Biharmonic Potential Theory and its Applications

by

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Abstract

Biharmonic functions can be represented in terms of harmonic or biharmonic potentials. These potentials can be generated by hypothetical continuous source densities distributed over suitable surfaces. Accordingly the boundary value problems of two-dimensional elastostatics can be formulated as integral equations to determine these sources. This thesis extends the analysis to cover multiply-connected and infinite exterior domains. A new theory of two-dimensional Volterra dislocations and two-dimensional point-forces have been formulated.

Introduction

Biharmonic analysis was systematically inagurated by J. Hadamard at the beginning of the present century. He proved the fundamental existence-uniqueness theorems for both interior and exterior problems. These theorems were then applied to solve certain problems arising from the linear theory of the transverse deflections of thin elastic plates. About fifty years earlier, G.B. Airy (1863) introduced a two-dimensional stress function which satisfies the biharmonic equation. He applied this to certain problems of the stretching of plates under conditions of plane strain and plane stress. Much of Hadamard's biharmonic analysis also applies to the Airy stress function. But nevertheless, important differences arise, because Hadamard's biharmonic function represents the transverse deflection whereas Airy's stress function has no direct physical significance.

A considerable advance in the representation of biharmonic functions was made by E. Almansi (1898), when he proved that any biharmonic function can be expressed in terms of two harmonic functions. This has been utilised to good advantage by Coker and Filon (1957), and more recently by I.C. Brown (1973). A powerful representation has been developed by the Russian School as summarised by N.I. Muskhelishvili (1963). Here the biharmonic function is expressed in terms of two complex potentials. These can often be determined on the boundary of a unit circle from the given boundary data, so enabling the biharmonic function to be continued into the interior or exterior domain of the circle. Accordingly, a wide class of biharmonic boundary-value problems may be solved for any domain which can be conformally mapped onto the unit circle.

Although Mushkelishvili's approach is very powerful, it is also rather restricted. In engineering practice, we often encounter shapes which cannot be mapped onto the unit circle. This has led in recent years, to a renewed interest in Almansi's representation. M.A. Jaswon (1963) introduced Almansi's harmonic functions as simple-layer potentials, generated by hypothetical sources on the boundary. Boundary conditions involving given tractions may then be formulated as boundary integral equations for determining the source densities. Generally speaking, these integral equations cannot be solved analytically. However, methods of numerical discretisation were initiated by G.T. Symm (1963), which enable numerical solutions to be obtained for a wide class of boundaries. This approach was applied to some problems by M.A. Jaswon *et al* (1967, 1968), by S.K. Chakrabarty (1971), and by I.C. Brown (1973).

Considerable complications arise in Almansi's representation when the boundary is doubly-connected or is an infinite exterior domain. This thesis deals with an extension of Almansi's representation to cope with these complications. A feature of the extension lies in the central role played by Volterra dislocations. In effect, a new theory of two-dimensional Volterra dislocations has been formulated. Integral equation methods may therefore now be applied to doubly-connected domains, using a natural extension of the procedure for simply-connected domains.

Another feature of doubly-connected domains is that the tractions need not be equilibrated separately on each boundary. This possibility requires the introduction of multi-valued biharmonic functions, which have not previously been studied in connection with Almansi's representation. The thesis provides an analysis of such functions and brings out their connection with resultant or point-forces. The thesis also provides a new method for determining the displacement field generated by a point-force and by a point couple. Accordingly, boundary displacement problems can now be formulated for doubly-connected domains as well as for simply-connected domains.

A direct representation of biharmonic functions in terms of biharmonic potentials has been introduced by S.K. Chakrabarty (1971). This provides an alternative type of representation to Almansi's representation. Here again the boundary conditions may be expressed as boundary integral equations, which determine hypothetical biharmonic sources on the boundary. These have been solved numerically by Chakrabarty for certain problems concerned with the bending of plates. The present thesis provides a fresh formulation of Chakrabarty's representation, and it also extends the representation to cope with doubly-connected domains (including infinite exterior domains). The scope and limitations of Chakrabarty's representation, as compared with Almansi's representation, has now become more clearly understood.

To summarise, this thesis provides a foundation for the application of integral equation methods to doubly-connected domains and infinite exterior domains of engineering interest. There would seem to be no difficulty in extending the work to cover multiply-connected domains.

PART I

SINGLE-VALUED BIHARMONIC FUNCTIONS

Chapter I

Almansi Representation of Biharmonic Functions

1.1 Introduction

A function ϕ is said to be harmonic within a domain B, bounded by a closed surface ∂B, if it satisfies the following conditions:

i) ϕ is continuous in $B + \partial B$;

ii) ϕ has continuous second order derivatives in B;

iii) ϕ satisfies Laplace's equation

$$\nabla^2\phi \equiv \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \text{ within B.}$$

A function χ is said to be biharmonic within a domain B, bounded by a closed surface ∂B if:

i) χ is continuous in $B + \partial B$;

ii) χ has continuous derivatives upto the fourth order within B;

iii) χ satisfies the equation

$$\nabla^4\chi = \nabla^2\nabla^2\chi = \nabla^2(\nabla^2\chi) = 0 \text{ in B.} \qquad (1\text{-}1\text{-}1)$$

According to Almansi (1897), an arbitrary biharmonic function χ may be represented in terms of two harmonic functions ϕ and ψ in any of the following forms:

$$\chi = x\phi + \psi,$$
$$\chi = y\phi + \psi, \qquad (1\text{-}1\text{-}2)$$
$$\chi = z\phi + \psi,$$

or in the form

$$\chi = r^2\phi + \psi \; ; \;\; r^2 = x^2 = y^2 + z^2, \qquad (1\text{-}1\text{-}3)$$

where $\nabla^2\phi = \nabla^2\psi = 0$. It may be verified by differentiation that the forms (1-1-2), (1-1-3) are biharmonic functions for an arbitrary choice of the harmonic functions ϕ and ψ.

Before giving the proofs of Almansi's theorems, we require a lemma which we state and prove below.

1.2 Lemma

An arbitrary harmonic function h may be written as $\frac{\partial\phi}{\partial x}$ where ϕ is a harmonic function.

Proof: Given $h(x,y,z)$ where $\nabla^2 h = 0$, we introduce the indefinite integral

$$\phi = \int h(x,y,z)dx + \eta(y,z), \qquad (1\text{-}2\text{-}1)$$

where η is a function of y and z to be determined.

Clearly

$$\frac{\partial\phi}{\partial x} = h$$

for any choice of η. Also, since ϕ is a harmonic function, we have

$$0 = \nabla^2\phi = \nabla^2 \int h(x,y,z)dx + \nabla^2\eta(y,z).$$

This shows that $\eta(y,z)$ satisfies the Poission's equation

$$\nabla^2\eta(y,z) = -\nabla^2 \int h(x,y,z)dx. \tag{1-2-2}$$

A solution always exists, so providing a possible ϕ.

1.3 Construction of ϕ

Our proof of the lemma provides a method of constructing a ϕ for a given h, although it should be noted that ϕ is only determined upto an arbitrary harmonic function of y and z. The method of construction is illustrated by the following two examples.

Example 1: $h = x$, in which case

$$\phi = \int x dx + \eta(y,z) = \tfrac{1}{2} x^2 + \eta(y,z).$$

Therefore

$$0 = \nabla^2\phi = 1 + \nabla^2\eta(y,z),$$

from which it follows that

$$\nabla^2\eta(y,z) = -1.$$

A possible solution is provided by

$$\eta = - \frac{1}{4} (y^2 + z^2),$$

so that

$$\phi = \tfrac{1}{2} x^2 - \frac{1}{4} (y^2 + z^2) = \frac{1}{4} (2x^2 - y^2 - z^2).$$

It may be verified that

$$\nabla^2\phi = 0 \quad \text{and} \quad \frac{\partial\phi}{\partial x} = x.$$

<u>Example 2</u>: $h = \log r$; $r \ = x \ + y$, in which case

$$\phi = \eta(y) + \int \log r \, dx$$

$$= \eta(y) + [(x\log r - y\theta) - x]$$

In this case

$$0 = \nabla^2\phi = \nabla^2[(x\log r - y\theta) - x] + \nabla^2\eta$$

$$= \nabla^2(x\log r - y\theta) - \nabla^2 x + \nabla^2\eta = \nabla^2\eta,$$

since $(x\log r - y \ )$ is a harmonic function, from which it follows $0 = \nabla^2\eta$. A possible solution is $\eta = 0$, hence

$$\phi = x\log r - y\theta - x.$$

We note that this expression for ϕ is not single-valued for any circuit which enclosed the origin.

1.4 Completeness of the x-form

Since χ is biharmonic, $\nabla^2\chi$ is harmonic and therefore, using the above lemma, it may be written as

$$\nabla^2\chi = 2\frac{\partial\phi}{\partial x}; \quad \nabla^2\phi = 0. \tag{1-4-1}$$

Equation (1-4-1) is a Possion's equation for χ in terms of ϕ. Since

$$\nabla^2(x\phi) = 2\frac{\partial\phi}{\partial x},$$

we see that $x\phi$ is a particular solution of the equation, which therefore has the general solution

$$\chi = x\phi + \psi; \quad \nabla^2\phi = \nabla^2\psi = 0.$$

This establishes the completeness of the x-form.

The representation

$$\chi = x\phi + \psi \tag{1-4-2}$$

is not unique, in the sense that we can undertake a transformation

$$\phi \to \phi + \eta\,(y,z), \quad \psi \to \psi - x\eta(y,z), \tag{1-4-3}$$

where η is any harmonic function of y,z which leaves χ invariant. Alternatively expressed, the homogeneous equation

$$x\phi + \psi = 0, \qquad (1\text{-}4\text{-}4)$$

has a class of non-trivial solutions of the form

$$\phi = \eta\ (y,z), \quad \psi = -x\eta\ (y,z); \quad \nabla^2\eta = 0. \qquad (1\text{-}4\text{-}5)$$

The same analysis holds for the other forms in (1-1-2).

1.5 Mutual transformability of x,y and z-forms

Although corresponding permutations of the analysis of the x-form may be applied to the y- and z- forms, the following transformation is of interest. Given the biharmonic form

$$\chi = yh; \quad \nabla^2 h = 0,$$

we may write

$$yh = y\,\frac{\partial\phi}{\partial x} \quad ; \quad \nabla^2\phi = 0$$

by virtue of the lemma 1.2. We then write identically

$$y\,\frac{\partial\phi}{\partial x} = x\frac{\partial\phi}{\partial y} + \left(y\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial y}\right),$$

where $\frac{\partial\phi}{\partial y}$, $y\frac{\partial\phi}{\partial x} - x\frac{\partial\phi}{\partial y}$ are harmonic functions since ϕ is harmonic. Accordingly the y-form yh can be transformed into the x-form. Similarly, the z-form can be transformed into the x-form.

Chapter 2

The r^2-form of Biharmonic functions

2.1 Existence and Uniqueness of the r^2-form

The representation

$$\chi = r^2\phi + \psi; \quad \nabla^2\phi = \nabla^2\psi = 0, \quad r^2 = x^2 + y^2 + z^2, \qquad (2\text{-}1\text{-}1)$$

is essentially equivalent to any of the forms (1-1-2), but it has the advantage that ϕ, ψ are unique for a given χ. First we prove that χ is biharmonic if ϕ, ψ are arbitrary harmonic functions. We note that

$$\nabla^2\chi = \nabla^2(r^2\phi) + \nabla^2\psi = 6\phi + 4(x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + z\frac{\partial\phi}{\partial z}). \qquad (2\text{-}1\text{-}2)$$

Since ϕ is harmonic, the expression in the bracket is also harmonic, so that the right-hand side of (2-1-2) is harmonic. Accordingly, χ is biharmonic.

We now prove that an arbitrary biharmonic function χ may be written in the form (2-1-1). Clearly this means proving that an arbitrary harmonic function $\nabla^2\chi$ can be written in the form (2-1-2). We suppose that χ is defined in some simply-connected domain B, in which case $\nabla^2\chi$ is also defined throughout this domain. In the neighbourhood of any interior point of B, we may express $\nabla^2\chi$ by the convergent series of spherical harmonics:

$$\nabla^2\chi = \sum_{m=-n}^{n} \sum_{n=o}^{\infty} a_{nm} r^n S_n; \quad S_n = P_n^m (\cos\theta) e^{\sin\psi} , \tag{2-1-3}$$

using the usual symbolisms (Webster, 1954), where a_{nm} are uniquely determined coefficients. This expansion converges within any interior sphere centred around this point. Also let us assume the corresponding expansion

$$\phi = \sum_{m=-n}^{h} \sum_{n=o}^{\infty} b_{nm} r^n S_n \tag{2-1-4}$$

within the same sphere, where the coefficients b_{nm} are to be determined in terms of the coefficients a_{nm}. Substituting (2-1-4) for ϕ into the right-hand side of (2-1-2), and equating the nm spherical harmonic on each side, we find

$$a_{nm} = (6 + 4n)\, b_{nm}, \quad \text{i.e.} \quad b_{nm} = a_{nm}/(6 + 4n), \tag{2-1-5}$$

bearing in mind that $r^n S_n$ is a homogeneous form of degree n in x,y,z for each allowed choice of m. We conclude from (2-1-5) that

i) the b_{nm} are uniquely determined;

ii) the b_{nm} remain finite as n runs through all the positive integers including zero;

iii) the expansion (2-1-4) for ϕ certainly converges within the sphere since (2-1-3) converges.

Accordingly, we have stablished the unique existence of ϕ within any interior sphere of B. Alternatively expressed, we have established a unique local solution of the differential equation (2-1-2) for ϕ in terms of $\nabla^2\chi$. Since such a local solution exists in the neighbourhood of any interior point of B, we may link them up into a unique global solution throughout B using the principle of analytic continuation. Therefore the unique existence of the r^2-form has been demonstrated for a simply-connected domain.

As an immediate corollary, the homogeneous equation

$$r^2\phi + \psi = o; \quad \nabla^2\phi = \nabla^2\psi = o, \tag{2-1-7}$$

has only the solution

$$\phi = \psi = o. \tag{2-1-8}$$

2.2 Interior and exterior spherical domains

There are two domains for which an immediate global solution is available. First if B happens to be a sphere, the previous expansions apply throughout the sphere. Also, if B is the infinite domain exterior to a sphere, we introduce the convergent expansion

$$\nabla^2\chi = r^{-1} \sum_{m=-n}^{n} \sum_{n=o}^{\infty} a_{nm}\, r^{n} S_n, \tag{2-2-1}$$

which holds throughout B. Similarly, we assume the expansion

$$\phi = r^{-1} \sum_{m=-n}^{n} \sum_{n=o}^{\infty} b_{nm} r^{n} S_{n}. \qquad (2\text{-}2\text{-}2)$$

By repetition of the previous analysis, we find

$$a_{nm} = [6 + 4(n - 1)] \ b_{nm}, \quad \text{i.e.} \quad b_{nm} = a_{nm}/(2 + 4n). \qquad (2\text{-}2\text{-}3)$$

We conclude that ϕ is unique and determined throughout the infinite exterior domain. Our results may easily be extended to a concentric spherical annulus.

2.3 Mutual transformability of the r^2-form and x-,y-,z-forms

We now show that the r^2-form can be transformed into combinations of the x-,y-,z- forms. Thus, given the biharmonic form

$$\chi = r^2 h; \quad \nabla^2 h = o, \qquad (2\text{-}3\text{-}1)$$

we may write

$$r^2 h = r^2 \frac{\partial \phi}{\partial z}; \quad \nabla^2 \phi = o, \qquad (2\text{-}3\text{-}2)$$

by lemma 1.2. Also identically,

$$r^2\frac{\partial\phi}{\partial z} = x(x\,\frac{\partial\phi}{\partial z} - z\,\frac{\partial\phi}{\partial x}) + y\,(\,y\,\frac{\partial\phi}{\partial z} - z\,\frac{\partial\phi}{\partial y}\,)$$

$$+\; x\;(x\,\frac{\partial\phi}{\partial x} + y\,\frac{\partial\phi}{\partial y} + z\,\frac{\partial\phi}{\partial z}\,),$$

i.e. $$r^2\frac{\partial\phi}{\partial z} = x\phi_1 + y\phi_2 + z\phi_3 \qquad (2\text{-}3\text{-}3)$$

where ϕ_i $(i = 1,2,3)$ is harmonic since ϕ is harmonic. Any two of the forms on the right-hand side may be transformed into the third by section 1.5, and therefore we conclude that r h can be transformed into the x-, y-, or z- forms.

Chapter 3

Two-dimensional Biharmonic Functions

3.1 Almansi x- and y- forms

Most of the results obtained in three-dimensions hold good for two-dimensional biharmonic functions, though important exceptions arise. The representations

$$\left.\begin{aligned} \chi &= x\phi + \psi \\ \chi &= y\phi + \psi \end{aligned}\right\} \quad \nabla^2\phi = \nabla^2\psi = 0, \qquad (3\text{-}1\text{-}1)$$

are not unique, in the sense that we can undertake a transformation

$$\phi \to \phi + A + By, \quad \psi \to \psi - Ax - Bxy, \qquad (3\text{-}1\text{-}2)$$

which leaves $\chi = x\phi + \psi$ invariant, where A and B are arbitrary constants. Alternatively expressed, the homogeneous equation

$$x\phi + \psi = 0. \qquad (3\text{-}1\text{-}3)$$

has a class of non-trivial solutions

$$\phi = A + By, \quad \psi = -Ax - Bxy. \qquad (3\text{-}1\text{-}4)$$

A similar analysis holds for the y-form.

The representations (3-1-1) are complete only if we admit multi-valued harmonic functions. For instance, we may write

$$y\log r = x(-\theta) + (y \log r + x\theta), \tag{3-1-5}$$

where the expressions in brackets are respectively the imaginary parts of the analytic functions -z and zlogz. However, these are multi-valued for a complete circuit round the origin. Similarly, we can write

$$x\log r = y\,\theta + (x\log r - y\theta), \tag{3-1-6}$$

where $(x\log r - y\theta)$ is the real part of the analytic function zlog z.

3.2 Almansi r -form

The analysis of the r^2-form

$$\chi = r^2\phi = \psi; \quad r^2 = x^2 + y^2, \tag{3-2-1}$$

differs significantly from the corresponding form in three-dimensions. This is because

$$\nabla^2\chi = \nabla^2 (r^2\phi) = 4\phi + 4\left(x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}\right) \tag{3-2-2}$$

where we note the appearance of 4ϕ instead of 6ϕ which appeared in (2-1-2). As before we assume that χ is given in some simply-connected domain B, in which case $\nabla^2\chi$ is also defined throughout this domain. To prove the completeness of the form (3-2-1), it is clearly necessary to examine whether an arbitrary harmonic function can be represented in the form (3-2-2). In the neighbourhood of any interior point of B, we may express $\nabla^2\chi$ by the convergent series of plane harmonics

$$\nabla^2\chi = \sum_{n=o}^{\infty} r^n(a_{nc}\cos n\theta + a_{ns}\sin n\theta), \qquad (3\text{-}2\text{-}3)$$

where a_{nc}, a_{ns} are uniquely determined coefficients. The expansion (3-2-3) converges within any interior circle centred around this point. Also, let us assume the corresponding expansion

$$\phi = \sum_{n=o}^{\infty} r^n (b_{nc} \cos n\theta + b_{ns} \sin n\theta) \qquad (3\text{-}2\text{-}4)$$

within the same circle, where the coefficients b_{nc}, b_{ns} are to be determined into terms of the coefficients a_{nc}, a_{ns}. Substituting (3-2-4) for ϕ into the right-hand side of (3-2-2), and equating the coefficients of $\cos n\theta$, $\sin n\theta$ on each side, we find

$$b_{nc} = a_{nc}/(4 + 4n), \quad b_{ns} = a_{ns}/(4 + 4n); \quad n = 0,1,2,.. \qquad (3\text{-}2\text{-}5)$$

bearing in mind that $r^n\cos n\theta$, $r^n\sin n\theta$ are homogeneous forms of degree n in x,y. We conclude from (3-2-5) that ϕ uniquely exists within this circle. Hence we have proved the existence of a local solution within any interior circle of B and thus by the principle of analytic continuation, we may construct a global solution ϕ throughout B.

There are two domains for which global solutions are available. First, if B happens to be a circle, the previous expansions apply throughout this circle. Second, if B is the infinite domain exterior to a circle, the expansion corresponding to (3-2-4) takes the form

$$\phi = \sum_{n=o}^{-\infty} r^n(b_{nc}\cos n\theta + b_{ns}\sin n\theta). \tag{3-2-6}$$

By substitution, we obtain the relations

$$b_{nc} = a_{nc}/(4 + 4n), \quad b_{ns} = a_{ns}/(4 + 4n); \qquad n = 0, -1, -2,.... \tag{3-2-7}$$

which are formally the same relations as (3-2-5). From (3-2-7) we draw the conclusion that a breakdown occurs when $n = -1$, i.e. when $\nabla^2\chi$ contains components proportioned to $r^{-1}\cos\theta$ or $r^{-1}\sin\theta$. Also, since

$$\nabla^2(x\log r) = 2r^{-1}\cos\theta, \; \nabla^2(y\log r) = 2r^{-1}\sin\theta. \tag{3-2-8}$$

it follows that the singular biharmonic functions $x \log r$, $y \log r$ cannot be represented by (3-2-1). These functions are inadmissible within any domain enclosing r = o, but they could exist within any finite domain which excludes r = o.

Another conclusion we draw from (3-2-7) is that b_{-1c}, b_{-1s} may be finite even is all the a_{nc} and a_{ns} are zeros. Alternatively expressed, this means that the homogeneous equation

$$0 = 4\phi + 4 \left(x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y}\right) \tag{3-2-9}$$

admits two independent non-trivial solutions

$$\phi = r^{-1}\cos\theta \quad , \quad r^{-1}\sin\theta \ , \tag{3-2-10}$$

as may be verified by differentiation. These also satisfy the homogeneous equation

$$r^2\phi + \psi = 0; \quad \nabla^2\phi = \nabla^2\psi = 0. \tag{3-2-11}$$

This may also be seen directly from the fact that the harmonic functions

$$\phi = r^{-1}\cos\theta, \quad \psi = -r^2.r^{-1}\cos\theta = -r\cos\theta = -x$$

$$\phi = r^{-1}\sin\theta, \quad \psi = -r^2.r^{-1}\sin\theta = -r\sin\theta = -y$$

identically satisfies the equation (3-2-11).

3.3. Completeness of the r^2-form

We note the identities

$$\left.\begin{aligned} x\log r &= \tfrac{1}{2} \quad r^2\left(\frac{x\log r + y\theta}{r^2}\right) + (x\log r - y\theta) \quad , \\ y\log r &= \tfrac{1}{2} \quad r^2\left(\frac{y\log r - x\theta}{r^2}\right) + (y\log r - x\theta) \quad , \end{aligned}\right\} \qquad (3\text{-}3\text{-}1)$$

where

$$\phi = \left(\frac{x\log r + y\theta}{2r^2}\right) \quad , \quad \left(\frac{y\log r - x\theta}{2r^2}\right) \qquad (3\text{-}3\text{-}2)$$

are multi-valued harmonic functions, i.e. they are the real and imaginary parts of the complex analytic function $(\log z)/z$. Accordingly the singular biharmonic functions $x\log r$, $y\log r$ can be represented in the r^2-form, if we admit multi-valued harmonic functions for any circuit round the origin. Operating on both sides of (3-3-1) by ∇^2, we infer that ϕ as given by (3-3-2) satisfies the equation (3-2-2), i.e.

$$\frac{2\cos\theta}{r},\ \frac{2\sin\theta}{r} = 4\phi + 4\ (x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}). \qquad (3\text{-}3\text{-}3)$$

The identity (3-3-3) may be verified by direct differentiation

The harmonic function ϕ remains single-valued within any domain which does not include the origin. If we construct a chain of overlapping circles B_1, B_2, B_3,..circuit the origin, we get a global ϕ in the multiply connected domain

$$B = B_1 \cup B_2 \cup B_3 \cup \ldots$$

even though it is single-valued within any particular circle. Multi-valued harmonic functions clearly lie outside the scope of the expansion (3-2-6), so that no inconsistency arises between (3-3-1) and the breakdown (3-2-7).

Accordingly, the Almansi r^2-form is not complete if we restrict harmonic functions concerned to be single-valued. The problem of making it complete, subject to this restriction, will be discussed in Chapter 4 .

3.4 Transformation of the r^2-form into x- and y- forms

Given the biharmonic function

$$\chi = r^2 h \ ; \quad \nabla^2 h = 0, \tag{3-4-1}$$

we can always write

$$h = \frac{\partial\phi}{\partial x} \ ; \ \nabla^2\phi = 0 \tag{3-4-2}$$

by lemma 1.2 . Also, we have identically

$$r^2 \frac{\partial\phi}{\partial x} = x \left(x \frac{\partial\phi}{\partial x} + y \frac{\partial\phi}{\partial y}\right) + y \left(y \frac{\partial\phi}{\partial x} - x \frac{\partial\phi}{\partial y}\right), \tag{3-4-3}$$

where the expressions in the paranthsis are harmonic functions. Thus the r^2-form transforms into a combination of the x- and y- forms which, in turn, can be transformed into either form.

For example, let us consider the biharmonic function $r^2 \log r$. We note

$$r^2 \log r = r^2 \frac{\partial\phi}{\partial x} \ ; \ \phi = (x \log r - y\theta) - x$$

so that, according to (3-4-3),

$$r^2\log r = x(x\log r - y\theta) + y(y\log r + x\theta)$$

as can also be seen directly. As before, this result involves multi-valued harmonic functions for any circuit round the origin. Accordingly, the transformation is not always admissible if we restrict the harmonic functions to be single-valued.

Chapter 4

Harmonic Potentials

4.1 Introduction

In this chapter we develop an alternative representation to the Almansi representation, which may be preferable for certain purposes. This depends upon the theory of biharmonic potentials, which is a close analogue to the theory of harmonic potentials. We therefore recapitulate briefly classical potential theory, as found for instance in Kellogg (1929), in a form that will be found convenient for subsequent applications.

4.2 Simple source potential

A unit simple source, located at a point $\underset{\sim}{q}$ referred to a suitable three-dimensional co-ordinate system, generates the Newtonian potential

$$g\,(\underset{\sim}{p},\underset{\sim}{q}) = g(\underset{\sim}{q},\underset{\sim}{p}) = |\underset{\sim}{p} - \underset{\sim}{q}|^{-1} \qquad (4\text{-}2\text{-}1)$$

at a field point $\underset{\sim}{p}$. This potential is a continuous function of $\underset{\sim}{p}$, differentiable to all orders, and it formally satisfies Laplace's equation

$$\nabla^2\, g(\underset{\sim}{p},\underset{\sim}{q}) = 0 \qquad (4\text{-}2\text{-}2)$$

everywhere except at $\underset{\sim}{p} = \underset{\sim}{q}$. It is therefore a harmonic function everywhere except at $\underset{\sim}{p} = \underset{\sim}{q}$. Alternatively expressed, it satisfies the Poission's equation

$$\nabla^2 g(\underset{\sim}{p},\underset{\sim}{q}) = -4\pi\delta(\underset{\sim}{p} - \underset{\sim}{q}) \qquad (4\text{-}2\text{-}3)$$

everywhere, where δ is Dirac's delta function centre upon $\underset{\sim}{q}$ (Dirac, 1930).

A discrete distribution of simple sources, of strengths $\sigma_1, \sigma_2, \ldots, \sigma_N$ located at $\underset{\sim}{q}_1, \underset{\sim}{q}_2, \ldots, \underset{\sim}{q}_N$ respectively, generates the Newtonian potential

$$V(\underset{\sim}{p}) = \sum_{\alpha=1}^{N} g(\underset{\sim}{p},\underset{\sim}{q}_\alpha)\sigma_\alpha \qquad (4\text{-}2\text{-}4)$$

at $\underset{\sim}{p}$. This has essentially the same properties as $g(\underset{\sim}{p},\underset{\sim}{q})$, in particular it is a harmonic function everywhere except at $\underset{\sim}{q}_1, \underset{\sim}{q}_2, \ldots, \underset{\sim}{q}_N$ and it formally satisfies Poission's equation

$$\nabla^2 V(\underset{\sim}{p}) = -4\pi \sum_{\alpha=1}^{N} \delta(\underset{\sim}{p} - \underset{\sim}{q}_\alpha)\sigma_\alpha \qquad (4\text{-}2\text{-}5)$$

everywhere. If the source points are located within a finite region of space B, then

$$g(\underset{\sim}{p},\underset{\sim}{q}_\alpha) = |\underset{\sim}{p}|^{-1} + |\underset{\sim}{p}|^{-3}(\underset{\sim}{p},\underset{\sim}{q}_\alpha) + O(|\underset{\sim}{p}|^{-3}) \text{ as } |\underset{\sim}{p}| \to \infty, \qquad (4\text{-}2\text{-}6)$$

and it follows that

$$V(\underset{\sim}{p}) = |\underset{\sim}{p}|^{-1} \sum_{\alpha=1}^{N} \sigma_\alpha + |\underset{\sim}{p}|^{-3} \sum_{\alpha=1}^{N} (\underset{\sim}{p}.\underset{\sim}{q}_\alpha)\sigma_\alpha + O(|\underset{\sim}{p}|^{-3}) \text{ as } |\underset{\sim}{p}| \to \infty . \quad (4\text{-}2\text{-}7)$$

4.3 Surface distribution of simple sources

A continuous distribution of simple sources extending over a Kellogg-regular surface ∂B (not necessarily closed), and of surface density $\sigma(\underset{\sim}{q})$ at $\underset{\sim}{q} \epsilon\ \partial B$, generates the simple-layer potential

$$V(\underset{\sim}{p}) = \int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq. \quad (4\text{-}3\text{-}1)$$

This potential is continuous everywhere provided that σ satisfies a Hölder condition, and it has the asymptotic behaviour

$$V(\underset{\sim}{p}) = |\underset{\sim}{p}|^{-1} \int_{\partial B} \sigma(\underset{\sim}{q})dq + O(|\underset{\sim}{p}|^{-2}) \text{ as } |\underset{\sim}{p}| \to \infty . \quad (4\text{-}3\text{-}2)$$

It is differentiable to the second order and satisfies Laplace's equation, and it is therefore a harmonic function everywhere, except at ∂B.

If ∂B is closed, V represented a harmonic function everywhere in the interior domain B_i bounded by ∂B. Conversely, an arbitrary harmonic function ϕ within B_i can generally be represented as a simple-layer potential (Jaswon, 1963), except in the two-dimensional case of a Γ - contour (App. 1).

By virtue of (4-3-2), V represents a regular harmonic function in the infinite domain B_e exterior to the closed surface ∂B. Conversely, an arbitrary regular harmonic function in B_e can generally be represented by a simple-layer potential, except in the case of a Γ - contour.

4.4 Normal derivatives of V

We erect a normal line to one side of ∂B at $\underset{\sim}{p}_o \varepsilon\, \partial B$, and we locate points on the normal by a variable n which increases moving away from ∂B. At any point $\underset{\sim}{p}$ on the normal other than the initial point, we have

$$\frac{\partial V(\underset{\sim}{p})}{\partial n} = \int_{\partial B} \frac{\partial g(\underset{\sim}{p},\underset{\sim}{q})}{\partial n} \sigma(\underset{\sim}{q})\, dq, \qquad (4\text{-}4\text{-}1)$$

where $\partial g(\underset{\sim}{p},\underset{\sim}{q})/\partial n$ denotes the derivative of g at $\underset{\sim}{p}$ keeping $\underset{\sim}{q}$ fixed. But, at the initial point $\underset{\sim}{p}_o$,

$$\frac{\partial V(\underset{\sim}{p}_o)}{\partial n} = \int_{\partial B} \frac{\partial g(\underset{\sim}{p}_o,\underset{\sim}{q})}{\partial n} \sigma(\underset{\sim}{q})dq - 2\pi\sigma(\underset{\sim}{p}_o);\quad \underset{\sim}{p}_o \varepsilon\ \partial B, \qquad (4\text{-}4\text{-}2)$$

because of the singularity which arises in g when $\underset{\sim}{q}$ coincides with $\underset{\sim}{p}_o$. An apparent indeterminary occurs in $\partial g/\partial n$ when $\underset{\sim}{q} = \underset{\sim}{p}_o$, but its behaviour there may be uniquely obtained by a limiting process discussed in section 4.5 below. For the sake of convenience, we shall use the notation

$$\left.\begin{aligned} \frac{\partial V(\underset{\sim}{p})}{\partial n} &= V'(\underset{\sim}{p}) \\ \frac{\partial g(\underset{\sim}{p},\underset{\sim}{q})}{\partial n} &= g'(\underset{\sim}{p},\underset{\sim}{q}) = g(\underset{\sim}{q},\underset{\sim}{p})' \end{aligned}\right\} \tag{4-4-3}$$

for normal derivatives, in which case dropping the subscript $\underset{\sim}{p}_o$, we can write (4-4-2) as

$$V'(\underset{\sim}{p}) = \int_{\partial B} g'(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq - 2\pi\sigma(\underset{\sim}{p}) \ ; \ \underset{\sim}{p} \varepsilon \ \partial B. \tag{4-4-4}$$

There exist two distinct normals at $\underset{\sim}{p}$, one on each side of ∂B. Following Jaswon (1963), we shall adopt the convention that these two normals have equal status, in the sense that the relevant variables n_i, n_e both increases moving away from ∂B. Correspondingly, we expand (4-4-4) into

$$V'_i(\underset{\sim}{p}) = \int_{\partial B} g'_i(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq - 2\pi\sigma(\underset{\sim}{p}) \ ; \ \underset{\sim}{p} \varepsilon \ \partial B, \tag{4-4-5}$$

$$V'_e(\underset{\sim}{p}) = \int_{\partial B} g'_e(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq - 2\pi\sigma(\underset{\sim}{p}) \ ; \ \underset{\sim}{p} \varepsilon \ \partial B. \tag{4-4-6}$$

Since $g(\underset{\sim}{p},\underset{\sim}{q})$ remains continuous as $\underset{\sim}{p}$ crosses ∂B, it follows that

$$g'_i(\underset{\sim}{p},\underset{\sim}{q}) + g'_e(\underset{\sim}{p},\underset{\sim}{q}) = 0 \ ; \ \underset{\sim}{p}, \underset{\sim}{q} \ \varepsilon \ \partial B, \tag{4-4-7}$$

whence

$$V'_i(\underset{\sim}{p}) + v'_e(\underset{\sim}{p}) = 0 \;-\; 4\pi\,\sigma(\underset{\sim}{p}) \;;\quad \underset{\sim}{p} \in \partial B. \tag{4-4-8}$$

It is interesting to note that the tangential derivatives of V exist and are continuous at any point $\underset{\sim}{p} \varepsilon\ \partial B$, provided that σ satisfies a Hölder condition at $\underset{\sim}{p}$.

4.5 Surface distribution of double sources

A unit double source at $q \varepsilon\ \partial B$ generates the potential $g(\underset{\sim}{p},\underset{\sim}{q})'_i$ at any other point $\underset{\sim}{p}$, where $g(\underset{\sim}{p},\underset{\sim}{q})'_i$ denotes the n_i-normal derivative of g at $\underset{\sim}{q}$ keeping $\underset{\sim}{p}$ fixed. This potential exists and is a continuous function of $\underset{\sim}{p}$, and is differentiable to all orders, and satisfies Laplace's equation everywhere, except at $\underset{\sim}{q}$. These properties characterise $g(\underset{\sim}{p},\underset{\sim}{q})'_i$ as a harmonic function everywhere except at $\underset{\sim}{q}$.

A discrete distribution of double sources on ∂B, of strength $\mu_1, \mu_2, \ldots, \mu_N$ located at $\underset{\sim}{q}_1, \underset{\sim}{q}_2, \ldots, \underset{\sim}{q}_N$, generates the potential

$$W(\underset{\sim}{p}) = \sum_{\alpha=1}^{N} g(\underset{\sim}{p},\underset{\sim}{q})'_i \mu_\alpha(\underset{\sim}{p}), \tag{4-5-1}$$

which has the same properties of $g(\underset{\sim}{p},\underset{\sim}{q})'_i$ everywhere except at $\underset{\sim}{q}_i, \underset{\sim}{q}_2, \ldots, \underset{\sim}{q}_N$. A continuous distribution of double sources on ∂B, of surface density $\mu(\underset{\sim}{q})$ at $\underset{\sim}{q}$, generates the potential

$$W(\underset{\sim}{p}) = \int_{\partial B} g(\underset{\sim}{p},\underset{\sim}{q})'_i \mu(\underset{\sim}{q})\,dq. \tag{4-5-2}$$

This exists and is continuous, and is differentiable to all orders, and satisfies Laplace's equation; and is therefore a harmonic function everywhere, except at ∂B, and it has the asymptotic behaviour

$$W(\underset{\sim}{p}) = O\,(|\underset{\sim}{p}|^{-1}) \quad \text{as} \quad |\underset{\sim}{p}| \to \infty . \qquad (4\text{-}5\text{-}3)$$

To define the discontinuity of W at $\underset{\sim}{p}_o \varepsilon\ \partial B$, let $\underset{\sim}{p}_i, \underset{\sim}{p}_e$ be points on the n_i - , n_e - normal lines respectively from $\underset{\sim}{p}_o$. then

$$\lim_{\underset{\sim}{p}_i \to \underset{\sim}{p}_o} W(\underset{\sim}{p}_i) = W\,(\underset{\sim}{p}_o) + 2\pi\mu(\underset{\sim}{p}_o), \qquad (4\text{-}5\text{-}4)$$

$$\lim_{\underset{\sim}{p}_e \to \underset{\sim}{p}_o} W(\underset{\sim}{p}_e) = W(\underset{\sim}{p}_o) - 2\pi\mu(\underset{\sim}{p}_o), \qquad (4\text{-}5\text{-}5)$$

provided μ is continuous at $\underset{\sim}{p}_o$.

The normal derivatives of W remain continuous at ∂B, which we express by

$$\lim_{\underset{\sim}{p}_i \to \underset{\sim}{p}_o} \frac{dW(\underset{\sim}{p}_i)}{dn_i} - \lim_{\underset{\sim}{p}_e \to \underset{\sim}{p}_o} \frac{dW(\underset{\sim}{p}_e)}{dn_e} = 0. \qquad (4\text{-}5\text{-}6)$$

The separate limits do not necessarily exist if μ is continuous. To ensure their existence, it is necessary to impose further smoothness restrictions on μ.

4.6 Logarithmic potential

In two dimensions, a continuous distribution of simple sources extending over a closed regular curve ∂B, and of density $\sigma(q)$ at $\underset{\sim}{q} \in \partial B$, generates the simple-layer logarithmic potentials

$$\begin{aligned} V(\underset{\sim}{p}) &= \int_{\partial B} \log|\underset{\sim}{p} - \underset{\sim}{q}|\sigma(\underset{\sim}{q})dq \; ; \quad \underset{\sim}{p} \in B_i + \partial B, \\ V(\underset{\sim}{p}) &= \int_{\partial B} \log|\underset{\sim}{p} - \underset{\sim}{q}|\sigma(\underset{\sim}{q})dq \; ; \quad \underset{\sim}{p} \in B_e + \partial B. \end{aligned} \tag{4-6-1}$$

These define distinct harmonic functions in B_i, B_e respectively which become equal at ∂B. Also,

$$V(\underset{\sim}{p}) = \log|\underset{\sim}{p}| \int_{\partial B} \sigma(\underset{\sim}{q})dq - |\underset{\sim}{p}|^{-2} \int (\underset{\sim}{p}.\underset{\sim}{q})\sigma(\underset{\sim}{q})dq + O\,(|\underset{\sim}{p}|^{-2}) \quad \text{as } |p| \to \infty. \tag{4-6-2}$$

All the formulae in section 4.4 may be easily adapted to the logarithmic potential by writing $-^{\pi}/_{2}$ in place of π. Thus the results corresponding to (4-4-5), (4-4-6) are

$$V_i'(\underset{\sim}{p}) = \int_{\partial B} \log_i' |\underset{\sim}{p} - \underset{\sim}{q}| \; \sigma(\underset{\sim}{q})dq + \pi\sigma(\underset{\sim}{p}) \; ; \quad \underset{\sim}{p} \in \partial B, \tag{4-6-3}$$

$$V_e'(\underset{\sim}{p}) = \int_{\partial B} \log_e' |\underset{\sim}{p} - \underset{\sim}{q}| \; \sigma(\underset{\sim}{q})dq + \pi\sigma(\underset{\sim}{p}) \; ; \quad \underset{\sim}{p} \in \partial B, \tag{4-6-4}$$

from which is immediately follows that

$$V_i'(\underset{\sim}{p}) + V_e'(\underset{\sim}{p}) = 2\pi\sigma(\underset{\sim}{p}) \; ; \; \underset{\sim}{p} \in \partial B. \tag{4-6-5}$$

Also the results corresponding to (4-5-4), (4-5-5) become

$$\lim_{\underset{\sim}{p}_i \to \underset{\sim}{p}_o} W(\underset{\sim}{p}_i) = W(\underset{\sim}{p}_o) - \pi\mu(\underset{\sim}{p}_o), \tag{4-6-6}$$

$$\lim_{\underset{\sim}{p}_e \to \underset{\sim}{p}_o} W(\underset{\sim}{p}_e) = W(\underset{\sim}{p}_o) + \pi\mu(\underset{\sim}{p}_o). \tag{4-6-7}$$

PART II

DISLOCATION SOLUTIONS OF THE BIHARMINIC EQUATION

Chapter 5

Biharmonic Potentials

5.1 Biharmonic potential in Almansi form

According to the analysis in Chapter 4 of simple-layer potentials, we may now represent the harmonic functions ϕ, ψ in the Almansi representation of a biharmonic function by

$$\phi(\underset{\sim}{p}) = \int_{\partial B} \log |\underset{\sim}{p} - \underset{\sim}{q}|\sigma(\underset{\sim}{q})dq \ ; \quad \underset{\sim}{p} \ \varepsilon \ B + \ \partial B$$

$$\psi(\underset{\sim}{p}) = \int_{\partial B} \log |\underset{\sim}{p} - \underset{\sim}{q}|\mu(\underset{\sim}{q})dq \ ; \quad \underset{\sim}{p} \ \varepsilon \ B + \ \partial B \qquad (5\text{-}1\text{-}1)$$

where ∂B is the boundary of any neighbourhood B within the domain of existence of ϕ and ψ, and σ,μ are two independent (Hölder) continuous source densities on ∂B. Then the biharmonic function χ, expressed for instance in the Almansi r^2-form, appears as

$$\chi(\underset{\sim}{p}) = |\underset{\sim}{p}|^2 \int_{\partial B} \log \ |\underset{\sim}{p}-\underset{\sim}{q}|\sigma(\underset{\sim}{q})dq + \int_{\partial B} \log \ |\underset{\sim}{p}-\underset{\sim}{q}|\mu(\underset{\sim}{q})dq; \qquad (5\text{-}1\text{-}2)$$

$$\underset{\sim}{p} \ \varepsilon \ B + \partial B$$

where $|p| = x^2 + y^2 + z^2$. Accordingly, we have a representation of a biharmonic function in terms of simple-layer potentials.

5.2 Unit biharmonic potential

The function

$$r^2 . r^{-1} = r = |p| \ ; \qquad r^2 = x^2 + y^2 + z^2, \tag{5-2-1}$$

is biharmonic everywhere except at $r = 0$. Also it satisfies the differential equations

$$\nabla^2 r = 2r^{-1}, \ \nabla^4 r = 2\nabla^2 r^{-1} = -8\pi\delta(\underset{\sim}{p}), \tag{5-2-2}$$

where $\delta(p)$ is the Dirac delta function centred around $p = \langle 0,0,0 \rangle$. More generally, the function

$$R = |\underset{\sim}{p} - \underset{\sim}{q}| \tag{5-2-3}$$

is biharmonic everywhere except at $\underset{\sim}{q}$. Also it satisfies the differential equations

$$\nabla^2 R = 2 \ |p-q|^{-1}, \ \nabla^4 R = -8\pi\delta(\underset{\sim}{p}-\underset{\sim}{q}) \tag{5-2-4}$$

where $\delta(p-q)$ is the Dirac delta function centred around $\underset{\sim}{p} = \underset{\sim}{q}$. By analogy with harmonic potential theory, we may regard (5-2-3) as the potential generated by a unit biharmonic source at $\underset{\sim}{q}$. Pursuing this analogy, we introduce the simple-layer biharmonic potential

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} |\underset{\sim}{p}-\underset{\sim}{q}|\sigma(\underset{\sim}{q})dq; \quad \underset{\sim}{p} \in B + \partial B, \tag{5-2-5}$$

where σ is a biharmonic simple source density on ∂B and is assumed to be Hölder continuous. This potential $\Omega(\underset{\sim}{p})$ has the following three properties:

i) $\Omega(\underset{\sim}{p})$ is a continuous function in $B + \partial B$;

ii) $\Omega(\underset{\sim}{p})$ has derivatives upto fourth order in B;

iii) $\Omega(\underset{\sim}{p})$ satisfies the equation

$$\nabla^2\Omega(\underset{\sim}{p}) = \nabla^2 \int_{\partial B} |\underset{\sim}{p}-\underset{\sim}{q}|\sigma(\underset{\sim}{q})dq = \int_{\partial B} \nabla^2|\underset{\sim}{p}-\underset{\sim}{q}| \sigma(\underset{\sim}{q})dq$$

$$= 2\int_{\partial B} |\underset{\sim}{p}-\underset{\sim}{q}|^{-1}\sigma(\underset{\sim}{q})dq \; ; \quad \underset{\sim}{p} \in B. \tag{5-2-6}$$

Since the right-hand side of (5-2-6) is a simple-layer potential, it follows that

$$\nabla^4\Omega(\underset{\sim}{p}) = 0 \; ; \quad \underset{\sim}{p} \in B,$$

i.e. Ω is a biharmonic function everywhere in B.

The property (5-2-6) has a very important consequence. For, given an arbitrary biharmonic function χ in B, it follows that $\nabla^2\chi$ is a known harmonic function therein. Now from section 4., any arbitrary harmonic function in B can be represented as a simple-layer potential generated by a continuous source distribution on ∂B. Thus we may write

$$\nabla^2\chi = 2\int_{\partial B} |\underset{\sim}{p}-\underset{\sim}{q}|^{-1}\sigma(\underset{\sim}{q})dq \ ; \quad \underset{\sim}{p} \in B + \partial B, \tag{5-2-7}$$

where σ is, in principle, known in terms of $\nabla^2\chi$. If we use this σ in (5-2-6), then

$$\nabla^2\chi = \nabla^2\Omega, \quad \text{i.e.} \quad \nabla^2 (\chi - \Omega) = 0$$

$$\text{i.e. } \chi = \Omega + f \ ; \quad \nabla^2 f = 0,$$

where f is an arbitrary harmonic function in B. So we have formulated a new general representation for biharmonic functions, which provides an alternative to the Almansi representation.

5.3 Two-dimensional biharmonic potential

In two dimensions, the function $r^2 \log r$ corresponds to (5-2-1). More generally, the biharmonic function

$$R^2 \log R = |p - q|^2 \log |p - q| \tag{5-3-1}$$

corresponds to (5-2-3). This satisfies the differential equations

$$\begin{aligned} \nabla^2 (R^2 \log R) &= 4 + 4\log R, \\ \nabla^4 (R^2 \log R) &= 4\nabla^2 \log R = 8\pi\delta(\underset{\sim}{p} - \underset{\sim}{q}). \end{aligned} \tag{5-3-2}$$

Accordingly, we may regard (5-3-2) as the two-dimensional potential generated by a unit biharmonic source at $\underset{\sim}{q}$.

An equivalent potential to (5-3-1) is

$$\begin{aligned} G(\underset{\sim}{p},\underset{\sim}{q}) &= -R^2 + R^2 \log R \\ &= |\underset{\sim}{p} - \underset{\sim}{q}| \; [-1 + \log|\underset{\sim}{p} - \underset{\sim}{q}|] \end{aligned} \tag{5-3-3}$$

which was introduced by Chakrabarty (1972). This satisfies the differential equations

$$\nabla^2 G = 4 \log R, \quad \nabla^4 G = 8\pi\delta(\underset{\sim}{p} - \underset{\sim}{q}), \tag{5-3-4}$$

in which cas the constant term "4" of (5-3-2) does not appear. Accordingly, it corresponds directly with (5-2-2).
Pursuing the analogy, we introduce the simple-layer biharmonic potential

$$\Omega(\underset{\sim}{p}) = \int_{\partial B} G(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq \; ; \quad \underset{\sim}{p} \in B + \partial B, \tag{5-3-5}$$

which has the convenient property

$$\nabla^2\Omega(\underset{\sim}{p}) = \nabla^2 \int_{\partial B} G(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq = \int_{\partial B} \nabla^2 G(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq$$

$$= 4 \int_{\partial B} \log |\underset{\sim}{p} - \underset{\sim}{q}|\sigma(\underset{\sim}{q})dq \; ; \quad \underset{\sim}{p} \in B. \tag{5-3-6}$$

Conversely, an arbitrary two-dimensional χ in B may be written in the form

$$\chi = \Omega + f \; ; \quad \nabla^2 f = 0, \tag{5-3-7}$$

where f is an arbitrary two-dimensional harmonic function in B.

5.4 Asymptotic behaviour of biharmonic potential

Given a closed two-dimensional contour ∂B, we introduce an arbitrary (Hölder) continuous distribution of biharmonic source on ∂B. This generates an interior potential $\Omega(\underset{\sim}{p})$, of the form (5-3-5), within the interior domain B_i bounded by ∂B. Also, it generates an exterior biharmonic potential of the same type, within the infinite domain B_e exterior to B, having the asymptotic behaviour

$$\Omega(\underset{\sim}{p}) = (-r^2 + r^2\log r)\int_{\partial B}\sigma(\underset{\sim}{q})dq - 2x\log r\int_{\partial B} q_1\sigma(\underset{\sim}{q})dq$$

$$-2y\log r\int_{\partial B} q_2\sigma(\underset{\sim}{q})dq + O(r) \text{ as } |\underset{\sim}{p}|=r\to\infty, \qquad (5\text{-}4\text{-}1)$$

where q_1,q_2 are the Cartesian co-ordinates of $\underset{\sim}{q}$. This may be demonstrated directly by examining the asymptotic behaviour of $|\underset{\sim}{p} - \underset{\sim}{q}|^2[-1 + \log|\underset{\sim}{p} - \underset{\sim}{q}|]$ as $|p|\to\infty$. Alternatively, we note from (5-3-6), bearing in mind (5-3-4), that $\nabla^2\Omega$ has the asymptotic behaviour

$$\nabla^2\Omega(\underset{\sim}{p}) = 4\log r\int_{\partial B}\sigma(\underset{\sim}{q})dq - 4r^{-1}\cos\theta\int_{\partial B} q_1\sigma(\underset{\sim}{q})dq$$

$$- 4r^{-1}\sin\theta\int_{\partial B} q_2\sigma(\underset{\sim}{q})dq + O(r^{-2}) \qquad (5\text{-}4\text{-}2)$$

$$\text{as } |\underset{\sim}{p}| = r\to\infty.$$

By an application of the results

$$\nabla^2(-r^2 + r^2\log r) = 4\log r,$$
$$\nabla^2(x\log r) = 2r^{-1}\cos\theta \;, \; \nabla^2(y\log r) = 2r^{-1}\sin\theta, \qquad (5\text{-}4\text{-}3)$$

we deduce (5-4-1) from (5-4-2).

The first three terms of (5-4-1) are generally not admissible within an infinite exterior domain, because they increase too rapidly on physical grounds. To eliminate these, it is necessary to introduce the side conditions

$$\int_{\partial B}\sigma(\underset{\sim}{q})dq = 0, \quad \int_{\partial B} q_1\sigma(\underset{\sim}{q})dq = 0, \quad \int_{\partial B} q_2\sigma(\underset{\sim}{q})dq = 0, \qquad (5\text{-}4\text{-}4)$$

on σ. How these can be implemented will be shown in a later chapter. From this point of view, by comparison with (5-1-2), the Chakrabarty representation is inferior to the Almansi representation for exterior problems.

Within a ring-shaped domain, such that the interior boundary enclosed r = 0, the functions $r^2\log r$, $x\log r$, $y\log r$ are perfectly admissible. Since these are covered by the

Chakrabarty representation, we see that this representation is complete within a ring-shaped domain. As already noted in section 3.3 , the Almansi representation is not complete within the ring unless supplemented by xlogr and ylogr. Accordingly, Chakrabarty representation is superior to the Almansi representation for a ring-shaped domain, even though it is inferior for an infinite exterior domain.

Chapter 6

Dislocation Fields

6.1 Introduction

The biharmonic equation (1-1-1) plays a fundamental role in two-dimensional linear elasticity, i.e. the stretching and bending of plates. In connection with stretching, χ is known as the Airy stress function since it was first introduced by Airy (1863). As already noted, there exists a general class of solutions for χ. It is well established (Coker and Filon, 1957) that the displacement components u, v corresponding to a given χ are provided by

$$2\mu u = -\frac{\partial\chi}{\partial x} + (1-\nu)H, \tag{6-1-1}$$

$$2\mu v = -\frac{\partial\chi}{\partial y} + (1-\nu)H, \tag{6-1-2}$$

where μ,ν denote the shear modulus and Poissions's ratio respectively, and H, H are conjugate harmonic functions satisfying the relations

$$\frac{\partial H}{\partial x} = \frac{\partial \overline{H}}{\partial y} = \nabla^2\chi. \tag{6-1-3}$$

The most convenient representation for χ in this context is

$$\chi = x\phi + \psi; \quad \nabla^2\phi = \nabla^2\psi = 0. \tag{6-1-4}$$

This is because (6-1-4) implies

$$\nabla^2\chi = 2\frac{\partial\phi}{\partial x}, \tag{6-1-5}$$

which, in turn, implies

$$H = 2\phi + Ay + B, \tag{6-1-6}$$

where A,B are arbitrary constants. Correspondingly

$$\overline{H} = 2\overline{\phi} - Ax + C, \tag{6-1-7}$$

where $\overline{H}$, $\overline{\phi}$ are the harmonic conjugates of H, ϕ respectively and C is a further arbitrary constant. Substituting from (6-1-5), (6-1-6) into (6-1-1), (6-1-2) we find

$$2\mu u = -\frac{\partial\chi}{\partial x} + 2(1 - \nu)\phi + ay + b, \tag{6-1-8}$$

$$2\mu v = -\frac{\partial\chi}{\partial y} + 2(1 - \nu)\overline{\phi} - ax + c, \tag{6-1-9}$$

where $a = A(1 - \nu)$, $b = B(1 - \nu)$, $c = C(1 - \nu)$.

The contributions ay, $-ax$ in (6-1-8), (6-1-9) provide the components of rigid-body rotation and b, c provide the components of rigid-body translation. Later on, it will be proved that we can put $a = b = c = 0$ without loss of generality. Accordingly, the field equations (6-1-8), (6-1-9) are henceforth written in the forms

$$2\mu u = -\frac{\partial\chi}{\partial x} + 2(1-\nu)\phi \quad , \tag{6-1-10}$$

$$2\mu v = -\frac{\partial\chi}{\partial y} + 2(1-\nu)\overline{\phi} \quad . \tag{6-1-11}$$

There exist singular displacement fields corresponding to the singular stress functions $r^2\log r$, $x\log r$, $y\log r$. These describe Volterra dislocations (Love, 1952). Also there exist multi-valued solutions $x\theta$, $y\theta$, θ which essentially describe concentrated forces. These singular displacement fields were constructed by Love without utilising χ. We shall recover his results by more efficient methods based on χ.

6.2 Volterra dislocations

A single-valued χ implies that $\partial\chi/\partial x$, $\partial\chi/\partial y$ are single-valued. However, it does not necessarily imply that ϕ or ψ are single-valued. For instance, it has been noted in (3-1-5) that

$$\chi = y\log r = x(-\theta) + (y\log r + x\theta), \tag{6-2-1}$$

where θ, $y\log r + x\theta$ are multi-valued harmonic functions for any circuit around the origin. By reference to (6-1-8) and (6-1-9), we find that the corresponding displacement components are provided by

$$2\mu u = -\frac{\partial}{\partial x}(y\log r) + 2(1-\nu)\theta, \tag{6-2-2}$$

$$2\mu v = -\frac{\partial}{\partial x}(y\log r) - 2(1-\nu)\log r, \tag{6-2-3}$$

since $\log r$, θ are harmonic conjugates. Therefore, the component u jumps by an amount $2(1-\nu)\pi/\mu$ for any anti-clockwise circuit around the origin, while the component v remains unaltered. Similarly, we note that the biharmonic function $\chi = x\log r$ produces the displacement components

$$2\mu u = -\frac{\partial}{\partial x}(x\log r) + 2(1-\nu)\log r, \tag{6-2-4}$$

$$2\mu v = -\frac{\partial}{\partial y}(x\log r) + 2(1-\nu)\theta. \tag{6-2-5}$$

In this case the component u remains unchanged, whereas v jumps by an amount $2(1-\nu)\pi/\mu$ for any circuit around the origin.

The local rotation ω associated with the displacement components u,v is defined by

$$\omega = \tfrac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right). \qquad (6\text{-}2\text{-}6)$$

If so, for a given χ, we find from (6-1-10), (6-1-11) that

$$2\mu\omega = -2(1 - \nu) \frac{\partial \phi}{\partial y} . \qquad (6\text{-}2\text{-}7)$$

A multi-valued ω is provided by the single-valued biharmonic function $\chi = r^2 \log r$. Thus we note

$$\chi = \tfrac{1}{2} r^2 \log r = x(x \log r - y\theta) + \left\{ \tfrac{1}{2}(y^2 - x^2) \log r + xy\theta \right\}, \qquad (6\text{-}2\text{-}8)$$

where the expressions

$$x \log r - y\theta, \quad \tfrac{1}{2}(y^2 - x^2) \log r + xy\theta \qquad (6\text{-}2\text{-}9)$$

are multi-valued harmonic functions, the latter being the real part of the analytic function $-(\tfrac{1}{2}z^2 + zz)\log z$. Inserting $\phi = (x \log r - y\theta)$ into (6-2-7), we find

$$2\mu\omega = -2(1 - \nu) \left\{ \frac{\partial}{\partial y} (x \log r) - y \frac{\partial \theta}{\partial y} - \theta \right\} . \qquad (6\text{-}2\text{-}10)$$

It is to be noted that ω jumps by an amount $2(1 - \nu)\pi/\mu$ for any anti-clockwise circuit around the origin. This dislocation is termed a radial fissure by Love.

6.3 Point forces

We now consider the multi-valued biharmonic functions $x\theta$, $y\theta$, θ. The function $\chi = x\theta$ provides the jumps

$$\left[\frac{\partial\chi}{\partial x}\right] = \left[\theta + x\frac{\partial\theta}{\partial x}\right] = \left[\theta\right] = 2\pi,$$

$$\left[\frac{\partial\chi}{\partial y}\right] = 0, \quad \left[\chi - x\frac{\partial\chi}{\partial x} - y\frac{\partial\chi}{\partial y}\right] = 0, \qquad (6\text{-}3\text{-}1)$$

on circuiting the origin in an anti-clockwise sense. Similarly, the function $\chi = y\theta$ provides the jumps

$$\left[\frac{\partial\chi}{\partial x}\right] = 0, \quad \left[\frac{\partial\chi}{\partial y}\right] = 2\pi \quad \left[\chi - x\frac{\partial\chi}{\partial x} - y\frac{\partial\chi}{\partial y}\right] = 0. \qquad (6\text{-}3\text{-}2)$$

Also, the function $\chi = \theta$ provides the jumps

$$\left[\frac{\partial\chi}{\partial x}\right] = 0, \quad \left[\frac{\partial\chi}{\partial y}\right] = 0, \quad \left[\chi - x\frac{\partial\chi}{\partial x} - y\frac{\partial\chi}{\partial y}\right] = 2\pi\ . \qquad (6\text{-}3\text{-}3)$$

We shall prove later that $[{}^{\partial\chi}/\partial x]$ gives the magnitude of a concentrated force acting at the origin in the y-direction. Similarly $[{}^{\partial\chi}/\partial y]$ gives the magnitude of a concentrated force acting at the origin in the x-direction. Also $[\chi - x\frac{\partial\chi}{\partial x} - y\frac{\partial\chi}{\partial y}]$ gives the magnitude of a concentrated couple acting at the origin with it axis parallel to the x,y - plane.

These particular stress functions include dislocations. To eliminate any unwanted dislocations inherent in $\chi = x\theta$, we introduce the function

$$\chi_1 = x\theta + My\log r, \tag{6-3-4}$$

$$= x(1 - M)\theta + M(x\theta + y\log r), \tag{6-3-5}$$

where M is a constant to be determined by the condition

$$2\mu[u_1] = -\left[\frac{\partial\chi_1}{\partial x}\right] + 2(1 - \nu)\ [\phi] = 0. \tag{6-3-6}$$

Bearing in mind from (6-3-4) that $[\frac{\partial\chi_1}{\partial x}] = [\theta]$ and from (6-3-5) that $[\phi] = [(1 - M)\theta]$, we find

$$-1 + 2(1 - \nu)\ (1 - M) = 0,$$

whence

$$M = (1 - 2\nu)/2(1 - \nu). \tag{6-3-7}$$

Similarly, to eliminate unwanted dislocations inherent in $\chi = y\theta$, we introduce the function

$$\chi_2 = y\theta - Nx\log r, \tag{6-3-8}$$

$$= y(1-N)\theta - N(x\log r - y\theta), \tag{6-3-9}$$

where N is a constant to be determined by the condition

$$2\mu[v_2] = - \left[\frac{\partial \chi_2}{\partial y}\right] + 2(1-\nu)\ [\phi] = 0. \tag{6-3-10}$$

Bearing in mind (6-3-8) and (6-3-9), we find

$$-1 + 2(1-\nu)\ (1-N) = 0,$$

whence

$$N = (1-2\nu)/2(1-\nu). \tag{6-3-11}$$

Finally, the components corresponding to

$$\chi_3 = \theta = x\,0 + \theta \tag{6-3-12}$$

are given by

$$2\mu u_3 = \frac{-\partial\theta}{\partial x},\quad 2\mu v_3 = \frac{-\partial\theta}{\partial y} \tag{6-3-13}$$

which are seen to be single-valued.

The local rotation ω associated with (6-3-4) or (6-3-8) is seen to be single-valued, and in (6-3-12) it is seen to be zero.

Within a ring-shaped domain B, a single-valued χ may not be sufficient because of the possibility of non-equilibtrated tractions acting on each boundary. Accordingly, we write

$$\chi = \chi_s + a\,(x\theta + My\log r) + b(y\theta - Ny\log r) + c\theta\ , \qquad (6\text{-}3\text{-}14)$$

where χ_s is a single-valued biharmonic function (possibly containing dislocations), expressed by either the Almansi or Chakrabarty forms, and a,b,c are constants chosen in accordance with the magnitudes of concentrated force and couple components.

Chapter 7

Formulation of Displacement Boundary-value Problems

7.1 Elimination of rigid-body translation

In order to make progress with the field equations (6-1-8) and (6-1-9), we write them in the expanded form

$$
\begin{aligned}
2\mu u &= -x\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x} + (1-2\nu)\phi + ay + b, \\
2\mu v &= -x\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial y} + 2(1-\nu)\overline{\phi} - ax + c.
\end{aligned}
\qquad (7\text{-}1\text{-}1)
$$

Also, we introduce the representations

$$\phi(\underset{\sim}{p}) = \int_{\partial B} \log|\underset{\sim}{p} - \underset{\sim}{q}|\sigma(\underset{\sim}{q})d\underset{\sim}{q}; \quad \underset{\sim}{p} \in B + \partial B, \qquad (7\text{-}1\text{-}2)$$

$$\frac{\partial\psi(\underset{\sim}{p})}{\partial x} = \int_{\partial B} \log|\underset{\sim}{p} - \underset{\sim}{q}|\eta(\underset{\sim}{q})d\underset{\sim}{q}; \quad \underset{\sim}{p} \in B + \partial B, \qquad (7\text{-}1\text{-}3)$$

where ∂B is any closed boundary within the domain of existence of the harmonic functions concerned, and where σ, η are two hypothetical source densities which are known in principle in terms of ϕ, $\frac{\partial\psi}{\partial x}$ on ∂B. If so, since $\overline{\phi}$, $-\partial\psi/\partial y$ are the harmonic conjugates of ϕ, $\partial\psi/\partial x$ respectively, we may write

$$\bar{\phi}(\underset{\sim}{p}) = \int_{\partial B} \theta(\underset{\sim}{p} - \underset{\sim}{q})\sigma(\underset{\sim}{q})dq; \quad \underset{\sim}{p} \in B + \partial B, \tag{7-1-4}$$

$$-\frac{\partial\psi(\underset{\sim}{p})}{\partial y} = \int_{\partial B} \theta(\underset{\sim}{p} - \underset{\sim}{q})\eta(\underset{\sim}{q})dq; \quad \underset{\sim}{p} \in B + \partial B, \tag{7-1-5}$$

where the kernel function $\theta(\underset{\sim}{p} - \underset{\sim}{q})$ is the harmonic conjugate of $\log|\underset{\sim}{p} - \underset{\sim}{q}|$. Details of this are discussed in the Appendix [2]. These representatives unambiguously define $\bar{\phi}$, $-\partial\psi/\partial y$ corresponding to ϕ, $\partial\psi/\partial x$. For instance, if $\phi = 1$ in $B + \partial B$, then σ satisfies the equation

$$\int_{\partial B} \log|\underset{\sim}{p} - \underset{\sim}{q}|\sigma(\underset{\sim}{q})dq = 1; \quad \underset{\sim}{p} \in \partial B. \tag{7-1-6}$$

It follows that σ generates a unit potential in $B + \partial B$, and it also generates the conjugate potential

$$\int_{\partial B} \theta(\underset{\sim}{p} - \underset{\sim}{q})\sigma(\underset{\sim}{q})dq = k ; \quad \underset{\sim}{p} \in B + \partial B, \tag{7-1-7}$$

where k is a constant specific to ∂B.

We may use the representations (7-1-2) - (7-1-5) to eliminate the constants b,c in (7-1-1). It is necessary to show that we may determine ϕ, $\overline{\phi}$, $\frac{\partial\psi}{\partial x}$, $-\frac{\partial\psi}{\partial y}$ which satisfy the equations

$$\left.\begin{aligned} -b &= -x\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x} + (1 - 2\nu)\phi, \\ -c &= -x\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial y} + 2(1 - \nu)\overline{\phi}, \end{aligned}\right\} \tag{7-1-8}$$

for an arbitrary choice of b,c. We try the solutions

$$\phi = m, \quad \frac{\partial\psi}{\partial x} = n \tag{7-1-9}$$

where m,n are parameters to be determined, in which case

$$\overline{\phi} = km, \quad -\frac{\partial\psi}{\partial y} = kn. \tag{7-1-10}$$

Substituting (7-1-9) and (7-1-10) into (7-1-8), we find

$$m = \frac{1}{k}(bk - c), \quad n = 2b(1 - \nu) - \frac{C}{k}(1 - 2\nu)$$

thereby providing the required harmonic functions

$$\left.\begin{aligned} \phi &= \frac{1}{k}(bk-c), \quad \frac{\partial\psi}{\partial x} = 2b(1 - \nu) - \frac{C}{k}(1 - 2\nu), \\ \overline{\phi} &= (bk - c), \quad \frac{-\partial\psi}{\partial y} = 2kb(1 - \nu) - c(1 - 2\nu). \end{aligned}\right\} \tag{7-1-11}$$

7.2 Elimination of rigid-body rotation

It is also possible to eliminate the rigid-body rotation (ay, -ax), i.e. it is possible to choose $\phi, \bar{\phi}, \frac{\partial\psi}{\partial x}, \frac{-\partial\psi}{\partial y}$ which satisfy the equations

$$\left.\begin{aligned} -ay &= -x\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x} + (1 - 2\nu)\phi, \\ ax &= -x\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial y} + 2(1 - \nu)\bar{\phi}, \end{aligned}\right\} \tag{7-2-1}$$

for an arbitrary choice of a. First we note that, if σ is chosen so that

$$\int_{\partial B} \log|\underset{\sim}{p} - \underset{\sim}{q}|\ \sigma(\underset{\sim}{q})dq = y; \quad \underset{\sim}{p} \in B + \partial B,$$

then

$$\int_{\partial} \theta(\underset{\sim}{p} - \underset{\sim}{q})\eta(\underset{\sim}{q})dq = -x + \alpha; \quad \underset{\sim}{p} \in B + \partial B, \tag{7-2-2}$$

where α is a constant specific to ∂B. We try the solutions

$$\left.\begin{aligned} \phi &= my & \frac{\partial\psi}{\partial x} &= ty, \\ \bar{\phi} &= m(-x+\alpha) & -\frac{\partial\psi}{\partial y} &= t(-x + \alpha), \end{aligned}\right\} \tag{7-2-3}$$

where m,t are parameters to be determined so substituting (7-2-3) into (7-2-1) yields

$$-ay = [(1 - 2\nu)m - t]\, y,$$
$$ax = -[(3 - 2\nu)m + t]\, x + [2(1 - \nu)m + t]\, \alpha. \qquad (7\text{-}2\text{-}4)$$

It is clear from (7-2-4) that m,t must satisfy the three equations

$$\begin{aligned}(1 - 2\nu)\, m - t + a &= 0,\\ (3 - 2\nu)\, m + t + a &= 0,\\ 2(1 - \nu)\, m + t \quad &= 0.\end{aligned} \qquad (7\text{-}2\text{-}5)$$

since $\alpha \neq 0$ generally. But the system (7-2-5) is not consistent, as the first two equations provide

$$m = -a/2(1 - \nu), \quad t = a/2(1 - \nu) \qquad (7\text{-}2\text{-}6)$$

which do not satisfy the third equation. This suggests that we should try

$$\begin{aligned}\phi &= my + n, \quad \frac{\partial\psi}{\partial x} = ty + \ell\\ \overline{\phi} &= m(-x + \alpha) + km, \quad \frac{-\partial\psi}{\partial y} = t(-x + \alpha) + k\ell\end{aligned} \qquad (7\text{-}2\text{-}7)$$

where m,n,t,ℓ are parameters to be determined. Substituting (7-2-7) into (7-2-1), we find

$$\left.\begin{aligned}-ay &= [(1 - 2\nu)m\ t]\ y + [(1 - 2\nu)n + \ell]\\ ax &= -[(3 - 2\nu)m + t]\ x + [2(1 - \nu)m + t]\alpha + [2(1 - \nu)n + \ell]\, k\, .\end{aligned}\right\} \qquad (7\text{-}2\text{-}8)$$

It is clear from (7-2-8) that

$$\left.\begin{aligned} (1 - 2\nu)m - t &= -a, \\ (3 - 2\nu)m + t &= -a, \end{aligned}\right\} \qquad (7\text{-}2\text{-}9)$$

and that

$$\left.\begin{aligned} (1 - 2\nu)n + \ell &= 0, \\ [2(1 - \nu)m + t]\alpha + [2(1 - \nu)\, n + \ell]\, k &= 0. \end{aligned}\right\} \qquad (7\text{-}2\text{-}10)$$

Equations (7-2-9) provide the values of m,t as defined by (7-2-6). Substituting these values in (7-2-10), we obtain

$$n = a(1 - 2\nu)/2k(1 - \nu), \; \ell = -a(1 - 2\nu)^2/2k(1 - \nu). \qquad (7\text{-}2\text{-}11)$$

Thus the required harmonic functions are

$$\begin{aligned} \phi &= \frac{a}{2(1 - \nu)} [-y + \frac{a}{k} (1 - 2\nu)] \; , \; \frac{\partial\psi}{\partial x} = \frac{a}{2(1 - \nu)}[y - \frac{a}{k}(1-2\nu)^2] \\ \phi &= \frac{a}{2(1 - \nu)} [x-\alpha + (1 - 2\nu)], \; - \frac{\partial\psi}{\partial y} = \frac{a}{2(1 - \nu)}[-x+\alpha - (1-2\nu)^3]. \end{aligned} \qquad (7\text{-}2\text{-}1$$

It is interesting to examine the possible trial solutions

$$\left.\begin{aligned} \phi &= my, & \frac{\partial\psi}{\partial x} &= tx \\ \phi &= m(-x + \alpha) & - \frac{\partial\psi}{\partial y} &= t(y + \beta) \end{aligned}\right\} \qquad (7\text{-}2\text{-}13)$$

where m,t are parameters to be determined, and β has a similar significance to α. Substituting (7-2-13) into (7-2-1) yields the equations

$$\left.\begin{aligned} -ay &= (1 - 2\nu)my - tx \\ ax &= -(3 - 2\nu)\, mx + ty + 2(1 - \nu)n\alpha + t\beta. \end{aligned}\right\} \qquad (7\text{-}2\text{-}14)$$

Clearly t=0 from the first equation. Substituting t=0 into the second equation, we obtain m=0, as $\alpha,\beta \neq 0$, in general. Accordingly, this possible trial solution is not consistent.

To summarise, we have shown that the components of a rigid-body translation and a rigid-body rotation may be absorbed into $\phi, \bar{\phi}$, $\partial\psi/\partial x$, $-\partial\psi/\partial y$ so justifying the expanded formulations

$$\begin{aligned} 2\mu u &= -x\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x} + (1 - 2\nu)\ \phi, \\ 2\mu v &= -x\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial y} + 2(1 - \nu)\ \bar{\phi}, \end{aligned} \qquad (7\text{-}2\text{-}15)$$

in places of (6-1-8) and (6-1-9).

7.3 Boundary formulation

The field equations (7-2-15) yield displacement components u,v corresponding to a given χ in B + ∂B, where $\chi = x\phi + \psi$. If u,v are given everywhere on ∂B, then they are in principle known everywhere in B (the second fundamental existence theorem of elastostatics). An interesting method of constructing u,v in B is to introduce the boundary equations

$$\left.\begin{aligned} -x\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x} + (1 - 2\nu)\phi &= 2\mu u, \\ -x\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial y} + 2(1 - \nu)\overline{\phi} &= 2\mu v, \end{aligned}\right\} \qquad (7\text{-}3\text{-}1)$$

which constitute a pair of functional relations for the quantities $\phi, \overline{\phi}$, $\partial\psi/\partial x$, $-\partial\psi/\partial y$ on ∂B. To make progress towards a solution of these equations, we introduce the representations (7-1-2) - (7-1-5) for ϕ, $\frac{\partial\psi}{\partial x}$, $\overline{\phi}$, $-\partial\psi/\partial y$ respectively. Substituting these representations into (7-3-1), we obtain two coupled boundary integral equations for σ,η. It is intuitively clear that solutions exist. Also, if they exist, they are unique (see section 7.4). The source densities generate $\phi, \overline{\phi}$, $\frac{\partial\phi}{\partial x}$, $\partial\phi/\partial y$, $\partial\phi/\partial x$, $\partial\psi/\partial y$ in B + ∂B, whence $2\mu u$, $2\mu v$ may be generated throughout B.

If ∂B is the internal boundary of an infinite exterior domain, or one boundary of a ring-shaped domain, we replace the right-hand side of equation (7-3-1) by

$$2\mu(u - \alpha u_1 - \beta u_2), \quad 2\mu\,(v - \alpha v_1 - \beta v_2) \qquad (7\text{-}3\text{-}2)$$

to allow for the possibility of non-equilibrated tractions acting on B. The two unknown coefficients α, β are compensated by the two side conditions

$$\int_{\partial B} \sigma(\underset{\sim}{q})dq = 0, \qquad \int_{\partial B} \eta(\underset{\sim}{q})dq = 0 \tag{7-3-3}$$

which ensure that

$$\left.\begin{aligned} &\phi, \frac{\partial\psi}{\partial x} = O(r^{-1}) \quad \text{as} \quad r \to \infty, \\ &\overline{\phi}, \frac{\partial\psi}{\partial y} = O(r^{-1}) \neq O(\theta) \text{ as } r \to \infty, \end{aligned}\right\} \tag{7-3-4}$$

thus eliminating the possibility of unwanted dislocations.

7.4 Uniqueness of solutions

The homogeneous equations

$$\left.\begin{aligned} &-x\frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial x} + (1 - 2\nu)\phi = 0, \\ &-x\frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial y} + 2(1 - \nu)\overline{\phi} = 0, \end{aligned}\right\} \tag{7-4-1}$$

corresponding to (7-3-1), have only the trivial solutions

$$\left.\begin{aligned} &\phi = 0, \quad \frac{\partial\psi}{\partial x} = 0, \\ &\overline{\phi} = 0, \quad \frac{\partial\psi}{\partial y} = 0. \end{aligned}\right\} \tag{7-4-2}$$

It would seem very difficult to prove this directly. However, assuming that there exist non-trivial solutions which are constants, we see from (7-1-11) that these constants must be zero. Also, if we assume non-trivial solutions of the form

$$\left.\begin{aligned} \phi &= \ell x + my, & \frac{\partial\psi}{\partial x} &= nx + ty, \\ \overline{\phi} &= (y+\beta)+m(-x+\alpha), & -\frac{\partial\psi}{\partial y} &= n(y+\beta)+t(-x+\alpha), \end{aligned}\right\} \qquad (7\text{-}4\text{-}3)$$

where ℓ,m,n,t are parameters to be determined and α,β have the same significance as in (7-2-2), we see that $\ell = m = n = \ell = 0$ by reference to (7-2-12), (7-2-14).

A general physical argument is that any non-trivial solution would imply a displacement field inside ∂B in the absence of any displacements on ∂B, which contradicts the second fundamental existence theorem of elastostatics.

PART III

FURTHER BOUNDARY-VALUE PROBLEMS

Chapter 8

Displacement field generated by a Point-force

8.1 Fundamental displacement dyadic

The most important example of a body-force field is a concentrated force acting at a specific point of the medium in a specific direction. A body-force of magnitude 4π acting in the η -direction at q may be formally represented by the field vector with components

$$f_\alpha(\underset{\sim}{p}) = 4\pi\delta(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\eta) \quad ; \left.\begin{array}{l}\alpha = 1,2,3 \\ \eta = 1 \text{ or } 2 \text{ or } 3\end{array}\right\} . \tag{8-1-1}$$

where $\delta(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\eta)$ is a Dirac delta function sufficiently defined by the properties

$$\begin{aligned} &\delta(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\eta) = 0 \text{ if } \underset{\sim}{p} = \underset{\sim}{q} \text{ or } \alpha = \eta \\ &\delta(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\eta) \to \infty \text{ as } \underset{\sim}{p} \to \underset{\sim}{q} \text{ , } \alpha = \eta \\ &\left.\begin{array}{l}\displaystyle\int_B \delta(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\eta)\, dP = 1 \quad \text{if } \alpha = \eta \\ \qquad\qquad\qquad\quad = 0 \quad \text{if } \alpha \neq \eta\end{array}\right\} \quad \underset{\sim}{q} \in B. \end{aligned} \tag{8-1-2}$$

This generates a displacement vector with components

$$\phi_\alpha(\underset{\sim}{p}) = g(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\eta) \quad ;\alpha = 1,2,3 \tag{8-1-3}$$

which becomes singular in the component $\alpha=\eta$ when the field point $\underset{\sim}{p}$ coincides with the source point $\underset{\sim}{q}$.

So far, η has been held at some fixed value defining the direction of the concentrated force at $\underset{\sim}{q}$. As η runs through 1,2,3 we obtain three such forces at $\underset{\sim}{q}$, each generating a singular vector of the type (8-1-3). These vectors may be displayed as columns of the array

$$\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q}) = \begin{bmatrix} g(\underset{\sim}{p}_1,\underset{\sim}{q}_1) & g(\underset{\sim}{p}_1,\underset{\sim}{q}_2) & g(\underset{\sim}{p}_1,\underset{\sim}{q}_3) \\ g(\underset{\sim}{p}_2,\underset{\sim}{q}_1) & g(\underset{\sim}{p}_2,\underset{\sim}{q}_2) & g(\underset{\sim}{p}_2,\underset{\sim}{q}_3) \\ g(\underset{\sim}{p}_3,\underset{\sim}{q}_1) & g(\underset{\sim}{p}_3,\underset{\sim}{q}_2) & g(\underset{\sim}{p}_3,\underset{\sim}{q}_3) \end{bmatrix} \qquad (8\text{-}1\text{-}4)$$

which constitutes the fundamental displacement dyadic of the medium. Our symbolism indicates that $\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q})$ plays an analogous role to the unit simple source potential $g(\underset{\sim}{p},\underset{\sim}{q})$ (Chapter 3). In line with $g(\underset{\sim}{p},\underset{\sim}{q}) = g(\underset{\sim}{q},\underset{\sim}{p})$, we may prove that

$$g(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\eta) = g(\underset{\sim}{q}_\eta,\underset{\sim}{p}_\alpha) , \qquad (8\text{-}1\text{-}5)$$

i.e. $g(\underset{\sim}{p}_\alpha,\underset{\sim}{q}_\eta)$ may be interpreted as the displacement in the -direction at $\underset{\sim}{q}$ generated by a concentrated force acting in the α-direction at $\underset{\sim}{p}$ (Jaswon and Symm, 1976). An immediate corollary is that each row of $\underset{\approx}{g}(\underset{\sim}{p},\underset{\sim}{q})$ defines a singular displacement vector generated at $\underset{\sim}{p}$, i.e. the vector

$$\phi_\eta(\underset{\sim}{q}) = g(\underset{\sim}{p}_\alpha, \underset{\sim}{q}_\eta) \; ; \; \eta = 1,2,3. \qquad (8\text{-}1\text{-}6)$$

8.2 Kelvin solution

In an isotropic linear elastic medium, the Cauchy-Navier equation governing the displacement vector $\underset{\sim}{\phi}$ in a source-free domain is

$$\mu\nabla^2\underset{\sim}{\phi} + (\lambda + \mu)\,\nabla(\nabla.\underset{\sim}{\phi}) = 0, \qquad (8\text{-}2\text{-}1)$$

where $\nabla.\phi$ is the dilatation associated with $\underset{\sim}{\phi}$ and x,μ are Lame's elastic constants **. This equation is preferably written as

$$\nabla^2\phi + \frac{1}{1-2\nu}\,\nabla(\nabla.\phi) = 0; \; \nu = \frac{\lambda}{2(\lambda+\mu)}, \qquad (8\text{-}2\text{-}2)$$

where ν $(0 < \nu < \frac{1}{2})$ is Poission's ratio. A general solution to equation (8-2-2) has been given by Papkovich (1932) and Neuber (1931) in the form

$$\underset{\sim}{\phi} = \underset{\sim}{h} - \kappa\nabla(\underset{\sim}{r}.\underset{\sim}{h} + f): \; \nabla^2\underset{\sim}{h} = 0, \; \nabla^2 f = 0, \;\; \kappa^{-1} = 4(1-\nu) \qquad (8\text{-}2\text{-}3)$$

where $\underset{\sim}{h}$ is any arbitrary harmonic vector and f is a harmonic scalar function. Very often it is possible to

---

** μ is the shear modulus and χ has no physical significance except through bulk modulus $\kappa = \frac{1}{3}(3\lambda + 2\mu)$ and Young's modulus $E = \mu(3\lambda + 2\mu)/(\lambda+\mu)$.

dispense with the function f (in 8-2-3) [Eubanks and Sternberg, 1956]. Alternatively expressed, it is generally possible to write

$$- \kappa \nabla f = \underset{\sim}{h} - \kappa \nabla(\underset{\sim}{r}.\underset{\sim}{h}) ; \; \nabla^2 \underset{\sim}{h} = 0, \; \nabla^2 f = 0 \qquad (8\text{-}2\text{-}4)$$

where $\underset{\sim}{h}$ can be determined in terms of f [Jaswon and Symm, 1976].

The vector $\underset{\sim}{h} = < \frac{1}{|p|}$, 0, 0 > is harmonic everywhere except at $\underset{\sim}{p} = 0$, and it therefore defines a displacement vector

$$\underset{\sim}{\phi}(\underset{\sim}{p}) = < \frac{1}{|\underset{\sim}{p}|},0,0,> \quad - \kappa \; \nabla(\frac{p_1}{|\underset{\sim}{p}|}) ; \; \underset{\sim}{p} = <p_1,p_2,p_3> \quad (8\text{-}2\text{-}5)$$

which becomes singular at $\underset{\sim}{p} = 0$. More generally, we may define the displacement vector

$$\underset{\sim}{\phi}(\underset{\sim}{p}) = < \frac{1}{R} ,0,0> - \kappa\nabla(\frac{p_1-q_1}{R}) ; \; R = |\underset{\sim}{p}-\underset{\sim}{q}| \qquad (8\text{-}2\text{-}6)$$

which becomes singular when $\underset{\sim}{p} = \underset{\sim}{q}$. This is Kelvin's solution for the displacement vector generated by a concentrated force acting in the 1-direction at $\underset{\sim}{q}$. It may be proved that the magnitude of the concentrated force in question is $4\pi\mu$. Analogous expressions hold for concentrated forces acting in the 2- and 3-directions at $\underset{\sim}{q}$. Separating into components and using the symbolism of (8-1-4), we arrive at the isotropic fundamental displacement dyadic with components

$$g(\underset{\sim}{p}_i \underset{\sim}{q}_j) = \frac{(1-\kappa)}{R}\,\delta_{ij} + \frac{\kappa}{\mu R}\,\frac{\partial R}{\partial p_i}\,\frac{\partial R}{p_j} \;;\quad i\ j = 1,2,3. \tag{8-2-7}$$

This expression defines the displacement components in the i-direction at $\underset{\sim}{p}$ generated by a concentrated force of magnitude 4π acting in the j-direction at $\underset{\sim}{q}$.

It will be seen from (8-2-7) that

$$g(\underset{\sim}{p}_i, \underset{\sim}{q}_j) = g(\underset{\sim}{p}_j, \underset{\sim}{q}_i) = g(\underset{\sim}{q}_j, \underset{\sim}{p}_i), \tag{8-2-8}$$

which verifies (8-1-5) for an isotropic continuum.

The two-dimensional expression corresponding to (8-2-7) is

$$g(\underset{\sim}{p}_i, \underset{\sim}{q}_j) = \frac{-(3-4\nu)}{4(1-\nu)\mu}\,\delta_{ij}\,\log R + \frac{1}{4(1-\nu)\mu}\,\frac{\partial R}{\partial p_i}\,\frac{\partial R}{\partial p_j} \;;\ i,j = 1,2. \tag{8-2-9}$$

This refers to a concentrated force of strength 2π per unit length of an infinite cylinder of isotropic elastic material.

8.3 Displacement generated by a point-force without dislocations

It has already been noted that the biharmonic function $\chi_1 = x\theta + M\, y \log r$, defined by (6-3-4), represents a point-force without dislocations for a suitable choice of M. This point force is of magnitude 2π acting in the y-direction at the origin. The displacement components u_1, v_1 associated with χ_1 are given by

$$2\mu u_1 = 2(1 - \nu)\phi - \frac{\partial \chi_1}{\partial x} , \qquad (8\text{-}3\text{-}1)$$

$$2\mu v_1 = 2(1 - \nu)\bar{\phi} - \frac{\partial \chi_1}{\partial y} , \qquad (8\text{-}3\text{-}2)$$

where $\phi = (1 - M)\theta, \bar{\phi} = -(1 - M) \log r$. These expressions reduce to

$$u_1 = \frac{xy}{4(1 - \nu)\, \mu r^2} , \qquad (8\text{-}3\text{-}3)$$

$$v_1 = \frac{-(3 - 4\nu)}{4(1 - \nu)\mu} \log r + \frac{y^2}{4(1 - \nu)\mu r^2} - \frac{1}{2\mu} , \qquad (8\text{-}3\text{-}4)$$

where the constant $^{-1}/2\mu$ in (8-3-4) may be eliminated by adding a suitable constant to ϕ. These agree with (8-2-7) if we identify

$$u_1 = g(p_1, q_2), \quad V_1 = g(p_2, q_2) \ ; \quad \underset{\sim}{q} = 0 \qquad (8\text{-}3\text{-}5)$$

Similarly the function $\chi_2 = y\theta - N\, x \log r$ yields the displacement components

$$u_2 = \frac{-(3 - 4\nu)}{4(1 - \nu)\mu} \log r + \frac{x}{4(1 - \nu)\mu r^2} - \frac{1}{2\mu} \tag{8-3-6}$$

$$v_2 = \frac{xy}{4(1 - \nu)\mu r^2} \tag{8-3-7}$$

By reference to the arbitrariness in ϕ, which is a feature of the Almansi x-form (1-1-2), we see that the constant in ϕ may be eliminated without loss of generality. As before, these agree with (8-2-7) if we identify

$$u_2 = g(p_1,q_1), \quad v_2 = g(p_2,q_1) \; ; \; \underset{\sim}{q} = 0. \tag{8-3-8}$$

Chapter 9

Bending of Thin Plates

9.1 Introduction

Hadamard (1908) proved that a function χ which is continuous and has continuous derivatives upto the fourth order within a finite two-dimensional domain B_i, and which satisfies the equation

$$\nabla^4\chi = 0$$

in B_i, is uniquely determined in B_i if it has a prescribed continuous set of values χ and set of normal derivative values χ' ($= d\chi/dn$) on ∂B. To construct χ in B_i, we utilise the two-dimensional representation (3-2-1), taking the origin $r = 0$ inside B_i. This yields the boundary functional relations

$$\chi = r^2\phi + \psi \qquad (9\text{-}1\text{-}1)$$

$$\chi' = (r^2\phi + \psi)' = 2rr'\phi + r^2\phi' + \psi' \qquad (9\text{-}1\text{-}2)$$

connecting ϕ, ϕ', ψ, ψ' on ∂B.

We now write, using the theory of part I,

$$\phi(\underset{\sim}{p}) = \int_{\partial B} \log|\underset{\sim}{p} - \underset{\sim}{q}| \; \sigma(\underset{\sim}{q})dq; \quad \underset{\sim}{p} \; \varepsilon \; B_i + \partial B \tag{9-1-3}$$

$$\psi(\underset{\sim}{p}) = \int_{\partial B} \log|\underset{\sim}{p} - \underset{\sim}{q}|\eta(\underset{\sim}{q})dq; \quad \underset{\sim}{p} \; \varepsilon \; B_i + \partial B \tag{9-1-4}$$

where σ,η are hypothetical source densities to be determined. We bear in mind the normal derivative expressions

$$\phi'(p) = \int_{\partial B} \log'|p - q|\sigma(q)dq + \pi\sigma(p); \quad p \; \varepsilon \; \partial B, \tag{9-1-5}$$

$$\psi'(p) = \int_{\partial B} \log'|p - q|\eta(q)dq + \pi\eta(p); \quad p \; \varepsilon \; \partial B. \tag{9-1-6}$$

as given in Chapter 4. Substituting from (9-1-3);(9-1-6) into (9-1-1), (9-1-2) yields two coupled integral equations for σ,η which may be solved by established numerical methods (Symm, 1970), so enabling ϕ and ψ, and hence also χ, to be generated throughout B_i.

A simple illustrative example is provided by the following boundary data:

$$\left.\begin{aligned} \chi &= \alpha x + \beta y + \gamma \\ \chi' &= \alpha x' + \beta y' \end{aligned}\right\} \text{ on } \partial B, \tag{9-1-7}$$

where α,β,γ are constants. Clearly

$$\chi = \alpha x + \beta y + \gamma \text{ in } B_i + \partial B, \qquad (9\text{-}1\text{-}8)$$

with the breakdown

$$= r^2 \cdot 0 + (\alpha x + \beta y + \gamma) \text{ in } B_i + \partial B.$$

Another simple example which can be used as a test problem in numerical solutions is

$$\left.\begin{aligned} \chi &= r^2 \\ \chi' &= 2r \end{aligned}\right\} \text{ on } \partial B. \qquad (9\text{-}1\text{-}9)$$

Clearly

$$\chi = r^2 \quad \text{in } B_i + \partial B \qquad (9\text{-}1\text{-}10)$$

with the breakdown

$$\chi = r^2.1 + 0.$$

9.2 Chakrabarty boundary-value formulation

An alternative formulation of boundary-value problems is to utilise the Chakrabarty representation (5-3-5) for χ. The boundary properties of χ, χ' on this basis are discussed in an Appendix. However, this formulation does not appear so suitable as (9-1-1), (9-1-2) for finite simply-connected domains.

Another type of boundary-value problems for which Chakrabarty's representation may be preferable is χ, $\nabla^2\chi$ given on ∂B. Hadamard proved that a unique χ exists in B_i under these conditions. To construct this, we write from (3-2-2) that

$$\nabla^2\chi = 4\phi + 4\left(x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}\right); \quad \underset{\sim}{p} \in B_i + \partial B, \tag{9-2-1}$$

Substituting (9-1-3) for ϕ in (9-2-1), we find complicated boundary integral equations for σ. On the other hand, utilising Chakrabarty's representation we have

$$\nabla^2\chi = 4\int_{\partial B} \log|\underset{\sim}{p} - \underset{\sim}{q}|\sigma(\underset{\sim}{q})d\underset{\sim}{q}; \quad \underset{\sim}{p} \in B + \partial B. \tag{9-2-2}$$

This leads to the boundary integral equation for σ in terms of $\nabla^2\chi$. With σ known, we may then generate Ω on ∂B and we therefore determine χ on ∂B from (5-2-8).

9.3 Bending of thin plates

The small transverse deflection, ω, of a thin isotropic elastic plate under the action of a transverse load T per unit area, satisfies the equation

$$\nabla^4 \omega = {}^T/D \text{ in } B_i, \tag{9-3-1}$$

where B_i is the domain of the plate and D is the flexural rigidity given by $D = 2h\,E/3(1 - \nu^2)$. Here E is Young's modulus, ν is Poission's ratio, and 2h is the plate thickness. Equation (9-3-1) has a general solution of the form

$$\omega = W + \chi, \tag{9-3-2}$$

where W is a particular solution and χ satisifes equation (1-1-1). For a uniform transverse load, i.e. T = k(a constant), we have

$$W(\underset{\sim}{p}) = \frac{k}{64D}(x^2 + y^2)^2 \text{or } W(\underset{\sim}{p}) = \frac{k}{48D}(x^4 + y^4) \tag{9-3-3}$$

at $\underset{\sim}{p} = \langle x, y \rangle$.

For a concentrated transverse load applied at some point $\underset{\sim}{q} \in B_i$, i.e. expressed mathematically

$$T(\underset{\sim}{p}) = k\delta(\underset{\sim}{p} - \underset{\sim}{q}); \quad k = \text{total load}, \tag{9-3-4}$$

we have

$$W(\underset{\sim}{p}) = \frac{k}{8\pi D} |\underset{\sim}{p} - \underset{\sim}{q}|^2 \log |\underset{\sim}{p} - \underset{\sim}{q}| \tag{9-3-5}$$

bearing in mind from (5-3-2) that

$$\nabla^4 \left\{ |\underset{\sim}{p} - \underset{\sim}{q}|^2 \log |\underset{\sim}{p} - \underset{\sim}{q}| \right\} = 4\nabla^2 \log |\underset{\sim}{p} - \underset{\sim}{q}| = 8\pi\delta(\underset{\sim}{p}-\underset{\sim}{q}). \tag{9-3-6}$$

The boundary conditions for a clamped pate are

$$\omega = 0, \quad \omega' = 0 \text{ on } \partial B, \tag{9-3-7}$$

i.e. $W + \chi = 0, \quad W' + \chi' = 0$ on ∂B,

$$\text{or} \quad \chi = -W, \ \chi' = -W' \text{ on } \partial B. \tag{9-3-8}$$

Since W, W' may be computed on ∂B, it follows that χ, χ' are known on ∂B, so enabling χ to be determined throughout B_i by any of the methods of section 9.1, whence may be generated throughout B_i from (9-3-2). Utilising an obvious notation, the associated bending and twisting moments on ∂B appear, respectively, as

$$M_{nn} = -D \left(\nabla^2\omega + \frac{1-\nu}{\rho} \frac{\partial\omega}{\partial n}\right.$$

$$M_{nt} = -D (1-\nu) \frac{\partial^2 \omega}{\partial n \, \partial t} \tag{9-3-9}$$

where ρ is the radius of curvature. These expressions simplify to

$$M_{nn} = - D \nabla^2\omega, \quad M_{nt} = 0 \tag{9-3-10}$$

on bearing in mind that $\partial\omega/\partial n = \partial\omega/\partial t = 0$ under clamped conditions [Jaswon, Maiti and Symm, 1967].

The boundary conditions for a simply-supported plate are

$$\omega = 0, \quad M_{nn} = 0. \tag{9-3-11}$$

Substituting $\omega = W + \chi$ in (9-3-11), we formulate two relations between χ, χ' on ∂B which suffice to determine σ, η and hence eventually throught B_i. The interesting boundary quantities $\partial\omega/\partial n$, M_{nt} may then be calculated. It will be noticed that $\rho^{-1}\, \partial\omega/\partial n = 0$ along a straight line boundary, in which case conditions (9-3-11) reduces to

$$\omega = 0, \quad \nabla^2\omega = 0.$$

i.e.

$$\chi = - W, \; \nabla^2\chi = -\nabla^2 W. \tag{9-3-12}$$

This greatly simplifies the problem for a polygonal boundaries on the assumption that corner effects may be neglected, since we may immediately apply the theory of section 9.2.

Chapter 10

Two-dimensional Exterior and Ring-shaped Domains

10.1 Exterior domain

The uniqueness of χ for an infinite exterior domain B_e requires (Hadamard, 1908)

$$\chi = O(r) \quad \text{as} \quad r \to \infty. \tag{10-1-1}$$

in addition to the requirements mentioned in the previous chapter for B_i. Requirement (10-1-1) is ensured by the representation

$$\chi = r^2\phi + \psi; \; \nabla^2\phi = \nabla^2\phi = 0,$$

provided that $\phi = O(r^{-1})$ as $r\to\infty$ and that ψ is suitably restricted. We therefore utilise (9-1-3) for ϕ subject to the condition

$$\int \sigma(q)\,dq = 0, \tag{10-1-2}$$

which ensures, by reference to (7-3-4), $\phi = O(r^{-1})$ as $r\to\infty$. If so,

$$r^2\phi = ax + by + O(1) \tag{10-1-3}$$

where a,b are coefficients provided by (   ), (   ), and O(1) does not include a constant.

It is advantageous to utilise (9-1-6) for ψ since this

(a) yields the acceptable behaviour $\psi = \log r + O(r^{-1})$ as $r \to \infty$, and it

(b) excludes the terms $ax + by$ already covered by $r^2\phi$, so avoiding complications due to non-uniqueness.

It should be noted, however, that neither $r^2\phi$ nor ψ, as we have defined them, covers the possibility of a constant component entering into χ. Accordingly, we write

$$\chi = r^2\phi + \psi + k \quad \text{in } B_e + \partial B \tag{10-1-4}$$

where k is a constant to be determined. This extra unknown balances the side condition (10-1-2), but otherwise the formulation follows similar lines to that of the interior problem (Chapter 9).

A specialised exterior problem is provided by

$$\chi = \alpha x + \beta y + \gamma \quad \text{on } \partial B$$

$$' = \alpha x' + \beta y' \quad \text{on } \partial B.$$

Clearly

$$\chi = \alpha x + \beta y + \gamma \text{ in } B_e + \partial B,$$

though, of course, the normal direction now points into B_e.
In this case we cannot write

$$\chi = r^2 .0 + (\alpha x + \beta y + \gamma),$$

because the breakdown

$$\phi = 0, \quad \psi = \alpha x + \beta y + \gamma$$

would mean $\psi = O(1)$ as $r \to \infty$, contrary to the behaviour $\psi = k\log r + O(r^{-1})$ as $r \to \infty$. The correct breakdown is

$$\chi = r^2 \left(\frac{\alpha\cos\theta}{r} \right) + \left(\frac{\beta\sin\theta}{r} \right) + \gamma \text{ in } B_e + \partial B \qquad (10\text{-}1\text{-}5)$$

where

$$\phi = \frac{\alpha\cos\theta}{r} + \frac{\beta\sin\theta}{r}, \; \psi = 0, \quad k = \gamma .$$

10.2 Ring-shaped domain

Within a ring-shaped domain B, bounded externally by ∂B_0 and internally by ∂B_1 which encloses $r = 0$, χ may include the special biharmonic functions $x\log r$, $y\log r$. These are inadmissible within an interior domain (which includes $r = 0$), and they must also be excluded from an infinite exterior domain because of the requirement (10-1-1).

As already noted (3-2-8), the biharmonic function $x\log r$, $y\log r$ cannot be covered by $r^2\phi + \psi$ if ϕ,ψ are single-valued harmonic functions. Clearly the potential representations for ϕ,ψ are inherently single-valued. Accordingly, within a ring-shaped domain, we must write

$$\chi = r^2\phi + \psi + ax\log r + by\log r \qquad (10\text{-}2\text{-}1)$$

where ϕ,ψ are defined by (5-1-1), (5-1-2) respectively, and a,b are coefficients to be determined. No constant need be added to the expression (10-2-1), because ψ covers such a constant within a ring-shaped domain.

The unknown coefficients a,b in (10-2-1) may be balanced by the introduction of suitable side conditions. These must be chosen to eliminate the non-uniqueness possibilities (3-2-7), i.e. to ensure that ϕ does not contain any components proportional to $r^{-1}\cos\theta$, $r^{-1}\sin\theta$. This is achieved (cf. section 5 4) by the introduction of the side conditions

$$\int_{\partial B_1} q_1\sigma(\underset{\sim}{q})dq = 0, \quad \int_{\partial B_1} q_2\sigma(\underset{\sim}{q})dq = 0; \quad \underset{\sim}{q} = \langle q_1,q_2\rangle, \qquad (10\text{-}2\text{-}2)$$

it being noted that asymptotic behaviour depends only on the source densities over ∂B_1. Coupling (10-2-2) with the generalised versions of (9-1-1), (9-1-2) corresponding to

(10-2-1), we have sufficient equations to determine a,b and σ, η; whence ϕ and ψ, and hence also χ, may be generated throughout B.

10.3 Chakrabarty representation for exterior domains

Interior biharmonic boundary-value problems may also be formulated by using the Chakrabarty representation (5-3-5). The boundary relations then become

$$\chi = \Omega + \psi, \ \chi' = \Omega' + \psi' \text{ on } \partial B, \tag{10-3-1}$$

in place of (9-1-1), (9-1-2).

For exterior problems, it is necessary to write χ as

$$\chi = \Omega + \psi + ax + by + c \text{ in } B_e + \partial B, \tag{10-3-2}$$

where Ω is given by (5-3-5) and a,b,c are constants to be determined subject to the side conditions

$$\int_{\partial B} \sigma(\underset{\sim}{q})dq = 0 \quad \int_{\partial B} q_1\sigma(\underset{\sim}{q})dq = 0, \quad \int_{\partial B} q_2\sigma(\underset{\sim}{q})dq = 0. \tag{10-3-3}$$

By reference to (5-4-1), these ensure the acceptable behaviour (10-1-1) of χ. Otherwise the boundary formulations on similar line to (10-3-1).

10.4 Chakrabarty representation for ring-shaped domains

Within a ring-shaped domain B, bounded externally by ∂B_o and internally by ∂B_1 which encloses r = 0, we write

$$\Omega(\underset{\sim}{p}) = \int_{\partial B_o} G(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq + \int_{\partial B_1} G(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq;\ \underset{\sim}{p} \in B + \partial B_o + \partial B_1. \quad (10\text{-}4\text{-}1)$$

The boundary equations (10-3-1) apply on the understanding that $\partial B = \partial B_o + \partial B_1$. It will be noted that the first of the biharmonic potentials in (10-4-1) is characterised, utilising (3-2-4), by

$$\nabla^2 \int_{\partial} G(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq = \int_{\partial B} \nabla^2 G(\underset{\sim}{p},\underset{\sim}{q})\ \sigma(\underset{\sim}{q})dq$$

$$= 4\int_{\partial B_o} \log|\underset{\sim}{p}-\underset{\sim}{q}|\sigma(\underset{\sim}{q})dq = \sum_{n=o}^{\infty} r^n(b_{nc}\cos n\theta + b_{ns}\sin n\theta);$$

$$\underset{\sim}{p} \in B, \quad (10\text{-}4\text{-}2)$$

because sources on ∂B_o necessarily generate harmonic potentials which remain continuous at the origin. Accordingly, bearing in mind (3-2-5), we note that

$$\int_{\partial B_o} G(\underset{\sim}{p},\underset{\sim}{q})\sigma(q) = \frac{r^2}{4}\sum_{n=o}^{\infty} \frac{r^n}{(n+1)}(b_{nc}\cos_\theta n\theta + b_{ns}\sin n\theta) + \psi_o;$$

$$\underset{\sim}{p} \in B + \partial B_o + \partial B_1, \qquad (10\text{-}4\text{-}3)$$

where ψ_o is a harmonic function in B of the form (10-4-2).

It will also be noted that the second biharmonic potential in (10-4-1) is characterised, utilising (3-2-4), by

$$\nabla^2 \int_{\partial B_1} G(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq = 4\log r \int_{\partial B_1} \sigma(\underset{\sim}{q})dq - \frac{4\cos\theta}{r}\int_{\partial B_1} q_1\sigma(\underset{\sim}{q})dq$$

$$\frac{4\sin\theta}{r}\int_{\partial B_1} q_2\sigma(\underset{\sim}{q})dq + \sum_{n=-2}^{\infty} r^n(b_{nc}\cos n\theta + b_{ns}\sin n\theta);$$

$$p \in B + \partial B_o + \partial B_1, \qquad (10\text{-}4\text{-}4)$$

because sources on ∂B_1 necessarily generate an exterior potential having $k\log r + O(r^{-1})$ behaviour at infinity. Therefore

$$\int_{\partial B_1} G(\underset{\sim}{p},\underset{\sim}{q})\sigma(\underset{\sim}{q})dq = (-r^2 + r^2\log r)\int_{\partial B_1} \sigma(\underset{\sim}{q})dq - 2x\log r\int_{\partial B_1} q_1\sigma(\underset{\sim}{q})dq$$

$$-2y\log r\int_{\partial B_1} q_2\sigma(\underset{\sim}{q})dq + \frac{r^2}{4}\sum_{n=-2}^{\infty} \frac{r^n}{n+1}(b_{nc}\cos n\theta + b_{ns}\sin\theta) + \quad ,$$

$$\underset{\sim}{p} \in B + \partial B_o + \partial B_1, \qquad (10\text{-}4\text{-}5)$$

where c_1 is a harmonic function having $O(r)$ behaviour at infinity.

An immediate consequence of the above analysis is that Ω automatically covers xlogr, ylogr, r^2logr within a ring-shaped domain B. In this respect, the Chakrabarty representation is superior to the Almansi representation for a ring-shaped domain though the situation reverses for an exterior domain.

As an exercise on the preceding, we show how to express xlogr in the form

$$x\log r = \Omega + \psi \ ; \quad \underset{\sim}{p} \in B + \partial B_o + \partial B_1 \tag{10-4-6}$$

where Ω has been defined as in (10-4-1). Clearly, equation (10-4-6) implies

$$\frac{2\cos\theta}{r} = 4\int \log|\underset{\sim}{p} - \underset{\sim}{q}|\sigma(\underset{\sim}{q})dq \ ; \quad \underset{\sim}{p} \in B + \partial B_o + \partial B_1$$

where σ satisfies the integral equation

$$\frac{2\cos\theta}{r} = 4\int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\sigma(\underset{\sim}{q})dq; \ \underset{\sim}{p} \in \partial B_1. \tag{10-4-7}$$

It will be noted that equation (10-4-7) is restricted only to the internal boundary, since evidently $\sigma = 0$ on B_o. Also, by reference to (5-4-4), σ satisfies the conditions

$$\int_{\partial B_1} \sigma(\underset{\sim}{q})dq = 0 \, , \quad \int_{\partial B_1} q_1\sigma(\underset{\sim}{q})dq = 1, \quad \int_{\partial B_1} q_2\sigma(\underset{\sim}{q})dq = 0. \qquad (10\text{-}4\text{-}8)$$

With σ known, we may generate Ω in $B + \partial B_o + \partial B_1$, and therefore we may generate the harmonic function

$$\psi = x\log r - \Omega \text{ in } B.$$

Chapter 11

Stretching of Thin Plates

11.1 Plane strain

The formulation of problems of two-dimensional classical isotropic elasticity in terms of the Airy stress function, χ, is well known (Timoshenko and Goodier, 1951). In particular, if an infinitely long cylinder composed of homogeneous isotropic elastic material is loaded by forces that are perpendicular to the axis, and which do not vary along its length, it may be assumed that all cross-sections suffer the same deformation. In practice we suppose that the end sections are confined between fixed smooth rigid planes, so that the displacement in the axial direction is prevented. Since there is no axial displacement at the ends and nor, by symmetry, at the midsection, it may be assumed that this holds at every cross-section. This state of strain is called plane strain. The displacement components take the form

$$u = u(x,y),\ v = v(x,y),\ w = 0, \qquad (11\text{-}1\text{-}1)$$

with the axis of the cylinder being identified as the z-axis. Clearly the only non-vanishing strain components are

$$e_{xx} = \frac{\partial u}{\partial x},\ e_{yy} = \frac{\partial v}{\partial y},\ e_{xy} = \tfrac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = e_{yx}. \qquad (11\text{-}1\text{-}2)$$

The only non-vanishing stress components are P_{xx}, P_{yy}, P_{xy}, P_{zz}, and these are related to the strain components by

$$P_{ij} = 2\mu\, e_{ij} + \lambda\, e_{ij}\delta_{ij}; \quad i,j = x,y,z \tag{11-1-3}$$

where μ is the rigidity modulus and λ is Lame's elastic constant. Alternatively, these relations may be written as

$$e_{ij} = \frac{1+\nu}{E} P_{ij} - \frac{\nu}{E}(P_{xx} + P_{yy} + P_{zz}); \; i,j = x,y,z \tag{11-1-4}$$

where E is Young's modulus and ν is Poission's ratio.
Since $e_{zz} = 0$, we see from (11-1-4) that

$$P_{zz} = \nu(P_{xx} + P_{yy}), \tag{11-1-5}$$

under conditions of plane strain.

The Cauchy-Navier equations of equilibrium in the absence of body forces are

$$\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} = 0, \quad \frac{\partial P_{xy}}{\partial x} + \frac{\partial P_{yy}}{\partial y} = 0. \tag{11-1-6}$$

These equations are automatically satisfied by writing

$$P_{xx} = \frac{\partial^2\chi}{\partial y^2}, \quad P_{yy} = \frac{\partial^2\chi}{\partial x^2}, \quad P_{xy} = -\frac{\partial^2\chi}{\partial x\partial y} \tag{11-1-7}$$

where χ is the Airy stress function. Since $P_{xx} + P_{yy} + P_{zz}$ is a harmonic function in the absence of body forces (Love), by reference to (11-1-5) it follows that $P_{xx} + P_{yy}$ is a harmonic function under conditions of plane strain. Accordingly,

$$\nabla^2(P_{xx} + P_{yy}) = 0, \tag{11-1-8}$$

i.e. $\nabla^2(\nabla^2\chi) = 0.$

Thus, the Airy stress function χ satisfies the biharmonic equation. If so, all our preceding analysis has immediate application to the construction of χ.

11.2 Generalised plain stress

For a thin plate in a state of generalised plane stress (Filon, 1903), the average values of elastic quantities (with respect to thickness) at each point provide a sufficiently complete picture of elastic equilibrium. Denoting average values by a bar placed over the symbols, it can be shown that

a) $\overline{P}_{xx}$, $\overline{P}_{yy}$, $\overline{P}_{xy}$ satisfy the same equilibrium equations (11-1-5) as the corresponding stress components in plane strain;

b) P_{xx}, P_{yy}, P_{xy} are related to e_{xx}, e_{yy}, e_{xy} by (11-1-3) except that g is replaced by $\lambda = 2\mu\lambda/(2\mu + \lambda)$.

Hence any problem of generalised plane stress can be solved by treating it as a problem of plain strain, and then substituting λ for λ in the results. We shall deal explicitly only with the problems of plane strain, the application of generalised plane stress being understood.

11.3 Tractions given on the boundary

We now express the elastostatic boundary conditions in terms of χ.

If traction components T_x, T_y per unit length of the boundary are given on ∂B, we have the relations

$$-T_x = P_{xx} \cos(x,n) + P_{xy} \cos(y,n), \tag{11-3-1}$$

$$-T_y = P_{yy} \cos(y,n) + P_{xy} \cos(x,n),$$

where $\cos(x,n) = dx/dn$, $\cos(y,n) = dy/dn$ are cosines of the angles between the positive co-ordinate axes and the inword normal $\underset{\sim}{n}$.

In theoretical elasticity, the normal is usually taken outward from the material. However, we are taking it to be inward to bring the theory in line with that of potential theory (see Chapter 4). Accordingly the tractions which appear in (11-3-1) have the opposite sign to the sign which would appear appropriate on the mechanical ground.

Directing the tangent to the boundary so as to keep the domain on the left, and using the Cauchy-Riemann equations

$$\frac{dx}{dn} = -\frac{dy}{dq}, \quad \frac{dy}{dn} = \frac{dx}{dq}, \tag{11-3-2}$$

we deduce from (11-2-1), (11-1-1) that

$$T_x = \frac{d}{dq}\left(\frac{\partial \chi}{\partial y}\right), \quad T_y = -\frac{d}{dq}\left(\frac{\partial \chi}{\partial x}\right). \tag{11-3-3}$$

For a material subject to given boundary tractions, these

On integration, they yield

$$\begin{aligned} \frac{\partial \chi}{\partial x} &= -\int_{\underset{\sim}{P}}^{\underset{\sim}{p}} T_y(\underset{\sim}{q})\,dq + \alpha, \\ \frac{\partial \chi}{\partial y} &= -\int_{\underset{\sim}{P}}^{\underset{\sim}{p}} T_x(\underset{\sim}{q})\,dq + \beta, \end{aligned} \tag{11-3-4}$$

where $\underset{\sim}{p}$ defines an arbitrary origin of integration on the boundary ∂B, $\underset{\sim}{p}$ signifies the current point on ∂B at which $\frac{\partial\chi}{\partial x}$, $\partial\chi/\partial y$ are being evaluated, and α,β are arbitrary constants of integration. Hence, we may evaluate

$$\chi(\underset{\sim}{p}) = \int_{\underset{\sim}{P}}^{\underset{\sim}{p}} \frac{\partial\chi}{\partial q}\,dq + \quad = \int_{\underset{\sim}{P}}^{\underset{\sim}{p}} \left(\frac{\partial\chi}{\partial x}\frac{dx}{dq} + \frac{\partial\chi}{\partial y}\frac{dy}{dq}\right) dq + \gamma, \tag{11-3-5}$$

where ν is a further constant of integration, and

$$\frac{d\chi}{dn} = \frac{\partial\chi}{\partial x}\frac{dx}{dn} + \frac{\partial\chi}{\partial y}\frac{dy}{dn} . \tag{11-3-6}$$

It follows that the traction problem leads directly to the cannonical biharmonic boundary-value problem, i.e. that of finding a function χ, satisfying equation (11-1-7) throughout a domain B, and such that $\chi(\underset{\sim}{p})$, $d\chi/dn$ assume prescribed values on the boundary ∂B of B.

Note that χ, $d\chi/dn$ include arbitrary terms $\alpha x(\underset{\sim}{p}) + \beta y(\underset{\sim}{p}) + \gamma$, $\alpha(dx/dn) + \beta(dy/dn)$ respectively. Reference to (8-1-8) shows that these boundary contributions define a biharmonic function in B which yields no stress components. Accordingly, neither the location of $\underset{\sim}{P}$ nor the values there of χ, $\partial\chi/\partial x$, $\partial\chi/\partial y$ have any physical significance. Without any loss of generality, we may put $\alpha = \beta = \gamma = 0$.

It will be seen from (11-3-4) that $[\partial\chi/\partial x]_{\underset{\sim}{P}}^{\underset{\sim}{p}}$ measures the y-component of the resultant force acting upon ∂B between P and $\underset{\sim}{p}$. If, therefore, the total resultant y-component acting on ∂B vanishes, as must be the case for a simply-connected domain B in statical equilibrium, $\partial\chi/\partial x$ will be a single-valued function of $\underset{\sim}{p}$ on ∂B. Similarly, if the total resultant x-component vanishes, $\partial\chi/\partial y$ will be a single-valued function of $\underset{\sim}{p}$. Furthermore, the resultant anti-clockwise couple acting on ∂B between P and $\underset{\sim}{p}$ is measured by

$$\int_{\underset{\sim}{P}}^{\underset{\sim}{p}} (xT_y - yT_x)dq = - \int_{\underset{\sim}{P}}^{\underset{\sim}{p}} \left\{ x \frac{d}{dq} \left(\frac{\partial\chi}{\partial x}\right) + y \frac{d}{dq} \left(\frac{\partial\chi}{\partial y}\right) \right\} dq$$

$$= [\chi - x\frac{\partial\chi}{n} - y\frac{\partial\chi}{y}]_{\underset{\sim}{P}}^{\underset{\sim}{p}} . \qquad (11\text{-}3\text{-}7)$$

If, therefore, the total resultant couple acting on B vanishes, $\chi(\underset{\sim}{p})$ also will be a single-valued function on ∂B and therefore also throughout B.

11.4 Tractions given on a ring-shaped domain

For a ring-shaped domain B bounded internally by ∂B_1 (which includes $r = 0$) and externally by ∂B_0, the possibility arises of a non-vanishing resultant force or couple to act upon ∂B_1, with a compensating force or couple acting upon ∂B_0, thus implying a multi-valued χ in B. Discounting this possibility, we now extend the preceding boundary analysis as follows.

Starting from an arbitrary origin $\underset{\sim}{P}_o$ on ∂B_o, we calculate χ, $\partial\chi/\partial x$, $\partial\chi/\partial y$ as before. These quantities resume their starting values, all taken to be zero, on completing a circuit of ∂B_o. We then jump from $\underset{\sim}{P}_o$ to some arbitrary origin $\underset{\sim}{P}$ on ∂B_1, it being supposed that $\chi, \frac{\partial\chi}{\partial x}, \frac{\partial\chi}{\partial y}$ jump to the values

$$\chi = \gamma, \frac{\partial\chi}{\partial x} = \alpha, \frac{\partial\chi}{\partial y} = \beta \text{ at } \underset{\sim}{P}_1 = < X_1 \; Y_1 > \in B. \qquad (11\text{-}4\text{-}1)$$

These yield contributions

$$\frac{\partial\chi}{\partial n} = \alpha \frac{dx}{dn} + \beta\frac{dy}{dn} ,$$

$$\chi = \alpha(x - X_1) + \beta(y - Y_1) + \gamma, \qquad (11\text{-}4\text{-}2)$$

to the cannonical boundary quantities on ∂B_1. Other contributions of course arise from the traction integrals of (11-3-4) taken along ∂B_1.

To understand the significance of α,β,γ suppose that ∂B_o, ∂B_1 are completely free from tractions. In this case the boundary conditions on ∂B_o are

$$\chi = 0, \;\; \frac{d\chi}{dn} = 0.$$

These combined with the boundary conditions (11-4-2) on ∂B_1 define a unique non-trivial biharmonic function χ within B, so implying the existence of a continuous purely elastic generated entirely by internal sources. These sources are the three independent two-dimensional Volterra dislocations (Love, 1952), each characterised by a discontinuity in a specific displacement or rotation component. Methods for constructing χ have been already noted in chapters 6 , 9 .

As already noted, the dislocation solutions of the biharmonic equation are xlogr, ylogr, r^2logr. To eliminate unwanted dislocations we put a = b = 0 in (10-2-1) so that the appropriate representation becomes

$$\chi = r^2\phi + \psi \tag{11-4-3}$$

where ϕ,ψ are single-valued harmonic functions. These are not unique as already shown in (3-2-8). However, if we represent ϕ,ψ as simple-layer potentials, uniqueness may be ensured by the introduction of the side conditions (10-2-2). Also we introduce the condition

$$\int \sigma(q)dq = 0 \tag{11-4-4}$$

in order to ensure that $r^2\phi$ contains no component proportional to r^2logr, which represents rotational dislocations.

11.5 Infinite exterior domain

If the outer boundary of the ring moves off to infinity, so that we are dealing with an infinite domain B containing an internal boundary ∂B subject to equilibrated tractions, the problem becomes more managable. This is because the behaviour of harmonic functions may be regulated at infinity in accordance with specific physical requirements, for instance so as to eliminate dislocations or, more generally, to ensure that stress components fall off sufficiently rapidly.

For an infinite exterior domain we may utilise the representation

$$\chi = r^2\phi + \psi + k$$

where ϕ, ψ are defined as logarithmic potentials. The constant k is to be determined subject to the side condition (11-4-4). This provides the acceptable behaviour $\chi = O(r)$ at infinity and also eliminates a possible dislocation component $r \log r$ in $r^2\phi$, and of course there are no other possibilities of dislocations. The construction of χ is similar to the methods outlined in Chapter 10.

11.6 Chakrabarty representation

Instead of using the representation (11-4-3) for the interior problem, we may also utilise the Chakrabarty representation (5-3-5), with the boundary condition (10-3-1)

For exterior problems, it is necessary to write

$$\chi = \Omega + \psi + \alpha x + \beta y + \gamma \text{ in } B_e + \partial B \qquad (11\text{-}6\text{-}1)$$

where Ω is given by (5-3-5) and has the asymptotic expansion (5-4-1). The coefficients α, β, γ are constants to be determined subject to the side conditions (10-3-2). Reference to the asymptotic expansion (5-4-1) shows that these eliminate all the possible dislocation components. Otherwise, the boundary conditions follow essentially the same lines as (10-3-1).

Within a ring-shaped domain B, we define Ω as in (10-4-1). This uniquely covers the functions xlogr, ylogr, r^2logr and therefore Chakrabarty representation (10-4-1) applies on the understanding that $\partial B = \partial B_o + \partial B$.

Appendix 1: Γ-contour

A distinguishing feature of two-dimensional theory is the existence of contours, termed Γ-contours, for which the equation

$$\int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\lambda(\underset{\sim}{q})dq = 1; \quad \underset{\sim}{p} \in \partial B \tag{A1-1}$$

does not exhibit a solution. For example, if ∂B is a circle of radius a, then $\lambda(\underset{\sim}{q}) = \lambda_o$ (a constant) by symmetry, in which case (A1-1) becomes

$$\lambda_o \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}| \, dq = \lambda_o . \, 2\pi a \log a = 1,$$

i.e. $\lambda_o = (2\pi a \log a)^{-1}$. (A1-2)

It follows that λ_o has no finite value when a = 1, and that any finite λ_o provides a non-trivial solution of the homogeneous equation

$$\int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}| \, \lambda(\underset{\sim}{q})dq = 0. \tag{A1-3}$$

The unit circle case may be generalised. Given a contour ∂B defined by the Cartesian equation $f(x,y)=0$, construct the family of similar contours defined by $f(x/m, y/m)= 0$, where $m > o$ is a continuously varying parameter. Then the equation (A1.1) admits a unique solution for every member of the family with one exception, i.e. the Γ-contour, which is characterised by the existence of a non-trivial solution of (A1.3). The proof is given by Jaswon (1963).

Appendix 2: Conjugate Harmonics

Here we consider the problem of constructing the harmonic conjugate

$$\bar{\phi}(\underset{\sim}{p}) = \int_{\partial B} \theta(\underset{\sim}{p}-\underset{\sim}{q})\sigma(\underset{\sim}{q})dq \tag{A2-1}$$

of the simple-layer logarithmic potential

$$\phi(\underset{\sim}{p}) = \int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|\sigma(\underset{\sim}{q})dq, \tag{A2-2}$$

where ∂B is an arbitrary closed contour. The angle $\theta(\underset{\sim}{p}-\underset{\sim}{q})$ associated with $\log|\underset{\sim}{p}-\underset{\sim}{q}|$ may be defined as follows:

Introduce a line through $\underset{\sim}{q}$ parallel to the positive x-axis of a Cartesian co-ordinate system. By our convention, θ is the angle moving in the anticlockwise sense between this line and the vector joining $\underset{\sim}{q}$ to $\underset{\sim}{p}$ (fig: 1). If so, $\log|\underset{\sim}{p}-\underset{\sim}{q}|$ and $\theta(\underset{\sim}{p}-\underset{\sim}{q})$ are conjugate harmonic functions of $\underset{\sim}{q}$. More generally, $\sigma(\underset{\sim}{q})\log|\underset{\sim}{p}-\underset{\sim}{q}|$ and $\sigma(\underset{\sim}{q})\theta(\underset{\sim}{p}-\underset{\sim}{q})$ are conjugate harmonic functions of $\underset{\sim}{p}$, which satisfy the Cauchy-Riemann equations

$$\frac{\partial}{\partial p_1}[\sigma(\underset{\sim}{q})\log|\underset{\sim}{p}-\underset{\sim}{q}|] = \frac{\partial}{\partial p_2}[\sigma(\underset{\sim}{q})\theta(\underset{\sim}{p}-\underset{\sim}{q})] , \tag{A2-3}$$

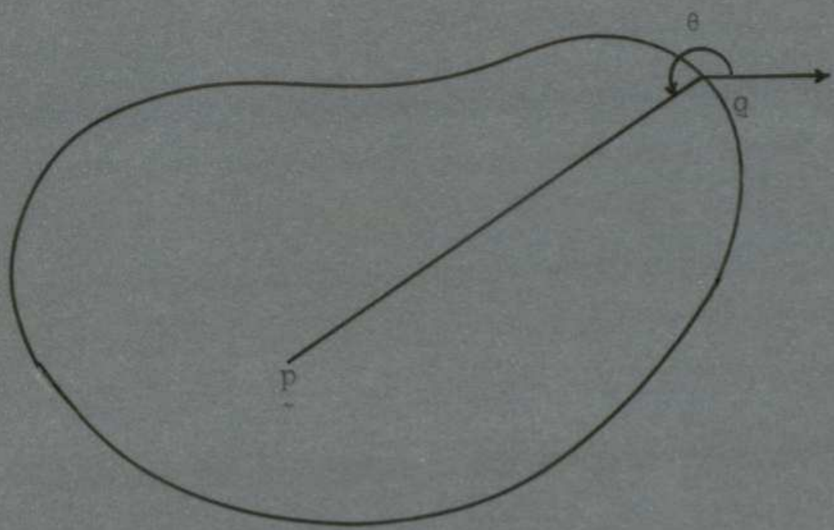

Fig: 1 $\theta(\underset{\sim}{p}-\underset{\sim}{q})$ for an arbitrary closed domain

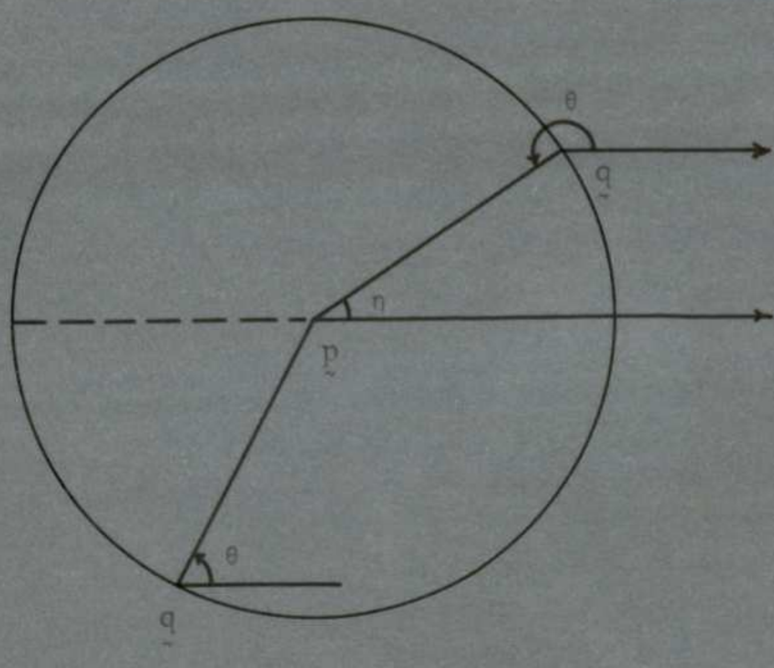

Fig: 2 $\underset{\sim}{p}$ at the centre for two choices of $\underset{\sim}{q}$.

$$\frac{\partial}{\partial p_2}\sigma(q)\log|\underset{\sim}{p}-\underset{\sim}{q}| = -\frac{\partial}{\partial p_1}\sigma(\underset{\sim}{q})\,\theta(\underset{\sim}{p}-\underset{\sim}{q}) \quad , \tag{A2-4}$$

where $\underset{\sim}{p} = (p_1,p_2)$. Since this holds for every $\underset{\sim}{q} \in \partial B$, we see that ϕ, $\bar{\phi}$ in (A2-1), (A2-2) are conjugate harmonic functions of $\underset{\sim}{p}$.

To ensure that θ is single-valued if $\underset{\sim}{p} \varepsilon\ \partial B$, we draw a fixed straight line which cuts ∂B at q_o, at which point we commence our anti-clockwise circuit of ∂B for every location of $\underset{\sim}{p}$. The angle θ jumps by -2π at q_o after completing one circuit on ∂B. The same applies if $\underset{\sim}{p}\ \varepsilon\ \partial B$, except that θ jumps by π as $\underset{\sim}{q}$ passes through $\underset{\sim}{p}$.

As an example of the above, we evaluate

$$\int_{\partial B}\theta(\underset{\sim}{p}-\underset{\sim}{q})\,dq \tag{A2-5}$$

on a circle of radius a. Three cases may now be distinguished.

(i) If $\underset{\sim}{p}$ is located at the centre of the circle, then we see that (fig:2)

$$\begin{aligned} \theta &= \eta + \pi; \qquad -\pi \lesssim \eta \lesssim 0 \\ &= \eta + \pi; \qquad 0 < \eta \lesssim \pi \end{aligned}$$

where $\underset{\sim}{q} = (a\cos\eta,\ a\sin\eta)$ and $dq = a\,d\eta$. We find that

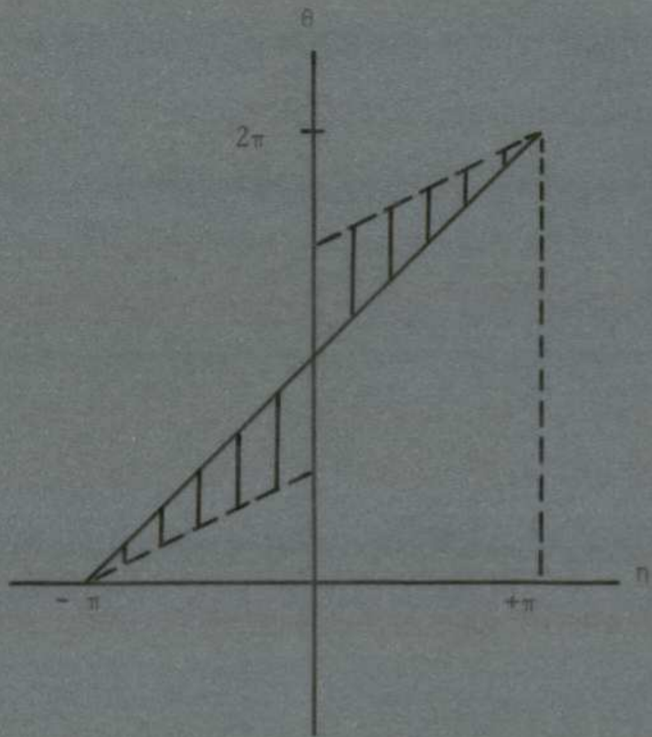

Fig: 3 graph $\theta = \eta + \pi$ (———)
$\theta = {}^{\pi}/_{2} + {}^{\eta}/_{2},\ \frac{3\pi}{2} + {}^{\eta}/_{2}$ (- - -).

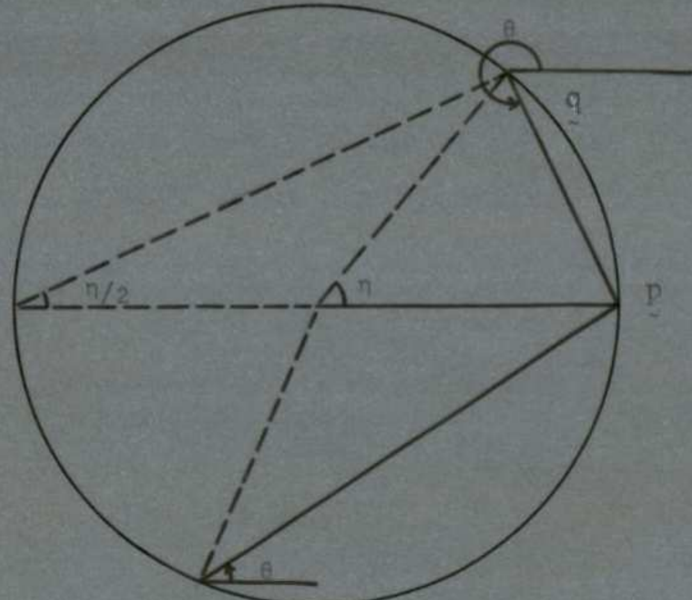

Fig: 4 $\underset{\sim}{p}$ on the circumference for two choices of $\underset{\sim}{q}$.
(Note that θ jumps by π as $\underset{\sim}{q}$ passes through $\underset{\sim}{p}$)

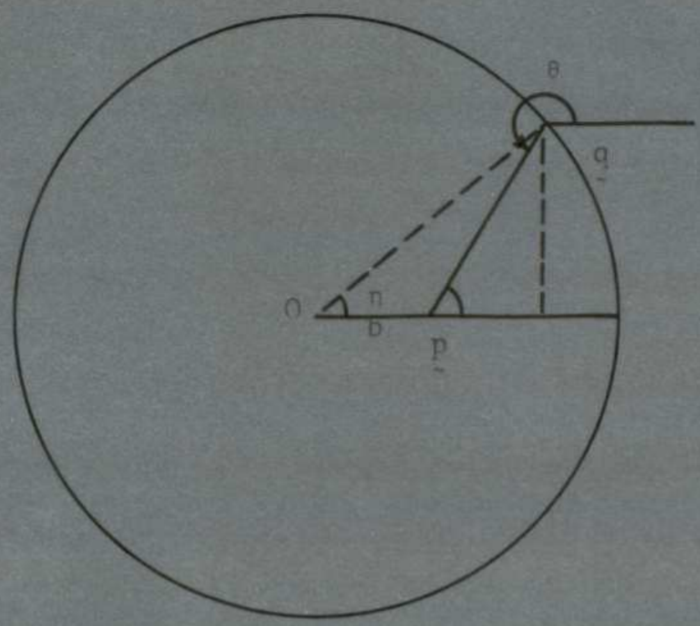

Fig: 5 $\underset{\sim}{p}$ as an interior point; $O\underset{\sim}{p} = b$.

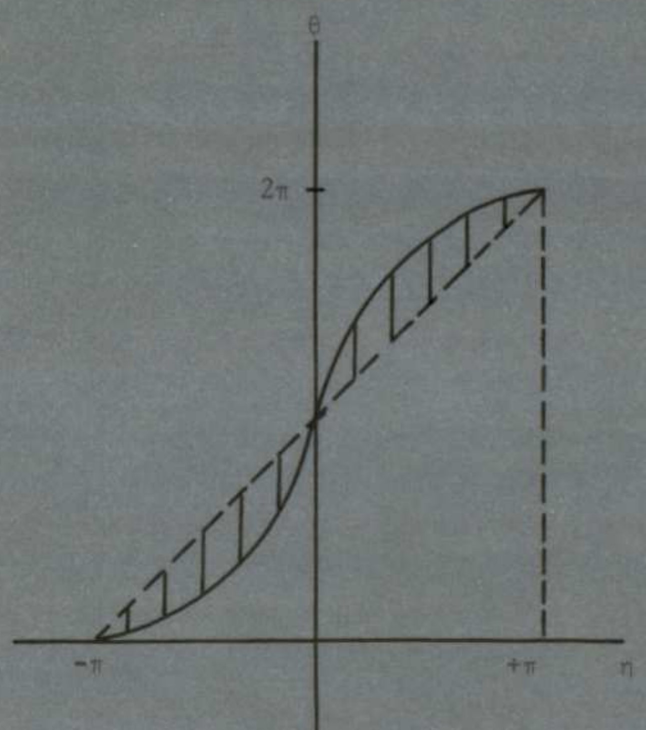

Fig: 6 graph $\theta = \theta(\eta)$ for an interior point $\underset{\sim}{p}$.

$$\int_{\partial B} \theta(\underset{\sim}{p}-\underset{\sim}{q})dq = a\int_{-\pi}^{\pi} (\pi + \eta)d\eta = 2\pi^2 a.$$

(ii) If $\underset{\sim}{p}$ lies on ∂B (fig:4), then

$$\theta = \frac{\pi}{2} + \frac{\eta}{2}\ ; \quad -\pi < \eta \leq 0$$

$$= 3\frac{\pi}{2} + \eta/_2\ ; \quad 0 \leq \eta \leq \pi.$$

in which case

$$\int_{\partial B} \theta(\underset{\sim}{p}-\underset{\sim}{q})dq = a\int_{-\pi}^{0} (\frac{\pi}{2} + \frac{\eta}{2})d\eta + a\int_{0}^{\pi} (\frac{3\pi}{2} + \frac{\eta}{2})d\eta = 2\pi^2 a$$

once again.

(iii) If $\underset{\sim}{p}$ is an interior point other than the centre, then

$$\theta = \pi + \arctan\left(\frac{a \sin\eta}{a \cos\eta - b}\right)$$

where OP = 6 (fig:5). We sketch θ as a function of η in fig: 6. Two shaded portions are equal, by symmetry, from which it follows that

$$\int_{\partial B} \theta(\underset{\sim}{p}-\underset{\sim}{q})dq = 2\pi^2 a.$$

We note that the integral (A2-5) has a value independent of the location $\underset{\sim}{p}$. This is expected since (A2-5) is the conjugate harmonic of $\int_{\partial B} \log|\underset{\sim}{p}-\underset{\sim}{q}|dq$, which has a constant value within the circle.

Appendix 3: Behaviour of Chakrabarty biharmonic potentials on the boundary

Since $|\underset{\sim}{p} - \underset{\sim}{q}|^2 \to 0$, $|\underset{\sim}{p} - \underset{\sim}{q}|^2 \log |\underset{\sim}{p} - \underset{\sim}{q}| \to 0$ as $|\underset{\sim}{p} - \underset{\sim}{q}| \to 0$,

we see that

$$|\underset{\sim}{p} - \underset{\sim}{q}|^2 \left\{ -1 + \log |\underset{\sim}{p} - \underset{\sim}{q}| \right\} \to 0 \quad \text{as } |\underset{\sim}{p} - \underset{\sim}{q}| \to 0.$$

However, it is necessary to evaluate the contribution to

$$\int_{\partial B} |p - q|^2 \left\{ -1 + \log|p - q| \right\} dq \; ; \quad p \in \partial B, \tag{A3-1}$$

from a small interval in the neighbourhood of p. This is achieved by setting up a local Cartesian co-ordinate system centred upon $\underset{\sim}{p}$ as origin, the x- and y-axes through $\underset{\sim}{p}$ being identified respectively with the tangential and normal directions through $\underset{\sim}{p}$ (fig: 7). To a first approximation, a small arc of the curve at $\underset{\sim}{p}$ may be taken as the interval $[-h,h]$ on the local x-axis. If so, we may write $\underset{\sim}{q} = (x,o)$ where $-h < x < h$. Clearly

$$|\underset{\sim}{p} - \underset{\sim}{q}|^2 = x^2, \quad |\underset{\sim}{p} - \underset{\sim}{q}|^2 \log|\underset{\sim}{p} - \underset{\sim}{q}| = x^2 \log|x|,$$

in which case the relevant contribution to the integral (A3-1) is seen to be

$$2\int_o^h x^2(-1 + \log|x|)dx = \frac{2}{3} h^3 \log h - \frac{8}{9} h^3. \tag{A3-2}$$

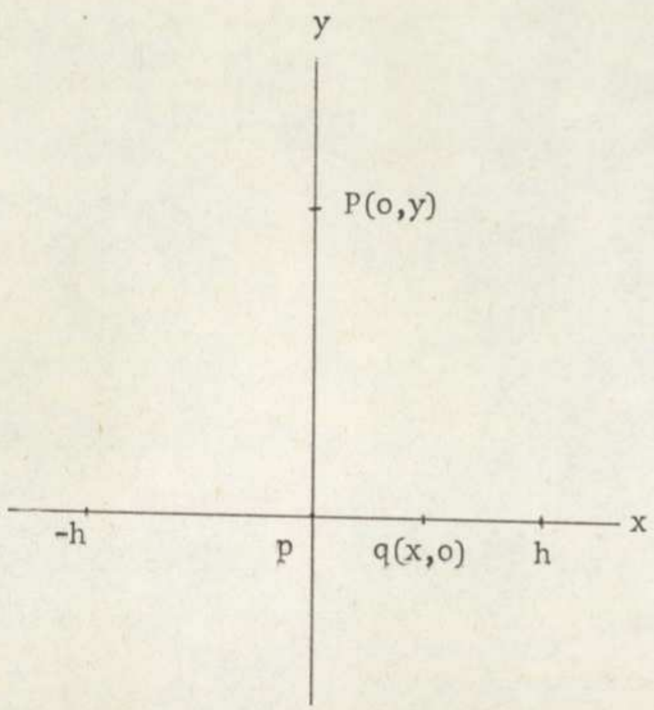

Fig: 7 small interval in the neighbourhood of $\underset{\sim}{p}$

To compute the contribution to

$$\lim_{\underset{\sim}{P} \to \underset{\sim}{p}} \frac{d}{dn_i} \int_{\partial B} |\underset{\sim}{P} - \underset{\sim}{q}|^2 \left\{ -1 + \log |\underset{\sim}{P} - \underset{\sim}{q}| \right\} dq \qquad \text{(A3-3)}$$

from this interval, we take $\underset{\sim}{P} = (0,y)$. If so, then

$$|p - q|^2 \left\{ -1 + \log |p - q| \right\} = \tfrac{1}{2}(x^2 + y^2)\left\{ -2 + \log(x^2 + y^2) \right\}$$

in which case the relevant contribution to (A3-3) is seen to be

$$\lim_{y \to 0} \frac{\partial}{\partial y} \int_0^h (x^2 + y^2)\left\{ -2 + \log (x^2 + y^2) \right\} dx = 0.$$

These results agree with earlier conclusions of Chakrabarty (1971).

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