Ext^1 for Weyl modules for q-$\mathrm{GL}(2,k)$

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In a recent paper [7], Erdmann has calculated Ext^1_G between Weyl modules for SL_2. In this paper we generalise this result to solve the corresponding problem for quantum GL_2 as defined by Dipper and Donkin in [2]. We also show how our result also holds for the Manin quantisation. To apply the methods of [7], it is necessary to determine the block structure of quantum GL_2, so the first main result of this paper is a description of this, derived from the analysis of the subcomodule structure of the symmetric powers in [10].

After an initial section of generalities, the next section consists of the determination of the block structure. We also need a quantum analogue of two short exact sequences from [11], which we give in the following section. With these results, the argument now follows much as in [7]; we consider the infinitesimal case, and then use the Lyndon-Hochschild-Serre spectral sequence to obtain the desired result. Finally we show how the result also holds for the Manin quantisation.

It should be noted that the result here uses the classical case, so is not independent of that in [7]. The only real difference in the arguments used occurs in Lemma 4.8 where the original methods do not generalise, so we use a more direct argument. There is also an unfortunate typographical error in the statement of the main result in [7].

1 Preliminaries

In this section we summarise very briefly some of the basic results that will be needed later. We consider the quantum general linear group defined by Dipper and Donkin in [2], over an algebraically closed field k. We consider the case where $n = 2$, and denote the quantum GL_2 by q-$\mathrm{GL}(2,k)$ (where q is the quantum parameter) or simply by G. It will be assumed that $q \neq 0$ and that the field k has characteristic $p > 0$.

This paper adopts the philosophy (and general notation) of [4, §1], to which the reader is referred for the basic homological definitions and results. In particular, we have both the generalised tensor identity and the Lyndon-Hochschild-Serre spectral sequence, which are essential for the results that follow. In the same paper, the quantum analogue to the Borel subgroup, denoted B, is defined. Thus one can consider the modules for G induced from one-dimensional B-modules. As in the classical case, the non-zero induced modules correspond to the dominant weights (see [4, Lemma 3.2]); and in the case $n = 2$ considered here these can be completely classified (see [4, Remark 3.7]).

If we denote the induced module corresponding to $\lambda = (\lambda_1, \lambda_2)$ by $\nabla(\lambda)$, then we have

$$\nabla(\lambda) \cong \mathrm{S}_q^{\lambda_1-\lambda_2}(E) \otimes q\text{-det}^{\lambda_2}$$

where $\mathrm{S}_q^r(E)$ is the quantum analogue of the rth symmetric power of the natural module (see [2, 2.1.8]) and q-det is the analogue of the determinant module. In general, the tensor product $U \otimes V$ is not isomorphic to $V \otimes U$, but the generalised tensor identity (see [4, 1.3]) gives that in this case $\nabla(r, 0) \otimes q\text{-det}^a \cong q\text{-det}^a \otimes \nabla(r, 0)$; a fact that will be used repeatedly in what follows. The Weyl modules $\Delta(\lambda)$ are defined as the duals of appropriate induced modules as in the classical case (see [4, §4]).

We have now defined the objects of interest, and can begin to consider the problem of determining when two Weyl modules have non-trivial extensions. As in the classical case (see [1, 3.2 Corollary]), it is easy to see that

$$\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\mu)) \neq 0 \ \text{ implies } \quad \lambda < \mu$$

so we will restrict to this case. By [4, 4(8)], for non-trivial extensions to exist we must have q a root of unity, so we assume also that q is a primitive lth root of unity. Note that we must have $(l, p) = 1$ for such a q to exist. If $l = 1$ then we are in the classical case, so we also assume that $l > 1$.

Finally we note that we can define an analogue of the first Frobenius kernel, denoted G_1, which will be an essential tool in what follows. The definition of this, along with some of its basic representation theory can be found in [5, §3]. We will also need the related factor group of G which defines G_1 (see [4, remark after Corollary 1.4]), which we denote by $\bar{G}$.

2 The blocks of q-GL$(2, k)$

The first part of this section depends on the submodule structure of the symmetric powers as described in [10]. We begin by recalling some notation from that paper. Let E be the quantum analogue of the natural module for GL_2, with basis $\{e_1, e_2\}$. Given a basis element $e^a = e_1^{a_1} e_2^{a_2} \in \mathrm{S}_q^r(\mathrm{E})$ we write:

$$\begin{array}{llll} a_i = a_i^1 l + a_i^0 & \text{with } 0 \le a_i^0 < l & \text{and } a_i^1 = \sum_j a_i^{1,j} p^j & \text{with } 0 \le a_i^{1,j} < p \quad \forall i,j \\ r = r_1 l + r_0 & \text{with } 0 \le r_0 < l & \text{and } r_1 = \sum_j r_1^j p^j & \text{with } 0 \le r_1^j < p \quad \forall j \end{array}$$

Set $m = \max\{0, j \mid r_1^j > 0\}$. We define the *carry pattern* $c(e^a) = (c_0(e^a), \ldots, c_m(e^a))$ recursively using:

$$\left.\begin{array}{c} a_1^0 + a_2^0 = c_0(e^a) l + r_0 \\ c_{t-1}(e^a) + a_1^{1,t-1} + a_2^{1,t-1} = c_t(e^a) p + r_1^{t-1} \end{array}\right\} \tag{1}$$

Let $\mathrm{C}(r) = \{c(e^a) \mid e^a \in \mathrm{S}_q^r(\mathrm{E})\}$. The submodules of $\mathrm{S}_q^r(\mathrm{E})$ correspond to order closed subsets of $\mathrm{C}(r)$, where $c \le c'$ if $c_i \le c'_i$ for all i. The results of [10], along with [6, Lemma 3], give $(\mathrm{c}_0, \ldots, \mathrm{c}_m) \in \mathrm{C}(r)$ if, and only if,

$$\begin{array}{cl} c_0 \in \{0, \ldots, \mathrm{M}\} & \\ 0 \le c_k \le \sum_{j \ge k} r_1^j p^{j-k} & \text{for } 1 \le k \le m \\ 0 \le r_1^k + p c_{k+1} - c_k \le 2p - 2 & \text{for } 0 \le k \le m \end{array} \tag{2}$$

where we set $c_{m+1} = 0$ and $\mathrm{M} = \begin{cases} 0 & \text{if } r < l-1 \\ 1 & \text{if } r > l-1 \quad \text{and } r_0 \ne l-1 \\ 0 & \text{otherwise.} \end{cases}$

From (1) it is easy to determine the highest weight $a = (a_1, a_2)$ such that $c(e^a) = c$; call this the *highest weight in c*. We obtain

$$\begin{array}{rcl} a_1^0 & = & \min\{l-1, r_0 + l c_0\} \\ a_1^{1,t-1} & = & \min\{p-1, r_1^{t-1} - c_{t-1} + p c_t\}. \end{array} \tag{3}$$

Theorem 2.1 *A weight $a = (a_1, a_2)$ is linked to $(r+d, d)$ if, and only if, the following conditions hold:*

i) $a_1 + a_2 = r + 2d$

ii) $\bar{a} \equiv \pm \bar{r} \pmod{2l}$

iii) If $\bar{a} \equiv 0 \pmod{l}$ *then* $\bar{a} \equiv \pm l p^t (r_1^t + 1) \pmod{p^{t+1}}$

where $\bar{a} := a_1 - a_2 + 1$, $\bar{r} := r + 1$ *and* $t := \max\{0, s \mid \bar{r} \equiv 0 \pmod{p^s}\}$.

Proof: The statement of the linkage condition in terms of equivalence classes under the relation generated by: $\lambda \sim \mu$ if $[\nabla(\lambda) : \mathrm{L}(\mu)] \ne 0$, implies that i) must hold. Note that i) implies i′) $\bar{a} \equiv \bar{r} \pmod 2$. For the necessity of ii) and iii), we show that $[\nabla(r+d, d) : \mathrm{L}(a)] \ne 0$ implies both ii) and iii); as then this must clearly be true for every element of the equivalence class generated by $(r+d,d)$ under $\sim$. Further we may assume that $d=0$ as we can tensor with an appropriate power of the q-determinant to get the general result.

Necessity of ii): Let $c \in \mathrm{C}(r)$, and a be the highest weight in c. We have

$$a_1^0 = \begin{cases} r_0 & \text{if } c_0 = 0 \\ l-1 & \text{if } c_0 = 1. \end{cases}$$

But $a_1 + a_2 = r$ implies $a_2^0 = \begin{cases} 0 & \text{if } c_0 = 0 \\ r_0+1 & \text{if } c_0 = 1. \end{cases}$ Hence we have

$$a_1^0 - a_2^0 = \begin{cases} r_0 & \text{if } c_0 = 0 \\ l-r_0-2 & \text{if } c_0 = 1. \end{cases} \tag{4}$$

If l is odd then (4) implies that $\bar{a} \equiv \pm\bar{r} \pmod{l}$, and this together with i$'$) gives the necessity of ii). If l is even then p is odd (as $(l,p)=1$). Now by (1) we have:

$$\begin{aligned} a_1 - a_2 + 1 &= a_1^0 - a_2^0 + l\left(\textstyle\sum_{j=0}^m p^j (a_1^{1,j} - a_2^{1,j})\right) + 1 \\ &= a_1^0 - a_2^0 + 1 + l\left(\textstyle\sum_{j=0}^m p^j (c_{j+1}p + r_1^j - c_j - 2a_2^{1,j})\right) \\ &= a_1^0 - a_2^0 + 1 + l\phi. \end{aligned}$$

where $\phi = \sum_{j=0}^m p^j(c_{j+1}p + r_1^j - c_j - 2a_2^{1,j})$. So using (4) we obtain

$$\bar{a} = \begin{cases} r_0 + 1 + l\phi & \text{if } c_0 = 0 \\ -(r_0+1) + l(\phi+1) & \text{if } c_0 = 1. \end{cases}$$

Also we have that

$$\bar{r} = \begin{cases} r_0 + 1 + l \pmod{2l} & \text{if } r_1 \text{ odd} \\ r_0 + 1 \pmod{2l} & \text{if } r_1 \text{ even.} \end{cases}$$

So it is enough to show that ϕ satisfies:

$$\phi \equiv \begin{cases} 1 \pmod{2} & \text{if } c_0 + r_1 \text{ odd} \\ 0 \pmod{2} & \text{if } c_0 + r_1 \text{ even.} \end{cases} \tag{5}$$

As we are only interested in ϕ mod 2, and p is odd, we can replace ϕ by $\hat{\phi}$ where

$$\hat{\phi} = \sum_{j=0}^m (c_{j+1} + r_1^j - c_j) = \sum_{j=0}^m r_1^j + c_{m+1} - c_0 = r_1 - c_0$$

which satisfies (5). So ii) is necessary.

Necessity of iii): If $r_0 = l-1$ then we have $c_0 = 0$. From (2) we have $0 \le p-1+pc_{s+1}-c_s \le 2p-2$ for all $s \le t-1$. So by induction we have $c_s = 0$ for all $s \le t$. Hence for a the highest weight in c we have:

$$\begin{aligned} a_1^{1,s} &= p-1 & & \forall s \le t-1 \\ a_1^{1,t} &= \min\{p-1, r_1^t + pc_{t+1}\} & &= \begin{cases} r_1^t & \text{if } c_{t+1} = 0 \\ p-1 & \text{otherwise.} \end{cases} \end{aligned} \tag{6}$$

Note that this implies that $a_2^{1,s} = 0$ for all $s \leq t-1$. Now $a_1 + a_2 = r$ implies that

$$\begin{array}{lll} a_1 + a_2 & \equiv r_0 + l(r_1^0 + pr_1^1 + \cdots + p^t r_1^t) & \pmod{p^{t+1}} \\ & \equiv l(1 + p - 1 + p^2 - \cdots + p^t - p^{t-1} + p^t r_1^t) - 1 & \pmod{p^{t+1}} \\ & \equiv lp^t(r_1^t + 1) - 1 & \pmod{p^{t+1}}. \end{array}$$

Similarly we have

$$\begin{array}{lll} a_1 + a_2 & \equiv a_1^0 + a_2^0 + l(a_1^{1,0} + a_2^{1,0} + \cdots + p^t a_1^{1,t} + p^t a_2^{1,t}) & \pmod{p^{t+1}} \\ & \equiv l(1 + p - 1 + p^2 - \cdots + p^t - p^{t-1} + p^t(a_1^{1,t} + a_2^{1,t})) - 1 & \pmod{p^{t+1}} \\ & \equiv lp^t(a_1^{1,t} + a_2^{1,t} + 1) - 1 & \pmod{p^{t+1}}. \end{array}$$

These give

$$\begin{array}{rll} lp^t(r_1^t + 1) - 1 & \equiv lp^t(a_1^{1,t} + a_2^{1,t} + 1) - 1 & \pmod{p^{t+1}} \\ lp^t r_1^t & \equiv lp^t(a_1^{1,t} + a_2^{1,t}) & \pmod{p^{t+1}} \\ r_1^t & \equiv a_1^{1,t} + a_2^{1,t} & \pmod{p}. \end{array}$$

Then (6) implies that $a_2^{1,t} \equiv \begin{cases} 0 & \pmod{p} \quad \text{if } c_{t+1} = 0 \\ r_1^t + 1 & \pmod{p} \quad \text{if } c_{t+1} \neq 0 \end{cases}$ and hence we get

$$\begin{array}{lll} a_1 - a_2 & \equiv a_1^0 - a_2^0 + l(a_1^{1,0} - a_2^{1,0} + \cdots + p^t(a_1^{1,t} - a_2^{1,t})) & \pmod{p^{t+1}} \\ & \equiv l(1 + p - 1 + p^2 - \cdots + p^t - p^{t-1} + p^t(a_1^{1,t} - a_2^{1,t})) - 1 & \pmod{p^{t+1}} \\ & \equiv lp^t(a_1^{1,t} - a_2^{1,t} + 1) - 1 & \pmod{p^{t+1}} \\ & \equiv \begin{cases} +lp^t(r_1^t + 1) - 1 & \text{if } c_{t+1} = 0 \\ -lp^t(r_1^t + 1) - 1 & \text{if } c_{t+1} \neq 0 \end{cases} & \pmod{p^{t+1}} \end{array}$$

as required. So i)–iii) are necessary.

For sufficiency: Consider $\nabla(a, b) \cong \nabla(a - b, 0) \otimes (q\text{-det})^b$. If this is not irreducible then its submodule structure is determined by that of $\nabla(a - b, 0)$. This must have a composition factor with highest weight (c, d) such that $0 \leq c - d < a - b$. Thus (a, b) is linked to whatever $(c + b, d + b)$ is; so it is enough to consider (c, d) and tensor up with an appropriate power of the q-determinant. Continuing this descent, the sequence must terminate in an irreducible module. Hence it is sufficient to show that there is a unique irreducible $\nabla(a_1, a_2)$ satisfying the conditions. In fact, we need only consider $\nabla(a_1 - a_2, 0)$ with i) replaced by i′), as if this is unique then tensoring up will give the result.

Let $r = a_1 - a_2 = r_0 + l(r_1^0 + \cdots + p^m r_1^m)$. It is necessary to determine which $\mathrm{S}_q^r(\mathrm{E})$ are irreducible. By Steinberg's Tensor Product Theorem, we have

$$\begin{array}{ll} \dim \mathrm{L}(r, 0) & = (r_0 + 1)(r_1^0 + 1) \cdots (r_1^m + 1) \\ & = 1 + r_0 + (r_0 + 1)[r_1^0 + (r_1^0 + 1)r_1^1 + \cdots + \left(\prod_{i=0}^{m-1}(r_1^i + 1)\right) r_1^m]. \end{array}$$

As $\mathrm{soc}\nabla(\lambda) = \mathrm{L}(\lambda)$ we require that $\dim \mathrm{S}_q^r(\mathrm{E}) = \dim \mathrm{L}(r,0) = r+1$. Hence we require that

$$r+1 = 1 + r_0 + (r_0+1)[r_1^0 + (r_1^0+1)r_1^1 + \cdots + \left(\prod_{i=0}^{m-1}(r_1^i+1)\right) r_1^m].$$

That is

$$r_0 + 1 + l[r_1^0 + \cdots + p^m r_1^m] = 1 + r_0 + (r_0+1)[r_1^0 + \cdots + \left(\prod_{i=0}^{m-1}(r_1^i+1)\right) r_1^m].$$

This holds if, and only if, either $r_1^i = 0$ for all i or $r_0 + 1 = l$ and $r_1^i + 1 = p$ for $0 \le i \le m-1$. Hence $\mathrm{S}_q^r(\mathrm{E})$ is irreducible precisely when $r \le l-1$ or $r = lp^m(r_1^m+1) - 1$. Amongst these r there is a unique one satisfying the required conditions, and so we are done.

We record from the above proof the following fact.

Corollary 2.2 *For all $r \ge 0$, we have $\mathrm{S}_q^r(\mathrm{E})$ is irreducible if, and only if $r \le l-1$ or $r = lp^m(r_1^m+1) - 1$.*

We also use the results of [10] to prove the following lemma, which will be needed later.

Lemma 2.3 *If $\lambda_1 + \lambda_2 = 2s$ then we have*

$$\mathrm{Hom}_G(\Delta(s,s), \Delta(\lambda)) \cong \begin{cases} k & \text{if } \lambda_1 - \lambda_2 = 2(lp^m - 1) \text{ or } 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Since $\Delta(\lambda_1,\lambda_2) \cong \nabla^*(-\lambda_2,-\lambda_1)$, this will follow from

$$\nabla(s,s) \text{ occurs in } \mathrm{hd}\nabla(\lambda) \text{ if, and only if, } \lambda_1 - \lambda_2 = 2(lp^m-1) \text{ or } 0$$

once we have shown that $\nabla(\lambda)$ has a simple head. Clearly, it is enough to show this when $\lambda_2 = 0$, as then the result follows by tensoring up with an appropriate power of the q-determinant. Hence we will work with $\nabla(r,0)$. By the last proposition we have $1 \equiv \pm(r+1) \pmod{2l}$; that is $r = 2lm$ or $2lm-2$ for some m. We first find a c maximal in C, say $c_{\max} = (c_0, \ldots, c_m)$. From (2) we have $c_0 \in \{0,1\}$ unless $r < l-1$, in which case we must have $r = 0$.

By induction on t we have that if $r \ne 0$ then $c_t \in \{0,1\}$ for all $t \le m$. This follows as for $1 \le t \le m$ the first condition of (2) is clearly satisfied by 0 and 1, while the second gives $0 \le pc_{t+1} \le 2p - 2 + c_t - r_1^t \le 2p-1$, by induction. Hence $0 \le c_t < 2$, as claimed. Suppose $c_t = 1$. Then $0 \le pc_{t+1} \le p + (p-1) - r_1^t = p + \epsilon$ with $\epsilon \ge 0$. So $c_t = 1$ implies that c_{t+1} can equal 1 (for $t < m$). Hence $c_{\max}$ is unique, and is either 0 or $\mathbf{1}= (1,\ldots,1)$; which implies that $\nabla(r,0)$ has a simple head. The zero case corresponds to $r=0$.

Suppose that $c_{\max} = \mathbf{1}$, and let a be the highest weight in $c_{\max}$. Then (3) implies that $a_1^0 = l-1$, and $a_1^{1,t} = \begin{cases} p-1 & \text{if } t \le m-1 \\ r_1^m - 1 & \text{if } t = m. \end{cases}$ We require that

$$\begin{aligned} r &= 2a_1 = 2a_1^0 + \textstyle\sum_{t=0}^m 2a_1^{1,t} p^t l \\ &= 2l - 2 + 2l \left(\textstyle\sum_{t=0}^{m-1} (p^{t+1} - p^t) + p^m (r_1^m - 1) \right) \\ &= 2(l r_1^m p^m - 1). \end{aligned}$$

Thus $r_0 = l-2$, and $\sum_{t=0}^m r_1^t p^t = 2r_1^m p^m - 1$, which implies that $\sum_{t=0}^{m-1} r_1^t p^t = r_1^m p^m - 1$. This forces

$$r_1^t = \begin{cases} p-1 & \text{if } 0 \le t \le m-1 \\ 1 & \text{if } t = m \end{cases}$$

which gives $r = 2lp^m - 2$ as required.

3 Two short exact sequences

This section, largely based on results in [11], will produce two short exact sequences of G-modules which are essential to our later results. We will assume from this point on that $l > 1$. This is no great restriction as we aim to prove a result already known in the $l = 1$ case. We shall also fix some notation that shall be used henceforth.

We set $\lambda = (\lambda_1, \lambda_2) = (\mu + \delta, \delta)$, where $0 \le \mu \le l-2$, and put $|\lambda| = \lambda_1 + \lambda_2$. Then $\bar{\mu}$ is defined to be the unique integer such that $\mu + \bar{\mu} = l-2$. We also set $\rho = (1,0)$. Finally, we define $\tilde{\lambda} = (\bar{\mu} + \delta, \delta) + (\mu - l + 1)(1,1) = (\lambda_2 - 1, \lambda_1 + 1 - l)$. Note that $\tilde{\tilde{\lambda}} = \lambda - l(1,1)$.

Proposition 3.1 *i) For $n > 0$ there exists a (non-split) short exact sequence of G-modules:*

$$0 \to \nabla(\lambda) \otimes \nabla(n\rho)^{\mathrm{F}} \to \nabla(\lambda + ln\rho) \to \nabla(\tilde{\lambda} + l(1,1)) \otimes \nabla((n-1)\rho)^{\mathrm{F}} \to 0.$$

ii) There is an isomorphism of G-modules:

$$\nabla(ln - 1 + \delta, \delta) \cong \nabla(l - 1 + \delta, \delta) \otimes \nabla((n-1)\rho)^{\mathrm{F}}.$$

Proof: Part i): It is enough to show that we have the short exact sequence:

$$0 \to \nabla(\mu, 0) \otimes \nabla(n\rho)^{\mathrm{F}} \to \nabla(\mu + ln, 0) \to \nabla(l-1, \mu+1) \otimes \nabla((n-1)\rho)^{\mathrm{F}} \to 0$$

since the result follows on tensoring up with an appropriate power of the q-determinant. Now we use the isomorphism noted in [4, 3.7] of $\nabla(\mu, 0)$ with k-span$\{c_{11}^{r_1} c_{12}^{r_2} \mid r_1 + r_2 = \mu\}$. This gives the first injection via the multiplication map.

Now consider

$$\phi : \ \nabla(\mu + l, 0) \otimes \nabla(n - 1, 0)^{\mathrm{F}} \xrightarrow{\mathbf{m}} \nabla(\mu + nl, 0) \xrightarrow{\mathbf{p}} \frac{\nabla(\mu + nl, 0)}{\nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}}}$$

where $\mathbf{m}$ is multiplication of polynomials and $\mathbf{p}$ is the natural projection. We first show that ϕ is surjective. Let $c_{11}^a c_{12}^b + \nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}}$ be a non-zero element of $\nabla(\mu+nl, 0)/\nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}}$. Suppose $a = a_1 + la_2, \quad b = b_1 + lb_2$, where $0 \leq a_1, b_1 \leq l - 1$. Then $ln + \mu = a + b$ implies that $a_1 + b_1 = \mu$ or $l + \mu$. The former is impossible as then $c_{11}^a c_{12}^b + \nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}} = 0$. Hence $a_1 + b_1 = l + \mu$. Then, under ϕ, the element $c_{11}^{a_1} c_{12}^{b_1} \otimes c_{11}^{la_2} c_{12}^{lb_2} \in \nabla(\mu + l, 0) \otimes \nabla((n-1)\rho)^{\mathrm{F}}$ has image $c_{11}^a c_{12}^b + \nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}}$. Hence ϕ is surjective as claimed.

Clearly $\nabla(\mu, 0) \otimes \nabla(1, 0)^{\mathrm{F}} \otimes \nabla(n - 1, 0)^{\mathrm{F}} \subseteq \ker \phi$. But then

$$\frac{\nabla(\mu + l, 0) \otimes \nabla(n - 1, 0)^{\mathrm{F}}}{\nabla(\mu, 0) \otimes \nabla(1, 0)^{\mathrm{F}} \otimes \nabla(n - 1, 0)^{\mathrm{F}}} \cong \left(\frac{\nabla(\mu + l, 0)}{\nabla(\mu, 0) \otimes \nabla(1, 0)^{\mathrm{F}}} \right) \otimes \nabla(n - 1, 0)^{\mathrm{F}}$$

has dimension $n(\bar{\mu} + 1)$. Also

$$\dim \frac{\nabla(\mu + nl, 0)}{\nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}}} = n(\bar{\mu} + 1).$$

Hence $\ker \phi = \nabla(\mu, 0) \otimes \nabla(1, 0)^{\mathrm{F}} \otimes \nabla(n - 1, 0)^{\mathrm{F}}$. So

$$\begin{aligned} \operatorname{Im} \phi &\cong \frac{\nabla(\mu + nl, 0)}{\nabla(\mu, 0) \otimes \nabla(n, 0)^{\mathrm{F}}} \\ &\cong \frac{\nabla(\mu + l, 0) \otimes \nabla(n - 1, 0)^{\mathrm{F}}}{\nabla(\mu, 0) \otimes \nabla(1, 0)^{\mathrm{F}} \otimes \nabla(n - 1, 0)^{\mathrm{F}}} \cong \left(\frac{\nabla(\mu + l, 0)}{\nabla(\mu, 0) \otimes \nabla(1, 0)^{\mathrm{F}}} \right) \otimes \nabla(n - 1, 0)^{\mathrm{F}}. \end{aligned}$$

So it remains to show that

$$\frac{\nabla(\mu + l, 0)}{\nabla(\mu, 0) \otimes \nabla(1, 0)^{\mathrm{F}}} \cong \nabla(l - 1, \mu + 1) \quad \left[\cong \nabla(\bar{\mu}, 0) \otimes q\text{-}\det{}^{\mu+1} \right].$$

As the right-hand side is simple, it is enough to show that these have the same character, which is a straight-forward calculation.

Part ii): Injectivity as in i), and the result then follows by dimension.

We will need the following properties, shown in [5, 3.3–3,4], of the modules there denoted $Q(\lambda)$; which are certain tilting modules whose restrictions to G_1 are the injective envelopes of the corresponding simples. There $Q(\lambda)$ is defined to be $T(\tilde{\lambda}+l\rho)$, where this is the indecomposable tilting module of highest weight $\tilde{\lambda}+l\rho$. Further, from the character formula for the $Q(\lambda)$'s we obtain that $\mathrm{ch}Q(\lambda)=\chi(\lambda)+\chi(\tilde{\lambda}+l\rho)$. From [10] we see that $\mathrm{hd}\nabla(\tilde{\lambda}+l\rho)\cong L(\lambda)$ and $\mathrm{soc}\nabla(\tilde{\lambda}+l\rho)=\mathrm{rad}\nabla(\tilde{\lambda}+l\rho)\cong L(\tilde{\lambda}+l\rho)$. We also have that $\mathrm{soc}Q(\lambda)\cong L(\lambda)$ and $Q(\lambda)^*\cong Q(\lambda)\otimes q\text{-}\det^{-|\lambda|}$. Finally, we note that $Q(\lambda)\otimes\nabla(n,0)^{\mathrm{F}}$ has a good filtration.

Proposition 3.2 *For $n\geq 0$ there exists a non-split short exact sequence of G-modules:*

$$0\to\nabla(\lambda+ln\rho)\to Q(\lambda)\otimes\nabla(n\rho)^{\mathrm{F}}\to\nabla(\tilde{\lambda}+l(n+1)\rho)\to 0.$$

Proof: Using [5, 3.3(5)] we have that $Q(\lambda)\otimes\nabla(n\rho)^{\mathrm{F}}$ is indecomposable. So, as we have that $\mathrm{Ext}^1(\nabla(\alpha),\nabla(\beta))\neq 0$ implies that $\alpha>\beta$, it is enough to prove the above at the level of characters. We use induction on n. The case $n=0$ is clear from the remarks above, while $n=1$ follows by direct calculation.

For $n>1$ recall that

$$\begin{aligned}\mathrm{ch}\nabla(n,0)&=e(n,0)+\cdots+e(0,n)\\&=e(n,0)+e(0,n)+\mathrm{ch}\nabla(n-2,0)\chi(1,1).\end{aligned}$$

Hence we have

$$\mathrm{ch}\left(Q(\lambda)\otimes\nabla(n,0)^{\mathrm{F}}\right)=\mathrm{ch}\left(Q(\lambda)\otimes\nabla(n-2,0)^{\mathrm{F}}\right)\chi(l,l)+\mathrm{ch}Q(\lambda)(e(ln,0)+e(0,ln))$$

$$=\mathrm{ch}\nabla(\lambda+(n-2)l\rho)\chi(l,l)+\mathrm{ch}\nabla(\tilde{\lambda}+(n-1)l\rho)\chi(l,l)+\mathrm{ch}Q(\lambda)(e(ln,0)+e(0,ln))$$

$$=\sum_{i=0}^{\mu+(n-2)l}e(\mu+\delta+(n-1)l-i,\delta+l+i)+\sum_{i=0}^{2l-2-\mu+(n-2)l}e(ln-1-i+\delta,\mu+1+i+\delta)$$

$$+\sum_{i=0}^{\mu}(e(\mu+\delta-i+nl,\delta+i)+e(\mu+\delta-i,\delta+i+nl))$$

$$+\sum_{i=0}^{2l-2-\mu}(e(\delta+(n+1)l-1-i,\mu+\delta+1+i-l)+e(\delta+l-1-i,\mu+\delta+1+i+(n-1)l)).$$

Taking the second and third terms we get $\mathrm{ch}\nabla(\lambda+ln\rho)$, and the rest give $\mathrm{ch}\nabla(\tilde{\lambda}+l(n+1)\rho)$, so the result follows by induction.

After dualising, and tensoring with appropriate powers of the q-determinant, we may rewrite the last two propositions in terms of Δ's as:

Proposition 3.3 *i) For $n > 0$ there exists a (non-split) short exact sequence of G-modules:*

$$0 \to \Delta((n-1)\rho)^{\mathrm{F}} \otimes \Delta(\tilde{\lambda} + (l,l)) \to \Delta(\lambda + ln\rho) \to \Delta(n\rho)^{\mathrm{F}} \otimes \Delta(\lambda) \to 0.$$

ii) There is an isomorphism of G-modules:

$$\Delta(ln-1+\delta, \delta) \cong \Delta((n-1)\rho)^{\mathrm{F}} \otimes \Delta(l-1+\delta, \delta).$$

Proposition 3.4 *For $n \geq 0$ there exists a non-split short exact sequence of G-modules:*

$$0 \to \Delta(\tilde{\lambda} + l(n+1)\rho) \to \Delta(n\rho)^{\mathrm{F}} \otimes Q(\lambda) \to \Delta(\lambda + ln\rho) \to 0.$$

Corollary 3.5 *Considered as G_1-modules, the central term of the above sequence is the projective cover (respectively injective envelope) of the right (respectively left) term.*

Proof: As G_1-modules, the $Q(\lambda)$'s are projective by [5, 3.3(2)], and hence also injective (as $Q(\lambda)^* \cong Q(\lambda) \otimes q\text{-det}^{-|\lambda|}$). Thus $Q(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}$ is also both projective and injective. To show that $Q(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}$ is the projective cover, respectively injective envelope, of the appropriate module in the last proposition, it thus suffices to prove:

$$\begin{array}{rll} \text{i)} & \mathrm{hd}_{G_1}(Q(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}) & \cong \mathrm{hd}_{G_1}\Delta(\lambda + ln\rho). \\ \text{ii)} & \mathrm{soc}_{G_1}(Q(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}) & \cong \mathrm{soc}_{G_1}\Delta(\tilde{\lambda} + l(n+1)\rho). \end{array}$$

In both cases the previous proposition gives one inclusion.

Consider i): As $\Delta(n\rho)^{\mathrm{F}}$ has trivial G_1 action we have

$$\begin{aligned} \mathrm{hd}_{G_1}(Q(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}) &\cong \mathrm{hd}_{G_1}(Q(\lambda)) \otimes \Delta(n\rho)^{\mathrm{F}} \\ &\cong \hat{L}_1(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}} \\ &\cong L_1(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}} \end{aligned}$$

and

$$\begin{aligned} \mathrm{hd}_{G_1}\Delta(\lambda + ln\rho) &\geq \mathrm{hd}_{G_1}(\Delta(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}) \qquad \text{(by (1.1)(i))} \\ &\cong \mathrm{hd}_{G_1}(\Delta(\lambda)) \otimes \Delta(n\rho)^{\mathrm{F}} \\ &\cong L_1(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}. \end{aligned}$$

Consider ii). By a similar argument we have

$$\begin{aligned} \mathrm{soc}_{G_1}(Q(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}) &\cong \mathrm{soc}_{G_1}(Q(\lambda)) \otimes \Delta(n\rho)^{\mathrm{F}} \\ &\cong \hat{L}_1(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}} \\ &\cong L_1(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}} \end{aligned}$$

and

$$\begin{aligned} \mathrm{soc}_{G_1}\Delta(\tilde{\lambda} + l(n+1)\rho) &\cong \mathrm{soc}_{G_1}\Delta((\lambda_2 - 1, \lambda_1 - l + 1) + l(n+1)\rho) \\ &\geq \mathrm{soc}_{G_1}(\Delta(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}) \qquad \text{(by (1.1)(i))} \\ &\cong \mathrm{soc}_{G_1}(\Delta(\lambda)) \otimes \Delta(n\rho)^{\mathrm{F}} \\ &\cong L_1(\lambda) \otimes \Delta(n\rho)^{\mathrm{F}}. \end{aligned}$$

These give the reverse inclusions.

4 Calculations for G_1

If M is an indecomposable, non-projective G_1-module, we denote the kernel of the projective cover by $\Omega(M)$, and the cokernel of the injective hull by $\Omega^{-1}(M)$. We have $\Omega\Omega^{-1}(M) \cong M \cong \Omega^{-1}\Omega(M)$, and $\mathrm{Ext}^1_{G_1}(A,B) \cong \mathrm{Ext}^1_{G_1}(\Omega^{-1}A, \Omega^{-1}B)$ for arbitrary G_1-modules A,B. From (3.4), along with the remark that $\tilde{\tilde{\lambda}} = \lambda - (l,l)$, we can determine $\Omega^n(\Delta(\lambda))$. We obtain

$$\Omega^n(\Delta(\lambda)) \cong \begin{cases} \Delta(\lambda - \frac{nl}{2}(1,1) + nl\rho) & \text{if } n \text{ even} \\ \Delta(\tilde{\lambda} - \frac{(n-1)l}{2}(1,1) + nl\rho) & \text{if } n \text{ odd.} \end{cases} \tag{7}$$

Lemma 4.1 *For $m \geq n \geq 0$ we have*

$$\Omega^{-n}\Delta(\tilde{\lambda} + l(m+1)\rho) \cong \begin{cases} \Delta(\tilde{\lambda} + \frac{nl}{2}(1,1) + (m+1-n)l\rho) & \text{if } n \text{ even} \\ \Delta(\lambda + \frac{(n-1)l}{2}(1,1) + (m+1-n)l\rho) & \text{if } n \text{ odd.} \end{cases}$$

Proof: We have

$$\Delta(\tilde{\lambda} + l(m+1)\rho) \cong \begin{cases} \Omega^{m+1}\Delta(\lambda + \frac{ml}{2}(1,1)) & \text{if } m \text{ even} \\ \Omega^{m+1}\Delta(\tilde{\lambda} + \frac{(m+1)l}{2}(1,1)) & \text{if } m \text{ odd.} \end{cases}$$

So

$$\Omega^{-n}\Delta(\tilde{\lambda} + l(m+1)\rho) \cong \begin{cases} \Omega^{m+1-n}\Delta(\lambda + \frac{ml}{2}(1,1)) & \text{if } m \text{ even} \\ \Omega^{m+1-n}\Delta(\tilde{\lambda} + \frac{(m+1)l}{2}(1,1)) & \text{if } m \text{ odd.} \end{cases}$$

The result now follows from (7), replacing λ by $\tilde{\lambda}$ for the case m odd.

The rest of this section is devoted to calculating Hom_{G_1} and $\mathrm{Ext}^1_{G_1}$ between various Weyl modules, for use in the next section. We write $\cong_{G_1}$ for an isomorphism of G_1-modules, and use t to denote an integer.

Lemma 4.2 *For $n \geq 0$ we have*

$$\mathrm{Hom}_{G_1}(\Delta(\lambda + t(1,1)), \Delta(\tilde{\lambda} + l(n+1)\rho)) \cong \begin{cases} \left(\Delta(n\rho) \otimes q\text{-}\det^{-u}\right)^{\mathrm{F}} & \text{if } t \equiv 0 \pmod{l} \\ 0 & \text{otherwise} \end{cases}$$

where $lu = t$.

Proof: As $\Delta(\lambda + t(1,1))$ is simple, and (3.4) gives the injective envelopes, we have

$$\begin{aligned} &\mathrm{Hom}_{G_1}(\Delta(\lambda + t(1,1)), \Delta(\tilde{\lambda} + l(n+1)\rho)) \\ &\quad\cong \mathrm{Hom}_{G_1}(\Delta(\lambda + t(1,1)), \Delta(n\rho)^{\mathrm{F}} \otimes Q(\lambda)) \\ &\quad\cong \Delta(n\rho)^{\mathrm{F}} \otimes \mathrm{Hom}_{G_1}(\Delta(\lambda + t(1,1)), Q(\lambda)). \end{aligned}$$

Now $\Delta(\lambda+t(1,1)) \cong_{G_1} L_1(\lambda+t(1,1))$, and $\mathrm{soc}_{G_1} Q(\lambda) \cong_{G_1} L_1(\lambda)$. Writing $t = s + lu$ with $0 \le s < l$, we have:

$$L_1(\lambda+t(1,1)) \cong_{G_1} L_1(\lambda) \otimes q\text{-det}^t \cong_{G_1} L_1(\lambda) \otimes q\text{-det}^s \cong_{G_1} L_1(\lambda+s(1,1)).$$

Hence $L_1(\lambda) \cong_{G_1} L_1(\lambda+s(1,1))$ if, and only if, $s=0$. If $s=0$ then

$$\begin{aligned}
&\Delta(n\rho)^{\mathrm{F}} \otimes \mathrm{Hom}_{G_1}(\Delta(\lambda+t(1,1)), Q(\lambda)) \\
&\quad \cong \Delta(n\rho)^{\mathrm{F}} \otimes \mathrm{Hom}_{G_1}(q\text{-det}^t \otimes L(\lambda), Q(\lambda)) \\
&\quad \cong \Delta(n\rho)^{\mathrm{F}} \otimes \mathrm{Hom}_{G_1}(L(\lambda), Q(\lambda)) \otimes (q\text{-det}^{-u})^{\mathrm{F}} \\
&\quad \cong \left(\Delta(n\rho) \otimes q\text{-det}^{-u}\right)^{\mathrm{F}}
\end{aligned}$$

as required.

Lemma 4.3 *For $n \ge 0$ we have*

$$\mathrm{Hom}_{G_1}(\Delta(\lambda+t(1,1)), \Delta(\lambda+ln\rho))$$
$$\cong \begin{cases} (q\text{-det}^{-u})^{\mathrm{F}} & \text{if } n=0 \quad \text{and } t \equiv 0 \pmod{l} \\ (q\text{-det}^{-v} \otimes \Delta((n-1)\rho))^{\mathrm{F}} & \text{if } n \ge 1, \quad 2\mu = l-2 \quad \text{and } t \equiv \frac{l}{2} \pmod{l} \\ 0 & \text{otherwise} \end{cases}$$

where $lu = t$ and $lv = t - \frac{l}{2}$.

Proof: Suppose $n = 0$, and consider $\mathrm{Hom}_{G_1}(\Delta(\lambda+t(1,1)), \Delta(\lambda))$. Then for this to be non-zero we require $\Delta(\lambda+t(1,1)) \cong_{G_1} \mathrm{soc}_{G_1}\Delta(\lambda)$. That is $L_1(\lambda+t(1,1)) \cong_{G_1} L_1(\lambda)$. As in the previous lemma, this requires $t \equiv 0 \pmod{l}$, say $t = lu$. Then the rest follows as in the previous lemma.

Suppose $n \ge 1$. The injective envelope of $\Delta(\lambda+nl\rho)$ is $\Delta((n-1)\rho)^{\mathrm{F}} \otimes Q(\tau)$, where $\lambda = \tilde{\tau}$ by (3.4). This implies that $\tau = \tilde{\lambda} + l(1,1)$. Then as in the previous lemma we have

$$\begin{aligned}
&\mathrm{Hom}_{G_1}(\Delta(\lambda+t(1,1)), \Delta(\lambda+ln\rho)) \\
&\quad \cong \mathrm{Hom}_{G_1}(\Delta(\lambda+t(1,1)), \Delta((n-1)\rho)^{\mathrm{F}} \otimes Q(\tilde{\lambda}+l(1,1))) \\
&\quad \cong \Delta((n-1)\rho)^{\mathrm{F}} \otimes \mathrm{Hom}_{G_1}(\Delta(\lambda+t(1,1)), Q(\tilde{\lambda}+l(1,1))).
\end{aligned}$$

As before we require $L_1(\lambda+t(1,1)) \cong_{G_1} L_1(\tilde{\lambda}+l(1,1))$. That is $L_1(\mu,0) \otimes q\text{-det}^{\lambda_2+t} \cong_{G_1} L_1(l-2-\mu,0) \otimes q\text{-det}^{\lambda_1+1}$. This holds if, and only if, $l-2 = 2\mu$ and $\lambda_2 + t \equiv \lambda_1 + 1 \pmod{l}$. When these conditions hold, set $lv = \lambda_2 + t - \lambda_1 - 1 = t - \mu - 1$, and then as before we obtain the required result.

Lemma 4.4 *We have*

$$\mathrm{Ext}^1_{G_1}(\Delta(\lambda+t(1,1)),\Delta(\tilde{\lambda}))\cong \mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda}+t(1,1)+l\rho),\Delta(\tilde{\lambda}))$$
$$\cong\begin{cases}\left(q\text{-det}^{-u}\otimes\Delta^*(\rho)\right)^{\mathrm{F}} & \text{if } t\equiv 0 \pmod{l}\\ 0 & \text{otherwise}\end{cases}$$

and

$$\mathrm{Ext}^1_{G_1}(\Delta(\lambda+t(1,1)),\Delta(\lambda))\cong \mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda}+t(1,1)+l\rho),\Delta(\lambda))$$
$$\cong\begin{cases}\left(q\text{-det}^{-v}\otimes\Delta^*(\rho)\right)^{\mathrm{F}} & \text{if } 2\mu=l-2 \quad\text{and } t\equiv \frac{l}{2} \pmod{l}\\ 0 & \text{otherwise}\end{cases}$$

where $lu=t$ and $lv=t-\frac{l}{2}$.

Proof: Applying $\mathrm{Hom}_{G_1}(-,\Delta(\tau))$ to the sequence in (3.4) gives:

$$0\to\mathrm{Hom}_{G_1}(\Delta(\lambda+t(1,1)),\Delta(\tau))\to\mathrm{Hom}_{G_1}(Q(\lambda+t(1,1)),\Delta(\tau))$$

$$\to\mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda}+l\rho+t(1,1)),\Delta(\tau))\to\mathrm{Ext}^1_{G_1}(\Delta(\lambda+t(1,1)),\Delta(\tau))\to 0.$$

Taking $\tau=\lambda$ or $\tau=\tilde{\lambda}$ we have that the first two terms are isomorphic, and hence the last two are. We have

$$0\to\Delta(\tilde{\lambda}+(t+l)(1,1))\to\Delta(\tilde{\lambda}+l\rho+t(1,1))\to\Delta(\rho)^{\mathrm{F}}\otimes\Delta(\tilde{\lambda}+t(1,1))\to 0$$

and this restricts to a Loewy series, as G_1-modules, for $\Delta(\tilde{\lambda}+l\rho+t(1,1))$; so

$$\begin{aligned}\mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda}+l\rho+t(1,1)),\Delta(\tau)) &\cong \mathrm{Hom}_{G_1}(\Delta(\rho)^{\mathrm{F}}\otimes\Delta(\tilde{\lambda}+t(1,1)),\Delta(\tau))\\ &\cong \mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda}+t(1,1)),\Delta(\tau))\otimes\Delta^*(\rho)^{\mathrm{F}}.\end{aligned}$$

Applying (4.3) with $\tau=\tilde{\lambda}$ gives the first result. For the second, take $\tau=\lambda$ and then the right-hand side above becomes

$$\mathrm{Hom}_{G_1}(\Delta(\tilde{\lambda}+t(1,1)),\Delta(\lambda))\otimes\Delta^*(\rho)^{\mathrm{F}}\cong\mathrm{Hom}_{G_1}(L_1(\tilde{\lambda})\otimes q\text{-det}^t,L_1(\lambda))\otimes\Delta^*(\rho)^{\mathrm{F}}.$$

For this to be non-zero we must have $\mu=\bar{\mu}$, that is $2\mu=l-2$, which implies that $L_1(\tilde{\lambda})\otimes q\text{-det}^t\cong L_1(\lambda)\otimes q\text{-det}^{t+l-1-\mu}$ which gives the rest of the condition, and the result.

Lemma 4.5 *For $m \geq n \geq 0$ we have*

$$\operatorname{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t(1,1)), \Delta(\lambda + lm\rho))$$

$$\cong \begin{cases} \operatorname{Ext}^1_{G_1}(\Delta(\lambda + \left(t + \frac{nl}{2}\right)(1,1)), \Delta(\lambda + \frac{nl}{2}(1,1) + (m-n)l\rho)) & \text{if } n \text{ even} \\ \operatorname{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + \left(t + \frac{(n+1)l}{2}\right)(1,1)), \Delta(\tilde{\lambda} + \frac{(n+1)l}{2}(1,1) + (m-n)l\rho)) & \text{if } n \text{ odd.} \end{cases}$$

$$\operatorname{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t(1,1)), \Delta(\tilde{\lambda} + lm\rho))$$

$$\cong \begin{cases} \operatorname{Ext}^1_{G_1}(\Delta(\lambda + \left(t + \frac{nl}{2}\right)(1,1)), \Delta(\tilde{\lambda} + \frac{nl}{2}(1,1) + (m-n)l\rho)) & \text{if } n \text{ even} \\ \operatorname{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + \left(t + \frac{(n+1)l}{2}\right)(1,1)), \Delta(\lambda + \frac{(n-1)l}{2}(1,1) + (m-n)l\rho)) & \text{if } n \text{ odd.} \end{cases}$$

Proof: Writing τ for λ or $\tilde{\lambda}$, we have

$$\begin{aligned} &\operatorname{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t(1,1)), \Delta(\tau + lm\rho)) \\ &\quad \cong \operatorname{Ext}^1_{G_1}(\Omega^{-n}\Delta(\lambda + ln\rho + t(1,1)), \Omega^{-n}\Delta(\tau + lm\rho)) \\ &\quad \cong \begin{cases} \operatorname{Ext}^1_{G_1}(\Delta(\lambda + \left(t + \frac{nl}{2}\right)(1,1)), \Omega^{m-n}\Delta(\tau + \frac{ml}{2}(1,1)) & \text{if } m, n \text{ even} \\ \operatorname{Ext}^1_{G_1}(\Delta(\lambda + \left(t + \frac{nl}{2}\right)(1,1)), \Omega^{m-n}\Delta(\tilde{\tau} + \frac{(m+1)l}{2}(1,1)) & \text{if } m \text{ odd}, n \text{ even} \\ \operatorname{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + \left(t + \frac{(n+1)l}{2}\right)(1,1)), \Omega^{m-n}\Delta(\tau + \frac{ml}{2}(1,1)) & \text{if } m \text{ even}, n \text{ odd} \\ \operatorname{Ext}^1_{G_1}(\Delta(\tilde{\lambda} + \left(t + \frac{(n+1)l}{2}\right)(1,1)), \Omega^{m-n}\Delta(\tilde{\tau} + \frac{(m+1)l}{2}(1,1)) & \text{if } m, n \text{ odd} \end{cases} \end{aligned}$$

using the results of Lemma (4.1). The result now follows using (7).

Lemma 4.6 *For $n \geq 0$ we have*

$$\operatorname{Ext}^1_{G_1}(\Delta(\lambda + t(1,1)), \Delta(\tilde{\lambda} + ln\rho))$$

$$\cong \begin{cases} \left(q\text{-det}^{-\alpha} \otimes \Delta^*(\rho)\right)^{\mathrm{F}} & \text{if } n = 0 \quad \text{and } t \equiv 0 \pmod{l} \\ \left(q\text{-det}^{-\alpha}\right)^{\mathrm{F}} & \text{if } n = 1 \quad \text{and } t \equiv 0 \pmod{l} \\ \left(q\text{-det}^{-\beta} \otimes \Delta((n-2)\rho)\right)^{\mathrm{F}} & \text{if } n \geq 2, \quad 2\mu = l - 2 \quad \text{and } t \equiv \frac{l}{2} \pmod{l} \\ 0 & \text{otherwise} \end{cases}$$

where $l\alpha = t$, $l\beta = t - \frac{l}{2}$.

Proof: The case $n = 0$ is done in (4.4). For $n \geq 1$ apply $\operatorname{Hom}_{G_1}(\Delta(\lambda + t(1,1)), -)$ to

$$0 \to \Delta(\tilde{\lambda} + ln\rho) \to \Delta((n-1)\rho)^{\mathrm{F}} \otimes Q(\lambda) \to \Delta(\lambda + l(n-1)\rho) \to 0$$

to obtain

$$0 \to \operatorname{Hom}_{G_1}(\Delta(\lambda + (t,t)), \Delta(\tilde{\lambda} + ln\rho)) \to \operatorname{Hom}_{G_1}(\Delta(\lambda + (t,t)), \Delta((n-1)\rho)^{\mathrm{F}} \otimes Q(\lambda))$$

$$\to \operatorname{Hom}_{G_1}(\Delta(\lambda + (t,t)), \Delta(\lambda + l(n-1)\rho)) \to \operatorname{Ext}^1_{G_1}(\Delta(\lambda + (t,t)), \Delta(\tilde{\lambda} + ln\rho)) \to 0.$$

As in earlier lemmas, the first two terms are isomorphic. Hence the next two are, and the result follows from (4.3).

Lemma 4.7 *For $n \geq 0$ we have*

$$\mathrm{Ext}^1_{G_1}(\Delta(\lambda + t(1,1)), \Delta(\lambda + ln\rho))$$

$$\cong \begin{cases} \left(q\text{-det}^{-\alpha} \otimes \Delta^*(\rho)\right)^{\mathrm{F}} & \text{if } n = 0, \quad 2\mu = l-2 \quad \text{and } t \equiv \frac{l}{2} \pmod{l} \\ \left(q\text{-det}^{-\alpha}\right)^{\mathrm{F}} & \text{if } n = 1, \quad 2\mu = l-2 \quad \text{and } t \equiv \frac{l}{2} \pmod{l} \\ \left(q\text{-det}^{-\gamma} \otimes \Delta((n-2)\rho)\right)^{\mathrm{F}} & \text{if } n \geq 2 \quad \text{and } t \equiv 0 \pmod{l} \\ 0 & \text{otherwise} \end{cases}$$

where $l\alpha = t - \frac{l}{2}$, $l\gamma = t - l$.

Proof: The case $n = 0$ is done in (4.4). For $n \geq 1$ apply $\mathrm{Hom}_{G_1}(\Delta(\lambda + t(1,1)), -)$ to

$$0 \to \Delta(\lambda + lnp) \to \Delta((n-1)\rho)^{\mathrm{F}} \otimes Q(\tilde{\lambda} + l(1,1)) \to \Delta(\tilde{\lambda} + l(1,1) + l(n-1)\rho) \to 0.$$

As in the previous lemma, the first two terms are isomorphic, and hence the next two are also; that is

$$\mathrm{Hom}_{G_1}(\Delta(\lambda + t(1,1)), \Delta(\tilde{\lambda} + (l,l) + l(n-1)\rho)) \cong \mathrm{Ext}^1_{G_1}(\Delta(\lambda + t(1,1)), \Delta(\lambda + ln\rho)).$$

For the case $n \geq 2$ write $\lambda' = \lambda + l(1,1)$ and $t' = t - l$. Then the left-hand side equals $\mathrm{Hom}_{G_1}(\Delta(\lambda' + t'(1,1)), \Delta(\tilde{\lambda'} + l(n-1)\rho))$, and the result follows from (4.2). For the case $n = 1$ consider $\mathrm{Hom}_{G_1}(\Delta(\lambda + t(1,1)), \Delta(\tilde{\lambda} + l(1,1))$. This is clearly zero unless $\mu = \bar{\mu}$, in which case it is isomorphic to $\mathrm{Hom}_{G_1}(\Delta(\lambda + t(1,1)), \Delta(\lambda + \frac{l}{2}(1,1)))$, when the result follows from (4.3).

For the next two lemmas, it is necessary to restrict to a specific value of t. However, as this condition will always hold in the cases of interest, this is of no great consequence.

Lemma 4.8 *For $m \geq n \geq 0$ and $t = \frac{l}{2}(m-n)$ we have*

$$\mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t(1,1)), \Delta(\tilde{\lambda} + l(m+1)\rho))$$

$$\cong \begin{cases} \left(q\text{-det}^{-u} \otimes \Delta(m\rho) \otimes \Delta^*(n\rho)\right)^{\mathrm{F}} & \text{if } t \equiv 0 \pmod{l} \\ 0 & \text{otherwise} \end{cases}$$

where $lu = t$.

Proof: Applying $\mathrm{Hom}_{G_1}(-, \Delta(\tilde{\lambda}+l(m+1)\rho))$ to (3.3(i)) we obtain

$$\begin{array}{rl} 0 & \to \mathrm{Hom}_{G_1}(\Delta(n\rho)^{\mathrm{F}} \otimes \Delta(\lambda+t(1,1)), \Delta(\tilde{\lambda}+l(m+1)\rho)) \\ & \to \mathrm{Hom}_{G_1}(\Delta(\lambda+ln\rho+t(1,1)), \Delta(\tilde{\lambda}+l(m+1)\rho)) \\ & \to \mathrm{Hom}_{G_1}(\Delta((n-1)\rho)^{\mathrm{F}} \otimes \Delta(\tilde{\lambda}+(l+t)(1,1)), \Delta(\tilde{\lambda}+l(m+1)\rho)) \\ & \to \mathrm{Ext}^1_{G_1}(\Delta(n\rho)^{\mathrm{F}} \otimes \Delta(\lambda+t(1,1)), \Delta(\tilde{\lambda}+l(m+1)\rho)) \\ & \to \mathrm{Ext}^1_{G_1}(\Delta(\lambda+ln\rho+t(1,1)), \Delta(\tilde{\lambda}+l(m+1)\rho)). \end{array}$$

We claim that the first two terms are isomorphic. With this we are done, as the first term is isomorphic to $\mathrm{Hom}_{G_1}(\Delta(\lambda+t(1,1)), \Delta(\tilde{\lambda}+l(m+1)\rho)) \otimes \Delta^*(n\rho)^{\mathrm{F}}$ and hence the result follows from (4.2).

Proof of the claim: Consider the third term. Setting $\lambda' = \tilde{\lambda}$ and $t' = t+l$, this is isomorphic to $\mathrm{Hom}_{G_1}(\Delta(\lambda'+t'(1,1)), \Delta(\lambda'+l(m+1)\rho)) \otimes \Delta^*((n-1)\rho)^{\mathrm{F}}$. By (4.3), this is zero unless $2\mu' = l-2$ and $t' \equiv \frac{l}{2} \pmod{l}$; that is $2\mu = l-2$ and $t \equiv \frac{l}{2} \pmod{l}$. If non-zero it has dimension $(m+1)n$. If this is zero we are done, so we may assume that $2\mu = l-2$, $t \equiv \frac{l}{2} \pmod{l}$. Hence $m-n$ is odd, so $m \geq 1$.

Term four is isomorphic to $\mathrm{Ext}^1_{G_1}(\Delta(\lambda+t(1,1)), \Delta(\tilde{\lambda}+l(m+1)\rho)) \otimes \Delta^*(n\rho)^{\mathrm{F}}$, which is isomorphic to $(q\text{-det}^{-\beta} \otimes \Delta((m-1)\rho)) \otimes \Delta^*(n\rho)^{\mathrm{F}}$ by (4.6). By (4.5) term five is isomorphic to

$$\begin{cases} \mathrm{Ext}^1_{G_1}(\Delta(\lambda+(t+\frac{nl}{2})(1,1)), \Delta(\tilde{\lambda}+\frac{nl}{2}(1,1)+(m+1-n)l\rho)) & \text{if } n \text{ even} \\ \mathrm{Ext}^1_{G_1}(\Delta(\tilde{\lambda}+(t+\frac{(n+1)l}{2})(1,1)), \Delta(\lambda+\frac{(n-1)l}{2}(1,1)+(m+1-n)l\rho)) & \text{if } n \text{ odd.} \end{cases}$$

For appropriate λ''s, both cases are isomorphic to

$$\mathrm{Ext}^1_{G_1}(\Delta(\lambda'+t(1,1)), \Delta(\tilde{\lambda}'+(m+1-n)l\rho)) \cong (q\text{-det}^{-\beta} \otimes \Delta((m-n-1)\rho))^{\mathrm{F}}$$

by (4.6), as $m+1-n \geq 2$ (since $m-n$ is odd). So the fourth and fifth terms have dimension $m(n+1)$ and $m-n$ respectively. Thus the dimension of the fourth term is the sum of the dimensions of the terms on either side; hence the map into it must be injective. This implies that the first two terms are isomorphic as required.

Lemma 4.9 *For $m > n \geq 0$ and $t = \frac{l}{2}(m-n)$ we have*

$$\mathrm{Hom}_{G_1}(\Delta(\lambda+ln\rho+t(1,1)), \Delta(\lambda+lm\rho))$$

$$\cong \begin{cases} \left(q\text{-det}^{-v} \otimes \Delta((m-1)\rho) \otimes \Delta^*(n\rho)\right)^{\mathrm{F}} & \text{if } 2\mu = l-2 \quad \text{and } t \equiv \frac{l}{2} \pmod{l} \\ 0 & \text{otherwise} \end{cases}$$

where $lv = t - \frac{l}{2}$.

Proof: If $2\mu = l-2$ then

$$\begin{aligned}&\mathrm{Hom}_{G_1}(\Delta(\lambda+ln\rho+t(1,1)),\Delta(\lambda+lm\rho))\\ &\quad\cong \mathrm{Hom}_{G_1}(\Delta(\lambda+ln\rho+t(1,1)),\Delta(\tilde{\lambda}+\tfrac{l}{2}(1,1)+lm\rho))\\ &\quad\cong \mathrm{Hom}_{G_1}(\Delta(\lambda'+ln\rho+\left(t-\tfrac{l}{2}\right)(1,1)),\Delta(\tilde{\lambda}'+lm\rho))\end{aligned}$$

where $\lambda'=\lambda+\frac{l}{2}(1,1)$, and the result follows from the previous lemma. So we may assume that $\mu\neq\bar{\mu}$. Applying $\mathrm{Hom}_{G_1}(\Delta(\lambda+ln\rho+t(1,1)),-)$ to (3.4) we obtain

$$\begin{aligned}0\ &\to \mathrm{Hom}_{G_1}(\Delta(\lambda+ln\rho+t(1,1)),\Delta(\tilde{\lambda}+l(m+1)\rho))\\ &\to \mathrm{Hom}_{G_1}(\Delta(\lambda+ln\rho+t(1,1)),\Delta(m\rho)^{\mathrm{F}}\otimes Q(\lambda))\\ &\to \mathrm{Hom}_{G_1}(\Delta(\lambda+ln\rho+t(1,1)),\Delta(\lambda+lm\rho))\\ &\to \mathrm{Ext}^1_{G_1}(\Delta(\lambda+ln\rho+t(1,1)),\Delta(\tilde{\lambda}+l(m+1)\rho))\to 0.\end{aligned}$$

As $\mu\neq\bar{\mu}$, any map $\Delta(\lambda+ln\rho+t(1,1))\to\Delta(m\rho)^{\mathrm{F}}\otimes Q(\lambda)$ has image in the socle. Hence the first two terms are isomorphic; and so the next two are also. By (4.5) we then have

$$\begin{aligned}&\mathrm{Hom}_{G_1}(\Delta(\lambda+ln\rho+t(1,1)),\Delta(\lambda+lm\rho))\\ &\cong\begin{cases}\mathrm{Ext}^1_{G_1}(\Delta(\lambda+\left(t+\frac{ln}{2}\right)(1,1)),\Delta(\tilde{\lambda}+\frac{ln}{2}(1,1)+(m+1-n)l\rho)) & \text{if } n \text{ even}\\ \mathrm{Ext}^1_{G_1}(\Delta(\tilde{\lambda}+\left(t+\frac{l(n+1)}{2}\right)(1,1)),\Delta(\lambda+\frac{l(n-1)}{2}(1,1)+(m+1-n)l\rho)) & \text{if } n \text{ odd.}\end{cases}\end{aligned}$$

Let $\lambda'=\lambda+\frac{nl}{2}(1,1)$ (respectively $(\lambda+\widetilde{\frac{(n+1)l}{2}(1,1)})$) for n even (respectively n odd). Then in both cases this is isomorphic to $\mathrm{Ext}^1_{G_1}(\Delta(\lambda'+t(1,1)),\Delta(\tilde{\lambda}'+l(m+1-n)\rho))$. Repeating the argument above, with $n=0$, $m=m-n$, this is isomorphic to $\mathrm{Hom}_{G_1}(\Delta(\lambda'+t(1,1)),\Delta(\lambda'+l(m-n)\rho))$ and now (as $\mu\neq\bar{\mu}$) the result follows from (4.3).

Lemma 4.10 *For $m,n\geq 1$ we have*

$$\begin{aligned}&\mathrm{Hom}_{G_1}(\Delta(ln-1+\delta+t,\delta+t),\Delta(lm-1+\delta,\delta))\\ &\cong\begin{cases}\left(q\text{-}\det^{-u}\otimes\Delta((m-1)\rho)\otimes\Delta^*((n-1)\rho)\right)^{\mathrm{F}} & \text{if } t\equiv 0 \pmod{l}\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

where $lu=t$.

Proof: By (3.3)(ii) applied twice we have

$$\begin{aligned}&\mathrm{Hom}_{G_1}(\Delta(ln-1+\delta+t,\delta+t),\Delta(lm-1+\delta,\delta))\\ &\quad\cong \mathrm{Hom}_{G_1}(\Delta((n-1)\rho)^{\mathrm{F}}\otimes\Delta(l-1+\delta+t,\delta+t),\Delta((m-1)\rho)^{\mathrm{F}}\otimes\Delta(l-1+\delta,\delta))\\ &\quad\cong \Delta((m-1)\rho)^{\mathrm{F}}\otimes\mathrm{Hom}_{G_1}(\Delta(l-1+\delta+t,\delta+t),\Delta(l-1+\delta,\delta))\otimes\Delta^*((n-1)\rho)^{\mathrm{F}}.\end{aligned}$$

Let $\tau=(l-1+\delta,\delta)$; then

$$\begin{aligned}&\mathrm{Hom}_{G_1}(\Delta((l-1)\rho+(\delta+t)(1,1)),\Delta(l-1+\delta,\delta))\\ &\quad=\mathrm{Hom}_{G_1}(\Delta(\tau+t(1,1)),\Delta(\tau))\cong\mathrm{Hom}_{G_1}(L_1(\tau+t(1,1)),L_1(\tau))\end{aligned}$$

and the result now clearly follows.

5 Ext^1_G for Weyl Modules

In this section we calculate $\mathrm{Ext}^1_G(\Delta,\Delta')$ for all possible Δ,Δ''s. This uses the results of the previous section, along with the Lyndon-Hochschild-Serre spectral sequence (see [4, 1.6]), which gives rise to the five term exact sequence

$$0 \to H^1(\bar{G}, V^{G_1}) \to H^1(G,V) \to H^1(G_1,V)^{\bar{G}} \to H^2(\bar{G}, V^{G_1}) \to H^2(G,V)$$

which will form the basis of the calculations in this section.

Consider $k[G]$ with the usual generators c_{ij}, and $k[\mathrm{GL}_2]$ with generators $\bar{c}_{ij}$. There is an isomorphism from GL_2 to $\bar{G}$ via the map $c^l_{ij} \longmapsto \bar{c}_{ij}$ (see [5, 3.2]). This gives rise to the following isomorphism:

$$H^i(\bar{G}, V^{\mathrm{F}}) \cong H^i(\mathrm{GL}_2, V).$$

This will allow us to use the existing result in [7] for the classical case.

Lemma 5.1 *For all λ,λ' such that $0 \le \mu$,$\mu' \le l-1$ we have*

$$\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\lambda')) = 0.$$

Proof: This is clear, as for all λ' such that $0 \le \mu' \le l-1$ we have

$$\Delta(\lambda') \cong L(\lambda') \cong \nabla(\lambda')$$

and by [4, 4(2)] we have that

$$\mathrm{Ext}^1_G(\Delta(\lambda), \nabla(\lambda')) = 0.$$

In the rest of this section we will frequently make use of the fact that $\mathrm{Ext}^1_{\mathrm{GL}_2}$ can be easily determined from $\mathrm{Ext}^1_{\mathrm{SL}_2}$. To be more precise, $\mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta(\alpha), \Delta(\beta)) = \mathrm{Ext}^1_{\mathrm{SL}_2}(\Delta(\alpha_1 - \alpha_2), \Delta(\beta_1 - \beta_2))$ provided that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$; else it is zero.

Lemma 5.2 *For n,$m > 0$ we have*

$$\begin{aligned} &\mathrm{Ext}^1_G(\Delta(ln-1+\delta+t, \delta+t), \Delta(lm-1+\delta, \delta)) \\ \cong &\begin{cases} \mathrm{Ext}^1_{\mathrm{SL}_2}(\Delta(n-1), \Delta(m-1)) & \text{if } m-n \text{ even and } t = \frac{l}{2}(m-n) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof: We may assume that $1 \leq n < m$. Set $V = \Delta(lm-1+\delta,\delta)\otimes\Delta^*(ln-1+\delta+t,\delta+t)$. Then we have

$$0 \to H^1(\bar{G}, V^{G_1}) \to H^1(G,V) \to H^1(G_1,V)^{\bar{G}}.$$

The third term is isomorphic to $\mathrm{Ext}^1_{G_1}(\Delta(ln-1+\delta+t,\delta+t),\Delta(lm-1+\delta,\delta))^{\bar{G}}$ which equals zero by (3.3(ii)) applied twice and (5.1). Hence the first two terms must be isomorphic. Now, by (4.10)

$$\begin{aligned} V^{G_1} &\cong \mathrm{Hom}_{G_1}(\Delta(ln-1+\delta+t,\delta+t),\Delta(lm-1+\delta,\delta)) \\ &\cong \begin{cases} (\Delta((m-1)\rho)\otimes\Delta^*((n-1)\rho))^{\mathrm{F}}\otimes q\text{-}\mathrm{det}^t & \text{if } m-n \text{ even} \\ 0 & \text{otherwise} \end{cases} \\ &\cong \begin{cases} (\Delta(m-1-t',-t')\otimes\Delta^*((n-1)\rho))^{\mathrm{F}} & \text{if } \quad m-n \text{ even} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $lt' = t$. So

$$\begin{aligned} &\mathrm{Ext}^1_G(\Delta(ln-1+\delta+t,\delta+t),\Delta(lm-1+\delta,\delta)) \\ &\quad\cong \begin{cases} \mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta((n-1)\rho),\Delta((m-1)\rho-t'(1,1))) & \text{if } m-n \text{ even} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

which, by the remark above, implies the result.

Lemma 5.3 *For $0 \leq n < m$ we have*

$$\mathrm{Ext}^1_G(\Delta(\lambda+ln\rho+t(1,1)),\Delta(\tilde{\lambda}+lm\rho))$$

$$\cong \begin{cases} k & \text{if } m-n = 2p^\alpha, \quad \alpha \geq 0, \quad 2\mu = l-2 \text{ and } t = \frac{l}{2}(m-n-1) \\ k & \text{if } m-n = 1 \text{ and } t = \frac{l}{2}(m-n-1) \\ \mathrm{Ext}^1_{\mathrm{SL}_2}(\Delta(n),\Delta(m-1)) & \text{if } m-n \text{ odd, } m-n \neq 1 \text{ and } t = \frac{l}{2}(m-n-1) \\ 0 & \text{otherwise.} \end{cases}$$

Proof: First note that in the first three cases t is an integer, as required. Let $V = \Delta(\tilde{\lambda}+lm\rho)\otimes\Delta^*(\lambda+ln\rho+t(1,1))$. Now

$$\begin{aligned} V^{G_1} &\cong \mathrm{Hom}_{G_1}(\Delta(\lambda+ln\rho+t(1,1)),\Delta(\tilde{\lambda}+lm\rho)) \\ &\cong \begin{cases} (q\text{-}\mathrm{det}^{-u}\otimes\Delta((m-1)\rho)\otimes\Delta^*(n\rho))^{\mathrm{F}} & \text{if } m-n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $lu = t$, by (4.8). By (4.5) we have

$$\begin{aligned} H^1(G_1,V) &\cong \mathrm{Ext}^1_{G_1}(\Delta(\lambda+ln\rho+t(1,1)),\Delta(\tilde{\lambda}+lm\rho)) \\ &\cong \mathrm{Ext}^1_{G_1}(\Delta(\lambda'+t(1,1)),\Delta(\tilde{\lambda'}+(m-n)l\rho)) \end{aligned}$$

where $\lambda' = \begin{cases} \lambda + \frac{nl}{2}(1,1) & \text{if } n \text{ even} \\ \tilde{\lambda} + \frac{(n+1)l}{2}(1,1) & \text{if } n \text{ odd.} \end{cases}$ Now by (4.6) this is isomorphic to

$$\begin{cases} k & \text{if } m-n=1 \\ (q\text{-det}^{-\beta} \otimes \Delta((m-n-2)\rho))^{\mathrm{F}} & \text{if } m-n \geq 2, \quad 2\mu = l-2 \quad \text{and } m-n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

where $\beta = \frac{1}{2}(m-n-2)$.

Consider the five term exact sequence. If $m-n$ is even then the first and fourth terms are zero by above. Hence

$$\begin{aligned} H^1(G,V) &\cong H^1(G_1,V)^{\bar{G}} \\ &\cong \begin{cases} \left(\Delta\left(\frac{m-n-2}{2}(1,-1)\right)^{\mathrm{F}}\right)^{\bar{G}} & \text{if } 2\mu = l-2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now the first case is isomorphic to $\mathrm{Hom}_{\mathrm{GL}_2}\left(\Delta(0), \Delta\left(\frac{m-n-2}{2}(1,-1)\right)\right)$ which, by (2.3) with $l=1$, is isomorphic to

$$\begin{cases} k & \text{if } m-n-2 = 2(p^\alpha - 1), \quad \alpha \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

which gives the result for $m-n$ even. If $m-n$ is odd, and $m-n \neq 1$, then $H^1(G_1,V)$ (and hence $H^1(G_1,V)^{\bar{G}}$)$=0$. Hence

$$\begin{aligned} H^1(G,V) &\cong H^1(\bar{G}, V^{G_1}) \\ &\cong \mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta(n\rho - u(1,1)), \Delta((m-1)\rho)) \\ &\cong \mathrm{Ext}^1_{\mathrm{SL}_2}(\Delta(n), \Delta(m-1)). \end{aligned}$$

If $m = n+1$ then $V^{G_1} \cong (\Delta(n\rho) \otimes \Delta^*(n\rho))^{\mathrm{F}}$. Now for $i > 0$,

$$H^i(\bar{G}, V^{G_1}) \cong \mathrm{Ext}^i_{\mathrm{GL}_2}(\Delta(n\rho), \Delta(n\rho)) = 0.$$

Thus $H^1(G,V) \cong H^1(G_1,V)^{\bar{G}} \cong k$, and we are done.

Lemma 5.4 *For $0 \leq n < m$ we have*

$$\mathrm{Ext}^1_G(\Delta(\lambda + ln\rho + t(1,1)), \Delta(\lambda + lm\rho))$$

$$\cong \begin{cases} k & \text{if } m-n = 2p^\alpha, \quad \alpha \geq 0 \quad \text{and } t = \frac{l}{2}(m-n) \\ k & \text{if } m-n=1,\ 2\mu = l-2 \text{ and } t = \frac{l}{2}(m-n) \\ \mathrm{Ext}^1_{\mathrm{SL}_2}(\Delta(n), \Delta(m-1)) & \text{if } m-n \text{ odd},\ m-n \neq 1,\ 2\mu = l-2 \text{ and } t = \frac{l}{2}(m-n) \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Again note that in the first three cases t is an integer as required. Let $V = \Delta(\lambda + lm\rho) \otimes \Delta^*(\lambda + ln\rho + t(1,1))$. Now

$$\begin{aligned} V^{G_1} &\cong \mathrm{Hom}_{G_1}(\Delta(\lambda + ln\rho + t(1,1)), \Delta(\lambda + lm\rho)) \\ &\cong \begin{cases} (q\text{-det}^{-v} \otimes \Delta((m-1\rho) \otimes \Delta^*(n\rho))^{\mathrm{F}} & \text{if } 2\mu = l-2 \quad \text{and } m-n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

by (4.9). First consider the case when this is zero. Then by the five term exact sequence we must have $H^1(G,V) \cong H^1(G_1,V)^{\bar{G}}$.

$$\begin{aligned} H^1(G_1,V) &\cong \mathrm{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t(1,1)), \Delta(\lambda + lm\rho)) \\ &\cong \mathrm{Ext}^1_{G_1}(\Delta(\lambda' + \tfrac{l}{2}(m-n)(1,1)), \Delta(\lambda' + (m-n)l\rho)) \end{aligned}$$

where $\lambda' = \lambda + \frac{nl}{2}(1,1)$ (respectively $\tilde{\lambda} + \frac{(n+1)l}{2}(1,1)$) for n even (respectively n odd), by (4.5). This, by (4.7), is isomorphic to

$$\begin{cases} \left(q\text{-det}^{-\alpha}\right)^{\mathrm{F}} & \text{if } m = n+1 \quad \text{and } t \equiv \mu' + 1 \pmod{l} \\ (q\text{-det}^{-\beta} \otimes \Delta((m-n-2)\rho))^{\mathrm{F}} & \text{if } m-n \geq 2 \quad \text{and } t \equiv 0 \pmod{l} \\ 0 & \text{otherwise} \end{cases}$$

where $l\alpha = t - \mu' - 1$ and $l\beta = t - l$. But $m = n+1$ implies that $t = \frac{l}{2}$. So $t \equiv \mu' + 1 \pmod{l}$ implies that $\mu' = \bar{\mu}'$, that is $\mu = \bar{\mu}$, so the first case is impossible.

Thus for $\mu \neq \bar{\mu}$ or $m-n$ even we have

$$H^1(G_1,V) \cong \begin{cases} \Delta\left(\frac{m-n-2}{2}(1,-1)\right)^{\mathrm{F}} & \text{if } m-n \geq 2 \quad \text{and } t \equiv 0 \pmod{l} \\ 0 & \text{otherwise.} \end{cases}$$

In the zero case we are done; if non-zero then

$$\begin{aligned} H^1(G_1,V)^{\bar{G}} &\cong H^0\left(\bar{G}, \Delta\left(\tfrac{m-n-2}{2}(1,-1)\right)^{\mathrm{F}}\right) \\ &\cong H^0\left(\mathrm{GL}_2, \Delta\left(\tfrac{m-n-2}{2}(1,-1)\right)\right) \\ &\cong \mathrm{Hom}_{\mathrm{GL}_2}\left(\Delta(0), \Delta\left(\tfrac{m-n-2}{2}(1,-1)\right)\right) \\ &\cong \begin{cases} k & \text{if } m-n-2 = 2(p^\alpha - 1), \quad \alpha \geq 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

by (2.3), with $l = 1$. Now if $\mu = \bar{\mu}$ and $m-n$ odd then we have

$$V^{G_1} \cong \left(q\text{-det}^{-u} \otimes \Delta((m-1)\rho) \otimes \Delta^*(n\rho)\right)^{\mathrm{F}}$$

where $lu = t - \frac{l}{2}$, by our earlier calculation. In this case

$$\begin{aligned} H^1(G_1,V) &\cong \mathrm{Ext}^1_{G_1}(\Delta(\lambda + ln\rho + t(1,1)), \Delta(\lambda + lm\rho)) \\ &\cong \begin{cases} (q\text{-det}^{-\beta})^{\mathrm{F}} & \text{if } m = n+1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $l\beta = t - \frac{l}{2}$ by (4.5) and (4.7). So if $m \neq n+1$ then from the five term exact sequence we have

$$\begin{aligned} H^1(G,V) &\cong H^1(\bar{G}, V^{G_1}) \\ &\cong H^1(\mathrm{GL}_2, \Delta((m-1)\rho - u(1,1)) \otimes \Delta^*(n\rho)) \\ &\cong \mathrm{Ext}^1_{\mathrm{GL}_2}(\Delta(n\rho), \Delta((m-1)\rho - u(1,1))) \\ &\cong \mathrm{Ext}^1_{\mathrm{SL}_2}(\Delta(n), \Delta(m-1)). \end{aligned}$$

If $m = n+1$ then $V^{G_1} \cong (\Delta(n\rho) \otimes \Delta^*(n\rho))^{\mathrm{F}}$. Hence, for $i \geq 1$,

$$H^i(\mathrm{GL}_2, V^{G_1}) \cong \mathrm{Ext}^i_{\mathrm{GL}_2}(\Delta(n\rho), \Delta(n\rho)) = 0$$

and so $H^1(G,V) \cong H^1(G_1,V)^{\bar{G}} \cong k^{\bar{G}} = k$ and this completes the proof.

By the characterisation of blocks calculated earlier, and as for $\mathrm{Ext}^1_G(\Delta(\tau), \Delta(\tau'))$ to be non-zero we must have $\tau < \tau'$, we see that these lemmas have exhausted all possible cases where a non-trivial extension could exist. Thus these, in conjunction with the results of [7], complete the calculation. The final result of this section now merely combines these into a more managable form.

Suppose that $l = 1$. Then for an integer a with $0 \leq a \leq p-1$ we define $\hat{a}$ by $a + \hat{a} = p-1$. If $r = \sum_{i \geq 0} r_i p^i$ with $0 \leq r_i \leq p-1$ then, as in [7] we define

$$\Psi(r)(= \Psi_p(r)) = \begin{array}{c} \left\{\sum_{i=0}^{u-1} \hat{r}_i p^i + p^{u+a} : \hat{r}_u \neq 0, a \geq 1, u \geq 0\right\} \\ \bigcup \left\{\sum_{i=0}^{u} \hat{r}_i p^i : \hat{r}_u \neq 0, u \geq 0\right\}. \end{array}$$

Now suppose that $l \geq 1$. Then if $r = r_{-1} + l\sum_{i \geq 0} r_i p^i$ with $0 \leq r_i \leq p-1$, for $i \geq 0$ and $0 \leq r_{-1} \leq l-1$, we define $\hat{r}_i$ as before for $i \geq 0$; while $\hat{r}_{-1}$ is defined by $r_{-1} + \hat{r}_{-1} = l-1$. With this we can now define a quantum version of the above set by

$$\tilde{\Psi}(r)(= \tilde{\Psi}_{l,p}(r)) = \begin{array}{c} \left\{\sum_{i=-1}^{u-1} \hat{r}_i \theta(i) + lp^{u+a} : \hat{r}_u \neq 0, a \geq 1, u \geq -1\right\} \\ \bigcup \left\{\sum_{i=-1}^{u} \hat{r}_i \theta(i) : \hat{r}_u \neq 0, u \geq -1\right\} \end{array}$$

where $\theta(i) = \begin{cases} lp^i & \text{if } i \geq 0 \\ 1 & \text{if } i = -1. \end{cases}$ We can now state the main result. Note that we now drop our long-standing restriction on λ.

Theorem 5.5 *Let $\lambda = (r+\delta, \delta)$ and $\tau = (s+\delta', \delta')$. Then*

$$\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\tau)) \cong \begin{cases} k & \text{if } r + 2\delta = s + 2\delta' \text{ and } s = r + 2e \text{ with } e \in \tilde{\Psi}(r) \\ 0 & \text{otherwise.} \end{cases}$$

Proof: It is clear that we require $r + 2\delta = s + 2\delta'$ by consideration of blocks. So we only need consider the cases that arise from lemmas (5.1–5.4). We consider when each of these could give a non-zero $\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\tau))$ in turn.

Firstly, suppose that $r = l - 1 + ln, s = l - 1 + lm$. By (5.2) and [7] we must have $s = r + 2dl$ with $d \in \Psi(n)$. Secondly, suppose that $r = \mu + ln, s = \mu + lm$. By (5.4) and [7] we must have

$$s = \begin{cases} r + 2lp^a & \text{if } a \geq 0 \\ r + l & \text{if } \mu = \bar{\mu} \\ r + l(2d+1) & \text{if } \mu = \bar{\mu} \quad \text{and } d \in \Psi(n). \end{cases}$$

Lastly, suppose that $r = \mu + ln, s = \bar{\mu} + lm$. By (5.3) and [7] we must have

$$s = \begin{cases} r + 2lp^a & \text{if } \mu = \bar{\mu} \quad \text{and } a \geq 0 \\ r + l + \bar{\mu} - \mu & \\ r + l(2d+1) + \bar{\mu} - \mu & \text{if } d \in \Psi(n). \end{cases}$$

Further, if r, s satisfy any of the above conditions then $\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\tau))$ is non-zero. Thus, if we allow $\mu = l - 1$, we can state the above results as $\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\tau))$ is non-zero if, and only if

$$s = \begin{cases} r + 2dl & \text{if } \mu = l - 1 \quad \text{and } d \in \Psi(n) \\ r + 2p^a l & \text{if } \mu \neq l - 1 \quad \text{and } a \geq 0 \\ r + 2\bar{\mu} + 2 & \text{if } \mu \neq l - 1 \\ r + 2\bar{\mu} + 2 + 2ld & \text{if } \mu \neq l - 1 \quad \text{and } d \in \Psi(n). \end{cases}$$

So in the form of the statement of the theorem we have that $\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\tau))$ is non-zero if, and only if

$$e = \begin{cases} ld & \text{if } \mu = l - 1 \quad \text{and } d \in \Psi(n) \\ lp^a & \text{if } \mu \neq l - 1 \quad \text{and } a \geq 0 \\ \bar{\mu} + 1 & \text{if } \mu \neq l - 1 \\ \bar{\mu} + 1 + ld & \text{if } \mu \neq l - 1 \quad \text{and } d \in \Psi(n). \end{cases}$$

It is now straight-forward to see that these give rise to the required result.

6 The Manin quantisation

There is another, non-isomorphic, quantum GL_2 due to Manin (see [8]) which we will denote by $\mathrm{GL}_q(2, k)$. In this section we will show how our previous result also holds in this case for q a primitive lth root of unity when l is odd. The key to this approach is the fact that both quantisations give rise to the q-Schur algebras of Dipper and James (see [3]), which allows us to translate from one quantisation to the other. We denote these algebras by $\mathrm{S}_q(2,r)$ where r runs over the natural numbers.

For the Manin quantisation there is also an analogue of the Borel subgroup, so we can again consider the modules induced up from the one-dimensional Borel modules (see [9, 8.3]). As before, the non-zero induced modules correspond to the dominant weights, and can again be described explicitly (see [9, (8.6.1)]). Again, we can define the Weyl modules as duals of appropriate induced modules (see [9, 8.10.1–2]).

Just as the Schur algebras are related to the general linear groups, there are deformations of these algebras related to each of our quantisations in a similar way. In the case of our first quantisation, this procedure yields the q-Schur algebras of Dipper and James (see [2, 3.2.6]), while for the Manin quantisation we obtain the q^{-2}-Schur algebras (see [9, 11.3]). Given two $\mathrm{S}_{q^{-2}}(2,r)$-modules V and W, they are also naturally modules for $\mathrm{GL}_q(2,k)$ and q^{-2}-$\mathrm{GL}(2,k)$. Further, by [4, 4(5)] and [9, (11.5.6)] we have

$$\mathrm{Ext}^1_{q^{-2}\text{-}\mathrm{GL}(2,k)}(V,W) \cong \mathrm{Ext}^1_{\mathrm{S}_{q^{-2}}(2,r)}(V,W) \cong \mathrm{Ext}^1_{\mathrm{GL}_q(2,k)}(V,W) \tag{8}$$

when either q is a non-zero non-root of unity, or q is a primitive lth root of unity with l odd.

Corollary 6.1 *The previous theorem also holds for the Manin quantisation, $GL_q(2,k)$, when q is a primitive lth root of unity with l odd.*

Proof: Consider the Weyl modules $\Delta(\lambda)$ and $\Delta(\tau)$ for $\mathrm{GL}_q(2,k)$. If these are not polynomial modules for $\mathrm{GL}_q(2,k)$, then there exists an $n>0$ such that $\Delta(\lambda)\otimes(\det_q)^n$ and $\Delta(\tau)\otimes(\det_q)^n$ are polynomial, where $\det_q$ is the analogue of q-det for the Manin quantisation. These modules are isomorphic to $\Delta(\lambda+n(1,1))$ and $\Delta(\tau+n(1,1))$ respectively. By [9, (11.1.1)], there is a non-trivial extension between them only if $\lambda_1+\lambda_2=\tau_1+\tau_2$. Thus the same is true for $\Delta(\lambda)$ and $\Delta(\tau)$, as implied by the theorem. So we may assume that there is an r such that $\Delta(\lambda+n(1,1))$ and $\Delta(\tau+n(1,1))$ are both $\mathrm{S}_{q^{-2}}(2,r)$-modules. Clearly extensions of $\Delta(\tau+n(1,1))$ by $\Delta(\lambda+n(1,1))$ correspond to extensions of $\Delta(\tau)$ by $\Delta(\lambda)$, and so the result follows from (8) (as if q is a primitive lth root of unity with l odd, then so is q^{-2}).

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