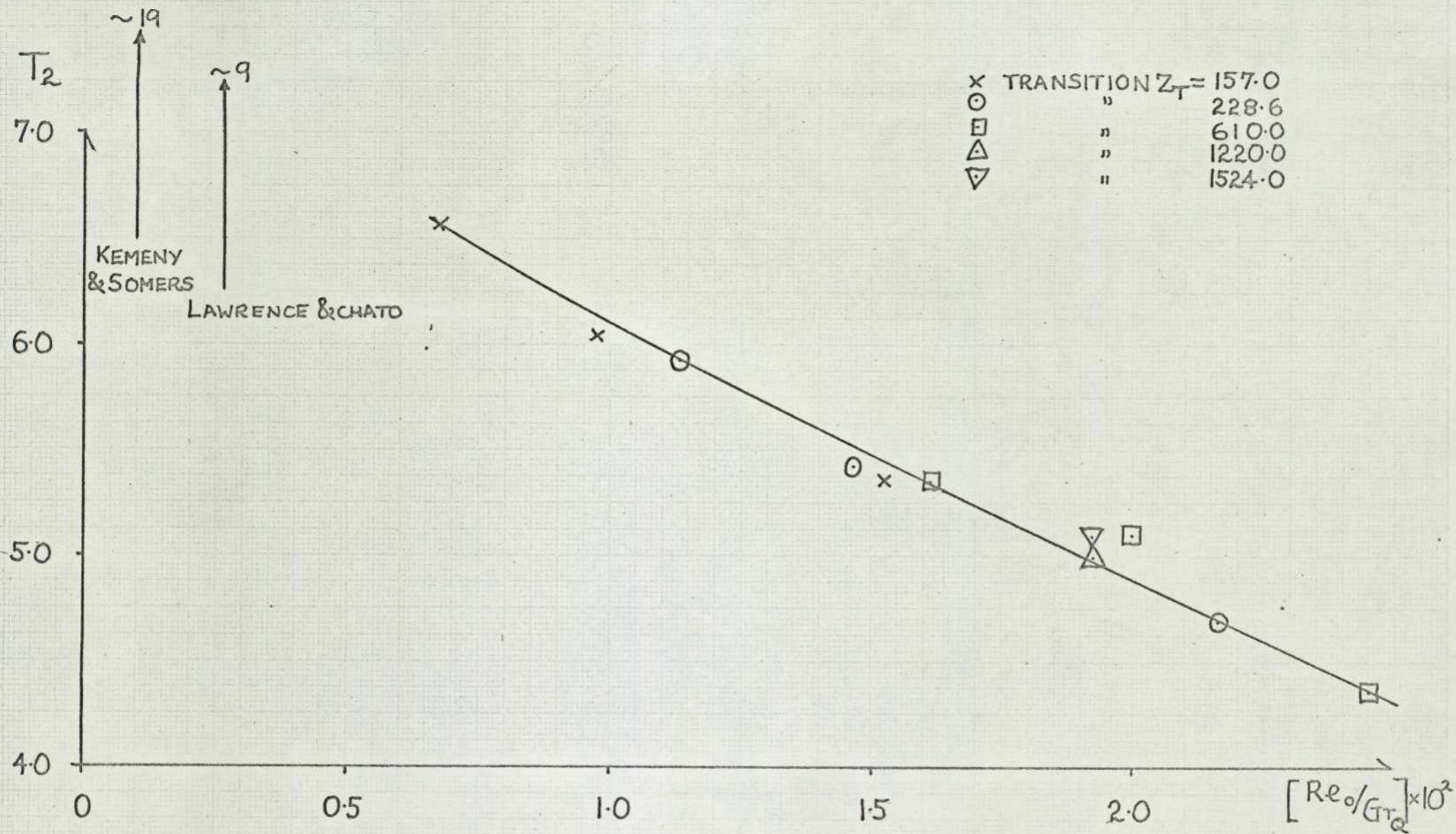


FIGURE 28 ~ MODIFIED TRANSITION PARAMETER

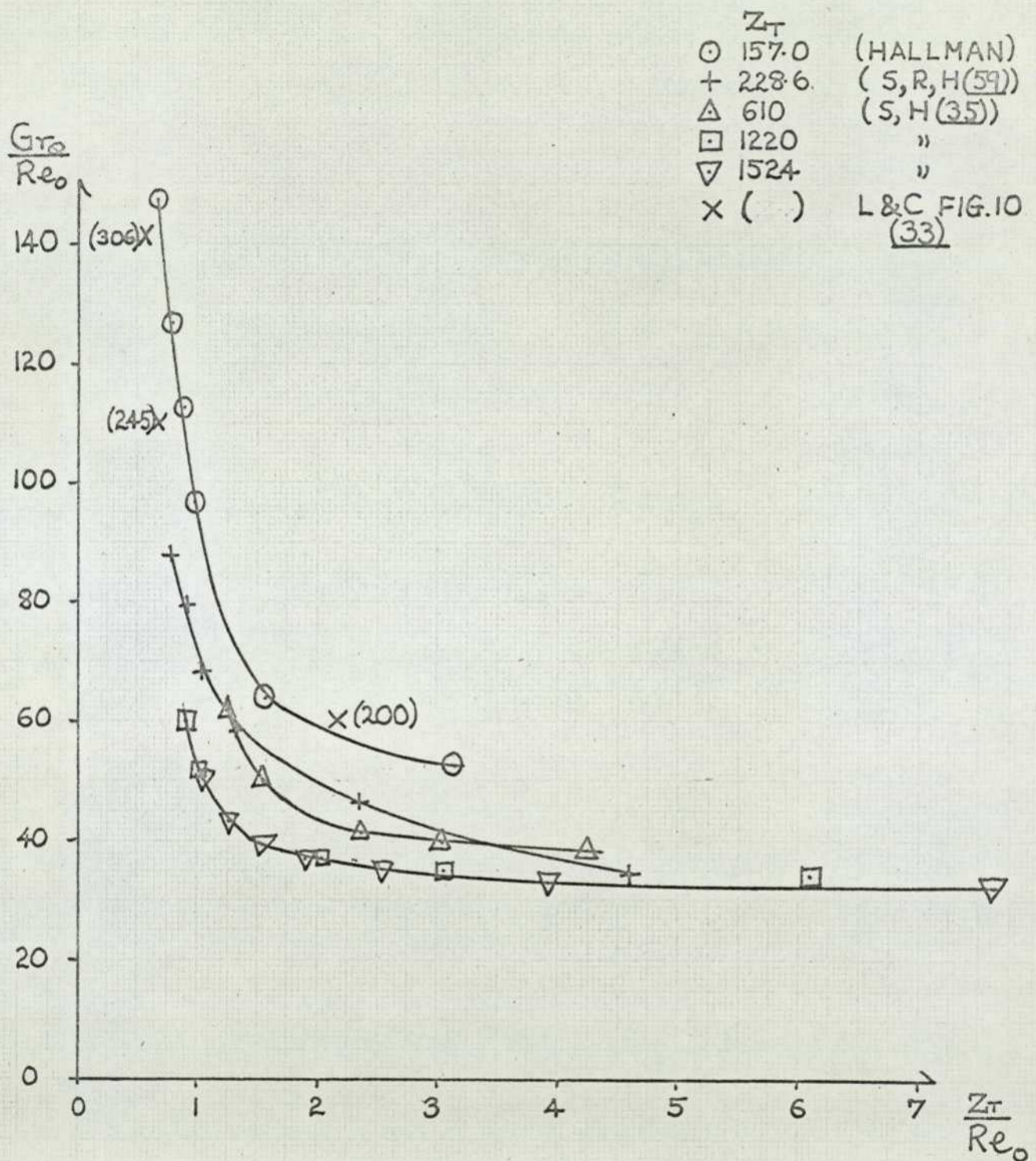


FIGURE 29 ~ TRANSITION CORRELATION USING (Gr_Q/Re_o)

- | | | | |
|-------------|-------|--------|-----------------|
| $\circ$ | Z_T | 157.0 | (HALLMAN) |
| $+$ | | 228.6 | (S,R,H(59)) |
| $\triangle$ | | 610.0 | (S,H(35)) |
| $\square$ | | 1220.0 | " |
| ∇ | | 1524.0 | " |
| $\times$ | | () | L&C FIG 10 (33) |

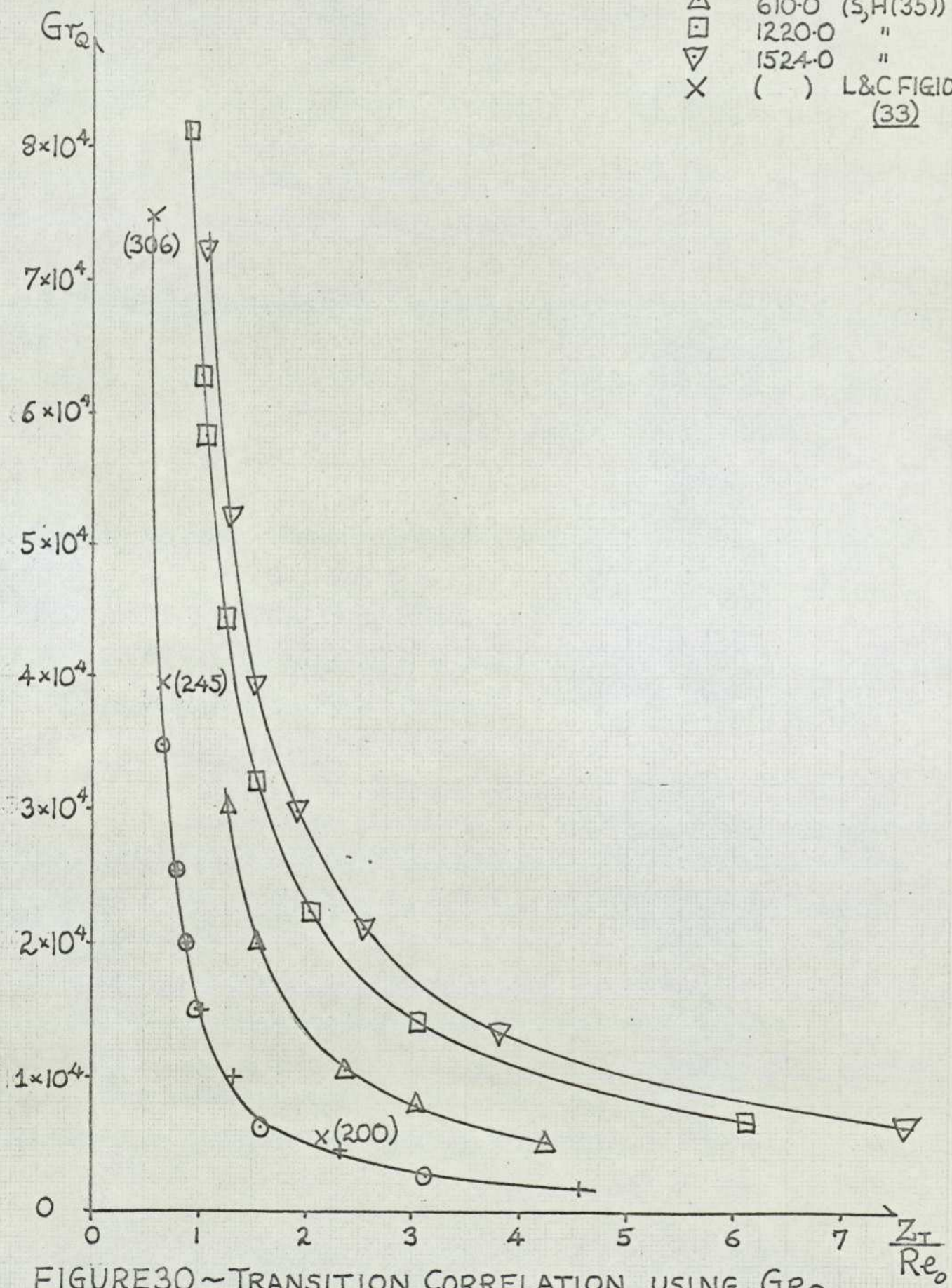


FIGURE 30 ~ TRANSITION CORRELATION USING Gr_Q

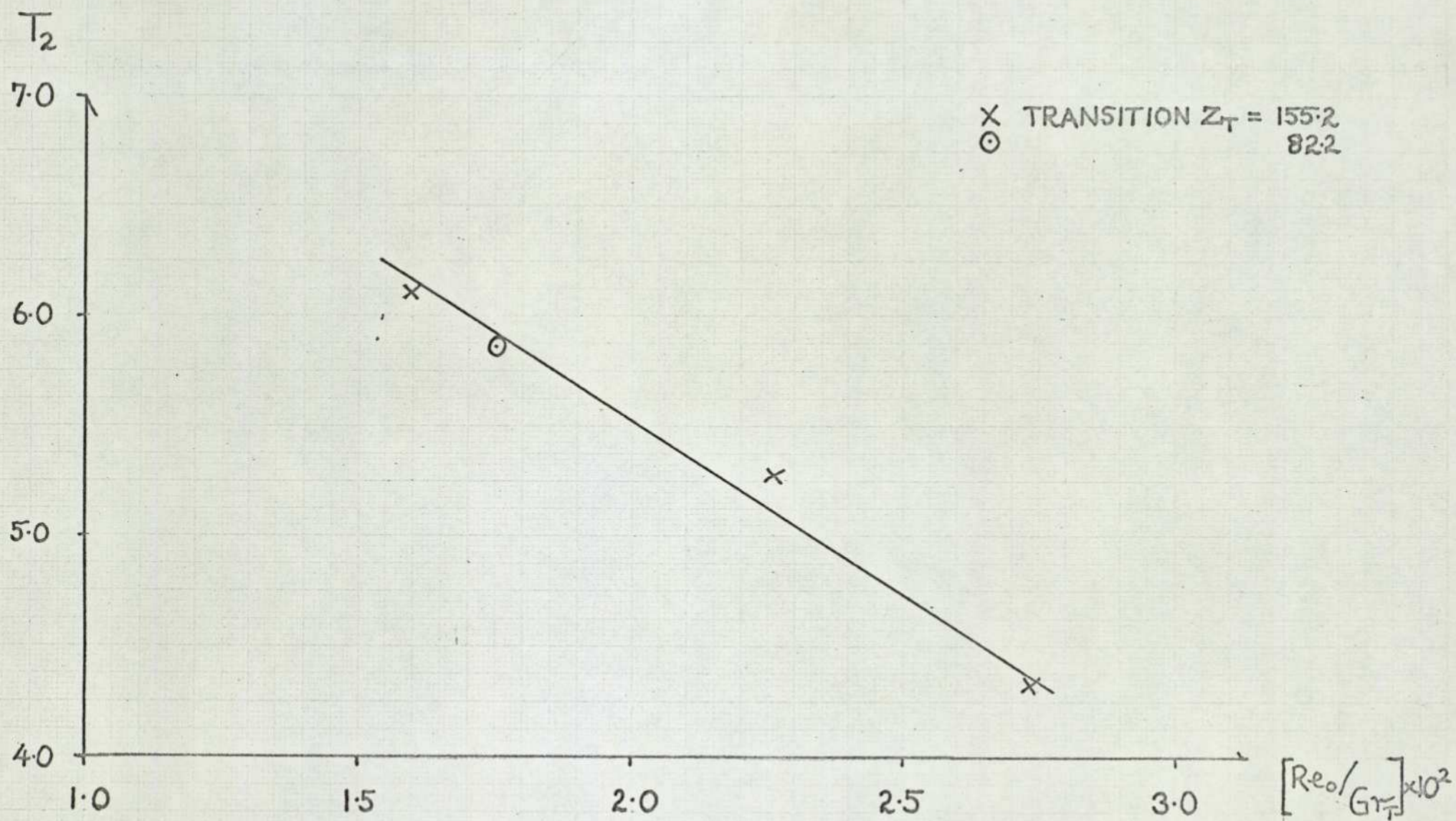


FIGURE 31 ~ MODIFIED TRANSITION PARAMETER (CONSTANT WALL TEMPERATURE)

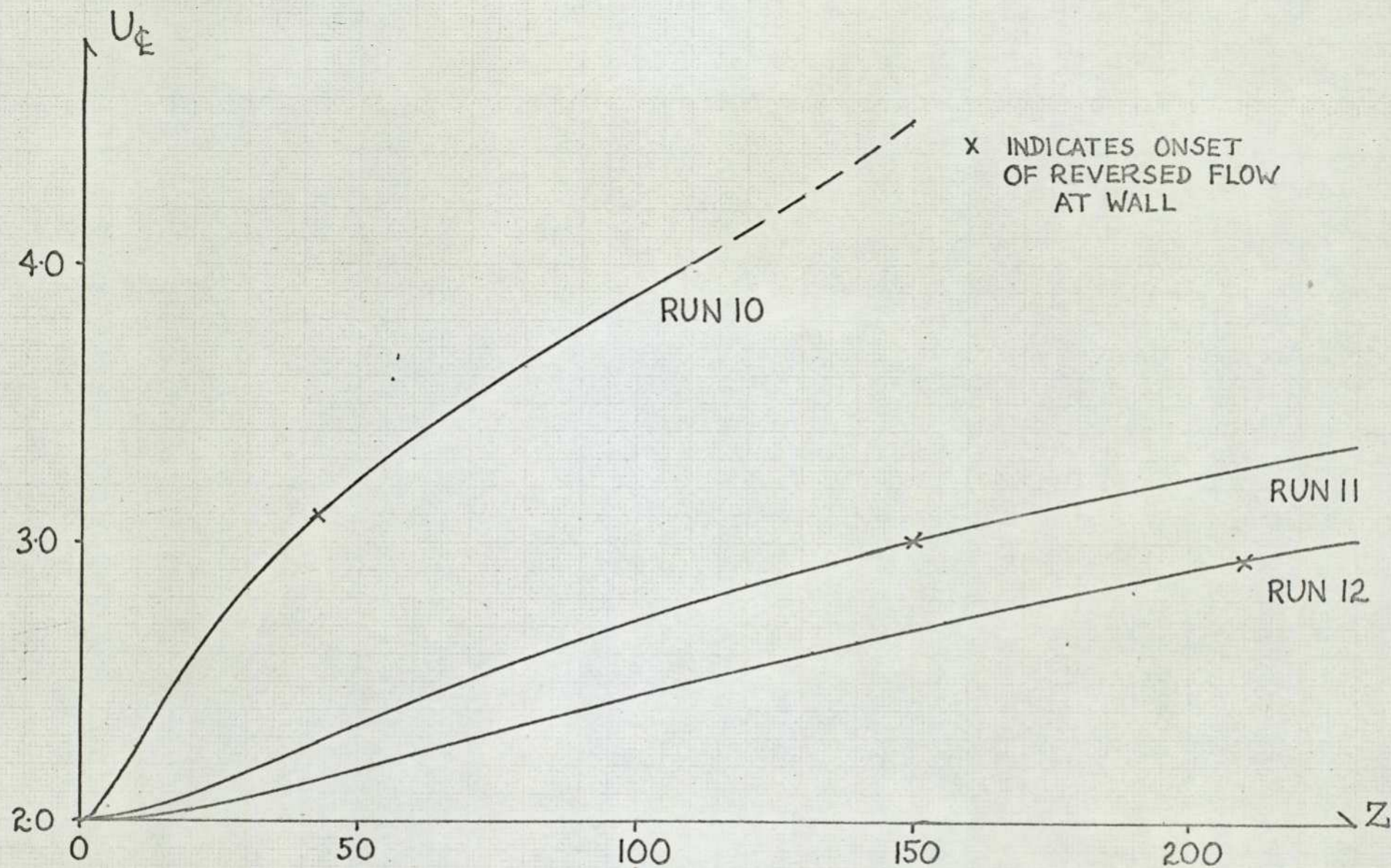


FIGURE 32 ~ U_δ PREDICTIONS FOR DOWNWARD HEATED FLOW (CONSTANT HEAT FLUX)

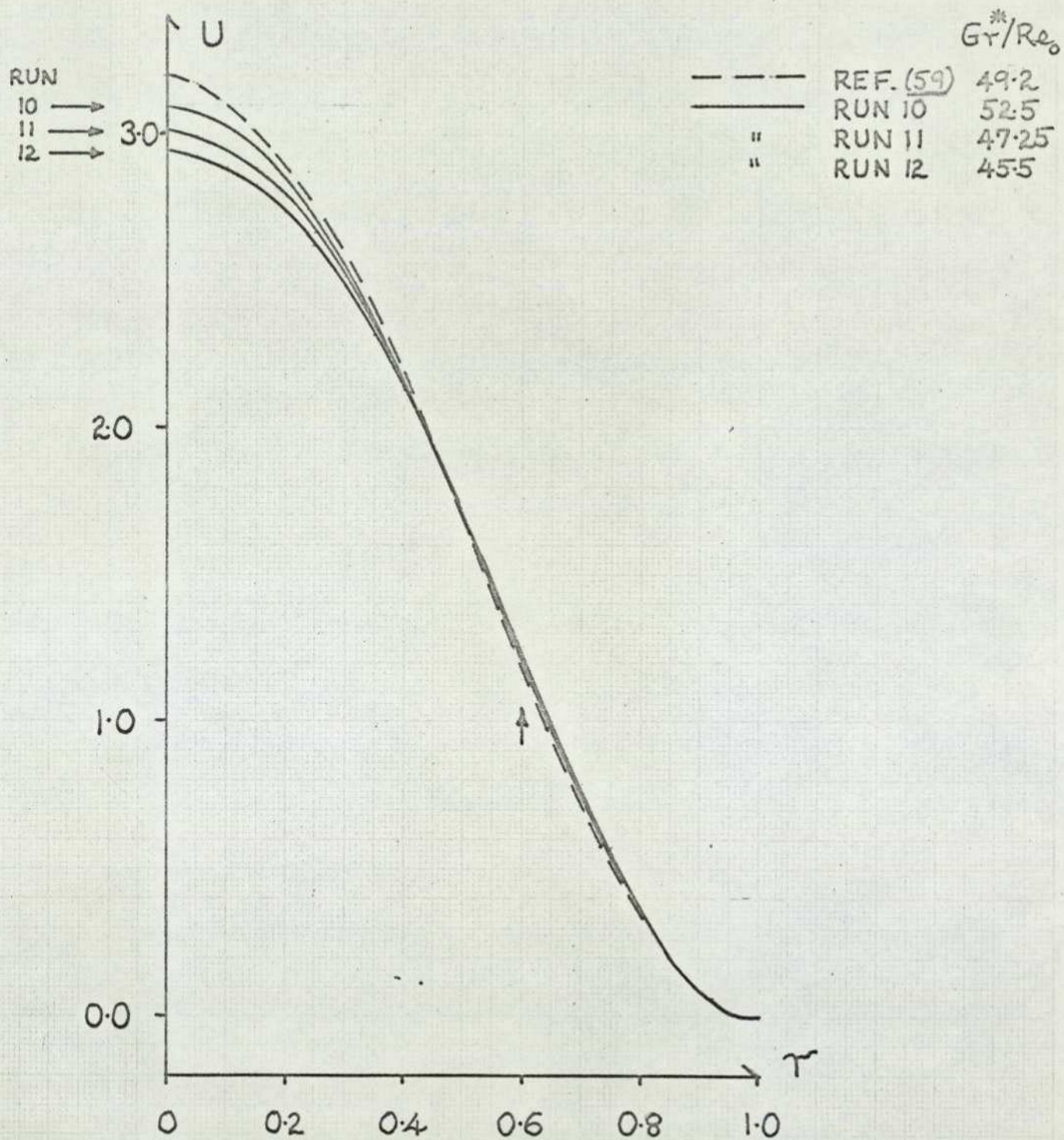


FIGURE 33 ~ VELOCITY PROFILE PREDICTIONS
(ONSET OF REVERSED FLOW)

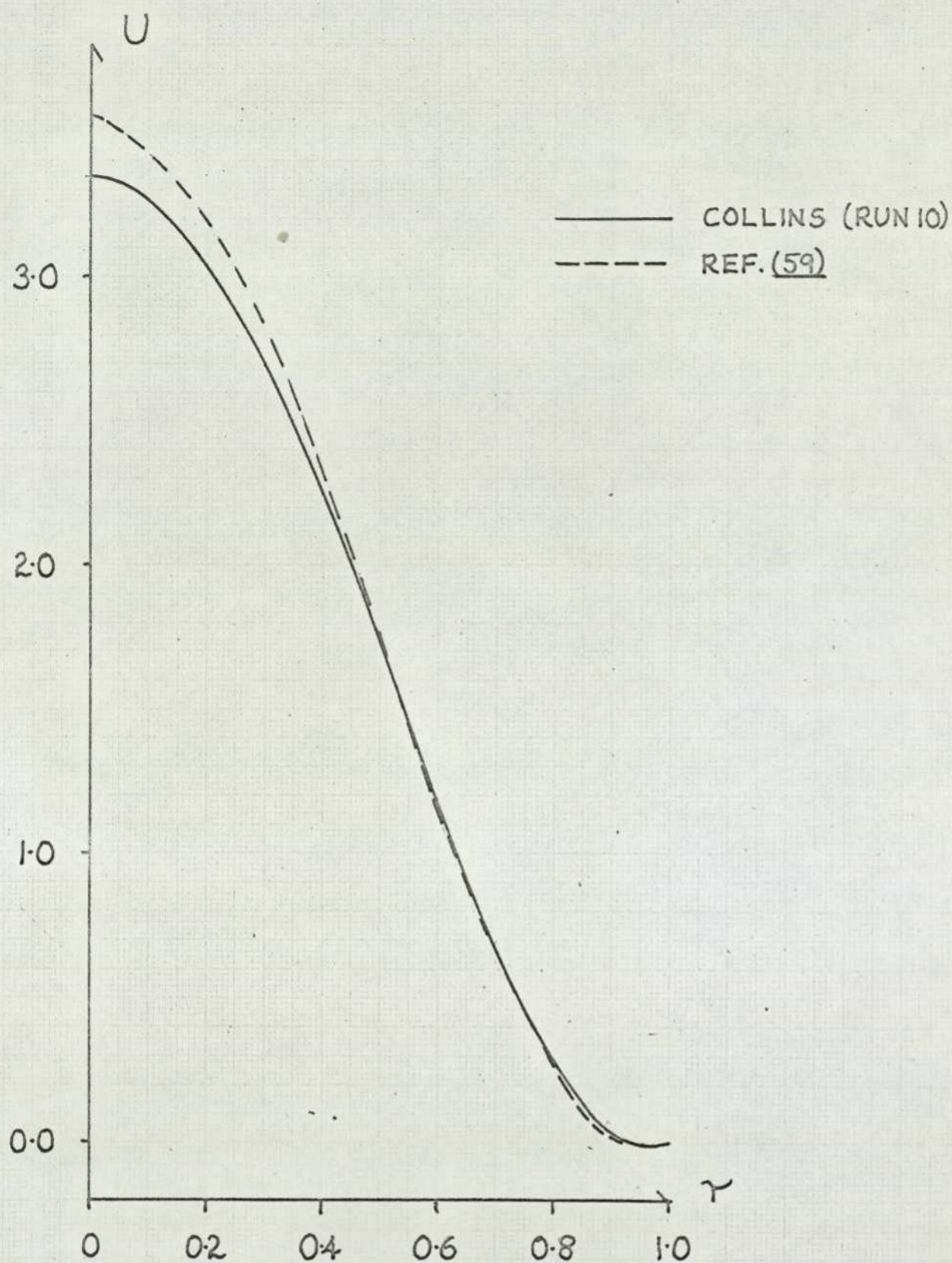


FIGURE 34 ~ VELOCITY PROFILE PREDICTIONS
($GR^*/RE_0 = 63.0$)

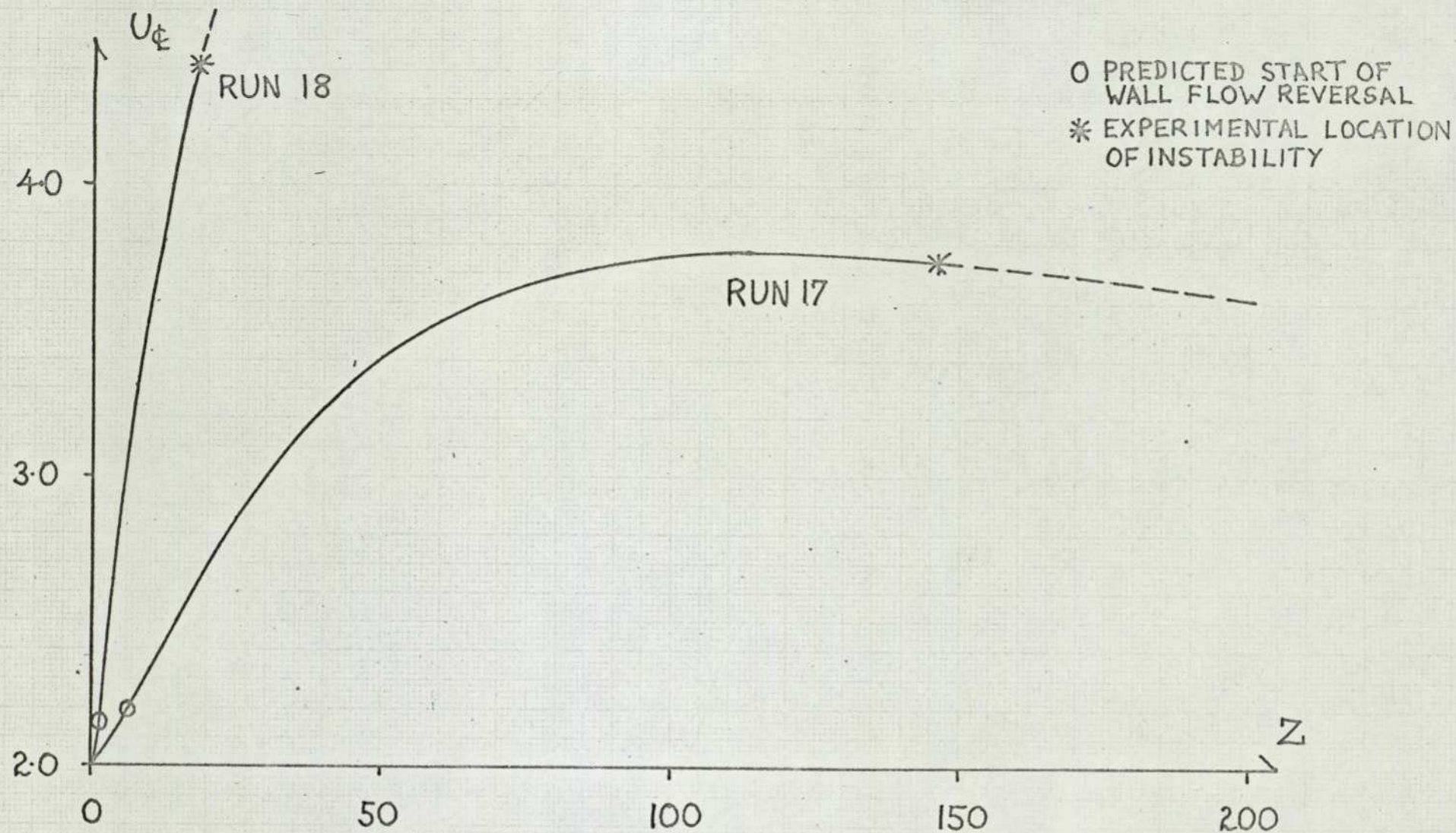


FIGURE 35 $\sim U_{\delta}$ PREDICTIONS FOR UPWARD COOLED FLOW (INSTABILITY CASES)

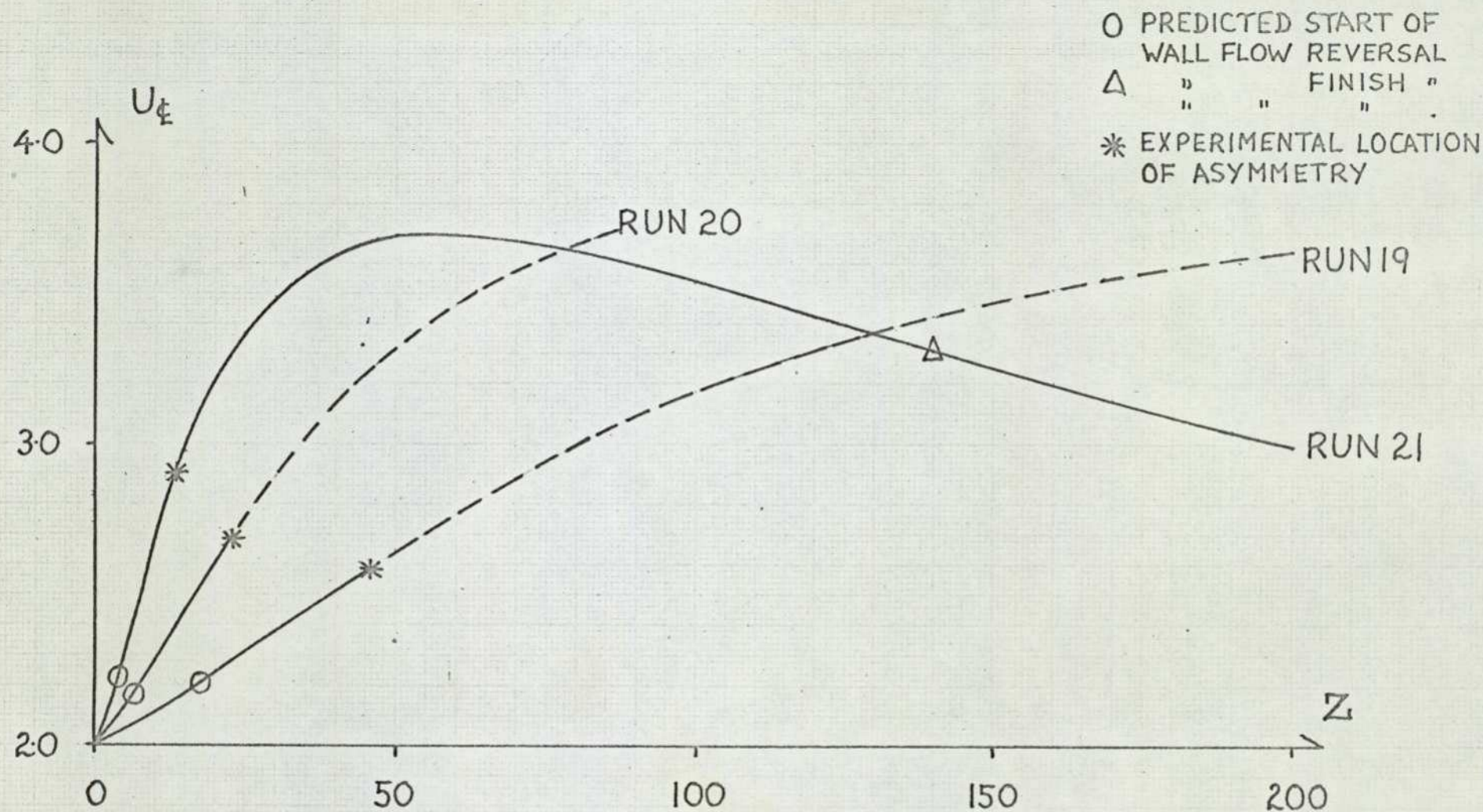


FIGURE 36 ~ U_ϕ PREDICTIONS FOR UPWARD COOLED FLOW (ASYMMETRY CASES)

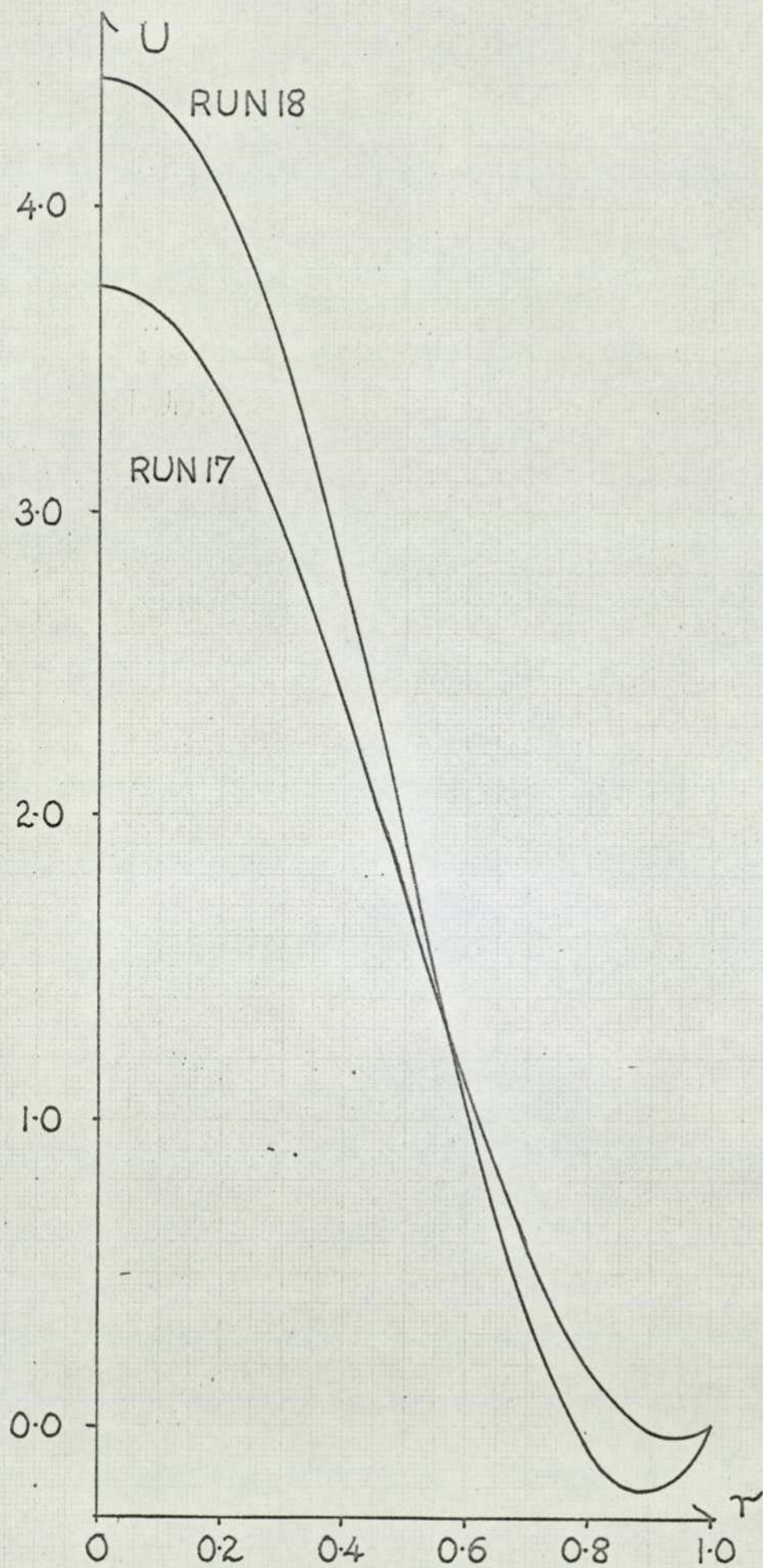


FIGURE 37 ~ VELOCITY PROFILES AT INSTABILITY
(UPWARD COOLED FLOWS)

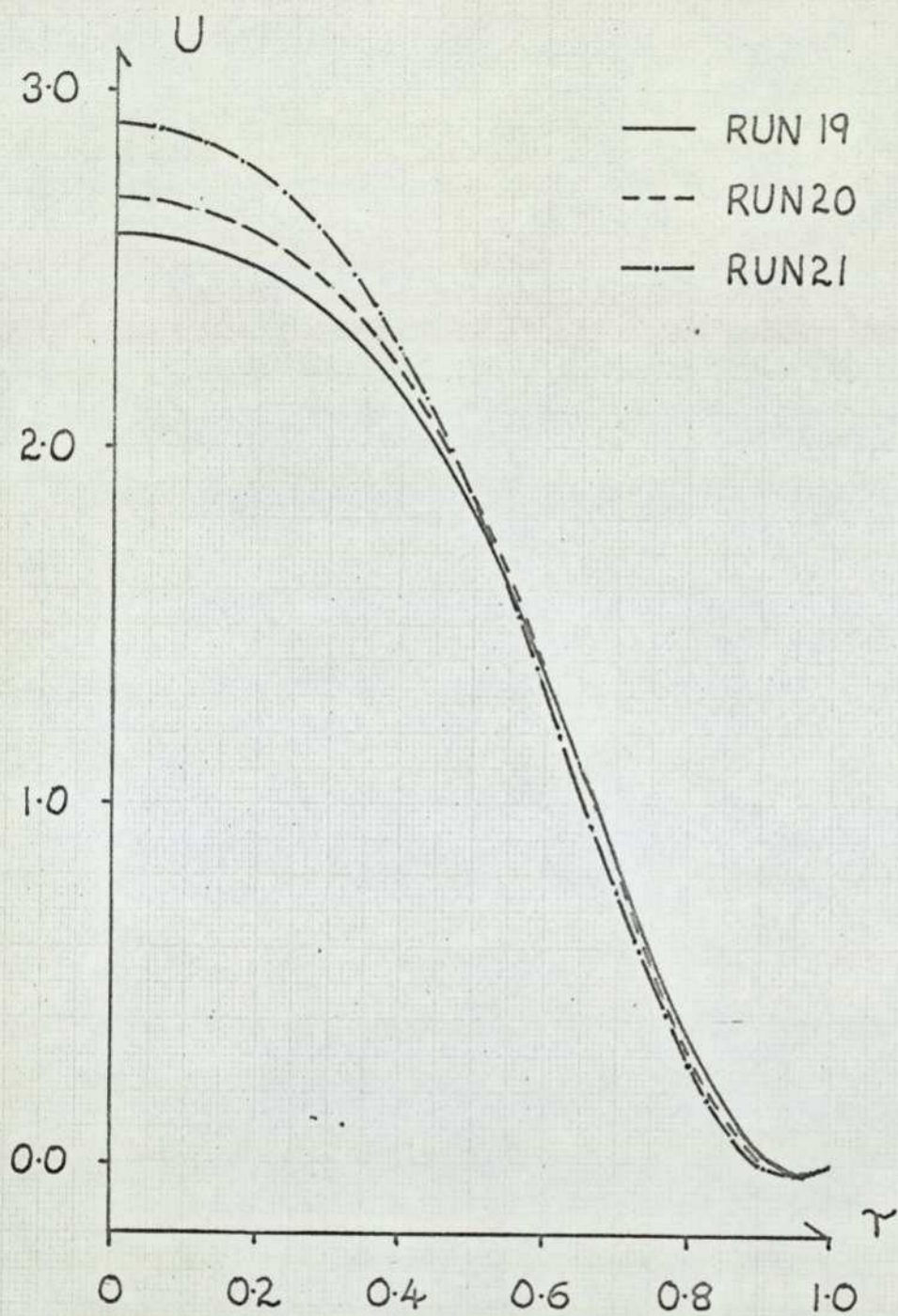


FIGURE 38 ~ VELOCITY PROFILES AT ASYMMETRY
(UPWARD COOLED FLOWS)

A P P E N D I X 1

Extract from Post Experience Course Notes,

The City University,

1974

DEPARTMENT OF MECHANICAL ENGINEERING

POST EXPERIENCE COURSE

FORTRAN AND ANALOGUE METHODS IN HEAT TRANSFER

THE BASIC EQUATIONS FOR CONVECTIVE HEAT TRANSFER
IN CYLINDRICAL CO-ORDINATES, WITH COMPUTER SOLUTIONS

USING FORTRAN

(CONTINUATION)

by

M.W. COLLINS, M.A., C.Eng., M.I.Mech.E., M.I.Nuc.E.

April, 1974

1. BASIC EQUATIONS OF FLUID FLOW

1.1. INTRODUCTION

The partial differential equations are obtained for:

- (i) Continuity (conservation of mass),
- (ii) Momentum,
- and (iii) Energy.

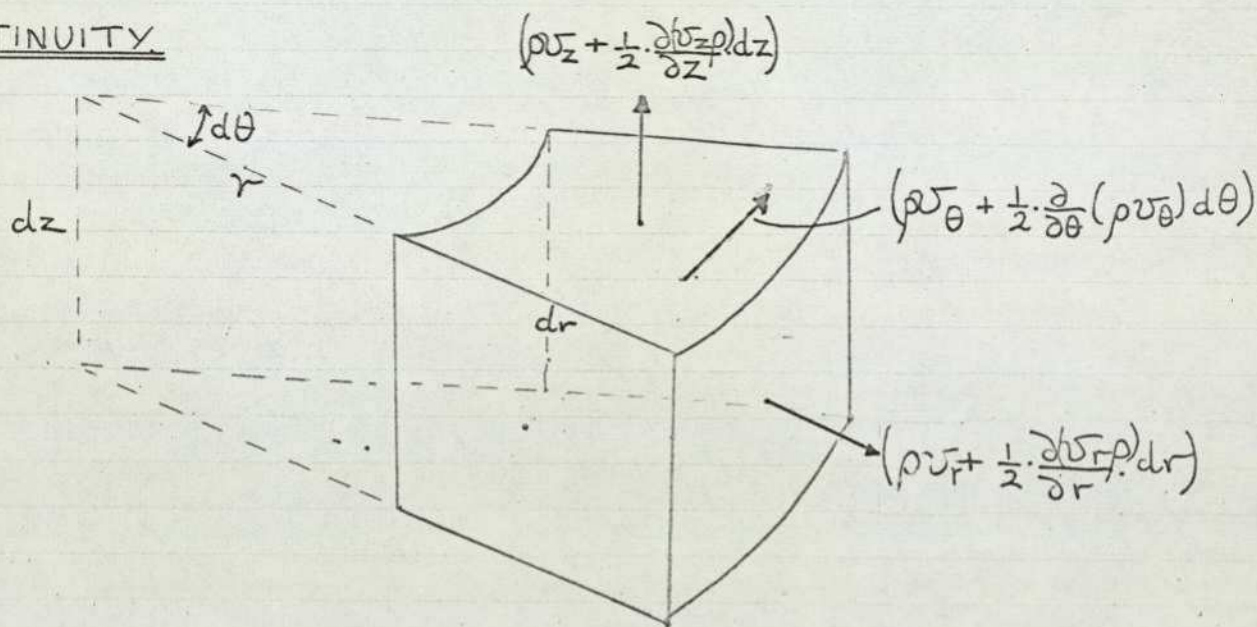
The cylindrical co-ordinate system is defined as r (radial), z (axial), θ (circumferential).

The limitations on generality are as follows:
flow is

- (i) Laminar,
- (ii) Steady (no variation with time),
- (iii) Symmetrical (no variation with θ),
- (iv) Incompressible (density constant) with certain exceptions,
- (v) Constant-property, except for viscosity.

These limitations will be discussed where appropriate.

1.2. CONTINUITY



The infinitesimal control volume illustrated is generated by the changes dr , dz and $d\theta$. (Outward velocities shown.) For this analysis only, limitations (ii), (iii) and (iv) will be lifted.

$$\begin{aligned} \text{Rate of flow in} &= \left\{ \rho v_r - \frac{1}{2} \frac{\partial}{\partial r} (\rho v_r) dr \right\} \cdot \underbrace{(r \cdot d\theta \cdot dz)}_{\text{area}} \quad r\text{-dir}^n \\ &+ \left\{ \rho v_z - \frac{1}{2} \frac{\partial}{\partial z} (\rho v_z) dz \right\} \cdot \underbrace{(r \cdot d\theta \cdot dr)}_{\text{area}} \quad z\text{-dir}^n \\ &+ \left\{ \rho v_\theta - \frac{1}{2} \frac{\partial}{\partial \theta} (\rho v_\theta) d\theta \right\} \cdot (dr \cdot dz) \quad \theta\text{-dir}^n \end{aligned}$$

$$\begin{aligned} \text{Rate of flow out} &= \left\{ \rho v_r + \frac{1}{2} \frac{\partial}{\partial r} (\rho v_r) \cdot dr \right\} \cdot (r + dr) \cdot d\theta \cdot dz \\ &+ \left\{ \rho v_z + \frac{1}{2} \frac{\partial}{\partial z} (\rho v_z) \cdot dz \right\} \cdot (r \cdot d\theta \cdot dr) \\ &+ \left\{ \rho v_\theta + \frac{1}{2} \frac{\partial}{\partial \theta} (\rho v_\theta) \cdot d\theta \right\} \cdot (dr \cdot dz) \end{aligned}$$

$$\begin{aligned} \text{Flow rate in} - \text{Flow rate out} &= \text{Rate of mass increase in control volume} \\ &= \frac{\partial}{\partial t} (\rho \cdot r \cdot d\theta \cdot dr \cdot dz) \end{aligned}$$

Substituting in this equation, dividing by $(r \cdot dr \cdot d\theta \cdot dz)$, and ignoring 2nd. order term formed by $\overset{*}{\curvearrowright}$, gives:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) + \frac{1}{r} (\rho v_r) + \underbrace{\frac{\partial}{\partial r} (\rho v_r)}_{\frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r)} = 0$$

$$\text{i.e. } \underline{\underline{\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) + \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) = 0.}}$$

Limitations

(ii) $\frac{\partial \rho}{\partial t}$ is zero for steady flow (and is also zero for incompressible flow)

(iii) $\frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta)$ is zero for axial symmetry.

(iv) for ρ constant equation becomes:

$$\frac{\partial}{\partial z} (\rho v_z) + \frac{1}{r} \frac{\partial}{\partial r} (r \rho v_r) = 0$$

$$\text{i.e. } \rho \frac{\partial v_z}{\partial z} + \rho \frac{v_r}{r} + \rho \frac{\partial v_r}{\partial r} = 0$$

Thus the continuity equation for these conditions is:

$$\underline{\underline{\frac{\partial v_r}{\partial r'} + \frac{v_r}{r'} + \frac{\partial v_z}{\partial z'} = 0 \quad \text{--- (1)}}}$$

The ' attached to r , z , and later T , is used in the numbered equations to distinguish them from non-dimensional quantities to be defined later.

1.3

MOMENTUM (NAVIER - STOKES EQUATIONS)

Newton's Second Law gives:

for a fixed mass,

Rate of change of linear momentum =
Nett external forces acting on mass

AXIAL DIRECTION

Above statement gives:

Rate of change of z direction momentum =
Axial component of nett external forces.

L.H.S.

The rate of change 'follows' the particle of fluid, the infinitesimal mass, because the vector velocity $\vec{V}$ (c.f. p.94, Ref. 3) is in general a function not only of time, but also of position; i.e. for cylindrical co-ordinates

$$\vec{V} = \vec{V}(r, z, \theta, t)$$

and for steady symmetrical flows, this becomes

$$\vec{V} = \vec{V}(r, z).$$

Rate of change of momentum is defined as:

$$\frac{D}{Dt} (dm \vec{V}) = dm \frac{D\vec{V}}{Dt} = dm \left[\frac{\partial}{\partial t} + v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r} \frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z} \right] \vec{V}$$

where $\frac{D}{Dt}$ (given in full above, for cylindrical co-ordinates) is

termed the 'material', 'substantial', 'particle', or 'Eulerian' derivative. $\frac{\partial}{\partial t}$ (the 'local' derivative) is change of a quantity with respect to time, at a fixed position. The 3 other terms ('convective' derivatives) are the change due to movement of the mass.

Now $dm = \rho \cdot r \cdot dr \cdot d\theta \cdot dz$,

Axial component of $\vec{V} = v_z$,

Hence for steady, symmetrical flow,

$$\underline{\underline{l. h. s. = \rho \cdot r \cdot dr \cdot d\theta \cdot dz \left(v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right)}}$$

T. h. s.

Forces which may act are:

(1) Body, viz. gravitational.

(2) Surface (a) direct stresses,

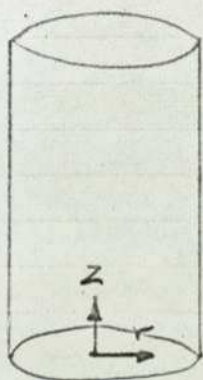
(b) shear stresses.

(1) Body Force - Gravitational

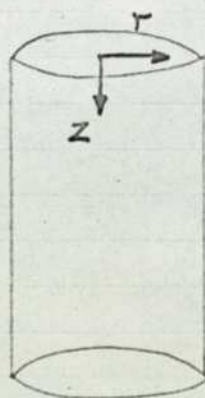
Gravitational force on element of volume $r \cdot dr \cdot d\theta \cdot dz$

$$= \pm \rho g \cdot r \cdot dr \cdot d\theta \cdot dz$$

where $\pm$ are for { downward / upward } flows respectively.



Upward Flow



Downward Flow

Co-ordinate Systems

(2) Surface Forces

Analysis is rather long, but is split into stages:

(1) A fluid, unlike a solid, is unable to resist strain and deforms continuously. We find the various 'rate of strain' components on the infinitesimal element.

(2) The strain-rates give rise to corresponding stresses. For Newtonian fluids a linear relationship is assumed, rather like Hooke's law for solids.

(3) The surface forces arise from these stresses.

(1) Rate of Strain Components.

(i) Linear distortion. In the three co-ordinate directions these strain-rates are defined as e_{rr} , $e_{\theta\theta}$, e_{zz} .

(ii) Angular distortion. Change of shape in a rotation sense. These strain-rates are $e_{r\theta}$, e_{rz} , $e_{\theta r}$, $e_{\theta z}$, e_{zr} , $e_{z\theta}$.

(iii) Volumetric distortion. Overall change in size of element. The dilatation (c.f. volumetric strain in solids) is defined as $e_{rr} + e_{\theta\theta} + e_{zz}$.

(i) Linear

A line segment in the radial direction will change from length $[v_r \cdot dt]$ to length $[(v_r + \frac{\partial v_r}{\partial r} \cdot dr) \cdot dt]$ over the length $[dr]$.

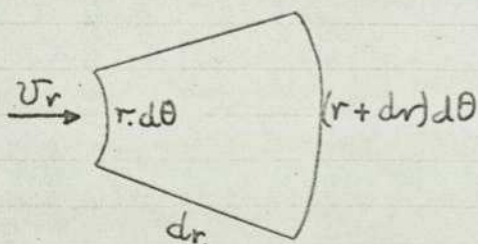
$$\text{The strain-rate } e_{rr} = \frac{1}{dr} \cdot \frac{[(v_r + \frac{\partial v_r}{\partial r} \cdot dr) \cdot dt] - [v_r \cdot dt]}{dt}$$

$$\text{i.e. } e_{rr} = \frac{\partial v_r}{\partial r}$$

$$\text{similarly } e_{zz} = \frac{\partial v_z}{\partial z}$$

$$\text{circumferential strain-rate is } \frac{[(v_\theta + \frac{\partial v_\theta}{\partial \theta} \cdot d\theta) \cdot dt] - [v_\theta \cdot dt]}{(r \cdot d\theta) \cdot (dt)}$$

$$= \frac{1}{r} \cdot \frac{\partial v_\theta}{\partial \theta}$$

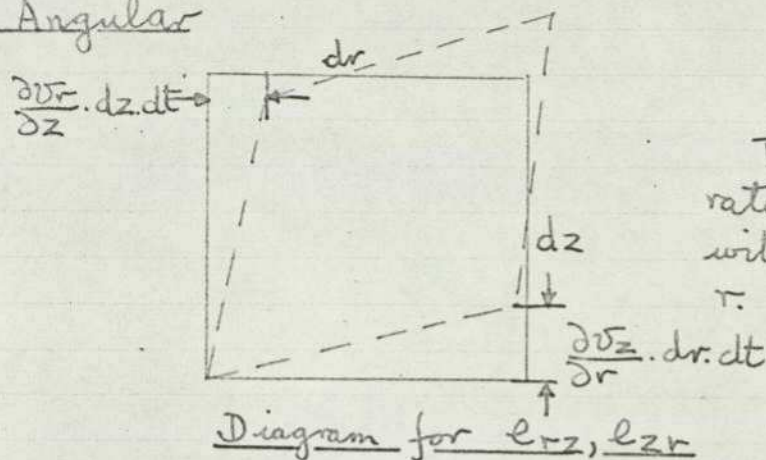


there is a circumferential strain-rate also because of the radial velocity v_r alone.

$$\text{it will be } \frac{[(r + dr) \cdot d\theta] - [r \cdot d\theta]}{(r \cdot d\theta) \cdot dt} = \frac{dr}{r \cdot dt} \rightarrow \frac{v_r}{r}$$

$$\text{i.e. } e_{\theta\theta} = \frac{1}{r} \cdot \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}$$

(ii) Angular

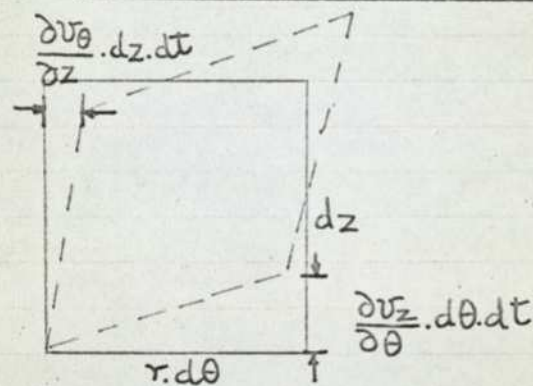


This combines strain-rates due to v_r varying with z , and v_z with r .

$$\begin{aligned} \text{Angular deformation (r-z plane)} &= \frac{\partial v_r}{\partial z} \cdot \frac{dz \cdot dt}{(dz)(dt)} + \frac{\partial v_z}{\partial r} \cdot \frac{dr \cdot dt}{(dr)(dt)} \\ &= \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \end{aligned}$$

But deformation is $e_{rz} + e_{zr}$ in the r-z plane (see Strain Tensor) and so we have for a consistent definition*

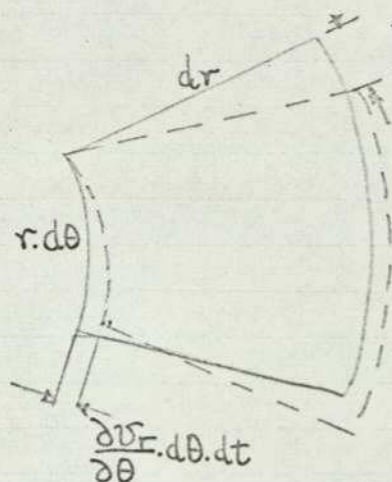
$$e_{rz} = e_{zr} = \frac{1}{2} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right)$$



$$\begin{aligned} \text{Deformation } (\theta-z) \text{ plane} &= \frac{\partial v_z}{\partial \theta} \cdot \frac{d\theta \cdot dt}{(r \cdot d\theta)(dt)} + \frac{\partial v_\theta}{\partial z} \cdot \frac{dz \cdot dt}{(dz) \cdot dt} \\ &= \frac{1}{r} \cdot \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \end{aligned}$$

$$\text{Hence } e_{z\theta} = e_{\theta z} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right)$$

* This is a point of some nicety. It is a property of the flow-field itself. Li and Lam (Ref. 2) having demonstrated that the velocity gradient field is a tensor (p 203), evaluate this tensor as the sum of an anti-symmetric tensor (representing rotation of the element), and a symmetric one (representing strain-rate). The latter requires $e_{rz} = e_{zr}$. Also see Shapiro (Ref. 5, p 2.8)



This analogous to $\frac{v_r}{r}$
component in $e_{\theta\theta}$.
A θ ve v_θ tends to reduce distortion

This gives
$$e_{r\theta} = e_{\theta r} = \frac{1}{2} \left[\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right]$$

(iii) Volumetric

Strain rate = $e_{rr} + e_{\theta\theta} + e_{zz}$
or
$$e_v = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$
 (This will be discussed later)

(iv) Strain-rate Tensor

The strain-rate may be written in tensor notation

$$e_{ij} = \begin{bmatrix} e_{rr} & e_{r\theta} & e_{rz} \\ e_{\theta r} & e_{\theta\theta} & e_{\theta z} \\ e_{zr} & e_{z\theta} & e_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) & \frac{1}{2} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \\ \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) & \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_\theta}{\partial z} + \frac{\partial v_z}{\partial \theta} \right) \\ \frac{1}{2} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) & \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_\theta}{\partial z} + \frac{\partial v_z}{\partial \theta} \right) & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

(v) Strain-rates for stated limitations

For circumferential velocities and changes absent, the strain-rates become:

$$e_{rr} = \frac{\partial v_r}{\partial r}$$

$$e_{\theta\theta} = \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} = \frac{v_r}{r}$$

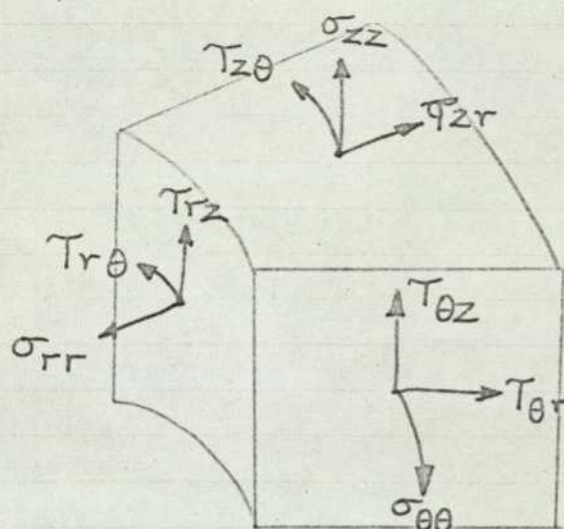
$$e_{zz} = \frac{\partial v_z}{\partial z}$$

$$e_{r\theta} = e_{\theta r} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) = 0$$

$$e_{rz} = e_{zr} = \frac{1}{2} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right)$$

$$e_{\theta z} = e_{z\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right) = 0$$

(2) Stress Components



There is a stress field and tensor, where σ represents a normal stress, and τ a shear stress, viz:

$$\text{Stress}_{ij} = \begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_{zz} \end{bmatrix}$$

This field is shown above, with a corresponding stress distribution on other faces. By taking moments about the mass centre, it may be shown that $\tau_{rz} = \tau_{zr}$ etc., i.e. like the rate-of-strain tensor, the stress tensor is symmetric.

We now have to relate 'stress' to 'rate-of-strain'.

Three assumptions form the basis of this relationship.

- (i) The stress components must reduce to the hydrostatic pressure p , if all velocity gradients, i.e. if all strain-rates are zero. Hence:

$$\text{hydrostatic} \begin{cases} \sigma_{ii} = -p \\ \tau_{ij} = 0 \end{cases}, \text{ since pressure is both a}$$

normal stress, and in the opposite sense (compressive) to σ .

(ii) The stress/strain-rate relationship must be independent of the co-ordinate system.

(iii) The stresses are linearly related to the strain-rate components, ("the simplest hypothesis one can frame is that a stress component is linearly related to the components of the rate of strain." Ref. 2, p 214), this relationship being independent of direction (i.e. the fluid is newtonian and isotropic, respectively, Ref. 3, p 149). Also note Shapiro's (Ref. 5, p. 2.13) summary of the assumptions.

With the above definitions, the following may be demonstrated (see Ref. 4, pp 97-100, for 2-dimensional Cartesian treatment):

$$\sigma_{rr} = -p + 2K_1 e_{rr} + K_2 e_v$$

$$\sigma_{\theta\theta} = -p + 2K_1 e_{\theta\theta} + K_2 e_v$$

$$\sigma_{zz} = -p + 2K_1 e_{zz} + K_2 e_v$$

$$\tau_{ij} = \tau_{ji} = K_1 e_{ij}$$

where K_1, K_2 are constants, and e_v is the volumetric strain-rate

The assumption involves putting $K_1 = \mu$ (dynamic viscosity) to obtain the momentum equations.

Second Coefficient of Viscosity.

(i) Constant K_2 , above, is usually replaced by $-\frac{2}{3}(\mu - K_3)$

where K_3 is the 'second coefficient of viscosity'. The $\frac{2}{3}\mu$ arises (c.f. Ref. 4, p 100) from equating mean of normal stresses for an incompressible fluid to the negative hydrostatic pressure.

(ii) If fluid is compressible K_3 (written, for example, as μ') is zero for monatomic gases (kinetic theory), and in general small. If K_3 is negligible, $K_2 = -\frac{2\mu}{3}$

(iii) If fluid is incompressible, e_v is zero (compare e_v , p C/1/8, where 3rd. term is zero, with equation (1) p C/1/3). Hence K_2 term is also zero.

The stress-tensor p C/1/9 is related to the strain-rate-tensor by:
 $(\text{Stress})_{ij} = 2\mu (\text{Strain-rate})_{ij}$, together with the pressure terms.

Taking the strain-rates for our limitations from p C/1/8 (bottom), we have for the stress-tensor:

$$\begin{bmatrix} -p + 2\mu \frac{\partial v_r}{\partial r} & 0 & \mu \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \\ 0 & -p + \frac{2\mu v_r}{r} & 0 \\ \mu \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) & 0 & -p + 2\mu \frac{\partial v_z}{\partial z} \end{bmatrix}$$

(3) Forces

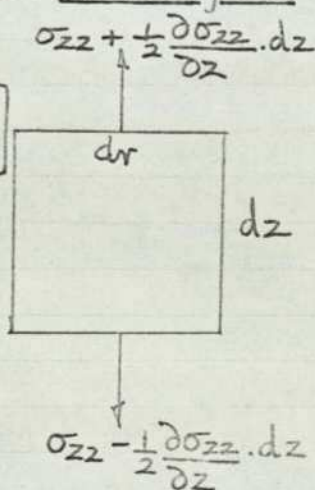
The stresses resulting in an axial force are:

(i) σ_{zz} .

$$\text{Nett force} = \left[\sigma_{zz} + \frac{1}{2} \frac{\partial \sigma_{zz}}{\partial z} dz \right] [r \cdot dr \cdot d\theta]$$

$$- \left[\sigma_{zz} + \frac{1}{2} \frac{\partial \sigma_{zz}}{\partial z} dz \right] [r \cdot dr \cdot d\theta]$$

$$= \left[\frac{\partial}{\partial z} (\sigma_{zz}) \right] r \cdot dr \cdot dz \cdot d\theta \quad \text{normal area}$$

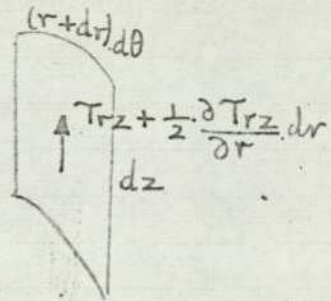
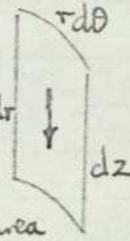


(ii) T_{rz} .

$$\text{Nett force} = \frac{\partial}{\partial r}(T_{rz}) \cdot r \cdot dr \cdot dz \cdot d\theta$$

$$+ \left(\frac{T_{rz}}{r} \right) (r \cdot dr \cdot dz \cdot d\theta)$$

because of increased outer area



Viscous Force is, therefore, due to σ_{zz} and T_{rz} in axial

direction, and equals:

$$(r \cdot dr \cdot d\theta \cdot dz) \left[\frac{\partial}{\partial z} \left(-p + 2\mu \frac{\partial v_z}{\partial z} \right) + \frac{1}{r} \frac{\partial}{\partial r} \left(r \mu \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \right) \right]$$

$$= (r \cdot dr \cdot d\theta \cdot dz) \left\{ \begin{aligned} & - \frac{\partial p}{\partial z} && \text{pressure term} \\ & + \mu \left(\frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r} \cdot \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial r^2} \right. && \text{constant viscosity term} \\ & \quad \left. + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r} \cdot \frac{\partial v_r}{\partial z} + \frac{\partial^2 v_r}{\partial r \partial z} \right)^* && \text{zero term (incompressible)} \\ & + \frac{\partial \mu}{\partial z} \left(2 \frac{\partial v_z}{\partial z} \right) + \frac{\partial \mu}{\partial r} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) && \text{variable viscosity term} \end{aligned} \right\}$$

Hence (also see pC/1/5) the axial momentum equation becomes, if divided by volume $r \cdot dr \cdot d\theta \cdot dz$:

$$\rho \left(v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) = \pm \rho g - \frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r} \cdot \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial r^2} \right) + \frac{\partial \mu}{\partial z} \left(2 \frac{\partial v_z}{\partial z} \right) + \frac{\partial \mu}{\partial r} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \quad (2)$$

* This term becomes $\mu \frac{\partial}{\partial z} \left(\frac{\partial v_z}{\partial z} + \frac{v_r}{r} + \frac{\partial v_r}{\partial r} \right) \rightarrow 0$

c.f. continuity equation C/1/3

RADIAL DIRECTION

Rate of change of r direction momentum
= Radial component of nett external forces.

l.h.s.

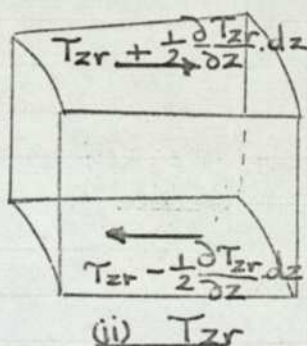
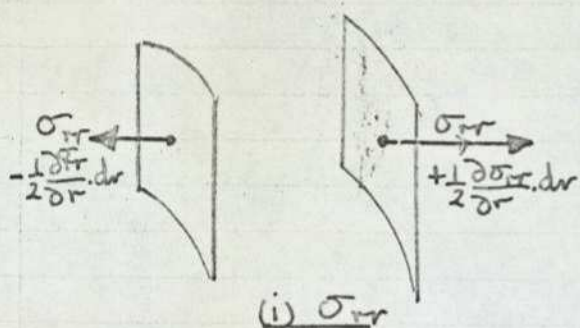
$$= \rho (r \cdot dr \cdot d\theta \cdot dz) \left(v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right) \quad \left[\text{as for axial equation} \right]$$

r.h.s.

No body force.

Surface Forces

These correspond to (3) Forces on p C/1/11 for Axial Equation

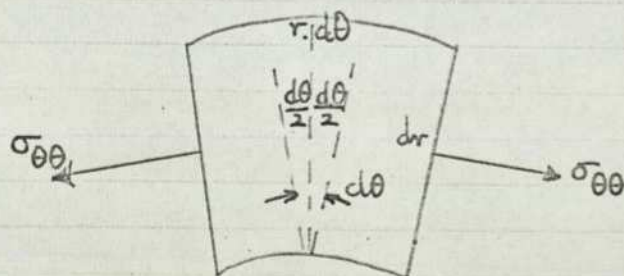


$$\text{Force} = (r \cdot dr \cdot d\theta \cdot dz) \left(\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \sigma_{rr} \right)$$

increased outer area

$$\text{Force} = (r \cdot dr \cdot d\theta \cdot dz) \left(\frac{\partial}{\partial z} T_{zr} \right)$$

(iii) $\sigma_{\theta\theta}$. There is also a radial component of the circumferential stress, as given below.



In the positive radial direction, stress component is

$$- 2\sigma_{\theta\theta} \sin \frac{\alpha}{2}$$

$$\text{i.e. force} = - 2\sigma_{\theta\theta} (dr \cdot dz) \cdot \frac{d\theta}{2}$$

where $\sin \alpha \rightarrow \alpha$ for small angles.

$$= - \frac{\sigma_{\theta\theta}}{r} (r \cdot dr \cdot d\theta \cdot dz)$$

Nett radial surface force is:

$$(r \cdot dr \cdot d\theta \cdot dz) \left[-\frac{\partial p}{\partial r} + \mu \left\{ 2 \frac{\partial^2 v_r}{\partial r^2} + \frac{2}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{\partial^2 v_z}{\partial r \partial z} \right\} \right. \\ \left. + \frac{\partial \mu}{\partial r} \left(2 \frac{\partial v_r}{\partial r} \right) + \frac{\partial \mu}{\partial z} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \right]$$

The {quantity} becomes:

$$\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right) + \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_z}{\partial r \partial z} \right) \\ \underbrace{\frac{\partial}{\partial r} \left(\frac{\partial v_r}{\partial r} \right) \quad \frac{\partial}{\partial r} \left(\frac{v_r}{r} \right) \quad \frac{\partial}{\partial r} \left(\frac{\partial v_z}{\partial z} \right)}_{\downarrow} \\ \frac{\partial}{\partial r} \left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \right) \rightarrow 0 \text{ (Continuity)}$$

$$\text{i.e. Nett Force is } (r \cdot dr \cdot d\theta \cdot dz) \left[\mu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right) + \frac{\partial \mu}{\partial r} \left(2 \frac{\partial v_r}{\partial r} \right) \right. \\ \left. + \frac{\partial \mu}{\partial z} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \right] \\ + (r \cdot dr \cdot d\theta \cdot dz) \left[-\frac{\partial p}{\partial r} \right]$$

Hence the radial momentum equation becomes:

$$\rho \left(v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \mu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right) \\ + \frac{\partial \mu}{\partial r} \left(2 \frac{\partial v_r}{\partial r} \right) + \frac{\partial \mu}{\partial z} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \quad (3)$$

1.4 ENERGY

This involves the application of the First Law of Thermodynamics to the infinitesimal fluid element. Heat (Q) and work (W) transfers are related to the energy (E) by:

$$\frac{dQ}{dt} - \frac{dW}{dt} = \frac{DE}{Dt}$$

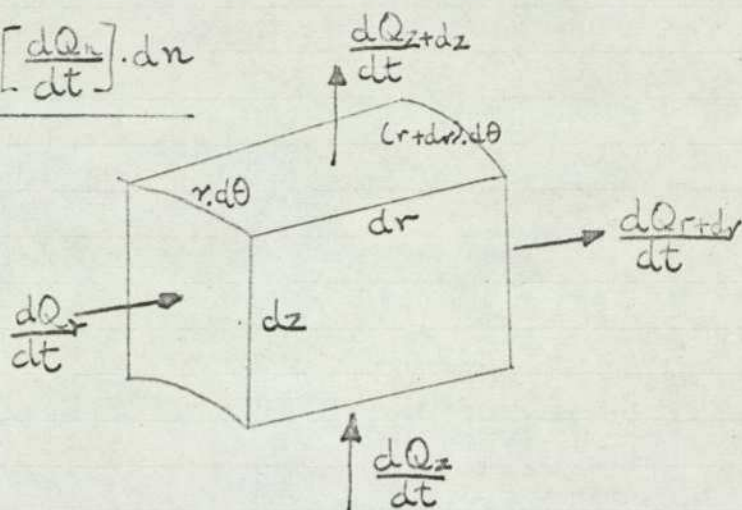
Heat Transfer

This is by conduction, and is given by Fourier's Law:

$$\frac{dQ_n}{dt} = -k A \frac{\partial T}{\partial n}, \text{ where } A = \text{area of cross-section} \\ n = \text{co-ordinate.}$$

Hence $\frac{d}{dt}(Q_n + dn) = \frac{dQ_n}{dt} + \frac{\partial}{\partial n} \left[\frac{dQ_n}{dt} \right] \cdot dn$

Applying this principle to heat flow rates in the axial ($n=z$) and radial ($n=r$) directions, as specified by the diagram:



Nett heat flow rate in axial direction:

$$\begin{aligned} &= \frac{dQ_z}{dt} - \frac{d}{dt}(Q_{z+dz}) = -\frac{\partial}{\partial z} \left[\frac{dQ_z}{dt} \right] \cdot dz \\ &= \frac{\partial}{\partial z} \left[k(r \cdot dr \cdot d\theta) \frac{\partial T}{\partial z} \right] dz \\ &= (r \cdot dr \cdot d\theta \cdot dz) \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) \end{aligned}$$

Nett heat flow rate in radial direction:

$$\begin{aligned} &= \frac{dQ_r}{dt} - \frac{d}{dt}(Q_{r+dr}) = -\frac{\partial}{\partial r} \left[\frac{dQ_r}{dt} \right] \cdot dr \\ &= \frac{\partial}{\partial r} \left[k(r \cdot d\theta \cdot dz) \cdot \frac{\partial T}{\partial r} \right] \cdot dr \\ &= (r \cdot dr \cdot d\theta \cdot dz) \frac{1}{r} \cdot \frac{\partial}{\partial r} \left(k r \frac{\partial T}{\partial r} \right) \end{aligned}$$

$$\begin{aligned} \therefore \frac{dQ}{dt} &= (r \cdot dr \cdot d\theta \cdot dz) \left\{ \frac{1}{r} \cdot \frac{\partial}{\partial r} \left(k r \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) \right\}^* \\ &= (r \cdot dr \cdot d\theta \cdot dz) k \left\{ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right\} \rightarrow \text{constant } k \text{ term} \\ &\quad + (r \cdot dr \cdot d\theta \cdot dz) \left[\frac{\partial T}{\partial r} \frac{\partial k}{\partial r} + \frac{\partial T}{\partial z} \frac{\partial k}{\partial z} \right] \rightarrow \text{variable } k \text{ term} \end{aligned}$$

for constant k , the second term is zero.

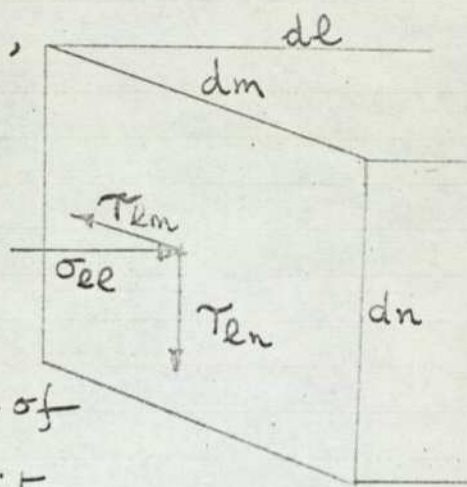
i.e. $\frac{dQ}{dt} = (r \cdot dr \cdot d\theta \cdot dz) k \left\{ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right\}^+$

Including circumferential terms for the sake of completeness,
* { } includes $\frac{1}{r^2} \frac{\partial}{\partial \theta} \left(k \frac{\partial T}{\partial \theta} \right)$, and

+ { } includes $\frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2}$

Work Transfer

$\frac{dW}{dt}$ is the total nett rate of work transfer done by the (normal and shear stresses of the) element on the surroundings. Reversing directions of stresses shown on p C/1/9 to obtain those exercised ON the surroundings BY the element, and generalising the co-ordinates as l, m, n , we have nett work done by virtue of changes in the l direction, due to σ_{ll} , T_{lm} , and T_{ln} .



In general $\frac{dW}{dt} = \text{force} \times \text{vel}^{\text{th}}$ in direction of force
 $= \text{stress} \times \text{area} \times \text{velocity}$

For normal stress $\frac{dW_l}{dt} = \sigma_{ll} \times A_{lm} \times v_l = \textcircled{A}$

and $\frac{dW_{l+dl}}{dt} = \frac{dW_l}{dt} + \frac{\partial}{\partial l} \left[\frac{dW_l}{dt} \right] \cdot dl = \textcircled{B}$

i.e. nett work transfer $= \textcircled{A} - \textcircled{B} = - \frac{\partial}{\partial l} \left[\frac{dW_l}{dt} \right] \cdot dl$

similarly for shear stress T_{lm} , and T_{ln} respectively:

nett work transfer. $= - \frac{\partial}{\partial l} \left[\frac{dW_m}{dt} \right] \cdot dl$, $\frac{dW_m}{dt} = T_{lm} \times A_{lm} \times v_m$

-and $- \frac{\partial}{\partial l} \left[\frac{dW_n}{dt} \right] \cdot dl$, $\frac{dW_n}{dt} = T_{ln} \times A_{lm} \times v_n$

i.e. total nett work in respect of ' l ' direction changes:

$$= - \frac{\partial}{\partial l} \left[(\sigma_{ll} \times A_{lm} \times v_l) + (T_{lm} \times A_{lm} \times v_m) + (T_{ln} \times A_{lm} \times v_n) \right] \cdot dl$$

$$= - \frac{\partial}{\partial l} \left\{ (A_{lm}) \times [\sigma_{ll} v_l + T_{lm} v_m + T_{ln} v_n] \right\} dl$$

change of area w.r.t. ' l ' is allowed for in cylindrical system

there will be similar symmetrical terms in m, n directions

Making substitution $\begin{matrix} l \rightarrow r \\ m \rightarrow \theta \\ n \rightarrow z \end{matrix}$, $\begin{matrix} A_{nm} = r \cdot d\theta \cdot dz \\ A_{ln} = dr \cdot dz \\ A_{lm} = r \cdot dr \cdot d\theta \end{matrix}$

we have following non-zero stresses (c.f. C/1/11)

$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \tau_{rz}, \tau_{zr}$, and since changes w.r.t. θ are absent, there are only $\frac{\partial}{\partial r}$, and $\frac{\partial}{\partial z}$ components.

This gives
$$\frac{dW}{dt} = - \left[\frac{\partial}{\partial r} \{ (r \cdot d\theta \cdot dz) (\sigma_{rr} v_r + \tau_{rz} v_z) \} \cdot dr + \frac{\partial}{\partial z} \{ (r \cdot dr \cdot d\theta) (\sigma_{zz} v_z + \tau_{zr} v_r) \} \cdot dz \right]$$

$$= - (r \cdot dr \cdot d\theta \cdot dz) \left[\frac{1}{r} \cdot \frac{\partial}{\partial r} (r \cdot (\sigma_{rr} v_r + \tau_{rz} v_z)) + \frac{\partial}{\partial z} (\sigma_{zz} v_z + \tau_{zr} v_r) \right]$$

Particle Energy Change

$$\frac{DE}{Dt} = m \frac{De}{Dt}$$

where $m = \text{mass of fluid element}$
 $= \rho(r \cdot dr \cdot d\theta \cdot dz)$

$$e = u + \frac{\bar{V}^2}{2} + gz_1 \quad \text{where } u = \text{internal energy}$$

$$\frac{\bar{V}^2}{2} = \text{kinetic energy}$$

$$gz_1 = \text{potential energy relative to arbitrary datum}$$

Hence:
$$\frac{DE}{Dt} = \rho(r \cdot dr \cdot d\theta \cdot dz) \left\{ \frac{Du}{Dt} + \frac{D}{Dt} (\bar{V}^2) + \frac{D}{Dt} (gz_1) \right\}$$

The kinetic and potential energy terms will now be re-expressed using the Navier-Stokes equations written in the following form:

Axial:
$$\underbrace{(\rho \cdot r \cdot dr \cdot d\theta \cdot dz) \frac{Dv_z}{Dt}}_{(C/1/4 \& 5)} + \underbrace{(\rho \cdot r \cdot dr \cdot d\theta \cdot dz) g}_{(C/1/5)} = \dots \dots \dots \text{PTO}$$

* To avoid excessive complexity, θ co-ordinate effects will be eliminated, and only stated at the end of the analysis.

$$= \dots\dots\dots (r \cdot dr \cdot d\theta \cdot dz) \left[\frac{\partial}{\partial z}(\sigma_{zz}) + \frac{1}{r} \cdot \frac{\partial}{\partial r}(r \cdot \tau_{rz}) \right]$$

$$\text{i.e. } \rho \frac{Dv_z}{Dt} + \rho g = \left[\frac{\partial}{\partial z}(\sigma_{zz}) + \frac{1}{r} \frac{\partial}{\partial r}(r \cdot \tau_{rz}) \right]$$

and multiplying by $\frac{v_z}{\rho} :-$

$$\left(v_z \frac{Dv_z}{Dt} + v_z g \right) = \frac{v_z}{\rho} \left[\frac{\partial}{\partial z}(\sigma_{zz}) + \frac{1}{r} \frac{\partial}{\partial r}(r \cdot \tau_{rz}) \right] \quad (2)A$$

Similarly for the radial direction :

$$\underbrace{(\rho \cdot r \cdot dr \cdot d\theta \cdot dz) \frac{Dv_r}{Dt}}_{(C/1/13)} = \underbrace{(r \cdot dr \cdot d\theta \cdot dz) \left[\frac{1}{r} \cdot \frac{\partial}{\partial r}(r \cdot \sigma_{rr}) + \frac{\partial}{\partial z}(\tau_{zr}) - \frac{\sigma_{\theta\theta}}{r} \right]}_{(C/1/13 \& 14)}$$

multiplying by $\left(\frac{v_r}{\text{mass}} \right) :$

$$\underline{v_r \frac{Dv_r}{Dt} = \frac{v_r}{\rho} \left[\frac{1}{r} \cdot \frac{\partial}{\partial r}(r \cdot \sigma_{rr}) + \frac{\partial}{\partial z}(\tau_{zr}) - \frac{\sigma_{\theta\theta}}{r} \right]} \quad (3)A$$

$$\text{However : } \underline{\text{k.e.} \frac{D}{Dt} (\bar{V}^2)} = \frac{D}{Dt} (v_r^2 + v_\theta^2 + v_z^2)$$

$$= \underline{v_r \frac{Dv_r}{Dt} + v_z \frac{Dv_z}{Dt}}$$

$$\text{p.e. } \& \frac{D}{Dt} (gz_1) = + \left(v_r \frac{\partial}{\partial r}(gz_1) + \cancel{\frac{v_\theta}{r} \frac{\partial}{\partial \theta}(gz_1)} + v_z \frac{\partial}{\partial z}(gz_1) \right)$$

since $z_1 = f(z)$

$$\underline{\frac{D}{Dt} (gz_1) = + v_z g}$$

Hence k.e. + p.e. terms become: $\frac{D}{Dt} (\bar{V}^2) + \frac{D}{Dt} (gz_1)$

$$= \left(v_z \frac{Dv_z}{Dt} + v_z g \right) + v_r \frac{Dv_r}{Dt}$$

$$= \frac{v_z}{\rho} \left[\frac{\partial}{\partial z}(\sigma_{zz}) + \frac{1}{r} \frac{\partial}{\partial r}(r \cdot \tau_{rz}) \right] + \frac{v_r}{\rho} \left[\frac{1}{r} \frac{\partial}{\partial r}(r \cdot \sigma_{rr}) + \frac{\partial}{\partial z}(\tau_{zr}) - \frac{\sigma_{\theta\theta}}{r} \right]$$

from (2)A and (3)A.

The energy equation (C/1/14) may be re-written:

$$m \frac{Du}{Dt} = \frac{dQ}{dt} - \left\{ m \frac{D}{Dt} (\text{k.e.} + \text{p.e.}) + \frac{dW}{dt} \right\}$$

It is convenient to determine the {bracketed terms} using C/1/17 and C/1/18 (bottom) as:

$$-(r.d\theta.dr.dz) \left\{ \frac{1}{r} \cdot \frac{\partial}{\partial r} (r(\sigma_{rr} v_r + T_{rz} v_z)) + \frac{\partial}{\partial z} (\sigma_{zz} v_z + T_{rz} v_r) - \sigma_{\theta\theta} v_r \right\} \\ - \rho \left\{ \frac{v_z}{r} \left(\frac{\partial}{\partial z} (\sigma_{zz}) + \frac{1}{r} \cdot \frac{\partial}{\partial r} (r T_{rz}) \right) + \frac{v_r}{r} \left(\frac{1}{r} \cdot \frac{\partial}{\partial r} (r \sigma_{rr}) + \frac{\partial}{\partial z} (T_{rz}) \right) \right\}$$

This expression looks complicated, but each term in the first {bracket} may be partially expanded into two terms, one of which cancels with a term in the second {bracket}.

For example: $\frac{1}{r} \frac{\partial}{\partial r} (r \sigma_{rr} v_r) = \frac{r \sigma_{rr}}{r} \frac{\partial v_r}{\partial r} + \frac{v_r}{r} \frac{\partial}{\partial r} (r \sigma_{rr})$,
and the second term cancels with an identical term above.
Hence the above expression becomes:-

$$-(r.dr.d\theta.dz) \left[\sigma_{rr} \frac{\partial v_r}{\partial r} + T_{rz} \frac{\partial v_z}{\partial r} + \sigma_{zz} \frac{\partial v_z}{\partial z} + T_{rz} \frac{\partial v_r}{\partial z} + \frac{v_r \sigma_{\theta\theta}}{r} \right] \\ = -(r.dr.d\theta.dz) \left[\sigma_{rr} \frac{\partial v_r}{\partial r} + \sigma_{\theta\theta} \frac{v_r}{r} + \sigma_{zz} \frac{\partial v_z}{\partial z} + T_{rz} \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) \right]$$

We may now substitute for the stresses from C/1/11, to give

$$-(r.dr.d\theta.dz) \left[-p \left(\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \right)^* + 2\mu \left[\left(\frac{\partial v_r}{\partial r} \right)^2 + \left(\frac{v_r}{r} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 + \frac{1}{2} \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right)^2 \right] \right] \\ = -(r.dr.d\theta.dz) \left[-p.e_v^* + \mu \phi \right]$$

where ϕ (the DISSIPATION FUNCTION) will be explained below

* The bracketed term is zero, for reasons of continuity in incomp. (C/1/3) possible flow; it is in fact the volumetric strain-rate (C/1/3)

Substituting in energy equation for $\frac{dQ}{dt}$ from C/1/15, gives:

$$\rho \frac{Du}{Dt} = k \left\{ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right\} - p \cdot e_v + \mu \phi \quad *$$

or, Rate of increase of internal energy =

- Heat transfer rate by conduction
- Reversible rate of (flow) work for compressible flow
- + Irreversible rate of (flow) work due to fluid friction

The viscous dissipation term is always positive (each term is squared) and represents an irreversible degradation of kinetic into thermal energy i.e. its effect is that of a non-uniformly distributed heat source. Its magnitude depends on (velocity gradients) squared, hence is important at high speed, effect being referred to as 'kinetic heating'. However, it has recently been postulated (e.g. Ref. 7) as affecting laminar flow heat transfer under certain conditions, and hence is included in analysis.

For heat transfer with fluid flow the enthalpy 'h' is used in preference to 'u', and we now adapt the equation accordingly (after Bayley, Owen & Turner (Ref. 3) p156).

$$h \equiv u + \frac{p}{\rho} \quad \text{by definition.}$$

$$\text{hence } \frac{Du}{Dt} = \frac{Dh}{Dt} - \frac{1}{\rho} \left(\frac{Dp}{Dt} - \frac{p}{\rho} \frac{D\rho}{Dt} \right) \quad \text{--- (i)}$$

$$\text{We also have: } dh = T \cdot ds + \frac{dp}{\rho} \quad \text{where } ds = \left(\frac{\partial s}{\partial T} \right)_p \cdot dT + \left(\frac{\partial s}{\partial p} \right)_T \cdot dp$$

--- (ii)

* See Pao (Ref. 6) pp 282-285 for a 3-D Cartesian/Vector analysis and statement of equation, corresponding to C/1/16 onwards

Since $\left(\frac{\partial s}{\partial T}\right)_p = \frac{1}{T} \left(\frac{\partial h}{\partial T}\right)_p$ and $\left(\frac{\partial s}{\partial p}\right)_T = \frac{1}{\rho^2} \left(\frac{\partial \rho}{\partial T}\right)_p$ [Maxwell's Relations]

and $\left(\frac{\partial h}{\partial T}\right)_p \equiv C_p$ and $-\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T}\right)_p \equiv \beta$ [By definition]

Then $ds = \frac{C_p \cdot dT}{T} - \frac{\beta \cdot dp}{\rho}$

and $dh = C_p \cdot dT - \frac{\beta T \cdot dp}{\rho} + \frac{dp}{\rho}$ from (ii)

or $\frac{Dh}{Dt} = C_p \frac{DT}{Dt} + \left(\frac{1 - \beta T}{\rho}\right) \frac{Dp}{Dt}$

hence: $\frac{Du}{Dt} = C_p \frac{DT}{Dt} + \frac{1}{\rho} \frac{Dp}{Dt} - \frac{\beta T \cdot Dp}{\rho \cdot Dt} - \frac{1}{\rho} \frac{Dp}{Dt} + \frac{p}{\rho^2} \frac{D\rho}{Dt}$ from (i)
 $= C_p \frac{DT}{Dt} - \frac{\beta T \cdot Dp}{\rho \cdot Dt} + \frac{p}{\rho^2} \cdot \frac{D\rho}{Dt}$ ----- (iii)

We obtain $\frac{D\rho}{Dt}$ from the general continuity equation (C/1/3) which may be expanded as:

$$\left(\frac{\partial \rho}{\partial t} + \frac{v_\theta}{\rho} \cdot \frac{\partial}{\partial \theta}(\rho) + v_z \frac{\partial}{\partial z}(\rho) + v_r \frac{\partial}{\partial r}(\rho)\right) + \rho \left(\frac{1}{T} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r}\right) = 0$$

or $\frac{D\rho}{Dt} + \rho \cdot e_v = 0$
 (C/1/4) (C/1/8)

hence $\frac{p}{\rho^2} \cdot \frac{D\rho}{Dt} = -\frac{p}{\rho} \cdot e_v$

and (iii) becomes $\frac{Du}{Dt} = C_p \frac{DT}{Dt} - \frac{\beta T \cdot Dp}{\rho \cdot Dt} - \frac{p}{\rho} \cdot e_v$

which may be written $\rho \frac{Du}{Dt} + p \cdot e_v = \rho C_p \frac{DT}{Dt} - \beta T \frac{Dp}{Dt}$

Hence 'internal energy' form of energy equation may be replaced by 'enthalpy form' as:

$$\rho C_p \frac{DT}{Dt} = \beta T \frac{Dp}{Dt} + k \left\{ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right\} + \mu \phi$$

where (i) only property assumed constant is 'k'

(ii) $\beta T \frac{Dp}{Dt}$ represents work term for variable density flow

For density constant $\beta = 0$, and expanding $\frac{D}{Dt}$, and putting $\frac{k}{\rho c_p} \equiv \alpha$ (thermal diffusivity), we have:

$$\underline{v_r \frac{\partial T'}{\partial r'} + v_z \frac{\partial T'}{\partial z'} = \alpha \left(\frac{\partial^2 T'}{\partial z'^2} + \frac{1}{r'} \frac{\partial T'}{\partial r'} + \frac{\partial^2 T'}{\partial r'^2} \right) + \frac{\mu}{\rho c_p} \phi} \quad (+)^*$$

$$\text{where } \phi = 2 \left\{ \left(\frac{\partial v_r}{\partial r'} \right)^2 + \left(\frac{v_r}{r'} \right)^2 + \left(\frac{\partial v_z}{\partial z'} \right)^2 \right\} + \left\{ \frac{\partial v_z}{\partial r'} + \frac{\partial v_r}{\partial z'} \right\}^2$$

Finally, it should be noted that two heat transfer terms have been omitted, viz: (i) Volumetric heat generation, usually in a uniform manner throughout the fluid. Hence, add to r.h.s. q_v (= heat generation rate per unit volume).

(ii) Radiation effects.

CONCLUDING NOTE ON EQUATIONS

The equations (1) - (4) have been derived from first principles, and the limitations noted where appropriate. In a number of basic fluids texts (e.g. Refs. 6, 8) the logic is similar, but normally confined to rectangular co-ordinates with transforms to cylindrical or spherical co-ordinates, or given inclusively in a vector statement. (See Yuan (Ref. 9) p115 for a partial cylindrical treatment). For complete statements of equations (1) - (4) for different co-ordinates and/or different limitations, see Hughes and Gaylord (Ref. 10).

*For constant properties, including θ -direction effects, (+) is

$$v_r \frac{\partial T'}{\partial r'} + \frac{v_\theta}{r'} \frac{\partial T'}{\partial \theta} + v_z \frac{\partial T'}{\partial z'} = \alpha \left(\frac{\partial^2 T'}{\partial z'^2} + \frac{1}{r'} \frac{\partial T'}{\partial r'} + \frac{\partial^2 T'}{\partial r'^2} + \frac{1}{r'^2} \frac{\partial^2 T'}{\partial \theta^2} \right) + \frac{\mu}{\rho c_p} \phi$$

$$\text{where } \phi = 2 \left\{ \left(\frac{\partial v_r}{\partial r'} \right)^2 + \left(\frac{1}{r'} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r'} \right)^2 + \left(\frac{\partial v_z}{\partial z'} \right)^2 \right\} + \left\{ \frac{\partial v_z}{\partial r'} + \frac{\partial v_r}{\partial z'} \right\}^2 + \left\{ \frac{\partial v_\theta}{\partial r'} - \frac{v_\theta}{r'} + \frac{1}{r'} \frac{\partial v_r}{\partial \theta} \right\}^2$$

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A P P E N D I X 2

Program print-out (Standard and Viscous Dissipation)

		PROGRAM COLL(INPUT,OUTPUT,TAPE1=INPUT,TAPE2=OUTPUT)	
		MIXED FREE AND FORCED CONVECTION IN VERTICAL TUBES	90
		REAL MTWO,MONE	100
		REAL NUSSLT	140
5		DIMENSION U(3,80),V(3,80),T(3,80),P(3,80),	
		1A(238,238),B(238),X1(79),X2(238),L(10)	
		COMMON /CHN/ A	
		LEVEL 2,A	
10		DATA PI/3.14159/	3H130
		CALL ITIME(ISEC1)	150
		INITIAL RECORDING OF TIME.	
	1	READ(1,2)ILAST,KLAST,MAXR,ISUNU,IFT00,C01,C02,C03,C04,C05,C06,	160
		1GNUMBR,PENTRY,DELTAZ,PRNOTL,RYNLOS	170
15	2	FORMAT(D13,F10.8,F17.15,F25.22,F10.8,F18.14/F20.17,F15.1,F5.1,	171
		1F6.2,F14.10,F10.4)	172
		IF (ILAST.LT.1)STOP	173
		READ(1,3)(L(K),K=1,KLAST)	174
	3	FORMAT(10I3)	175
		READ(1,5)S,RATIO0	180
20	5	FORMAT(F4.1,F10.6)	190
		WRITE(2,6)S,RATIO0	200
	6	FORMAT(1H1,3H S=,F5.2,22H + SIGN=UP;- SIGN=DOWN,15H DENSITY RATIO=	210
		1,F8.7)	211
25		READS INPUT CARDS 1-5.	
		CARD 173 STOPS PROGRAM WHEN NO MORE RUNS ARE NEEDED.	
		EFFECT OF VISCOUS DISSIPATION	
		READ(1,41)QDISS	V01
	41	FORMAT(F15.12)	V02
		WRITE(2,42)QDISS	V03
30	42	FORMAT(1H ,7H QDISS=,F15.12)	V04
		READS AND WRITES INPUT CARD 6.	V05
		XMAXP=MAXP	580
		SQ42=MAXR**2/2	660
35		G1=C01/GNUMBR	PA6
		GO TO(4,8,4),ISUNU	212
	4	ZBARX=0.0	A212
	7	CONTINUE	213
		STATEMENT 7 STARTS HEATED SECTION FOR P.A.D. (PARTIAL ADIABATIC	
		DEVELOPMENT), OTHERWISE FOLLOW STATEMENT--.	
40	9	FORMAT(F10.1)	214
		GO TO(10,12),IFT00	220
	10	READ(1,9)TWALL	221
		TALPHA=(G1*TWALL+C02)**(-C03)	222
		WRITE(2,11)TWALL,MAXR	230
45	11	FORMAT(1H ,25HCONSTANT WALL TEMP.TWALL=,F10.3,6H MAXR=,16)	240
		GO TO 14	250
	12	READ(1,9)HFLUX	251
		XNUX=2.0*HFLUX*XMAXR	3570
		WRITE(2,13)HFLUX,MAXR	260
50	13	FORMAT(1H ,25HCONSTANT HEAT FLUX.HFLUX=,F10.3,6H MAXR=,16)	270
	14	CONTINUE	280
		READS AND WRITES INPUT CARD 8.	
		CARD 222 IS TALPHA AT WALL FOR CONSTANT TWALL (VISCOSITY TYPE A).	
		TALPHA=(TWALL/GNUMBR+1.0)**(-C01)	222
55		TYPE B VISCOSITY.	
		TALPHA=C01+(C02+C03*TWALL)*TWALL	222
		TYPE C VISCOSITY.	

		GO TO(8,60,8),ISUNU	281
	C	ENSURES, WITH CARD 212, FOR ANY ISUNU CHOICE, PROGRAM INITIALLY	
50	C	GOES TO STATEMENT 8.	
	8	WRITE(2,15) (L(K),K=1,KLAST)	290
	15	FORMAT(1H,16H VALUES OF L(K),/(7I4))	300
		WRITE(2,16) ILAST,DELTZ	310
	16	FOR 4AT(1H,7H ILAST=,16,7H DELTZ=,F10.6)	320
60		WRITE(2,17) CO1,CO2,CO3,CO4,CO5,CO6,RYNLDS,PRNDTL,GNUMBR,PENTRY	330
	17	FORMAT(1H,5H CO1=,F8.6,5H CO2=,F18.15,5H CO3=,F21.18,	340
		15H CO4=,F9.6,5H CO5=,F9.6,5H CO6=,F15.12,	350
		28H RYNLDS=,F10.1,8H PRNDTL=,F15.6,8H GNUMBR=,	360
		3F15.2,8H PENTRY=,F5.3)	370
70	C	WRITES REMAINING INPUT DATA.	
		GO TO(23,25,35),ISUNU	380
	20	READ(1,21) ZBARX	390
	21	FORMAT(F10.6)	400
		WRITE(2,22) ZBARX	410
75	22	FORMAT(1H,24H ADIABATIC ENTRY LENGTH=,F10.6)	420
	C	READS AND WRITES INPUT CARD 7 FOR P.A.D.	
		NUS3LT=0.0	421
		GO TO(26,27) IFT00	422
80	26	THALL=0.0	423
		TALPH=0.0	424
		GO TO 23	425
	27	HFLUX=0.0	426
	23	DO 24 M=2,MAXR	427
		U(2,M)=1.0	428
85	24	CONTINUE	429
		GO TO 25	430
	C	FOR P.A.D. NU, AND THALL OR 0, ARE ZERO.	
	C	STATEMENT 23 DEFINES UNIFORM ENTRY PROFILE, FOR ISUNU=1 OR 2.	
	30	DO 31 I=2,MAXR	470
90		XI=I-1	480
	C	DEFINES PARABOLIC VELOCITY PROFILE, FOR ISUNU=3.	
		U(2,1)=2.0-XI**2/SQM2	481
	31	CONTINUE	482
95	25	DO 40 M=2,MAXR	483
		U(1,M)=U(2,M)	484
		V(1,M)=0.0	485
		V(2,M)=0.0	486
		T(1,M)=0.0	487
		I(2,M)=0.0	488
100		T(3,M)=0.0	489
		P(2,M)=PENTRY	490
	40	CONTINUE	491
		P(2,1)=PENTRY	492
		P(2,MAXR+1)=PENTRY	570
105	C	PMEAN=0.0	
	C	ADDITIONAL INPUT DATA.	
	C	CARDS 500, ZERO ACCELERATION, 510, 520, ZERO RADIAL VELOCITIES,	
	C	530, 540, 550, 560, 570, ZERO TEMPERATURE AND GRADIENT, 580, 590, PRESSURES.	
		NUMBR=MAXR/2	590
110	50	ZBAR=0.0	600
		TBAR=0.0	610
		ITALLY=0	620
		K=1	630
	C	CARDS 600-620 DEFINE START OF RUNNING VARIABLES.	


```

115      NITER=0
      C NITER IS SEMI-ITERATIVE LOOP VARIABLE.
      60 CONTINUE
      C STARTING POINT FOR EACH AXIAL POSITION.
120      DIZSQ=1.0/DELTZ**2
      DPRRE=1.0/(PRNDIL*RYNLDS)
      FUNC1=DPRRE/DELTZ
      HALFM=MAXR/2
      DOZM=XMAXR/12.0
      UCZM8=DCZM*8.0
125      FMSQ=SQM2*DPRRE
      MAX1=MAXR-1
      MAXP1=MAXR+1
      C CALCULATION OF FREQUENTLY-USED VARIABLES.
      90 CONTINUE
130      DTWOZ=0.5/DELTZ
      DIVZ=1.0/DELTZ
      MAX32=3*MAXR-2
      MAX21=2*MAXR-1
      MAX2=2*MAXR
      NIWO=DIZSQ-1.0*SQM2
      DIVRE=1.0/RYNLDS
      MONE=SQM2/DIVRE
      UPORDN=3*RATIO0/RYNLDS**2
      UP1=UPORDN*GNUMBR*CO4
140      UP2=UPORDN*CO5
      UP3=UPORDN*CO5
      DIVRE2=2.0/RYNLDS
      DIVRE3=DIVRE*DIZSQ
      DIVRE4=HALFM/(DELTZ*RYNLDS)
145      C INITIAL CALCULATION OF VARIABLES.
      DO 110 I=1,MAX32
      B(I)=0.0
      DO 110 J=1,MAX32
      A(I,J)=0.0
150      110 B(MAX32)=0.75*SQM2
      C LOOP CLEARS ALL A,S AND B,S.
      C STATEMENT 110 GIVES B FOR INTEGRAL CONTINUITY EQUATION.
      B(MAX21)=DTWOZ*(U(2,MAX1)+U(2,MAXR))
155      B(MAX2)=U(2,2)/DELTZ
      A(MAX21,MAX32-1)=DTWOZ
      A(MAX21,MAX32)=DTWOZ
      A(MAX2,MAX2)=1.0/DELTZ
      A(MAX21,MAX21-1)=-XMAXR
      A(MAX21,MAX21)=XMAXR**2/(XMAXR-1.0)
160      A(MAX2,MAXP1)=XMAXR
      C DEFINES BOUNDARY CONDITION EQUATIONS FOR ROWS (2M-1=MAX21)
      C AND (2M=MAX2) OF UVP MATRIX.
      A(MAX1-1,MAX1-1)=-DIVZ
      A(MAX1,MAX1)=-DIVZ
165      A(MAXR,MAXP-2)=HALFM
      A(MAXR,MAXR)=-HALFM
      A(MAXP1,1)=HALFM*3.0
      A(MAXP1,2)=-HALFM*4.0
      A(MAXP1,3)=HALFM
170      C OTHER BOUNDARY CONDITION A VALUES.
      G2=-G1*CO3

```

T7

PA7

		CO8=CO3+1.5		PA8
		TCOMN=G1*T(2,2)+CO2		PA9
		TALPHA=TCOMN*(-CO3)		PA10
175		TBETA=G2*TCOMN*(-CO5)		PA11
	C	PROGRAM IS FOR VISCOSITY TYPE A (CARDS PA7-PA11).		
	C	TALPHA AND TBETA STATED FOR INNER BOUNDARY CONDITION.		
	C	TALPHA=(T(2,2)/GNUMB+1.0)**(-CO1)	1360	
	C	TBETA=(-CO1/GNUMB)*T(2,2)/GNUMB+1.0)**(-CO2)	1370	
180	C	ALTERNATIVE VISCOSITY TYPE B.		
	C	TALPHA=CO1+(CO2+CO3*T(2,2))*T(2,2)	1360	
	C	TBETA=CO2+2.0*CO3*T(2,2)	1370	
	C	ALTERNATIVE VISCOSITY TYPE C.		
		TONE=TBETA*(T(2,3)-T(2,2))	1380	
185		TTWO=TBETA*(T(2,2)-T(1,2))*DIZSQ	1390	
		VONE=HALFM*V(2,2)	1400	
		TFOUR=TALPHA*3.0	1410	
		TSIX=TALPHA*DIZSQ	1420	
		A(1,MAX2+1)=MONE*(TFOUR+0.5*TONE)-VONE	1430	
190		UONE=DIVZ*U(2,2)	1440	
		A(1,MAX2)=MONE*(TALPHA-0.5*TONE)+VONE+DIVRE*(TALPHA*MTWO+2.0*TTWO)	1450	
		1-UONE	1460	
		A(MAXP1,MAXP1+1)=MONE*(TFOUR+TONE)-VONE	1470	
		VARIT=TALPHA*(DIZSQ-SQM2*6.0)	1480	
195		A(MAXP1,MAXP1)=MONE*(TALPHA-TONE)+VONE+DIVRE*(VARIT+TTWO)-UONE	1490	
		SONE=V(2,2)*(-U(2,2)*DIVZ+DIVRE*(2.0*TSIX+TTWO))	1500	
		STWO=-DIVRE*(V(1,2)*TSIX+HALFM*DELIZ*TTWO*(U(2,3)-U(2,2)))	1510	
		B(MAXP1)=SONE+STWO	1520	
		UPORDN2=UP1+T(2,2)*(UP2+UP3*T(2,2))	1530	
200		BONE=UPORDN2-DIVZ*(P(2,2)+U(2,2)**2)-U(1,2)*TALPHA*DIVRE3	1540	
		BTWO=DIVRE2*U(2,2)*(TSIX+TTWO)-DIVRE4*TONE*(V(2,2)-V(1,2))	1550	
		B(1)=BONE+BTWO	1560	
	C	COMPLETES A AND B SPECIFICATION FOR INNER BOUNDARY CONDITION.		
205		TCOMN=G1*T(2,MAXR)+CO2		PA12
		TALPHA=TCOMN*(-CO3)		PA13
		TBETA=G2*TCOMN*(-CO5)		PA14
	C	RE-DEFINES TALPHA AND TBETA FOR OUTER (WALL) B.C.		
	C	TALPHA=(T(2,MAXR)/GNUMB+1.0)**(-CO1)	1570	
	C	TBETA=(-CO1/GNUMB)*T(2,MAXR)/GNUMB+1.0)**(-CO2)	1580	
210	C	TYPE B VISCOSITY.		
	C	TALPHA=CO1+(CO2+CO3*T(2,MAXR))*T(2,MAXR)	1570	
	C	TBETA=CO2+2.0*CO3*T(2,MAXR)	1580	
	C	TYPE C VISCOSITY.		
		GO TO(111,112),IFTOO	1590	
215	111	TONE=TBETA*(TWALL-T(2,MAX1))	1600	
		GO TO 113	1610	
	112	TONE=TBETA*(T(2,MAXR)+HFLUX-T(2,MAX1))	1620	
	113	CONTINUE	1630	
		TTWO=TBETA*(T(2,MAXR)-T(1,MAXR))*DIZSQ	1640	
220		VONE=HALFM*V(2,MAXR)	1650	
		TTTHREE=TALPHA/(XMAXR-1.0)	1660	
		TFOUR=TTTHREE*(2.0*XMAXR-1.0)	1670	
		TFIVE=TTTHREE*(2.0*XMAXR-3.0)	1680	
		TSIX=TALPHA*DIZSQ	1690	
225		UONE=DIVZ*U(2,MAXR)	1700	
		A(MAX1,MAX32-1)=MONE*(TFIVE-0.5*TONE)+VONE	1710	
		A(MAX1,MAX32)=DIVRE*(TALPHA*MTWO+2.0*TTWO)-UONE	1720	
		A(MAXR,MAX21-1)=MONE*(TFIVE-TONE)+VONE	1730	

230

VARIT=TALPHA*(DIZSQ-SQM2*(4.2+2.5/(XMAXR-1.0)**2))
 A(MAXR,MAX21)=DIVRE*(VARIT+TTWO)-VONE
 BONE=V(2,MAXR)*(-U(2,MAXR)*DIVZ+DIVRE*(2.0*TSIX+TTWO))
 BTWO=-DIVRE*(V(1,MAXR)*TSIX-HALFM*DELTZ*TTWO*U(2,MAX1))
 B(MAXP)=BONE+BTWO
 UPORDN3=UP1+T(2,MAXP)*(UP2+UP3*T(2,MAXR))

235

BONE=UPORDN3-DIVZ*(P(2,MAXR)
 1+U(2,MAXR)**2)-U(1,MAXR)*TALPHA*DIVRE3
 BTWO=DIVRE2*U(2,MAXP)*(TSIX+TTWO)
 1-DIVRE4*TONE*(V(2,MAXR)-V(1,MAXR))

240

C SPECIFIES ALL A,S AND B,S FOR OUTER B.C. EQUATIONS.

MAXR3=MAXR-3

DO 115 I=1,MAXR3

C OTHER A,S AND B,S MAY BE SPECIFIED IN SINGLE LOOP.

MAXI1=MAX2+I

245

XI=I
 VARI=2*I+1
 MAXI2=MAXR+I
 XI=I+2
 XM1=I+1

250

MAXI3=MAXP1+I
 MAXI4=MAX21+I
 I1=I+1
 I2=I+2
 I3=I+3

255

B(MAXI1)=DTWOZ*(U(2,I2)+U(2,I1))
 A(MAXI1,MAXI1-1)=DTWOZ
 A(MAXI1,MAXI1)=DTWOZ
 A(MAXI1,MAXI2)=-XMAXR
 A(MAXI1,MAXI2+1)=XMAXR*XM/XM1
 A(MAXI3,I1)=DOZM8
 A(MAXI3,I2)=-DOZM8

260

A(MAXI3,I3)=DOZM
 A(MAXI3+1,I1)=-DOZM

265

A(I,I)=-DIVZ
 TCOMMN=G1*T(2,I2)+CO2
 TALPHA=TCOMMN*(-CO3)
 TBETA=G2*TCOMMN*(-CO8)
 CARDS PA16-PA18 RE-DEFINE TALPHA AND TBETA FOR TYPE A.
 TALPHA=(T(2,I2)/GNUMBER+1.0)**(-CO1)
 TBETA=-(CO1/GNUMBER)*T(2,I2)/GNUMBER+1.0)**(-CO2)
 TALPHA=CO1+(CO2+CO3*T(2,I2))*T(2,I2)
 TBETA=CO2+2.3*CO3*T(2,I2)
 TYPES B AND C ALSO GIVEN.

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TONE=TBETA*(T(2,I3)-T(2,I1))
 TTWO=TBETA*(T(2,I2)-T(1,I2))*DIZSQ
 VONE=HALFM*V(2,I2)

275

ITHREE=TALPHA/XM1
 TFOUR=ITHREE*(VARI+2.0)
 TFIVE=ITHREE*VARI
 TSIX=TALPHA*DIZSQ
 A(I1,MAXI4)=MONE*(TFIVE-0.5*TONE)+VONE
 A(I1,MAXI4+2)=MONE*(TFOUR+0.5*TONE)-VONE
 A(MAXI3,MAXI2)=MONE*(TFIVE-TONE)+VONE
 A(MAXI3,MAXI2+2)=MONE*(TFOUR+TONE)-VONE
 UONE=DIVZ*U(2,I2)

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A(I1,MAXI1)=DIVRE*(TALPHA*MTWO+2.0*TTWO)-UCNE
VARIT=TALPHA*(DIZSQ-SOM2*(4.0+2.0/XM1**2))
A(MAXI3,MAXI3)=DIVRE*(VARIT+TTWO)-UCNE
BONE=V(2,I2)*(-U(2,I2)*DIVZ+DIVRE*(2.0*TSIX+TTWO))
290 BTWO=-DIVRE*(V(1,I2)*TSIX+
1 HALF*DELIZ*TTWO*(U(2,I3)-U(2,I1)))
B(MAXI3)=BONE+BTWO
UPORDN4=UP1+T(2,I2)*(UP2+UP3*T(2,I2))
BONE=UPORDN4-DIVZ*(P(2,I2)+U(2,I2)**2)
295 1-U(1,I2)*TALPHA*DIVRE3
BTWO=DIVRE2*U(2,I2)*(TSIX+TTWO)-
1 DIVRE4*TONE*(V(2,I2)-V(1,I2))
B(I1)=BONE+BTWO
115 CONTINUE
300 C END OF LOOP.
A(MAXR+2,1)=DOZH*7.0
A(MAX21,MAXR-3)=0.0
C TWO OTHER A VALUES.
MHALF=MAXR/2
305 DO 120 M=1,MHALF
M1=2*M-1
M2=MAX21+M1
A(MAX32,M2)=M1
120 CONTINUE
MHAL1=MHALF-1
310 DO 121 M=1,MHAL1
M1=MAX21+2*M
A(MAX32,M1)=M
121 CONTINUE
315 C SPECIFIES A VALUES, INTEGRAL CONTINUITY EQUATION.
IF(NITER-1)98,96,96
96 DO 95 M=1,MAX1
T(1,M+1)=X2(M)
MA=MAX1+M
320 T(2,M+1)=X2(MA)
95 CONTINUE
C FOR NITER=1 FOR ITERATION, ABOVE RETURNS TRUE T(1),S AND T(2),S TO
C CORRECT STORES, RELEASING X2 AS STORE FOR SOLUTION OF UVP MATRIX.
98 CALL MAT(MAX32,A,B,X2)
325 C CALLS FOR SOLUTION OF UVP EQUATIONS.
MAXF=MAXR-1
DO 100 M=1,MAXF
P(3,M+1)=X2(M)
MA=MAXR+M
330 V(3,M+1)=X2(MA)
MB=MA+MAXF
U(3,M+1)=X2(MB)
100 CONTINUE
P(3,MAXR+1)=X2(MAXR)
335 C DEFINES LEVEL 3 VALUES FROM SOLUTION IN STORE Y2.
U(3,1)=U(3,2)
V(3,1)=0.0
P(3,1)=P(3,2)
C DEFINES TUBE AXIS VALUES.
340 DO 501 I=1,MAX1
DO 501 J=1,MAX1
A(I,J)=0.0

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H03A

H04

H05


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501 FUNC2=DPRE*(4.0*SOM2-DISSQ) H06
C LOOP CLEARS A VALUES ( B,S ARE REPLACED). FUNC2 IS VARIABLE FOR
345 C T MATRIX.
DO 502 I=2,MAX1
I1=I+1
XI=1
XI1=I1
353 A(I,I-1)=FMSQ*(3.0-2.0*XI1)/XI-HALFM*V(3,I1) H13
502 CONTINUE
DO 505 I=1,MAX1
I1=I+1
XI=1
XI1=I1
355 B(I)=(U(3,I1)-2.0*FUNC1)*T(2,I1)+FUNC1*T(1,I1)/DELTZ
A(I,I)=U(3,I1)/DELTZ+FUNC2
A(I,I1)=HALFM*V(3,I1)+FMSQ*(1.0-2.0*XI1)/XI
505 FUNC3=HALFM*V(3,MAXR)+FMSQ*(1.0-2.0*XMAXR)/(XMAXR-1.0) H07
360 C SPECIFIES A AND B VALUES FOR T EQUATIONS. LEVEL 3 U,S AND V,S MAY H08
C BE USED. H09
A(1,1)=A(1,1)-FMSQ-HALFM*V(3,2) H10
C RECTIFIES A VALUE FOR INNER B.C. EQUATION. H11
DO 522 M=2,MAXR H12
XM=M H13
I1=1.0+T(3,M)*(CO5+CO6*T(3,M))/(CO4*GNUMBF) H14
A(MAX1,M-1)=(XM-1.0)*U(3,M)*I1 H15
522 CONTINUE
C DEFINES A VALUES FOR INTEGRAL ENERGY EQUATION.
MAXR2=MAXR-2
VSUM=0.0
DO 80 I=1,MAXR2
XI=I
I1=I+1
I2=I+2
370 QDISS2=QDISS*(T(3,I1)/GNUMBR+1.0)**(-CO1)
QDISS3=XI*(T(3,I1)/GNUMBR+1.0)**(-CO1)
Q1=((V(3,I2)-V(3,I1))*XMAXR)**2
Q2=(XMAXR*V(3,I1)/XI)**2
380 Q3=((U(3,I1)-U(2,I1))/DELTZ)**2
Q4=((V(3,I1)-V(2,I1))/DELTZ+XMAXR*(U(3,I2)-U(3,I1)))**2
VSUM=VSUM+QDISS3*(2.0*(Q1+Q2+Q3)+Q4)
B(I)=B(I)+QDISS2*(2.0*(Q1+Q2+Q3)+Q4)
C ADAPTS B VALUES FOR VISC. DISS. CARDS V69 AND V26FOR TYPE B
385 C VISCOSITY,IRRELEVANT IF QDISS=0.0.
C VSUM IS OVERALL ENERGY INPUT FOR VISC. DISS.
80 QDISS2=(T(3,MAXR)/GNUMBR+1.0)**(-CO1)
Q1=((V(3,MAXR)-V(2,MAXR))*XMAXR)**2
Q2=(XMAXR*V(3,MAXR))**2
390 Q3=((U(3,MAXR)-U(2,MAXR))/DELTZ)**2
Q4=((V(3,MAXR)-V(2,MAXR))/DELTZ+XMAXR*(-U(3,MAXR)))**2
VSUM=VSUM+QDISS3*(2.0*(Q1+Q2+Q3)+Q4)*(XMAXR-1.0)
C UPDATES VSUM FOR FLUID STATION NEAREST WALL.
GO TO(510,515),IFT00
395 C CHOICE OF WALL THERMAL B.C. TO FIND B VALUE FOR INTEGRAL ENERGY
C EQUATION.
510 TWEB=XMAXR*DELTZ*DPRE*RATIO/3.0
TBAR1=TBAR+TWEB*12.5*TWALL
TWEB1=TWEB*SQM2

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H15(22)

EB1

EB2

EB3

EB4

V20

V21

V22

V23

V24

V25

V26

V27

V28

V29

V30

V31

V14

V32

V33

V34

V35

V36

V37

WEB01

WEB02

WEB03

WEB05

400		A(MAX1,MAX1)=A(MAX1,MAX1)+24.0*TWEB1	WEB06
		A(MAX1,MAXR2)=A(MAX1,MAXR2)-18.0*TWEB1	WEB06
		A(MAX1,MAXR3)=A(MAX1,MAXR3)+8.0*TWEB1	WEB07
		A(MAX1,MAXR4)=A(MAX1,MAXR4)-1.5*TWEB1	WEB08
		GO TO 520	WEB09
405	C	RECTIFIES SOME A VALUES FOR CONSTANT WALL TEMP.	
	510	TBAR1=TBAR+2.0*HFLUX*XM*MAXR*DELTZ*OPRRE*RATIO	EB5
	C	THIS IS TBAR FOR CONSTANT HFLUX. TBAR1 IS USED BECAUSE TBAR	
	C	IS ONLY UPDATED ONCE PER AXIAL STEP.	
	520	CONTINUE	WEB10
410		IF(TBAR.GT.0.0)GO TO 81	EB6
		TBAR1=TBAR1+1.5*DELTZ*QDISS*VSUM/SQM2	EB10
		GO TO 82	EB11
	81	TBAR1=TBAR1+0.5*(VRATE+DELTZ*QDISS*VSUM/SQM2)	EB12
	C	UPDATES TBAR1 TO ACCOUNT FOR VISC. DISS.	
415	82	B(MAX1)=SQM2*TBAR1	EB7
	C	SPECIFIES B VALUE FOR INTEGRAL EQUATION.	
		CALL MAT(MAX1,A,B,X1)	EB8
	C	CALLS FOR SOLUTION OF T EQUATIONS.	
		DO 97 M=1,MAXF	3440A
420		T(3,M+1)=X1(M)	3360
		.	
	97	CONTINUE	3440B
		T(3,1)=T(3,2)	3460
	C	DEFINES LEVEL 3 T VALUES FROM SOLUTION IN STORE X1.	
		GO TO(181,182),IFTOO	WEB11
425	181	TBAR1=TBAR1+TWEB* (-24.0*T(3,MAXR)+16.0*T(3,MAX1)-8.0*T(3,MAXR2)	WEB12
		+1.5*T(3,MAXR3))	WEB14
	C	CORRECTS TBAR1 FOR CONSTANT TWALL, TO TRUE ENERGY BALANCE.	
	C	IN CARD WEB03 TBAR1 DOES NOT INCLUDE LEVEL 3 FLUID TEMPS.	
	182	WRITE(2,187)TBAR1	EB9
430	187	FORMAT(1H,6H TBAR=,F15.5)	2607
	C	WRITES TRUE TBAR AS TBAR1, WHICH MAY NOT COINCIDE FOR ITERATIVE	
	C	CHECK.	
		SUMT=0.0	3510
		DO130 M=2,MAXR	3520
435		XM=M	3530
		T1=1.0+T(3,M)*(C05+C06*T(3,M))/(C04*GNUMBR)	
		SUMT=SUMT+(XM-1.0)*U(3,M)*T(3,M)*T1	3540
	130	CONTINUE	3550
		TMEAN=SUMT*2.0/XMAXR**2	3570
440	C	CALCULATES TMEAN FROM LEVEL 3 U, T PROFILES.	
		XF1=(2.0*XM*MAXR)/RYNLOS	FF1
		XF2=((18.0*U(3,MAXR)-9.0*U(3,MAXF)+2.0*U(3,MAXR2))/6.0	FF3
	C	CALCULATES 3RD ORDER WALL VELOCITY GRADIENT.	
		IF(NITER-1)126,127,127	3590
445	126	ITALLY=ITALLY+1	2560
		ZBAR=ZBAR+DELTZ	2570
	127	WRITE(2,125)DELTZ,ZBAR	2580
	125	FORMAT(1H,7H DELTZ=,F10.6,6H ZBAR=,F10.6)	2590
	C	UPDATES ZBAR (ONCE PER AXIAL STEP). WRITES DELTZ, ZBAR.	
450		IF(ZBAR-ZBARX)116,116,122	3571
	116	XFRW1=XF1*XF2	FF1
		TALPHW=1.0	FF10
		GO TO 117	FF15
	C	FOR P.A.D., HEAT TRANSFER EFFECTS OMITTED. JUMP TO STATEMENT 117.	
455	122	CONTINUE	3572
		GO TO(124,123),IFTOO	3573


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124  XFRW1=XF1*TALPHW*XF2
      XNUX=(TBAR1-TBAR)/(DELTZ*DPRE*RATIO)
      GO TO 117
460  C   FOR CONSTANT TWALL, ONLY 4TH ORDER NU CALCULATED, TO BE
      C   CONSISTENT WITH ENERGY BALANCE EQUATION. XNUX IS NU NUMERATOR.
      C   JUMP TO STATEMENT 117.
123  TWALL=(2.0*HFLUX+4.0*T(3,MAXR)-T(3,MAXF))/3.0
      NUSSLT=XNUX/(TWALL-TMEAN)
465  WRITE(2,135)TWALL,NUSSLT
      3574
      2576
      3577
185  FORMAT(1H,7H TWALL=,F15.5,8H NUSSLT=,F10.5)
      3578
      C   CALCULATES AND WRITES 2ND ORDER TWALL AND NU FOR CONSTANT HFLUX.
      TWALL=(6.0*HFLUX+18.0*T(3,MAXR)-9.0*T(3,MAXF)+
470  12.0*T(3,MAXF-1))/11.0
      TALPHW=(G1*TWALL+CO2)**(-CO3)
      3578
      3580
      3581
      3582
      C   TALPHW=(TWALL/GNUMBR+1.0)**(-CO1)
      C   TALPHWYCO1=(CO2+CO3*TWALL)*TWALL
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132 WRITE(2,140) (ITALLY,TMEAN,TWALL,MUSSLT,(U(3,M),M=2,MAXR))
515 140 FORMAT(1H,7H TALLY=,I4,7H TMEAN=,F15.5,7H TWALL=,F15.5,8H MUSSLT=
1,F15.5/44H AXIAL VELOCITY,RADIAL VELOCITY,TEMPERATURE,
29HPRESSURE,/(12F10.5))
150 WRITE(2,150) (V(3,M),M=2,MAXR)
150 FORMAT(1H,12F10.7)
520 160 WRITE(2,160) (T(3,M),M=2,MAXR)
160 FORMAT(1H,12F10.4)
170 WRITE(2,170) (P(1,M),M=2,MAXP1)
170 FORMAT(1H,12F10.4)
C WRITES 4TH ORDER TWALL AND NU, AND U,V,T,P PROFILES.
525 IF(QDISS.GT.0.0)GO TO 192
ZB=ZBAR-ZBARX
IF(ZB.GT.0.0)GO TO 192
GO TO 188
192 IF(NITER-1)191,188,188
530 191 NITER=NITER+1
DO 89 M=1,MAX1
X2(M)=T(1,M+1)
MA=MAX1+M
X2(MA)=T(2,M+1)
535 T(1,M+1)=T(2,M+1)
T(2,M+1)=X1(M)
89 CONTINUE
GO TO 90
C IF NITER=0, STATEMENT 191 UPDATES IT, AND T,S ARE SHIFTED TO LET
540 U,V,P MATRIX USE SOLVED VALUES OF T. IF NITER=1, T,S ARE NOT
C SHIFTED, AND CARD H08 CALLED FOR. WITH P.A.O. ITERATION IS
C POINTLESS, UNLESS VISC. DISS. IS PRESENT.
188 NITER=0
TBAR=TBAR1
545 PMEAN=PMEAN1
C END OF ITERATION, AND ANALYSIS FOR AXIAL STEP. NITER=0, TBAR
C ASSUMES TBAR1, READY FOR NEXT STEP. PMEAN ASSUMES PMEAN1.
VRATE=TBAR1-TBAR
IF(ILAST-ITALLY)400,400,200
550 IF(L(K)-ITALLY)215,215,200
C STATEMENT 400 ENDS A RUN. STATEMENT 200 DOUBLES OR KEEPS DELTZ.
205 DO 210 M=1,MAXR
U(1,M)=U(2,M)
555 V(1,M)=V(2,M)
T(1,M)=T(2,M)
210 CONTINUE
GO TO 220
215 K=K+1
DELTZ=2.0*DELTZ
560 C IF DELTZ IS KEPT, NEW LEVEL 1 ASSUMES OLD LEVEL 2 DATA.
C IFDELTZ IS DOUBLED K AND DELTZ ARE UPDATED.
220 DO 300 M=1,MAXR
U(2,M)=U(3,M)
565 V(2,M)=V(3,M)
T(2,M)=T(3,M)
300 CONTINUE
DO 350 M=1,MAXP1
P(2,M)=P(3,M)
350 CONTINUE
570 C FOR ANY DELTZ, NEW LEVEL 2 ASSUMES OLD LEVEL 3 DATA.

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A2600
B2600
C2600
D2600
E2600
F2600
G2600
H2600
J2600
K2600

V36
PA01
PA02
PA03

T1
T2
T3
T4
T5
T6
T6A

H07

H08
H09
FF27

E014
2010
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		GO TO(60,370,60),ISUNUN	2790
		IF(ZBARX-ZBAR)60,7,60	2800
		NETT EFFECT TO RESTART NEXT STEP AT STATEMENT 60,	
		EXCEPT AT END OF P.A.D. WHEN STATEMENT 7 IS NEEDED.	
575	400	WRITE(2,430)	2900
	430	FORMAT(1H,12H END OF RUN.)	2850
		END OF RUN ROUTINE STARTS.	
		CALL ITIME(ISEC2)	2870
580		ISEC=ISEC2-ISEC1	2890
		WRITE(2,450)ISEC	2880
	450	FORMAT(1H,6H TIME=,I6,4HSEC.)	2840
		TIME READ. NETT TIME WRITTEN.	
		GO TO 1	
585		RETURN TO STATEMENT 1 FOR NEXT RUN.	
		END	2830
		SUBROUTINE MAT(N,A,Y,X)	3090
		GAUSSIAN ELIMINATION SUBROUTINE.	
		DIMENSION A(238,238),Y(238),X(238)	
		LEVEL 2,A	
5	5	M=N-1	3110
		DO 10 I=1,M	3120
		L=I+1	3130
		DO 10 J=L,N	3140
		IF(A(J,I).EQ.0.)GOTO10	3150
10	6	DO 8 K=L,N	3160
	8	A(J,K)=A(J,K)-A(I,K)*A(J,I)/A(I,I)	3170
		Y(J)=Y(J)-Y(I)*A(J,I)/A(I,I)	3180
	10	CONTINUE	3190
		X(N)=Y(N)/A(N,N)	3200
15		DO 30 I=1,N	3210
		K=N-I	3220
		L=K+1	3230
		DO 20 J=L,N	3240
20	20	Y(K)=Y(K)-X(J)*A(K,J)	3250
		X(K)=Y(K)/A(K,K)	3260
	30	CONTINUE	3270
		RETURN	3280
		END	3290
		SUBROUTINE ITIME(I)	
		TIME READING SUBROUTINE.	
		CALL SECOND(T)	
5		I = IFIX(T)	
		RETURN	
		END	

A P P E N D I X 3

Author's paper,
Salford Symposium on Internal Flows,
1971

only assumptions made are that C_p and k are constant, that viscous dissipation may be neglected, and that p is constant except in the buoyancy term of equation (2).

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0 \quad (1)$$

$$\rho \left(v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial p}{\partial z} \pm \rho g + \mu \left(\frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial r^2} \right) + 2 \frac{\partial \mu}{\partial z} \frac{\partial v_z}{\partial z} + \frac{\partial \mu}{\partial r} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \quad (2)$$

(where + or - ρg holds for downward or upward flow respectively)

$$\rho \left(v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} \right) = - \frac{\partial p}{\partial r} + \mu \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial z^2} \right) + 2 \frac{\partial \mu}{\partial r} \frac{\partial v_r}{\partial r} + \frac{\partial \mu}{\partial z} \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right) \quad (3)$$

$$v_r \frac{\partial T'}{\partial r} + v_z \frac{\partial T'}{\partial z} = \frac{k}{\rho C_p} \left(\frac{\partial^2 T'}{\partial z^2} + \frac{1}{r} \frac{\partial T'}{\partial r} + \frac{\partial^2 T'}{\partial r^2} \right) \quad (4)$$

The viscosity and density variations with temperature are taken to be:

$$\mu = \mu_A (C_1 + C_2 T' - C_3 T'^2) \quad (5)$$

$$\text{and } \rho = \rho_A (C_4 - C_5 T' + C_6 T'^2) \quad (6)$$

In addition, the integral continuity equation may also be used for a numerical solution, viz:

$$2. \text{ Boundary Conditions} \quad \pi R^2 U_0 = \int_0^R 2\pi r' v_z dr' \quad (7)$$

$$(i) \text{ Entry condition } v_z = U_0, v_r = 0, p = p_0, T' = T_0$$

$$(ii) \text{ Tube axis } v_r = 0, \frac{\partial v_z}{\partial r} = 0, \frac{\partial T'}{\partial r} = 0$$

$$(iii) \text{ Wall flow condition } v_z = 0, v_r = 0$$

$$(iv) \text{ Wall thermal condition, either (a) or (b)}$$

$$(a) \text{ Constant heat flux } Q = kA \left(\frac{\partial T'}{\partial r} \right)_R = \text{const.}$$

$$(b) \text{ Constant temperature } T' = T'_R = \text{const.}$$

3. Dimensionless Equations

The above equations may be made dimensionless, using the parameters as defined in the Nomenclature. Equations (1) - (7) become (8) - (13) respectively.

$$\frac{\partial V}{\partial r} + \frac{V}{r} + \frac{\partial U}{\partial z} = 0 \quad (8)$$

$$V \frac{\partial U}{\partial r} + U \frac{\partial U}{\partial z} = - \frac{\partial P}{\partial z} \pm \frac{1}{Re^2} [G.C_{04} + T(C_{05} + C_{06}T)] + \frac{1}{Re} \left\{ [C_{01} + T(C_{02} + C_{03}T)] \left(\frac{\partial^2 U}{\partial z^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial r^2} \right) + (C_{02} + 2C_{03}T) \left[2 \frac{\partial T}{\partial z} \frac{\partial U}{\partial z} + \frac{\partial T}{\partial r} \left(\frac{\partial V}{\partial z} + \frac{\partial U}{\partial r} \right) \right] \right\} \quad (9)$$

$$V \frac{\partial V}{\partial r} + U \frac{\partial V}{\partial z} = - \frac{\partial P}{\partial r} + \frac{1}{Re} \left\{ [C_{01} + T(C_{02} + C_{03}T)] \left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} - \frac{V}{r^2} + \frac{\partial^2 V}{\partial z^2} \right) + (C_{02} + 2C_{03}T) \left[2 \frac{\partial T}{\partial r} \frac{\partial V}{\partial r} + \frac{\partial T}{\partial z} \left(\frac{\partial V}{\partial z} + \frac{\partial U}{\partial r} \right) \right] \right\} \quad (10)$$

$$V \frac{\partial T}{\partial r} + U \frac{\partial T}{\partial z} = \frac{1}{Pr.Re} \left(\frac{\partial^2 T}{\partial z^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \right) \quad (11)$$

$$\mu = \mu_A (C_{01} + C_{02}T + C_{03}T^2) \quad (12)$$

$$\rho = \rho_A \left(C_{04} + \frac{C_{05}T}{G} + \frac{C_{06}T^2}{G} \right) \quad (13)$$

$$\frac{1}{2} = \int_0^1 U.r.dr \quad (14)$$

The boundary conditions become:

- (i) $U = 1, V = 0, P = P_0, T = T_0$
(ii) $V = 0, \frac{\partial U}{\partial r} = 0, \frac{\partial T}{\partial r} = 0$
(iii) $U = 0, V = 0$
(iv) (a) $Q = kA \left(\frac{r^2}{R^4 C_r g} \right) \cdot \left(\frac{\partial T}{\partial r} \right)$
(b) $T = T_R$

4. Finite Difference Equations

The above equations may be re-expressed using standard finite difference techniques, as applied by Lawrence and Chato (2). The dimensionless radius 'r' is re-expressed as $r = \frac{m-1}{M}$, for $m = 1, 2 \dots M+1$ i.e. for $m = 1, r = 0$ (axis), and for $m = M+1, r = 1$ (wall).

This gives a rectangular, axial and radial, mesh, with suffices 'n' and 'm' respectively. Equations (8) - (11) and (14) become (15) - (19) respectively, written in linear form with suffix 'n+1' representing the unknown axial position to be solved. The (known) constants relevant to each variable are in square brackets, as are the 'known' right hand sides of equations.

$$\left[\frac{2M_m \Delta Z}{m-1} \right] V_{n+1,m} - [2\Delta Z M] V_{n+1,m-1} + [1] U_{n+1,m} + [1] U_{n+1,m-1} = [U_{n,m} + U_{n,m-1}] \quad (15)$$

$$\begin{aligned} & \left[\frac{M^2}{2Re} \left\{ \frac{T_A(2m-1)}{m-1} + \frac{T_B}{2} (T_{n,m+1} - T_{n,m-1}) \right\} - \frac{M}{2} V_{n,m} \right] U_{n+1,m+1} + \left[\frac{1}{Re} \left\{ T_A \left(\frac{1}{\Delta Z^2} - 2M^2 \right) + \frac{2T_B}{\Delta Z^2} (T_{n,m} - T_{n-1,m}) \right\} \right. \\ & \left. - \frac{U_{n,m}}{\Delta Z} \right] U_{n+1,m} + \left[\frac{M^2}{2Re} \left\{ T_A \frac{(2m-3)}{(m-1)} - \frac{T_B}{2} (T_{n,m+1} - T_{n,m-1}) \right\} + \frac{M}{2} V_{n,m} \right] U_{n+1,m-1} - \frac{P_{n+1,m}}{[\Delta Z]} = \\ & \left[-\frac{1}{Re^2} \{ G.C_{04} + T_{n,m} (C_{05} + C_{06} T_{n,m}) \} - \frac{P_{n,m}}{\Delta Z} - \frac{U_{n,m}^2}{\Delta Z} + \frac{2U_{n,m}}{\Delta Z^2 Re} [T_A + T_B (T_{n,m} - T_{n-1,m})] \right. \\ & \left. - \frac{U_{n-1,m}}{\Delta Z^2 Re} T_A - \frac{M T_B}{2 Re \Delta Z} (T_{n,m+1} - T_{n,m-1}) (V_{n,m} - V_{n-1,m}) \right] \quad (16) \end{aligned}$$

$$\begin{aligned} & \left[\frac{M^2}{2Re} \left\{ \frac{T_A(2m-1)}{(m-1)} + T_B (T_{n,m+1} - T_{n,m-1}) \right\} - \frac{M}{2} V_{n,m} \right] V_{n+1,m+1} + \left[\frac{1}{Re} \left\{ T_A \left(\frac{1}{\Delta Z^2} - \frac{M^2}{(m-1)^2} (2m^2 - 4m + 3) \right) \right. \right. \\ & \left. \left. + \frac{T_B}{\Delta Z^2} (T_{n,m} - T_{n-1,m}) \right\} - \frac{U_{n,m}}{\Delta Z} \right] V_{n+1,m} + \left[\frac{M^2}{2Re} \left\{ T_A \frac{(2m-3)}{(m-1)} - T_B (T_{n,m+1} - T_{n,m-1}) \right\} + \frac{M}{2} V_{n,m} \right] V_{n+1,m-1} \\ & + \left[\frac{M}{12} \right] P_{n+1,m+2} - \left[\frac{2M}{3} \right] P_{n+1,m+1} + \left[\frac{2M}{3} \right] P_{n+1,m-1} - \left[\frac{M}{12} \right] P_{n+1,m-2} = \left[-\frac{U_{n,m} V_{n,m}}{\Delta Z} + \frac{2V_{n,m} T_A}{Re \Delta Z^2} \right. \\ & \left. + \frac{V_{n-1,m}}{\Delta Z^2 Re} \left\{ \frac{T_B}{2} (T_{n,m} - T_{n-1,m}) - T_A \right\} - \frac{M T_B}{2 Re \Delta Z} (T_{n,m} - T_{n-1,m}) (U_{n,m+1} - U_{n,m-1}) \right] \quad (17) \end{aligned}$$

$$T_A = [C_{01} + T_{n,m} (C_{02} + C_{03} T_{n,m})] \quad T_B = (C_{02} + 2C_{03} T_{n,m})$$

$$\begin{aligned} & \left[\frac{M}{2} V_{n,m} + \frac{M^2(1-2m)}{2Pr Re (m-1)} \right] T_{n+1,m+1} + \left[\frac{U_{n,m}}{\Delta Z} + \frac{1}{Pr Re} \left(2M^2 - \frac{1}{\Delta Z^2} \right) \right] T_{n+1,m} \\ & + \left[\frac{M^2(3-2m)}{2Pr Re (m-1)} - \frac{M}{2} V_{n,m} \right] T_{n+1,m-1} = \left[\left\{ U_{n,m} - \frac{2}{Pr Re \Delta Z} \right\} \frac{T_{n,m}}{\Delta Z} + \frac{T_{n-1,m}}{\Delta Z^2 Pr Re} \right] \quad (18) \end{aligned}$$

$$\frac{3M^2}{8} = \sum_{m=1}^{\frac{M}{2}} (2m-1) U_{n+1,2m} + \sum_{m=1}^{\frac{M}{2}-1} m U_{n+1,2m+1} \quad (19)$$

Equation (19) is obtained by applying Simpson's Rule to the integral continuity equation.

The boundary conditions become:

- (i) $U_{1,m} = 1, V_{1,m} = 0, P_{1,m} = P_0, T_{1,m} = 0$
- (ii) $U_{n,1} = U_{n,2}, V_{n,1} = 0, P_{n,1} = P_{n,2}, T_{n,1} = T_{n,2}$
- (iii) $U_{n,M+1} = 0, V_{n,M+1} = 0$
- (iv) (a) $T_{n,M+1} = T_{n,M} + \frac{QR^4 C_5 g}{A z^2 M h}$
 (b) $T_{n,M+1} = T_R$

COMPUTER PROGRAM

The computer program was written in Fortran IV, some of the techniques used by Lawrence and Chato (2) being applicable, and the work was carried out on the City University's ICT 1900 Computer. For details of the program see Collins (3). The program was in the form of 64 character punched cards; as will subsequently appear, this gave a high degree of flexibility in making program changes.

The basic method was to solve for one axial position at a time. Equation (18) was written for each radial position, giving $M-1$ equations for the same number of unknowns, viz. the temperature profile. This was solved by a standard matrix inversion technique (Gauss elimination method - McCormack and Salvadori (4)). Then equations (15), (16) and (17) were similarly written and solved, as $3M-2$ equations (including equation (19)) for the unknown U, V and P values. The next axial position was tackled, and use made of the above solved profiles in calculating the fresh set of coefficients.

To eliminate writing errors etc. more easily, as much as possible of the program was written in independent Subroutines (four in all). The two matrix inversions, for example, were carried out by a single Subroutine. These were checked as they were completed against simple Master programs, written for the purpose. By this means, there were fewer difficulties encountered when the complete program was tested.

The presence of second derivatives of velocity and temperature axially, meant that there were three axial positions involved, 'n-1', 'n' (both known) and 'n+1' (to be solved). After solving one step, the 'old' values at 'n' and 'n+1' became the new values for 'n-1' and 'n' respectively. When axial changes in velocities etc. became small, it was advantageous to increase the axial step ΔZ . In this program ΔZ may be conveniently increased by doubling by causing old 'n-1' and 'n+1' values to become new 'n-1' and 'n' values respectively. (The old 'n' values are redundant). This doubling may be repeated any number of times.

VELOCITY RESULTS

1. Introduction

The main output of the program for the adiabatic case was the series of developing velocity profiles. These were obtained for a range of Reynolds Numbers between 300 and 2700, and, as expected, found to be identical when plotted as in Fig. 1.

2. Nikuradse's Data

The best-known experimental data is that of Nikuradse given by Prandtl and Tietjens (5), and this is also shown on Fig. 1. The results are virtually co-incident at the upstream end (about half-way along the total entrance length) but increasingly divergent towards the entry.

A great hindrance to comment occurs here, as Prandtl and Tietjens give no experimental details, so a careful search was made for the published or unpublished report of Nikuradse. Eventually, from a private communication (Nikuradse (6)) it transpired that no such report existed, and that the original research material had been lost.

3. Effect of Program Changes

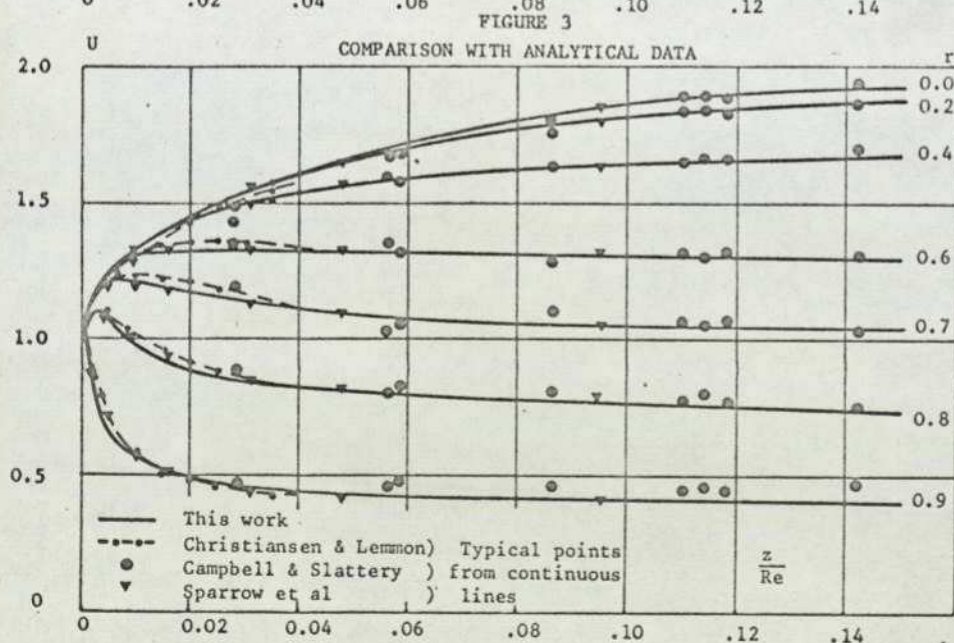
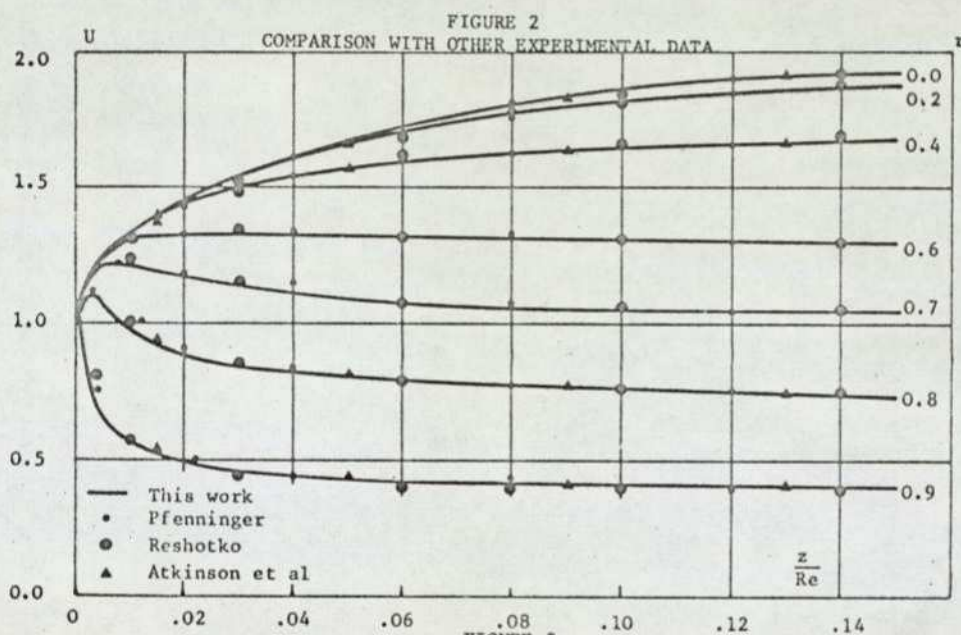
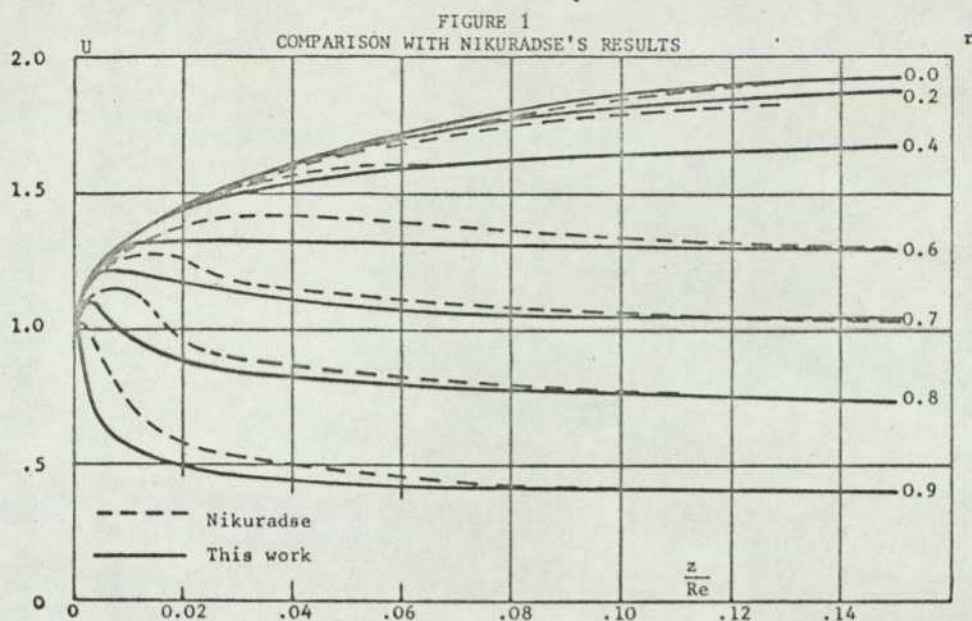
In view of the divergences mentioned, it was decided to investigate whether the program results were reliable near the tube entry, and a considerable effort was made in this direction.

Consideration was first given to the integral continuity equation, which is replaced by a Simpson's Rule summation. The accuracy of this approximation may be seen by applying equation (19) to the uniform velocity profile at entry. For $M = 30$, both sides of the equation give the value 337.5. It was found, by arbitrarily reducing the $\frac{3M^2}{8}$ to 330.0, that somewhat better agreement was achieved near entry, to the

detriment of agreement further downstream. To complete the study, the Simpson's Rule was then replaced entirely with a better approximation, viz. the closed 6-interval Newton-Coates rule of reference (7). This resulted in an almost imperceptible improvement, and it was concluded that the Simpson's Rule (which takes less storage space) was sufficiently correct.

The next attempt was to increase accuracy by decreasing the initial value of ΔZ , the axial distance increment. This is considered later in more detail, but for the present purposes was unhelpful.

One of the variables in the $3M-2$ size matrix was P_{M+1} , the wall pressure. If this were eliminated as a variable, the number of equations required decreases by one, and two alternative alterations of the matrix were made which took advantage of this.



In the first alteration, the radial momentum equation (17) for $m = 2$ was dropped. This gave increasingly meaningless results as the number of axial steps grew. It was felt that this was due to the omission of what was, in effect, a boundary point.

In the second alteration, the integral continuity equation (19) was dropped. This gave velocities for the first two steps which were positive but less than the entrance velocity, and from then on, a large proportion of negative velocities, or impossibly high positive velocities. The conclusion from this was that the equation (19) was essential as an overall bound to the solution.

The last possible alteration was to the finite-difference approximation to the radial pressure gradient $\frac{\partial p}{\partial r}$, written as $\frac{p_{m+1} - p_{m-1}}{(2/M)}$ in equation

(17). Alternative expressions covering 4 or 3 radial-intervals (instead of 2) were used where possible. Again, this had virtually no effect on the solution but equation (17) as written incorporates the changes.

The net result of each change was negative. It does, however, serve to illustrate the flexibility of the program and the finite-difference technique, and the ease of changes using punched cards.

4. Other Experimental Data

The results in Fig. 1 cannot be reconciled, therefore, and in fact the reliability of Nikuradse's data has been questioned by, for example, Shapiro (Discussion Appendix, Campbell and Slattery (8)). The present investigation appears to confirm these reservations. More recent experimental work has been carried out, by Reshotko (9), Pfenninger (10), and Atkinson et al (11), and results are plotted in Fig. 2, being directly comparable with Fig. 1. Pfenninger's results are given, like Nikuradse's, as a set of smooth curves. Over the region of interest, Reshotko's data differs somewhat from the other results. Atkinson suggests that this may be due to the difficulty of determining the exact entry position. The maximum difference between the computer results and those of Refs. (10) and (11) is around 4%, below z/Re of 0.03, reducing to about 1½% at 0.095.

Consideration of all available experimental velocity data, therefore, leads to the conclusion that the program results are reliable, and that the Nikuradse curves close to entry should be treated with caution.

5. Analytical Results

A large number of analytical treatments of this problem exists, and no attempt will be made to duplicate the discussions which may be found in the following selection of more recent papers: Campbell and Slattery (8), Sparrow et al (12), Hornbeck (13), Christiansen and Lemmon (14), Schmidt and Zeldin (15). The first two pages are modifications of Schiller's (16) integral technique and Langhaar's (17) linearization solution respectively, and the others use finite difference methods. Fig. 3 gives a comparison of the present work with that of Refs. (8), (12) and (14); the agreement is close.

As a general comment from Figures 2 and 3, it appears that between z/Re of 0.01 and 0.04, the present work slightly overestimates the velocities near the duct centre, and hence underestimates them closer to the wall (but not at $r = 0.9$). This implies that the velocity profile should be somewhat flatter than in fact is calculated. Apart from this, agreement is very favourable.

EFFECT OF STEP SIZE ALTERATION

Work was carried out to investigate the effect upon accuracy of changing both the radial and axial distance increments. For greatest accuracy in principle, each is made as small as possible, within the limits of computer storage and time. For economy, the reverse applies.

1. Radial Increment

A convenient maximum for the number of radial positions (M) was 30, and this required 21952 storage places for a typical program (Case I of Table 1). For adiabatic work, temperature calculations were unnecessary, and it was a straight forward matter to remove or change the card input relating to these, again, incidentally, demonstrating the ease of programming changes. This, with all input unchanged, reduced the storage required and time taken by 2% and 3% respectively (Case II). Then cases were run with values of M down to 12, and otherwise the same input. The strong dependence of computer size and time upon M was because the latter determined the size of matrix, which in turn dominated the program.

Case	M	Time (secs)	Storage
I *	30	745	21952
II	30	722	21568
III	24	429	15680
IV	20	354	12416
V	16	146	9792
VI	12	72	7744

TABLE 1 Alteration of Radial Increment

* With temperature matrix in

The effect upon accuracy is small, and not discernible on a graph. Comparing Cases 3 and 6, for instance, the maximum velocity differences are about 2% near the tube wall and about 1% in the centre. The latter is partly due to the error inherent in assuming the central velocity to be the same as that at the first radial point.

2. Axial Increment

Effort here was concentrated on finding the lowest possible initial value of the axial increment Δz , since velocity changes were greatest at extreme entry. Using a constant value of M of 24, a range of Δz was tested, including a change in Reynolds Number as specified in Table 2.

Case	Initial Δz	Reynolds No.
VII	2.0	900.0
VIII	1.5	900.0
IX	1.0	900.0
X	0.75	900.0
XI	0.5	900.0
XII	1.0	450.0
XIII	0.25	450.0

TABLE 2 Alteration of Axial Increment

In the standard case 7, the velocity profiles decrease very slightly (from the central value), from the entrance. With case 8, some of the initial profiles increase from the centre. This is accentuated, until with case 11, the centre line values begin to oscillate from step to step, instead of continuously increasing. As this oscillation is up to about 9% in velocity, it

implies a lower limit of Δz for accuracy. It should, however, be noted that these effects 'die out' as the axial distance increases, and even case 11 results in the same developed profile (at the same position) as case 7 (maximum difference under 0.1%).

The possibility of this effect being similar for proportionately different Reynolds Numbers and Δz was tested, because the normal profile sets are identical. Cases 12 and 13 were run and compared with 7 and 11 respectively. In fact, case 12 was somewhat worse than 7, and 13 gave meaningless results along the whole tube.

This instability was not altogether unexpected, as it is a major factor in finite difference equations for transient problems (see Dusinberre (18)).

PRESSURE RESULTS

1. Radial Variation

The radial variation of pressure is rather weak, as might be expected, and reduces to below 0.5% of the centre-line value by $z/Re = 0.005$.

It has already been mentioned that varying Δz affects extreme entry velocity profiles; the effects on pressure profiles are just as marked, and it is less easy to draw correct conclusions. Irrespective of Reynolds Number and Δz the first profile is always a descending one from the centre-line value. The sharpness of reduction increases as Δz is reduced. Provided the instability condition has not been reached, the profile changes (at the next step) to an increasing one, which quickly dies out, as has been mentioned. If, on the other hand, instability exists the oscillation of the velocity profiles also gives alternating ascending and descending pressure profiles. Because of this, the ascending profiles are felt to be the reliable ones, and a typical example is shown in Fig. 4.

A concluding point is whether the profiles depend on Reynolds Number. Within the limits of the technique and the range of Re investigated (300 - 2700) no significant trend could be discerned. Commenting on Schmidt and Zeldin's results (15) their trends may be due to the relatively heavy differences in pressure profile at entry itself, as compared with uniform profiles used in this work.

2. Axial Variation

The axial variation of pressure may be expressed in terms of K , the pressure drop coefficient, where:

$$\frac{P - P_0}{\frac{1}{2}\rho U_0^2} = f z + K$$

f is the laminar flow friction factor, hence K represents the additional pressure loss due to entry effects. Non-dimensionally this gives:

$$2P = \frac{16z}{Re} + K$$

Values of K are plotted on Fig. 5 as a continuous line. No discernible trend with Reynolds Number was apparent. As with the velocity comparison, a selection of the more recent data is shown. For fuller tabulations, see Refs. (8), (14) or (15), which indicate that the value obtained is partly dependent on the mode of theoretical treatment. Apart from Sparrow et al (linearization of momentum equation), the comparison is with other numerical solutions.

When the flow is fully-developed, K reaches an asymptotic value, e.g. the point data of Hornbeck, and Christiansen and Lemmon. The latter, by including some radial momentum terms (in the equivalent of our equation (2)) obtained the value of K shown, as opposed to a value of 1.015 obtained when omitting the terms. It is therefore to be expected that the pressure loss could increase with a still more comprehensive solution.

The range of Schmidt and Zeldin's results is shown as data (3), at a particular value of $\frac{z}{Re}$. The

K values overlap those of this work, lending credence to the view expressed above. In addition, however, one or two reservations must be stated. Firstly, the fully-developed length appears to be 0.40 rather than about 0.25, and secondly a definite trend with Reynolds Number is obtained. The latter difference could be due to the same cause as that discussed for the radial variation.

The overall conclusion is, therefore, that taking all radial terms into account increases the pressure loss. Finally, it should be noted that since K only accounts for entry losses, percentage differences reduce sharply when converted to total loss (e.g. for Hornbeck, -6.3% in K , -2% in pressure loss).

TEMPERATURE EFFECTS

A relevant extension of the adiabatic results was to investigate the effect of temperature on the development length (i.e. where the centre-line velocity reaches 99% of its parabolic value). Four cases were run (using upward flow of air) as described in Table 2. Air properties were taken from Ref. 20. (The contents of the Appendix should be noted.) With a constant wall temperature, the heat transfer to the fluid is larger at entry, but decreases as the fluid temperature rises. In these cases, however, the overall heating effect is small and only tends to deter the normal developing process. This it does quite significantly. Table 3 shows, as

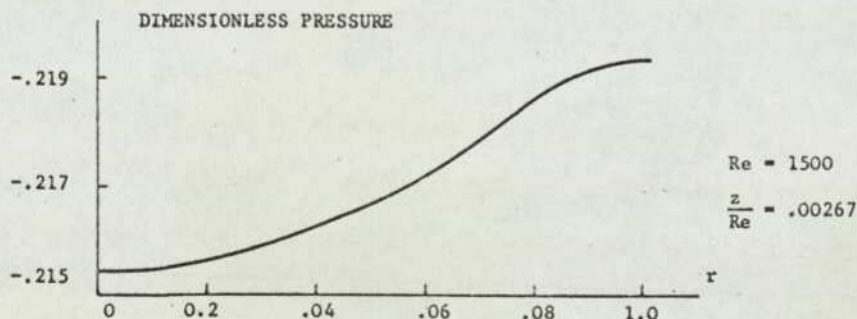


FIGURE 4
PRESSURE PROFILE

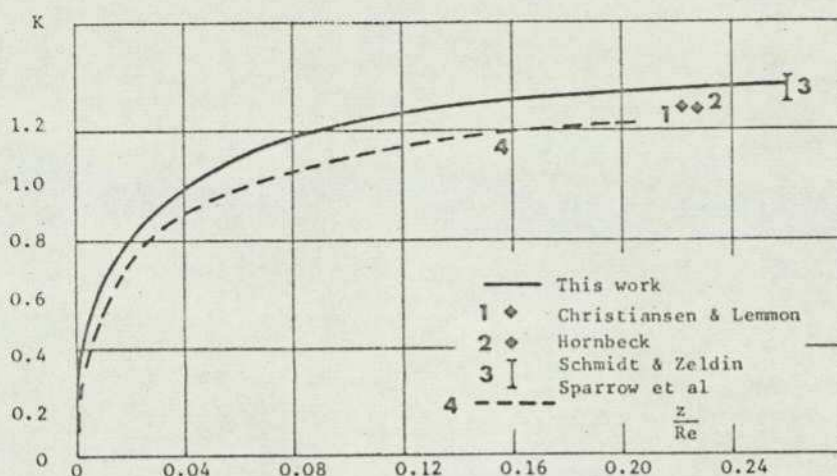


FIGURE 5
PRESSURE DROP COEFFICIENT

might be expected, that the effect increases with increasing temperature difference and decreasing Reynolds Number.

Case	Reynolds Number	Temp. Diff. °C	Development Length $\frac{z}{Re}$
1	1500	2.0	0.356
2	1500	5.0	0.413
3	300	2.0	0.648
4	300	5.0	0.705

TABLE 3 Temperature Effects

FUTURE WORK

Apart from the main purpose of the program, in investigating heated and cooled mixed convection in upward and downward flows, the following developments have considerable potential, and emphasise the flexibility of the approach.

It should be feasible, by including appropriate eddy and thermal diffusivity values (c.f. Ref. 19) to adapt the program to deal with turbulent flows.

Secondly, the cylindrical co-ordinate flow field is the same as that for a vertical annulus, and with only a change in the inner boundary conditions, comprehensive results for this important geometry should be equally obtainable.

CONCLUSIONS

A finite-difference approximation to the complete set of partial differential equations for internal cylindrical pipe flow has been compiled as a digital computer program written in Fortran IV.

The adiabatic velocity profiles have been compared with the experimental data of Nikuradse, but give closer agreement with more recent data near the pipe entry.

A study has been made of the effect upon accuracy of reducing both the radial step size from the maximum obtainable, and the axial distance increment. A lower limit to the latter is reached when instability occurs.

The case has been studied of the pipe-wall being somewhat higher in temperature than the entering fluid.

The flexibility of the program has been demonstrated, and its further potential discussed.

ACKNOWLEDGMENTS

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Mr. B. Girling (Mathematics Department) during the writing and subsequent use of the program.

Mr. L.D. Will (Information Section, Library) for the attempt to find source material for Nikuradse's data.

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APPENDIX

Further work has resulted in two improvements to the treatment of the equations; they only affect non-adiabatic results, and have been incorporated to give the results summarised in Table 3.

Firstly, density variation with temperature has been allowed for in the integral continuity equation. The physical meaning is that density differences are large along a finite tube length compared with differences radially, and axially from step to step.

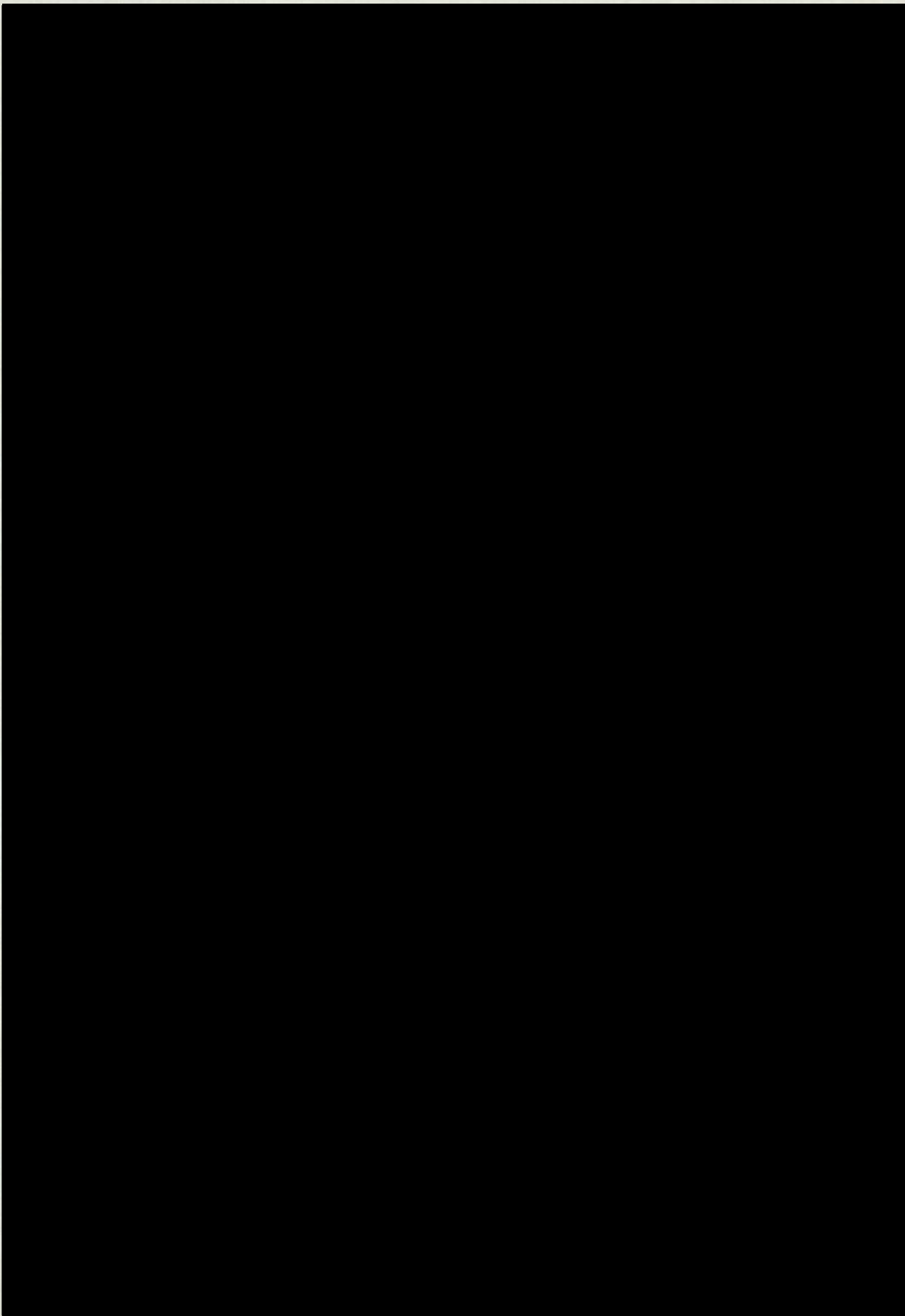
Equations (7), (14) and (19) become respectively:

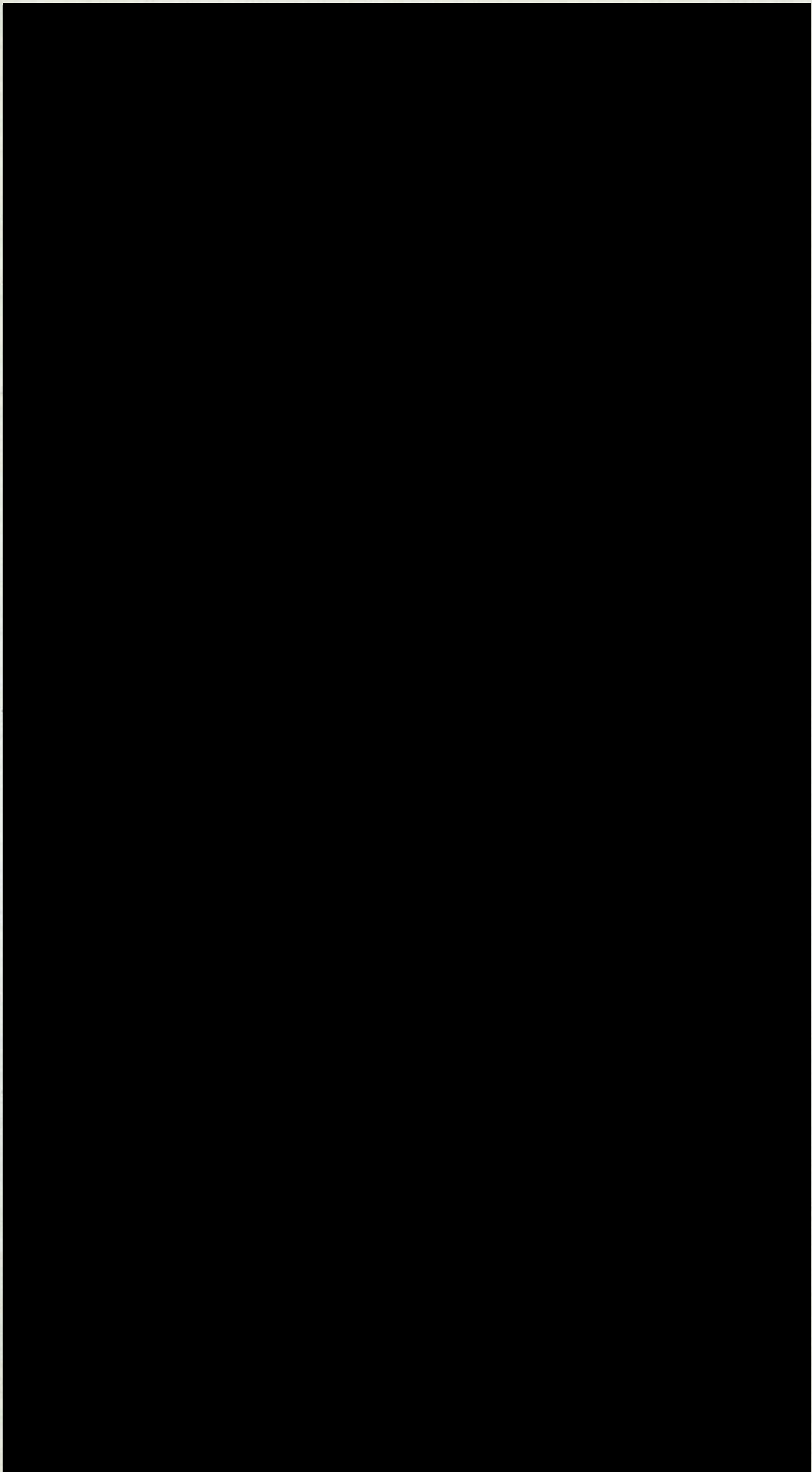
$$\pi R^2 \rho_0 U_0 = \int_0^R 2\pi r' \rho \cdot v_z \cdot dr' \quad (7A)$$

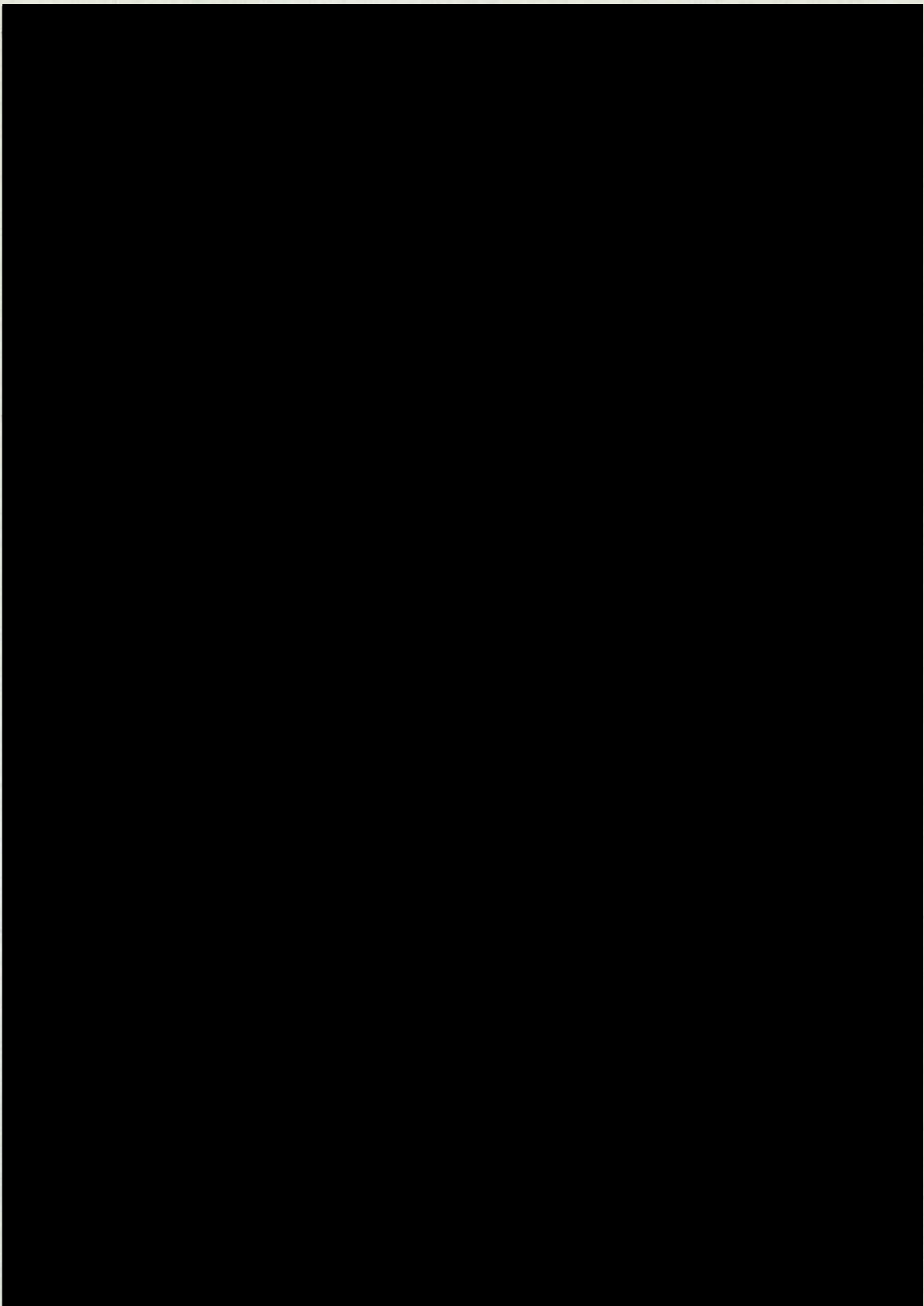
$$\frac{C_{04}}{2} = \int_0^1 U \cdot r' \left\{ C_{04} + \frac{C_{05} T}{G} + \frac{C_{06} T^2}{G} \right\} \cdot dr' \quad (14A)$$

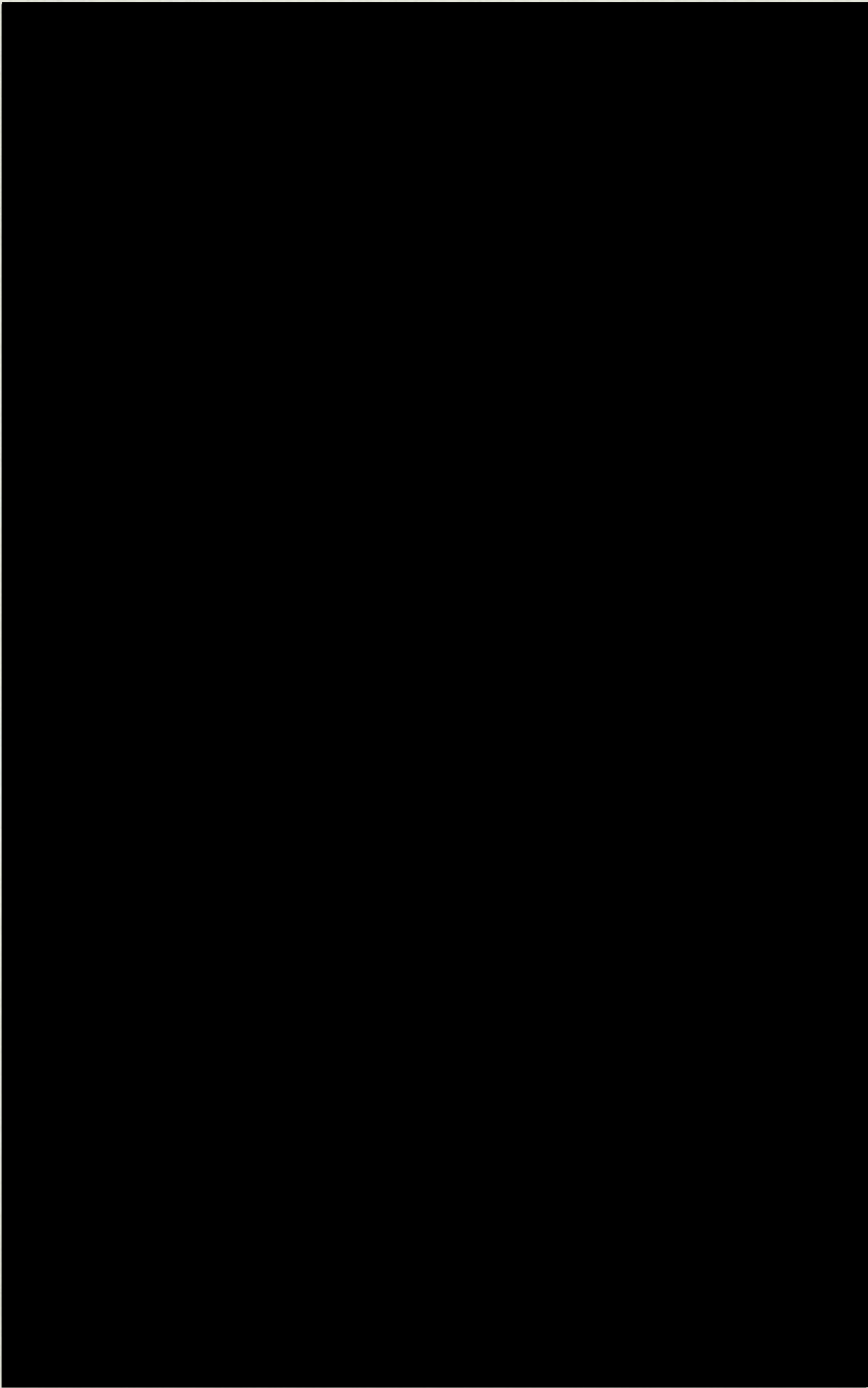
$$\begin{aligned} \frac{3M^2 C_{04}}{8} = & \sum_{m=1}^{\frac{M}{2}} \left[(2m-1) U_{n+1,2m} \left\{ C_{04} + \frac{C_{05} T_{n+1,2m}}{G} + \frac{C_{06} T_{n+1,2m}^2}{G} \right\} \right. \\ & \left. + \sum_{m=1}^{\frac{M}{2}-1} \left[m \cdot U_{n+1,2m+1} \cdot \left\{ C_{04} + \frac{C_{05} T_{n+1,2m+1}}{G} + \frac{C_{06} T_{n+1,2m+1}^2}{G} \right\} \right] \right] \quad (19A) \end{aligned}$$

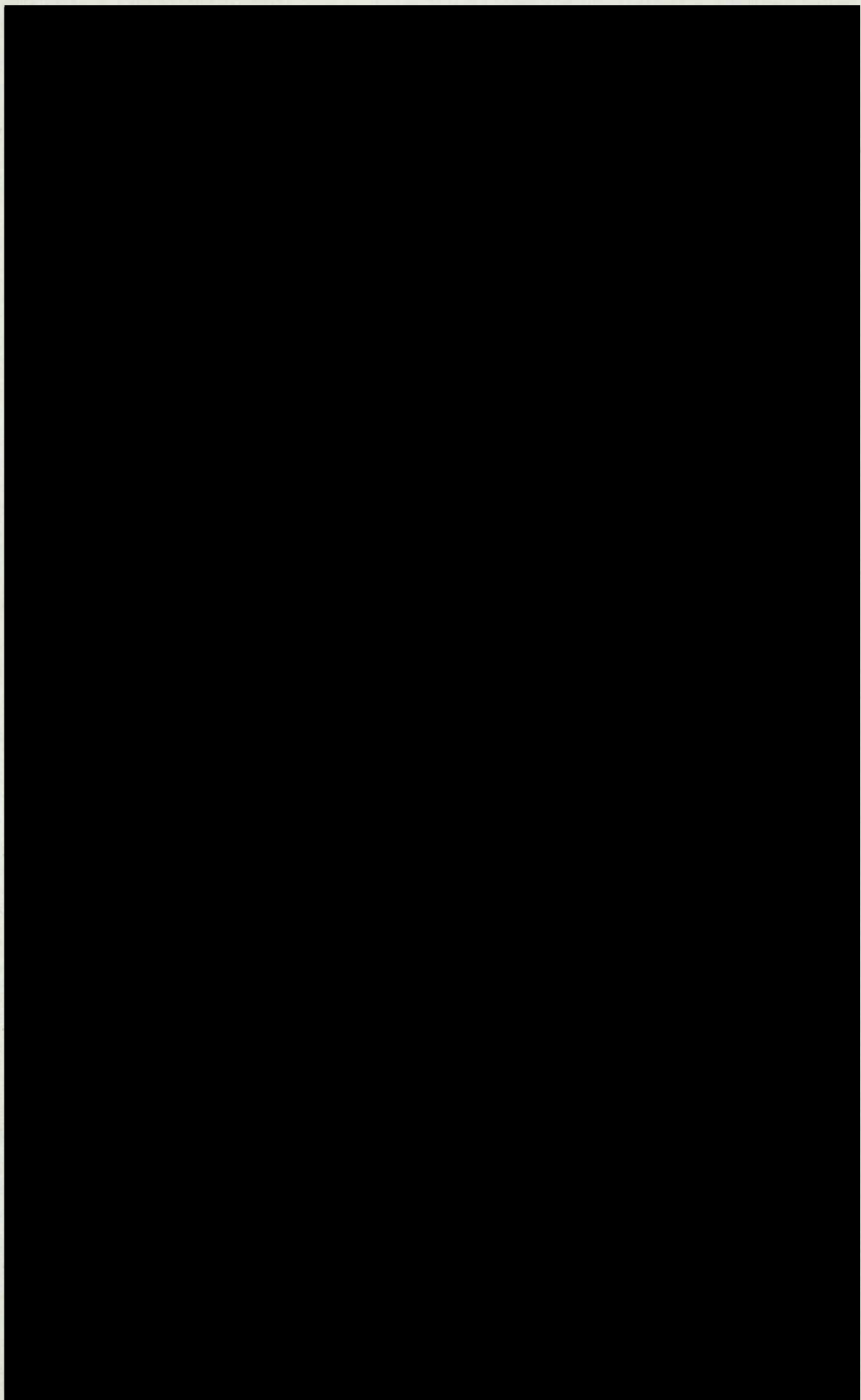
The second improvement was to use the output from the temperature matrix inversion as input to the U,V,P matrix. This removes the linearisation approximation due to temperature in equations (15) - (17). For example, T_n may be replaced by T_{n+1} , and T_{n-1} by T_n , respectively. Equation (19A) includes this change.







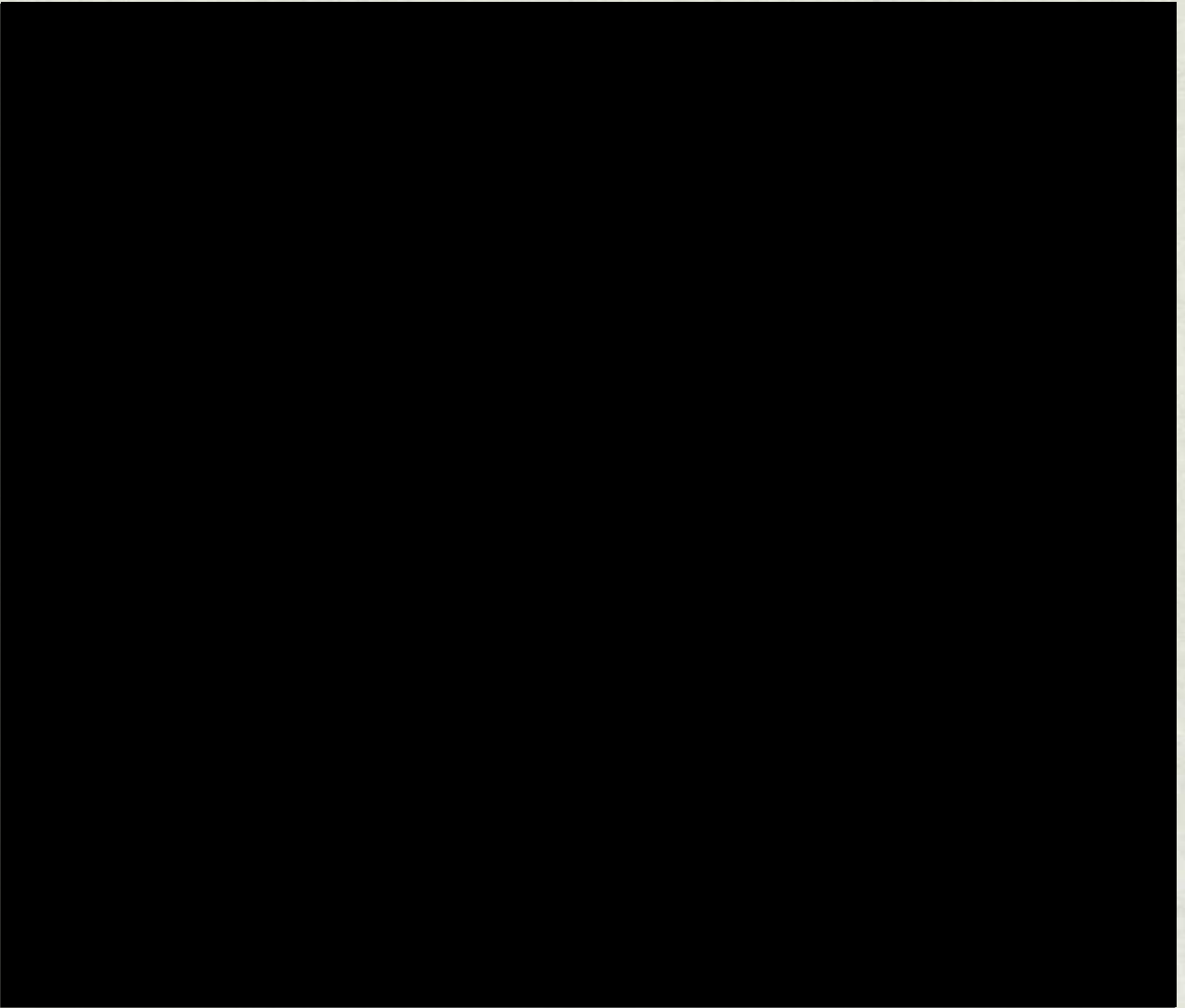




A P P E N D I X 4

Letter from Professor J. Nikuradse,

1970



A P P E N D I X 5

Author's contribution to written discussion of two papers,

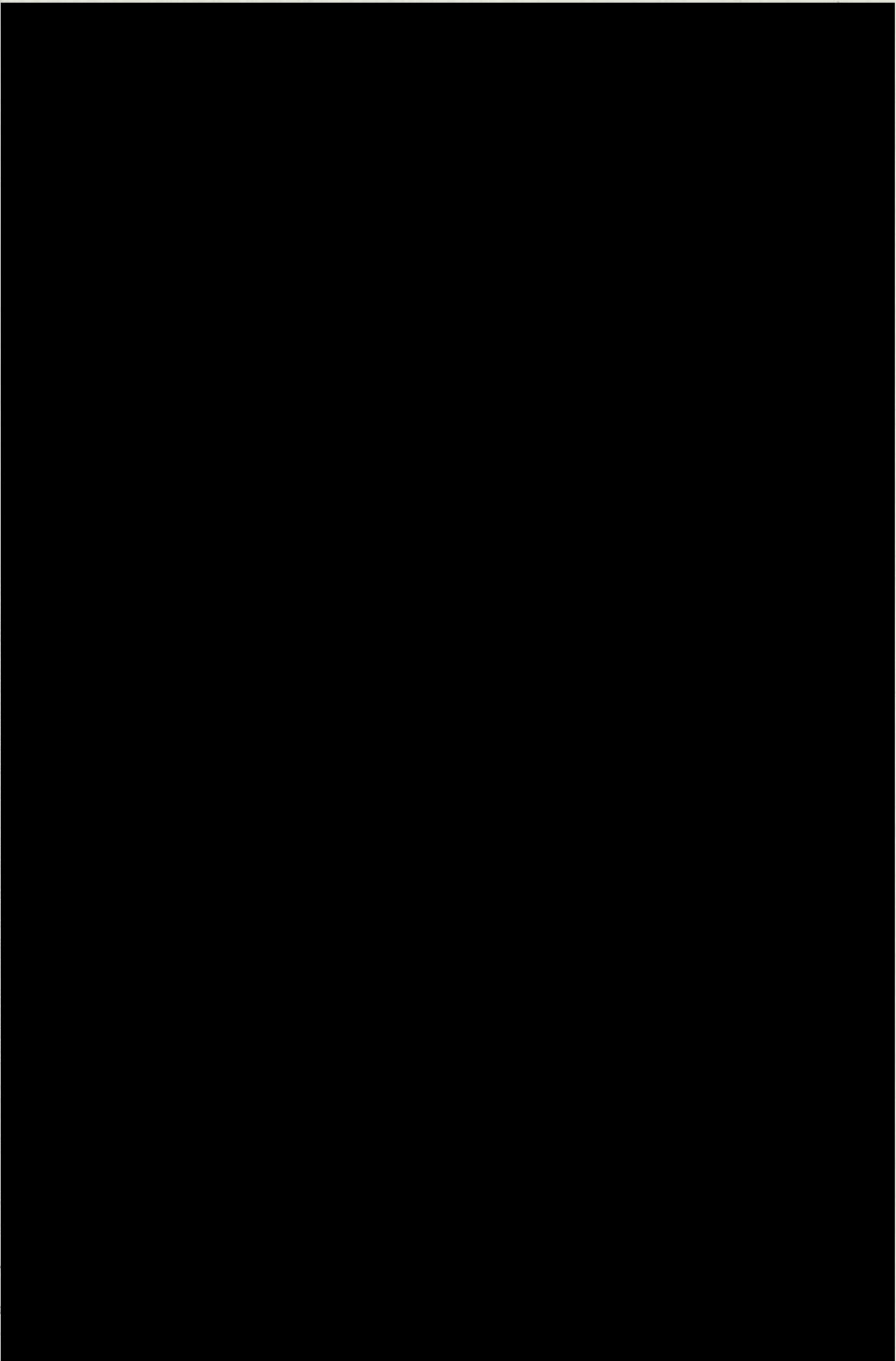
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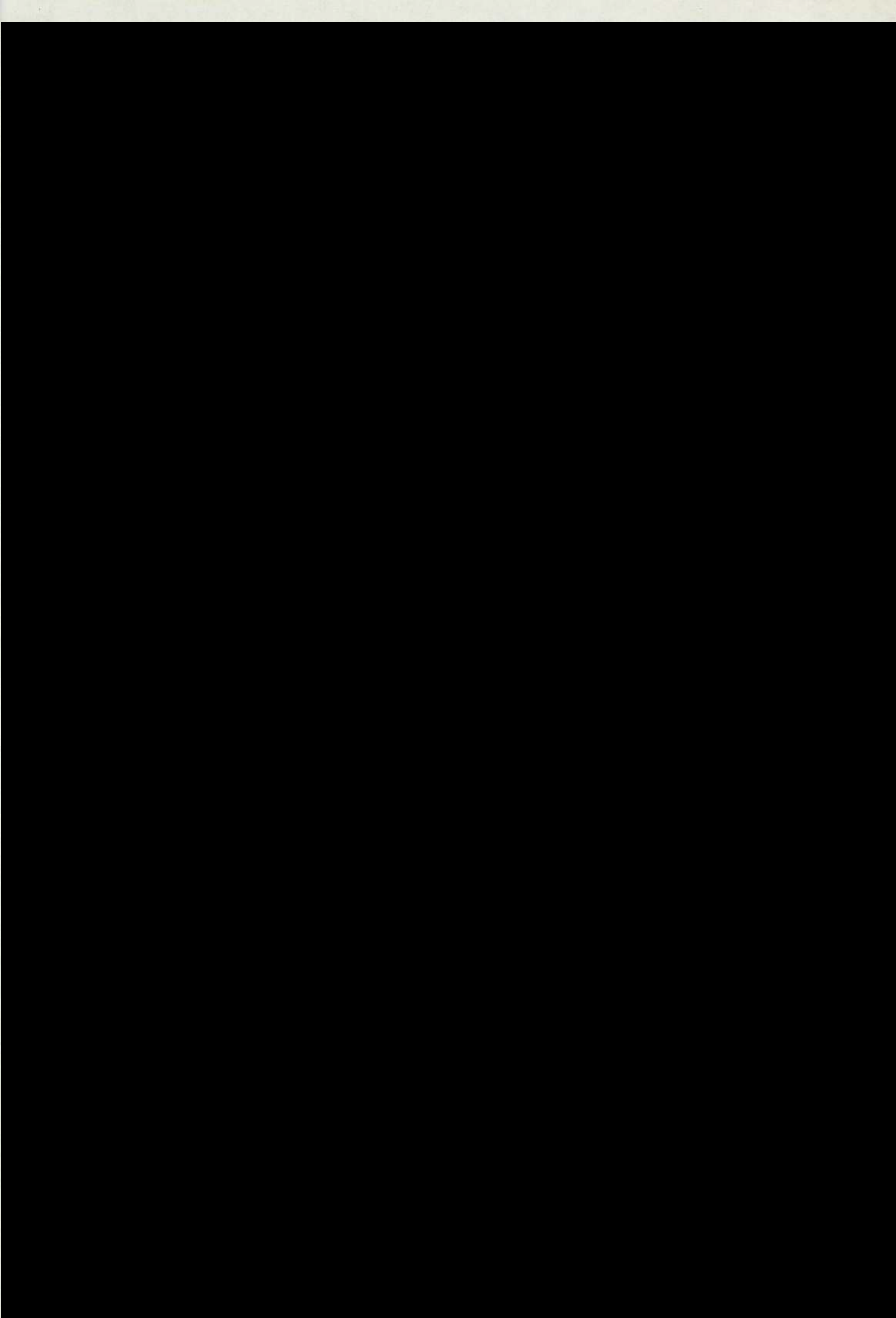
5.A 1972

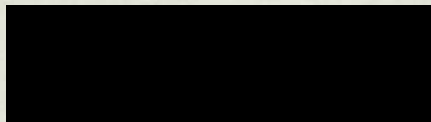
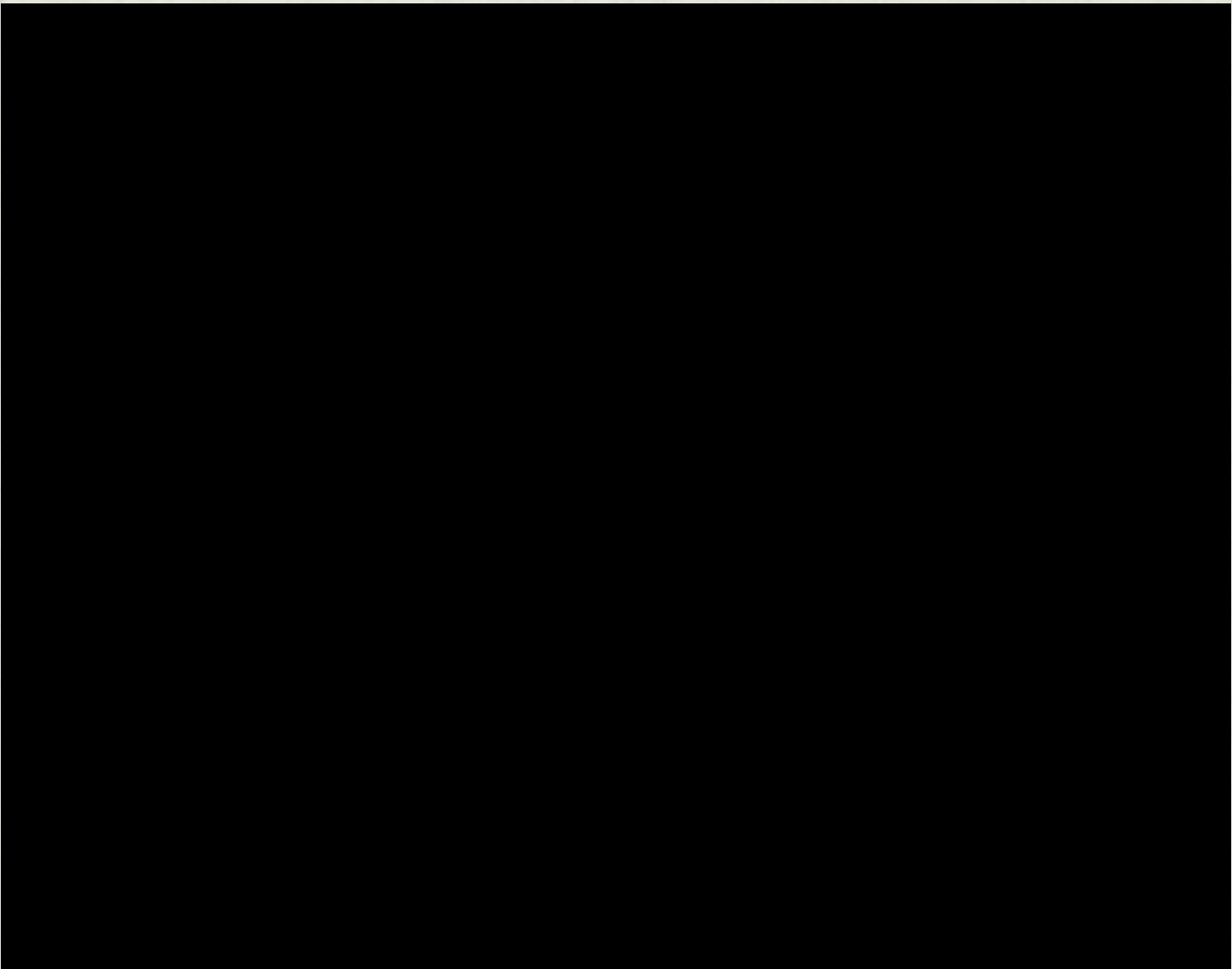
5.B 1973.

A P P E N D I X 6

Author's paper,
Institution of Mechanical Engineers,
1975







A P P E N D I X 7

Program print-out (Additional Property Variation)

PROGRAM CELL(INPUT,OUTPUT,TAPE1=INPUT,TAPE2=OUTPUT)
MIXED FREE AND FORCED CONVECTION IN VERTICAL TUBES

90

C
C

P H G A

REAL MTWO,MONE

100

REAL NUSSLT

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DIMENSION U(3,80),V(3,80),T(3,80),P(3,81),
1A(238,238),B(238),X1(79),X2(238),L(10)

COMMON /CMN/ A

LEVEL 2,A

DATA PI/3.14159/

SH130

CALL ITIME(ISEC1)

150

1 READ(1,2)ILAST,KLAST,MAXR,ISUNU,IFT00,C01,C02,C03,C04,C05,C06,

160

1GNUMBR,PENTRY,DELTAZ,PRNDTL,RYNLDS

170

2 FORMAT(5I3/F10.8,F17.15,F25.22,F10.8,F18.14/F20.17,F15.1,F5.1,

171

1F6.2,F14.10,F10.4)

172

IF (ILAST.LT.0)STOP

173

READ(1,3)(L(K),K=1,KLAST)

174

3 FORMAT(10I3)

175

READ(1,5)S,RATIO0,CTEN,CTEN1,CTEN2

180

FORMAT(F4.1,F11.6,F15.12,F15.12,F15.12)

190

WRITE(2,6)S,RATIO0,CTEN,CTEN1,CTEN2

200

6 FORMAT(1H1,3H S=,F5.2,22H + SIGN=UP,- SIGN=DOWN,15H DENSITY RATIO=

210

1,F10.7,5H C10=,F15.12,5H C11=,F15.12,5H C12=,F15.12)

211

EFFECT OF VISCIOUS DISSIPATION

V01

READ(1,41)QDISS

V02

FORMAT(F15.12)

V03

WRITE(2,42)QDISS

V04

FORMAT(1H ,7H QDISS=,F15.12)

V05

SGM2=MAXR**2/2

650

GO TO(4,8,4),ISUNU

610

ZBARX=0.0

ANN12

CONTINUE

FORMAT(F10.1)

613

GO TO(10,12),IFT00

614

READ(1,9)TWALL

620

WRITE(2,11)TWALL,MAXR

621

FORMAT(1H ,25HCONSTANT WALL TEMP.TWALL=,F10.3,6H MAXR=,I6)

640

GO TO 14

650

12 READ(1,9)HFLUX

651

WRITE(2,13)HFLUX,MAXR

652

13 FORMAT(1H ,25HCONSTANT HEAT FLUX.HFLUX=,F10.3,6H MAXR=,I6)

653

CONTINUE

654

GO TO(8,62,8),ISUNU

655

WRITE(2,15)(L(K),K=1,KLAST)

656

8 FORMAT(1H ,16H VALUES OF L(K),/(7I4))

657

WRITE(2,16)ILAST,DELTAZ

658

FORMAT(1H ,7H ILAST=,I6,7H DELTAZ=,F10.6)

659

WRITE(2,17)C01,C02,C03,C04,C05,C06,RYNLDS,PRNDTL,GNUMBR,PENTRY

660

FORMAT(1H ,5H C01=,F8.6,5H C02=,F18.15,5H C03=,F21.18,

661

15H C04=,F9.6,5H C05=,F9.6,5H C06=,F15.12/

662

28H RYNLDS=,F10.1,5H PRNDTL=,F15.6,5H GNUMBR=,

663

7F15.2,8H PENTRY=,F3.3)

664

CC7=CC2**(-CC3)

PA1

PR=PRNDTL*CC7

PA2

RE=RYNLDS/CC7

PA3

WRITE(2,18)PR,RE

PA4

FORMAT(1H ,35H ENTRY PRANDTL AND REYNOLDS NUMBERS,2F15.6)

PA5


```

GO TO(23,20,30),ISUNU
READ(1,21)ZBARX
60 21  FORMAT(F10.6)
WRITE(2,22)ZBARX
22  FORMAT(1H,24H ADIABATIC ENTRY LENGTH=,F10.6)
NUSSLT=0.0
GO TO(26,27)IFT00
65 26  TWALL=0.0
GO TO 23
27  HFLUX=0.0
23  DO 24 M=2,MAXR
U(2,M)=1.0
70 24  CONTINUE
GO TO 25
30  DO 31 I=2,MAXR
XI=I-1
U(2,I)=2.0-XI**2/SQM2
75 31  CONTINUE
25  DO 40 M=2,MAXR
U(1,M)=U(2,M)
V(1,M)=0.0
V(2,M)=0.0
80 T(1,M)=0.0
T(2,M)=0.0
T(3,M)=0.0
P(2,M)=PENTRY
40  CONTINUE
85 P(2,1)=PENTRY
P(2,MAXR+1)=PENTRY
T(3,1)=0.0
TBAR1=0.0
PMEAN=0.0
90 XMAXR=MAXR
NUMBR=MAXR/2
50 ZBAR=0.0
TBAR=0.0
ITALLY=0
95 K=1
NITER=0
60  CONTINUE
DIZSO=1.0/DELTZ**2
OPRRE=RATIO0/(PRNDTL*RYNLCS)
100 FUNC1=OPRRE/DELTZ
HALFM=MAXR/2
DOZM=XMAXR/12.0
DOZM8=DOZM**8.0
FMSQ=SQM2*OPRRE
{VARC1=FMSQ*CTEN1
{VARC2=CTEN1*OPRRE*DIZSO
MAX1=MAXR-1
MAXP1=MAXR+1
90  CONTINUE
110 DTMOZ=0.5/DELTZ
DIVZ=1.0/DELTZ
MAX32=3*MAXR-2
MAX21=2*MAXR-1
MAX2=2*MAXR

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115	MTWO=DI ZSQ-4.0*SQM2	1060
	DIVRE=1.0/RYNLOS	1070
	MONE=SQM2*DIVRE	1080
	UPORDN=CC*PATIOO/RYNLOS**2	1090
	UP1=UPORDN*GNUMBR*CC4	1100
120	UP2=UPORDN*CC5	1110
	UP3=UPORDN*CC6	1120
	DIVRE2=2.0/RYNLOS	1130
	DIVRE3=DIVRE*DIZSQ	1140
	DIVRE4=HALFM/(DELTZ*RYNLOS)	1150
125	DO 110 I=1, MAX32	1160
	B(I)=0.0	1170
	DO 110 J=1, MAX32	1180
	A(I,J)=0.0	1190
130	110 B(MAX32)=0.75*SQM2	1200
	B(MAX21)=DTWOZ*(U(2,MAX1)+U(2,MAXR))	1210
	B(MAX2)=U(2,2)/DELTZ	1220
	A(MAX21,MAX32-1)=DTWOZ	1230
	A(MAX21,MAX32)=DTWOZ	1240
	A(MAX2,MAX2)=1.0/DELTZ	1250
135	A(MAX21,MAX21-1)=-XMAXR	1260
	A(MAX21,MAX21)=XMAXR**2/(XMAXR-1.0)	1270
	A(MAX2,MAXP1)=XMAXR	1280
	A(MAX1-1,MAX1-1)=-DIVZ	1290
	A(MAX1,MAX1)=-DIVZ	1300
140	A(MAXR,MAXR-2)=HALFM	1310
	A(MAXR,MAXR)=-HALFM	1320
	A(MAXP1,1)=HALFM*3.0	1330
	A(MAXP1,2)=-HALFM*4.0	1340
	A(MAXP1,3)=HALFM	1350
145	G1=CO1/GNUMBR	1360
	G2=-G1*CC3	1370
	CC3=CC3+1.0	1380
	TCOMMN=G1*T(2,2)+CC2	1390
	TALPHA=TCOMMN*(-CC3)	1400
150	TBETA=G2*TCOMMN*(-CC3)	1410
	TONE=TBETA*(T(2,3)-T(2,2))	1420
	TTWO=TBETA*(T(2,2)-T(1,2))*DIZSQ	1430
	VONE=HALFM*V(2,2)	1440
	TFOUR=TALPHA*3.0	1450
155	TSIX=TALPHA*DIZSQ	1460
	A(1,MAX2+1)=MONE*(TFOUR+0.5*TONE)-VONE	1470
	UONE=DIVZ*U(2,2)	1480
	A(1,MAX2)=MONE*(TALPHA-0.5*TCNE)+VONE+DIVRE*(TALPHA*MTWO+2.0*TTWO)	1490
160	1-UONE	1500
	A(MAXP1,MAXP1+1)=MONE*(TFOUR+TONE)-VONE	1510
	VARIT=TALPHA*(DIZSQ-SQM2*6.0)	1520
	A(MAXP1,MAXP1)=MONE*(TALPHA-TONE)+VONE+DIVRE*(VARIT+TTWO)-UONE	1530
	SCNE=V(2,2)*(-U(2,2)*DIVZ+DIVRE*(2.0*TSIX+TTWO))	1540
	STNO=-DIVRE*(V(1,2)*TSIX+HALFM*DELTZ*TTWO*(U(2,3)-U(2,2)))	1550
165	B(MAXP1)=SCNE+STNO	1560
	UPORDN2=UP1+T(2,2)*(UP2+UP3*T(2,2))	1570
	BONE=UPORDN2-DIVZ*(B(2,2)+U(2,2)**2)-U(1,2)*TALPHA*DIVRE3	1580
	BTNO=DIVRE2*U(2,2)*(TSIX+TTWO)-DIVRE4*TONE*(V(2,2)-V(1,2))	1590
	B(1)=BONE+BTNO	1600
170	TCOMMN=G1*T(2,MAXR)+CC2	1610
	TALPHA=TCOMMN*(-CC3)	1620

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230      TFIVE=TTHREE*VARI
      TSIX=TALPHA*DIZSO
      A(I1,MAXI4)=MONE*(TFIVE-0.5*TONE)+VONE
      A(I1,MAXI4+2)=MONE*(TFOUR+0.5*TONE)-VONE
      A(MAXI3,MAXI2)=MONE*(TFIVE-TONE)+VONE
235      A(MAXI3,MAXI2+2)=MONE*(TFOUR+TONE)-VONE
      UONE=DIVZ*U(2,I2)
      A(I1,MAXI1)=DIVRE*(TALPHA*MTWC+2.0*TTHO)-UONE
      VARIT=TALPHA*(DIZSO-SQM2*(4.0+2.0/XM1**2))
      A(MAXI3,MAXI3)=DIVRE*(VARIT+TTWC)-UONE
240      BCNE=V(2,I2)*(-U(2,I2)*DIVZ+DIVRE*(2.0*TSIX+TTWC))
      BTWO=-DIVRE*(V(1,I2)*TSIX+
1HALFM*DELIZ*TTWO*(U(2,I3)-U(2,I1)))
      G(MAXI3)=BCNE+BTWO
      UPOORDN4=UP1+T(2,I2)*(UP2+UP3*T(2,I2))
      ECNE=LPOORDN4-DIVZ*(P(2,I2)+U(2,I2)**2)
245      1-U(1,I2)*TALPHA*DIVRE3
      BTWO=DIVRE2*U(2,I2)*(TSIX+TTWC)-
1DIVRE4*TONE*(V(2,I2)-V(1,I2))
      G(I1)=BCNE+BTWO
250      115      CONTINUE
      A(MAXR+2,1)=DOZN*7.0
      A(MAX21,MAXR-3)=0.0
      MHALF=MAXR/2
      DO 120 M=1,MHALF
255      M1=2*M-1
      M2=MAX21+M1
      A(MAX32,M2)=M1
      120      CONTINUE
      MHAL1=MHALF-1
      DO 121 M=1,MHAL1
260      M1=MAX21+2*M
      A(MAX32,M1)=M
      121      CONTINUE
      IF(NITER-1)93,96,96
265      96      DO 95 M=1,MAX1
      T(1,M+1)=X2(M)
      MA=MAX1+M
      T(2,M+1)=X2(MA)
      95      CONTINUE
270      98      CALL MAT(MAX32,A,P,X2)
      MAXF=MAXR-1
      DO 100 M=1,MAXF
      P(3,M+1)=X2(M)
      MA=MAXR+M
      V(3,M+1)=X2(MA)
275      MB=MA+MAXF
      100      U(3,M+1)=X2(MB)
      CONTINUE
      P(1,MAXR+1)=X2(MAXR)
      U(3,1)=U(3,2)
      V(3,1)=0.0
      P(3,1)=P(3,2)
      DO 501 I=1,MAX1
      DO 501 J=1,MAX1
      A(I,J)=0.0

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501  FUNC2=DPRRE*(4.5*SQM2-DIZSQ)
DO 502 I=2,MAX1
  I1=I+1
  XI=I
  XI1=I1
290  A(T,I-1)=FMSQ*(3.0-2.5*XI1)*(1.0+CTEN*T(3,I1))/XI-HALFM*V(3,I1)
1+VARC1*(T(3,I1)-T(3,I))/(1.0+CTEN2*T(3,I1))
502  CONTINUE
DO 505 I=1,MAX1
  I1=I+1
  XI=I
  XI1=I1
  VARC3=(1.0+CTEN*T(3,I1))
  VARC4=1.5/(1.0+CTEN2*T(3,I1))
300  VARC5=VARC2*VARC4*(T(3,I1)-T(2,I1))
  B(I)=T(2,I1)*(U(3,I1)/DELTZ-VARC5)
1-VARC3*DPRRE*DIZSQ*(2.0*T(3,I1)-T(2,I1))
  A(T,I)=U(3,I1)/DELTZ+FUNC2*VARC3-VARCF
  A(I,I1)=HALFM*V(3,I1)+FMSQ*(1.0-2.5*XI1)*VARC3/XI
305  1-VARC1*VARC4*(T(3,I1)-T(3,I))
505  A(1,1)=A(1,1)-FMSQ-HALFM*V(3,2)
  SUMT1=0.0
DO 522 M=2,MAXR
  XM=M
  T1=1.0+CTEN2*T(3,M)
  A(MAX1,M-1)=(XM-1.0)*U(3,M)*T1
  SUMT1=SUMT1+(XM-1.0)*U(3,M)*T1
522  CONTINUE
  MAXR2=MAXR-2
  VSUM=0.0
DO 80 I=1,MAXF
  XI=I
  I1=I+1
  I2=I+2
310  TALPH=(G1*T(3,I1)+C02)**(-C03)
  QDISS2=QDISS*TALPH
  QDISS3=XI*TALPH
  Q1=((V(3,I1)-V(3,I))*XMAXF)**2
  Q2=(XMAXF*V(3,I1)/XI)**2
325  Q3=((U(3,I1)-U(2,I1))/DELTZ)**2
  Q4=((V(3,I1)-V(2,I1))/DELTZ+XMAXF*(U(3,I1)-U(3,I)))**2
  VSUM=VSUM+QDISS3*(2.0*(Q1+Q2+Q3)+Q4)
  B(I)=B(I)+QDISS2*(2.0*(Q1+Q2+Q3)+Q4)
80  TBAR1=TBAR+2.0*HFLUX*XMAXR*DELTZ*DPRRE/(1.0+CTEN2*TBAR1)
1+DELTZ*QDISS*VSUM/SQM2
  B(MAX1)=SUMT1*TBAR1
  CALL MAT(MAX1,A,B,X1)
  WRITE(2,187) TBAR1
187  FORMAT(1H,6H TBAR=,F15.5)
335  DO 97 M=1,MAXF
  T(3,M+1)=X1(M)
97  CONTINUE
  T(3,1)=T(3,2)
  SUMT1=0.0
  SUMT=0.0
340  DO130 M=2,MAXR
  XM=M

```

H06

H13

AK03

H07

H08

H09

H10

AK04

AK05

AK06

AK07

AK08

AK09

AK10

AK11

H16(22)

AK12

EB1

EB2

AK13

EB3

AK14

EB4

V20

V21

V22

V23

V24

V25

V26

V27

V28

V29

V30

V31

V14

EB5

EB6

EB7

EB8

EB9

26000

34400

33600

34600

AK15

33400

35200

35800

		T1=1.0+CTEN2*T(3,M)		AK16
		SUMT=SUMT+(XM-1.0)*U(3,M)*T(3,M)*T1		3540
345		SUMT1=SUMT1+(XM-1.0)*U(3,M)*T1		AK17
	130	CONTINUE		3540
		TMEAN=SUMT/SUMT1		AK18
		XF1=(2.0*XMAGR)/RYNLOS	FF1	
		XF2=(18.0*U(3,MAXP)-9.0*U(3,MAXF)+2.0*U(3,MAXR2))/6.0	FF2	
350		IF(NITER-1)126,127,127		3590
	126	ITALLY=ITALLY+1		2575
		ZBAR=ZBAR+DELTZ		2570
	127	WRITE(2,125)DELTZ,ZBAR		2580
	125	FORMAT(1H,7H DELTZ=,F10.6,6H ZBAR=,F10.6)		2590
355		IF(ZBAR-ZBARX)116,116,122		3571
	116	XFRW1=XF1*XF2	FF14	
		TALPHW=1.0	FF16	
		GO TO 117	FF18	
	122	CONTINUE		3572
360		GO TO(124,123)IFTOQ		3573
	123	HFLUXA=HFLUX		AK19
		TWALL=(2.0*HFLUX+4.0*T(3,MAXF)-T(3,MAXF))/3.0		AK21
		HFLUX=HFLUXA/(1.0+CTEN1*TWALL)		AK20
		TWALL=(2.0*HFLUX+4.0*T(3,MAXF)-T(3,MAXF))/3.0		3574
365		XNUX=2.0*HFLUX*XMAGR		3575
		NUSSLT=XNUX/(TWALL-TMEAN)		2576
		WRITE(2,185)TWALL,NUSSLT		3577
	185	FORMAT(1H,7H TWALL=,F10.5,8H NUSSLT=,F10.5)		3578
		HFLUX=HFLUXA/(1.0+CTEN1*TWALL)		AK21
370		TWALL=(6.0*HFLUX+18.0*T(3,MAXR)-9.0*T(3,MAXF)+		3579
		12.0*T(3,MAXF-1))/11.0		3580
		TALPHW=(G1*TWALL+CO2)**(-CO3)	FF2	
		XFRW1=XF1*TALPHW*XF2	FF4	
		XNUX=2.0*HFLUX*XMAGR		AK24
375		NUSSLT=XNUX/(TWALL-TMEAN)		3581
		WRITE(2,185)TWALL,NUSSLT		3582
		HFLUX=HFLUXA/(1.0+CTEN1*TWALL)		AK22
		TWALL=(12.0*HFLUX+48.0*(T(3,MAXR)-36.0*T(3,MAXF)+16.0*T(3,MAXF-1)		3583
		1-3.0*T(3,MAXF-2))/25.0		3584
380		HFLUX=HFLUXA		AK23
		TALPHW=(G1*TWALL+CO2)**(-CO3)		3585
	117	XF2=(48.0*U(3,MAXR)-36.0*U(3,MAXF)+16.0*U(3,MAXR2)-3.0*U(3,MAXR3-	FF6	
		1))/12.0	FF7	
		XFRW2=XF1*TALPHW*XF2	FF8	
385	124	CONTINUE		3586
		DO 133 M=1,MAXF		3587
		P(1,M+1)=P(3,M+1)+ZBAR*(LP1+T(3,M+1)*(UP2+UP3*T(3,M+1)))		3588
	133	CONTINUE		3589
		P(1,MAXF1)=P(3,MAXP1)+ZBAR*(UP1+TWALL*(UP2+UP3*TWALL))		3590
390		SUMP=0.0		3591
		DO 134 M=2,MAXP1		3592
		XM=M		3593
		SUMP=SUMP+(XM-1.0)*P(1,M)		3594
	134	CONTINUE		3595
395		PMEAN1=SUMP/SGM2		3596
		XFRP=(PMEAN1-PMEAN)/DELTZ		3597
		XFRAP=PMEAN1/ZBAR		3598
		WRITE(2,155)(XFRP,XFRAP,XFRW1,XFRW2,PMEAN1)		3599
	155	FORMAT(1H,26H FRICTION FACTORS,PRESSURE,F10.5,8H AVERAGE,F10.5,	FF12	

400		110H WALL(3RD),F10.5,10H WALL(4TH),F10.5,6H PMEAN,F10.5)	FF13
	131	IF (ZBAR-ZBARX)132,132,131	A26000
		XSIGMA=ZBAR/RYNLD5	B26000
		XRANDH=(ZBAR-ZBARX)*DBRRE	C26000
		GRAETZ=PI/XRANDH	D26000
405		WRITE(2,128)XSIGMA,GRAETZ,XRANDH	E26000
	128	FORMAT(1H,7H SIGMA=,F10.6,8H GRAETZ=,F14.4,3H X=,F15.10)	F26000
		XNUX=2.0*HFLUX*XMAXP	G26000
		NUSSLT=XNUX/(TWALL-TMEAN)	H26000
	132	WRITE(2,140)(ITALLY,TMEAN,TWALL,NUSSLT,(U(3,M),M=2,MAXR))	I26000
410	140	FORMAT(1H,2H TALLY=,I4,7H TMEAN=,F15.5,7H TWALL=,F15.5,8H NUSSLT=	J26000
		1,F10.5/44H AXIAL VELOCITY,RADIAL VELOCITY,TEMPERATURE,	K26000
		20HPPRESSURE,/(12F10.5))	L26000
		WRITE(2,150)(V(3,M),M=2,MAXR)	M26000
	150	FORMAT(1H,12F10.7)	N26000
415		WRITE(2,160)(T(3,M),M=2,MAXR)	O26000
	160	FORMAT(1H,12F10.4)	P26000
		WRITE(2,170)(P(1,M),M=2,MAXP1)	Q26000
	170	FORMAT(1H,12F10.4)	R26000
		IF(QDISS.GT.0.0)GO TO 192	S26000
420		ZB=ZBAR-ZBARX	T26000
		IF(ZB.GT.0.0)GO TO 192	U26000
		GO TO 188	V26000
	192	IF(NITER-1)191,188,188	W26000
	191	NITER=NITER+1	X26000
425		DO 89 M=1,MAX1	Y26000
		X2(M)=T(1,M+1)	Z26000
		MA=MAX1+M	AA26000
		X2(MA)=T(2,M+1)	AB26000
		T(1,M+1)=T(2,M+1)	AC26000
430		T(2,M+1)=X1(M)	AD26000
	89	CONTINUE	AE26000
		GO TO 90	AF26000
	188	NITER=0	AG26000
		TBAR=TBAR1	AH26000
435		PMEAN=PMEAN1	AI26000
		VRATE=TBAR1-TBAR	AJ26000
		IF(ILAST-ITALLY)400,400,200	AK26000
	200	IF(L(K)-ITALLY)215,215,205	AL26000
	205	DO 210M=1,MAXP	AM26000
440		U(1,M)=U(2,M)	AN26000
		V(1,M)=V(2,M)	AO26000
		T(1,M)=T(2,M)	AP26000
	210	CONTINUE	AQ26000
		GO TO 220	AR26000
445	215	K=K+1	AS26000
		DELTZ=2.0*DELTZ	AT26000
	220	DO 300 M=1,MAXR	AV26000
		U(2,M)=U(3,M)	AW26000
		V(2,M)=V(3,M)	AX26000
450		T(2,M)=T(3,M)	AY26000
	300	CONTINUE	AZ26000
		DO 350 M=1,MAXP1	BA26000
		P(2,M)=P(3,M)	BB26000
	350	CONTINUE	BC26000
455		GO TO(60,370,60),ISUM	BD26000
	370	IF(ZBARX-ZBAR)60,7,60	BE26000

T1
T2
T3
T4
T5
T6
T6A

H01
H02
H03

F01
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F00

	400	WRITE (2,430)	2800
	430	FORMAT(1H ,12H END OF RUN.)	2860
		CALL ITIME(ISEC2)	2870
460		ISEC=ISEC2-ISEC1	2880
		WRITE (2,450) ISEC	2880
	450	FORMAT(1H ,6H TIME=,I6,4HSEC.)	2880
		GO TO 1	
		END	2830
465			
		SUBROUTINE MAT(N,A,Y,X)	3090
		DIMENSION A(238,238),Y(238),X(238)	
		LEVEL 2,A	
	5	M=N-1	3110
5		DO 10 I=1,M	3120
		L=I+1	3130
		DO 10 J=L,N	3140
		IF(A(J,I)/EQ.0.)GOTO10	3150
	6	DO 8 K=L,N	3160
10	8	A(J,K)=A(J,K)-A(I,K)*A(J,I)/A(I,I)	3170
		Y(J)=Y(J)-Y(I)*A(J,I)/A(I,I)	3180
	10	CONTINUE	3190
		X(N)=Y(N)/A(N,N)	3200
		DO 30 I=1,M	3210
15		K=N-I	3220
		L=K+1	3230
		DO 20 J=L,N	3250
	20	Y(K)=Y(K)-X(J)*A(K,J)	3260
		X(K)=Y(K)/A(K,K)	3270
20	30	CONTINUE	3280
		RETURN	3290
		END	3300
		SUBROUTINE ITIME(I)	
		CALL SECOND(T)	
		T = IFIX(T)	
5		RETURN	
		END	

A P P E N D I X 8

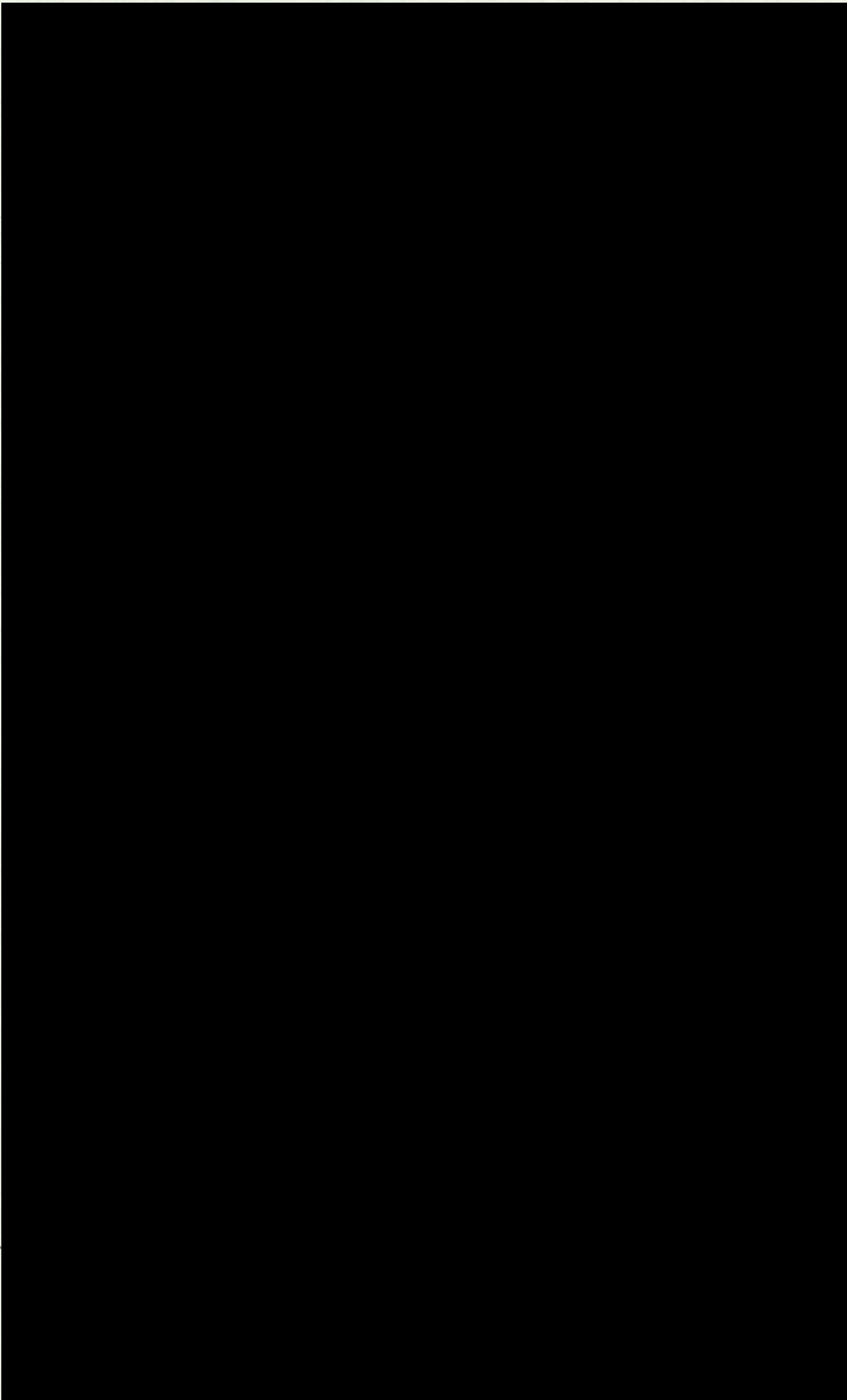
Letter from Professor M. Iqbal,
1975

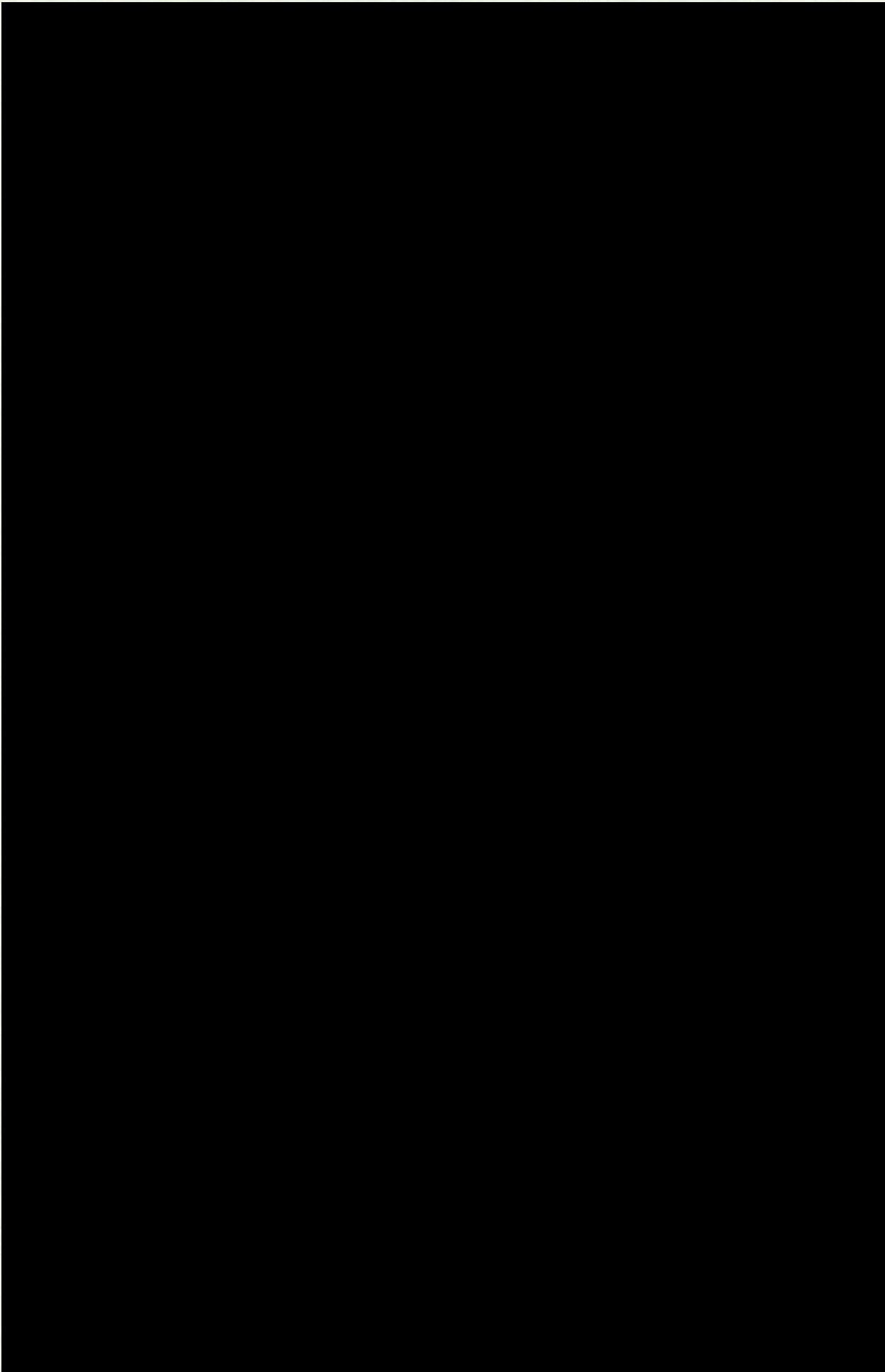
A P P E N D I X 9

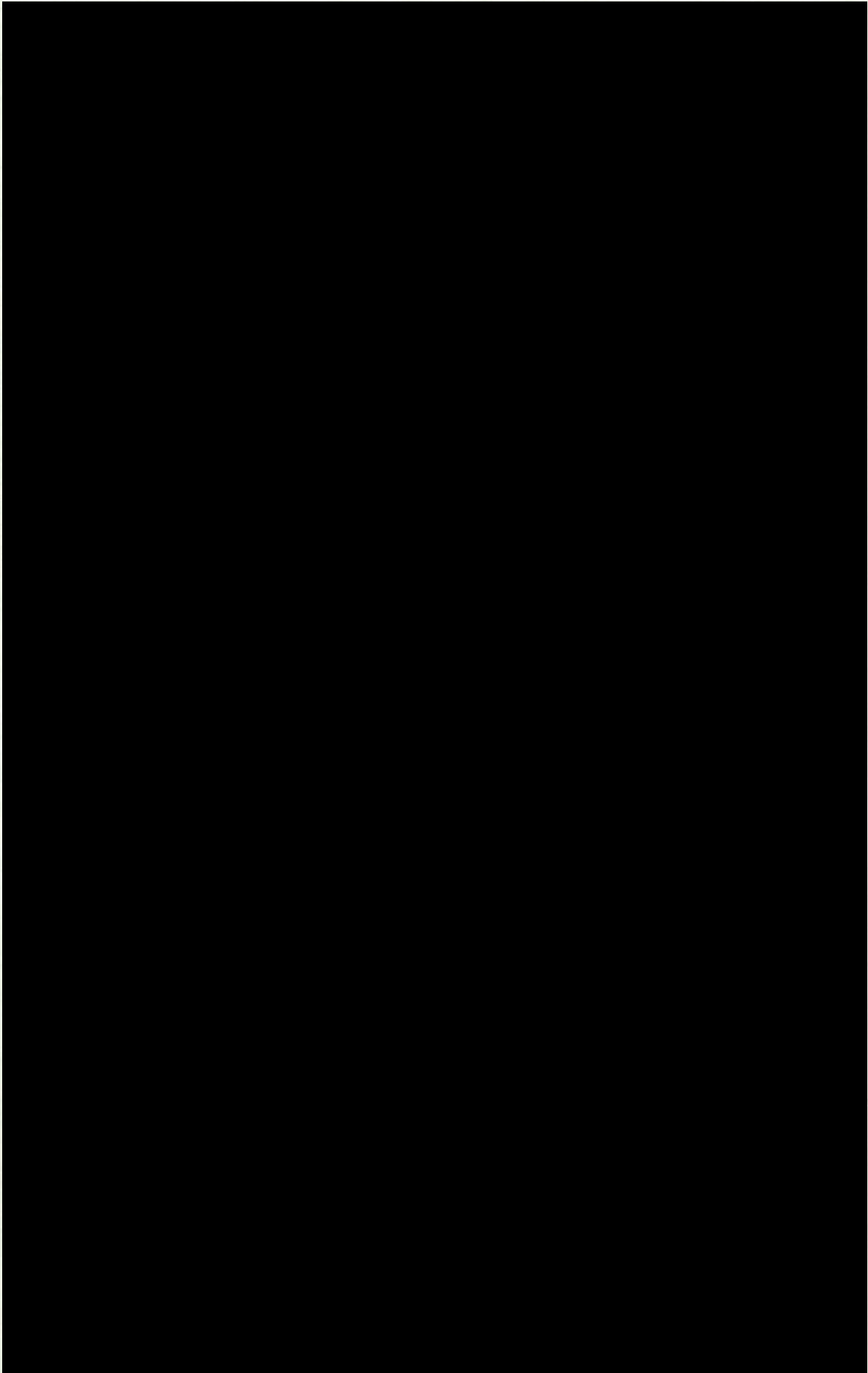
M/S submitted for publication by Author,

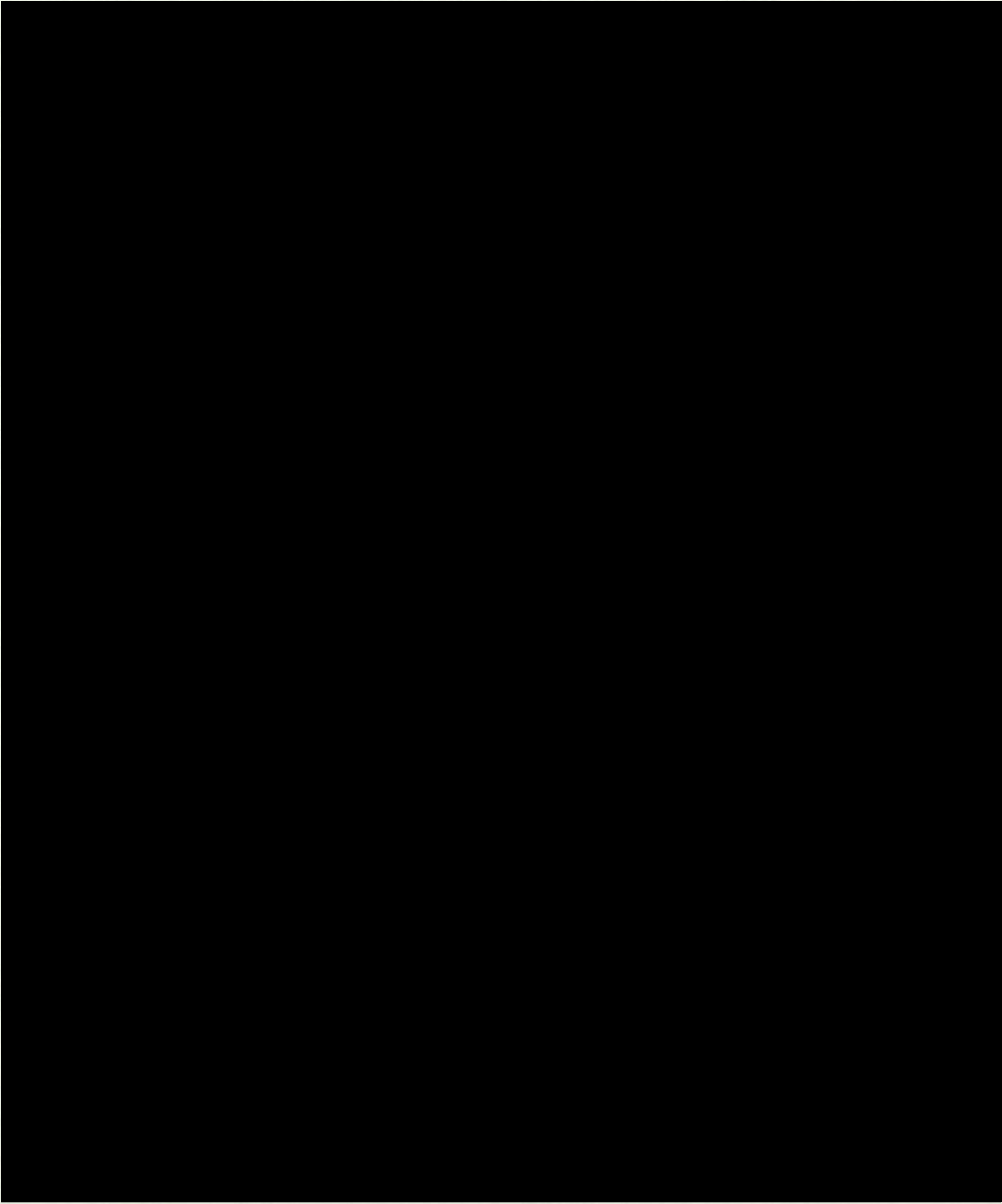
P.H.G. Allen, and O. Szpiro.

1975

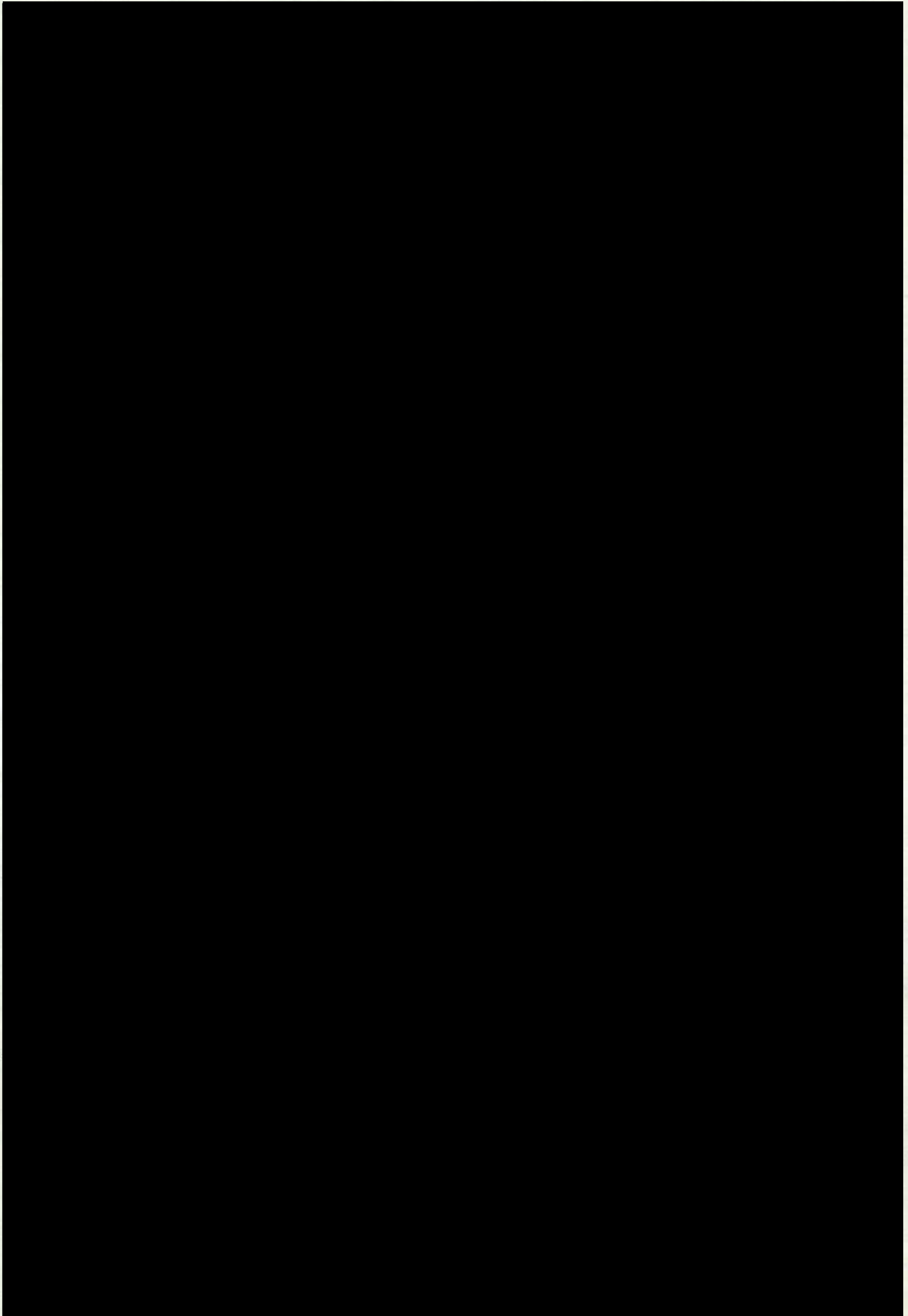


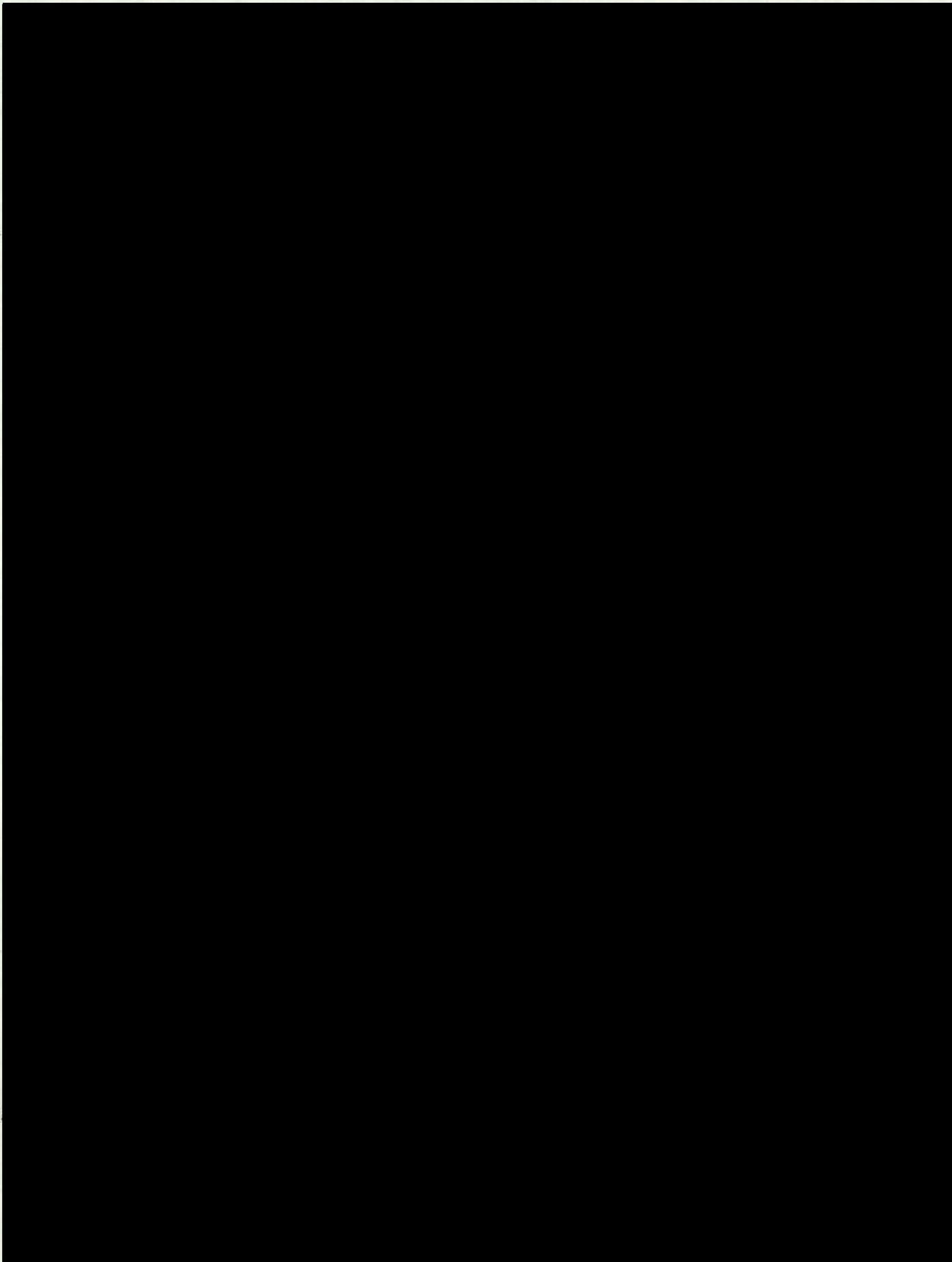


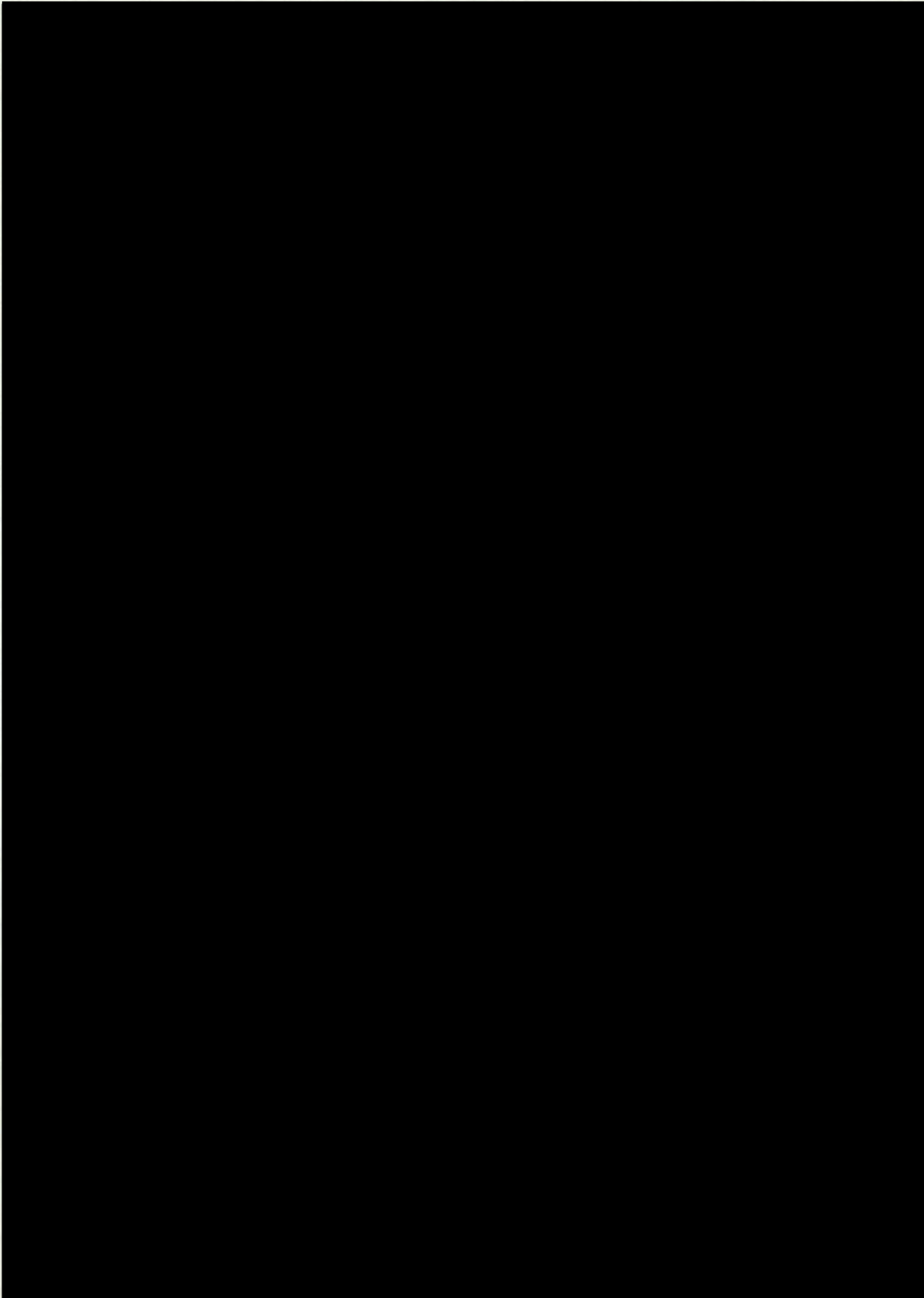


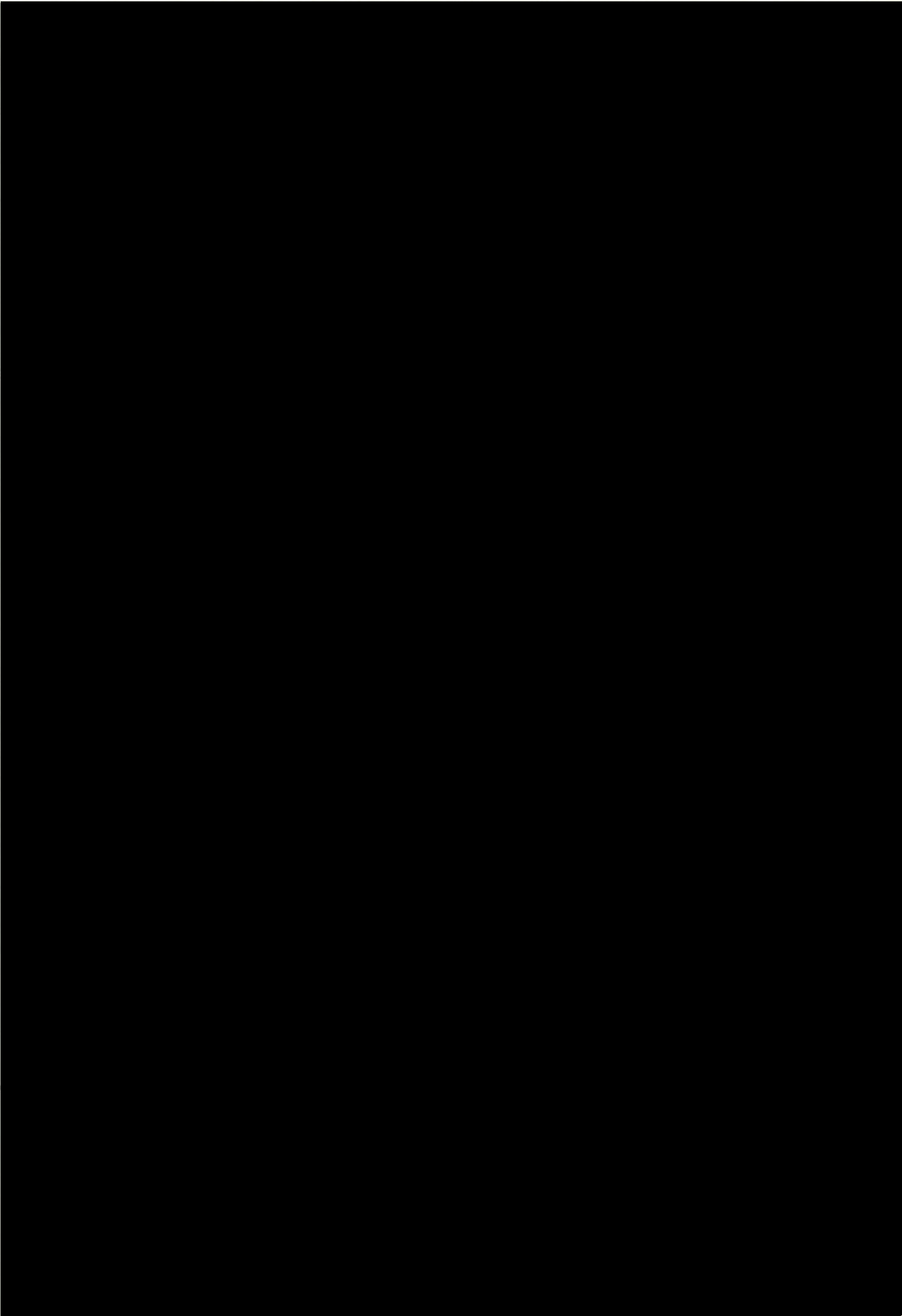


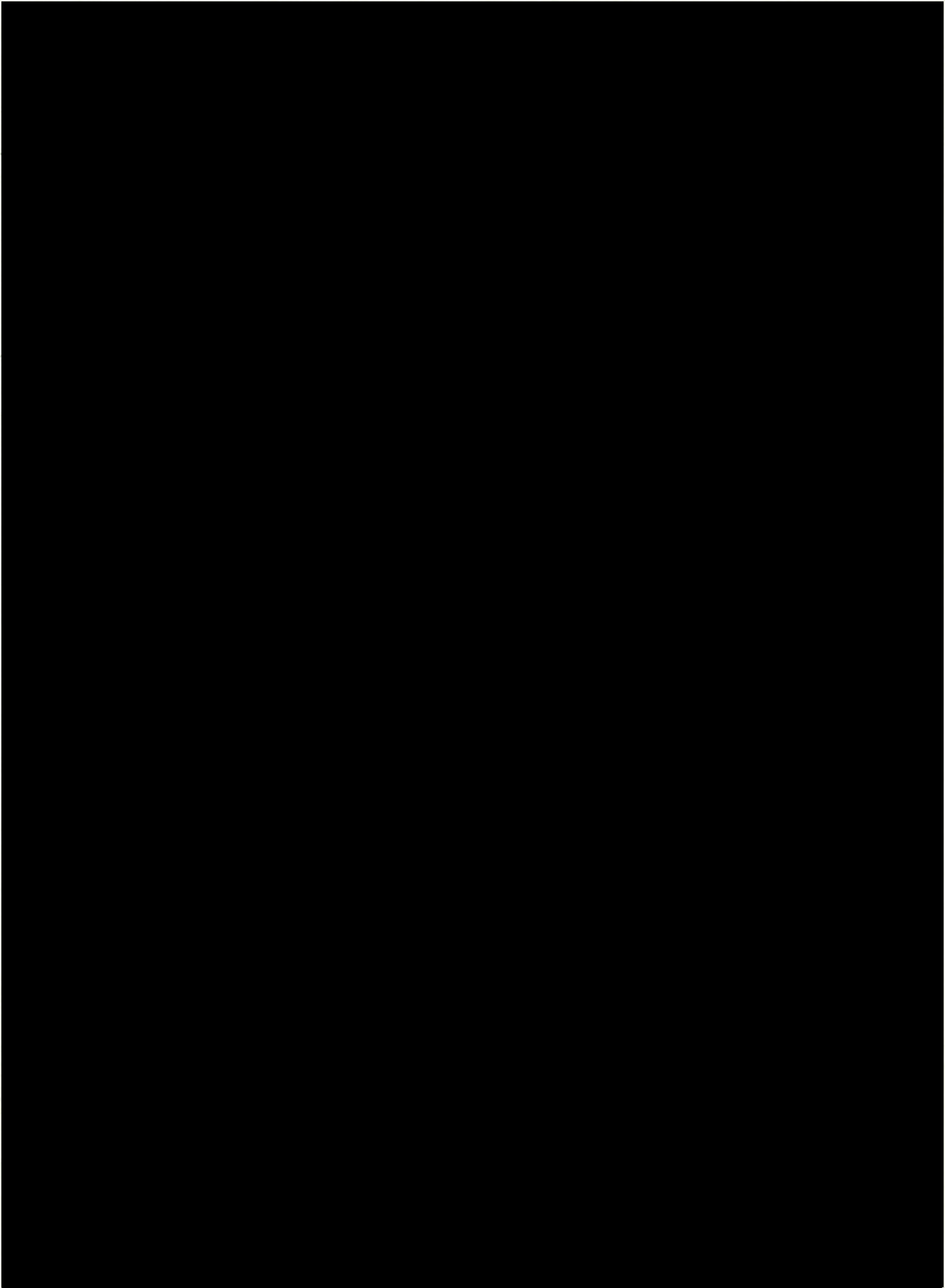
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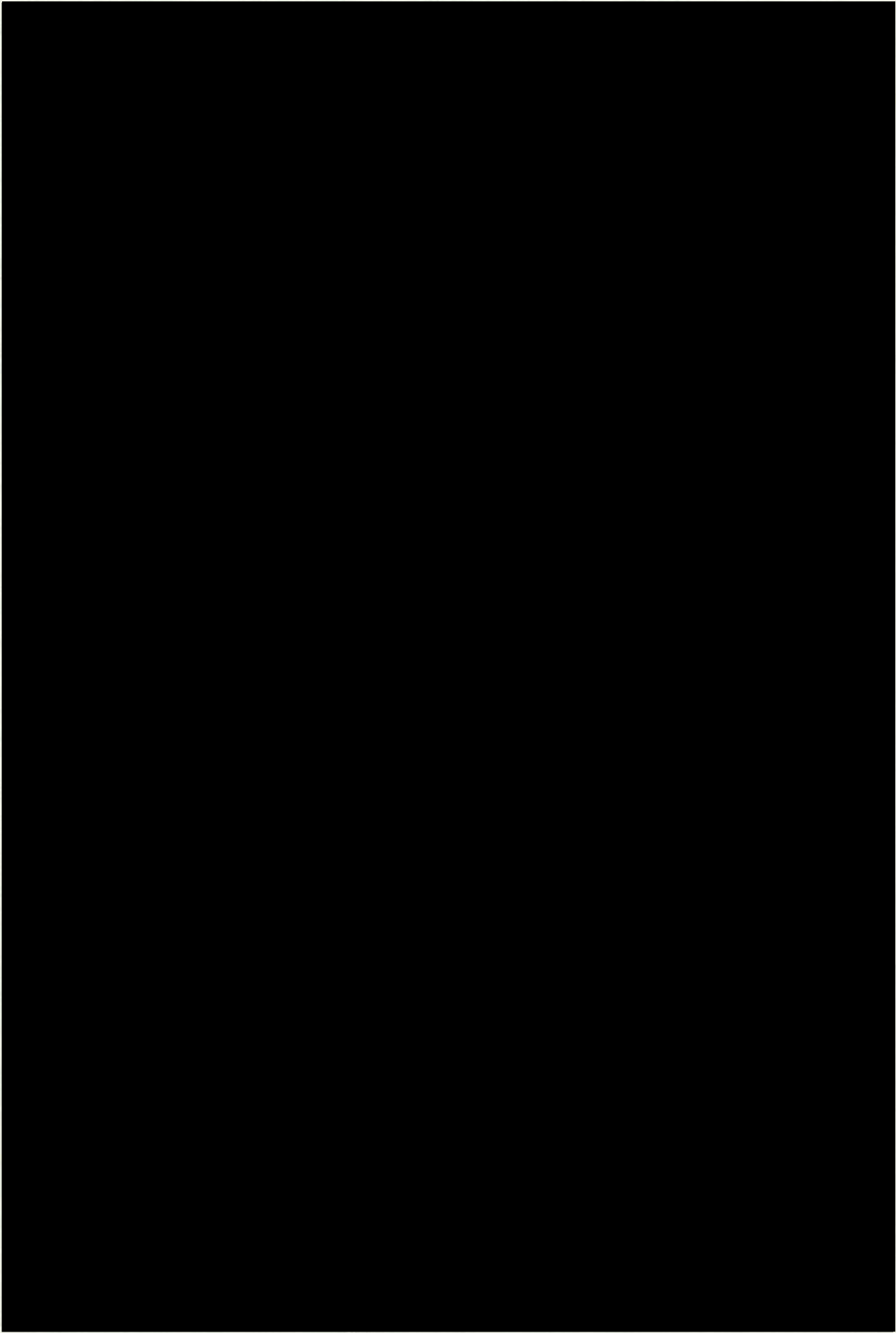


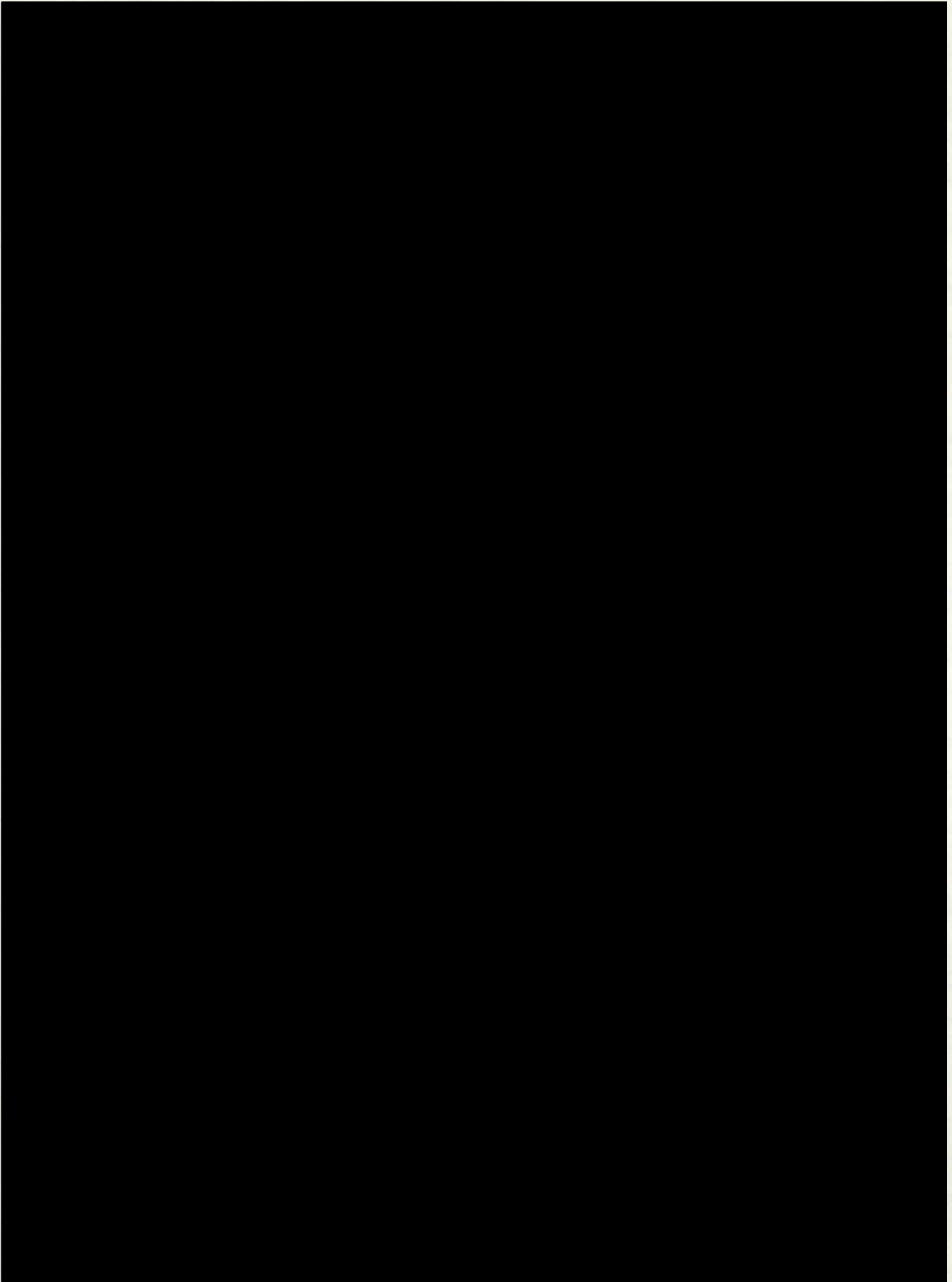


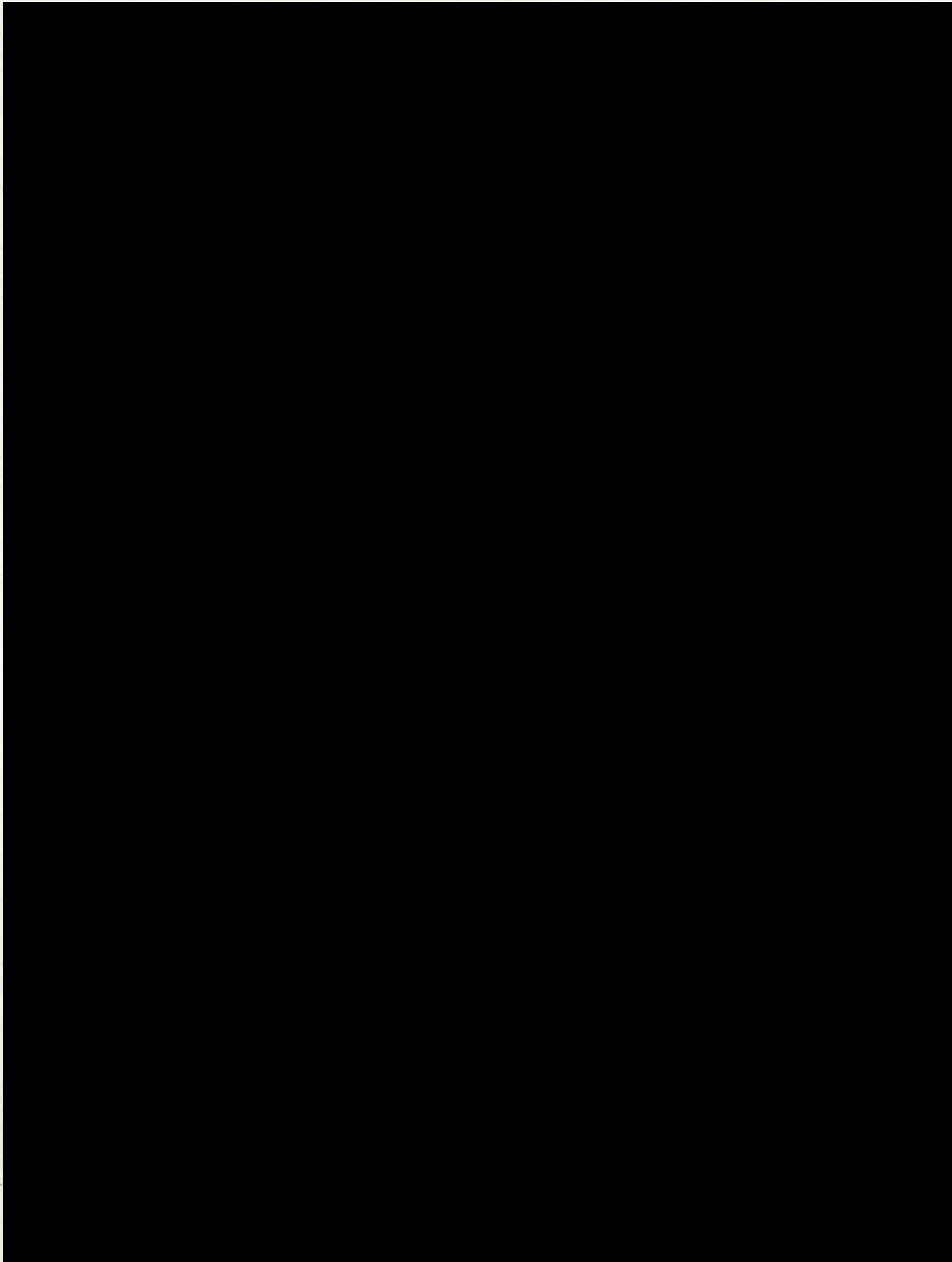




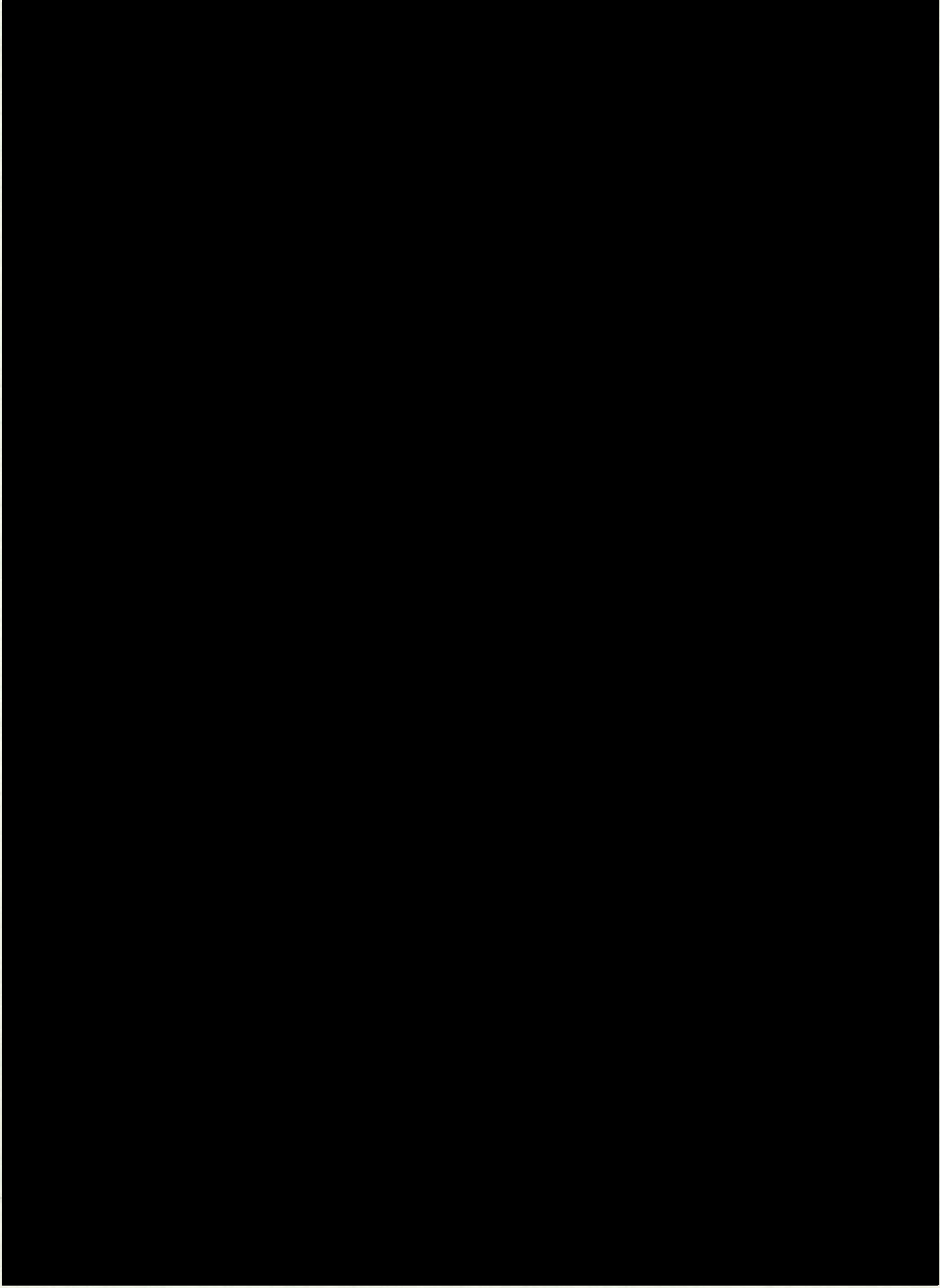


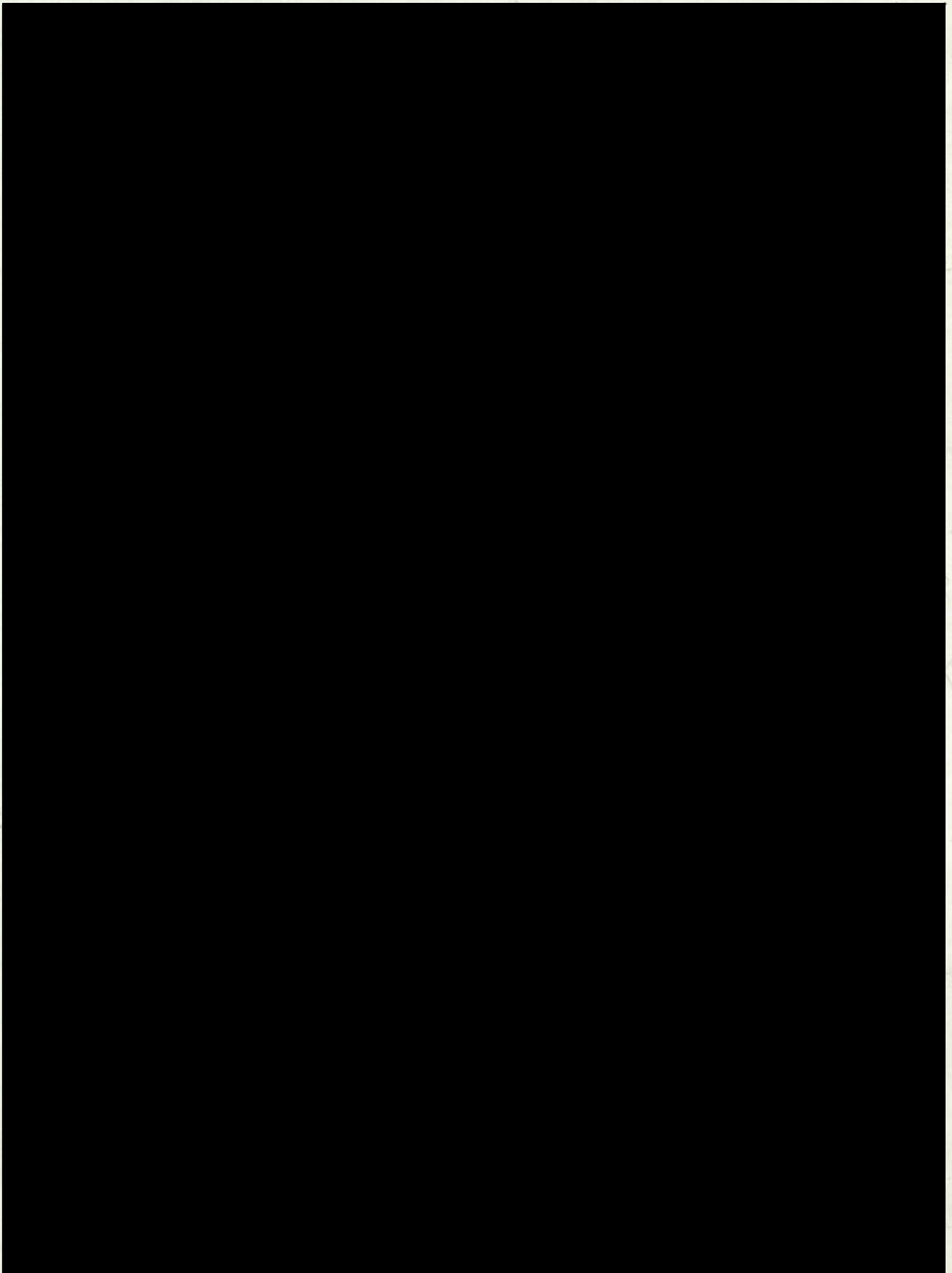


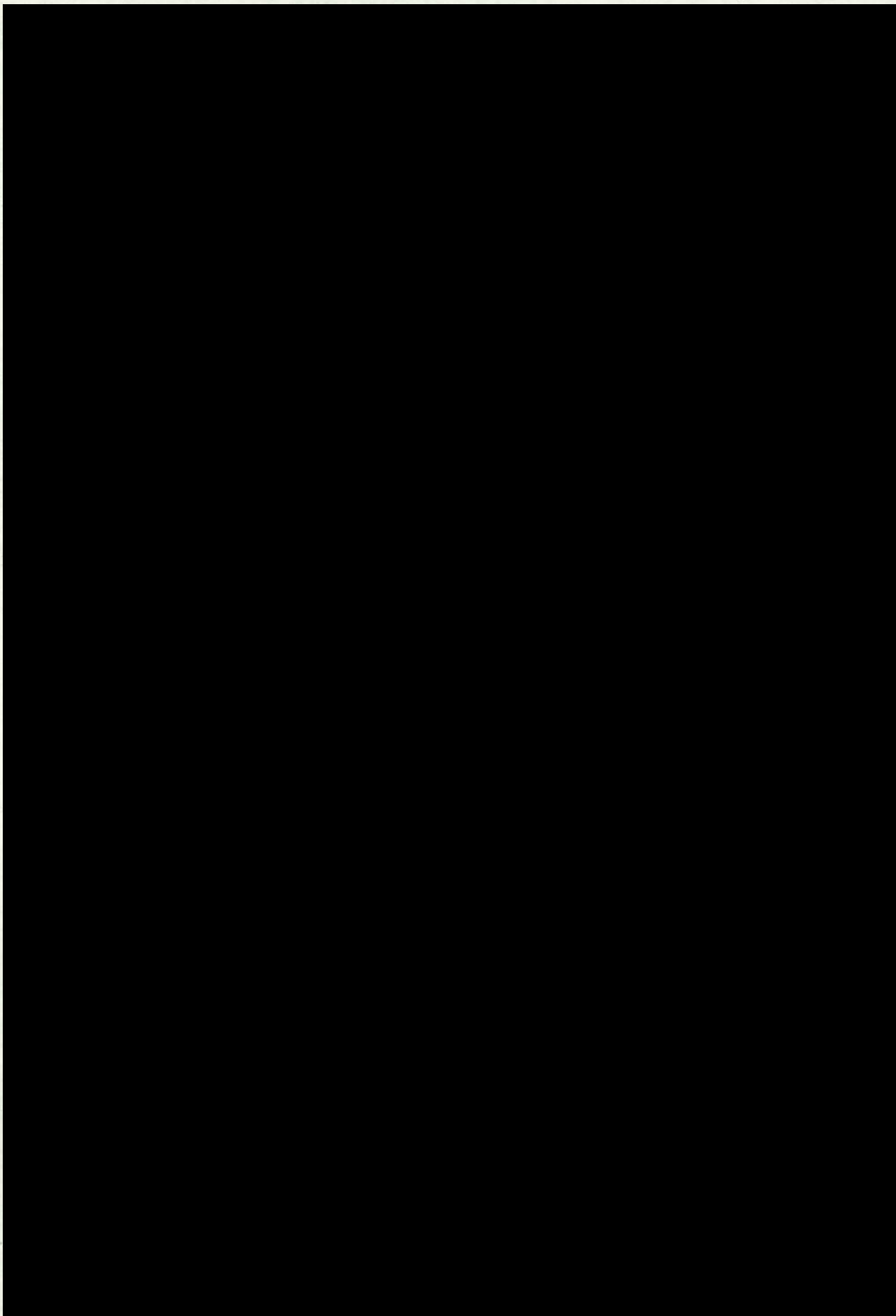


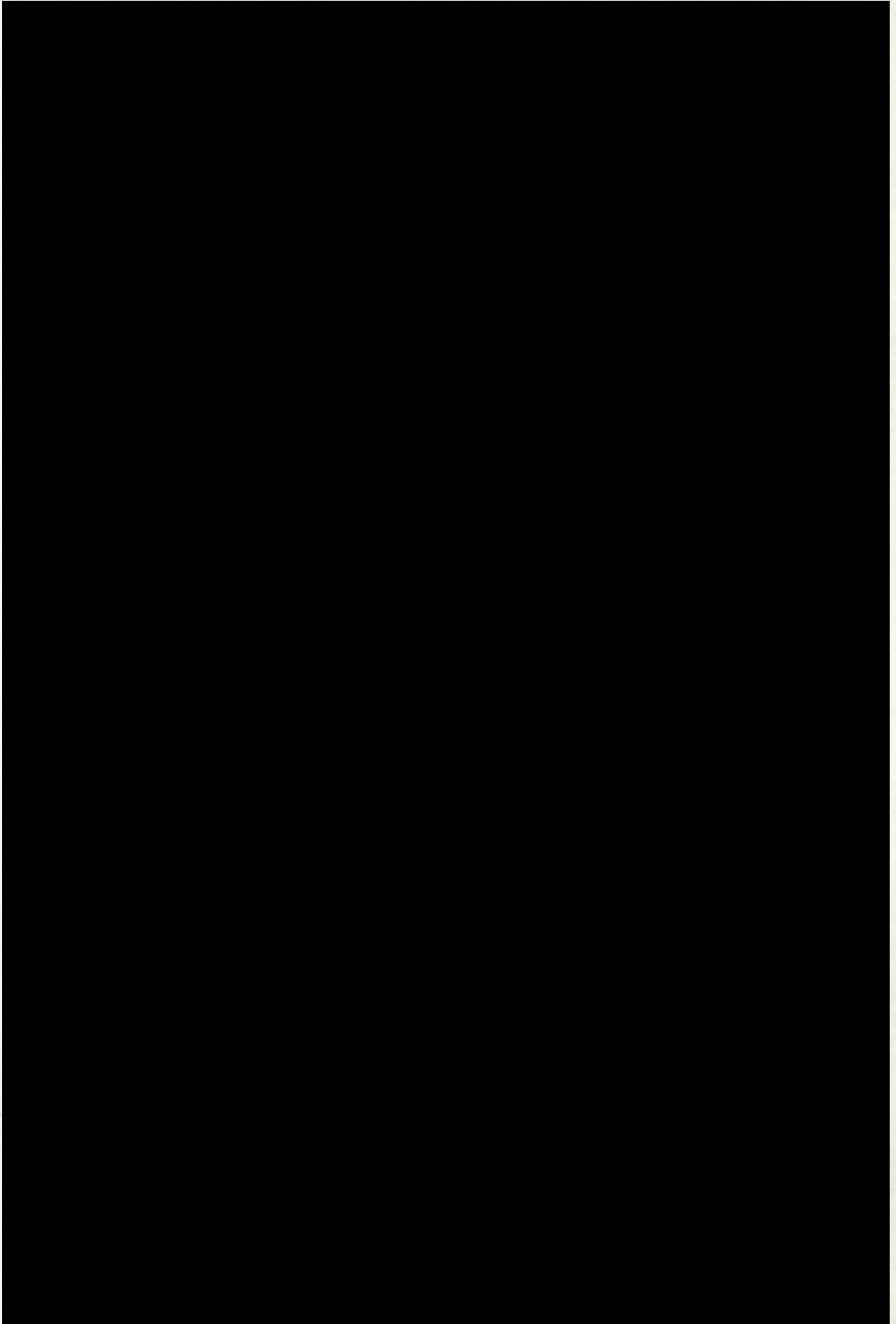


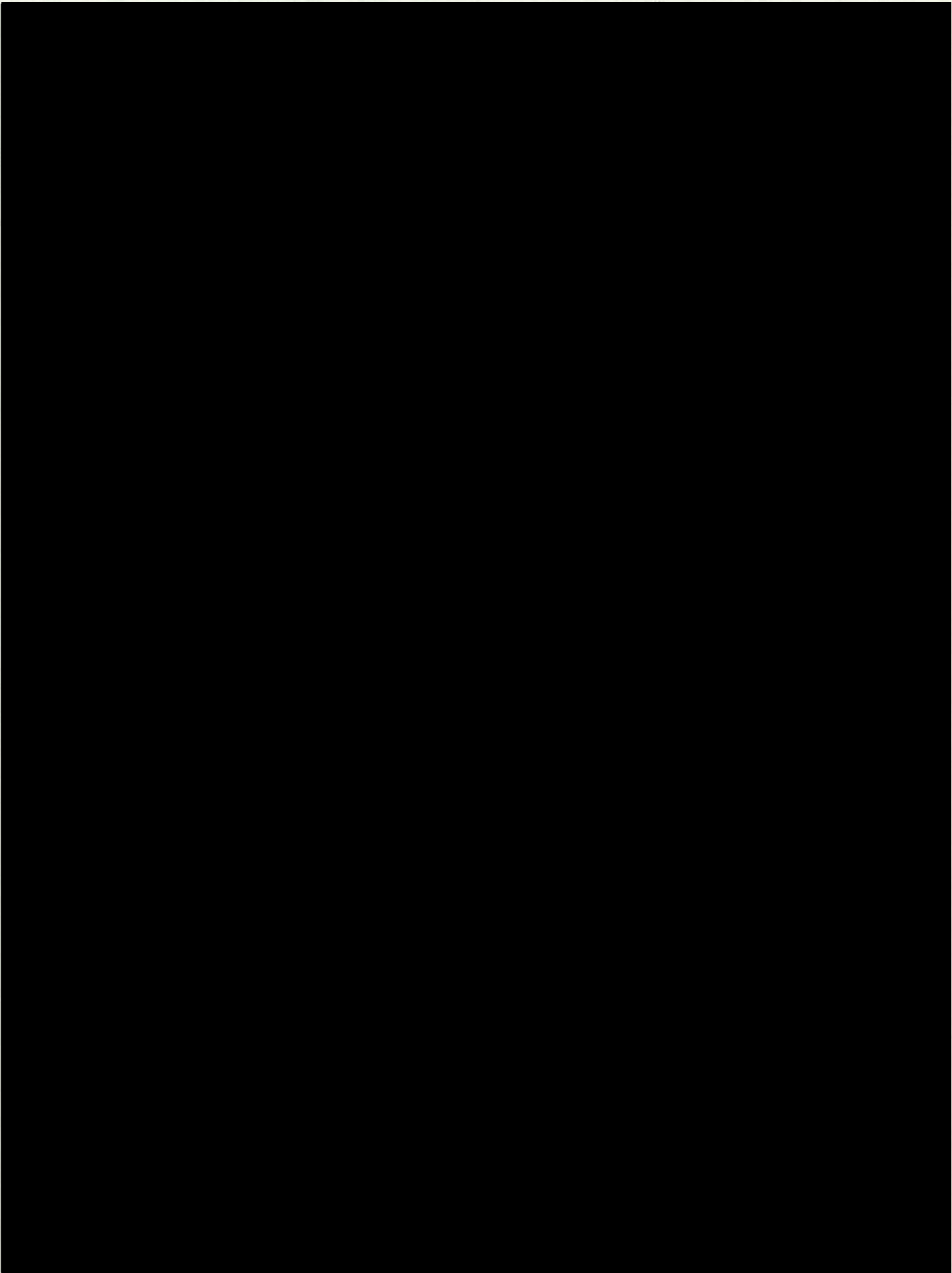
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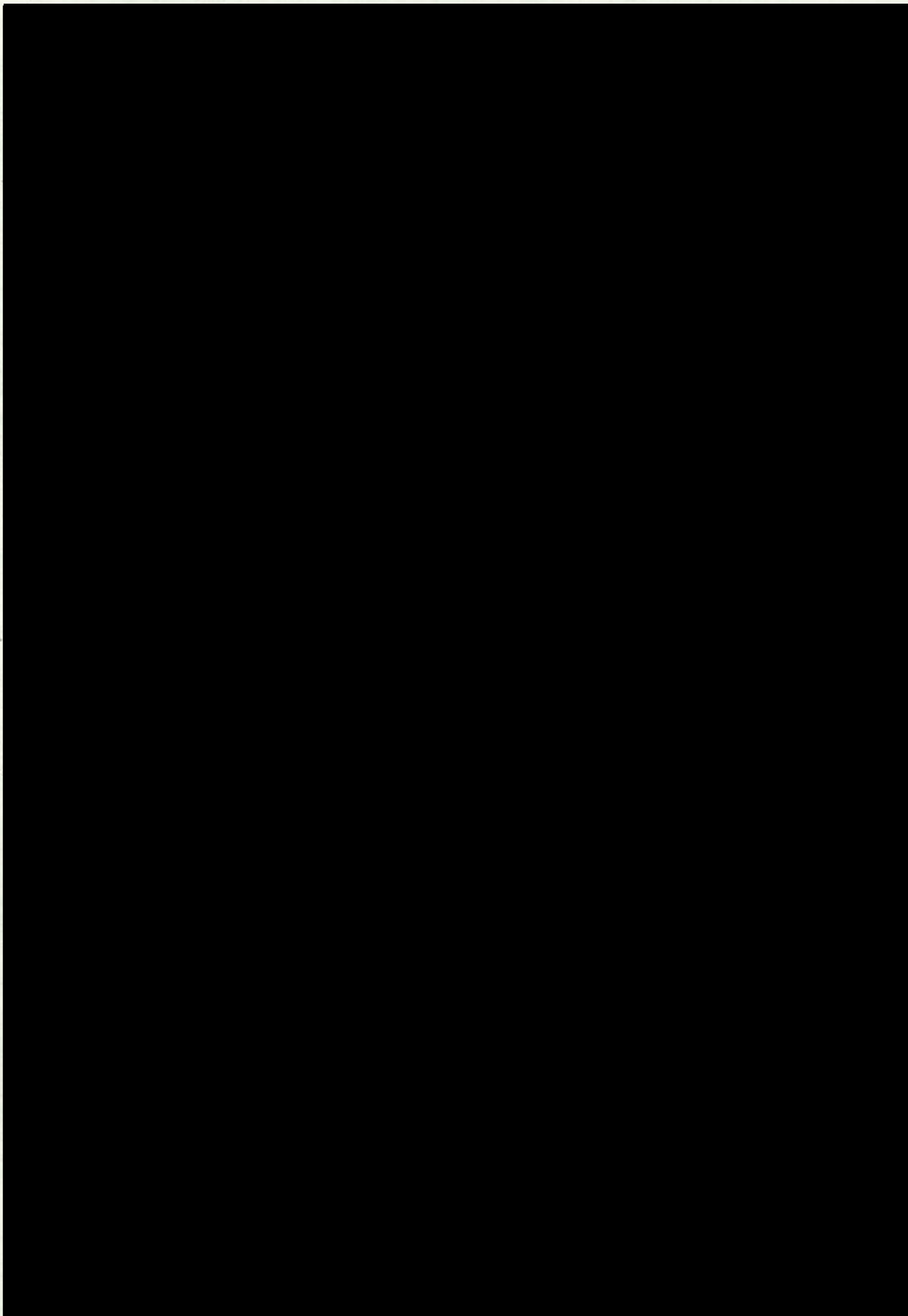


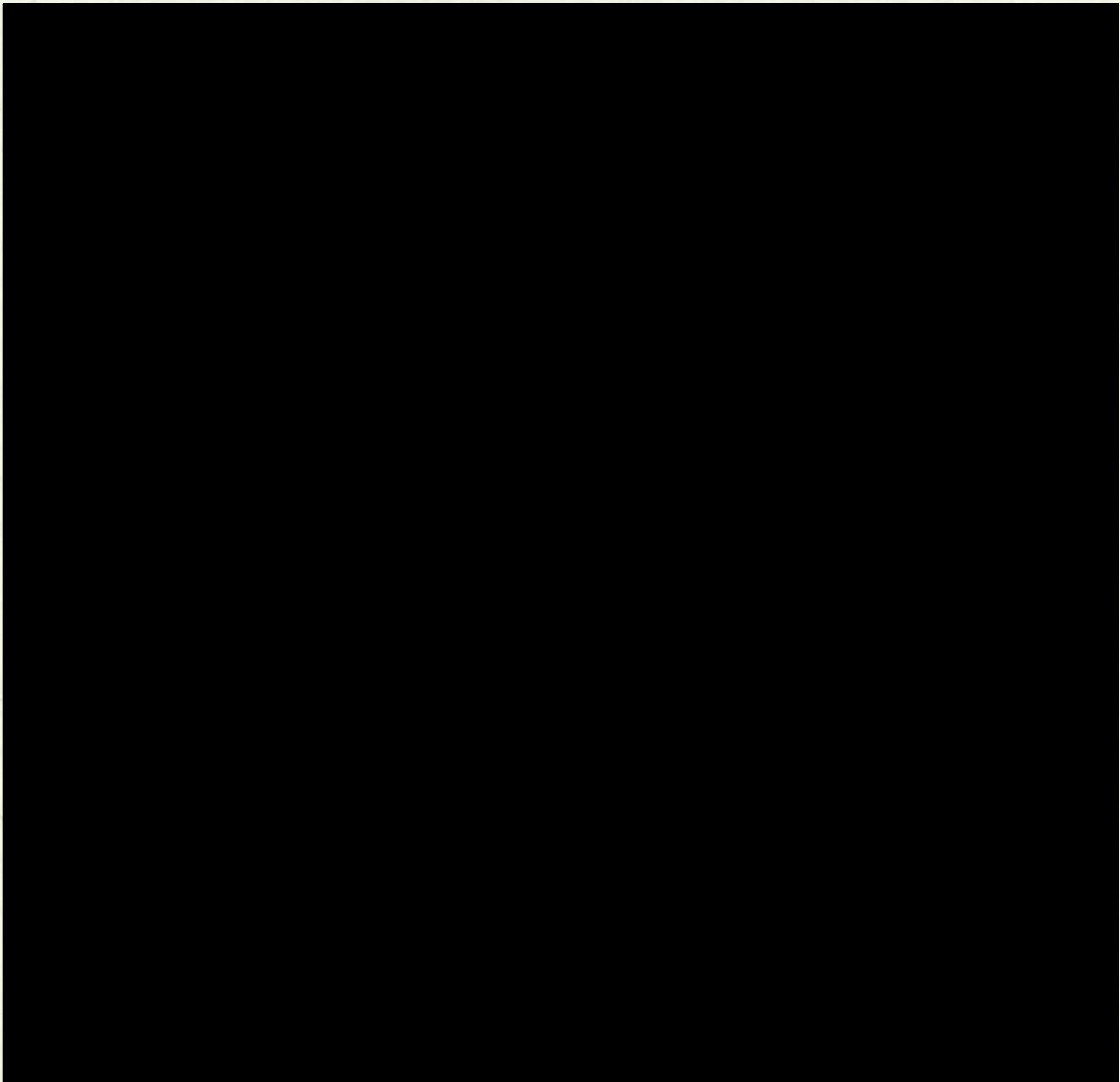




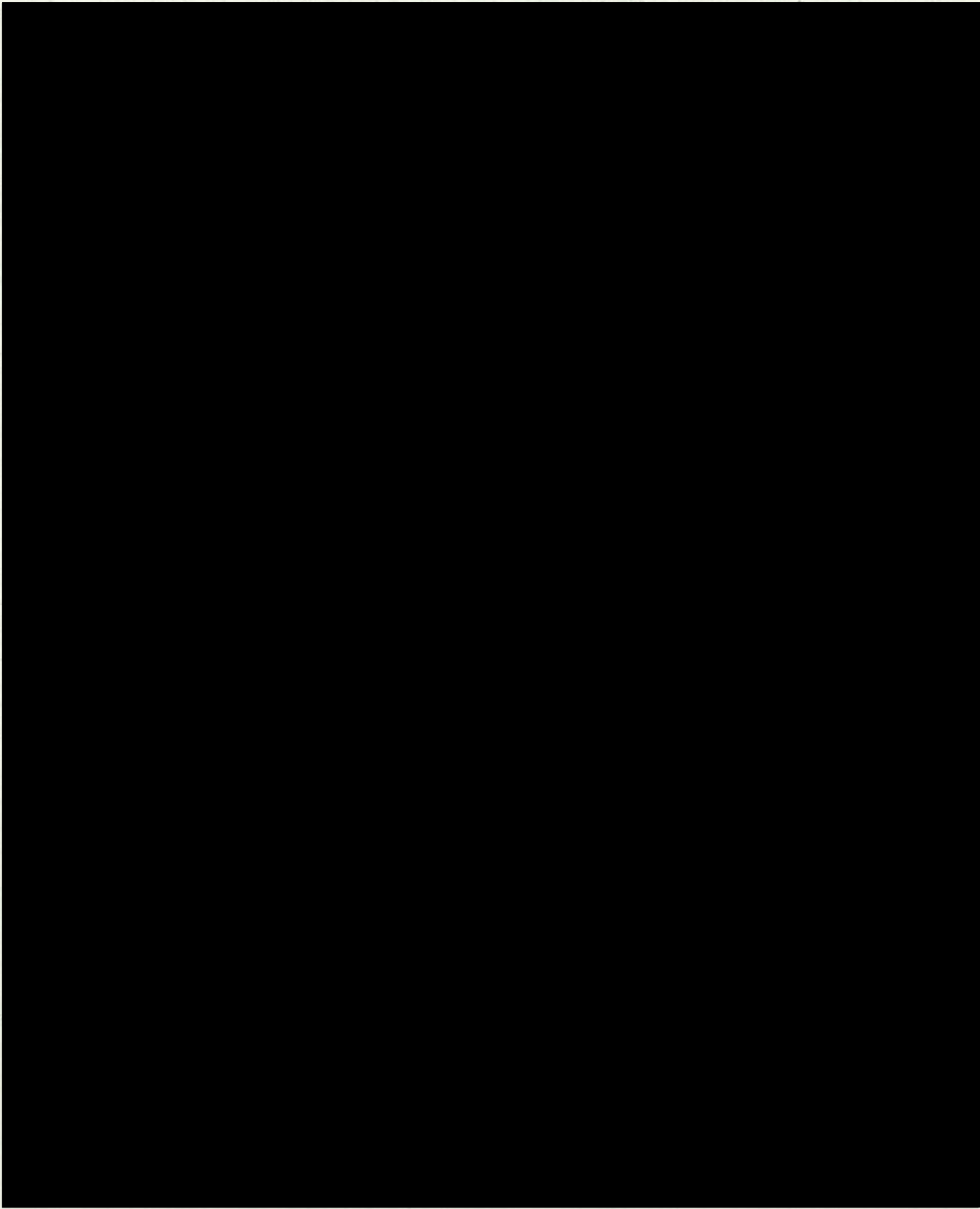








(3)



A P P E N D I X 10

Author's paper,
Manchester Symposium on Combined Convection,
1971

A P P E N D I X 11

Letter from Professor F.L. Test,

1975

A P P E N D I X 12

Second extract from Post Experience Course Notes,
The City University,
1974.

(Reference List may be found at end of Appendix 1).

II. ANNULUS ~ ANALYSIS

III. ANNULUS GEOMETRY

The annulus is an important cylindrical co-ordinate internal flow application. The flow-field is two-dimensional if the inner boundary is concentric. This inner wall replaces the tube axis as a boundary condition.

The formation of the partial differential equations, non-dimensional and finite-difference forms corresponding to them, is identical to the basic treatment of course. The boundary conditions become:

(i) At entry: $v_z = U_0, v_r = 0, p = p_0, T' = T_0'$

(ii) At inner wall: $v_z = 0, v_r = 0$

and either $T' = T'_{w1} = \text{const.}$ (constant wall temp.)
or $\frac{Q_1}{A_1} = k \left(\frac{\partial T'}{\partial r'} \right)_{r'=R_1} = \text{const.}$ (constant heat flux)

(iii) At outer wall: $v_z = 0, v_r = 0$

and either $T' = T'_{w2} = \text{const.}$ (constant wall temp.)
or $\frac{Q_2}{A_2} = k \left(\frac{\partial T'}{\partial r'} \right)_{r'=R_2} = \text{const.}$ (constant heat flux)

The integral continuity equation becomes:

$$\pi U_0 (R_2^2 - R_1^2) \rho_0 = \int_{R_1}^{R_2} 2\pi r' \rho v_z dr' \quad (5) \text{ ANN}$$

III. DIMENSIONLESS QUANTITIES

The dimensionless boundary conditions become:

(i) $U=1, V=0, P=P_0, T=T_0$

$$(ii) \quad U=0, V=0, \text{ and either } T=T_{w1} \\ \text{or } Q_1 = kA_1 \left(\frac{z_A^2}{R_1 + C_5 g} \right) \cdot \left(\frac{\partial T}{\partial r} \right)_{r=\bar{R}}$$

$$(iii) \quad U=0, V=0, \text{ and either } T=T_{w2} \\ \text{or } Q_2 = kA_2 \left(\frac{z_A^2}{R_2 + C_5 g} \right) \cdot \left(\frac{\partial T}{\partial r} \right)_{r=1}$$

and the continuity equation becomes:

$$\frac{1-\bar{R}^2}{2} = \int_{\bar{R}}^1 U \cdot r \cdot \left\{ C_0 + \frac{C_1 T}{G} + \frac{C_2 T^2}{G} \right\} \cdot dr \quad (14) \text{ ANN}$$

where $\bar{R} = \frac{R_1}{R_2}$ i.e. radius ratio.

for constant density:

$$\frac{1-\bar{R}^2}{2} = \int_{\bar{R}}^1 U \cdot r \cdot dr \quad A(14) \text{ ANN}$$

11.3 FINITE-DIFFERENCE QUANTITIES

Finally, the finite-difference boundary conditions become:

$$(i) \quad U_{1,m} = 1, V_{1,m} = 0, T_{1,m} = 0, P_{1,m} = P_0.$$

$$(ii) \quad U_{n,MIN+1} = 0, V_{n,MIN+1} = 0, \\ \text{and either } T_{n,MIN+1} = T_{w1} \\ \text{or } T_{n,MIN+1} = T_{n,MIN} + \frac{Q_1 R_1 + C_5 g}{A_1 z_A^2 (MIN) k} *$$

$$(iii) \quad U_{n,MR+1} = 0, V_{n,MR+1} = 0, \\ \text{and either } T_{n,MR+1} = T_{w2} \\ \text{or } T_{n,MR+1} = T_{n,MR} + \frac{Q_2 R_2 + C_5 g}{A_2 z_A^2 (MR) k} *$$

The finite-difference continuity equation is:

$$\frac{3(MR)^2}{8} (1-\bar{R}^2) = \sum_{n=1}^{\frac{MAXR}{2}} [2n + MIN - 1] \cdot U_{n+1,2n} + \\ \sum_{n=1}^{\frac{MAXR}{2}-1} \left[n + \frac{MIN}{2} \right] \cdot U_{n+1,2n+1} \quad (19) \text{ ANN}$$

The finite-difference radial mesh spacing that this represents is explained on the following page c.f. the tube case.

* This represents the original form of $\left(\frac{\partial T}{\partial r} \right)_{\text{wall}}$ as used with the tube (C/2/51 footnote)

11.4 PROGRAM ALTERATIONSInput Section

This is analogous to the Tube program, except that ISUNU only has two values 1 ~ square, 2 ~ adiabatic development. (However, it would be a straightforward matter to have ISUNU = 3 for the fully-developed adiabatic laminar profile ~ see Ref. 16)

Matrix

The number of equations for the T and U, V, P matrices are the same, provided that an inner wall condition is defined for the pressure.* Basically, the subscripts for the terms remain the same as before; where a value of the radius, and hence 'm', is involved in the calculations, this must be increased by MIN, as in the Figure on C/3/75. Tube 'm' becomes annulus 'm + MIN'.

The outer boundary condition is the same, except that, for example, the tube IFTOQ becomes the annulus IFTOQ(2). However, the cards specifying the inner wall boundary conditions have to be changed.

Apart from the above, and with the proviso that the pro-

* This replaces the tube axis condition $P_{n,1} = P_{n,2}$ with the inner wall condition $P_{n,2} - P_{n,1} = P_{n,3} - P_{n,2}$ i.e. constant pressure gradient. This is quite consistent with known tube results.

gram is for the City University computer only at present, the 'annulus' is essentially a development of the 'tube' program. However, one or two improvements in the tube program (notably the energy balance iteration) are not yet reflected in the annulus version.

The program print out in entirety is given as Section 11.6. A detailed explanation is not given, however, as the effects may be deduced by comparison with the corresponding parts in the tube program. It might be mentioned that the Nusselt Numbers (cards A3390 and A3400) are set at zero for the adiabatic investigation so far made.

11.5 INPUT DATA

The input data for each run requires a maximum of eight cards, as below.

CARD 1 (Integers)

ILAST	Total number of axial positions.
KLAST	Number of times Δz is doubled; it must be the same as the number of integers on card 4.
MAXR	The number of radial positions.
MR	Maximum value of 'm' in the fluid field. (See Figure 3 p C/3/75, where $R = (MR - MAXR) / MR$)
ISUNU	= 1, uniform velocity profile; = 2, adiabatic development.
IFTQ (1)	Inner boundary thermal choice = 1, Temp; = 2, heat flux
IFTQ (2)	As for above, but outer boundary choice.

CARD 2 (Real Numbers)

CO1 - CO5 Temperature dependent property variables.

CARD 3 (Real Numbers)

CO6	Property variable.
GNUMBR	Dimensionless group representing geometry.
PENTRY	Dimensionless entry pressure, usually 0.0.

DELTZ The initial value of Δz

PRNDTL Prandtl Number } See Nomenclature

RYNLDS Reynolds Number }

CARD 4 (Integers)

L(K) The number of steps at which the step size is doubled
K varies from 1 to KLAST.

CARD 5 (Real Numbers)

S Determines sign and magnitude of $UP\phi RDN$, as in tube program.

RATIO Bulk density ratio ρ_m/ρ_o .

CARD 6 (Real Number)

ZBARX Adiabatic entry length when $ISUNU=2$.
If $ISUNU=1$ card is omitted.

CARD 7 (Real Number)

TWALL(1) or } Inner wall boundary condition { $IFT\phi Q(1)=1$
HFLOX(1) } $IFT\phi Q(1)=2$

CARD 8 (Real Number)

TWALL(2) or } Outer wall boundary condition { $IFT\phi Q(2)=1$
HFLOX(2) } $IFT\phi Q(2)=2$

12 ANNULUS - PROGRAM

FORTRAN COMPILATION BY XXFAE MK 4D DATE 28/09/72 TIME 03.32.45

```
LIST(LP)
PROGRAM(CULL 520022)
INPUT 1=CKO
OUTPUT 2=LPU
TRACE 0
END
```

This introductory set is relevant to the TCW computer.

	MASTER COLL	90
	COMBINED CONVECTION IN ANNULI	100
	REAL MTWO,HDNE,NUSSLT(2)	110
	DIMENSION U(3,30),V(3,30),T(3,30),P(3,31),A(88,66),B(88),X1(29),	120
	X2(88),L(10),IFTOU(2),TWALL(2),HFLUX(2)	130
	CALL ITIME(1SEC1)	140
1	READ(1,2)ILAST,KLAST,MAXR,MR,ISUNU,(IFTOU(I),I=1,2),CO1,CO2,CO3,	150
	CO4,CO5,CO6,GNUMBR,PENTRY,DELTZ,PRNDTL,RYNLDS	160
2	FORMAT(7I5/F10.6,F17.15,F25.22,F15.13,F10.7/F20.17,F15.1,F5.1,	170
	F10.2,F14.10,F17.4)	180
	IF (ILAST.LT.0)STOP	181
	READ(1,12)(L(K),K=1,KLAST)	182
12	FORMAT(10I3)	190
	READ(1,3)S,RATIOU	200
3	FORMAT(F4.1,F10.6)	210
	WRITE(2,4)S,RATIO	220
4	FORMAT(1H1,3H S=,F5.2,22H + SIGN=UP,- SIGN=DOWN,15H DENSITY RATIO=	230
	1,F9.6)	240
	GO TO(13,15),ISUNU	250
13	DO 14 M=1,2	251
	IFTOU1=IFTOU(M)	260
	GO TO(5,8),IFTOU1	270
5	READ(1,6)TWALL(M)	280
6	FORMAT(F10.1)	290
	WRITE(2,7)TWALL(M)	300
7	FORMAT(1H ,25HCONSTANT WALL TEMP.TWALL=,F12.3)	310
	GO TO 11	320
8	READ(1,9)HFLUX(M)	330
9	FORMAT(F10.1)	340
	WRITE(2,10)HFLUX(M)	350
10	FORMAT(1H ,25HCONSTANT HEAT FLUX.HFLUX=,F12.3)	360
11	CONTINUE	370
14	CONTINUE	371
	GO TO(15,60),ISUNU	380
15	WRITE(2,16)(L(K),K=1,KLAST)	390
16	FORMAT(1H ,16H VALUES OF L(K),/(7I4))	400
	WRITE(2,17)ILAST,MAXR,MR,CO1,CO2,CO3,CO4,CO5,CO6,RYNLDS,PRNDTL,	410
	GNUMBR,PENTRY	420
17	FORMAT(1H ,7H ILAST=,I6,6H MAXR=,I6,4H MR=,I6,5H CO1=,F8.6,5H CO2=	430
	1,F13.10,5H CO3=,F21.18,5H CO4=,F9.6,5H CO5=,F9.6,5H CO6=,F15.12/	440
	26H RYNLDS=,F10.1,4H PRNDTL=,F8.6,8H GNUMBR=,I11.2,6H PENTRY=,F5.3)	450
	GO TO(26,20),ISUNU	460
20	READ(1,21)ZBARX	470
21	FORMAT(F10.6)	480
	WRITE(2,22)ZBARX	490
22	FORMAT(1H ,24H ADIABATIC ENTRY LENGTH=,F10.6)	500
	DO 25 M=1,2	510
	TWALL(M)=0.0	520
	HFLUX(M)=0.0	521
25	CONTINUE	530
26	DO 30 M=2,MAXR	540
	U(1,M)=1.0	550
	U(2,M)=U(1,M)	560
	V(1,M)=0.0	570
	V(2,M)=0.0	580
	T(1,M)=0.0	590
	T(2,M)=0.0	600
	P(2,M)=PENTRY	610
30	P(2,MAXR+1)=PENTRY	620
	NIN=MR-MAXR	630
	XMAXR=MR	640
	NUMBR=MAXR/2	650
	ZBAR=0.0	660
	ITALLY=0	

	K=1	670
60	DIZSQ=1.0/DELIZ**2	680
	DPRRE=1.0/(PRNDTL*RYNLDS)	690
	FUNC1=DPRRE/DELIZ	700
	SQM2=XMAXR**2/2.0	710
	HALFM=XMAXR/2.0	720
	DUZM=XMAXR/12.0	730
	DUZM8=DUZM*8.0	740
	FMSQ=SQM2*DPRRE	750
	MAX1=MAXR-1	760
	MAXP1=MAXR+1	770
	DO 101 I=1,MAX1	780
	DO 101 J=1,MAX1	790
	A(I,J)=0.0	800
101	FUNC2=DPRRE*(4.0*SQM2-DIZSQ)	810
	DO 102 I=1,MAX1	820
	I1=I+1	830
	B(I)=((U(2,I1)-2.0*FUNC1)*T(2,I1)+FUNC1*T(1,I1))/DELIZ	840
	X1=I+MIN	850
	A(I,I1)=HALFM*V(2,I1)+FMSQ*(1.0-2.0*(X1+1.0))/X1	860
	A(I,I)=U(2,I+1)/DELIZ+FUNC2	870
102	CONTINUE	871
	DO 103 I=2,MAX1	880
	X1=I+1+MIN	890
	A(I,I-1)=FMSQ*(3.0-2.0*X1)/(X1-1.0)-HALFM*V(2,I+1)	900
103	XMIN=MIN	910
	FUNC3=HALFM*V(2,MAXR)+FMSQ*(1.0-2.0*XMAXR)/(XMAXR-1.0)	920
	IFTOQB=IFTOQ(2)	930
	GO TO(106,107),IFTOQB	931
107	B(MAX1)=B(MAX1)-HFLUX(2)*FUNC3	940
	A(MAX1,MAX1)=A(MAX1,MAX1)+FUNC3	950
	GO TO 108	960
106	B(MAX1)=B(MAX1)-TWALL(2)*FUNC3	970
106	F4=-FMSQ*(1.0+2.0*XMIN)/(XMIN+1.0)-HALFM*V(2,2)	980
	IFTOQA=IFTOQ(1)	990
	GO TO(103,104),IFTOQA	1091
105	B(1)=B(1)-F4*TWALL(1)	991
	GO TO 109	992
104	B(1)=B(1)-F4*HFLUX(1)	993
	A(1,1)=A(1,1)+F4	994
109	CALL MAT(MAX1,A,B,X1)	1000
	DO 90 M=1,MAX1	
	X2(M)=T(1,M+1)	T1
	MA=MAX1+M	T2
	X2(MA)=T(2,M+1)	T3
	T(1,M+1)=T(2,M+1)	T4
	T(2,M+1)=X1(M)	T5
90	CONTINUE	T6
	BTWO7=0.5/DELIZ	T7
	DIVZ=1.0/DELIZ	1010
	MAX32=3*MAXR-2	1020
	MAX21=2*MAXR-1	1030
	MAX2=2*MAXR	1040
	MTWO=DIZSQ-4.0*SQM2	1050
	DIVRE=1.0/RYNLDS	1060
	MUNE=SQM2*DIVRE	1070
	UPORDN=S*RATIO/D/RYNLDS**2	1080
	UP1=UPORDN*GNUMBR*CO4	1090
	UP2=UPORDN*CO5	1100
	UP3=UPORDN*CO6	1110
	DIVRE2=2.0/RYNLDS	1120
	DIVRE3=DIVRE*DIZSQ	1130
	DIVRE4=HALFM/(DELIZ*RYNLDS)	1140
	DO 110 I=1,MAX32	1150
	B(I)=0.0	1160
	DO 110 J=1,MAX32	1170
		1180


```

110  A(I,1)=0.0 1190
      B(MAX32)=0.75*SQM2*(1.0-(XMIN/XMAXR)**2) 1200
      B(MAX21)=DTW0Z*(U(2,MAX1)+U(2,MAXR)) 1220
      C(MAX2)=U(2,2)*DTW0Z 1230
      A(MAX21,MAX32-1)=DTW0Z 1240
      A(MAX21,MAX32)=DTW0Z 1250
      A(MAY2,MAX2)=DTW0Z 1260
      A(MAX21,MAX21-1)=-XMAXR 1270
      A(MAY21,MAX21)=XMAXR**2/(XMAXR-1.0) 1280
      XMA=XMIN+1.0 1290
      XMR=YMIN+2.0 1291
      A(MAY2,MAXP1)=XMAXR*XMB/XMA 1292
      A(MAY1-1,MAX1-1)=-DIVZ 1300
      A(MAY1,MAX1)=-DIVZ 1310
      A(MAXR,MAXR-2)=HALFM 1320
      A(MAXR,MAXP)=-HALFM 1330
      A(MAXP1,1)=HALFM*3.0
      A(MAXP1,2)=-HALFM*4.0
      A(MAXP1,3)=HALFM
      TALPHA=CO1+(CO2+CO3*T(2,2))*T(2,2) 1360
      TBETA=CO2+2.0*CO3*T(2,2) 1370
      GU TC(112,111),IF T0QA 1380
111  TWALL(1)=T(2,2)+HFLUX(1) 1381
112  GU TC(114,113),IF T0QB A1382
113  TWALL(2)=T(2,MAXR)+HFLUX(2) 1383
114  TONE=TBETA*(T(2,3)-TWALL(1)) 1384
      TTWO=TBETA*(T(2,2)-T(1,2))*DIZSQ 1390
      VONE=HALFM*V(2,2) 1400
      TFOUR=TALPHA*(2.0*XMIN+3.0)/XMA 1410
      TSIX=TALPHA*DIZSQ 1420
      A(1,MAX2+1)=MONE*(TFOUR+0.5*TONE)-VONE 1430
      UONE=DIVZ*U(2,2) 1440
      A(1,MAX2)=DIVRE*(TALPHA*MTWO+2.0*TTWO) 1450
1-UONE 1460
      A(MAXP1,MAXP1+1)=MONE*(TFOUR+TONE)-VONE 1470
      VARIT=TALPHA*(DIZSQ-2.0*SQM2*(2.0*XMB**2-4.0*XMB+3.0)/XMA**2) 1480
      A(MAXP1,MAXP1)=DIVRE*(VARIT+TTWO)-UONE 1490
      SONE=V(2,2)*(-U(2,2)*DIVZ+DIVRE*(2.0*TSIX+TTWO)) 1500
      STWO=-DIVRE*(V(1,2)*TSIX+HALFM*DELTZ*TTWO*U(2,3)) 1510
      B(MAXP1)=SONE+STWO 1520
      UPORLN2=UP1+T(2,2)*(UP3*(2.0*T(2,2)-T(1,2))-UP2) 1530
      BONE=UPORLN2-DIVZ*(P(2,2)+U(2,2)**2)-U(1,2)*TALPHA*DIVRE 1540
      BTWO=DIVRE*U(2,2)*(TSIX+TTWO)-DIVRE*4*TONE*(V(2,2)-V(1,2)) 1550
      B(1)=BONE+BTWO 1560
      TALPHA=CO1+(CO2+CO3*T(2,MAXR))*T(2,MAXR) 1570
      TBETA=CO2+2.0*CO3*T(2,MAXR) 1580
      TONE=TBETA*(TWALL(2)-T(2,MAX1)) 1630
      TTWO=TBETA*(T(2,MAXR)-T(1,MAXR))*DIZSQ 1640
      VONE=HALFM*V(2,MAXR) 1650
      TTHREE=TALPHA/(XMAXR-1.0) 1660
      TFOUR=TTHREE*(2.0*XMAXR-1.0) 1670
      TFIVE=TTHREE*(2.0*XMAXR-3.0) 1680
      TSIX=TALPHA*DIZSQ 1690
      UONE=DIVZ*U(2,MAXR) 1700
      A(MAY1,MAX32-1)=MONE*(TFIVE-0.5*TONE)+VONE 1710
      A(MAY1,MAX32)=DIVRE*(TALPHA*MTWO+2.0*TTWO)-UONE 1720
      A(MAYR,MAX21-1)=MONE*(TFIVE-TONE)+VONE 1730
      VARIT=TALPHA*(DIZSQ-SQM2*(4.0+2.0/(XMAXR-1.0)**2)) 1740
      A(MAYR,MAX21)=DIVRE*(VARIT+TTWO)-UONE 1750
      BONE=V(2,MAXR)*(-U(2,MAXR)*DIVZ+DIVRE*(2.0*TSIX+TTWO)) 1760
      BTWO=-DIVRE*(V(1,MAXR)*TSIX+HALFM*DELTZ*TTWO*U(2,MAX1)) 1770
      B(MAXR)=BONE+BTWO 1780
      UPORLN3=UP1+T(2,MAXR)*(UP3*(2.0*T(2,MAXR)-T(1,MAXR))+UP2) 1790
      BONE=UPORLN3-DIVZ*(P(2,MAXR) 1800
      +U(2,MAXR)**2)-U(1,MAXR)*TALPHA*DIVRE3 1810
      BTWO=DIVRE*U(2,MAXR)*(TSIX+TTWO) 1820

```



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1-DIVRE4*TONE*(V(2,MAXR)-V(1,MAXR))
B(MAX1)=BONE+BTWO
MAXR3=MAXR-3
DO 115 I=1,MAXR3
MAXI1=MAX2+I
XI=I+MIN
VARI=2*(MIN+I)+1
MAXI2=MAXR+I
XM=I+2+MIN
XM1=I+1+MIN
MAXI3=MAXP1+I
MAXI4=MAX21+I
I1=I+1
I2=I+2
I3=I+3
E(MAXI1)=DTWOZ*(U(2,I2)+U(2,I1))
A(MAXI1,MAXI1-1)=DTWOZ
A(MAXI1,MAXI1)=DTWOZ
A(MAXI1,MAXI2)=-XMAXR
A(MAXI1,MAXI2+1)=XMAXR*XM/XM1
A(MAXI3,I)=DOZM0
A(MAXI3,I2)=-DOZM0
A(MAXI3,I3)=DOZM
A(MAXI3+1,I)=-DOZM
A(I,I)=-DIVZ
TALPHA=CO1+(CO2+CO3*T(2,I2))*T(2,I2)
TBETA=CO2+2.0*CO3*T(2,I2)
TONE=TBETA*(T(2,I3)-T(2,I1))
TTWO=TBETA*(T(2,I2)-T(1,I2))*DIZSQ
VONE=HALFM*V(2,I2)
TTHREE=TALPHA/XM1
TFOUR=TTHREE*(VARI+2.0)
TFIVE=TTHREE*VARI
TSIX=TALPHA*DIZSQ
A(I1,MAXI4)=MONE*(TFIVE-0.5*TONE)+VONE
A(I1,MAXI4+2)=MONE*(TFOUR+0.5*TONE)-VONE
A(MAXI3,MAXI2)=MONE*(TFIVE-TONE)+VONE
A(MAXI3,MAXI2+2)=MONE*(TFOUR+TONE)-VONE
UONE=DIVZ*U(2,I2)
A(I1,MAXI1)=DIVRE*(TALPHA*MTWO+2.0*TTWO)-UONE
VARIT=TALPHA*(DIZSQ-SQM2*(4.0+2.0/XM1**2))
A(MAXI3,MAXI3)=DIVRE*(VARIT+TTWO)-UONE
BONE=V(2,I2)*(-U(2,I2)*DIVZ+DIVRE*(2.0*TSIX+TTWO))
BTWO=-DIVRE*(V(1,I2)*TSIX+
1HALFM*DELTA*TTWO*(U(2,I3)-U(2,I1)))
B(MAXI3)=BONE+BTWO
UPORDN4=UP1+T(2,I2)*(UP3*(2.0*T(2,I2)-T(1,I2))+UP2)
BONE=UPORDN4-DIVZ*(P(2,I2)+U(2,I2)**2)
1-U(1,I2)*TALPHA*DIVRE3
BTWO=DIVRE2*U(2,I2)*(TSIX+TTWO)-
1DIVRE4*TONE*(V(2,I2)-V(1,I2))
B(I1)=BONE+BTWO
115 A(MAXR+2,1)=DOZM*6.0
A(MAXR+2,2)=DOZM
A(MAX21,MAXR-3)=0.0
MHALF=MAXR/2
DO 120 M=1,MHALF
M1=2*M+MIN-1
M2=MAX21+2*M-1
A(MAX32,M2)=M1
MHAL1=MHALF-1
DO 121 M=1,MHAL1
XM=M
M1=MAX21+2*M
A(MAX32,M1)=XM+0.5*XM*IN_
121 CONTINUE

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      DO 95 M=1,MAX1
      T(1,M+1)=X2(M)
      MA=MAX1+M
      T(2,M+1)=X2(MA)
95  CONTINUE
      CALL MAT(MAX32,A,B,X2)
      MAXF=MAXR-1
      DO 100 M=1,MAXF
      T(3,M+1)=X1(M)
      P(3,M+1)=X2(M)
      MA=MAXR+M
      V(3,M+1)=X2(MA)
      MB=MA+MAXF
      U(3,M+1)=X2(MB)
100 CONTINUE
      P(3,MAXR+1)=X2(MAXR)
      U(3,1)=0.0
      V(3,1)=0.0
      T(3,1)=TWALL(1)
      P(3,1)=P(3,2)*2.0-P(3,3)
      SUMT=0.0
      DO 130 M=2,MAXR
      XM=M+MIN
      SUMT=SUMT+(XM-1.0)*U(3,M)*T(3,M)
130  TMEAN=SUMT*2.0/XMAXR**2
      NUSSLT(1)=0.0
      NUSSLT(2)=0.0
      ITALLY=ITALLY+1
      ZBAR=ZBAR+DELTZ
      WRITE(2,140)DELTZ,ZBAR
140  FORMAT(1H,7H DELTZ=,F10.6,6H ZBAR=,F10.6)
      WRITE(2,141)ITALLY,TMEAN,(TWALL(M),M=1,2),(NUSSLT(M),M=1,2),(U(3,M),M=2,MAXR)
141  FORMAT(1H,7H TALLY=,I4,7H TMEAN=,F11.2,6H TWALLS:,2F11.2,
      19H NUSSLTS:,2F10.5/44H AXIAL VELOCITY,RADIAL VELOCITY,TEMPERATURE,
      29HPRESSURE,/(12F10.5))
      WRITE(2,142)(V(3,M),M=2,MAXR)
142  FORMAT(1H,12F10.7)
      WRITE(2,143)(T(3,M),M=2,MAXR)
143  FORMAT(1H,12F10.2)
      MAXR1=MAXR+1
      WRITE(2,144)(P(3,M),M=2,MAXR1)
144  FORMAT(1H,12F10.4)
      IF (ILAST-ITALLY)400,400,200
200  IF (L(K)-ITALLY)215,215,205
205  DO 210 M=1,MAXR
      U(1,M)=U(2,M)
      V(1,M)=V(2,M)
      T(1,M)=T(2,M)
210  CONTINUE
      GO TO 220
215  K=K+1
      DELTZ=2.0*DELTZ
220  DO 300 M=1,MAXR
      U(2,M)=U(3,M)
      V(2,M)=V(3,M)
      T(2,M)=T(3,M)
300  CONTINUE
      DO 350 M=1,MAXR1
      P(2,M)=P(3,M)
350  CONTINUE
      GO TO(60,370),ISUNU
370  IF (ZPARX-ZBAR)60,13,60
400  GO TO(440,410),IFTUQA
410  GO TO(440,420),IFTUQB
420  TBAR=2.0*(HFLUX(1)+HFLUX(2))*XMAXR*(ZBAR-ZBARX)*LPRRERATIO

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	WRITE(2,430)TBAR	3770
430	FORMAT(1H ,6H TBAR=,F12.2)	3780
440	WRITE(2,450)	3790
450	FORMAT(1H ,12H END OF RUN.)	3800
	CALL ITIME(ISEC2)	3810
	ISEC=ISEC2-ISEC1	3820
	WRITE(2,460)ISEC	3830
460	FORMAT(1H ,6H TIME=,16,4HSEC.)	3840
	GO TO 1	3850
	END	3860

END OF SEGMENT, LENGTH 2832, NAME COLL

	SUBROUTINE MAT(N,A,Y,X)	3090
	DIMENSION A(88,88),Y(88),X(29)	
5	M=N-1	3110
	DO 10 I=1,M	3120
	L=I+1	3130
	DO 10 J=L,N	3140
	IF(A(J,I).EQ.0.)GOTO10	3150
6	DO 8 K=L,N	3160
8	A(J,K)=A(J,K)-A(I,K)*A(J,I)/A(I,I)	3170
	Y(J)=Y(J)-Y(I)*A(J,I)/A(I,I)	3180
10	CONTINUE	3190
	X(N)=Y(N)/A(N,N)	3200
	DO 30 I=1,M	3210
	K=N-I	3220
	L=K+1	3230
	DO 20 J=L,N	3240
20	Y(K)=Y(K)-X(J)*A(K,J)	3250
	X(K)=Y(K)/A(K,K)	3260
30	CONTINUE	3270
	RETURN	3280
	END	3290

END OF SEGMENT, LENGTH 248, NAME MAT

There is no time call Subroutine necessary with this computer

NOMENCLATURE

A	wall area
C_1, C_2, C_3	constants defined by equation (5)
C_4, C_5, C_6	constants defined by equation (6)
C_{01}	$= C_1 + C_2 T'_0 - C_3 T'^2_0$
C_{02}	$= \frac{1}{G \cdot C_5} (C_2 - 2C_3 T'_0)$
C_{03}	$= -C_3 / G^2 \cdot C_5^2$
C_p	specific heat of fluid at constant pressure
G	$= \frac{R^3_g}{v_A^2}$ (dimensionless)
g	acceleration due to gravity
k	thermal conductivity of fluid
M	number of radial divisions
m, n	radial and axial positions
p	pressure
P	$= \frac{p - p_0}{\rho_m U_o^2}$ dimensionless pressure
Pr	$= \mu_A \left(\frac{C_p}{k} \right)_m$ Prandtl number
Q	wall heat flow rate
R	tube radius (equivalent) $= R_2 - R_1$
Re	$= \frac{v_z \cdot R_{om}}{\mu_A}$ Reynolds Number
r'	radial co-ordinate
r	$= r' / R_z$ dimensionless radial co-ordinate
T'	temperature
T	$= G \cdot C_5 (T' - T'_0)$, dimensionless temperature
U	$= v_z / U_o$, dimensionless axial velocity
U_r, U_z	radial and axial velocities
V	$= v_r / U_o$, dimensionless radial velocity
z'	axial co-ordinate
z	$= z' / R_z$ dimensionless axial co-ordinate
μ, ν	dynamic, kinematic viscosities
ρ	density

Subscripts

A	property reference point, equations (5), (6)
m	mean value
o	entrance condition
R	wall condition
r, z	radial, axial directions