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State estimation using a network of distributed observers with switching communication topology ☆

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ABSTRACT

State estimation of linear time-invariant (LTI) systems by using a network of distributed observers is studied in this paper. We assume that each observer has access to a local measurement which may be insufficient to provide the observability of the system, but the ensemble of all measurements in the network guarantees the observability. In this condition, the objective is to design a distributed state estimation approach such that, while the observers can exchange their estimated state vectors under a communication network, the estimated state vector of each observer converges to the state vector of the system. We consider a scenario when the communication links may fail and rebuild over time and the communication network does not stay connected constantly. Accordingly, the main contribution of the paper is to propose a distributed approach (with guarantees on the feasibility of the design) such that the state vector of the system is estimated by each observer if the union/joint of communication links in bounded intervals of time makes the network communication graph connected. Moreover, we also consider a scenario when the LTI system is subject to external disturbances and measurement noise. In this case, we derive sufficient conditions on the proposed approach such that if the communication topology stays connected during links failure, a desired $\mathscr{H}_\infty$ performance to attenuate the effect of external disturbances and measurement noise on estimation errors is guaranteed. Simulation results show the effectiveness of the proposed estimation approach.

1. Introduction

This paper deals with the state estimation problem using a network of distributed measurements. Suppose that the output of a dynamical system is measured using a group of sensors distributed across N nodes, where the nodes are connected via a communication network. We consider an observer at each node, and assume that each observer only gets access to a local measurement and the information from the neighboring nodes. Under these conditions, by properly designing the observers, we intend to estimate the whole state vector of the dynamical system at each observer. This problem is termed by the *distributed estimation problem*.

Let the measurements at different nodes together monitor a linear time-invariant (LTI) system, which is described by

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y_i &= C_i x, \; i \in \mathbf{N}, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ represents the state vector, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{n\times n}$ is the state matrix, $B \in \mathbb{R}^{n\times m}$ denotes the input matrix, and $\mathbf{N} := \{1, 2, \ldots, N\}$ is the set of the nodes. Moreover, $y_i \in \mathbb{R}^{p_i}$ is the measurement output at the ith node, and $C_i \in \mathbb{R}^{p_i\times n}$ is the associated output matrix where $\sum_{i=1}^{N} p_i = p$. Accordingly, the collection of all the outputs can be considered as

$$y = \begin{bmatrix} y_1^\top & y_2^\top & \ldots & y_N^\top \end{bmatrix}^\top,$$

which is corresponding to the following integrated output matrix:

$$C = \begin{bmatrix} C_1^\top & C_2^\top & \cdots & C_N^\top \end{bmatrix}^\top.$$

The goal of distributed state estimation is to build a network of observers such that the estimated state vector of the ith observer

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denoted by $\hat{x}_i$ converges to x, by only accessing the input u, the local measurement y_i and the estimated state vectors of neighboring nodes. To guarantee the observability of the system, the LTI system is assumed to be *jointly observable*, that is, the pair (C, A) is observable, while the pair (C_i, A) is not necessarily observable for all $i \in \mathbf{N}$.

Since only the assembly of the measurements guarantees the estimation of the dynamical system state vector, it is necessary for the nodes to communicate with each other. However, due to many problems such as obstacles, malfunction of communication devices, etc., the connection among nodes may not be steady. In other words, the communication links can fail and rebuild at some time instants, and the communication topology can be switching over time. This issue influences state estimation and can even cause the whole estimation scheme to fail. In this condition, it is necessary to devise a distributed estimation scheme to guarantee state estimation in the presence of communication links failure.

1.1. Literature review and existing problems

The general idea of distributed state estimation in the existing literature is to extend the classical linear observers to distributed networks of observers by considering cooperative terms in the design. For instance, in Kamgarpour and Tomlin (2008) and Olfati-Saber (2007, 2009), the classical Kalman filter is extended to the distributed estimation problem. In Han et al. (2019) and Kim et al. (2016), Luenberger-based distributed state estimation schemes were proposed. In Shen et al. (2010), $\mathscr{H}_\infty$-based consensus filtering was extended to a distributed scheme. Again, this idea was addressed in more complex scenarios such as resilient distributed state estimation (Mitra et al., 2019; Mitra & Sundaram, 2019), distributed state estimation of nonlinear systems (He et al., 2020; Yang et al., 2020), distributed state estimation with prescribed convergence rate (Wang et al., 2019; Wang & Morse, 2018), etc.

All the aforementioned studies focused on the distributed estimation, where the network enjoys a steady and fixed communication topology. To extend the state estimation problem to networks of observers with communication links failure, in Aclkmese et al. (2008), a class of distributed estimators in the presence of switching topologies was developed. However, based on that design, distributed estimation relied on the observability at all nodes (i.e., (C_i, A) should be observable for all $i \in \mathbf{N}$). Moreover, in that study, all the switching topologies were supposed to be connected, which might not have been satisfied in practice. In Ugrinovskii (2013), a distributed robust estimation strategy in the presence of Markovian switching topologies with spanning trees was introduced. The effectiveness of that approach was conditional to some rank and inequality constraints with respect to the systems and observer parameters, which limited the application of the proposed strategy to specific scenarios. A similar problem affected the strategy addressed in Xu et al. (2020), in which distributed state estimation in the presence of jointly connected switching topologies was studied. Indeed, in Xu et al. (2020), it was shown that the asymptotic state estimation can be achieved if there are some constraints on the dynamical system parameters and the set of switching topologies, whose feasibility remained unknown. Therefore, under that design, distributed state estimation was not guaranteed for all scenarios. Similar to Ugrinovskii (2013), a distributed estimation design in the presence of random switching topologies described by a Markov chain was addressed in Zhang and Zhang (2012). However, that scheme focused on the estimation of unknown parameters rather than state vectors, which is different from the problem considered in the current study.

1.2. Objectives and contributions

By investigating the existing results in the literature, it can be said that proposing a systematic design framework that guarantees state estimation of jointly observable LTI systems in the presence of jointly connected switching topologies is an open problem to be studied. This paper proposes a design methodology that solves the distributed state estimation problem in LTI systems in the presence of communication links failure. Compared with the existing results in the literature, the main contributions of the paper are as follows:

- Different from the estimation scheme introduced in Aclkmese et al. (2008) which was only applicable to connected switching topologies, the strategy proposed in this paper allows to perform state estimation in the presence of jointly connected switching topologies. Moreover, in spite of Aclkmese et al. (2008), in the current study the local measurement at each node is not required to guarantee the observability of the system, and the joint observability of the nodes is sufficient for estimation of the state vector of the system.
- Despite the distributed estimation scheme introduced in Ugrinovskii (2013) which was only applicable for classes of switching networks with rooted spanning tree, the scheme proposed in this paper is well suited to be applied to networks with jointly connected switching topologies, which is a weaker condition on the communication network.
- In Xu et al. (2020), the problem of state estimation in the presence of jointly connected switching topologies was addressed, where the pairs $(A, C_i^\top)$, $i \in \mathbf{N}$, were marginally stabilizable (that is, there exists a matrix L_i such that $A + L_iC_i$ is stable or marginally stable). In that study, for state estimation, some inequality assumptions on the system parameters and observer gains were considered (which might not have been satisfied by design). However, in this paper the marginal stabilizability is the only condition to guarantee state estimation in the presence of jointly connected switching topologies, and hence the proposed strategy is applicable to a wider range of LTI systems.

We combine a group of observers distributed over N nodes such that each observer has access to its own local measurement and the estimated state vectors of neighboring observers. First, we show that under some linear matrix inequality (LMI) conditions, if the union of all the measurements at all the nodes makes the system observable, and if the network communication graph is jointly connected within bounded time intervals, the proposed distributed scheme guarantees that the estimated state vector of each observer asymptotically converges to the state vector of the system. Then, by canonical observable decomposition of the dynamical system, we show that the LMI conditions always have solution just if the pairs $(A, C_i^\top)$, $i \in \mathbf{N}$, are marginally stabilizable. Moreover, we modify the proposed strategy such that in the presence of external disturbances and measurement noise, a desired performance for state estimation is ensured. More specifically, we modify the proposed estimation strategy such that if the switching communication topologies stay connected during links failure, a desired $\mathscr{H}_\infty$ performance in order to attenuate the effect of external disturbances and measurement noise on estimation errors is achieved. It should be noted that, although a similar problem is considered in Shen et al. (2010) and Ugrinovskii (2013), the design methods of Shen et al. (2010) and Ugrinovskii (2013) rely on conditions for which feasibility is not guaranteed.

The organization of this paper is as follows. In Section 2, some notations and basic information on graph theory are provided. The problem is formulated in Section 3. The distributed state estimation scheme in the presence of jointly connected switching topologies is proposed in Section 4. The $\mathscr{H}_\infty$-based distributed state estimation scheme under connected switching topologies is proposed in Section 5. Simulation results are shown in Section 6. Finally, concluding remarks and future work are stated in Section 7.

2. Preliminaries

Notation as well as some concepts and definitions of graph theory are presented in this section.

2.1. Notation

$\mathbb{N}$ denotes the set of non-negative integer numbers. $\mathbb{R}$ is the set of real numbers. $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ are the sets of positive and non-negative real numbers, respectively. I_n is an $n \times n$ identity matrix. $\mathbf{0}_{n\times m}$ is an $n \times m$ all-zeros matrix. $\mathbf{1}_n$ is an $n \times 1$ all-ones vector. $\mathscr{L}_2$ is the space of square integrable functions $t \longmapsto \delta(t) \in \mathbb{R}$ such that $\int_0^\infty \delta(t)^2 dt < \infty$. $|\cdot|$ is the standard Euclidean norm. $|\cdot|_2$ is the $\mathscr{L}_2$ norm, i.e., $|\delta(t)|_2 = \sqrt{\int_0^\infty \delta(t)^2 dt}$. $\otimes$ stands for the Kronecker product. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^{-r} \in \mathbb{R}^{m\times n}$ represents the right inverse of A such that $AA^{-r} = I_n$. min stands for the minimum value among a set of real numbers. $\lambda_{\min}(\cdot)$ and $\lambda_2(\cdot)$ are the smallest and second smallest eigenvalues of a real symmetric matrix, respectively. For a square matrix M, let $M \succ 0$ or $M \succeq 0$ if it is symmetric positive definite or symmetric positive semidefinite, and let $M \prec 0$ or $M \preceq 0$ if it is symmetric negative definite or symmetric negative semidefinite. For a nonsingular square matrix M, M^* and M^{-1*} stand for the conjugate transpose and the inverse of the conjugate transpose, respectively. $\mathrm{diag}(M_1, M_2, \ldots, M_n)$ represents a block diagonal matrix composed of the matrices $M_1, M_2, \ldots, M_n$. Im and Ker are the image and kernel of a matrix, respectively. $\dim(\mathscr{V})$ is the dimension of the space $\mathscr{V}$. Moreover, if $\mathscr{R}, \mathscr{S} \subseteq \mathscr{X}$, we define the subspace $\mathscr{R} + \mathscr{S} \subseteq \mathscr{X}$ as $\mathscr{R} + \mathscr{S} = \{r + s : r \in \mathscr{R} \text{ and } s \in \mathscr{S}\}$. Accordingly, the symbol $\oplus$ indicates that the subspaces being added are independent.

2.2. Graph theory

Communication among the observers is described by an unweighted undirected graph $\mathcal{G} = (\mathbf{N}, \mathcal{E}, \mathcal{A})$ where $\mathbf{N} = \{1, 2, \ldots, N\}$ is the set of nodes (denoting N observers with local sensors), $\mathcal{E} \subseteq \mathbf{N} \times \mathbf{N}$ is the set of edges (denoting communication links) where $(i,j) \in \mathcal{E}$ denotes an edge between Node i and Node j. Moreover, $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ denotes the adjacency matrix where $a_{ij} = a_{ji} = 1$ if $(i,j) \in \mathcal{E}$, and $a_{ij} = a_{ji} = 0$ otherwise. Now, the neighboring set of Node i is described as $\mathcal{N}_i = \{j|(i,j) \in \mathcal{E}\}$.

We define an undirected graph as *connected* if there exists a path of edges connecting each pair of the nodes of the graph. Accordingly, a set of undirected graphs $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$ with the node set $\mathbf{N}$ is defined to be *jointly connected* if their union is connected, while none of them may be connected.

The Laplacian matrix associated with the undirected graph $\mathcal{G}$ is a symmetric matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{N\times N}$ described as

$$\ell_{ij} = \begin{cases} \sum_{j=1, i\neq j}^{N} a_{ij} & i = j \\ -a_{ij} & i \neq j, \end{cases}$$

having rows/columns with zero entries summation. Therefore, $\mathcal{L}$ always has a zero eigenvalue, and if $\mathcal{G}$ is connected, all the other eigenvalues are positive. Moreover, if $\mathcal{G}$ is connected, the right and left eigenvectors associated with the zero eigenvalue are $\mathbf{1}_N/\sqrt{N}$ (Ren et al., 2007). Furthermore, the Laplacian matrix associated with any undirected graph is positive semidefinite (Chen et al., 2016).

For any connected graph $\mathcal{G}$, let $\lambda_2(\mathcal{L})$ denote the *algebraic connectivity* of the graph, which is a positive real number (Olfati-Saber & Murray, 2004). Accordingly, we define $\mathcal{C}(N)$ as a lower bound for the algebraic connectivity of all graphs with N nodes. It should be noted that there exist several ways in the literature to obtain a $\mathcal{C}(N)$ based on N (for instance, see Chung (1997) and Pirani and Sundaram (2016)).

3. Problem statement

We consider the distributed state estimation problem over the LTI system described in (1) where the observers along with their sensors are placed in N nodes, when the pair (C_i, A), $i \in \mathbf{N}$, may not be observable. Accordingly, for distributed state estimation, the observers are considered connected via a communication network through which information can be exchanged. One of important concerns in communication among the observers is discontinuity in the communication links such that the links may not stay connected constantly, and they can fail and rebuild over time. In this condition, the network communication graph is switching expressed by $\mathcal{G}(t) = (\mathbf{N}, \mathcal{E}(t), \mathcal{A}(t))$, where $\mathcal{A}(t) = [a_{ij}(t)]$ such that $a_{ij}(t) = 1$ describes a communication link between nodes i and j at time t, and it is zero otherwise. Now, we consider two estimation scenarios as follows:

- In the first scenario, we consider an infinite time sequence $t_0, t_1, t_2, \ldots$ starting at $t_0 = 0$, where the intervals $[t_0, t_1)$, $[t_1, t_2)$, $[t_2, t_3)$, … are bounded and in each interval, the graph is switching and *jointly connected*. In other words, in the interval $[t_k, t_{k+1})$, we consider a finite time sequence t_k^0, $t_k^1, \ldots, t_k^{m_k}$ without accumulation where $t_k = t_k^0$ and $t_{k+1} = t_k^{m_k}$ such that $\mathcal{G}(t)$ switches to $\mathcal{G}_k^j$ at $t_k^j, j \in \{0, 1, \ldots, m_k - 1\}$, and stays fixed during the subinterval $[t_k^j, t_k^{j+1})$. Now, the set including the graphs $\mathcal{G}_k^j, j \in \{0, 1, \ldots, m_k - 1\}$, is assumed to be jointly connected. In this condition, considering the LTI system (1), the objective is to exploit the joint observability of the nodes along with the joint connectivity of the communication topology to design a network of distributed observers such that the estimated state vector of each observer converges to the state vector of the system as
$$\lim_{t\to\infty} \big(\hat{x}_i(t) - x(t)\big) = \mathbf{0}_{n\times 1},\ i \in \mathbf{N}. \tag{2}$$
- In this scenario, we consider infinite time sequence $t_0, t_1, t_2, \ldots$ starting at $t_0 = 0$, at which $\mathcal{G}(t)$ switches to $\mathcal{G}_k, k = 0, 1, 2, \ldots$, but stays connected, while the LTI system is subject to external disturbances and measurement noise. In the presence of external disturbance and measurement noise, the LTI system described in (1) can be restated as follows:
$$\begin{aligned} \dot{x} &= Ax + Bu + Dw_0, \\ y_i &= C_i x + E_i w_i,\ i \in \mathbf{N}, \end{aligned} \tag{3}$$
where $w_0 \in \mathbb{R}^{n_0}$ denotes the system external disturbance, $w_i \in \mathbb{R}^{n_i}$ denotes the measurement noise at the ith node, $D \in \mathbb{R}^{n\times n_0}$, and $E_i \in \mathbb{R}^{p_i\times n_i}$. Under these conditions, we intend to exploit the joint observability property of the system to design a network of distributed observers such that the estimated state vector of the ith observer converges to the state vector in a noise-free setting. Indeed, by defining $e_i = \hat{x}_i - x, i \in \mathbf{N}$, and by considering $e \in \mathbb{R}^{Nn}$ and $w \in \mathbb{R}^{Nn+p}$ as
$$\begin{aligned} e &= \begin{bmatrix} e_1^\top & e_2^\top & \ldots & e_N^\top \end{bmatrix}^\top, \\ w &= \begin{bmatrix} w_0^\top D^\top & w_1^\top E_1^\top & \ldots & w_0^\top D^\top & w_N^\top E_N^\top \end{bmatrix}^\top, \end{aligned} \tag{4}$$
the objective is to design a set of distributed observers such that the effect of the exogenous signals vector w on e is attenuated. To achieve this goal, we minimize the $\mathscr{H}_\infty$ norm of

the mapping from w to e as the induced matrix norm of the $\mathscr{L}_2$ norm of e with respect to the $\mathscr{L}_2$ norm of w as follows:

$$\text{minimize } \mu \text{ s.t. } \left(|e|_2 - \mu |w|_2\right) < 0, \tag{5}$$

where $\mu \in \mathbb{R}_{>0}$.

The main results are presented in the next section.

4. Distributed state estimation under jointly connected switching topologies

The proposed distributed estimation strategy for the system described in (1) is presented in this section such that the objective (2) is achieved. To this end, the following assumption on the LTI system (1) is considered.

Assumption 1. The pairs $(A, C_i^\top)$, $i \in \mathbf{N}$, are marginally stabilizable.

Assumption 1 implies that if the state matrix A is unstable, there should exist an output gain L_i such that $A + L_iC_i$ is stable or marginally stable. However, the assumption is always satisfied if the state matrix A is stable or marginally stable.

The main idea behind distributed state estimation is that the ith observer updates its estimated state vector based on its local measurement and relative estimated state vectors of neighboring observers. This innovation term is given by $\sum_{j=1}^N a_{ij}(\hat{x}_j - \hat{x}_i)$ (for instance, see Mitra and Sundaram (2018, 2019), Park and Martins (2017) and Wang et al. (2019)). Based on the mentioned general idea, the proposed observer for the system described in (1) at Node i is as follows:

$$\begin{aligned}\dot{\hat{x}}_i &= A\hat{x}_i + L_i(C_i\hat{x}_i - y_i) + Bu \\ &\quad + P_i^{-1}\sum_{j=1}^N a_{ij}(t)(\hat{x}_j - \hat{x}_i),\ i \in \mathbf{N},\end{aligned} \tag{6}$$

in which $L_i \in \mathbb{R}^{n\times p_i}$ and $P_i^{-1} \in \mathbb{R}^{n\times n}$ denote the observer gains to be determined.

The mathematical analysis of the proposed distributed observers is given in Theorem 1. Before presentation of the theorem, the following lemmas are presented.

Lemma 1 (*Hunter, 2014, Chap. 5*). *Let $v(t_0), v(t_1), v(t_2), \ldots$ be a converging sequence of numbers. For all $\zeta \in \mathbb{R}_+$, there exists $k_\zeta \in \mathbb{N}$ such that for all $k \geq k_\zeta$, $|v(t_{k+1}) - v(t_k)| < \zeta$ (Cauchy's convergence criterion).*

Lemma 2. *Consider two LTI autonomous systems described by*

$$\begin{aligned}\dot{\xi}_1 &= A_1\xi_1, \\ \dot{\xi}_2 &= A_2\xi_2,\end{aligned} \tag{7}$$

where $\xi_1, \xi_2 \in \mathbb{R}^n$ are state vectors and $A_1, A_2 \in \mathbb{R}^{n\times n}$ are state matrices. Moreover, let $[t_1, t_2)$ be a finite time interval with $0 \leq t_1 < t_2$. In this condition, if $\xi_1(t) = \xi_2(t)$ for $t \in [t_1, t_2)$; then, $\xi_1(t) = \xi_2(t)$ for all $t \geq t_2$.

Proof. By defining $\xi_{12} = \xi_1 - \xi_2$ and by considering (7), one gets

$$\dot{\xi}_{12} = A_2\xi_{12} + (A_1 - A_2)\xi_1. \tag{8}$$

Since $\xi_{12} = \mathbf{0}_{n\times 1}$ for $t \in [t_1, t_2)$, from (8) it follows that

$$\xi_1 \in \mathrm{Ker}(A_1 - A_2),\ t \in [t_1, t_2). \tag{9}$$

Now, we prove the lemma by contradiction. Let us consider the case when $\xi_1(t) \neq \xi_2(t)$ for some time $t \geq t_2$, while $\xi_{12}(t) = \mathbf{0}_{n\times 1}$ for $t \in [t_1, t_2)$. According to (8) and (9), this can happen only if $\xi_1(t)$ leaves the kernel of $A_1 - A_2$ for some time $t \geq t_2$. This issue implies the occurrence of impulses in at least one of the derivatives of ξ_1, which is in conflict with (7). Thus, $\xi_1(t) = \xi_2(t)$ for all $t \geq t_2$. ■

Theorem 1. *Consider the jointly observable LTI system described in (1) under Assumption 1 and the distributed observers given in (6). Moreover, let $\mathcal{G}(t)$ be switching, but jointly connected within infinite sequence of bounded time intervals $[t_0, t_1)$, $[t_1, t_2)$, $[t_2, t_3)$, $\ldots$ starting at t_0. Under these conditions, the estimation errors e_i, $i \in \mathbf{N}$, converge to zero by obtaining $P_i \succ 0$ and $L_i = P_i^{-1}Y_i$ from the solution of the following LMIs:*

$$A^\top P_i + C_i^\top Y_i^\top + P_iA + Y_iC_i \preceq 0,\ i \in \mathbf{N}, \tag{10a}$$

$$\sum_{i=1}^N (A^\top P_i + C_i^\top Y_i^\top + P_iA + Y_iC_i) \prec 0. \tag{10b}$$

Proof. Considering the LTI system with the form of (1) and based on the devised observer in (6), the differential equation describing $e_i = \hat{x}_i - x$, $i \in \mathbf{N}$, is given by

$$\begin{aligned}\dot{e}_i &= (A + L_iC_i)e_i \\ &\quad \underbrace{+ P_i^{-1}\sum_{j=1}^N a_{ij}(t)(e_j - e_i)}_{=:\ \varepsilon_i},\ i \in \mathbf{N}.\end{aligned} \tag{11}$$

To analyze the evolution of the observers estimation errors, we consider the following Lyapunov function:

$$V = \sum_{i=1}^N e_i^\top P_i e_i.$$

The time derivation of V along (11) yields

$$\dot{V} = \sum_{i=1}^N e_i^\top \Lambda_i e_i + 2\sum_{i=1}^N\sum_{j=1}^N a_{ij}(t)\left(e_j - e_i\right)^\top e_i, \tag{12}$$

where

$$\Lambda_i := A^\top P_i + C_i^\top Y_i^\top + P_iA + Y_iC_i. \tag{13}$$

Now, by considering $\mathcal{L}_k^j$ as the fixed Laplacian matrix associated with $\mathcal{G}_k^j$ in the subinterval $[t_k^j, t_k^{j+1})$, (12) can be restated as follows:

$$\dot{V} = e^\top \Lambda e - \underbrace{2e^\top\left(\mathcal{L}_k^j \otimes I_n\right)e}_{=:\ \epsilon}, \tag{14}$$

where e is defined in (4) and

$$\Lambda = \mathrm{diag}\left(\Lambda_1, \Lambda_2, \ldots, \Lambda_N\right). \tag{15}$$

According to (10a), Λ is negative semidefinite, and since $-\mathcal{L}_k^j$ is negative semidefinite, from (14) it follows the boundedness of e_i, $i \in \mathbf{N}$. We continue the proof in two steps. In the first step, by invoking Lemma 1 we show that the term $\sum_{j=1}^N a_{ij}(t)(e_j - e_i)$ in (11) converges to zero. In the next step, by invoking Lemma 2, we show that $e_j - e_i$, $i, j, \in \mathbf{N}$, converge to zero, which implies that e_i, $i \in \mathbf{N}$, converge to a common variable $\omega \in \mathbb{R}^n$. Then, from (10b) and (14), we conclude that $\dot{V}$ is negative definite with respect to e.

Step 1: According to (10a), Λ is negative semi-definite. Therefore, from (14) it follows that

$$\dot{V} \leq -2e^\top\left(\mathcal{L}_k^j \otimes I_n\right)e. \tag{16}$$

Since $\dot{V} \leq 0$, based on Lemma 1, it can be said that for any $\zeta \in \mathbb{R}_+$, there exists a k_ζ such that, for all $k \geq k_\zeta$, it follows $|V(t_{k+1}) - V(t_k)| < \zeta$, which implies that

$$\left| \int_{t_k}^{t_{k+1}} \dot{V}(t)\, dt \right| < \zeta. \tag{17}$$

By considering the time sequence $t_k^0, t_k^1, \ldots, t_k^{m_k}$ in the interval $[t_k, t_{k+1})$ and since $\dot{V} \leq 0$, from (17), one gets

$$-\sum_{j=0}^{m_k-1} \int_{t_k^j}^{t_k^{j+1}} \dot{V}(t)\, dt < \zeta,$$

where, by defining $\tau_k := \min_{j\in\{0,1,2,\ldots,m_k-1\}} t_k^{j+1} - t_k^j$, it yields

$$-\sum_{j=0}^{m_k-1} \int_{t_k^j}^{t_k^j+\tau_k} \dot{V}(t)\, dt < \zeta.$$

By considering (16) and Lemma 1, as $\zeta \to 0$, it follows that

$$\lim_{t\to\infty} \sum_{j=0}^{m_k-1} \int_t^{t+\tau_k} e(\tau)^\top \left(\mathcal{L}_k^j \otimes I_n\right) e(\tau)\, d\tau = 0. \tag{18}$$

Note that during the interval $[t, t+\tau_k)$, the topology of the graph $\mathcal{G}_k^j$ is not switching. By using an orthogonal similarity transformation matrix P_k^j, we diagonalize $\mathcal{L}_k^j$ as $\mathcal{L}_k^j = P_k^j J_k^j P_k^{j\top}$, where $J_k^j = \mathrm{diag}(0, \eta_k^j)$ is the diagonal form of $\mathcal{L}_k^j$ in which η_k^j has a diagonal form with nonnegative entries and P_k^j is composed of the right eigenvectors of $\mathcal{L}_k^j$. Thus, if we define the new variable $q = (P_k^{j\top} \otimes I_n)e$, (18) can be restated as follows:

$$\lim_{t\to\infty} \sum_{j=0}^{m_k-1} \int_t^{t+\tau_k} q(\tau)^\top \left(J_k^j \otimes I_n\right) q(\tau)\, d\tau = 0. \tag{19}$$

If we decompose q as $q = \left[\varrho^\top \rho^\top\right]^\top$ where $\varrho \in \mathbb{R}^n$ and $\rho \in \mathbb{R}^{(N-1)n}$, since $J_k^j = \mathrm{diag}\left(0, \eta_k^j\right)$, (19) yields

$$\lim_{t\to\infty} \sum_{j=0}^{m_k-1} \int_t^{t+\tau_k} \rho(\tau)^\top \left(\eta_k^j \otimes I_n\right) \rho(\tau)\, d\tau = 0. \tag{20}$$

Now, by decomposing ρ as $\rho = [\rho_1^\top \;\; \rho_2^\top \;\; \ldots \;\; \rho_{N-1}^\top]^\top$ and by joint connectivity arguments, from (20) it follows that

$$\lim_{t\to\infty} \int_t^{t+\tau_k} \rho_i(\tau)^\top \rho_i(\tau)\, d\tau = 0\,,\ i = 1, \ldots, N-1. \tag{21}$$

According to the definition of τ_k, there is no graph switching from t to $t+\tau_k$. Moreover, we obtain that e is bounded and hence from (11) it follows that also $\dot{e}$ is bounded. Now, since $\dot{e}$ is bounded, $\rho_i^\top \rho_i$ is uniformly continuous from t to $t+\tau_k$. Hence, by considering (21) and invoking the Barbalat lemma (Tao, 1997), we have

$$\lim_{t\to\infty} \rho_i(t)^\top \rho_i(t) = 0,$$

implying that

$$\lim_{t\to\infty} \rho(t) = \mathbf{0}_{(N-1)n\times 1}. \tag{22}$$

Since $e = \left(P_k^j \otimes I_n\right) q$ and $q = \left[\varrho^\top \quad \rho^\top\right]^\top$, from (22) it follows that

$$\lim_{t\to\infty} (e(t) - p_k^j \otimes \varrho(t)) = \mathbf{0}_{Nn\times 1},$$

where p_k^j is the first column of P_k^j, which is the right eigenvector associated with a zero eigenvalue of $\mathcal{L}_k^j$. If $\mathcal{G}_k^j$ is connected, p_k^j spans $\mathbf{1}_N$. Otherwise, $\mathcal{G}_k^j$ can be considered as the combination of some subgraphs which are connected or have no links. Note that the right eigenvector associated with the zero eigenvalue of the Laplacian matrix of each connected subgraph has identical entries. Hence, if $\mathcal{G}_k^j$ is the combination of some subgraphs which are connected or have no links, p_k^j can be considered as the combination of some coefficients of $\mathbf{1}$ vectors with compatible dimensions. In this condition, for each $i, j \in \mathbf{N}$, if $a_{ij}(t) = 1$, we have $\lim_{t\to\infty}(e_i(t) - e_j(t)) = \mathbf{0}_{n\times 1}$. Moreover, $a_{ij}(t) = 0$ if $a_{ij}(t) \neq 1$. As a result

$$\lim_{t\to\infty} P_i^{-1} \sum_{j=1}^N a_{ij}(t)(e_j(t) - e_i(t)) = \mathbf{0}_{n\times 1}. \tag{23}$$

Step 2: According to (11) and (23), we have

$$\dot{e}_i = (A + L_i C_i)e_i + \varepsilon_i,\ i \in \mathbf{N},$$

where ε_i (defined in (11)) is bounded and converges to zero and $\lim_{t\to\infty}(e_i(t) - e_j(t)) = \mathbf{0}_{n\times 1}$ if $a_{ij}(t) = 1$. Based on Lemma 2 and since $\mathcal{G}(t)$ is jointly connected within bounded time intervals, one gets

$$\lim_{t\to\infty} (e_i(t) - e_j(t)) = \mathbf{0}_{n\times 1},\ i, j \in \mathbf{N},$$

implying that there exists an $\omega \in \mathbb{R}^n$ such that

$$\lim_{t\to\infty} (e_i(t) - \omega(t)) = \mathbf{0}_{n\times 1},\ i \in \mathbf{N}. \tag{24}$$

Now, from (14) and (24) we have

$$\dot{V} = e^\top \Lambda e - \epsilon, \tag{25}$$

where e has the form of $\mathbf{1}_N \otimes \omega$, and ϵ (defined in (14)) is bounded and converges to zero. By considering (15) and, since e in (25) has the form of $\mathbf{1}_N \otimes \omega$, one gets

$$e^\top \Lambda e = \omega^\top \left(\sum_{i=1}^N \Lambda_i\right) \omega,$$

where according to (10b), $\sum_{i=1}^N \Lambda_i$ is negative definite. Therefore, $e^\top \Lambda e$ in (25) is negative definite with respect to e and, since ϵ is bounded and converges to zero, the convergence of e to $\mathbf{0}_{Nn\times 1}$ follows from (25). ■

The effectiveness of the proposed distributed state estimation strategy depends on the observer gains that result from the solution of the LMIs (10). Next, based on observable canonical decomposition, we show that the LMI conditions of Theorem 1 are always feasible. Let us define a similarity transformation matrix $T_i \in \mathbb{R}^{n\times n}$ as

$$T_i = \begin{bmatrix} T_{io} & T_{iu} \end{bmatrix}, \tag{26}$$

in which $T_{iu} \in \mathbb{R}^{n\times v_i}$ is an orthonormal basis of the unobservable subspace of (C_i, A) where v_i is the dimension of the unobservable subspace, and $T_{io} \in \mathbb{R}^{n\times(n-v_i)}$ is an orthonormal basis such that $\mathrm{Im}\, T_{io}$ is orthogonal to $\mathrm{Im}\, T_{iu}$. According to the structure of T_i, we perform the coordinate transform adapted to $\mathscr{X} = \mathrm{Im}\, T_{io} \oplus \mathrm{Im}\, T_{iu}$ as follows (Kim et al., 2016) ($\mathscr{X}$ is the n-dimensional state space of the system):

$$\begin{aligned} T_i^\top A T_i &= \begin{bmatrix} A_{io} & \mathbf{0}_{(n-v_i)\times v_i} \\ A_{ir} & A_{iu} \end{bmatrix}, \\ C_i T_i &= \begin{bmatrix} C_{io} & \mathbf{0}_{p_i\times v_i} \end{bmatrix}, \end{aligned} \tag{27}$$

where the pair (C_{io}, A_{io}) is observable. By using the aforementioned formulation, the feasibility of the LMIs (10) is proved in Theorem 2. Before presenting the theorem, we introduce the following lemmas.

Lemma 3. *For any marginally stable real matrix $\mathcal{M}$, the LMI*

$$\mathcal{M}^\top \mathcal{P} + \mathcal{P}\mathcal{M} \preceq 0 \tag{28}$$

has a symmetric positive definite solution for $\mathcal{P}$.

Proof. There is a similarity transformation matrix $\mathcal{T}$ such that $\mathcal{M} = \mathcal{T}\mathcal{J}\mathcal{T}^{-1}$ where $\mathcal{J}$ is the Jordan form of $\mathcal{M}$. Since $\mathcal{M}$ is marginally stable, $\mathcal{J}$ can be stated as follows:

$$\mathcal{J} = \mathrm{diag}(\mathcal{W}, \mathcal{H}),$$

where $\mathcal{W}$ is a diagonal matrix containing the eigenvalues of $\mathcal{M}$ on the imaginary axis, and $\mathcal{H}$ is a Jordan matrix containing the eigenvalues of $\mathcal{M}$ on the open left half plane. Since the eigenvalues of $\mathcal{H}$ lie on the open left half plane, according to the Lyapunov stability criterion, for a Hermitian positive definite matrix $\tilde{\mathcal{Q}}$, there exists a Hermitian positive definite matrix $\tilde{\mathcal{P}}$ such that (Laffey & Smigoc, 2007, p. 2)

$$\mathcal{H}^* \tilde{\mathcal{P}} + \tilde{\mathcal{P}}\mathcal{H} = -\tilde{\mathcal{Q}}. \tag{29}$$

We define a matrix $\acute{\mathcal{Q}} := \mathrm{diag}(\mathbf{0}, \tilde{\mathcal{Q}})$, where $\mathbf{0}$ is a zeros matrix whose dimension is the same as the dimension of $\mathcal{W}$ and define a matrix $\acute{\mathcal{P}} := \mathrm{diag}(I, \tilde{\mathcal{P}})$, where I is an identity matrix whose dimension is the same as the dimension of $\mathcal{W}$. Now, according to (29), it follows

$$\mathcal{J}^* \acute{\mathcal{P}} + \acute{\mathcal{P}}\mathcal{J} = -\acute{\mathcal{Q}}. \tag{30}$$

By pre and post multiplication of the both sides of (30) by $\mathcal{T}^{-1*}$ and $\mathcal{T}^{-1}$, one gets

$$\mathcal{T}^{-1*}\mathcal{J}^*\acute{\mathcal{P}}\mathcal{T}^{-1} + \mathcal{T}^{-1*}\acute{\mathcal{P}}\mathcal{J}\mathcal{T}^{-1} = -\mathcal{T}^{-1*}\acute{\mathcal{Q}}\mathcal{T}^{-1},$$

which since $\mathcal{M} = \mathcal{T}\mathcal{J}\mathcal{T}^{-1}$, can be restated as follows:

$$\mathcal{M}^\top \mathcal{T}^{-1*}\acute{\mathcal{P}}\mathcal{T}^{-1} + \mathcal{T}^{-1*}\acute{\mathcal{P}}\mathcal{T}^{-1}\mathcal{M} = -\mathcal{T}^{-1*}\acute{\mathcal{Q}}\mathcal{T}^{-1}. \tag{31}$$

By defining $\bar{\mathcal{P}} = \mathcal{T}^{-1*}\acute{\mathcal{P}}\mathcal{T}^{-1}$ and $\bar{\mathcal{Q}} = \mathcal{T}^{-1*}\acute{\mathcal{Q}}\mathcal{T}^{-1}$, we rewrite (31) as

$$\mathcal{M}^\top \bar{\mathcal{P}} + \bar{\mathcal{P}}\mathcal{M} = -\bar{\mathcal{Q}}. \tag{32}$$

Now, from (32) we obtain

$$\mathcal{M}^\top(\bar{\mathcal{P}} + \bar{\mathcal{P}}^\top) + (\bar{\mathcal{P}} + \bar{\mathcal{P}}^\top)\mathcal{M} = -(\bar{\mathcal{Q}} + \bar{\mathcal{Q}}^\top). \tag{33}$$

Since $\acute{\mathcal{P}}$ is Hermitian positive definite, one can derive that $\bar{\mathcal{P}} = \mathcal{T}^{-1*}\acute{\mathcal{P}}\mathcal{T}^{-1}$ is Hermitian positive definite implying that $\bar{\mathcal{P}} + \bar{\mathcal{P}}^\top \succ 0$. Moreover, since $\acute{\mathcal{Q}}$ is Hermitian positive semidefinite, $\bar{\mathcal{Q}} = \mathcal{T}^{-1*}\acute{\mathcal{Q}}\mathcal{T}^{-1}$ is Hermitian positive semidefinite implying that $\bar{\mathcal{Q}} + \bar{\mathcal{Q}}^\top \succeq 0$. Now, by considering (33) and by defining $\mathcal{P} = \bar{\mathcal{P}} + \bar{\mathcal{P}}^\top$, the LMI (28) is satisfied. Based on all the above-mentioned issues, (28) has a solution for $\mathcal{P} \succ 0$. ∎

Lemma 4 (Yang et al., 2020)**.** *Consider the jointly observable system given in (1). By letting*

$$T_o = \begin{bmatrix} T_{1o} & T_{2o} & \dots & T_{No} \end{bmatrix}, \tag{34}$$

we have $\mathrm{Im}\, T_o = \mathscr{X}$.

Theorem 2. *Consider the jointly observable pairs (C_i, A), $i \in \mathbf{N}$, described in (1) under Assumption 1. Then, the LMIs (10) always have solution for $P_i \succ 0$ and $Y_i = P_i L_i$, $i \in \mathbf{N}$.*

Proof. Let the observer gains $L_i \in \mathbb{R}^{n\times p_i}$ and $P_i^{-1} \in \mathbb{R}^{n\times n}$ be as follows:

$$L_i = T_i \begin{bmatrix} L_{io} \\ \mathbf{0}_{v_i\times p_i} \end{bmatrix}, \qquad P_i = T_i \begin{bmatrix} P_{io} & P_{ir}^\top \\ P_{ir} & P_{iu} \end{bmatrix} T_i^\top, \tag{35}$$

where T_i is the similarity transformation matrix defined in (26), and $L_{io} \in \mathbb{R}^{(n-v_i)\times p_i}$, $P_{io} \in \mathbb{R}^{(n-v_i)\times(n-v_i)} \succ 0$, $P_{ir} \in \mathbb{R}^{v_i\times(n-v_i)}$, and $P_{iu} \in \mathbb{R}^{v_i\times v_i} \succ 0$ to be determined. From the definition of Λ_i in (13), the definition of L_i and P_i in (35), and the observable canonical decomposition in (27), we have

$$\Lambda_i = T_i \begin{bmatrix} \Lambda_{io} & \Lambda_{ir}^\top \\ \Lambda_{ir} & \Lambda_{iu} \end{bmatrix} T_i^\top, \tag{36}$$

where $\Lambda_{io} \in \mathbb{R}^{(n-v_i)\times(n-v_i)}$, $\Lambda_{ir} \in \mathbb{R}^{v_i\times(n-v_i)}$, and $\Lambda_{iu} \in \mathbb{R}^{v_i\times v_i}$ are as follows:

$$\Lambda_{io} = M_{io}^\top P_{io} + P_{io}M_{io} + P_{ir}^\top A_{ir} + A_{ir}^\top P_{ir},$$
$$\Lambda_{ir} = P_{ir}M_{io} + A_{iu}^\top P_{ir} + P_{iu}A_{ir},$$
$$\Lambda_{iu} = A_{iu}^\top P_{iu} + P_{iu}A_{iu},$$

in which $M_{io} = A_{io} + L_{io}C_{io}$. Recall that A_{iu} is stable or marginally stable (from Assumption 1), by invoking Lemma 3, there always exists $P_{iu} \succ 0$ such that $\Lambda_{iu} = A_{iu}^\top P_{iu} + P_{iu}A_{iu} \preceq 0$. Moreover, since (C_{io}, A_{io}) is observable for all $i \in \mathbf{N}$, the eigenvalues of M_{io} can be arbitrarily assigned by selecting L_{io}. Therefore, M_{io} is set to be Hurwitz while its eigenvalues are designed to be distinct with the eigenvalues of A_{iu}. In this condition, according to the criterion of the Sylvester equation, there exists a P_{ir} such that $\Lambda_{ir} = \mathbf{0}_{v_i\times(n-v_i)}$ (Sylvester, 1884). Thus, by selecting P_{iu} and P_{ir}, we have

$$\Lambda_i = T_i \begin{bmatrix} \Lambda_{io} & \mathbf{0}_{(n-v_i)\times v_i} \\ \mathbf{0}_{v_i\times(n-v_i)} & \Lambda_{iu} \end{bmatrix} T_i^\top,$$

where $\Lambda_{iu} \preceq 0$ for all $i \in \mathbf{N}$. Now, as T_i is of full rank, according to (36), the LMIs (10) have solution if and only if the following LMIs have solution:

$$\begin{bmatrix} \Lambda_{io} & \mathbf{0}_{(n-v_i)\times v_i} \\ \mathbf{0}_{v_i\times(n-v_i)} & \Lambda_{iu} \end{bmatrix} \preceq 0,\ i \in \mathbf{N}, \tag{37a}$$

$$\sum_{i=1}^{N} \left(T_i \begin{bmatrix} \Lambda_{io} & \mathbf{0}_{(n-v_i)\times v_i} \\ \mathbf{0}_{v_i\times(n-v_i)} & \Lambda_{iu} \end{bmatrix} T_i^\top \right) \prec 0. \tag{37b}$$

Since $\Lambda_{iu} \preceq 0$, (37a) is satisfied if and only if $\Lambda_{io} \preceq 0$ for all $i \in \mathbf{N}$. That implies

$$\tilde{\Lambda}_{io} + P_{ir}^\top A_{ir} + A_{ir}^\top P_{ir} \preceq 0, \tag{38}$$

where $\tilde{\Lambda}_{io} := M_{io}^\top P_{io} + P_{io}M_{io}$, whose eigenvalues can be designed negative and large enough for a $P_{io} \succ 0$, according to the Lyapunov stability criterion (as M_{io} has been Hurwitz). Since $T_i = \begin{bmatrix} T_{io} & T_{iu} \end{bmatrix}$, one gets

$$\begin{aligned} & T_i \begin{bmatrix} \Lambda_{io} & \mathbf{0}_{(n-v_i)\times v_i} \\ \mathbf{0}_{v_i\times(n-v_i)} & \Lambda_{iu} \end{bmatrix} T_i^\top \\ &= T_{io}\Lambda_{io}T_{io}^\top + T_{iu}\Lambda_{iu}T_{iu}^\top \\ &= T_{io}\tilde{\Lambda}_{io}T_{io}^\top + T_{io}\left(P_{ir}^\top A_{ir} + A_{ir}^\top P_{ir}\right)T_{io}^\top + T_{iu}\Lambda_{iu}T_{iu}^\top. \end{aligned}$$

Then, we have

$$\begin{aligned} & \sum_{i=1}^{N} \left(T_i \begin{bmatrix} \Lambda_{io} & \mathbf{0}_{(n-v_i)\times v_i} \\ \mathbf{0}_{v_i\times(n-v_i)} & \Lambda_{iu} \end{bmatrix} T_i^\top \right) \\ &= T_o\tilde{\Lambda}_o T_o^\top + \\ & \sum_{i=1}^{N} T_{io}\left(P_{ir}^\top A_{ir} + A_{ir}^\top P_{ir}\right)T_{io}^\top + T_{iu}\Lambda_{iu}T_{iu}^\top, \end{aligned} \tag{39}$$

where T_o is defined as in (34), and Λ_o is a block diagonal matrix as follows:

$$\tilde{\Lambda}_o = \mathrm{diag}(\tilde{\Lambda}_{1o}, \tilde{\Lambda}_{2o}, \dots, \tilde{\Lambda}_{No}).$$

As stated in Lemma 4, $\mathrm{rank}(T_o) = n$. Hence, according to (39), the LMI (37b) is satisfied if the following LMI is satisfied:

$$T_o\left(\tilde{\Lambda}_o + H\right)T_o^\top \prec 0, \tag{40}$$

where

$$H = T_o^{-r}\left(\sum_{i=1}^{N} T_{io}\left(P_{ir}^\top A_{ir} + A_{ir}^\top P_{ir}\right) T_{io}^\top + T_{iu}\Lambda_{iu}T_{iu}^\top\right) T_o^{-r\top}.$$

According to (38) and (40), the LMIs (10) have solution if the following LMIs have solution:

$$\begin{aligned}&\tilde{\Lambda}_{io} + \left(P_{ir}^\top A_{ir} + A_{ir}^\top P_{ir}\right) \preceq 0,\\ &\tilde{\Lambda}_o + H \prec 0.\end{aligned} \tag{41}$$

Therefore, it is straight forward to see that for $\Lambda_{io}, i \in \mathbf{N}$, with large enough negative eigenvalues, the LMIs (41) hold. Moreover, to let $P_i \succ 0$, by Schur complement (Boyd et al., 1994), P_{io} can be selected such that

$$P_{io} - P_{ir}^\top P_{iu}^{-1} P_{ir} \succ 0,\ \ i \in \mathbf{N},$$

which is always feasible while satisfying (41). Based on the aforementioned issues, there exist some L_{io}, P_{io}, P_{ir}, and P_{iu} such that the LMIs (10) hold. Then, we construct L_i and P_i as in (35) where P_i can be symmetric positive definite. Therefore, there exist $P_i \succ 0$ and $Y_i = P_i L_i$, $i \in \mathbf{N}$, such that the LMIs (10) are satisfied. ■

Remark 1. The current study is devoted to the distributed state estimation problem when the local measurement at each node may not be sufficient to estimate the state vector of the system. Although this problem is also considered in the literature, the main assumption of the existing results is communication among the nodes under steady and fixed topologies (Han et al., 2019; He et al., 2020; Kamgarpour & Tomlin, 2008; Kim et al., 2016; Mitra et al., 2019; Mitra & Sundaram, 2019; Olfati-Saber, 2007; Shen et al., 2010; Wang et al., 2019; Wang & Morse, 2018; Yang et al., 2020). In few studies in this area, the problem of communication under switching topologies was addressed as well. However, those studies were based on some constraints on the system and observers parameters (Xu et al., 2020), or the set of switching topologies (Ugrinovskii, 2013), whereas the feasibility of the conditions of the design for distributed state estimation in the presence of jointly connected topologies are one of the contributions of this paper.

5. $\mathscr{H}_\infty$-Based distributed state estimation under connected switching topologies

In this section, the design of the proposed distributed state observers (6) is addressed in the presence of external disturbances and measurement noise such that the objective (5) can be achieved. Toward this end, we assume the following.

Assumption 2. The exogenous signals $t \longmapsto w_i(t)$, $i \in \{0, 1, \ldots, N\}$, belong to the $\mathscr{L}_2$ space.

Remark 2. The assumption above allows to analyze how the noise affects the estimation errors. More specifically, the $\mathscr{H}_\infty$ approach is considered, which is in general aimed at attenuating the effect of the $\mathscr{L}_2$ norm of some exogenous disturbances (w in our case) on the $\mathscr{L}_2$ norm of some regulated outputs (i.e., e) as much as possible. Therefore, the main assumption behind the $\mathscr{H}_\infty$ design is that the exogenous signals are $\mathscr{L}_2$ functions of time. In practice, this means that the noise has bounded energy, which does not appear to be particularly restrictive.

Now, the distributed state observers proposed in (6) are modified as follows:

$$\begin{aligned}\dot{\hat{x}}_i &= A\hat{x}_i + L_i(C_i\hat{x}_i - y_i) + Bu\\ &\quad + \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(t)(\hat{x}_j - \hat{x}_i),\ i \in \mathbf{N},\end{aligned} \tag{42}$$

where $\chi \in \mathbb{R}_{>0}$. The observer gains L_i, P_i, and χ can be selected by solving an $\mathscr{H}_\infty$ optimization problem in such a way as to attenuate the effect of the exogenous signals vector w on the observer errors vector e in terms of the induced matrix norm of the $\mathscr{L}_2$ norm of e with respect to the $\mathscr{L}_2$ norm of w as in (5).

Theorem 3. *Consider the jointly observable LTI system described in (3) under Assumption 2 and the distributed observers given in (42). Moreover, let $\mathcal{G}(t)$ be switching, but connected within infinite sequence of bounded time intervals $[t_0, t_1)$, $[t_1, t_2)$, $[t_2, t_3)$, ... starting at t_0. Under these conditions, the $\mathscr{H}_\infty$ performance (5) is satisfied if $\acute{\mu}$, $\chi \in \mathbb{R}_{>0}$, $P_i \succ 0$, and Y_i, $i \in \mathbf{N}$, are obtained from the solution of the following LMI:*

$$\underset{Y_i, P_i, \chi}{\text{minimize}}\ \acute{\mu}\ \ \text{subject to} \tag{43a}$$

$$\begin{bmatrix} -\sum_{i=1}^{N}\Lambda_i - NI_n & -\Lambda_P & \Xi_P \\ -\Lambda_P^\top & \begin{matrix}2\chi\, \mathcal{C}(N) I_{Nn}\\ -\Lambda - I_{Nn}\end{matrix} & \Xi \\ \Xi_P^\top & \Xi^\top & \acute{\mu}\, I_{p+Nn}\end{bmatrix} \succ 0, \tag{43b}$$

in which Λ is defined in (15) and $\Lambda_P = \begin{bmatrix}\Lambda_1 & \Lambda_2 & \ldots & \Lambda_N\end{bmatrix}$, where Λ_i is defined in (13), and $\Xi = \operatorname{diag}(\Xi_1, \Xi_2, \ldots, \Xi_N)$ and $\Xi_P = \begin{bmatrix}\Xi_1 & \Xi_2 & \ldots & \Xi_N\end{bmatrix}$, where $\Xi_i = \begin{bmatrix}P_i & Y_i\end{bmatrix}$. In this condition, $\mu = \sqrt{\acute{\mu}}$ and $L_i = P_i^{-1} Y_i$, $i \in \mathbf{N}$.

Proof. Considering the LTI system with the form of (3) and based on the observers (42), the differential equation describing e_i, $i \in \mathbf{N}$, is given by

$$\begin{aligned}\dot{e}_i &= (A + L_iC_i)e_i - L_iE_iw_i - Dw_0\\ &\quad + \chi P_i^{-1}\sum_{j=1}^{N} a_{ij}(t)(e_j - e_i),\ i \in \mathbf{N}.\end{aligned} \tag{44}$$

To analyze the evolution of the estimation errors of all the nodes, we consider the following Lyapunov function:

$$V = \sum_{i=1}^{N} e_i^\top P_i e_i.$$

The time derivation of V along (44) yields

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N} e_i^\top \Lambda_i e_i + 2\chi \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(t)(e_j - e_i)^\top e_i\\ &\quad - 2\sum_{i=1}^{N} e_i^\top P_i(L_iE_iw_i + Dw_0).\end{aligned} \tag{45}$$

From (45) and according to the definition of Λ, Ξ, e, and w, it follows that

$$\dot{V} \le e^\top \Lambda e - 2\chi e^\top(\mathcal{L}_k \otimes I_n)e - 2e^\top \Xi w, \tag{46}$$

where $\mathcal{L}_k$ is the Laplacian matrix associated with $\mathcal{G}_k$. Let us denote by $\mathscr{E}$ the Nn dimensional error space. Since $\mathcal{G}_k$ is connected, according to the properties of the Laplacian matrix of a connected undirected graph, the kernel of $\mathcal{L}_k \otimes I_n$ has the form of $\mathbf{1}_N \otimes \omega$, where $\omega \in \mathbb{R}^n$ is an arbitrary vector. Now, by letting $\mathscr{E}_c \subseteq \mathscr{E}$

($\dim(\mathscr{E}_c) = n$) as the null space of $\mathcal{L}_k \otimes I_n$, we consider another subspace $\mathscr{E}_r \subseteq \mathscr{E}$ ($\dim(\mathscr{E}_r) = Nn - n$), which is the orthogonal complement of $\mathscr{E}_c$, i.e., $\mathscr{E}_c \oplus \mathscr{E}_r = \mathscr{E}$. Let the vectors e_c and e_r be the elements of the subspaces $\mathscr{E}_c$ and $\mathscr{E}_r$ respectively, i.e., $e_c \in \mathscr{E}_c$ and $e_r \in \mathscr{E}_r$. Therefore, from (46) one gets

$$\begin{aligned}\dot{V} &\le e_c^\top \Lambda e_c + 2e_r^\top \Lambda e_c + e_r^\top \Lambda e_r \\ &\quad - 2\chi e_r^\top(\mathcal{L}_k \otimes I_n)e_r - 2e_c^\top \Xi w - 2e_r^\top \Xi w.\end{aligned} \tag{47}$$

A necessary condition to guarantee the negative definiteness of $\dot{V}$ is to ensure that $e_c^\top \Lambda e_c$ is negative definite with respect to $e_c \in \mathscr{E}_c$. Since e_c has the form of $\mathbf{1}_N \otimes \omega$, from the definition of Λ we have

$$e_c^\top \Lambda e_c = \omega^\top \left(\sum_{i=1}^N \Lambda_i\right)\omega. \tag{48}$$

Considering the LMI condition (43b), $\sum_{i=1}^N \Lambda_i$ should be negative definite. Thus, according to (48), $e_c^\top \Lambda e_c$ is negative definite with respect to $e_c \in \mathscr{E}_c$. Based on the definition of Λ_P and since $e_c = \mathbf{1}_N \otimes \omega$, we have

$$e_r^\top \Lambda e_c = e_r^\top \Lambda_P^\top \omega = \omega^\top \Lambda_P e_r. \tag{49}$$

Moreover, since $\mathscr{E}_r$ is the orthogonal complement of $\mathscr{E}_c$, it follows that $\forall e_r \in \mathscr{E}_r$ and $\forall \omega \in \mathbb{R}^n$,

$$e_r^\top(\mathbf{1}_N \otimes \omega) = 0.$$

Thus, as $\mathbf{1}_N$ is the eigenvector associated with the zero eigenvalue of $\mathcal{L}_k$, one can get that $\forall e_r \in \mathscr{E}_r$ (Olfati-Saber & Murray, 2004),

$$e_r^\top(\mathcal{L}_k \otimes I_n)e_r \ge \lambda_2(\mathcal{L}_k)e_r^\top e_r. \tag{50}$$

Now, according to (48), (49), and (50) and since $e_c = \mathbf{1}_N \otimes \omega$, from (47) it follows that

$$\begin{aligned}\dot{V} &\le -\begin{bmatrix}\omega^\top & e_r^\top\end{bmatrix}\begin{bmatrix}-\sum_{i=1}^N \Lambda_i & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\,\lambda_2(\mathcal{L}_k)I_{Nn} - \Lambda\end{bmatrix}\begin{bmatrix}\omega \\ e_r\end{bmatrix} \\ &\quad - 2(\mathbf{1}_N \otimes \omega)^\top \Xi w - 2e_r^\top \Xi w,\end{aligned}$$

which using the definition of Ξ_P and since $\lambda_2(\mathcal{L}_k) \ge \mathcal{C}(N)$, can be restated as follows:

$$\begin{aligned}\dot{V} &\le -\begin{bmatrix}\omega^\top & e_r^\top\end{bmatrix}\begin{bmatrix}-\sum_{i=1}^N \Lambda_i & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\,\mathcal{C}(N)I_{Nn} - \Lambda\end{bmatrix}\begin{bmatrix}\omega \\ e_r\end{bmatrix} \\ &\quad - 2\omega^\top \Xi_P w - 2e_r^\top \Xi w.\end{aligned} \tag{51}$$

Now, by invoking (van der Schaft, 1992), we need to impose

$$J = \dot{V} + e^\top e - \mu^2 w^\top w < 0. \tag{52}$$

According to the definition of e_c and e_r and by defining $\acute{\mu} = \mu^2$, from (52) it follows that

$$J = \dot{V} + e_c^\top e_c + e_r^\top e_r + 2e_c^\top e_r - \acute{\mu} w^\top w. \tag{53}$$

Since $e_c = \mathbf{1}_N \otimes \omega$ and $e_r^\top e_c = 0$, (53) can be rewritten as

$$J = \dot{V} + N\omega^\top\omega + e_r^\top e_r - \acute{\mu} w^\top w. \tag{54}$$

Thus, by considering (51) and (54), we have

$$J \le -\begin{bmatrix}\omega^\top & e_r^\top & w^\top\end{bmatrix} M \begin{bmatrix}\omega^\top & e_r^\top & w^\top\end{bmatrix}^\top, \tag{55}$$

where

$$M = \begin{bmatrix}-\sum_{i=1}^N \Lambda_i - NI_n & -\Lambda_P & \Xi_P \\ -\Lambda_P^\top & 2\chi\,\mathcal{C}(N)I_{Nn} - \Lambda - I_{Nn} & \Xi \\ \Xi_P^\top & \Xi^\top & \acute{\mu} I_{p+Nn}\end{bmatrix}.$$

According to the LMI condition (43b), we have $M \succ 0$. Thus, from (55) it follows that $J < 0$, which implies that the $\mathscr{H}_\infty$ performance (5) is satisfied. Thus, the proof is completed. ■

Remark 3. From (51), it follows that for each $\varepsilon \in \mathbb{R}_{>0}$,

$$\begin{aligned}\dot{V} &\le -\begin{bmatrix}\omega^\top & e_r^\top\end{bmatrix}\begin{bmatrix}-\sum_{i=1}^N \Lambda_i & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\mathcal{C}(N)I_{Nn} - \Lambda\end{bmatrix}\begin{bmatrix}\omega \\ e_r\end{bmatrix} \\ &\quad + \frac{1}{\varepsilon}\omega^\top\omega + \varepsilon w^\top \Xi_P^\top \Xi_P w + \frac{1}{\varepsilon} e_r^\top e_r + \varepsilon w^\top \Xi^\top \Xi w,\end{aligned}$$

which can be restated as follows:

$$\dot{V} \le -\begin{bmatrix}\omega^\top & e_r^\top\end{bmatrix} Q \begin{bmatrix}\omega \\ e_r\end{bmatrix} + d, \tag{56}$$

where

$$Q = \begin{bmatrix}-\sum_{i=1}^N \Lambda_i - \frac{1}{\varepsilon}I_n & -\Lambda_P \\ -\Lambda_P^\top & 2\chi\,\mathcal{C}(N)I_{Nn} - \Lambda - \frac{1}{\varepsilon}I_{Nn}\end{bmatrix},$$

$$d = \varepsilon w^\top \Xi_P^\top \Xi_P w + \varepsilon w^\top \Xi^\top \Xi w.$$

Note that if the LMI (43b) is satisfied, then there exists ε such that $Q \succ 0$. Now, if $w_i, i \in \{0, 1, \dots, N\}$, are bounded, d is bounded as well. Therefore, by defining $\bar{d}$ as a supremum of d, according to (56), the estimation errors are uniformly ultimately bounded such that

$$\limsup_{t\to\infty} \left|\begin{bmatrix}\omega(t)^\top & e_r(t)^\top\end{bmatrix}\right| \le \left(\frac{\bar{d}}{\lambda_{\min}(Q)}\right)^{\frac{1}{2}},$$

which (since $e_c = \mathbf{1}_N \otimes \omega$) implies that $|e|$ is uniformly ultimately bounded as follows:

$$\limsup_{t\to\infty} |e(t)| \le \left(\frac{N\bar{d}}{\lambda_{\min}(Q)}\right)^{\frac{1}{2}}.$$

The effectiveness of the proposed distributed state estimation strategy depends on the successful selection of the observer gains according to (43b). In the following theorem, we show that the LMI condition of Theorem 3 is always feasible.

Theorem 4. *Consider the jointly observable pairs (C_i, A), $i \in \mathbf{N}$, described in (3). Then, for each $\acute{\mu} \in \mathbb{R}_{>0}$, there exist $\chi \in \mathbb{R}_{>0}$ and matrices $P_i \succ 0$ and $Y_i = P_i L_i$, $i \in \mathbf{N}$, satisfying the LMI (43b).*

Proof. Let us consider the matrix M as follows:

$$M = \begin{bmatrix}M_{11} & M_{12} & M_{13} \\ M_{12}^\top & M_{22} & M_{23} \\ M_{13}^\top & M_{23}^\top & M_{33}\end{bmatrix},$$

where

$$\begin{aligned}&M_{11} = -\sum_{i=1}^N \Lambda_i - NI_n, \quad M_{12} = -\Lambda_P, \\ &M_{13} = \Xi_P, \quad M_{22} = 2\,\chi\,\mathcal{C}(N)I_{Nn} - \Lambda - I_{Nn}, \\ &M_{23} = \Xi, \quad M_{33} = \acute{\mu} I_{p+Nn}.\end{aligned} \tag{57}$$

Based on the Schur complement, M is positive definite if and only if

$$M_{33} \succ 0, \tag{58a}$$

$$M_{22} - M_{23}M_{33}^{-1}M_{23}^\top \succ 0, \tag{58b}$$

$$M_{11} - \begin{bmatrix}M_{12} & M_{13}\end{bmatrix}\begin{bmatrix}M_{22} & M_{23} \\ M_{23}^\top & M_{33}\end{bmatrix}^{-1}\begin{bmatrix}M_{12}^\top \\ M_{13}^\top\end{bmatrix} \succ 0. \tag{58c}$$

For any $\acute{\mu} \in \mathbb{R}_{>0}$, the inequality (58a) is satisfied. Since the term $M_{23}M_{33}^{-1}M_{23}^\top$ does not depend on χ and $M_{22} = 2\,\chi\,\mathcal{C}(N)I_{Nn} - \Lambda - I_{Nn}$, there exists a large enough $\chi \in \mathbb{R}_{>0}$ such that (58b) is satisfied as well. Moreover, for some $\alpha \in \mathbb{R}_{>0}$,

$$\begin{bmatrix}M_{12} & M_{13}\end{bmatrix}\begin{bmatrix}M_{22} & M_{23} \\ M_{23}^\top & M_{33}\end{bmatrix}^{-1}\begin{bmatrix}M_{12}^\top \\ M_{13}^\top\end{bmatrix} \preceq \alpha I_n.$$

Now, to show that the inequality (58c) has solution, we show that $\forall \alpha \in \mathbb{R}_{>0}$, there exist $P_i \succ 0$ and $L_i = P_i^{-1} Y_i$, $i \in \mathbf{N}$, such that

$$M_{11} \succ \alpha I_n,$$

which according to (57), can be rewritten as follows:

$$-\sum_{i=1}^{N} \Lambda_i \succ \alpha I_n + N I_n. \tag{59}$$

By considering $L_i \in \mathbb{R}^{n \times p_i}$ and $P_i \in \mathbb{R}^{n \times n}$ as (35), following a procedure similar to the proof of Theorem 2, one gets there exist P_{iu}, L_{io}, and P_{ir} such that (59) is equivalent to

$$-T_o \left(\tilde{\Lambda}_o + H \right) T_o^\top \succ \alpha I_n + N I_n. \tag{60}$$

Moreover, it was shown that P_{io} can be selected symmetric positive definite such that $P_i \succ 0$ and the eigenvalues of Λ_{io} can be large enough. Hence, (60) holds with Λ_{io}, $i \in \mathbf{N}$, having large enough negative eigenvalues, which implies that (59) also holds. Therefore, there exist P_{iu}, L_{io}, P_{ir}, and P_{io} such that (59) holds. Then, we construct L_i and $P_i \succ 0$ as given in (35). By considering all the above-mentioned issues, for each $\acute{\mu} \in \mathbb{R}_{>0}$, there exist $\chi \in \mathbb{R}_{>0}$, $P_i \succ 0$, and $Y_i = P_i L_i$, $i \in \mathbf{N}$, satisfying the LMIs (58), which implies that the LMI (43b) is satisfied. Therefore, the proof is completed. ■

Remark 4. It should be noted that according to Theorem 4, for any performance index $\acute{\mu} > 0$, there always exists a set of gains for the proposed distributed observers satisfying the LMI (43b) with a dimension that scales quadratically with the number of the nodes N. Therefore, for any size of the network, by employing the proposed distributed estimation scheme, a desired performance can be guaranteed.

The effectiveness of the proposed distributed observers design is illustrated via numerical examples in the next section.

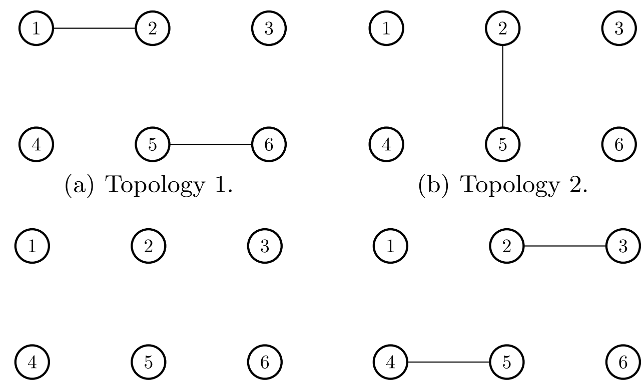
6. Simulation results

We consider a jointly observable LTI system as in (1) with six measurement nodes described by

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, C_3 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$
$$C_4 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, C_5 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, C_6 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix},$$

where $x = [x^1 \ x^2 \ x^3]^\top$ is the state vector and u is the input, which without loss of generality is selected as $u = F\hat{x}^1 + 10\cos(\hat{x}^1)^\top \sin(\hat{x}^1)\sin(t)$, where $F = [-6.3750 \ 1.1250 \ -1.7500]$ and the trigonometric functions are regarded as vectors for the sake of brevity. The proposed distributed estimation strategies are evaluated in two scenarios corresponding to noise-free and noisy settings. The solution of the LMI-based design conditions is obtained by using the CVX toolbox (Boyd & Vandenberghe, 2004).

Scenario 1. In the first scenario, it is assumed that the observers exchange their estimated state vectors under the jointly connected switching communication topologies depicted in Fig. 1, such that the information exchange starts from Topology 1 and switches to the next topology every 1 s. It is worth noticing that the graph in Topology 3 is set to be fully disconnected deliberately.



(a) Topology 1. (b) Topology 2. (c) Topology 3. (d) Topology 4.

Fig. 1. Switching set of the network topologies in Scenario 1.

The distributed observers (6) with the corresponding gains were chosen by solving the LMIs (10), thus obtaining

$$L_1 = \begin{bmatrix} 0 & 0 & -0.8325 \end{bmatrix}^\top, \ L_2 = \begin{bmatrix} 0 & -0.4182 & -0.4182 \end{bmatrix}^\top,$$
$$L_3 = \mathbf{0}_{3\times 1}, \ L_4 = \begin{bmatrix} -0.5813 & 0 & 0 \end{bmatrix}^\top,$$
$$L_5 = \begin{bmatrix} 0 & -0.8260 & 0 \end{bmatrix}^\top, \ L_6 = \mathbf{0}_{3\times 1},$$
$$P_i = \mathrm{diag}\,(0.1000, 0.1000, 0.1000)\,, i \in \{1, 2, \ldots, 6\}.$$

The results of a simulation run are shown in Fig. 2. Notice that, although the communication topology is not connected at each time instant and the pairs (C_i, A), $i \in \{1, 2, \ldots, 6\}$, are not observable, all the local observers are capable of providing asymptotic state estimates following the proposed distributed observers design.

Scenario 2. In the second scenario, the LTI system is subject to external disturbances, and the output measurements are considered noisy as described in (3) with $D = [1 \ 1 \ 1]^\top, E_i = 1, i \in \{1, 2, \ldots, 6\}$. Moreover, to model the external disturbance and noise, the white noise available in Matlab with power of 0.05 is used. It is assumed that the observers exchange their estimated state vectors under the connected switching communication topologies depicted in Fig. 3, such that the information exchange starts from Topology 1 and switches to the next topology every 1 s.

The distributed observers (42) were obtained from the solution of (43) as follows:

$$L_1 = \begin{bmatrix} 0 & -48.7298 & -42.6593 \end{bmatrix}^\top,$$
$$L_2 = \begin{bmatrix} 0 & -60.0379 & 5.6320 \end{bmatrix}^\top,$$
$$L_5 = \begin{bmatrix} 0 & -44.7048 & 49.3226 \end{bmatrix}^\top,$$
$$L_3 = L_4 = L_6 = \mathbf{0}_{3\times 1},$$
$$P_1 = \mathrm{diag}\left(27.7321, \begin{bmatrix} 2.6308 & -3.0058 \\ -3.0058 & 3.6700 \end{bmatrix}\right),$$
$$P_2 = \mathrm{diag}\left(27.7321, \begin{bmatrix} 0.2007 & 0.9322 \\ 0.9322 & 8.7297 \end{bmatrix}\right),$$
$$P_5 = \mathrm{diag}\left(27.7321, \begin{bmatrix} 3.5605 & 3.0227 \\ 3.0227 & 2.7403 \end{bmatrix}\right),$$
$$P_3 = P_4 = P_6 = \mathrm{diag}\left(27.7321, \begin{bmatrix} 0.1000 & 0.0001 \\ 0.0001 & 0.1369 \end{bmatrix}\right),$$

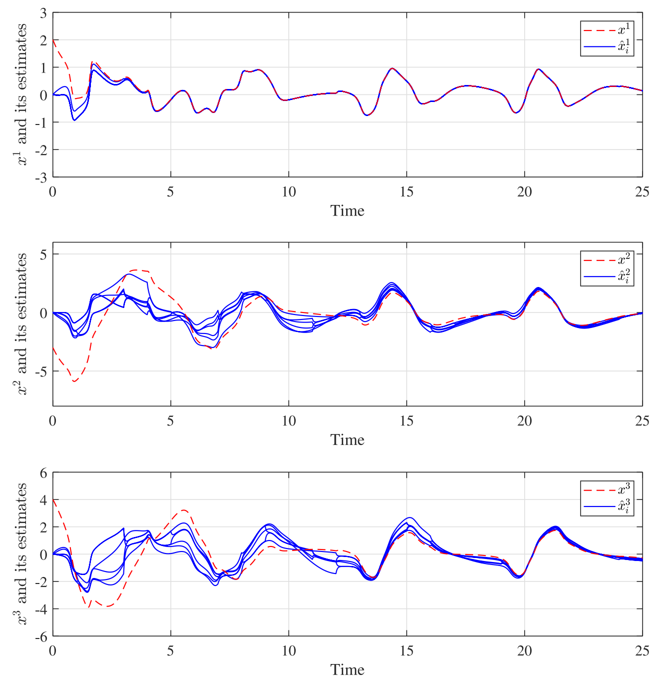


Fig. 2. Evolution of the state and estimated state vectors in Scenario 1 with initial conditions $x(0) = [2\ \ -3\ \ 4]^\top$ and $\hat{x}_i(0) = [0\ 0\ 0]^\top$, $i \in \mathbf{N}$.

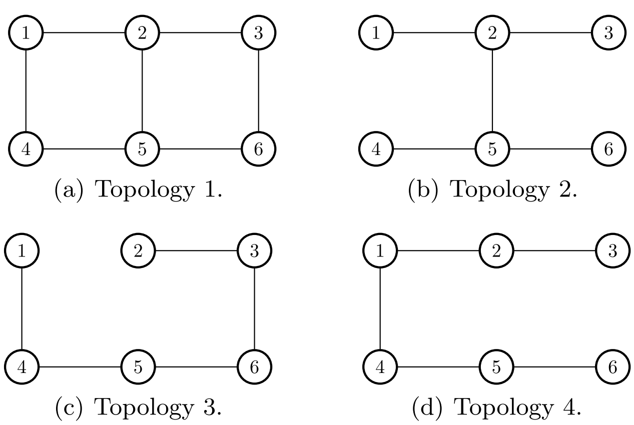


Fig. 3. Switching set of the network topologies in Scenario 2.

with $\chi = 177.4640$ and $\acute{\mu} = 20.5277$ ($\mu = 4.5308$). By employing the designed distributed observers, the estimated state vectors of the observers along with the real state vector of the system are illustrated in Fig. 4, which shows the effectiveness of the proposed approach in a noisy setting.

7. Conclusions and future work

Distributed estimation problem for LTI systems is addressed in this paper. We have developed a method to design a network of distributed observers where the output measurement available for each observer may not be sufficient for the observability of the system. We have shown that if the observers can exchange their estimated state vectors with other neighboring observers, the observability of the system under the joint of all the measurements in the network is sufficient for each observer to estimate the full state vector of the system. We have also

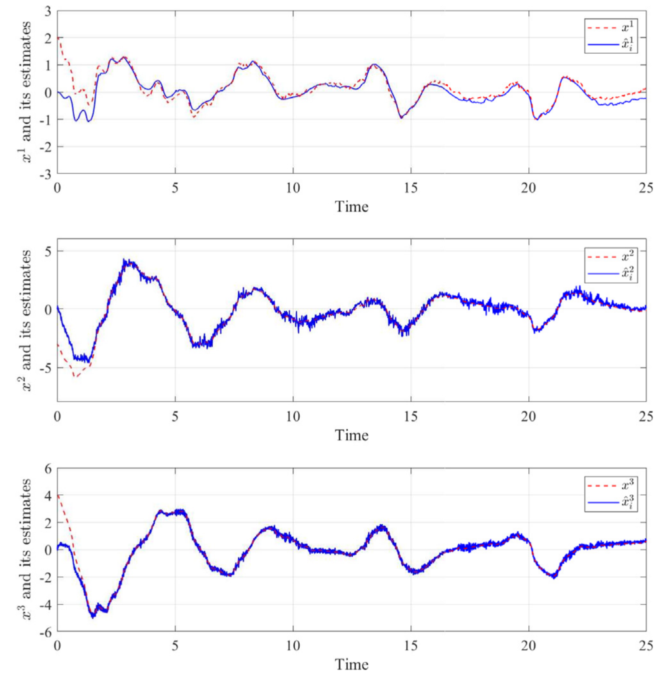


Fig. 4. Evolution of the state and estimated state vectors in Scenario 2 with initial conditions $x(0) = [2\ \ -3\ \ 4]^\top$ and $\hat{x}_i(0) = [0\ 0\ 0]^\top$, $i \in \mathbf{N}$.

shown that, to guarantee the performance of the state estimation under the proposed strategy, communication among the agents is not required to be continuous, and distributed state estimation can be performed if the joint of communication links over finite intervals of time makes the network communication topology connected. Despite similar existing studies limited to LTI systems under certain assumptions and conditions in terms of systems parameters and communication topologies (whose feasibility is not guaranteed for many scenarios), distributed state estimation for a wider class of LTI systems in the presence of jointly connected switching topologies is guaranteed in this paper. Moreover, the results are extended to a scenario when the LTI system is subject to $\mathscr{L}_2$ external disturbances and measurement noise. Accordingly, observer gains are designed such that a desired $\mathscr{H}_\infty$ performance to attenuate the effect of external disturbances and measurement noise is guaranteed. Despite similar existing results on $\mathscr{H}_\infty$-based distributed state estimation, limited to LTI systems under certain conditions (not guaranteed by design), distributed state estimation of all LTI systems in the presence of connected switching topologies is ensured in this paper. Extension of the proposed schemes to distributed state estimation of nonlinear systems is another problem to be studied as a future work.

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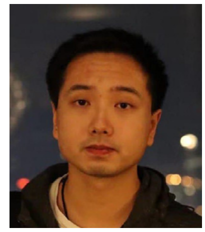

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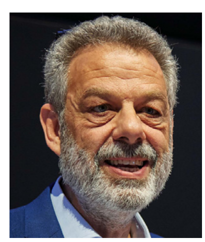

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