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ASPECTS OF BINOCULAR VISION: DERIVATION AND PROPERTIES OF THE RECTILINEAR AND SPACE HOROPTERS

by

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A thesis submitted in fulfillment
of the requirements for the degree
of
DOCTOR OF PHILOSOPHY
at
THE CITY UNIVERSITY
LONDON

Department of Ophthalmic Optics
and Visual Science

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ABSTRACT

The concepts of binocular correspondence, and the horopter - the locus of self-corresponding points - have been reviewed, from the earliest records to the present day.

Since current workers in this field have restricted their investigations to two dimensions, their interpretation of their findings have been limited in at least three important respects:

1) The horopter is found to be a plane curve.

2) The deviation of the horopter, (the so-called Hering-Hillebrand deviation) can only be interpreted by assuming that the corresponding points of the two retinas are unsymmetrically disposed, an assertion that is not consistent with the recent findings of the neuro-physiologists.

3) There has been no satisfactory explanation for the experimentally determined result that this deviation changes systematically with the distance of fixation.

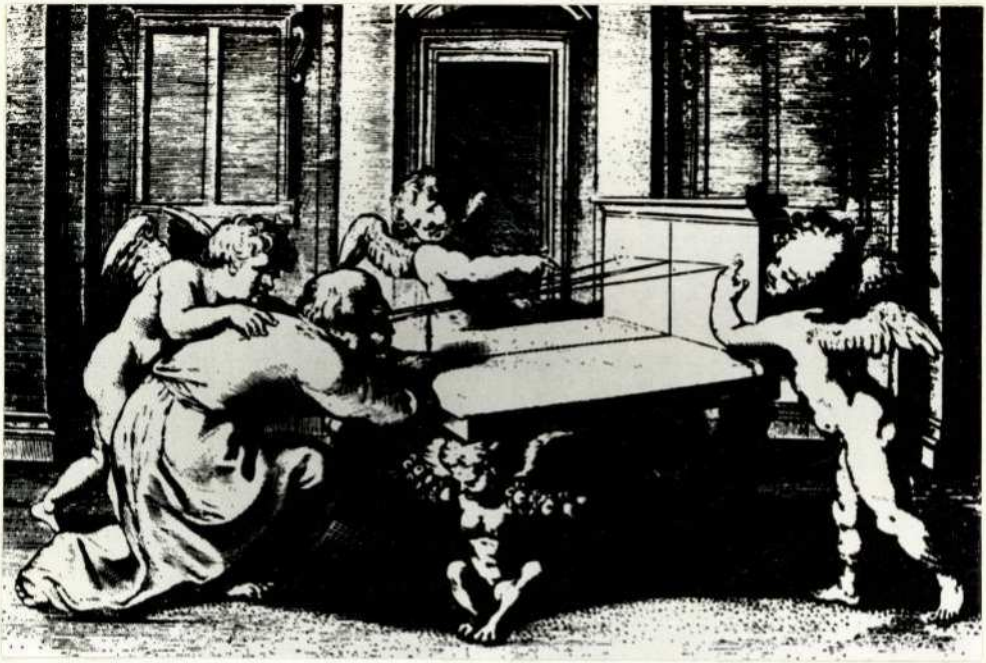
Using the criterion of the identity of visual directions and the techniques of linear algebra, it has been possible to extend the investigation of the horopter to the third dimension, whereupon the three anomalies referred to above have been resolved.

The horopter is not the plane curve that previous workers have indicated, but is a twisted cubic curve in space. This curve has been called the *space horopter*, its explicit parametric representation has been found and its properties explored in a variety of circumstances.

The chords of the space horopter are found to be self-corresponding lines, called *nonius-lines*, and it is argued that it is these lines that the experimental workers have unknowingly been determining. The nonius-lines generate a family of quadric surfaces that has been named the *rectilinear horopter*, and its various properties have been explored.

On this basis, the horopter deviation is seen not to imply a relative asymmetry of the two retinas, but is found to be an explicit function of the fixation distance and the angles of torsion (i.e. of cyclo-rotation) of the two eyes.

For this reason a kinematic analysis of the rotations of the eye have been included, so that the angles of torsion can be expressed in terms of known parameters.



Frontispiece The First Horopter Experiment

Engraving from Franciscus Aguilonius, *Opticorum Libri Sex*, Antwerp, 1613, showing an idealized experiment to determine the horopter.

INTRODUCTION

The horopter can be defined as the locus in space of those points that, for a given fixation point, will stimulate corresponding retinal points. That is to say, those points in space that will maximise the cortical response of those neurons sensitive to zero disparity.[1]

Current understanding of the horopter derives principally from a series of papers initiated by the Dartmouth Institute in 1932. These experiments were reported by AMES, GLIDDEN and OGLE[2] [3], and their theoretical justification given in a paper by OGLE[4].

This last mathematical analysis, despite some development by Linksz[5], is still virtually without any change, considered the theoretical justification for work being carried out today. The experimental technique currently being used[6] [7] [8], although refined in detail is still essentially the same as that described by Ames *et al.*, in 1932.

This technique reveals the horopter to be a curve lying in the plane of fixation, which, in all the work reported, is the horizontal plane.

A naive prediction of the form of this curve, dating back to Vieth (1818)[9] and Müller (1826)[10], is that of the circle in the fixation

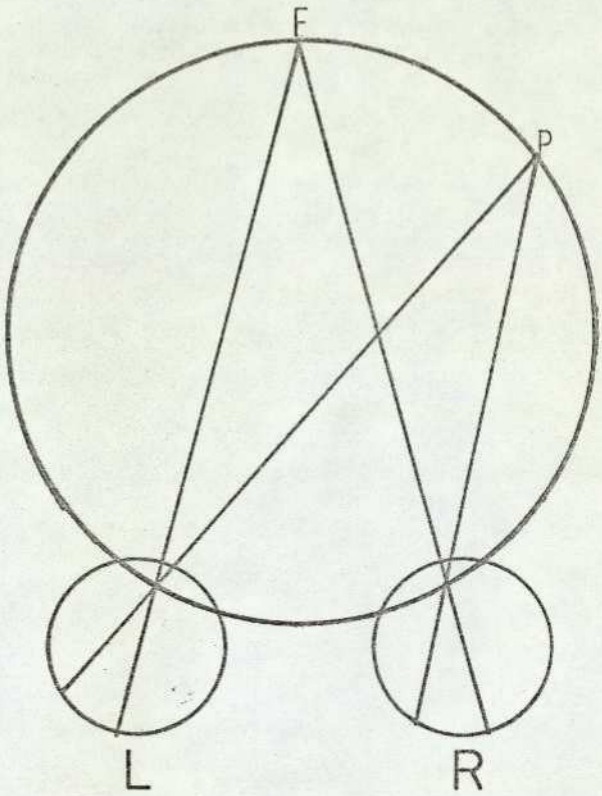


Fig. 1 The Vieth-Müller Circle

This is the circle passing through the fixation point F, and the nodal points of the two eyes. If P is any other point on this circle, then, F and P subtend equal angles at the two nodal points.

plane (fig. 1) passing through the fixation point and the two eyes, the so-called Vieth-Muller Circle.

In his theoretical work, Ogle[11 12 13 14] deduces that the horopter is a curve lying in the horizontal plane in the form of a conic section. This conic also passes through the fixation point and the two eyes. The amount by which the conic is distorted from the Vieth-Muller Circle is measured by an empirical constant H, called the Hering-Hillebrand deviation, which is a function of the curvature of the conic at the fixation point.

The explanation advanced for the existence of the Hering-Hillebrand deviation was originally the asymmetry of the points of the two retinas; (This lead to a long series of papers on Aniseikonia, see, for example[15 16 17 18 19 20].) more recently an optical explanation has been offered[21].

It is also found[22] that the Hering-Hillebrand deviation varies with the distance of fixation. All explanations of this phenomenon[23 24], which have to account for retinal correspondence varying with fixation, have been acknowledged as unsatisfactory.

Being aware of the limitations of restricting his analysis to two dimensions, Ogle described his horopter curve as the "Longitudinal Horopter". Current workers[25] use the term "Nonius Horoper", the adjective referring to the experimental method used in its' determination.

The nonius method can be summarized as follows:- The two eyes regard a fixation point in part of the visual field that is visible to both eyes. A vertical rod is so placed at an eccentric point in the field that the upper half is seen by one eye and the lower half by the other. In a general position, due to physiological diplopia, the two half rods will appear laterally displaced with respect to each other. The rod is then moved along a fixed track until a position is found such that the two half rods appear to the observer to be in line. (The fusion reflex is not initiated.) The point where the rod in this position meets the horizontal plane is called a point of the (nonius) horopter.

It is argued in this thesis - which considers the retina as a two-dimensional surface in 3-space, rather than a one-dimensional surface in 2-space as in most previous work - that the nonius technique is not determining the horopter at all, that is to say, it is not locating those points in space that will stimulate corresponding retinal points. In fact, it is merely locating the position of a certain line that is seen singly. It is shown in the analysis that follows that there is a unique such line through every general point of space. It is found that the envelope of these "single-lines" is a family of quadric surfaces. The totality of all such lines we can call the RECTILINEAR HOROPTER.

It can further be shown that each of these "single-lines" is a chord through a three-dimensional curve of degree three, i.e. a twisted cubic curve, and that each point on this curve will stimulate corresponding points on the two retinas; thus the twisted cubic, as predicted by Helmholtz[2,6] is the true form of the horopter.

There is, however, a relationship between the true space horopter and the nonius horopter as classically plotted. It is found that the locus in which parallel chords to the space horopter meets the horizontal plane is a conic curve and it is this *locus of parallel chords*, it is argued, that is determined by the experimenters. Relating the two conic sections, i.e. Ogle"s theoretical longitudinal horopter and the "locus of parallel chords", it is found that the Hering-Hillebrand deviation constant, H, which enters Ogle's equation as an experimentally determined parameter, emerges as an explicit function of the fixation distance and the angles of cyclo-rotation of the eyes around their anterio-posterior axes; i.e. the respective angles of torsion.

This explicit relationship explains the two unresolved problems of interpretation of the Hering-Hillebrand deviation referred to above. H is indeed a function of the asymmetry of binocular vision, but this asymmetry can be

explained by unsymmetrical degrees of cyclo-rotation of the eye, and need not involve non-empirical interpretations of retinal structure. Further, the various arguments put forward to explain the variation of H with the fixation distance are made redundant by this interpretation.

PART I REVIEW

Chapter 1 Historical Development

i) The chiasma and binocular correspondence

A review of the historical development of knowledge of the optic chiasma and of theories of correspondence between the two eyes, from the sixth century B.C. to the middle of the nineteenth century.

ii) The Horopter

A summary of the various forms and theories of the mathematical horopter from 1613 until the middle of the nineteenth century.

Chapter 2 Current view on the horopter

i) Helmholtz

A review of Helmholtz' contributions to the theory of the horopter, particularly the concepts of the "line-horopter", and the "horopter-curve".

ii) The Hering Tradition

A summary of currently accepted views on the horopter, paying particular attention to the work of Hering, Tschermak, Ames, Ogle and the most recent work.

1 HISTORICAL DEVELOPMENT

(i) The Chiasma and Binocular Correspondence

The first observer to record that the

> "channels from the two eyes"[27]

were connected was ALCMAEON, in the 6th century B.C. Tradition regards him as the first man to

> "dare to undertake the excision of an eye"[28]

for the purpose of discovering truth.

He argued that:

> "all the senses are connected in some way with the brain"[29]

and, according to him:

> "The eye itself is enclosed in certain transparent membranes and it is connected with the brain by two 'light bearing paths' which join behind the forehead. That these paths join, he showed by dissection, and supported his view by the observation that the eyes move together, not separately ...."[30]

Hence, almost as early as men have been recording their observations, they have been aware of the chiasma of the optic nerves.

Aristotle, in the 4th century B.C., wrote:

> "From the eye there go three ducts to the brain; the largest and medium sized to the cerebellum, the least to the brain itself; and the least is the one situated nearest to the nostril. The two largest ones then, run side by side and do not meet; the medium sized ones meet - and this is particularly visible in fishes - for they lie nearer than the larger one to the brain ...."[31] (see: fig. 1.1)

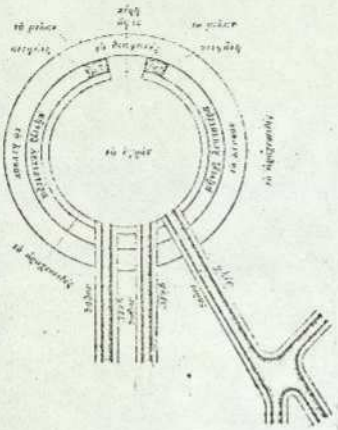


Fig. 1.1 The Eye According to Aristotle

Reconstructed by H. Magnus[32]

from the ancient texts.

These channels have been tentatively identified[33]; the largest with the trigeminal nerve, the medium sized with the optic nerve, and the thinnest with the olfactory nerve. The chiasma is clearly identified by the meeting of the middle sized channels.

In the spurious work included in the *Corpus Aristotelian*, the *Problemata** there are many observations on binocular vision:

The author wrote:

> "... the origins of sight in the two eyes are connected at one source, so that when one eye moves, the common source of sight is also set in motion, and when this moves, the other eye moves also ..."[34]

He is led to the question:

> "Why is it that we can turn the gaze of both eyes simultaneously towards the right and left and in the direction of the nose ...?"[35]

But not outwards? And that we can direct them downwards and upwards, but not separately.[36]

This book contains the first observation that the eyes converge to a fixation point. He notices that an old man, who has difficulty with his vision, will:

> "remove the object at which he is looking to the point at which the vision of the two eyes meet, expecting to be able to see it best in this position."[37]

---

*This book is certainly later than Aristotle, but it is probably based upon his writings or lectures; certainly on his tradition. See E.S. Forster[38].

There are quite a few obscure references that suggest that the necessity for some kind of binocular correspondence between the eyes was required for binocular vision. This, of course, was not retinal correspondence. The retinal coat was not identified for another 100 years, and its function for another 2000.

However, there is the passage:

> "Why is it that we see other things better with both eyes, but we can judge the straightness of lines of writing better with one eye, putting it close to the letters? Do both eyes falling on the same point cause confusion, as the writers on optics say."[39]

He noticed that drunkards experience double vision[40] [41], and suggests that this may be due to the same cause as the double vision that occurs when the eyes are displaced from their natural position. That is:

> "because the movement does not reach the same point on each of the eyes. So the mind thinks that it sees two objects when it really sees one twice ..."[42]

This can be induced voluntarily:

> "The same thing happens if one presses the eye from below; for this disturbs the source of vision, so that it no longer falls upon the same point as the other eye ..."[43]

(See also[44] [45])

And quite explicitly:

> "... When, therefore, they /the eyes/ are so placed that they are in a similar position to one another and midway between an upward and a downward and an oblique movement, the two eyes catch the visual rays on corresponding points of themselves..."[46]

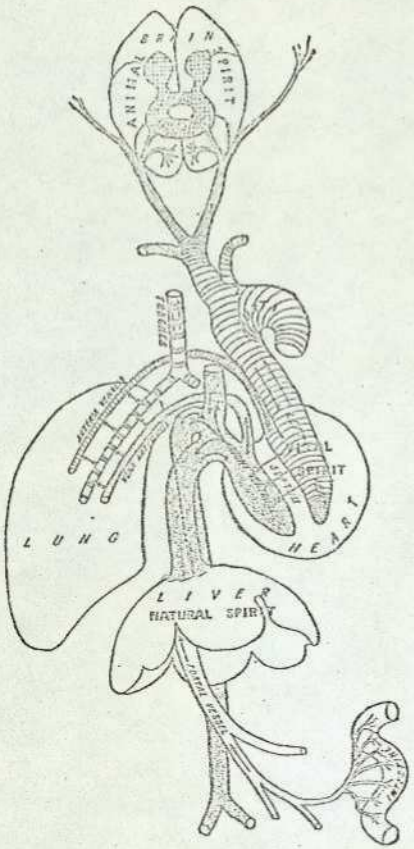
then vision is not distorted, but, if these conditions are not fulfilled, then vision is distorted.[47]

In the third century B.C., the anatomists of Alexandria, in particular HEROPHILOS and ERASISTRATOS, probably using human vivisection,[48] [49] named the retina, and identified most of the parts of the eye that are visible without magnification. Their work is summarized in the writings of RUFOS OF EPHESOS[50] (c. 100 A.D.) who considered the nerves to be hollow channels (ποροι). The optic nerves emerge from the base of the brain and divide into two branches, which, by coursing obliquely, enter the eyes.

The most influential writer who has ever lived in the field of medicine, and particularly anatomy and physiology was GALEN (c. 129-199). For nearly 1500 years and through three civilizations, men who wished to study the structure and function of the human body turned to the writings of GALEN[51] rather than to the object of study itself. For this reason it is neccessary to digress and describe GALEN's physiological system, which influenced visual theories well into the modern era.

GALEN's anatomy was reasonably sound, in so far as it could be studied without magnification, and despite the limitation of only dissecting animals, but on this structure he wove his fantastic physiology that was integrated into his general philosophy.

For a detailed account of Aristotle's contribution to visual science see H. Solomons[52]



Reconstructed by C. Singer[52] from ancient texts

Fig. 1.2 Diagram Illustrating Galen's Physiological System

The "Animal Spirit" that flowed from the brain along the optic nerves was called the "visual Spirit", a concept that was to dominate visual ideas until the seventeenth century.

According to him, the basic principle of life was the spirit (*pneuma*) that was drawn from the general "World Spirit" in the act of respiration. It entered the body through the trachea, and passed to the lung, and then through the venous artery (pulmonary vein) to the left ventricle, where it encountered the blood, (see fig. 1.2).

Now on this hypothesis, the blood originates in the liver. He tells us that the food substance from the intestines is carried as CHYLE from the alimentary tract, through the 'gate' of the liver. (Gate = Latin porta; hence our portal vein) In the liver this chyle was elaborated into VENOUS BLOOD by being imbued with a special pneuma called the NATURAL SPIRIT, which was possessed by all living substances. Natural spirit bestowed the power of growth and nutrition. Charged with this natural spirit derived from the liver and the nutritive material from the intestine, the blood was distributed by the liver through the VENA CAVA and the venous system to all the tissues of the body, where it ebbed and flowed in the veins. A portion of this venous blood entered the right ventricle of the heart.

For this blood there were two possible fates. Most of it remained for a while in the ventricle parting with its impurities, which were carried off by the arterial vein (pulmonary artery) to the lung, and there exhaled, after which the venous blood ebbed back into the general venous

system. A small portion of the venous blood followed a different course; it trickled through minute pores in the intra-ventricle septum one drop at a time. There it encountered the pneuma brought in by the venous artery and became charged into a higher type of pneuma, the VITAL SPIRIT, which was distributed, with the arterial blood, through the arterial system.

Of these arteries, some went to the head and thereby the vital spirit was brought to the base of the brain. On the under surface of the brain there was a network of minute vessels, the *rete mirabile* (wonderful network), and in this mysterious organ the blood became charged with yet a third pneuma, the ANIMAL SPIRIT.

This third pneuma, the noblest of all, was distributed to all the organs of the body by the network of (hollow) nerves. The animal spirit conducted along the optic nerves had a special function and was called the VISUAL SPIRIT.

The concept of the visual spirit lingered long after the other spirits had been abandonded.

The visual spirit travelled along the optic nerves divided at the chiasma then followed each optic nerve, passed through the retina (which was felt to be a network of hollow tubes), then traversed the fine hollow vessels of the zonule and so reach the "photoreceptor", the CRYSTALLINE, the essential organ of vision.

From here it could be emitted into the outer space in the form of a cone of rays and come into contact with the visual objects. At the same time, somehow, the impression mingled with the outer light in the ANTERIOR SPACE (see fig. 1.3) to be returned along the same path to that part of the brain that is the abode of the soul.[55]

GALEN held that the visual axes, the pupils, the optic nerves from the discs - which he held to be the apices of the visual cones - to the chiasma, for both eyes, all lie in a single plane, which explains why the two eyes see a single object. He also ascribes this to the chiasma which , he says, is

"worthy to be ascribed to some of the gods"[56]

He also suggested that the chiasma enabled the visual spirit from the brain to be distributed properly. In the case of shutting (or losing) one eye, all the pneuma could pass over to the other and would compensate for the loss. He confirmed this view by experiment.

His suggestion that the layout of the eye, the optic nerves and the chiasma could explain binocular single vision is the earliest attempt to interpret this phenomenon in terms of anatomical structure.[57]

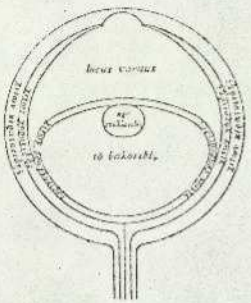


Fig. 1.3a

The Eye According to Celsus

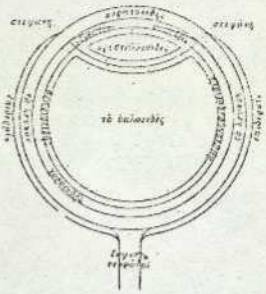


Fig. 1.3b

The Eye According to Rufos of Ephesos

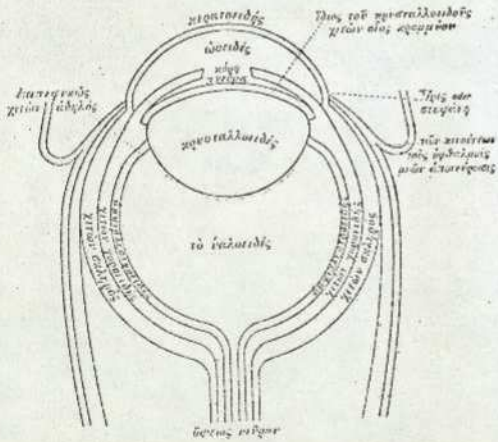


Fig. 1.3c

The Eye According to Galen

Reconstructed by H. Magnus[54] from the ancient texts

After GALEN's death there was a general decline of learning in Europe as men's minds turned away from seeking after knowledge and as part of this lack of interest, there was a consequent failure to improve the understanding of the visual processes.

However, after about 800 A.D., there was a renaissance in the Arabic speaking countries, as the writings of the Greek tradition became available in translation. Unfortunately, the culture in the Caliphate did not permit the dissection of human bodies and so the only source materials for research in human anatomy were the writings of Aristotle and Galen.

In spite of the lack of facilities for anatomical observation, there were important advances in the understanding of the visual processes during this time, particularly in the work of the great physicist ibn al-Haitham, who became known to the Latins in the West as Alhazen (965-1039). It has only been since his work that there has been general agreement that vision occurs as a result of a stimulus entering the eye rather than leaving it.*

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*Empedocles (5th century B.C.) was the first to postulate an emission theory of vision. His near contemporary, Democritos, suggested that an *eidola* or image was transmitted from an object to the eye. Plato (4th century B.C.) combined the two views with a kind of intermingling of the inner and outer light. Aristotle was inconsistent. He wrote[58] of light as a state of being that is

Al-Haitham wrote:

> "The act of vision is not accomplished by means of rays emitted from the visual organ .... The belief of those who think that something emerges from the visual organ is false."[61]

but

> "Vision is accomplished by rays coming from the external objects and entering the visual organ"[62]

We can also find this explicit reference to the existence of corresponding points:

> "Distinct vision is accomplished by means of perpendicular lines drawn from the object seen to the surface of the organ of sight. In this way each point of the object is related to a point of the surface of the visual organ having the same position as in the object seen."[63]

It is no doubt this passage that has resulted in al-Haitham being credited with the discovery of retinal correspondence[64]. However, he was unaware of the function of the retina. For him, as for Galen, the principal organ of sight was the lens, now called the "glacial sphere"; a view that prevailed for another six hundred years. It is worth noting just how this term came to be used for the crystalline lens. There had been no anatomical dissection and hence no-one had seen the crystalline lens for over six hundred years in al-Haitham's time. Thus the only source of information available for the

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actualized in the presence of fire, and denied that it could have a finite velocity, but elsewhere[59] he spoke of "rays of sight", and of the eyes "catching the rays". However, it was the Platonic view that had the most credence throughout later antiquity. See, H. Solomons[60].

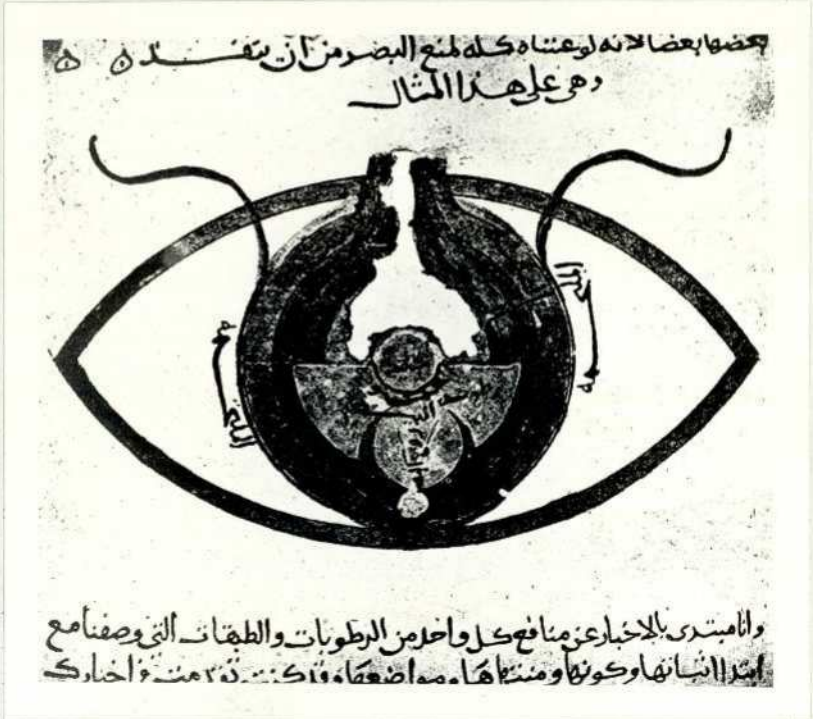


Fig. 1.4a The Eye of Hunain ibn Ishak (c 860 A.D.)

The oldest extant diagram of the anatomy of the eye, from a manuscript dated 1197 cited by Polyak *op. cit.* fig. 2. (see: 1.4b).

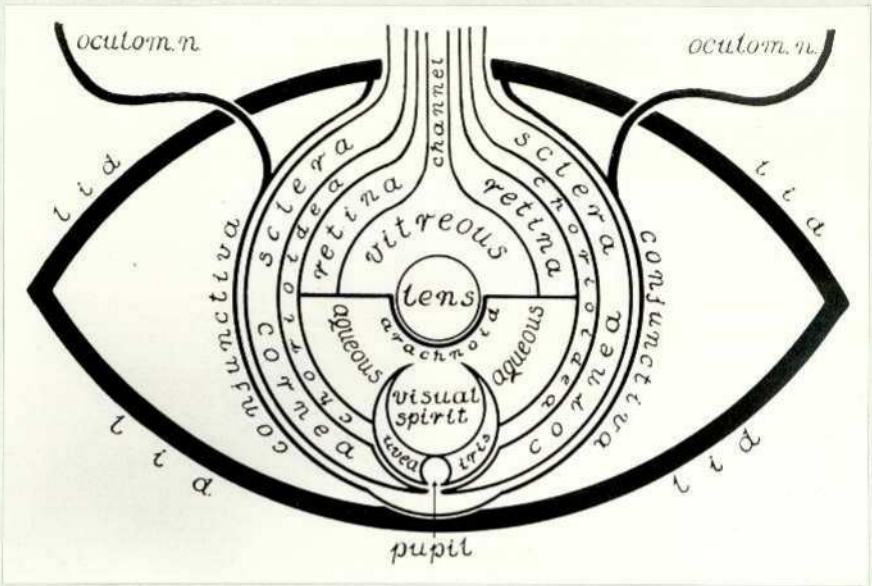


Fig 1.4b Reconstruction and English translation of the Eye of Fig. 1.4a

Based on a corruption of Galen's text.

Note the central spherical lens.

contents of the eye, were the illustrations of Galen's textbooks. These, of course, were not printed but were redrawn in each generation by ignorant scribes and had gradually become corrupted until the lens was represented as a perfect sphere occupying the centre of the eye. (See fig. 1.4)

It was on this principal element, the "glacial sphere", that al-Haitham put the end points of the perpendicular lines referred to above. However in this passage there is the first enunciation of the possibility of a topographical relationship existing between the points of physical space and the points within a single eye.

He contemplated further that there was an unaltered transmission of these "figures" from the posterior surface of the lens, along the retina and hollow optic nerve to the brain (see fig 1.5) where it was interpreted, so there was some sort of mapping of the points of physical space onto the cerebellum. The functional role of the chiasma was essential for the preservation of a single visual experience, as it was there that the "figures" arising in the two eyes were superimposed.[66] But, it is said, he was also aware of the existence of physiological diplopia.[67]

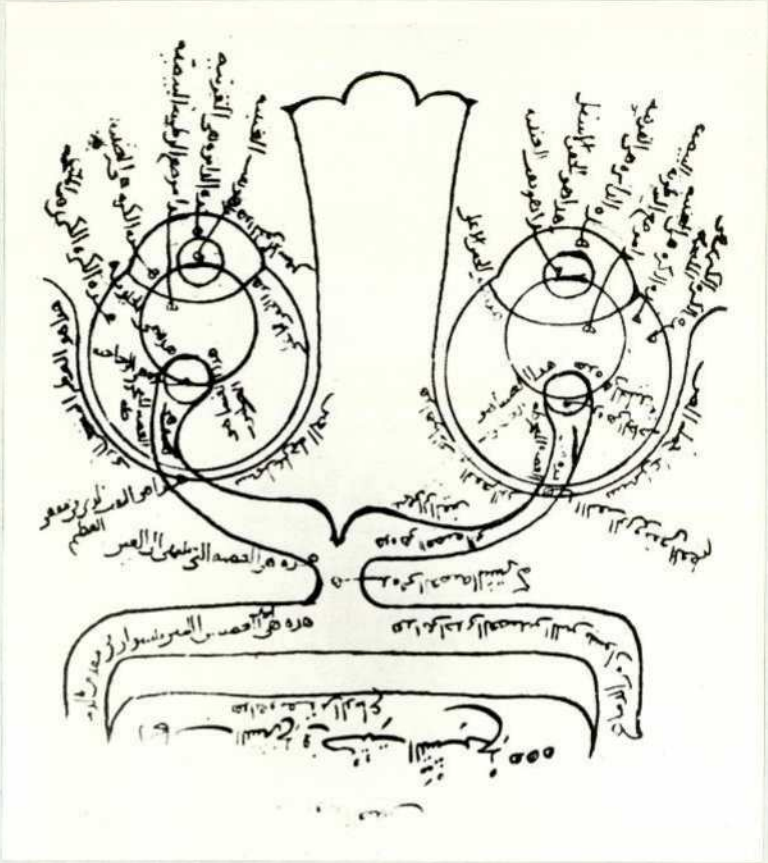


Fig 1.5 Ibn al-Haitham's diagram of the visual pathway

From the oldest existing copy of the *Thesaurus of Optics*, 1083

cited by Polyak *op. cit.* fig. 56

In the thirteenth century, Arabian work on optics - in particular, more or less corrupt Latin translations of ibn al-Haitham - became available to the Europeans. This led to a revival of interest in the subject, particularly at Oxford.[68 69 70 71 72 73] However, as far as the anatomy of the eye was concerned, there was no improvement in knowledge.[74] The errors of their predecessors, i.e., the centrally located "glacial sphere" as the essential organ of vision, the centrally located and hollow optic nerve and the concept of the "visual spirit" continued to be promulgated from generation to generation. (see figs. 1.6, 1.7)

These errors persisted to a remarkable extent; even in the work of such skillful anatomists and careful observers as Leonardo (1452-1519)[75] and Vesalius (1514-64)[76] (see figs. 1.8, 1.9).

Except for the last, these misconceptions were cleared up by the early years of the seventeenth century. MAUROLYCUS suggested that the crystalline functioned as an optical device in 1575.[77] Felix Plater[78], in 1583, was the first to realize that it was the retina that acted as the photoreceptor. The correct position of the lens was found by the anatomist Aquapendente[79] in 1600. These discoveries were related to each other by Kepler[80] in 1611, with his discovery of the formation of the retinal image, whose existence was demonstrated experimentally by

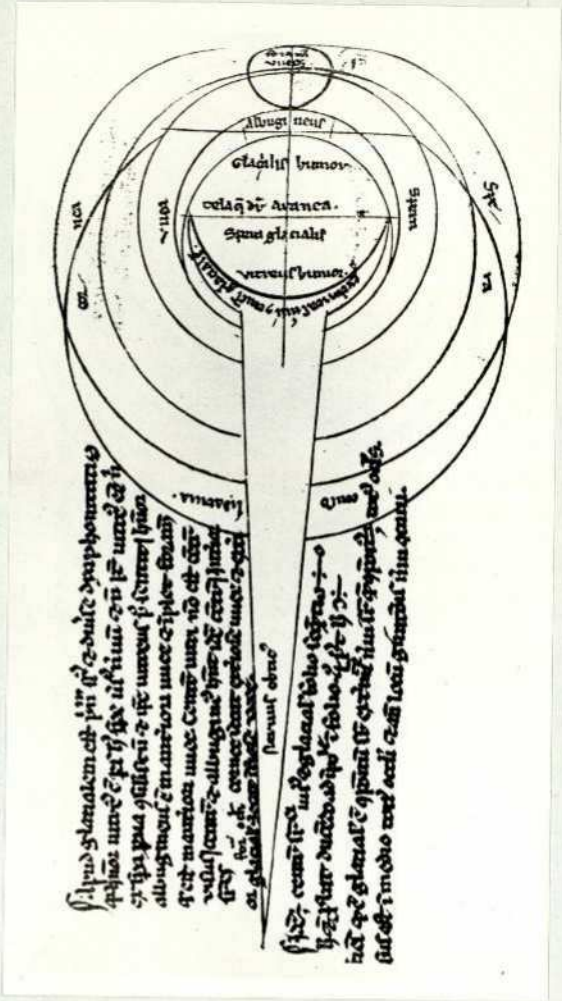


Fig. 1.6 The Anatomy of the Eye in the 13th century

This diagram from a Latin translation of ibn al-Haitham shows just how corrupt ideas can become after 1200 years of copying. All the Latin diagrams of this time, including those of Roger Bacon, were mere geometrical figures.

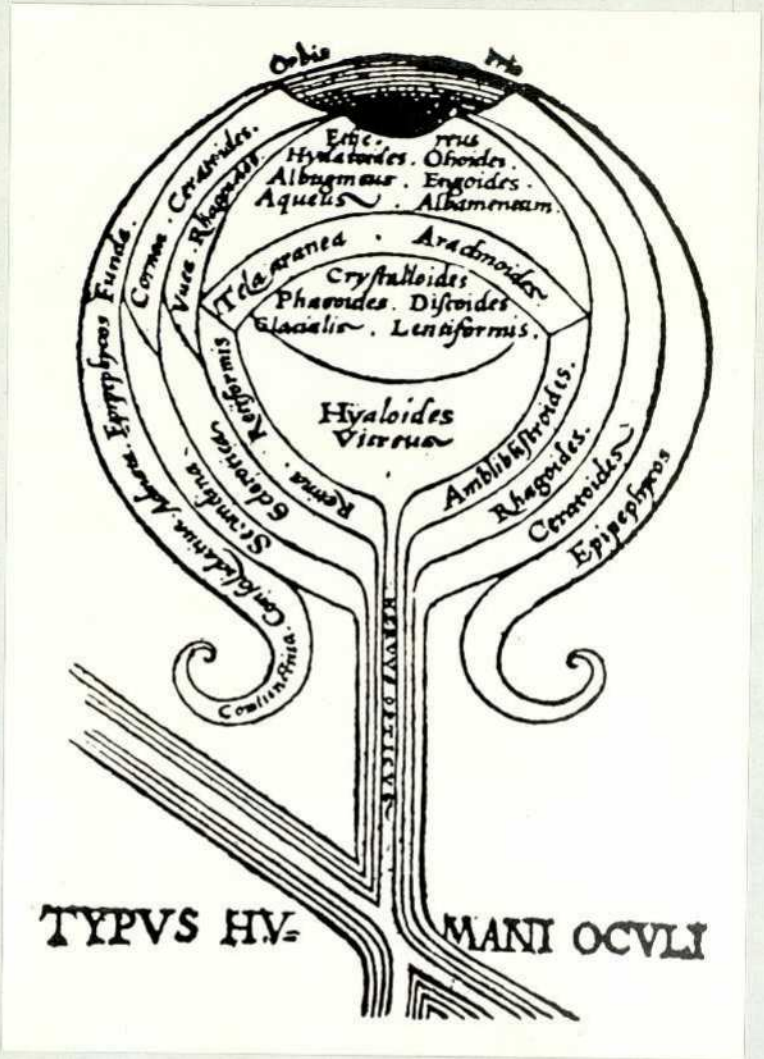


Fig. 1.7 Anatomy of the Eye in 1541

The hollow optic nerves can be seen to meet at the chiasma; just before Vesalius' *Fabrica* (1543).

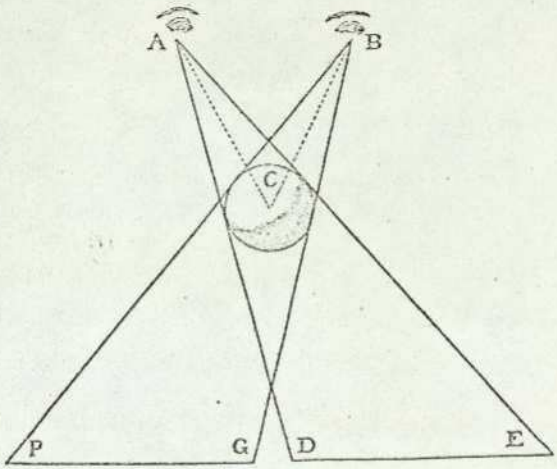


Fig. 1.8 Leonardo's Paradox

Leonardo da Vinci was the earliest writer to be aware that a painting cannot represent the appearance of objects as seen by the two eyes. He said:

> "If nature is seen with two eyes, it will be impossible to imitate it upon a picture so as to appear with the same relief, though the lines, the lights, shades and colour be perfectly imitated"[78a]

The eye at A sees the background PD, and the eye at B the background GE; hence both eyes see all the background and the opaque object C is transparent to binocular vision. This drawing was made by L.B. Alberti, in the publication of Leonardo's notebook in 1651.

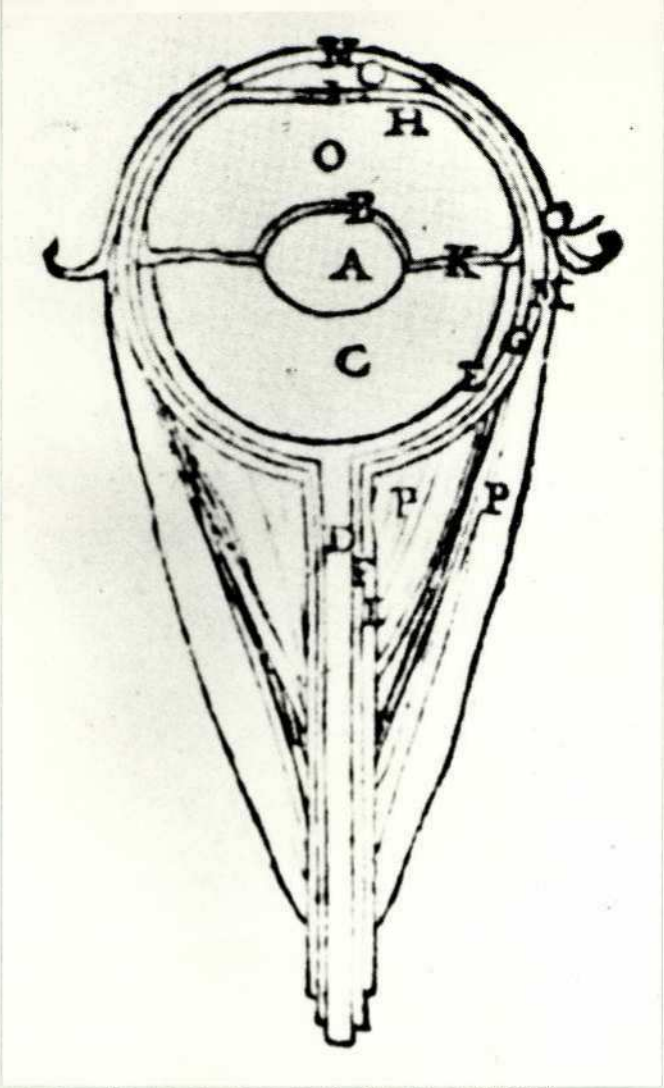


Fig. 1.9a Vesalius' Anatomy of the Eye (1543)

This illustration from the *Fabrica*[76] shows a great improvement over any previous drawings, as Vesalius did many dissections of the eye. It is incredible how such a great observer could still perpetuate such errors as the central location of the lens; the centrally placed optic nerve (at least no longer hollow), and the flatness of the cornea.

from the copy in the British Museum

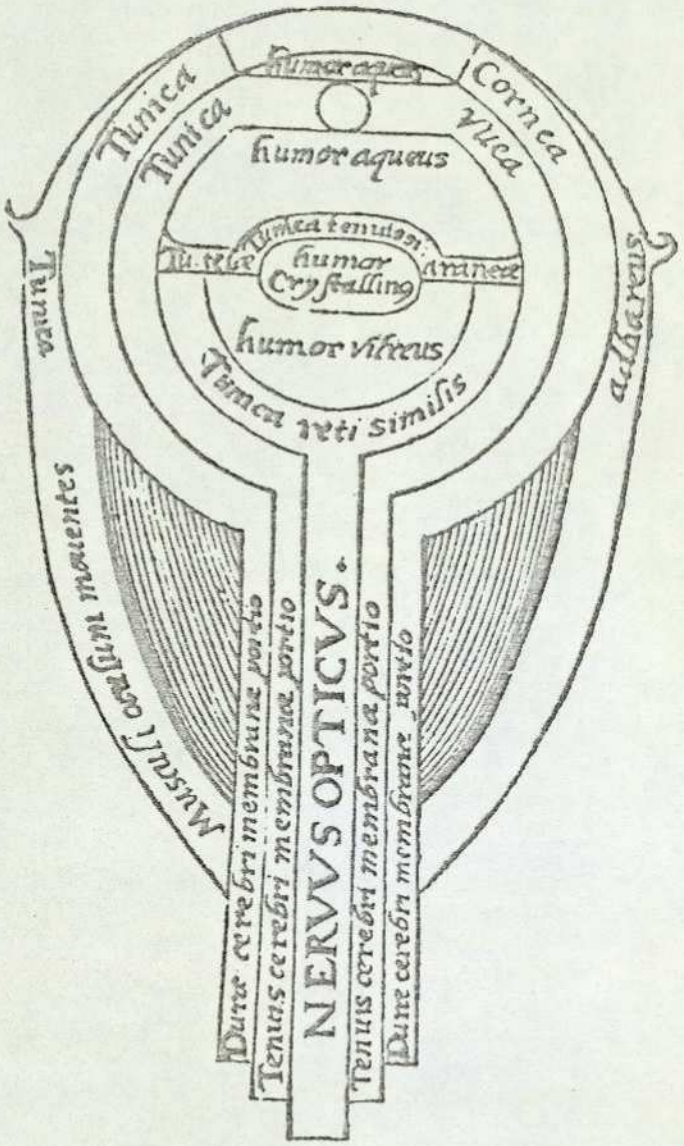


Fig. 1.9b Risner's Eye in 1572

This diagram from Risner's edition[61] of ibn al-Haitham's *Thesaurus of Optics*, was obviously based on Vesalius, rather than the original text. It is, perhaps, worth noting that this book was published in 1572 as a reference book, not as an historical document.

from the copy in the British Museum

Scheiner [81] in 1619. (see fig. 1.11)

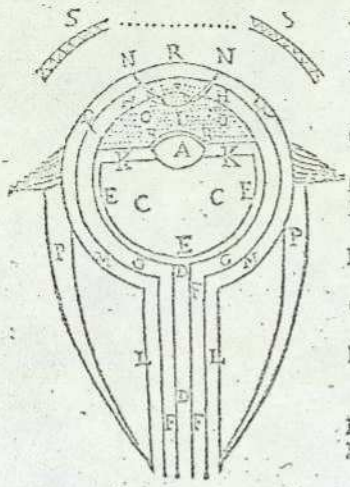
However all these writers still believed in a visual spirit pervading the ventricles of the brain.

Fig 1.12, from Vesalius' *Fabrica*, shows his observations of the chiasma.[82] The optic nerves are now solid, but he felt that the connection of the two nerves was only apparent, each nerve remaining on its own side.

The first modern writer to offer an explanation of the singleness of binocular vision was della Porta[83] in 1593. He suggested that this occurred by the sight alternating, that is, that vision occurs first with one eye and then the other.

Descartes[84 85] was the first to offer a mechanical and anatomical explanation of the phenomena of binocular single vision. The single monocular impression is painted upon the retinas of the two eyes. The image is located upon identical corresponding points in the two retinas (see fig 1.13). From the retina the impression was transmitted by the visual spirit through the 'pores' in the optic nerve until it united with the impression originating from the identical point of the other eye. This union did not take place at the chiasma (see fig 1.14a), but in the pineal gland (H in fig. 1.14), which for Descartes was the "seat of the soul", where the two impressions were unified to give rise to a single "idea" of an external object.

Visualis organi theoria.



A Humor chrystallinus, glacialis, pupillæ.

B Aranea pellucida, glacialem vestiens, pellucida sicut cæparum pellis.

C Humor vitreus, chrystallini nutrimentum.

D Neruus opticus, visorius.

E Retina, retiformis pellicula à visorio neruo procedens.

F Piæ matris, seu tenuis meningis pellis dictum neruum vestiens.

G Vuea tunica, secundina à dicta pelle procedens.

H Finis vueæ adumbrantis opacæ villositate humores ad perfectiorem visum.

I Foramen vueæ radios admittens.

K Tunica villosa, ab vuea deriuata, vitrei &, aquei humoris discrimen.

L Duræ matris, seu crassæ meningis pellis secundo vestiens neruum opticum, similis palpebrarum pelli.

M Sclerotica, posterior corneæ pars, à dicta pelle procedens, dura.

N Cornea tunica ex quatuor pelliculis perspicuis, ac tenuibus ad tutandos humores compacta; cornu instar pellucida.

O Humor aqueus siue albugineus, tutamen & excrementum glacialis.

P Musculi oculũ mouentes, è diuerso foramine quàm opticus.

Q Consolidatiua, tunica alba, densa, ex pericranio, seu pellicranio progenita, oculum totum vestiens.

R Foramen consolidatiuæ, transitum visui ad corneam, reliquasq. tunicas præbens.

S Palpebræ cum ciliis, clausura oculum complentes.

Hæc ex Anatomia Andreæ Vessalij Bruxellensis, viri ætatis nostra perspicacissimi ad pleniorem eorum, quæ dicta sunt de

oculo

Fig. 1.10 The Eye according to Maurolycus (1575)[77]

From the copy in the British Museum.

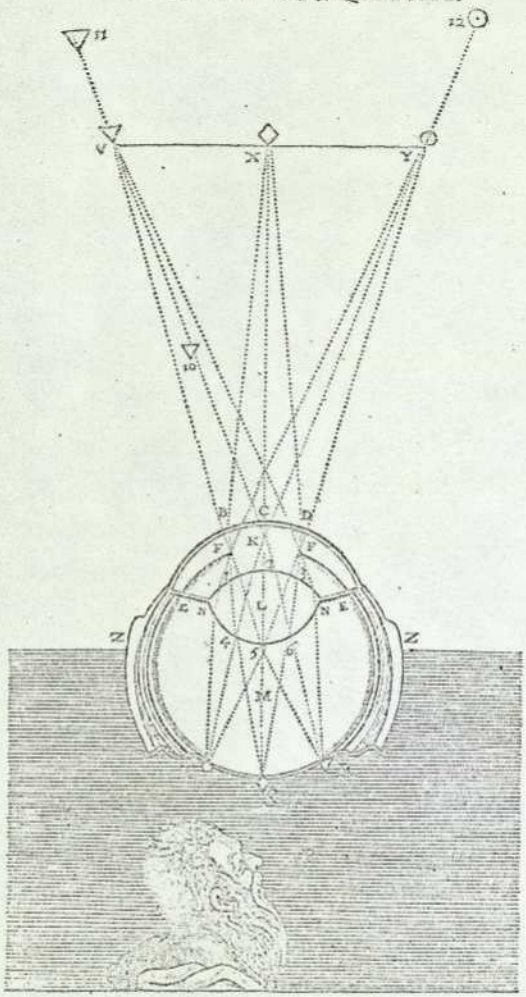


Fig. 1.11 Decartes: The formation of the Retinal image

Famous drawing, 1637[84] illustrating his and Christopher Scheiner's experimental observation of the retinal image by cutting away the outer coats of a bull's eye.

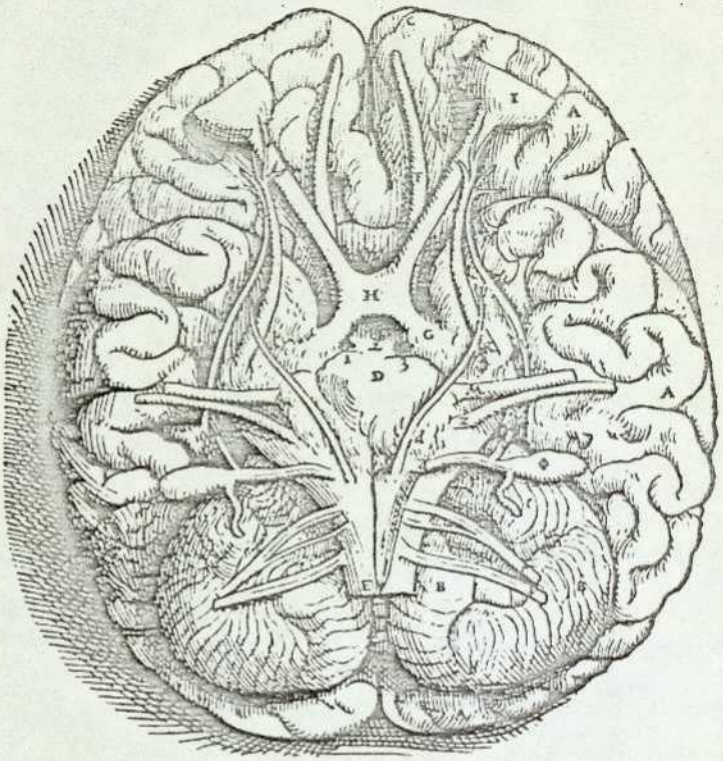


Fig. 1.12 Vesalius on the Visual Pathway

This diagram shows the optic chiasma (H) the retina (I), the optic nerves and the optic tract

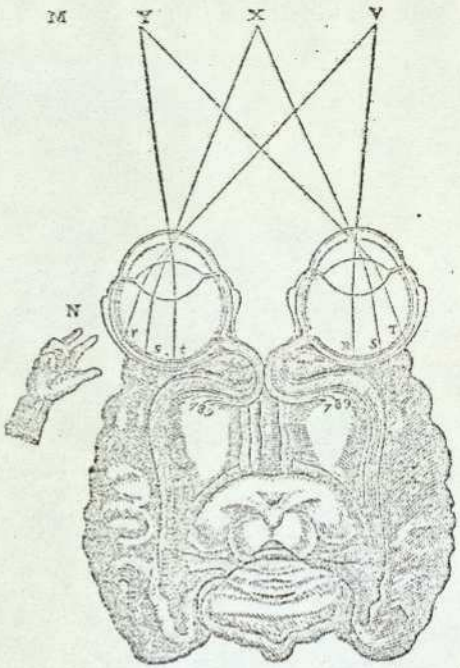


Fig. 1.13 Descartes on Corresponding Retinal Points

Illustrating that points in external space are pictured on to corresponding points in the two retinas.

Note the lack of decussation at the chiasma.

From: Descartes, R. *La Dioptrice*, 1637[illegible]84

(*From the copy in the British Museum*)

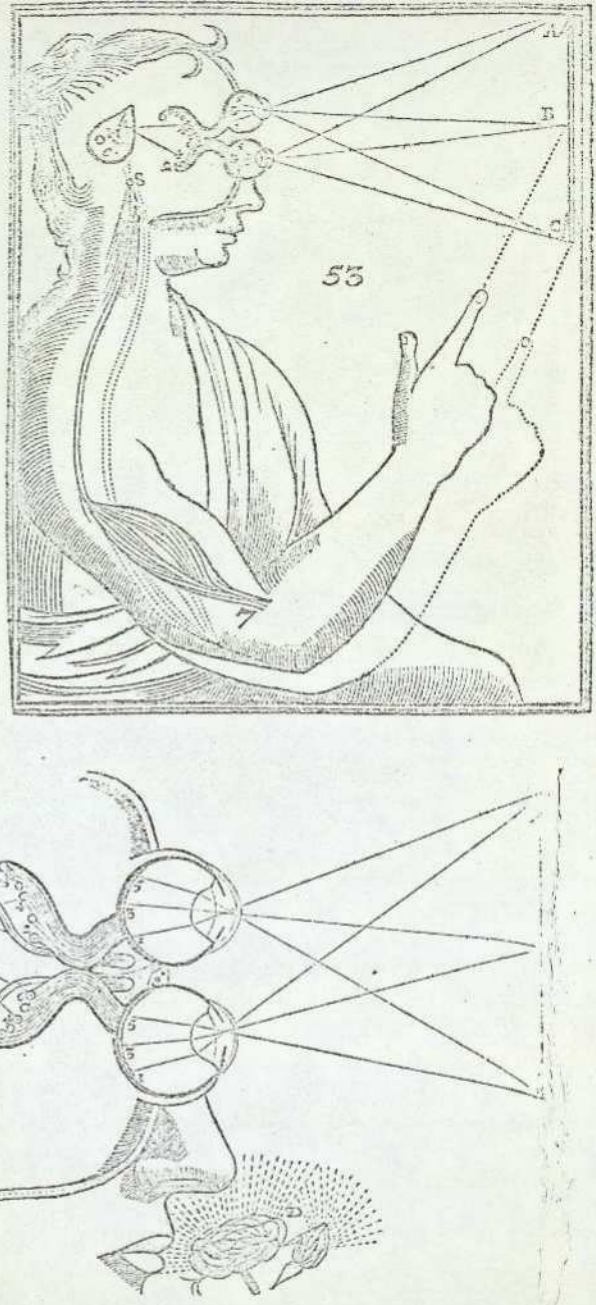


Fig. 1.14 Two diagrams from Descartes, R. (1664)[85] *Les Traitez de l'Homme*, which demonstrate that each point in external space is imaged onto a unique point within the pineal gland and this is true for all the senses.

From the copy in the British Museum

Thus, Descartes was not only the first to postulate the existence of corresponding retinal points, he also suggested a topological mapping of single points in external space with single points within the brain.

The first announcement of the partial decussation of the optic nerves at the chiasma, as the mechanism of binocular single vision, was by Newton in 1704.[86] He said:

> "Are not the Species of Objects seen with both Eyes united where the optick Nerves meet before they come into the Brain, the fibres on the right side of both Nerves uniting there, and after union going thence into the Brain in the Nerve which is on the right side of the Head, and these fibres on the left side of both Nerves uniting in the same place, and after union going into the Brain in the Nerve which is on the left side of the Head, and these two nerves meeting in the Brain in such a manner that their fibres make but one entire Species or Picture, half of which on the right side of the Sensorium comes from the right side of both Eyes through the right side of both optick Nerves to the place where the Nerves meet, and from thence on the right side of the Head into the Brain, and the other half on the left side of the Sensorium comes in a like manner from the left side of both Eyes.
>
> "For the optick Nerves of such Animals as look the same way with both Eyes (as of Men, Dogs, Sheep, Oxen, &c.) meet before they come into the Brain, but the optick Nerves of such Animals as do not look the same way with both Eyes (as of Fishes and of the Chameleon) do not meet, if I am rightly informed."

So far as we know, Newton did not undertake any anatomical experiments. It is unfortunate that he did not quote the source of his information.

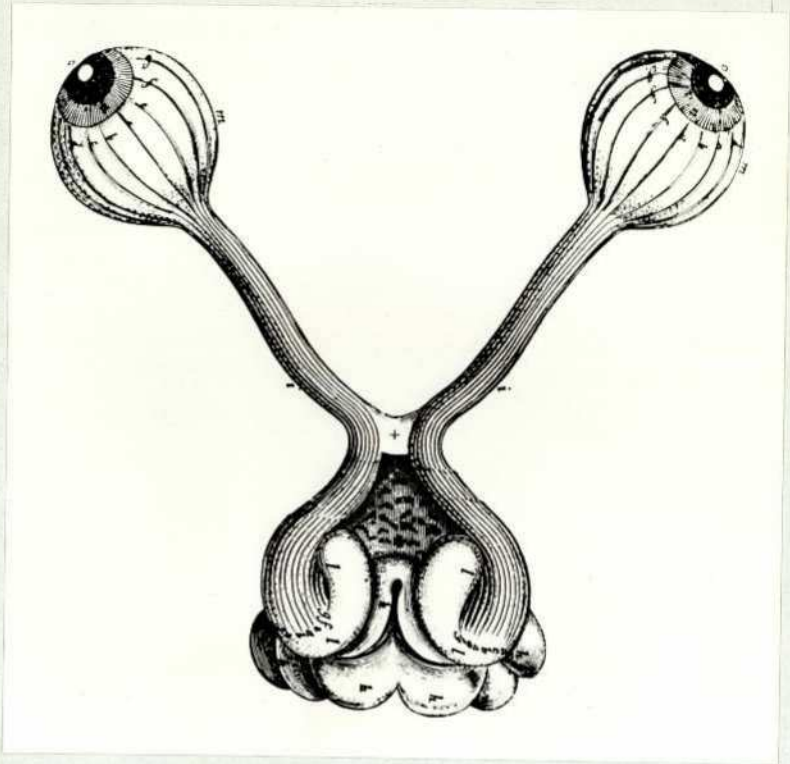


Fig. 1.15 The Visual Pathway according to Briggs[87] (1685) Illustrating the lack of contact between the nerves of the two eyes, but also suggesting a topological relationship between localized areas of the retina and localized areas of the brain.

The semi-decussation is clearly indicated in fig. 1.16[88].

The notion of corresponding retinal points was clearly enunciated by Huygens at about the same time:

> "Nature has provided for us in a very particular manner so that our two eyes would not show us objects double. She has done so, that each point of the fundus of the eye has its corresponding point in the fundus of the other, in order that if a point of the object falls on any two of these corresponding points, then it appears single as it is."[89]

Taylor (1727) offered an anatomical explanation for this, suggesting that those optic nerve fibres that are 'akin' unite in the brain. Thus, each point in one retina becomes anatomically correlated to a similar point in the other; this correlation occurring as a result of the fibres from the homonymous halves of the two eyes meeting and semi-decussating at the chiasma.[90]

Smith (1738) defined corresponding retinal points with regard to their location:

> "When the optick axes are parallel or meet in a point, the two middle points which are equally distant from them, and lye on the same sides of them either towards the right hand or left hand, or upwards or downwards, or in any oblique direction, are called corresponding points."[91]

These arise, he explains, because:

> "When we view an object steadily, we have acquired a habit of directing the optick axes to the point in view; because its pictures falling upon the middle points of the retinas, are then distincter than if they fell upon any other places; and since the pictures of the whole object are equal to one another, and are both inverted with respect to the optic axes, it follows that the pictures of any collateral

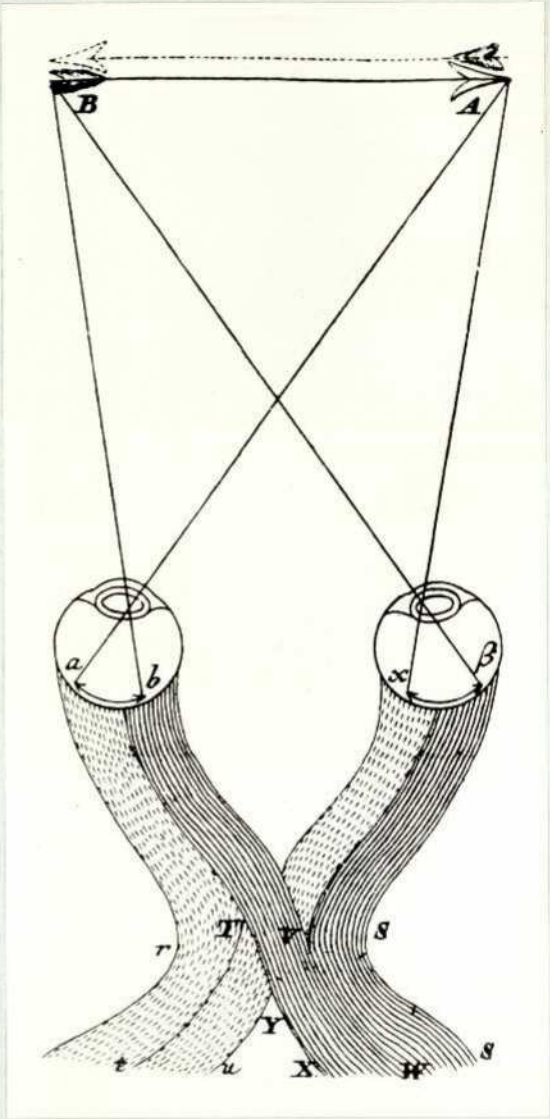


Fig. 1.16 Semi-decussation of the Optic Nerves

The earliest illustration of the semi-decussation of the optic nerves from John Taylor (1727) [90]

point are painted upon corresponding points of the retinas."[92]

And we learn by experience, he claims, that objects are seen as single when they fall upon corresponding points, and double otherwise. (see fig. 1.17)

Whereas Smith defined his retinal corresponding points in terms of position, Reid (1745) defined them in terms of function. Points correspond, he says, whatever their position may be if they give rise to single vision.[93] Furthermore, whereas Smith makes the correspondence acquired[94]; for Reid it is innate[95]. A distinction that was to dominate nineteenth century philosophy.[96]

Reid states:

> ".... that by an original property of human eyes, objects painted upon the centres of the two retinas, or upon points similarly situated with regard to the centres, appear in the same visible place; that the most plausible attempts to account for this property of the eyes have been unsuccessful; and therefore, that it must be either a primary law of our constitution, or the consequence of some more general law which is not yet discovered."[97]

Porterfield (1759) denies the claim that the union of the two optic nerves at the chiasma is to produce single binocular vision, For:

> "If this Conjunction were necessary, to the end that Objects might appear single, the same would also be necessary in all our other Senses, and every simple Sound would be heard double, because our Ears are double, and our auditory Nerves quite distinct. But this is contrary to Experience; every simple Sound being heard simple as it is; tho' our auditory Nerves are so far from being united, that they take a different and almost opposite Course."[98]

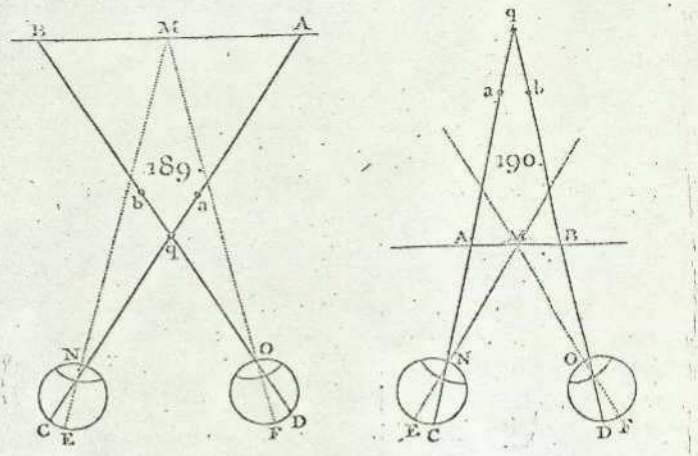


Fig. 1.17 Smith on Physiological Diplopia (1738)

Illustration from Smith (1738) *op. cit.* showing that objects that stimulate non-corresponding points are seen doubled.

His explanation for the chiasmal conjunction of the optic nerves is that this path enables the nerve to enter the eyeball perpendicularly, and consequently reduces in size the "insensible area", i.e., the optic disc.

The problem of the singleness of vision, he says:

> "... admits of an easy Solution ... there being nothing more required, but that we be possessed of a Power of seeing Things in the Place where they are; For tho' the Object is seen by both Eyes, yet being seen in the same Place at the same Time; and that we are possessed of a Power of seeing Objects in the Place where they are is a Matter of Experience..."[100]

Hartley[101], in 1749, advanced the anatomical argument, basing singleness of vision on the semi-decussation at the chiasma. He even suggested that an impression from the retinal image could be passed across to corresponding points in the other eye.

Wells(1792)[102], while:

> "Admitting... that objects are represented single, when their pictures fall upon the centres of the retinas, or upon any other two points which are equally distant from the centres, and both upon the same side, it appears to me, notwithstanding, to be in violation of all analogy, to ascribe this effect ... to any peculiar property which they possess from nature."

The analogy to which he refers is that organs in one species of animal have the same properties as the corresponding organ in another. Even more likely, then, he says, that there should be a correspondence between the different organs found in a particular individual.

He continues:

> "Now it is an universal fact, that if an animal be divided into a right and left half, the corresponding parts of those organs, which exist in pairs, are found at equal distances from the plane of partition. Thus, for instance, in respect to the eyes, the two optic nerves penetrate their outward coat at the same distance from this plane. Their muscles, blood-vessels, and every other of their component parts and appendages are arranged in the like manner; those nearest to the dividing plane, or the innermost, in the one, being similar in structure to the innermost in the other, the outermost to the outermost, and the intermediate to the intermediate.
>
> "It is surely natural therefore, to expect that such parts should also be similar in their properties; and we find in fact this similarity to exist, whenever it can be clearly ascertained what the properties are. Every person, for example, admits, that the internal straight muscle of the right eye performs the same office, with respect to that eye, as the other internal straight muscle does with respect to the left eye.
>
> "What judgment are we then to form of the opinion of Dr. Reid[103], which attributes the same original properties, or rather the joint possession of one original property, to places in the retinas situated at unequal distances from the general plane of partition; which makes an *external* point in one to correspond, in use, with an *internal* point in the other, and this too by a principle implanted by nature ? If such things exist, they may at least be said to stand opposed to a most extensive analogy."[104]

and Wells argued that therefore the correspondence must be acquired by experience and was not based on anatomical linkages. He supports this view by quoting the experience of Cheselden [105], whose patient at first found things doubled following an accident but, by degrees they became single, although there was still a manifest squint.

The anatomical view received support from Wollaston[105] in 1824, who wrote that he had twice lost the sensitivity of one half of his visual field; and ascribed the loss to the "semi-decussation" of the optic nerves at the chiasma in the way Newton had described.

In spite of arguments as to the mechanism then, in the early years of the nineteenth century there was general agreement that single vision arose as a result of the simultanous stimulation of corresponding retinal points. Then, in 1838, Wheatstone wrote:

> "When an object is viewed at so great a distance that the optic axes of both eyes are sensibly parallel when directed towards it, the perspective projections of it, seen by each eye separately, are similar, and the appearance to the two eyes is precisely the same as when the object is seen by one eye only ..... But this similarity no longer exists when the object is placed so near the eyes that to view it the optic axes must converge; under these conditions a different perspective projection of it is seen by each eye, and these perspectives are more dissimilar as the convergence of the optic axes becomes greater."[107]

That is to say, Wheatstone was the first to realize that an essential part of binocular vision is the formation of non-identical images on the retinas of the two eyes. And the fact that two non-identical images can be united into a single percept is an even greater problem than the problem of the unification of two identical ones that had concerned previous writers.

In this paper Wheatstone describes his new invention, the stereoscope (fig. 1.18), with which he presents dissimilar pictures to the two eyes and sees them to be unified into, not only a single perception, but a single perception in relief.

Then, having proved that objects

> "whose pictures do not fall on corresponding retinal points may still appear single"[108]

He goes on to prove

> "that similar pictures falling on corresponding points of the two retinae may appear double and in different places."[109]

and is led to the conclusion:

> "that there is no necessary physiological connection between the corresponding points of the two retinae, - a doctrine which has been maintained by so many authors."[110]

What then, he asks, does determine singularity and suggests: that each point of a figure is seen as single when it is at the point of

> "intersection of the two lines of visible direction in which it is seen by each eye separately, whether these lines of visible direction terminate at corresponding points of the two retinae or not."[111]

He goes on to point out though, that on this supposition points before and beyond the point of intersection of the optic axes should never appear double; but it is evident that they do.

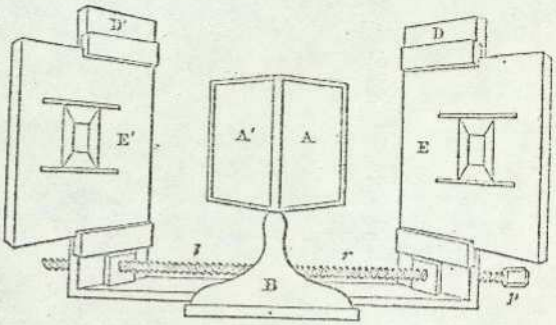


Fig. 1.18a Wheatstone's Stereoscope

E and E' are the disparate drawings that Wheatstone showed could be fused into a single three dimensional percept. Thus showing that two disparate images can give rise to a single percetion.

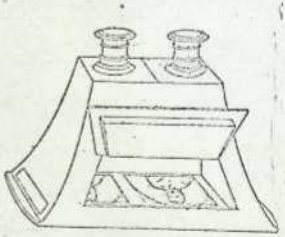


Fig. 1.18b Brewster's Refracting Stereoscope

Stereoscopes like this became the playthings of Victorian parlours.

So according to Wheatstone, the criterion for single vision becomes the identity of visual direction. But this suggestion had already been advanced by Wells:

> "For, if the question be concerning an object at the concourse of the optic axes, I say it is seen single, because its two similar appearances in regard to size, shape and colour, are seen by both eyes in one and the same direction"[112]

But, how is this visual direction to be defined? On this, Wells wrote:

> "Since Kepler's great discovery of the seat and manner of vision, there have been, as far as I know, only two theories offered respecting the apparent direction of objects. One is, that they are perceived in the direction of lines passing from their pictures on the retina through the centre of the eye; the other, that their apparent directions coincide with their visual rays."[113]

The first of these views had been advanced by Reid[114] and Porterfield[115]; and Smith, who said:

> "the visible direction of an object coincides with the visual ray, or the principal ray of the pencil which flows from it to the eye"[116]

D'Alembert had said that the visual direction was in the line joining the point itself and its retinal image.[117]

But Wells had had the same doubts as Wheatstone:

> "But both of these theories are inconsistent with the phenomena of single vision with the two eyes. For according to neither of them can an object, placed at the concourse of the optic axes, be seen single, unless we have the most accurate knowledge of its distance; nor will

> either admit two objects to be seen as one, which are situated in the optic axes, whether on this side, or beyond where they meet, unless the united object be referred by sight to their very point of intersection; both of which conclusions are contradicted by experience"[118]

However, Wheatstone's contemporary, Sir David Brewster pointed out that these three lines so nearly coincide with each other, that

> "at an inclination of 30^{0}, a line perpendicular to the point of impression on the retina passes through the common centre, and does not deviate from the real line of visible direction more than half a degree, a quantity too small to interfere with the purposes of vision."[119]

He goes on:

> "As the interior eyeball is as nearly as possible a perfect sphere, lines perpendicular to the surface of the retina must all pass through one single point, namely the centre of its spherical surface. This point may be called the centre of visible direction, because every point of a visible object will be seen in the direction of a line drawn from this centre to the visible point."[120]

As we have seen, it was Wheatstone who first realized that the two eyes are able to fuse non-corresponding points and see them as single, but it was Brewster[121], in 1844, who emphasized that non-corresponding points are a necessary condition of stereopsis. But the credit for realizing that binocular parallax accounts for the perception of depth belongs to Joseph Harris (1775), who wrote:

> "And by the parallax on account of the distance between our eyes, we can distinguish besides the front part of the two sides of a near object, not thicker than the said distance; and this gives a visible *relievo* to such objects, which helps greatly to raise or detach

them from the plane, on which they lie. Thus, the nose on a face is the more remarkably raised by our seeing each side of it at once."[122]

Wells and his immediate predecessors were aware that single vision could be achieved, even when non-corresponding retinal points were stimulated. Their various theories of binocular vision were attempts to try to explain this away, without their being aware that this was the basis of the perception of depth.

Wheatstone and Brewster realized that this phenomena - named disparity by Fechner[123] (1838) - was an essential ingredient of stereopsis, but they still had no satisfactory way of accounting for its mechanism. They abandoned the concept of corresponding points and adopted the concept of visual direction, but they still could not satisfactorily explain why disparate images gave rise to a single perception.

Brücke[124] in 1841 attempted to restore the idea of corresponding points by arguing that an extended solid object is not seen as three dimensional until the eyes have shifted their point of fixation many times; converging first on one point and then another; doubled images being ignored. Dove[125] disproved this suggestion immediately, by observing that a stereogram could still be perceived as solid when illuminated by an electric spark lasting less than a millionth of a second.

Experiments with the stereoscope confirmed that it was possible to achieve binocular single vision, even when the retinal points stimulated were disparate, but it was found that if the disparity was greater than a certain amount, then single vision broke down and diplopia supervened.

Wheatstone found that:

> "... if the inequality of the pictures be greater than the difference which exists between the two projections of the same object when seen in the most oblique position of the eyes normally employed, they do not coalesce."[126]

This led Panum[127] in 1858 to postulate the existence the existence of "corresponding sensory circles" suggesting that a point in one retina corresponded with a small circle of retinal points in the other.

Using a pair of stereograms as in fig. 1.19[128], he found that the two pairs of vertical lines could be fused if the difference of separation was less than 25 minutes of arc, but not otherwise. In fact, if the difference of separation was greater than 25 minutes, then the impression was one of three vertical lines. These zones of correspondence are today called "Panum's fusional areas".

Helmholtz disputed this interpretation:

> "... I have maintained that the reason why it was possible to fuse the double images was because the eye was not reliable enough or accurate enough to estimate the corresponding dimensions of the two images without possibility of errors, and that any error of this sort is fostered by the appercetion of the material object which is present before us, or which we fancy is present before us."[128a]

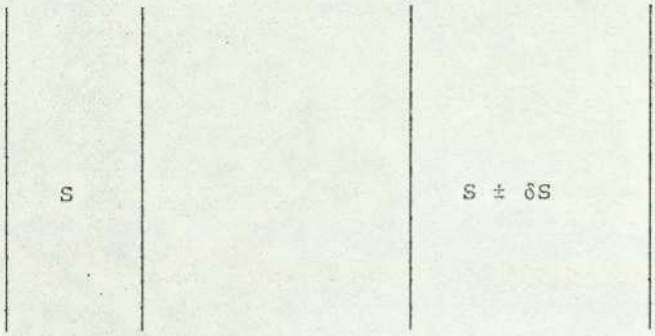


Fig. 1.19 Panum's Stereogram

A similar pair of stereograms to those described by Panum[127]. The separation S of the pair of lines seen by the left eye (in this case) is kept constant, and the separation of the lines seen by the other eye is varied. Then, this stereogram can determine δS, the maximum difference of separation for which single vision can be maintained. (See Mitchell[126])

This view was supported by Bourdon[129], who suggested that diplopia is compatible with a very fine stereoscopic acuity. That is to say, if the ability to recognise double images is much cruder than the threshold for stereopsis, then the stimulation of disparate points, close together, will not give rise to any perceived diplopia.

This controversy is not completely resolved today[128]. The confusion seems to arise from expecting physiology to exhibit the precision of physics. Nowwithstanding, it is sufficient to be aware that the relationship between a given retinal point, and the points in the other retina, with which it can give rise to a single percept, is a "one-many" relationship, a view that has been given anatomical justification by the recent work on binocular interaction in the striate cortex.[130]

The concept of corresponding retinal points has been restored by the consideration that each receptor element has associated with it a certain subjective visual direction. Corresponding retinal points can then be defined as the particular points in each eye that will give rise to the same such direction. This idea is pursued in much greater detail in the next chapter.

(ii) The Horopter

In 1613, soon after Kepler's discovery of the retinal image, an obscure Jesuit named Franciscus Aguilonius, coined the word *horopter*.[131] [132] He proposed that all visual images were projected onto a certain plane passing through the point of fixation. He called this plane "the horopter", and deduced that points located on this plane would be seen singly by the two eyes, and that other points would be seen doubled.

Wells[133] informs us:

> "If a line be drawn through the point of mutual intersection of the optic axes, parallel to the interval between the eyes, Aguilonius calls it, from its office, the *horopter*; and if through this line, a plane be made to pass at right angles to that of the optic axes, he names it the *plane of the horopter*. After defining these terms, he asserts, that, by a law of our constitution, all bodies which we see with one glance or look, whatever are their real places, appear to each eye to be situated in this plane. - And if this be granted to him, he easily and satisfactorily shows, why some should be seen single with the two eyes, and others double.
>
> "For since, according to a second opinion maintained by him, and not contradicted, I believe, by any other writer on vision, the two lines of direction, in which an object is seen when we employ both eyes, can meet each other only in one point, it follows, that all bodies which are really situated in the plane of the horopter, must necessarily appear single, as the lines of direction in which any one of them is perceived by the two eyes, coincide in that plane and nowhere else; and that all bodies, which are not situated in the plane of the horopter, must as

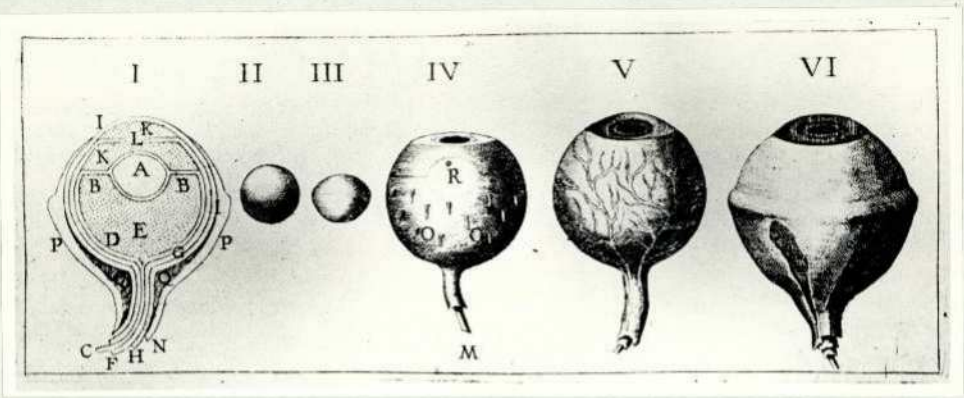


Fig. 1.20 Aguilonius on the Eye

Diagram of the eye and the dissection of various parts, from Aguilonius (1613)[129]

> necessarily appear double, since in this case, the lines of their visible directions intersect each other, either before or after they pass through it."[133]

It is of interest to note just how early the idea of visible or visual directions occur in horopter theory in view of its current significance.

His view of the horopter, as a vertical plane through the fixation point,¶ is beautifully illustrated in the frontispiece, which shows an idealized experiment to plot it. His horopter diagram is given in fig. 1.21.

Aguilonius also describes a circular pseudo-horopter, or false horopter (fig. 1.22), which clearly anticipates the horopter circles of Vieth[135] and Müller[136].

Aguilonius was aware that solid objects, which cannot lie completely in the plane of the horopter should appear confused and doubled according to his proposal. He wondered:

> "how it happens that bodies seen by both eyes are not all confused and shapeless, though we view them with optical axes fixed on the bodies themselves .... Since the images in each eye are dissimilar, the representation of the object must appear confused and disturbed to the primary sense ...."[137]

Thus, so soon after the discovery of the retinal image, it was noticed that the images are dissimilar, a phenomena unnoticed again until Wheatstone.

He goes on to say, that what is "truly

¶A similar view is implicit in the remark of Pettigrew (1972)[138], where he suggests that zero disparity sensitive neurons would fire in response to stimuli lying in a plane lamina, and not otherwise.

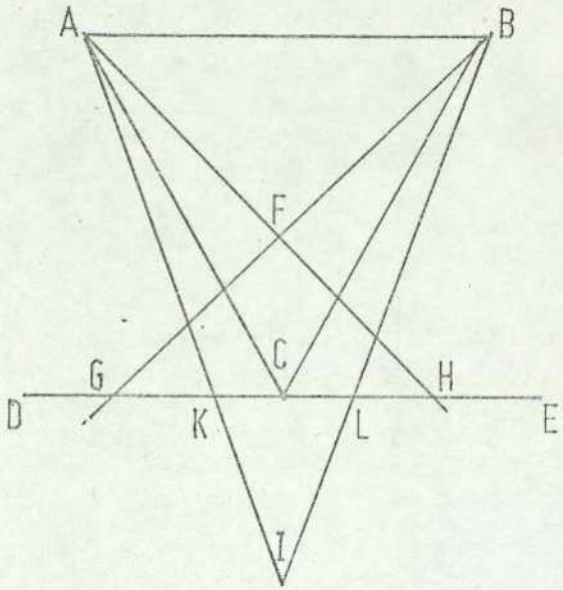


Fig. 1.21 Aguilonius' Horopter Diagram

A and B represent the two eyes, and DE the horopter plane, which passes through the fixation point, C, perpendicular to both the fixation and the median plane.

Aguilonius asserts that all visible objects are projected and seen as if they lay in this plane. Hence, if an object is at F, it will be seen doubled, apparently situated at G and H. Similarly an object located at I, will appear to be located at both K and L. It is only objects actually located in the plane of the horopter that will be seen in their true position, and be seen singly.

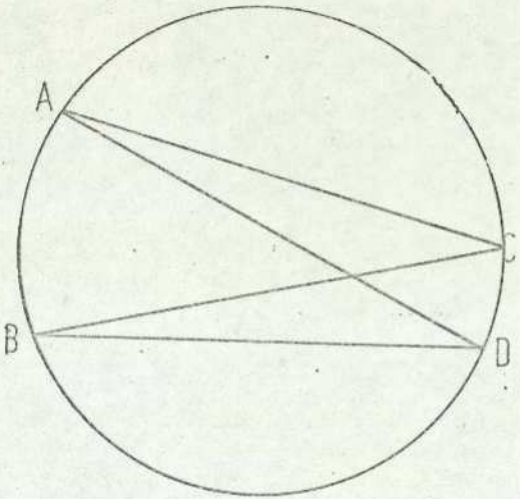


Fig. 1.22 Aguilonius' Pseudo-Horopter

Aguilonius says:

> "If objects fall upon different common radii, it can happen that things unequally distant can nevertheless be perceived at equal angles. Let point C be placed directly opposite the visual points A and B, and then let a circle be drawn through the three points A, B and C. Then, by No. 5 of Euclid's Fourth, any other point whatsoever taken on its circumference - say D - which necessarily lies closer to the observer than C, will include and angle ADB which will be equal to angle ACB by No. 21 of Euclid's Third. Consequently, the observer will believe that objects at C and D are equally far from the eye. But this is false, since it is agreed that point C is farther away than D."
>
> From Aguilonius[156], translation: Shipley and Rawlings[139]

wonderful", is that objects actually are:

> "seen clearly and distinctly with both eyes, when the optic axes are converged upon them."[140]

How, he asks, is this contradiction explained ?

And answers, by the action of "common sense":

> "Whatever body ... each eye sees with the eyes conjoined, the common sense makes a single notion, not composed of the two which belong to each eye, but belonging and accommodated to the imaginative faculty to which the common sense assigns it"[141]

Aguilonius' frontal plane was formulated as an explanation of what he called the "illusion of number", i.e. there are apparent double images visible even when it is "known" that there is really only one. And further:

> ".... we draw a mental line through the intersection of the optic axes by which ... our gaze is nevertheless delimited and, somehow constrained from variable positions into definite limits ..."[142]

Smith (1738)[143](see fig. 1.23) agreed with Porterfield (1759), who said:

> "... we always refer the Place of Objects to the *Choropter*, or Plan on which our eyes are fixed."[144]

that the horopter was Aguilonius' fronto-parallel plane through the point of fixation, and also that all images were projected to, and seen to lie in this plane.

On the other hand, Wells (1792), while agreeing that the Aguilonius plane was the horopter plane, denied that all images where seen to lie in it.

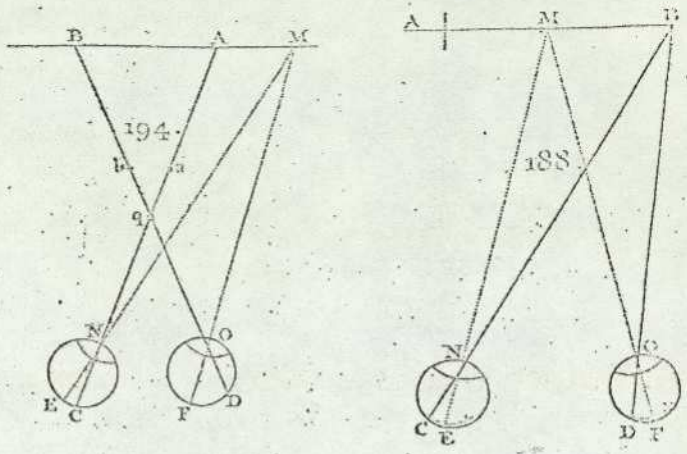


Fig. 1.23 Smith on the Horopter (1738)

Diagram from Smith (1738) *op. cit.* illustrating his concept that all images appeared to lie on the horopter in the fronto-parallel plane through the fixation point.

From the copy in the British Museum

More significantly, he introduced the now vital concept of visual direction into horopter studies. He said:

> "Objects anywhere in the horopter, will be seen single, because their apparent direction to the two eyes will then completely co-incide. And, for a contrary reason, those placed in any other part of the plane of the optic axes will appear double."[145]

but

> ".... it is not necessary that the united appearance should be judged to be at any particular distance ... in opposition to the plainest observations."[146]

Wells also describes an experiment that, he says, exemplifies the principal property of the horopter plane:

> "I suspend a fine chord at right angles to the horizon, and retreating a step or two, I looked steadily at a point in it which was upon a level with my eyes. The chord in these circumstances appeared single; but whenever I directed my eyes to any other point of it, either above or below the former, two chords would appear, crossing each other at the part, to which the eyes were directed. In the first case, the whole chord was in the plane of the horopter, but in every other, only that point of it to which both eyes happened to be turned."[147]

The first writer to discard the fronto-parallel plane as the horopter was Pierre Prévost[148] in 1804, who stated:

> "It follows from the stated law, that in the plane of the optic axes, the position of those points seen single with the two eyes is a circumference of a circle which passes through the two centres of the eyes and the intersection of their axes. I refrain here from demonstrating this proposition which is easy to deduce from principle...."

The principle referred to, is indeed, a simple one, and its explicit formulation fourteen years later gave Vieth the credit for priority. It is worth noting that this circle and the principle itself had already been put forward by Aguilonius (see: fig. 1.22). However, arguing from the work of Smith and Reid, Prévost also presented the case for another horopter; which was a vertical circle generated if the fixation point was rotated about the line joining the centres of the two eyes. Prévost himself said that he found difficulty in choosing the correct horopter between these two however, he:

> "preferred the last, because it ties the phenomena of double vision to those of single vision, in immediately deriving the position of other points seen single with the two eyes from the law of projection...."[149]

In 1818, Vieth[150] related corresponding retinal points to the shape of the horopter, and produced the diagram in fig. 1.24, based on the elementary geometric proposition, known to Euclid[151] (c. 300 B.C.), that the angles in the same segment of a circle are equal. His argument shows that if corresponding retinal points are equidistant from the fovea in each eye, then the locus of points which will simultaneously stimulate such points is the circle through O, U, and P, i.e. through the centres of the two eyes and the fixation point.

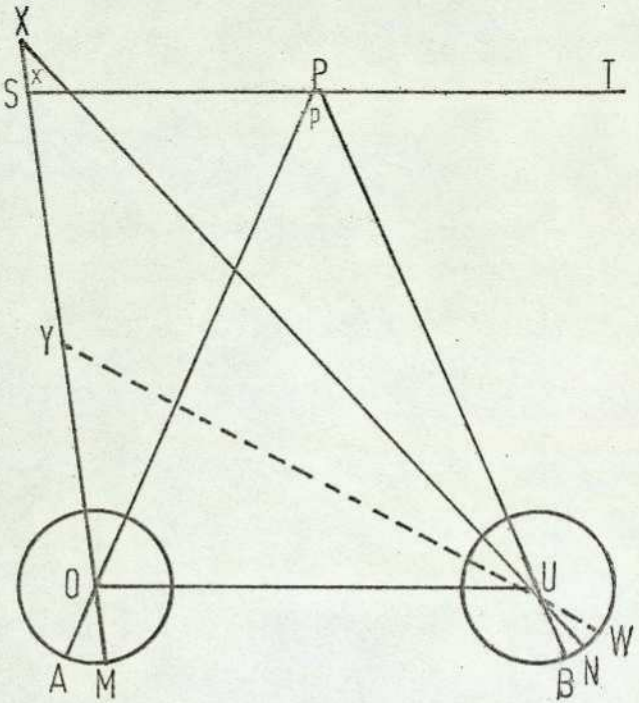


Fig. 1.24 Vieth's diagram of the Horopter[152]

Vieth stated:

"The images M and N, in this case, will lie equally far from A and B, when angle p and angle x are equal ... and X lies on the circumference of a circle which goes through O, U and P, such that all these angles at O and U subtend the same chords on this same circumference ..."[153]

In 1826, Johann Müller[154] independently published fig. 1.25, which he described:

> "... let *a* and *b* be identical central /i.e. corresponding foveal/ points of the retinas *mn* and *xy*. All parts in eye A which lie between *a* and *m* are identical to parts in eye B which are equally spaced between *b* and x. Moreover, *an* is identical to *by*, in accordance with the principles previously developed from subjective visual phenomena."[155]

That is, identical points are similarly situated in the two eyes.

> "If the visual axes intersect at c, then c appears single on the identical points *a* and *b*, and *d* also appears single on identical points *e* and *f*. When the convergence of the visual axes is at c, then all other objects at different distances appear double. In eye A, *i* appears on *g*; in eye B, *i* appears on *h*. But *g* and *h* are not identical. Rather, *g* is identical to *o* in eye B because both points are equally far from the central point of the retina. Consequently, the double images of *i* appear on *g* and *h*, separated from each other by the actual sum *ag+hb*. That is - the separation of the double images is to the whole visual field as *ag+hb* is to the size of the retina or, in relation to the point of convergence, the double images lie to the side of the visual exes in the magnitudes of the visual angles *irc* and *csq*. Therefore *i* appears as *p* and *q* in the plane of the convergence point."[156]

This circle, that is, the circle passing through the fixation point and two points locating the eyes*, is today called the Vieth-Müller circle. If it is agreed that corresponding retinal points,

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*Vieth located the eye at the pupil, and Müller at the centre of the lens. The problem of the proper location of the eye in these studies still gives rise to controversy today.[157]

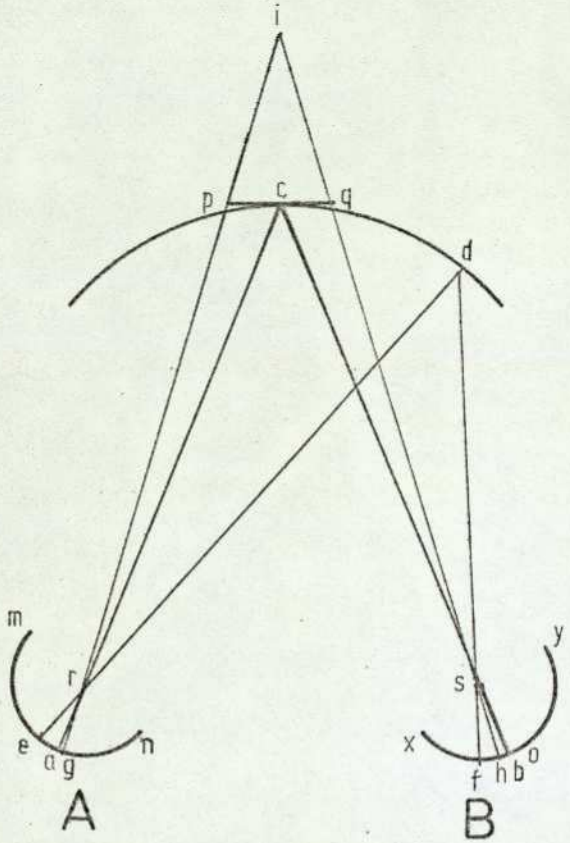


Fig. 1.25 Müller's diagram of the Horopter[154]

are points equidistant from the two foveas, as Müller had asserted, then this circle follows from the (two-dimensional) geometric analysis of these points. For this reason, it is often nowadays, referred to as the theoretical horopter[158], or, the mathematical horopter[159], and it is generally shown in the form (see fig. 1.26) first published by Volkmann[160] in 1836.

It is worth noting that Müller did not take "single-vision" as the criterion for his horopter, but rather defined it from the notion that corresponding retinal points are equidistant, and in the same direction, from the foveas in the two eyes. He further suggested that there must be an anatomical basis for this correspondence.

The third dimension was introduced into the analysis in 1843 by A.P. Prévost[161] and later by F. Burckhardt (1854)[162], who agreed that this involved taking into account cyclo-rotational movements of the eyes. They showed that if the eyes were adjusted without torsional rotations, then a straight line, not lying in the fixation plane, should be added to the Vieth-Müller circle.

The first attempt to verify the shape of the horopter curve by experiment, was made by Meissner[163] in 1854, who tried to arrange threads in positions where they could be seen as single.

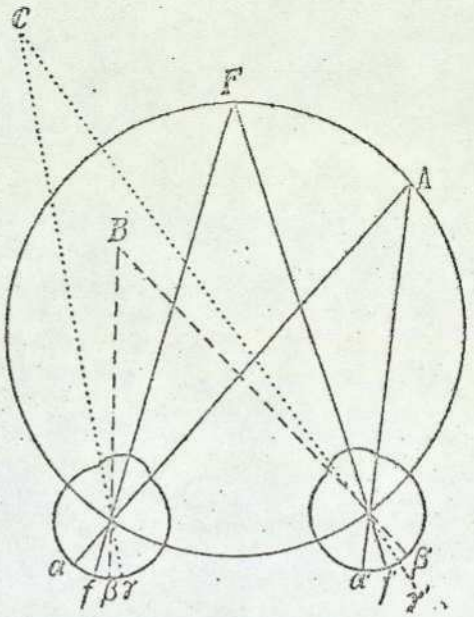


Fig. 1.26 A Current Represenation of the Vieth-Müller Circle

The points F and A which lie on the circle are imaged on corresponding retinal points f and f' and a and a' respectively. B and C, lying inside and outside the horopter, are imaged on non-corresponding (disparate) points β and β' and γ and γ'.

Illustration and caption from Adler op. cit fig. 409.

In a series of papers, Serre d'Uzès[164] [165] [166], tried to disprove the current theory of retinal correspondence by setting up a model of the binocular retinal system, apropos of which he said:

> "... as the reflected ray is physically stopped by the limiting screen, so in real vision, sight is virtually, or imaginatively stopped, for the particular direction that it travels in space by an invisible surface; which I have given the name *rideau physiologique* (limiting visual surface). [167]

He emphasizes that this was not a hypothetical surface but an important sensory phenomena.

He extended the horopter to three dimensions by saying:

> "In a plane, the geometric locus is an arc of a circle; in space, this is a surface of revolution formed by the rotation of this arc around the axis connecting the centres of the two crystalline lenses. This is the surface of a *torus* not that of a sphere as Porterfield* would have it ...." [168]

This, he calls the limiting torus.

These papers were violently criticized by Claparède[169] [170] [171], who describes them as reporting phenomena that are visible to the eyes of M. Serre d'Uzès alone. The limiting torus, he says, cannot exist. In fact, the horopter cannot be either a torus or a sphere, and quotes the work of A. Prévost and the experiments of

---

*This, surely should have been Prévost, who did say this. There is nothing like this in Porterfield.

Meissner, to prove that the horopter cannot be any form of surface.

His analysis of physiological diplopia, and the introduction of the third dimension to the horopter can be seen from the following quotation; (see: fig 1.27) where

> "... the eyes at I and II, are fixating a point A, in the horizontal visual plane. The points B and B' are situated in the same plane, one behind, and the other in front of the point A. They will be seen doubled; that is the images α and β of the point B will be seen uncrossed (that is to say, the left image α, belongs to the left eye I, and the right image β, to the right eye II), whereas the images γ and δ of the point B' will be crossed.
>
> "If we keep the optical axes converged onto the point A, and gradually raise the point B vertically, we will see the two images α and β slowly approach each other until they are seen as single, once B has been raised to a sufficient height ... If on the other hand, we lower the point B below the visual plane, we will see the two images α and β diverge until they finally leave the visual field."[172]

Whereas, raising and lowering the point B', has precisely the reverse effect. In fig. 1.27b,

> "... the visual plane I II A is represented in perpsective, whereas the vertical plane through RB'AB coincides with the plane of the paper. The reciprocal of the distance between the pairs of doubled images is proportional to the distance that separates the object from the horopter. It is thus clear, that when we raise the point B vertically, we approach the horopter at the same rate as the two images approach one another, and, at the moment that the two images meet, B is located on the horopter. If the point B becomes single when it arrives at the point ω, then it must lie on the line of intersection of the horopter and the vertical plane.

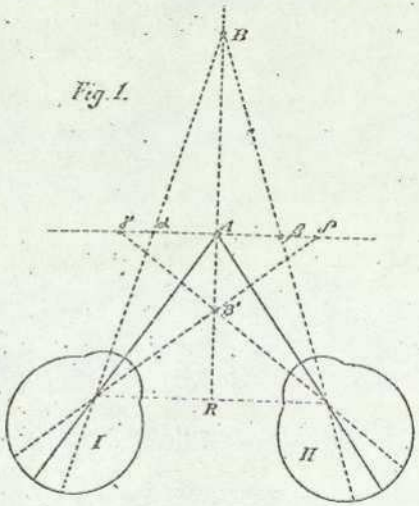


Fig. 1.27a Claparède's analysis of physiological diplopia

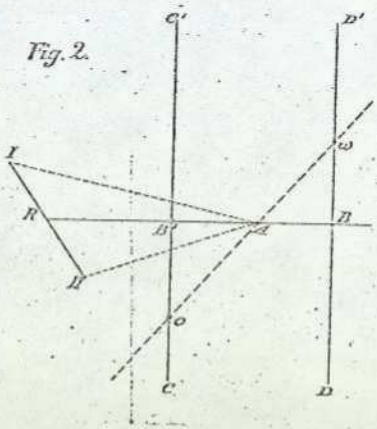


Fig. 1.27b Claparède's demonstration of the skew horopter line

"In the same way for the point B', it can be demonstrated that the point o must be on the line of intersection of the horopter and the vertical plane. And so, the three points o, A and ω, belong to both the horopter and the vertical plane, thus the line oAω, which passes through these three points is the line of intersection of the horopter and the vertical plane."[173]

Claparède, utilized Meissner's results, and added to them with some similar experiments of his own. He erected a thread through the fixation point, and inclined it at various angles in the vertical plane (the plane RB'AB of fig. 1.27b), he found that it was only seen as single along its whole length when it was inclined at 45^{o} to the vertical, and he added this line to the horizontal circle, giving a three dimensional horopter (see fig. 1.28). This explanation has been criticized[174] but this phenomena can be explained by the analysis of Part III (see below).

During the 1860s both Helmholtz and Hering published many works on the form of the horopter. These works alternated with each other, and involved a great deal of refutation and arguments over priorities. A concise, if somewhat partisan view of this can be given by quoting Helmholtz' own summary of the controversy.[175]

"My first contribution on this subject was made before the Heidelberg Medical Society of Natural History, October 24, 1862; the manuscript being transmitted on November 8, 1862. In this paper the equations of the form of the horopter in general were given for the first time, although they were not expressed in their simplest forms, because the horopter was regarded as being a curve made by the intersection of a surface of the second degree and one of the fourth degree.

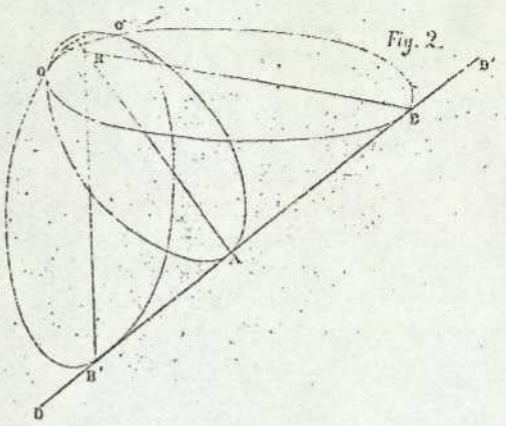


Fig. 1.28 Claparède's Horopter

The horopter consists of the Vieth-Müller circle O O' B, (which Claparède calls the Prévost circle) together with the line D'BB'D, in the median plane at 45^{o} to the vertical.

"Nor was the deviation of the apparently vertical meridian taken into account in this paper. The form of the horopter in the general case was briefly described. Before this merely preliminary contribution had been actually published (which was not until the autumn of 1863), the third part of Mr. E. HERING's *Beiträge zur Physiologie*[176] appeared, and there the proof was given that the horopter must certainly be always a line at least (if not a surface); but the form of the horopter was not actually determined except in the simpler cases which had previously been considered. Then following my article on the horopter in the *Archiv für Ophthalmologie*, X, 1, pp.1-60[177], the proof of which had been corrected by the middle of March 1864; and in this paper the horopter was considered as being the intersection of two surfaces of the second degree, and the effect of the deviations of the apparently vertical meridians was taken into account. Without being aware of this publication, Mr. E. Hering had sent the printers the fourth[178] instalment of his work in June 1864, in which also the horopter was considered as being the intersection of two surfaces of the second degree, and where the method of STEINER's geometry was used, which is particularly adapted to this problem.

"HERING's criticism here of my earliest contribution was due mainly to his failing to see that what I was talking about was what I have referred to above as the "horopter", whereas what he himself had in mind was the "horopter-curve". The two are not exactly identical, as I have explained in POGGENDORFFS *Annalen*, CXXIII, pp. 158-161[179]. Finally, the fifth instalment of HERING's *Beiträge zur Physiologie*[180] contained a further criticism of my second paper on this subject. I shall allude here to only one point in this latter criticism (see p. 350), about which Mr. E. HERING was really correct, namely, where he says that on page 44 of my paper the angle η has been generally put equal to η_1. I must confess that this was an oversight made when I was just about to start on a journey; I was going over the paper very hurriedly for the last time, trying to condense the mathematical work as much as possible. Formerly, I had treated separately the two cases in which the statement objected to was correct; and so the mistake did not have any effect on the correctness of the subsequent deductions. Some of the

other strictures which Mr. HERING makes are matters of personal interest only and may be answered by readers who wish to do so, without need of further reply from me."

i) Helmholtz

In fact it was Hering's, rather than Helmholtz', view of the horopter that was to prevail as the mainstream. There is an unbroken teacher-pupil lineage: Hering; Hillebrand; Tschermak; Ogle and Ames; to the currently accepted view. For this reason, discussion of his work will be deferred to the next section.

In 1863, both Volkmann[181] and Helmholtz[182] published reports of a phenomenon that they had noticed with after-image experiments. They observed that the apparently vertical meridian seemed to deviate from the true vertical meridian. That is:

> "... if the retinal horizons are in the visual plane, the apparently vertical meridians diverge a little upwards, or converge a little downwards."[183]

That this declination exists has been agreed by some[184 185 186], and disputed by others[187 188 189 190]. It is incorporated into Helmholtz' horopter analysis, but is in no way fundamental to it. The fact that not all subsequent workers have been able to reproduce this deviation does not invalidate Helmholtz' analysis, as has been claimed[191]. In fact, this controversy indicates that there is at least some variability in this phenomenon; and it is this variation that is argued below.[192]

Helmholtz defines the term *visual globe*[193] as the exterior projection of the retina. That is to say, it is the visual field of the fixed eye and moving with it. Hence the vascular system, the blind-spot, the fixation point and after-images will always occupy the same positions on this globe. The horizontal meridian of this fixed globe, he calls the *retinal horizon*.

He bases his horopter on the following criteria:

1) "In the normal eye the points of fixation of the two visual globes are pairs of corresponding points"[194]

2) "The retinal horizons of the two eyes correspond to one another."[195]

3) "The meridians which are apparently vertical with respect to the retinal horizons are a pair of corresponding meridians."[196]

4) "Points in the retinal horizon at equal distances from the point of fixation are pairs of corresponding points."[197]

5) "Points equally distant from the retinal horizons, which are on the apparently vertical corresponding lines, are a pair of corresponding points."[198]

All of which can be taken to imply:

6) "Any pair of points on the two visual globes, which are at equal distances in the same direction from the apparently horizontal and the apparently vertical pairs of corresponding lines, will be a pair of corresponding points."[199]

This last definition is not very different from Smith's definition of retinal correspondence in 1738.[200]

These conclusions, using the criterion of single vision, were confirmed in a series of experiments published by Volkmann in 1863[201] and 1864[202]. Helmholtz[203] describes an experiment where the attempt is made to overcome the effects of fusion. Using a stereogram such as fig. 2.1 with the centres of the two figures at the same distance as the observer's inter pupillary distance, it is found, if the figures are adjusted so that the axes of the eyes are parallel, that all the lines of one figure appear to fall on the corresponding lines in the other. He used black and white lines as they would be less easily fused.

Helmholtz uses the definition (6), as the basis on which to calculate the:

> "positions of those points in external space, whose images are depicted on corresponding places on the two retinas, and which therefore are seen single. The locus of all these points is called the *horopter*.[204]

Wheatstone had shown[205] that under certain circumstances, disparate (i.e. non-corresponding) points are seen single. However it is still true that corresponding points too are seen single, and so the above statement is true. The above experiment also indicates that Helmholtz was aware that as a consequence of fusion, there are points not on the horopter but defined in this way that are also seen as single.

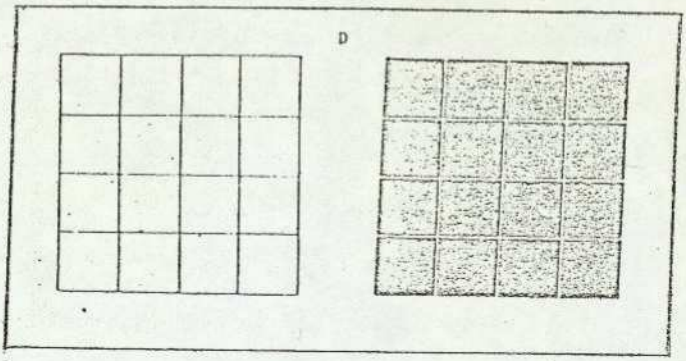


Fig. 2.1 Helmholtz' Stereogram

Helmholtz' experiment to demonstrate that the apparently vertical meridians fall onto corresponding retinal lines; even when the fusion reflex is inoperative.

from Helmholtz (1925) *op. cit.* Plate D

From this starting point, Helmholtz deduces that the horopter is the curve of intersection of two surfaces of the second degree (i.e. quadric surfaces). I will not reproduce his analysis, as it is extremely tedious, and is incorporated, in a more modern form, in the analysis of Part III, below.

He shows further, that this curve of intersection would, in general, break down into a straight line, and a curve of the third degree (i.e. a twisted cubic). However, he nowhere derives an explicit formulation of this curve, but to illustrate it, he produced perhaps the most widely misunderstood drawing [206] (fig. 2.2) in the history of horopter studies.

This curve, as the text makes clear, was meant to be an illustration to the reader of the sort of shape that a twisted cubic curve could take. He never claimed it to be an explicit representation of the horopter curve. However Shipley and Rawlings[207] for example, publish it as Helmholtz' horopter and criticize its obvious inadequacies. Troland[208] (fig.2.3) and Boring[209] (fig. 2.4), roll the paper up in different directions and published the perspective views as two different horopters.

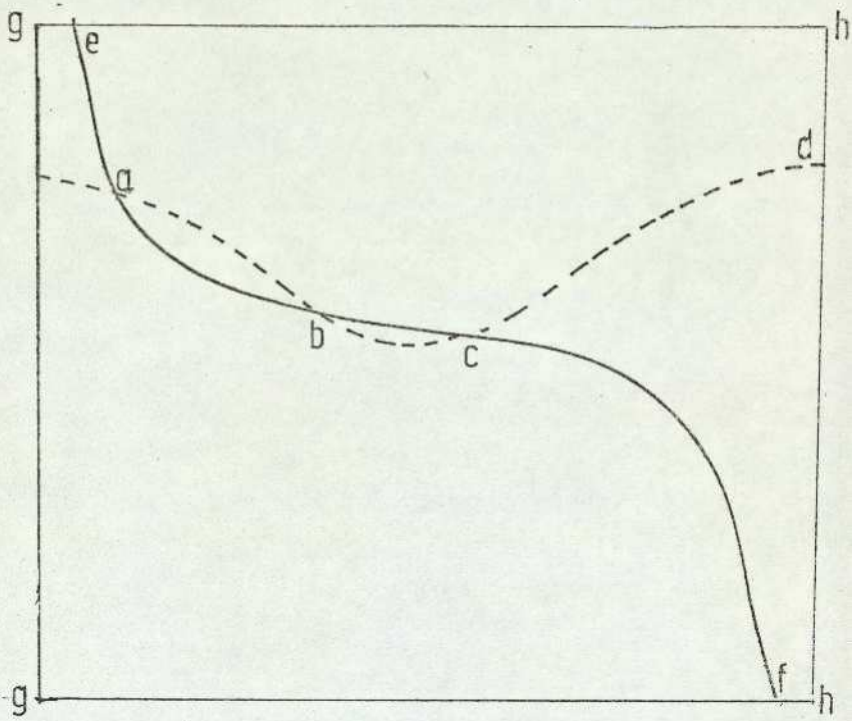


Fig. 2.2 Helmholtz' Twisted Cubic

Helmholtz' description of this curve, reads:

"An idea of the form of such a curve of the third degree can be obtained by supposing that it is drawn on the surface of a cylinder and the cylinder is then unrolled on a plane. The form of the curve would be then like that shown by the continuous line *eabcf* in the figure. When the paper is rolled into a circular cylinder so as to bring the edges *gg* and *hh* together, this curve will have the form of a curve of the third degree"[210]

showing that Helmholtz was merely trying to show his readers one of the forms that a twisted cubic can take.

Redrawn from Helmholtz (1925) op. cit. 3, fig. 69

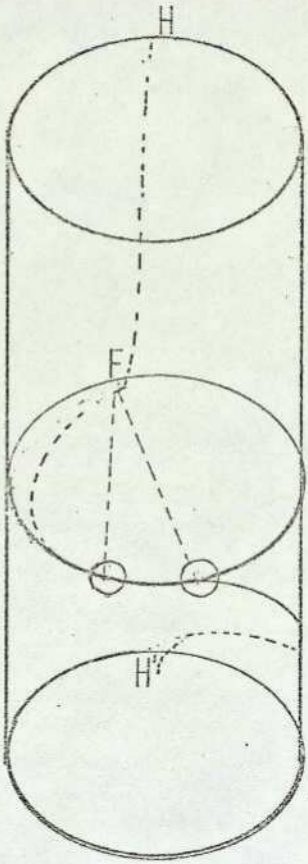


Fig. 2.3 Troland's (1929) interpretation of Helmholtz' Cubic

Troland rolled up Helmholtz' diagram (see: fig. 2.2) away from the reader and published the result as "the horopter".

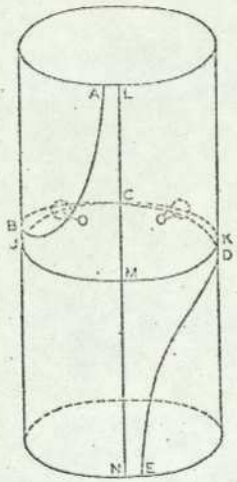


Fig. 2.4 Boring's (1942) interpretation of Helmholtz' cubic

Boring rolled up Helmholtz' diagram (see; fig. 2.2) towards the reader and published the result as the "Generalized Horopter-Line".

from: Boring, E. (1942) *op. cit.* fig. 28

Helmholtz, then, did not determine the explicit form of the horopter-curve in the general case - i.e. the twisted cubic - but he did determine its shape for certain special cases.

He found, that when the point of fixation is at a finite distance, and the two retinal horizons, are equally inclined to the visual planes (extending in opposite directions)*, then the cubic degenerates into a conic section with a straight line through it. This occurs he says,-if the eyes behave in accordance with Listing's law - when the fixation point is in either the median plane, or in the primary position of the visual plane.

He also claimed that:

> "In one single instance the horopter is a surface, which in fact is a plane; and that is when the point of fixation is in the median plane and at an infinite distance, and the retinal horizons, as is usually the case, or practically so at any rate if the eyes are normal, are both in the visual plane. Then this horopter-plane will be parallel to the visual plane, its distance from the latter depending on the amount of divergence of the apparently vertical meridians of the visual globes of the two eyes; that is, it will contain the line of intersection of these two meridian planes and will usually be practically the same as the horizontal plane on which the observer is standing, provided his eyes are normal and directed straight toward the horizon."[211]

---

*This, we call the quasi-symmetric case, in Part III below.

He suggested that the reason for the ground plane being part of the horopter is so that points thereon would be seen singly while walking. Unfortunately, it is difficult to see in what circumstances such a curve can degenerate into a plane, and the assertion is not born out by the analysis. It seems likely, that he is confusing here the horopter-curve, with the line-horopter, which he later defines.

A line, he says, may be seen single if the two retinal image lines that it stimulates correspond. The do not need to correspond point by point as a straight line displaced along itself is still the same line. He shows that, in general, there is such a single line through every point of space. He continues:

> "the surface on which straight line, having a certain definite direction, must lie in order to produce two images that correspond in this fashion is call a *line horopter*."[212]

The lines on the two visual globes that are apparently perpendicular to the retinal horizons he calls the *vertical horopter*, and for lines that are apparently parallel to the retinal horizons, the surface is called the *horizontal horopter*.

He deduced the horizontal and vertical horopters, and thus the horopter-curve, which is their curve of intersection, for two special cases.

a) When the point of fixation is in the median plane. In this case, he found that the vertical horopter was a cone and that the horizontal horopter consisted of two intersecting planes. From which he found that the horopter-curve was composed of a straight line and a conic section.

b) When the point of fixation is in the primary position of the fixation plane. In this case, the vertical horopter is a hyperboloid that intersects the visual plane in the Vieth-Müller circle and the horizontal horopter consists of the visual plane and a plane perpendicular to it. Their common curve of intersection, i.e. the horopter curve in this case, consists of the Vieth-Müller circle and a straight line, which is the form that A.P. Prévost and Burckhardt[213] had both predicted.

The concept of the horizontal and vertical horopter is not very useful, as they are merely two particular instances of the more general "rectilinear horopter" defined and developed in Part III.

Due, perhaps, to the obscurity of his mathematical analysis and even more likely to misunderstanding of his ideas, Helmholtz' work on this subject has had virtually no influence on the subsequent development of horopter studies. However, the mathematical derivation of the three dimensional horopter presented in this thesis is based extensively on the work of Helmholtz summarized above. Modern techniques of linear algebra have enabled the ideas to be extended farther than

Helmholtz cared to take them; in particular to develop the explicit equations for the "rectilinear horopter" and the horopter-curve, in the general case.

ii) The Hering Tradition

Hering[214] introduced two important concepts into horopter studies, viz: the so-called law of identical visual directions and the notion of the apparent fronto-parallel plane.

The first followed from the postulate by Lotze[215] that every retinal element was identified with a unique relative direction called "local sign".

Ogle[216] describes the classical experiment:

> "Suppose one stands about 2 feet before a window /see fig. 2.5/ through which he has a view out-of-doors. While holding his head steady, with the right eye closed, he directs his left eye to a distant object out-of-doors somewhat to the right. Suppose this object is an evergreen tree, and the more this is set off from the surrounding objects the more vivid will be the experiment. While fixating the tree he makes a small crayon mark on the window pane in line with his eye and the tree. Now the left eye is closed and the right eye is opened and directed to the same mark on the pane, and out-of-doors in line with it a distant object is identified, for example, a chimney.
>
> "Both eyes are then opened and directed to the mark on the pane. Immediately the images of this mark "fuse"; the mark is seen as *one* straightforward direction (the principal visual direction) relative to the body. Thus both of the foveas give rise to the *same* subjective direction. While fixating the mark on the pane, one observes that the mark also appears in line with both the tree and the chimney. Both will be seen more or less at the same time.... One sees, therefore, the mark, the tree and the chimney in the *same* straightforward subjective direction....."

This is, perhaps, well illustrated by

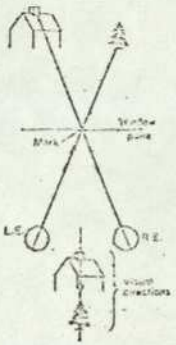


Fig. 2.5 The Identity of Visual Directions

Diagram illustrating Ogle's description of Hering's classic experiment for the identity of visual directions of the fixation point

From Ogle (1962) *op. cit.* p. 222 fig. 1

fig. 2.6, from Adler[217], which illustrates the more general situation and suggests the definition that retinal elements of each eye which give rise to the same primary subjective direction be designated corresponding retinal points.

Hering also postulated that the binocular system had an inbuilt ability to judge whether a point was "nearer than" or "farther than" the point of fixation. A sort of local-sign for depth. Consequently, an observer can judge and select a series of points that would be neither "nearer than" or farther than" this fixation point.

He then identified the horopter as the locus of points with this property, that is, those points which appear to be at the same distance as the fixation point. He further asserted that such points would appear to lie in an apparent frontal plane parallel to the chest.

Ogle[218] pointed out the ambiguity in this i.e. points that are apparently equidistant would not appear to lie on a plane at all. Nevertheless, he said:

> "... the criterion of the apparent fronto-parallel plane is the most used experimentally because it is the easiest and the most accurate."[219]

and he did use this criterion in much of his experimental work.

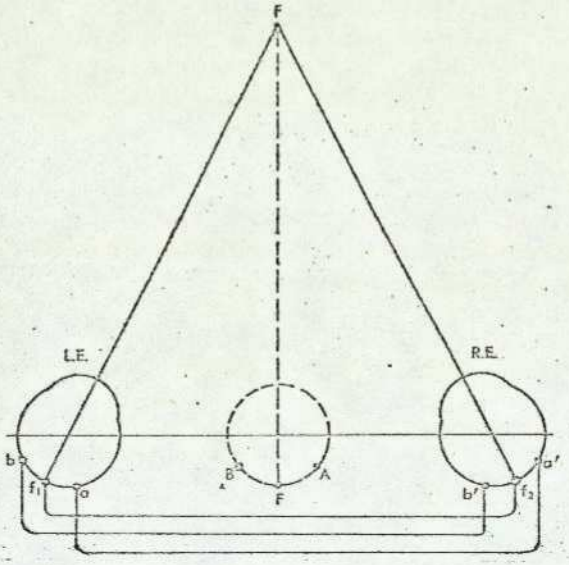


Fig. 2.6 Identical Visual Directions and Normal Correspondence

Visual directions of f_1 and f_2 proceed from F, the fovea of an imaginary cyclopean eye. A and B are points from which the visual directions belonging to a and a' and b and b' would proceed. (see also fig. 2.14)

From: Moses, R.A. (1970) *Adler's Physiology of the Eye*, 5th edition, St. Louis, Mosby, Fig. 23-7.

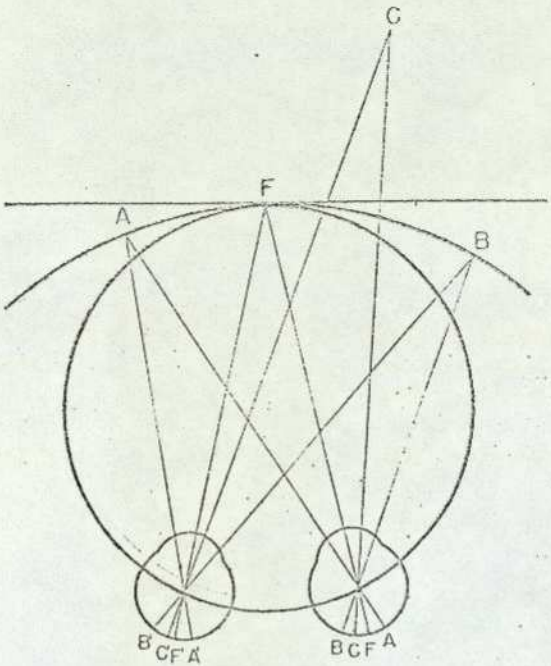


Fig. 2.7 The Horopter Deviation

Diagram to show that the experimentally determined horopter curve lies outside the Vieth-Müller circle. This deviation is called the Hering-Hillebrand deviation.

Experiments to determine the apparent fronto-parallel plane revealed that the curve in which this "apparent-plane" intersected the visual-plane did not lie on the Vieth-Müller circle, as had been expected, but somewhere outside it. (see: fig. 2.7) Furthermore, there was found to be extreme variations in this deviation from the Vieth-Müller circle between one individual and another.

This variation was taken by some to be evidence that the correspondence between the two retinas was not stable. However, Hillebrand[220] showed that the deviation could be explained by a relative asymmetry in the location of the corresponding elements in the two eyes.*(see: fig. 2.8) This departure of the experimentally plotted horopter curve from the Vieth-Müller circle is consequently called the Hering-Hillebrand deviation.

In 1900 Tschermak[221] defined the "longitudinal" horopter, which is a device that reduces considerations of the horopter to two dimensions. That is, the longitudinal horopter is a two-dimensional curve lying in the visual plane.

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* Some evidence for this view was found in the partition experiments of Kundt[222], who showed that a person using one eye to try to locate the mid-point of a line, invariably made his mark too far over towards the nasal side of the eye being used.

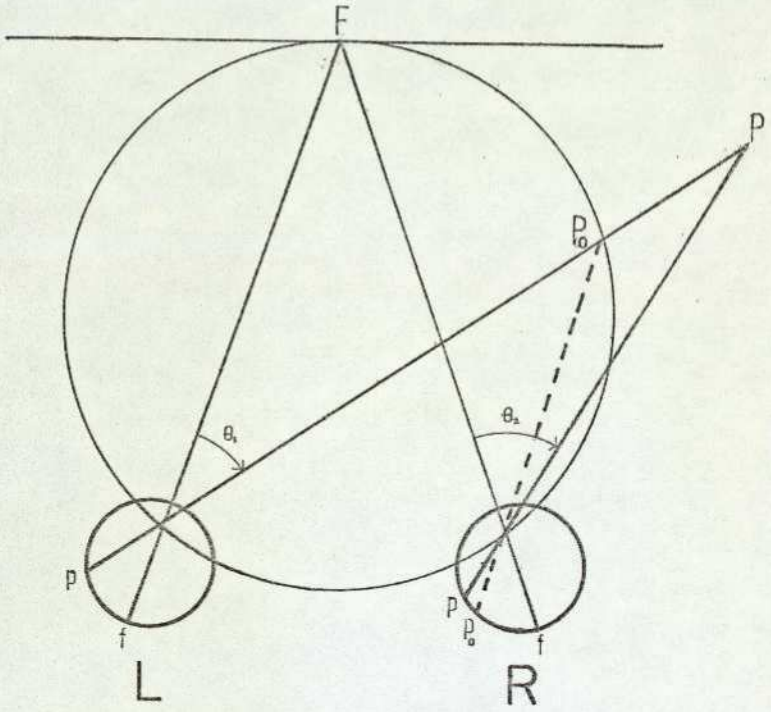


Fig. 2.8 Hillebrand's explanation of the Horopter Deviation as a result of retinal asymmetry

If the point P lies on the horopter, then its image falls on corresponding retinal points. It can be seen from the figure that the angular separation of the images of P from the images of the fixation point F is greater in the right eye than in the left. Only if the point fell on the Vieth-Müller circle at P_0 will these angular separations be equal. Thus, for a horopter curve that falls outside the circle, these separations will be greater in the eye on the same side as the particular horopter point than in the other.

Redrawn from Ogle (1950) *op. cit.* fig. 8

Ogle (1962)[223] gives the rationale:

> "... one can assume that the eyes are so orientated cyclotorsionally that the images of vertical lines in space fall on longitudinal sections of the retinas - which is to say on vertical rows of retinal elements so that the image of any point on the line would give rise to the same primary subjective breadth direction."

and elsewhere[224], he says:

> "Of more practical interest /than Helmholtz' approach/ is the longitudinal horopter, which depends only on the longitudinal sections of the retinas. If these sections are considered as being short enough, the longitudinal horopter approaches as a limit the intersection between the visual plane and the larger horopter surface. This is also the simplest approach to the study of the horopter, since it involves only plane geometry."

On the basis of this argument, there has been no attempt to investigate, mathematically, the third dimension of the horopter during this century. The properties of the "larger horopter surface" described in the passage above have not been explored anywhere in the literature, except, perhaps, by Linskz[225] who in 1954, produced a toroidal surface (see: fig. 2.9) by rotating the Vieth-Müller circle about the inter-ocular base-line; an interesting, if somewhat futile pastime.

Tschermak[226] listed four criteria, or operational definitions of the horopter. These are:

1) the positions of all the points appear to lie in a fronto-parallel plane that includes the point of fixation.

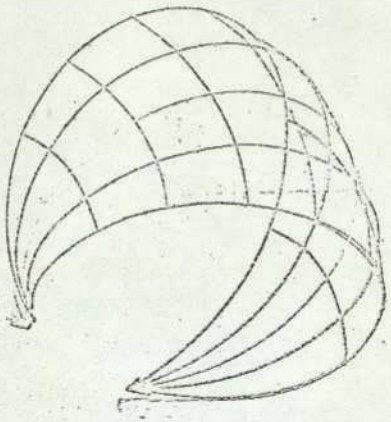


Fig. 2.9 Linksz' Toroidal Horopter Surface

Linksz (1954) *op. cit.* argued, that, if the angle of convergence of the eyes remained constant, then for various positions of gaze, the Vieth-Müller circle would sweep out the above toroidal surface.

2) the position of each of the points is such that the primary subjective visual directions of the images as formed in the two eyes are identical.

3) the positions of the points will lie in the "centre" of the region of binocular single vision.

4) the positions of the points will be such that the sterescopic sensitivity to changes in position will be a maximum.

To which Ogle[227] has added

5) the positions of the points are such that none provides a stimulus for a fusional movement of the eyes.

While the criteria (4) and (5), are operationally true, there has been no attempt to validate them experimentally, because of the difficulty in designing suitable experiments.

The third criterion was used by Fischer[228], using Tschermak's apparatus (see: fig. 2,10), but the results have been criticized[229 230 231], as the technique equated a monocular direction with a binocular one.

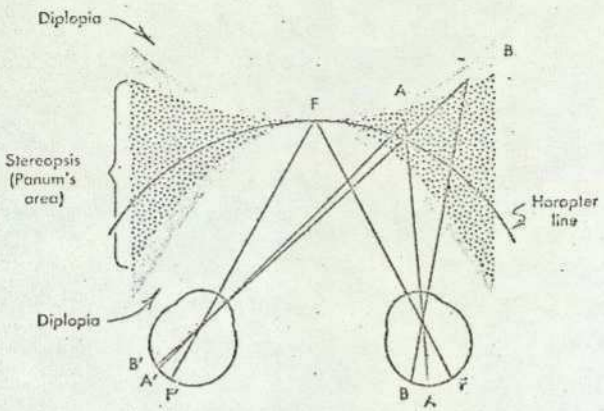


Fig. 2.10 Fischer's Horopter Line

Fischer (1924) *op. cit.* determination of the region of binocular single vision, using Tschermak's apparatus. The horopter line is then determined according to Tschermak's criterion (3).

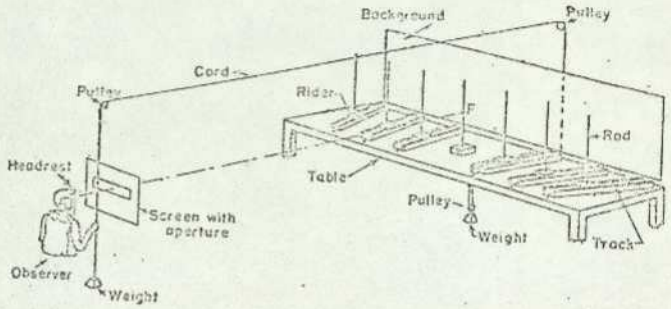
In their classical papers of 1932, Ames, Gliddon and Ogle[232 233] used the first two criteria. The work in these papers and as reassessed more recently by Ogle[234 235] is, with some qualification[236 237 238], the accepted view of the horopter at the present time.

To determine the apparent fronto-parallel plane, an apparatus is used such as that shown in fig. 2.11. While fixating the centre rod, the subject is required to adjust the lateral vertical rods until they appear to lie on the apparent fronto-parallel plane that includes the fixation rod.

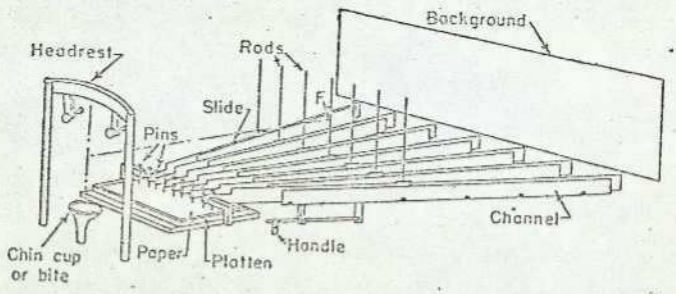
A typical set of data, as published by Ogle is given in fig. 2.12.

It was found that the actual shape of the apparent frontoparallel plane (i.e. the Hering-Hillebrand deviation from the Vieth-Müller circle) changed with the fixation distance, becoming flatter and possibly even concave to the observer at some point. (see: fig. 2.13)

Tschermak[239], in 1924, had shown that there were different apparent fronto-parallel planes with different coloured test stimuli. Linksz[240], and even more strongly, Shipley and Rawlings[241], have argued against the validity of the apparent fronto-parallel plane criterion. They claim that the second criterion, the so-called "nonius" method, is the only acceptable experimental technique for determining the shape of the longitudinal horopter.



(a) For an observation distance of 6 metres



(b) For near fixation points

Fig. 2.11 Ogle's sketches of the horopter apparatus
Experimental apparatus for determining the apparent fronto-parallel plane.

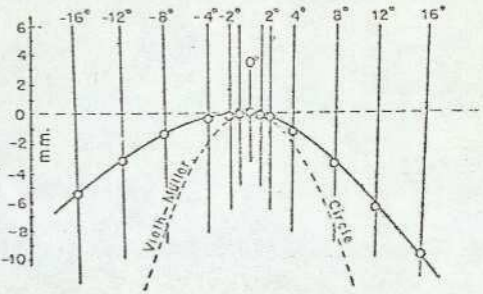


Fig. 2.12 Apparent fronto-parallel Plane; Experimental Plot.

A typical output for the apparent fronto-parallel plane for an observation distance of 40 cm. The vertical dimensions have been increased tenfold.

From Ogle (1952) *op. cit.* fig. 11.

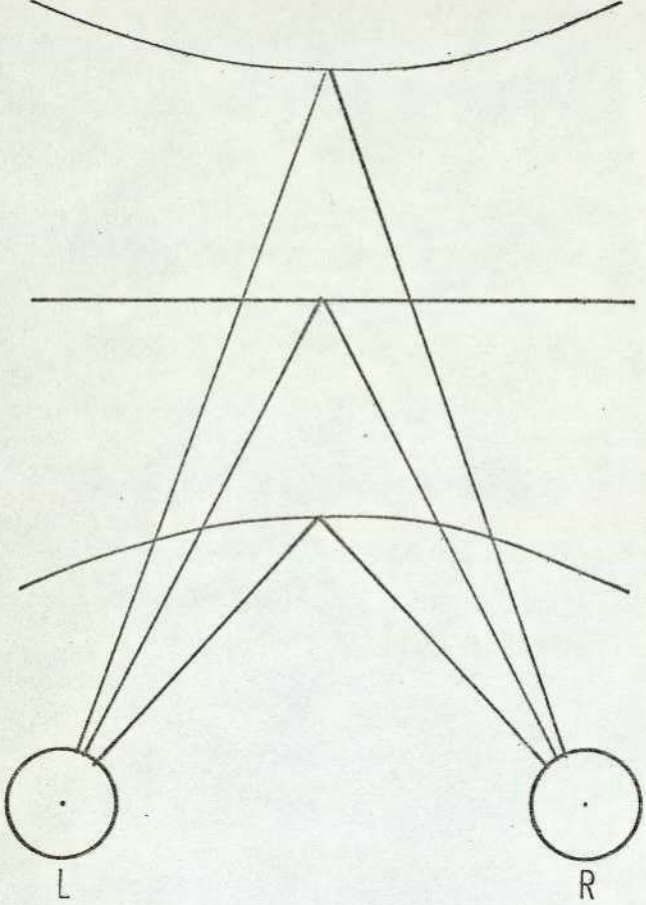


Fig. 2.13 Variations in the shape of the apparent fronto-parallel plane, with fixation distance

The nonius* method is an experimental technique for locating those points in the visual plane that will appear to lie in an identical subjective direction to each eye; i.e. to satisfy the second criterion, above. It is currently held to be the most valid technique:

> "There is only one horopter that is the *nonius*."[242]

The first to enunciate the principle of identical visual direction in relation to the horopter was Wells[243] in 1792. The nonius method was first used by Volkmann and described by Helmholtz[244] for determining the spatial congruency of corresponding points. It was first applied to horopter experiments by van der Meulen[245] in 1873.

The method depends on the <u>definition</u> that corresponding retinal points share an identical monocular direction in the sense described above. (see; fig. 2,14)

Fig. 2.15 shows the original Ames *et al*[246] nonius apparatus. Specially shaped screens are set before the eyes so that, except for the fixation line, the upper portions of the test rods¶ are seen by one eye, and the lower portions by

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*Nonius is the Latinized form of Nunes, a Portuguese mathematician, who invented a device for graduating instruments that was later improved into the vernier.

¶ Current methods[247] utilize the same principle, but the image separation is achieved by polarizing filters.

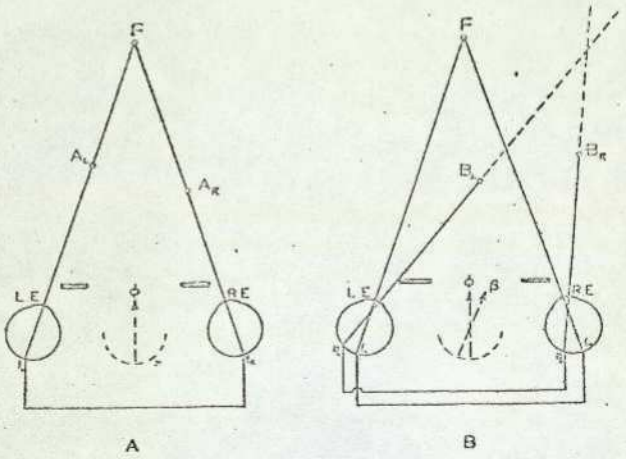


Fig. 2.14 Criterion of Identical Visual Direction

$f_L F$, $f_R F$ are the lines of direction of the foveas. $b_L B_L$, $b_R B_R$ are the lines of direction of the retinal elements b_L and b_R.

ϕ and β, are the common visual directions of the foveas f_L and f_R and the retinal elements b_L and b_R respectively.

In A, the object points F, A_L and A_R appear to lie behind each other in the direction ϕ.

In B, the points B_L and B_R appear to lie behind each other in the direction β.

From Adler (1965) *op. cit.* Fig. 408

the other. The screens are so illuminated as to have the same luminance as the background. The observer, while fixating the central rod, adjusts each of the lateral rods until the upper and lower images are seen aligned as in fig. 2.15b.

When the rod is in this position, the images perceived by the two eyes are seen to lie in the same subjective directions, without the fusion reflex being provoked, thus satisfying the criterion (2).

It is found that the location of the nonius horopter is affected by the observer's oculo-motor imbalance, and will only actually pass through the fixation point if he is orthophoric. (see: fig. 2.16)

It is worth noting that in all the experiments described fixation is limited to both the horizontal and the median plane; an extremely limiting restriction.

Except for the work of Amigo (1965)[248] who extended Ogle's analysis for the case of asymmetric convergence, the only mathematical analysis of the longitudinal horopter is that of Ogle[249] in 1932. This work has never been republished but its results are constantly quoted throughout the literature.

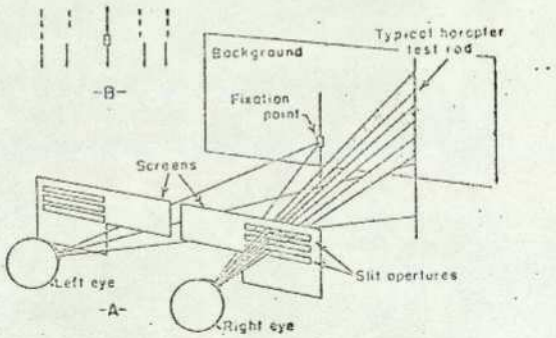


Fig. 2.15 A) The Original Nonius Apparatus

In this apparatus, the screens are arranged so that the lower half of each rod is visible to one eye only, while the other eye can only see the upper half, through the "grid". But both eyes can see the fixation rod. This is consequently, sometimes referred to as the "grid-nonius".

B) Nonius appearance of the rods

As the rod moves along its track, this is the sort of appearance of the rods that the observer sees.

From Ogle (1950) *op. cit.* Fig. 21

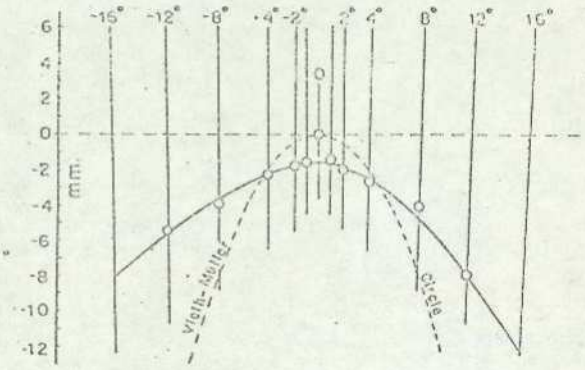


Fig. 2.16 Nonius Horopter Esophoric Case

Observation distance: 40 cm.

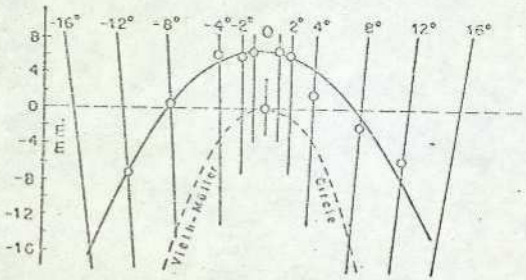


Fig 2.17 Nonius Horopter Exophoric Case

Observation distance: 76 cm.

A typical output for the nonius horopter on observers with different oculo-motor imbalances.

The Vertical dimensions have been increased tenfold.

From Ogle, K.N. (1950) *op. cit.* Figs. 22, 23

Ogle's theoretical analysis to determine the shape of the longitudinal horopter, can be followed by referring to fig. 2.18.

Let R and L represent the positions of the mean nodal points of the right and left eyes respectively with the eyes fixating the point F in the median plane.

Let, O, the origin of the co-ordinate system be taken as the mid-point of the segment LR.

Take OR as the x-axis;

OF as the y-axis

of the co-ordinate system.

Let* R $\equiv$ (b,0)

L $\equiv$ (-b,0)

F $\equiv$ (0,a)

Let θ_1, θ_2 be the angles between the left and right fixation lines respectively, measured anti-clockwise.

Let P be the point of intersection of the lines from a pair of corresponding retinal points, P_1 and P_2, projected out through the mean nodal points L and R.

Then P $\equiv$ P(x,y) is a point of the horopter.

---

*The lettering on the diagram (fig. 2.18), and the co-ordinates, have been modified from Ogle's original (1932 fig. 2). In particular, "a", the fixation distance, and "2b", the inter-ocular distance have been interchanged, to give consistency to the usage in Part III.

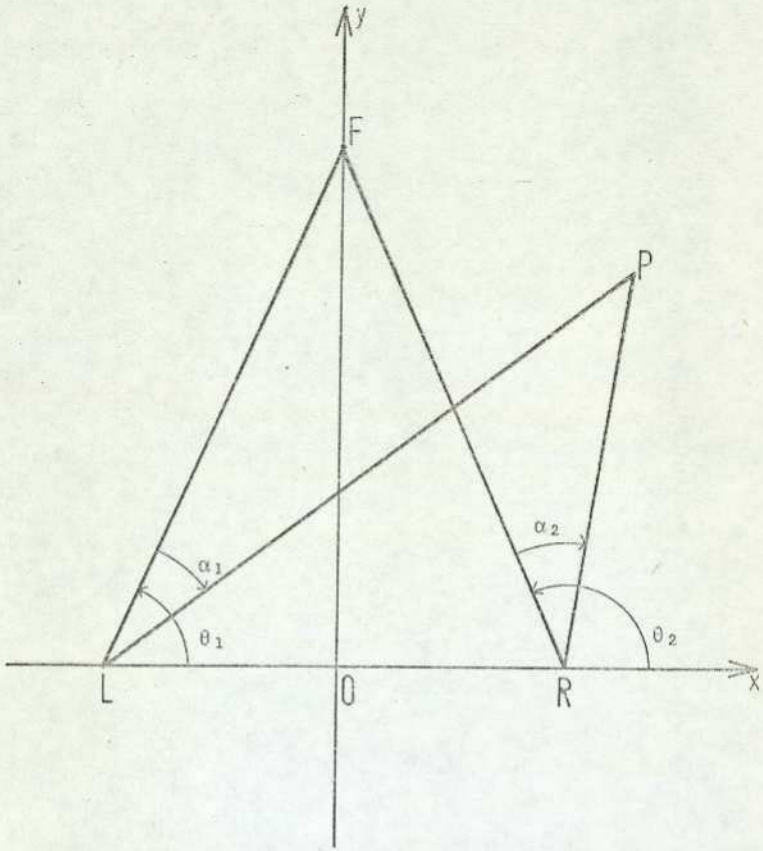


Fig. 2.18 Ogle's Analytical Diagram

Redrawn from Ogle (1932) *op. cit.* Fig. 2.

Let $\angle FLP = \alpha_1$

$\angle FRP = \alpha_2$

measured *clockwise*.

Thus positive values of α_1 indicate angles for which the retinal elements are on the temporal side of the left eye; while positive values of α_2 indicate angles for which the retinal elements are on the nasal side of the right eye. That is, in general, positive values of α_1 and α_2 pertain to the right spatial field and negative values to the left.

Ogle describes the horopter trace in terms of a relationship between these angles.

From elementary considerations, we can see that the equation of the line:

$$PR \equiv y = (x-b)\tan(\theta_2-\alpha_2) \qquad 2.1$$

and the line

$$PL \equiv y = (x+b)\tan(\theta_1-\alpha_1) \qquad 2.2$$

Further, if we put

$$d = \tan\theta_1 = \frac{a}{b} = -\tan\theta_2 \qquad 2.3$$

and

$$w_i = \tan\alpha_i \quad (i=1,2) \qquad 2.4$$

Then, (2.1), can be written

$$y = -(x-b)\frac{d+w_2}{1-dw_2} \qquad 2.5$$

and (2.2) =>

$$y = (x+b)\frac{d-w_1}{1+dw_1} \qquad 2.6$$

Ogle, then makes this assumption:

> "... that there exists a distance "a" for a given observer for which the trace of the longitudinal horopter, whether determinable experimentally or not, is a straight line lying in the fixation frontal-parallel plane."[250]

If this is the case, we are then required to find the relationship between the angles α_1 and α_2, for which this special situation is true.

This relationship can be found by substituting the value

$$y = a \qquad 2.7$$

into (2.5), and (2.6), and eliminating x.

This gives:

$$a\{2d(1+w_1w_2) + (d^2-1)(w_1-w_2)\} = 2b\{d^2 - d(w_1-w_2) - w_1w_2\}$$

and, substituting for d, from (2.3), gives:

$$w_1w_2 + \frac{a}{2b}(w_1-w_2) = 0$$

which can be written:

$$\frac{1}{w_1} - \frac{1}{w_2} = \frac{2b}{a} \qquad 2.8$$

So the expression (2.8) gives the relationship between the angles α_1 and α_2, if the line (2.7), is to be part of the horopter. That is to say, if the relationship (2.8) holds for a particular fixation distance "a", then the longitudinal horopter will lie in the objective fixation fronto-parallel plane. Or, as Ogle expressed it, this equation:

> "... states the relative distortion gradient or distribution of the

> optically projected longitudinal values of corresponding elements of one eye relative to the other, for the individual, who, at a fixation distance "a", sets the longitudinal horopter in the fixation frontal plane."[251]

In general, however, the experimentally determined longitudinal horopter does not lie in this plane but takes a form concave or convex towards the observer.

Such a curve can be described by an expansion of a power series in terms of w_2 (say), and could be written in the form

$$\frac{1}{w_1} - \frac{1}{w_2} = H(1 + Aw_2 + Bw_2^2 + \ldots .) \qquad 2.9$$

where H, A,B etc. are constants.

Now, if the equation of the longitudinal horopter was expressed in such a general form, then its shape would be determined by the value of the expression on the right-hand side of (2.9) and this value would depend on the relative distribution of the corresponding retinal points of the two eyes.

If the angles α_1 and α_2 are equal, then the geometric arguments advanced by Vieth and Müller would apply and the trace of the curve (2.9) should be the Vieth-Müller circle.

Thus, the family of curves represented by (2.9), should include the Vieth-Müller circle, and the objective fronto-parallel plane (2.7), as members. They should also, evidently, be symmetric with respect to the median line,

$$x = 0$$

On the basis of the experimental curves found[252], Ogle argues that these criteria are best satisfied by a family of conic sections passing through the fixation point, and the mean nodal points of the two eyes.

For equation (2.9) to represent a conic, the right hand term must be a constant, and accordingly, Ogle writes the fundamental equation of the longitudinal symmetric horopter as

$$\frac{1}{w_1} - \frac{1}{w_2} = H \qquad 2.10$$

where, for a given individual, H is a constant for the fixation distance a.

This is Ogle's oft quoted and well cited expression for the equation of the longitudinal horopter.

Now, (2.5) can be written

$$\frac{1}{w_2} = \frac{dy - x + b}{y + dx - db}$$

and (2.6) =>

$$\frac{1}{w_1} = \frac{dy + x + b}{dx + db - y}$$

if we insert the value of d from (2.3), and substitute in (2.10)

we find

$$y^2\{1+\frac{Hb}{2a}\} + x^2\{1-\frac{Ha}{2b}\} - y\{\frac{a^2-b^2}{a} + Hb\} - b^2 + \frac{Hab}{2} = 0 \qquad 2.11$$

This equation, then, is the explicit formulation of the longitudinal horopter.

For a given fixation distance, a, and a given inter-ocular distance, 2b, (2.11) represents a family of conic sections, each member of which is determined by ascribing a particular value to the undetermined parameter, H.

For example, if

$$H = \frac{2b}{a}$$

then (2.11) reduces to

$$y(y-a) = 0 \qquad 2.12$$

i.e. the conic degenerates into the x-axis, and the parallel line through the fixation point. (I.e. the fronto-parallel as determined by (2.8))

If, on the other hand, we take

$$H = 0$$

(2.11) becomes

$$x^2 + y^2 - \frac{(a^2-b^2)y}{a} - b^2 = 0 \qquad 2.13$$

which is the circle passing through R, L and F. i.e. the Vieth-Müller circle

This is evident from (2.10), since, if H is 0 in this equation, then

$$w_1 = w_2$$

which implies

$$\alpha_1 = \alpha_2$$

and corresponding points subtending equal angles with the foveas at the nodal point is the condition for the Vieth-Müller circle.

The parameter H, then, is a measure of the departure of the horopter curve from the Vieth-Müller circle. It is therefore a quantitative measure of the Hering-Hillebrand deviation and has been designated[253] the Hering-Hillebrand horopter deviation coefficient.

As H varies from $-\infty$ to $+\infty$, all the possible conics passing through the points F, R and L will be traced out. These variations are shown in fig. 2.19.

In his later work[254], Ogle modified the analysis, and introduced a second parameter R_0; modifying (2.10) to:

$$\frac{1}{w_1} - R_0\frac{1}{w_2} = H \qquad 2.14$$

to allow for the fact that the horopter sometimes appeared to rotate about the median line, as in fig. 2.20

If $R_0 = 1$* and the longitudinal angles are small, (2.14) can be written in approximate form as:

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* R_0 is the ratio of the effective magnifications of the images of the two eyes, and cannot be very different from 1.

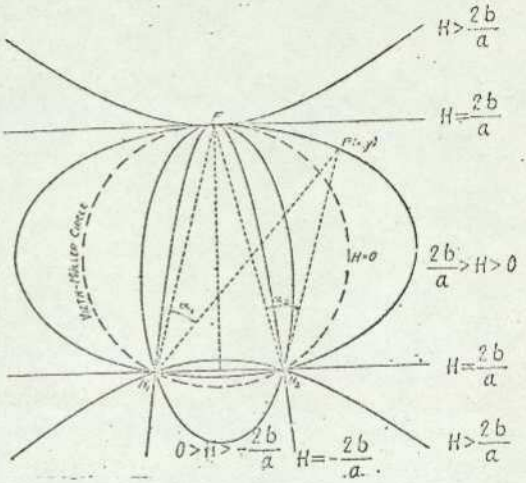


Fig. 2.19 Variations in Eqn. (2.11) with H

H	Shape of Eqn. (2.11)
$> \frac{2b}{a}$	a hyperbola
$= \frac{2b}{a}$	two parallel straight lines
$\frac{2b}{a}>H>0$	an ellipse (horizontal major axis)
$= 0$	the Vieth-Müller circle
$0>H>\frac{2b}{a}$	an ellipse (major axis vertical)
$=-\frac{2b}{a}$	a parabola
$<-\frac{2b}{a}$	a hyperbola

Modified from Ogle(1932) fig. 4 and Table 1.

$$\alpha_2 - \alpha_2 = H\alpha^2 \qquad 2.15$$

where α is the mean of α_1 and α_2;
showing that there is a progressive increase in the difference between the longitudinal angles with the peripheral angle.

If we write

$$R = \frac{w_2}{w_1}$$

then (2.14) can be re-written, as:

$$R = Hw_2 + R_0 \qquad 2.16$$

and if R is plotted against w_2 for a set of experimental results, then, if the analysis is valid, the points plotted should fall on to a straight line of slope H and intercept R_0.

Fig. 2.21 shows Ogle's plot for this data, and the straight line is a justification for the selection of conic curves for the analysis, and the graph reveals;

$$H = 0.120$$

$$R_0 = 0.9966$$

Ogle has shown[255] by placing afocal magnifying lenses before one eye, that the horopter does indeed rotate by the predicted value of R_0 (for up to 4% magnification), but H remains constant, which is indicative of the stability of the horopter.

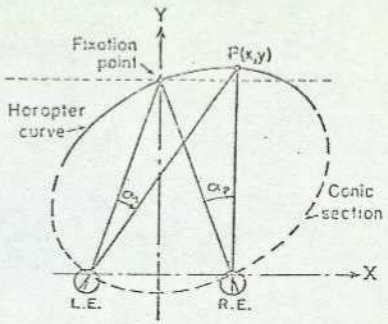


Fig. 2.20 Rotation of the horopter

Rotated position of the conic section due to a difference in the magnification of the two eyes.

From Ogle (1950) *op. cit.* fig. 9

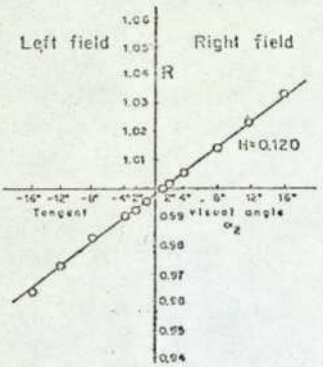


Fig. 2.21 Graph to determine H and R_0

The "goodness of fit" of the data points to a straight line, is a justification of Ogle's selection of a conic section for the horopter.

From Ogle (1962) *op. cit.* fig. 12

A most important, unresolved, problem of horopter studies is indicated in fig. 2.22, which shows the variation of the Hering-Hillebrand horopter deviation coefficient with fixation distance.

If, as has been generally accepted, H is truly an index of retinal correspondence and if retinal correspondence is ultimately cortical as the most recent work [256] [257] indicates, then it is difficult to understand how it can change with fixation distance.

Ogle (1950) observed:

> "... the functional binocular asymmetry between the two retinas increases markedly as the visual distance decreases"[258]

and concluded:

> "We must agree that more data will be necessary before the matter of the determination of the empirical longitudinal horopter, its functional significance and the extent to which it does or does not indicate the nature of the stability of the optical spatial counterparts of corresponding points can be known."[259]

and in his latest work (1962)[260], Ogle left the problem unresolved.

Experiments on the apparent fronto-parallel plane caused Tutisaka[261] (1958) to deny that the Hering-Hillebrand deviation can be explained by the asymmetry of receptor elements.

Hallden[262] offered an alternative explanation for the deviation of the horopter to the generally accepted one. He suggested

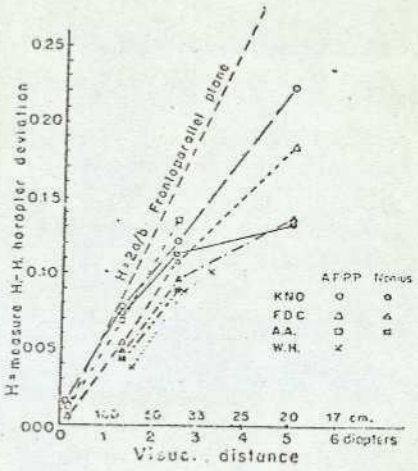


Fig. 2.22 Variation of H with Fixation Distance

Empirical results showing how the value of H varies for different observers, and for different distances.

From Ogle (1950) *op. cit.* fig. 18

that the Hering-Hillebrand deviation might be caused by dioptric asymmetries, rather than retinal ones. He produced an approximate formula for H,

$$H = 2\frac{n-r}{r}\sin\delta \qquad 2.17$$

where n is the posterior nodal distance of the eye; r is the radius of curvature of the retina; and δ is the angle between the optic axis of the eye and the geometric axis of the retina. But if this formula is true, it is difficult to see how it could account for the variation of H.

Shipley and Rawling (1970), in their review paper comment:

> "The rate of increase of nasal image disparity with eccentricity appears to decrease with the physical distance. If confirmed, this would force us to invoke some mechanism which acts as if to change correspondence - though this need not be implasticity in the classical sense."[263]

and, in their Experimental Report, they

> "... assert that correspondence changes in all observers during decreased convergence."[264]

In the analysis developed in Part III below, H emerges as an explicit function of the fixation distance, which, it is felt, offers an explanation of the hitherto unresolved problem just described.

PART II OCULAR KINEMATICS

Chapter 3 Derivation of the angle of torsion

3 Derivation of the Angle of Torsion

The concept of the horopter is based on the assumption of a correspondence between the retinal elements of the two eyes. These points are regarded as being identified by their retinal location, that is, their co-ordinates with respect to a co-ordinate system defined in each retina.

However, if the horopter analysis is to extend to three dimensions, allowance must be made for the fact that the eyes can, to some extent, rotate independently. That is to say they can make vergence movements as distinct from merely parallel version movements, and, in particular, it is possible for them to rotate about their anterio-posterior axes.

The relevance derives from the following consideration. If we consider a pair of corresponding retinal points before such an anterio-posterior rotation - or cyclo-rotation - occurs, they will have some specific relative configuration with respect to some arbitrary, fixed, external co-ordinate system. After the rotation their respective relative positions, with respect to this co-ordinate system would be different. If it is not to be argued - and it is not - that the correspondence changes as a result of the rotation, then specific allowance must be made for this in the analysis

For this reason an analysis of the phenomenon of torsion has been included in this study.

The eye-ball could be regarded as a spherical object that can move quite freely in its orbit; a sort of ball and socket with three degrees of freedom. However, the extra-ocular muscles and the fatty tissue of the orbit impose a most elaborate series of constraints. There are two nineteenth century experimental "Laws" of ocular movement that limit the number of parameters involved and make the question amenable to kinematic analysis. These are Donders Law and Listing's Law.

Donders' Law was established by Donders[265] in 1846, but it was named as such by Helmholtz[266]. It enunciates the principle that for any given direction of regard the eye always assumes the same position. That is, for any fixation point, the orienation of the eye is unique and thus, if any torsion is present for any particular direction of regard, it is always the same.

This Law means that the retinal image of any object in the external field will, for a given position of the head and fixation of the eye always fall upon the same retinal points. Thus the eye cannot spin in an arbitrary way around its anterio-posterior axis.

Listing's Law was formulated, and thus named by Reute[267] in 1853; apparently Listing never published the Law himself.

It states that when the eye moves to any position from the primary position, it rotates about an axis that is perpendicular to the initial and final lines of regard at their point of intersection.

Thus, Donders' Law shows that the actual position of an eye is fixed for any given direction of regard, and Listing's Law specifies what this position is; it is -independently of the path that it actually took - the position that it would have if it had moved directly from the primary position according to the above Law.

There have been many definition of the angle of torsion[268 269 270 271 272 273]

It will be defined here as the angle between the subjective horizontal plane (Fry's[274] horizontal meridian plane) and the plane of regard.[275] Where the subjective horizontal plane is the plane, fixed in the eye, that coincides with the plane of regard when the line of sight is straight forward, the non-fixating eye is occluded, the accommodation is relaxed, and the head is held with the inter-ocular base-line horizontal and the face-plane* vertical.[276 277] (see: Fig. 3.1)

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*a plane tangential to the chin and superciliary ridges[278]

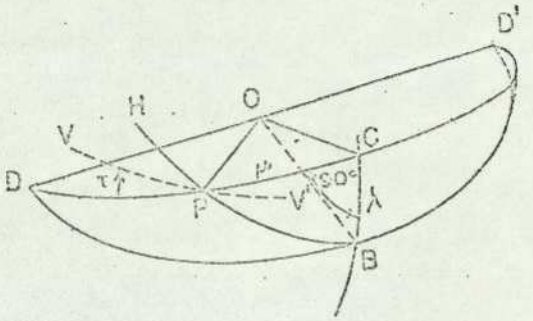


Fig. 3.1 The Angle of Torsion

O is the centre of rotation of the eye

OB is the straight-forward direction

DD' is the inter-ocular base-line

DBD' is the primary position of the plane of regard

OB is perpendicular to DD'

VV' is the subjective horizontal plane of the eye when the fixation is directed to P

τ, the angle between the plane OVV', and the elevated plane of regard DPD', is the angle of torsion of the eye

This angle has been derived, in different forms, by Helmholtz[279], Lamb[280], Fischer[281] and Schubert[282]; and, more recently, by Westheimer[283], who used the rather obscure notion of quaternions.

The following derivation, describes the rotations of the eye by means of matrices, and derives the angle of torsion in terms of the direction-cosines of the direction of regard.

Let O, the origin of co-ordinates be at the centre of rotation of the eye; that is, at the point of zero velocity during eye movements. This assumes that as the eye moves so that the direction of regard changes from one position to another, the movement takes place without translation. However, it has been shown quite clearly[284] that there is no unique centre of rotation. In fact, the eye undergoes a "screw" motion as well as a rotational one, and that the instantaneous centre of rotation changes continuously.

However, to the order of accuracy implicit in horopter experiments - which invariably * rely on an observer's subjective response - it is reasonable to accept the approximation that such a point does exist; and this point is chosen as the origin. It has been located as the point 13.5mm behind the cornea, on the line of sight when the eye is in the straight-forward position.[285, 286]

*See below §21.

Let the z-axis (see: fig. 3.2), be the vertical line through O (positive upwards), so that the horizontal plane is z = O.

Let the x-axis be the line through O, in the plane z = O, parallel to the median plane (positive forward).

Let the y-axis be the line through O, such that (x,y,z) form a right-handed orthogonal set.

When the eye is directed along the x-axis, it is said to be in the primary position.

Let (x',y',z') be a second co-ordinate system, fixed with respect to the eye, such that the vectors $\underline{x}$ and $\underline{x}'$ coincide when the eye is in the primary position.

The plane x' = O is known as Listing's plane.

Let the vectors $\underline{x}$ and $\underline{x}'$ have direction cosines u_1,u_2,u_3 and u_1',u_2',u_3' respectively.

If the eye now undergoes a series of rotations, the resultant can always be expressed as a single rotation through some angle about a particular axis.[287] Let us call this angle ω and let the axis of rotation be designated by direction cosines (ℓ,m,n).

As a consequence of this rotation the co-ordinate-sytem $\underline{x}'$ is no longer identical with the fixed system $\underline{x}$, but they are related to each other by a linear transformation, which we can write:

$$\underline{u}' = B\underline{u} \qquad 3.1$$

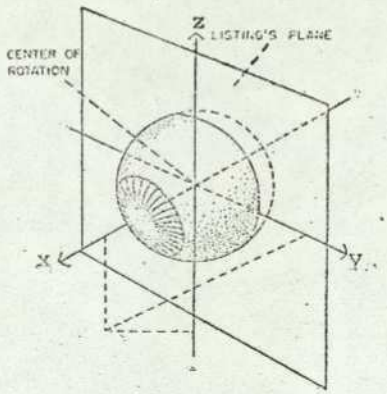


Fig 3.2 The co-ordinate axes for eye movements

When the eye is in the primary position, the axes fixed with respect to the head and those fixed with respect to the eye co-incide.

where B is a 3 x 3 matrix.

We can determine the elements of B by the following considerations

First, rotate the co-ordinate system to a new system $\underline{x}_1$, by rotating it through an angle θ_1 about the x-axis. (see fig. 3.3)

i.e. put

$$\underline{u}_1 = R_1\underline{u} \qquad 3.2$$

where

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta_1 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \qquad 3.3$$

Let this be followed by a further rotation of θ_2 about the new z_1-axis (see fig. 3.3)

i.e.

$$\underline{u}_2 = R_2\underline{u}_1 \qquad 3.4$$

where

$$R_2 = \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad 3.5$$

Thus, we have

$$\underline{u}_2 = R_2R_1\underline{u} = R\underline{u} \text{ (say)} \qquad 3.6$$

where

$$R = R_2R_1$$

$$= \begin{pmatrix} \cos\theta_2 & -\cos\theta_1\sin\theta_2 & \sin\theta_1\sin\theta_2 \\ \sin\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_1\cos\theta_2 \\ 0 & \sin\theta_1 & \cos\theta_1 \end{pmatrix} \qquad 3.7$$

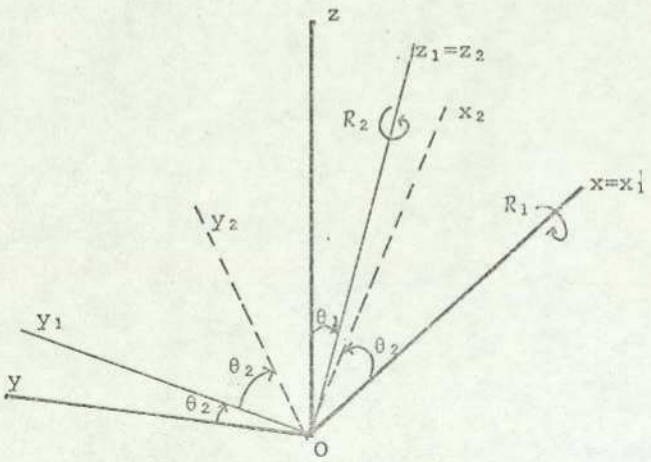


Fig. 3.3 The rotations R_1 and R_2

The rotation R_1 is a rotation through an angle θ_1 with the x-axis held fixed, bringing the co-ordinate axes to x_1, y_1, z_1.

The rotation R_2 is a rotation through an angle θ_2 with the new z -axis held fixed, bringing the coordinate axes to the new position x_2, y_2, z_2.

So R is a rotation after which the original y-axis has direction cosines

$$(\sin\theta_2,\ \cos\theta_1\cos\theta_2,\ -\sin\theta_1\cos\theta_2) \qquad 3.8$$

If we wish this line to represent the axis of rotation of the rotation S, we can put

$$\ell = \sin\theta_2 \qquad 3.9$$

$$m = \cos\theta_1\cos\theta_2 \qquad 3.10$$

$$n = -\sin\theta_1\cos\theta_2 \qquad 3.11$$

which implies

if we write $p^2 = m^2 + n^2$ 3.12

$$\sin\theta_1 = -\frac{n}{p} \qquad 3.13$$

$$\cos\theta_1 = \frac{m}{p} \qquad 3.14$$

$$\sin\theta_2 = \ell \qquad 3.15$$

$$\cos\theta_2 = p \qquad 3.16$$

Inserting these values in (3.7), we find

$$R = \frac{1}{p}\begin{pmatrix} p^2 & -m\ell & -n\ell \\ \ell p & mp & np \\ 0 & -n & m \end{pmatrix} \qquad 3.17$$

Thus R is a rotation that will bring the line with direction cosines (ℓ,m,n) into a line with direction cosines $(0,1,0)$

Hence, if we apply the rotation R to our original co-ordinate system, then the axis of rotation (ℓ,m,n) of the rotation S, will be the y-axis $(0,1,0)$ in the new system.

The original rotation B was a rotation through an angle ω about this axis. So, if we first apply the rotation R to the co-ordinate system so that the axis of B becomes the new y-axis, and then rotate through an angle ω, about this new y-axis, i.e.

$$R_0 = \begin{pmatrix} \cos\omega & 0 & -\sin\omega \\ 0 & 1 & 0 \\ \sin\omega & 0 & \cos\omega \end{pmatrix} \qquad 3.18$$

and then rotate by R^{-1} to restore the original co-ordinates.

Hence:

$$B = R^{-1}R_0R \qquad 3.19$$

is the matrix representing a rotation through an angle ω about an axis with direction cosines (ℓ,m,n).

i.e.

$$B = \frac{1}{p^2}\begin{pmatrix} p^2 & \ell p & 0 \\ -m\ell & mp & -n \\ -n\ell & np & m \end{pmatrix}\begin{pmatrix} \cos\omega & 0 & -\sin\omega \\ 0 & 1 & 0 \\ \sin\omega & 0 & \cos\omega \end{pmatrix}\begin{pmatrix} p^2 & -m\ell & -n\ell \\ \ell p & mp & np \\ 0 & -n & m \end{pmatrix} \qquad 3.20$$

$$= \begin{pmatrix} (m^2+n^2)\cos\omega+\ell^2 & m\ell(1-\cos\omega)+n\sin\omega & n\ell(1-\cos\omega)-m\sin\omega \\ m\ell(1-\cos\omega)-n\sin\omega & (\ell^2+n^2)\cos\omega+m^2 & mn(1-\cos\omega)+\ell\sin\omega \\ n\ell(1-\cos\omega)+m\sin\omega & mn(1-\cos\omega)-\ell\sin\omega & (\ell^2+m^2)\cos\omega+n^2 \end{pmatrix} \qquad 3.21$$

As a result of the rotation, B, the line of fixation (i.e. the x'-axis) is now along the line OF (see fig. 3.4), where $\underline{OF}$ is the unit vector along x'-axis.

Let the plane containing OF and the y-axis make an angle λ with the horizontal plane, and, in this plane, let the line OF make an angle α' with the y-axis.

Thus, the fixation line OF, is completely determine by the two angles α' and λ.*

In the plane YOF, we have

$$OP' = \cos(180-\alpha') = -\cos\alpha' \qquad 3.22$$

$$P'F = \sin(180-\alpha') = \sin\alpha' \qquad 3.23$$

where P' is the perpendicular from F on OY and in the vertical plane P'FP, we have

$$P'P = P'F\cos(-\lambda) = \sin\alpha'\cos\lambda \qquad 3.24$$

$$PF = P'F\sin(-\lambda) = -\sin\alpha'\sin\lambda \qquad 3.25$$

Hence, the direction cosines of the fixation line are

$$(\sin\alpha'\cos\lambda, -\cos\alpha', -\sin\alpha'\sin\lambda) \qquad 3.26$$

Further, from fig. 3.4, we can see that the equation of the plane YOF is given by

$$\sin\lambda x + \cos\lambda z = 0 \qquad 3.27$$

---

*The reason for identifying the fixation line by these two particular angles is determined by the treatment in Chapter 12 , below.

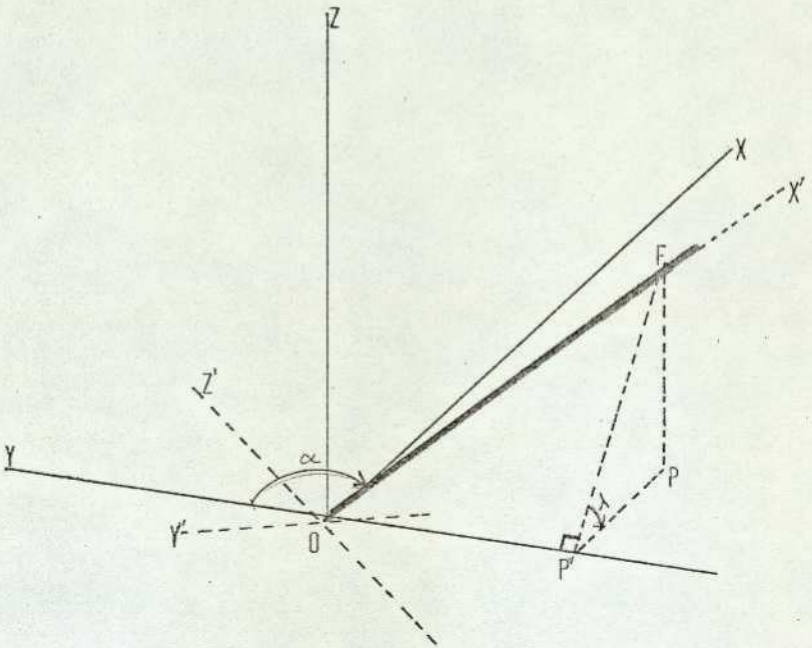


Fig. 3.4 The Rotation B

This rotation causes the fixation line of the eye (the x'-axis) to take up the line OF, with direction-cosines:

$$(\sin\alpha'\cos\lambda,\ -\cos\alpha',\ -\sin\alpha'\sin\lambda)$$

by rotating through an angle ω, from the original x-axis, about a line with direction-cosines (ℓ,m,n)

So, in the space described by $\underline{x}$, the fixed coordinate system, let the direction OF, be given by a vector $\underline{u}_1$

i.e.

$$\underline{u}_1 = (\sin\alpha'\cos\lambda, -\cos\alpha', -\sin\alpha'\sin\lambda) \quad 3.28$$

and the primary (original) fixation line, is given by (say)

$$\underline{u}_2 = (1,0,0) \quad 3.29$$

Hence, the angle ω, turned through by the eye to take up the fixation $\underline{u}_1$, is given by

$$\cos\omega = \underline{u}_1.\underline{u}_2 \qquad \text{(inner product)} \quad 3.30$$

$$= \sin\alpha'\cos\lambda \quad 3.31$$

from (3.28), (3.29).

If the axis of the rotation, B, is denoted by the vector $\underline{u}$

i.e.

$$\underline{u} = (\ell,m,n) \quad 3.32$$

then, if the eye moves in accordance with Listing's Law, $\underline{u}$ must be perpendicular to both the initial and final directions of the line of fixation; i.e. to both $\underline{u}_1$ and $\underline{u}_2$.

That is, we must have:

$$\underline{u}.\underline{u}_2 = 0 \quad 3.33$$

$\Rightarrow$

$$\ell = 0 \quad 3.34$$

from (3.29), (3.32)

And also

$$\underline{u}.\underline{u}_1 = 0 \qquad 3.35$$

=>

$$\frac{m}{\sin\alpha'\sin\lambda} = -\frac{n}{\cos\alpha'} \qquad 3.36$$

from (3.28),(3.32),(3.34).

Thus, we can write

$$\underline{u} = \frac{1}{k}(0,\ \sin\alpha'\sin\lambda,\ -\cos\alpha') \qquad 3.37$$

where k is a normalizing factor, such that

$$m^2 + n^2 = 1 \qquad 3.38$$

i.e.

$$k^2 = \sin^2\alpha'\sin^2\lambda + \cos^2\alpha'$$

$$= 1 - \sin^2\alpha'\cos^2\lambda \qquad 3.39$$

$$= \sin^2\omega \quad \text{(from 3.31)} \qquad 3.40$$

And so, we have

$$k = \sqrt{1 - \sin^2\alpha'\cos^2\lambda} \qquad 3.41$$

$$\cos\omega = \sin\alpha'\cos\lambda \qquad 3.42$$

$$\sin\omega = k \qquad 3.43$$

$$\ell = 0 \qquad 3.44$$

$$m = \frac{\sin\alpha'\sin\lambda}{k} \qquad 3.45$$

$$n = -\frac{\cos\alpha'}{k} \qquad 3.46$$

Now the subjective horizontal plane* is given by the equation

$$\dot{z}' = 0 \tag{3.47}$$

which can be written in plane co-ordinates as $\underline{u}_1'$ (say), whence

$$\underline{u}_1' = (0,0,1) \tag{3.48}$$

Inverting equation (3.1), we can write

$$\underline{u}_1 = \mathcal{B}^{-1}\underline{u}_1' \tag{3.49}$$

and so the plane (3.47) is represented by the vector

$$\underline{u}_1 = \mathcal{B}^{-1}\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{3.50}$$

in the original fixed co-ordinates.

But, since $\mathcal{B}$ is an orthogonal matrix, we have

$$\mathcal{B}^{-1} = \mathcal{B}^{T} \tag{3.51}$$

where $\mathcal{B}^{T}$ is the transpose of the matrix $\mathcal{B}$.

So, (3.50) becomes

$$\underline{u}_1 = \mathcal{B}^{T}\begin{pmatrix}0\\0\\1\end{pmatrix} \tag{3.52}$$

$$= \begin{pmatrix} n\ell(1-\cos\omega)+m\sin\omega \\ mn(1-\cos\omega)-\ell\sin\omega \\ (\ell^2+m^2)\cos\omega+n^2 \end{pmatrix} \tag{3.53}$$

from (3.21)

*This term is defined below §4.

From (3.27), we have, in the fixed co-ordinate system $\underline{x}$, the visual plane* can be written

$$x\sin\lambda + z\cos\lambda = 0 \qquad 3.54$$

that is, it can be represented by the vector

$$\underline{u}_0 = (\sin\lambda, 0, \cos\lambda) \qquad 3.55$$

Now, by definition, the angle of torsion* τ is the angle between these two planes and hence it is given by

$$\cos\tau = \underline{u}_1 . \underline{u}_0 \qquad 3.56$$

$$= \{n\ell(1-\cos\omega)+m\sin\omega\}\sin\lambda + \{(\ell^2+m^2)\cos\omega+n^2\}\cos\lambda \qquad 3.57$$

Substituting from (3.44), (3.45), (3.46), we find

$$\cos\tau =$$

$$\frac{\sin\alpha'\sin^2\lambda\sin\omega}{k} + \frac{\sin^2\alpha'\sin^2\lambda\cos\omega\cos\lambda}{k^2} + \frac{\cos^2\alpha'\cos\lambda}{k^2}$$

$$= \frac{\sin\alpha'\sin^2\lambda(1-\sin^2\alpha'\cos^2\lambda)+ \sin^3\alpha'\sin^2\lambda\cos^2\lambda + \cos^2\alpha'\cos\lambda}{1 - \sin^2\alpha'\cos^2\lambda}$$

by substituting (3.41), (3.42), (3.43).

Simplifying, we have

$$\cos\tau = \frac{\sin\alpha'\sin^2\lambda + \cos^2\alpha'\cos\lambda}{1 - \sin^2\alpha'\cos^2\lambda}$$

$$= \frac{(\sin\alpha'+\cos\lambda)(1-\sin\alpha'\cos\lambda)}{1 - \sin^2\alpha'\cos^2\lambda}$$

So, $$\cos\tau = \frac{\sin\alpha' + \cos\lambda}{1 + \sin\alpha'\cos\lambda} \qquad 3.58$$

=>

$$\sin\tau = \frac{\cos\alpha'\sin\lambda}{1 + \sin\alpha'\cos\lambda} \qquad 3.59$$

*This term is defined below §4.

and

$$\tan\tau = \frac{\cos\alpha'\sin\lambda}{\sin\alpha' + \cos\lambda} \qquad 3.60$$

Thus, the angle of torsion τ can be expressed as a function of α' and λ, the angles defining the direction of the fixation line; and, if Donder's Law is true, then this angle is uniquely defined.

From the equations (3.58), (3.59), (3.60), it can be seen, that if

$$\alpha' = \tfrac{1}{2}\pi \qquad 3.61$$

i.e., the fixation line is in the vertical plane through the primary fixation line,

or if

$$\lambda = 0 \qquad 3.62$$

i.e., the fixation line is in the horizontal plane, then

$$\cos\tau = 1 \qquad 3.63$$

and

$$\sin\tau = \tan\tau = 0 \qquad 3.64$$

=>

$$\tau = 0 \qquad 3.65$$

Positions of the eye, where the fixation lines are in these two planes, are called secondary positions of the eye; other positions are called tertiary. Equation (3.65) shows that in secondary positions of the eye, the angle of torsion is zero. This fact can be deduced from

Listing's Law directly, without the necessity of the above analysis.

The expressions (3.41)-(3.46) above depend on the validity of Listing's Law. Hence, under conditions when this Law breaks down, the equation (3.57) would be a true expression of the angle of torsion, but not necessarily the later ones.

There is no general agreement as to the validity of Listing's Law.

Helmholtz noticed that there were individual variations, and said:

> "... we must not expect quite the same precision in the eye as in a scientific instrument, although, under ordinary conditions normal eyes do obey the laws of Donders and Listing pretty accurately."[288]

Quereau, using a blind-spot plotting technique[289], found that, except for extreme positions of gaze, Listing's Law was generally valid. However, Weale[290], using a photographic technique, demonstrated the contrary assertion.

However, when the eyes are converged there is more agreement that Listing's Law breaks down. We have Landolt's classic experiment,[291] (1876), recently confirmed by Allen in 1954.[292] In fact, Allen and Carter (1967) have shown[293] that there is a torsional movement of the eyes on convergence, at all elevations, even when no other movement of the eye occurs leading them to consider a torsional component as part of the near reflex, a phenomenon that had already been

commented on by Tschermak-Seysenegg[294] (1952).

Analysis of these writings leads to the conclusion that, as with Helmholtz' "apparently vertical meridian", with which they are closely associated, there is a large element of individual variation in torsional eye-movements. Nevertheless, if Lisitng's Law is approximately true, then the equations (3.58)-(3.60) are a valid expression of the angle of torsion as a function of the direction-cosines of the line of fixation.

PART III

DERIVATION AND PROPERTIES OF THE RECTILINEAR AND SPACE HOROPTERS

4 Definitions

Definitions of most of the terms used to locate the position of an eye, and its relationship to the point of fixation and to the vertical and horizontal planes have been given by Helmholtz[295], Fry *et al.*[296], Alpern[297], etc. However because of ambiguities and inconsistencies none of these sets of definitions are suitable as the basis of a mathematical analysis and it is felt that the terms used should be clearly defined afresh.

From any point P in external space we can consider a pencil of rays entering the eye. If we limit our consideration to the principal ray of this pencil we can say that the incident ray is given by the line PR where R is the axial point of the entrance pupil.

We can describe this ray by the angles ϕ and θ where θ is the angle between this ray and the principal axis of the system and ϕ is the inclination of the plane of incidence to the horizontal.

After refraction at the various media this ray will meet the retina as if from R', the centre of the exit pupil of the eye, and described by the angles θ' and ϕ, where

$$\theta' = F(\theta) \qquad 4.1$$

i.e. θ' is a well defined function of the initial angle θ, which can be determined by considering the laws of refraction at each surface.

Hence if the pupil of the eye is of a size that justifies restricting our consideration to the principal rays, we need only consider those rays of light incident on the eye that pass through the point R. We can call R the *nodal point** of the system.

We can thus argue that the optical system of the eye induces a 1-1 transformation T on the set of lines $\{\ell\}$ radiating from R in object space to the set of lines $\{\ell'\}$ radiating from R' in image space, such that

$$\ell(\theta,\phi) \rightarrow \ell'(\theta',\phi)$$

can be written

$$\ell' = T(\ell) \qquad 4.2$$

Thus we can describe the formation of the retinal image by the transformation T, and because of the reversibility of optical paths, the projection of a retinal point into external space can be represented by the transformation T^{-1}.

Since each ray ℓ' in image space will meet the retina in a unique point P', and if the ray in object space that transforms to ℓ' is given by PR, we can write the transformation

$$T(PR) = P' \qquad 4.3$$

$$T^{-1}(P') = RP \qquad 4.4$$

---

*The term nodal point is used in this context to mean the apex of the pencil of all the rays incident on the eye. It should not be confused with either of the nodal points of the optical system of the eye.

A line from any point in object space to the nodal point we can call a *line of sight.**

The line joining the fovea of the eye to the point R' transforms under T^{-1} into a line of sight called the *fixation line.**

The fixation lines of the two eyes intersect at the *fixation point.*

The line joining the nodal points of the two eyes is called the *base line.*

The horizontal line that perpendicularly bisects the base line is called the *median line.*

The vertical plane that contains the median line is called the *median plane.*

If the head is erect and the fixation lines are both horizontal and parallel to the median plane, the eyes are said to be in the *primary position.*

A plane in object space through the nodal point can be defined by two lines of sight ℓ_1 and ℓ_2. These two lines transform under T to ℓ_1' and ℓ_2' and thus define a plane through the point R' within the eye. This plane will intersect the retina in a set of points that we will define as a *retinal line.*

---

*These definitions do not agree with those of Fry *et al.*[298] and Alpern[299].

We can then say that under the transformation T any plane through the nodal point "*intersects*" the retina in a retinal line, where "intersects" implies the effect of the transformation T.

When the eye is in the primary position the fixation plane "intersects" the retina of each eye in a retina line, fixed in the eye, that we will call the *retinal horizon*.*

The plane in object space that is the projection (under T^{-1}) of the retinal horizon is called the *subjective horizontal plane*.

When the eye is in the primary position, the plane that contains the fixation line and the line through the fixation point that appears vertical to the observer "intersects" the retina in a line, fixed in the eye, called the *apparently vertical meridian*.

This line may, or may not, be identical with the true vertical[300], and its actual inclination may vary from one individual to another. However it is assumed that for a given fixation and for a given observer, it is unique.

The plane in object space that is the projection (under (T^{-1}) of the apparently vertical meridian is called the *subjective vertical plane*.

---

*This does not agree with Helmholtz[301] who used this term sometimes for a line and sometimes for a plane.

The angle between the two lines of fixation is called the *angle of convergence*.

The angle between the plane of fixation and the horizontal plane is called the *angle of elevation* (= λ, fig. 3.4).

The angle between the vertical plane through the fixation line and the median plane is called the *angle of azimuth* (= $90^{0}-\alpha'$, fig.3.4).

The angle between the subjective horizontal plane and the plane of fixation is called the *angle of torsion*.

5 RELATED AND CORRESPONDING RETINAL POINTS

It is intended that the subsequent mathematical analysis be based on the following postulates.

POSTULATE I For a particular fixation point F, the orientation of the eye is uniquely determined. (Donders' Law)[302]

This assertion has never been disputed and without it no further analysis is possible.

POSTULATE II For a particular fixation point F, there is for each retinal receptor a functional relationship such that when stimulated each receptor element will give rise, through cortical connections, to a sensation of a particular subjective visual direction.

This relationship can be represented by the transformation

$$T^{-1}(P') = RP \qquad 5.1$$

Let the nodal points for the right and left eyes be located at the points R and L respectively.

Then Postulate I implies that for a particular fixation point the configuration of the eyes is fixed, and a stimulus at any point in space P' will form images at two uniquely determined points P_i (i=1,2) on the left and right retinas respectively.

Where

$$P_1 = T_1(P'L) \qquad 5.2$$

$$P_2 = T_2(P'R) \qquad 5.3$$

(see: fig. 5.1).*

P_1 and P_2 are said to be *related retinal points*.

However this relationship is not reciprocal, that is to say, given P_1 the point P_2 is not uniquely determined. This follows since although the point P' uniquely determines the point P_1

i.e. $P' \Rightarrow P_1 = T_1(P'L)$

the point P_1 does not determine the point P'

since $P_1 \Rightarrow LP' = T_1^{-1}(P_1)$

but only that P' must lie on LP', the projection of P_1.

Thus P_1 determines that the point P' must lie on the line $T_1^{-1}(P_1)$. But for each point P' on this line a unique point P_2 in the right retina is determined according to (5.3). Thus if P' is considered as travelling along the line LP' (fig. 5.1), then P_2 will trace out a unique retinal line p_2 in the right retina.

Thus there exists a retinal line p_2 in the right retina, each point of which is a related retinal point to the point P_1 in the left.

---

*In the figures (5.1) etc. the entrance and exit pupils are shown as coinciding so that the ray path P'LL'P_1 etc. is shown as the single ray PLP_1. This is intended to be a schematic representation of the relationships described by the transformations (5.2),(5.3) etc.

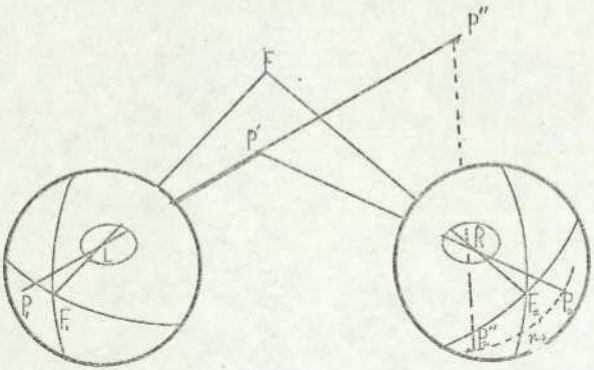


Fig. 5.1 Related retinal points

As the point P' moves along the line P'P" defined by the direction P_1L so the image of P' in the right eye traces out the retinal line $p_2 = P''_2P_2$, located by P'R. Points on the line p_2 are said to be related points to the point P_1.

This is the retinal line where the plane $T_1^{-1}(P_1)R$, i.e. the plane LP'R "intersects" the right retina.

Such a transformation κ, which carries points into lines is called a plane correlation[303].

Thus we can write

$$p_2 = \kappa P_1 \qquad 5.4$$

i.e. the point P_1 is transformed by the transformation κ into the line p_2.

Similarly

$$p_1 = \kappa P_2$$

Implying

$$\begin{aligned} P_1 &= \kappa^{-1} p_2 \\ P_2 &= \kappa^{-1} p_1 \end{aligned} \qquad 5.5$$

When the two eyes are directed simultaneously to a given fixation point large central portions of the monocular visual fields overlap. It therefore follows from Postulate II, that for any retinal point in one eye, there is a point in the other, which, when stimulated separately, would give rise to the same subjective visual direction.

These two points are said to be *corresponding retinal points*.

That there is such an anatomical connection between the elements of the two retinas has been demonstrated by the discovery of neurons in the striate cortex that respond to stimuli in both retinas (see: Bishop *et al.*[304], Nikara *et al.*[305], Pettigrew[306]).

If the points P_1 and P_2 of (5.5) are corresponding retinal points, then the correlation κ will reduce to a non-singular collineation τ (say) of points to points which can be represented

$$P_2 = \tau P_1 \qquad 5.6$$

Those points in space that for a given fixation point will stimulate corresponding points in the retinas of the two eyes are called *self-corresponding points*.

The locus in space of all self-corresponding points is called *the horopter*.

Since the points of any simple surface are isomorphic to the points of a plane, we can represent the points of the two retinas by the points of two particular planes.

Let us then select the two planes Π_1 and Π_2 perpendicular to the lines LF and RF respectively, and equidistant from the two nodal points (fig. 5.2).

Let any point P_1 in the left retina be represented in the plane Π_1 by the point where the line $T_1^{-1}(P_1)$ meets Π_1, and let a similar representation prevail for the points of the right retina.

Let the co-ordinates in the two planes be given by (x_1,y_1) and (x_2,y_2) respectively, and let each line of fixation meet its respective plane in the point $(0,0)$.

Now any retinal line is projected under T^{-1} into a plane which will intersect the plane Π_i in a straight line. In particular the retinal

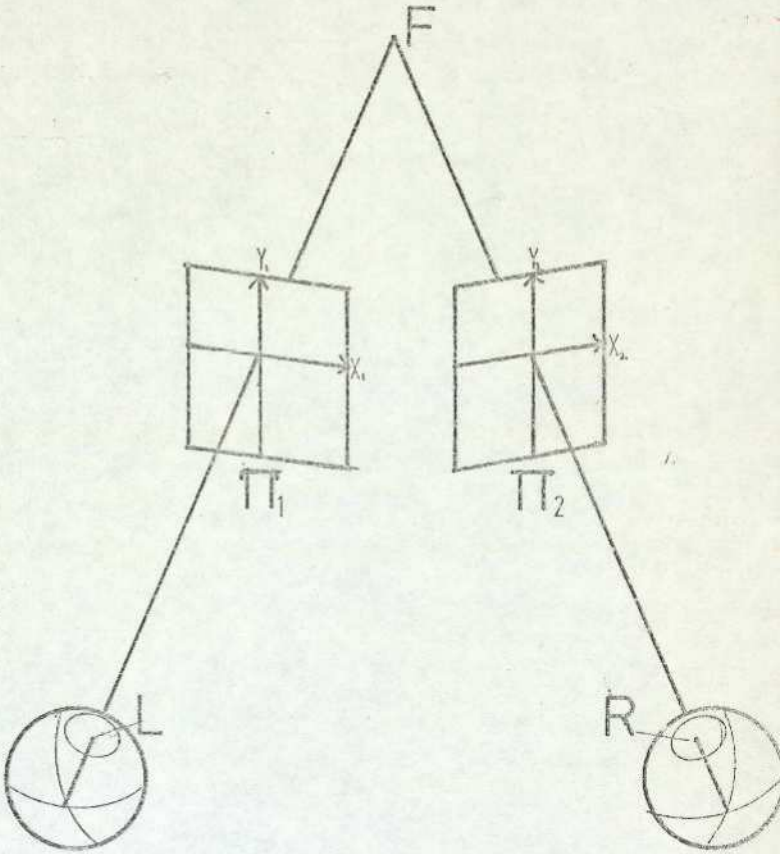


Fig. 5.2 The planes Π_1 and Π_2

horizon will project into a straight line. Let this line make an angle θ_i with the x_i-axis (see: fig. 5.3). Also let the projection of the apparently vertical meridian make an angle ϕ_i with the x_i-axis.

Hence the projection of the retinal horizon will have the equation

$$\sin\theta_i x_i + \cos\theta_i y_i = 0 \quad (i=1,2) \qquad 5.7$$

and the projection of the apparently vertical meridian will be given by

$$\sin\phi_i x_i + \cos\phi_i y_i = 0 \quad (i=1,2) \qquad 5.8$$

The operational definition of corresponding retinal points is that their projections lie in the same subjective visual direction, i.e. if P_1 and P_2 are corresponding retinal points then

$$T_1^{-1}(P_1) = LP'$$
$$T_2^{-1}(P_2) = RP''$$

have the same subjective visual direction.

Further if LP' meets Π_1 at P_1' and RP'' meets Π_2 at P_2', then Helmholtz[307] in a series of experiments has shown that P_1' and P_2' are equidistant from the two lines (5.7) and from the two lines (5.8).

We can take this as a third postulate and call such points *corresponding points in the two planes* Π_i. :

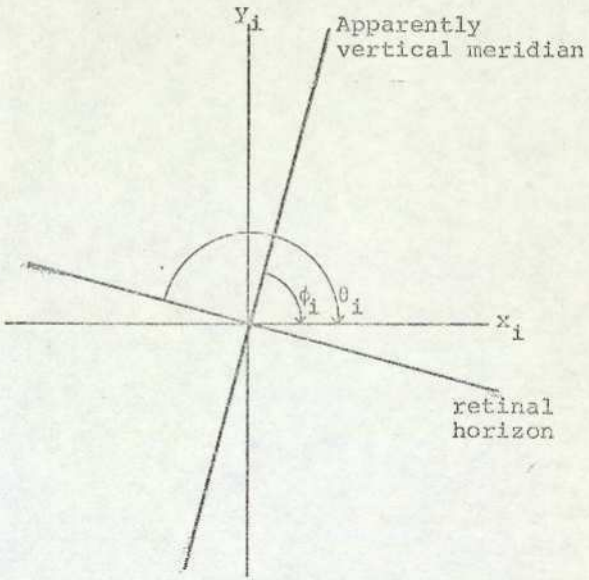


Fig. 5.3 The inclination of the principal meridians

the equation of the retinal horizon is

$$\sin\theta_i x_i + \cos\theta_i y_i = 0$$

the equation of the apparently vertical meridian is

$$\sin\phi_i x_i + \cos\phi_i y_i = 0$$

$(i = 1,2)$

A consequence of this postulate is that if T_1 and T_2 are equal, that is if the optical construction of the two eyes is identical, then the two corresponding retinal points

$$P_1 = T_1(P'L)$$

and $$P_2 = T_1(P''R)$$

will have identical situations on the two retinas with respect to their own principal lines.

That is to say corresponding retinal points are identically situated on the two retinas if the optical configurations of the two eyes are the same. If the optical components of the two eyes are not identical the corresponding retinal points will not be similarly situated, but will be placed such that their projections under T_1^{-1} and T_2^{-1} on to the planes Π_1 and Π_2 are equally located with respect to the lines (5.7) and (5.8).

This would suggest that the anatomical linking of corresponding retinal elements occurs as a result of experiencing identical visual directions, that is as a consequence of learning, as recent neuro-physiological work[308] would seem to confirm.

The distinction between the two cases of identical retinal distribution of corresponding retinal points and the non-identical distribution depends only on the operator T. This is a purely optical operator which depends only on the curvature,

position and refractive indices of the various media, the pupil characteristics and the position of the retina, but not on the distribution of the retinal elements, as has been claimed by Hillebrand[309] and later writers[310 311 312].

We know from elementary co-ordinate geometry that if

$$a^2 + b^2 = 1$$

then the distance D(X,Y) of a point P(X,Y) from the line

$$ax + by = 0$$

is

$$D(X,Y) = aX + bY$$

So the distance D(X,Y) of any point P(X,Y) from the projection of the retinal horizon in Π_i is from (5.7)

$$D(X,Y) = \sin\theta_i X + \cos\theta_i Y \quad (i=1,2) \qquad 5.9$$

with similar expressions for the other principal lines.

Thus for corresponding points we have:

$$\sin\phi_1 x_1 + \cos\phi_1 y_1 = \sin\phi_2 x_2 + \cos\phi_2 y_2 \qquad 5.10$$

$$\sin\theta_1 x_1 + \cos\theta_1 y_1 = \sin\theta_2 x_2 + \cos\theta_2 y_2 \qquad 5.11$$

(5.10),(5.11) can be written

$$\mathcal{D}_1 \underline{x}_1 = \mathcal{D}_2 \underline{x}_2 \qquad 5.12$$

where

$$\underline{x}_i = (x_i, y_i) \quad (i=1,2) \qquad 5.13$$

and

$$\mathcal{D}_i = \begin{pmatrix} \sin\phi_i & \cos\phi_i \\ \sin\theta_i & \cos\theta_i \end{pmatrix} \quad (i=1,2) \qquad 5.14$$

and (5.12) =>

$$\underline{x}_2 = \mathcal{D}_2^{-1}\mathcal{D}_1\underline{x}_1 \qquad 5.15$$

If now the collineation τ (equation 5.6) be represented by a matrix A, i.e. if

$$\tau \equiv \underline{x}_2 = A\underline{x}_1 \qquad 5.16$$

Then (5.15),(5.16) =>

$$A = \mathcal{D}_2^{-1}\mathcal{D}_1 \qquad 5.17$$

$$= \frac{1}{\sin(\phi_2-\theta_2)}\begin{pmatrix}\cos\theta_2 & -\cos\phi_2\\ -\sin\theta_2 & \sin\phi_2\end{pmatrix}\begin{pmatrix}\sin\phi_1 & \cos\phi_1\\ \sin\theta_1 & \cos\theta_1\end{pmatrix}$$

$$= \frac{1}{\sin(\phi_2-\theta_2)}\begin{pmatrix}\sin\phi_1\cos\theta_2-\sin\theta_1\cos\phi_2 & \cos\phi_1\cos\theta_2-\cos\theta_1\cos\phi_2\\ \sin\theta_1\sin\phi_2-\sin\phi_1\sin\theta_2 & \cos\theta_1\sin\phi_2-\cos\phi_1\sin\theta_2\end{pmatrix} 5.$$

which expresses the relationship between the corresponding points of the two eyes in terms of the inclinations of their principal lines.

If we now put

$$X_i = \sin\theta_i x_i + \cos\theta_i y_i \qquad 5.19$$

$$Y_i = \sin\phi_i x_i + \cos\phi_i y_i \qquad 5.20$$

$$(i = 1,2)$$

then from (5.10),(5.11) we can describe corresponding points by the relations

$$X_1 = X_2 \qquad 5.21$$

$$Y_1 = Y_2$$

6 CORRESPONDING RETINAL LINES

Any line in Π_1

$$a_1x_1 + b_1y_1 + c_1 = 0 \qquad 6.1$$

can be written

$$\ell X_1 + mY_1 + n = 0 \qquad 6.2$$

where, substituting from (5.19) and (5.20) and equating coefficients, we have:

$$\ell\sin\theta_1 + m\sin\phi_1 = a_1 \qquad 6.3$$

$$\ell\cos\theta_1 + m\cos\phi_1 = b_1 \qquad 6.4$$

$$n = c_1 \qquad 6.5$$

Thus ℓ,m,n, are determined from:

$$\ell = \frac{b_1\sin\phi_1 - a_1\cos\phi_1}{\sin(\phi_1-\theta_1)} \qquad 6.6$$

$$m = \frac{a_1\cos\theta_1 - b_1\sin\theta_1}{\sin(\phi_1-\theta_1)} \qquad 6.7$$

$$n = c_1 \qquad 6.8$$

But, for corresponding points, (5.21), (6.2) imply

$$\ell X_2 + mY_2 + n = 0 \qquad 6.9$$

Thus, any line (6.1), in Π_1, can be written as (6.2); where ℓ,m,n are determined by (6.6),(6.7),(6.8). These values determine a unique line (6.9) in Π_2, consisting of corresponding points. The lines (6.2) and (6.9) are said to be CORRESPONDING LINES in the planes Π_i.

Alternatively, we can say, that for any fixed values of ℓ,m,n, the lines

$$\ell X_i + mY_i + n = 0 \quad (i=1,2) \qquad 6.10$$

are corresponding lines.

If we write

$$k_i^2 = a_i^2 + b_i^2 \qquad 6.11$$

then, (6.1) can be expressed in normal form,

i.e.

$$\frac{a_i}{k_i}x_i + \frac{b_i}{k_i}y_i + \frac{c_i}{k_i} = 0 \qquad 6.12$$

in this form, the distance from the origin

is $\frac{c_i}{k_i}$.

But, (6.8) implies that for corresponding points, $c_1 = c_2$; thus, for corresponding points, we must have:

$$k_1^2 = k_2^2 \qquad 6.13$$

Substituting from (6.11), with (6.3), (6.4)

we have:

$$k_i^2 = (\ell\sin\theta_i + m\sin\phi_i)^2 + (\ell\cos\theta_i + m\cos\phi_i)^2$$
$$= \ell^2 + m^2 + 2\ell m\cos(\theta_i - \phi_i) \qquad 6.14$$

Hence, (6.13) implies

$$\ell m\cos(\theta_1 - \phi_1) = \ell m\cos(\theta_2 - \phi_2) \qquad 6.15$$

Thus, for corresponding lines, we have,

either: $\ell = 0$ 6.16

or: $m = 0$ 6.17

$(\theta_1 - \phi_1) = (\theta_2 - \phi_2)$ 6.18

And so, the corresponding lines (6.10), are either parallel to the principal lines (5.15), (5.16) or they are inclined at equal angles in the planes Π_1 and Π_2.

7 Corresponding planes in space and corresponding lines of sight

The discussion can now be extended to three dimensions by keeping x_i and y_i as defined, and introducing a third co-ordinate into each co-ordinate system, z_i, such that the nodal point of each eye has the co-ordinates $(0,0,-e)$ in both co-ordinate systems (see fig. 7.1).

Each plane Π_i is thus

$$z_i = 0 \qquad 7.1$$

and, we can write:

$$X_i = x_i \sin\theta_i + y_i \cos\theta_i \qquad 7.2$$

$$Y_i = x_i \sin\phi_i + y \cos\phi_i \qquad 7.3$$

$$(i=1,2)$$

from (5.29), (5.30); and define

$$Z_i = z_i + e \qquad 7.4$$

Consider the two planes:

$$\ell X_i + mY_i + nZ_i = 0 \qquad 7.5$$

$$(i=1,2)$$

which both contain the nodal point $(0,0,-e)$.

These planes intersect Π_i, when $z_i = 0$, i.e. when

$$\ell X_i + mY_i + ne = 0 \qquad 7.6$$

$$(i=1,2)$$

Thus, comparing with (6.10), we can see, that

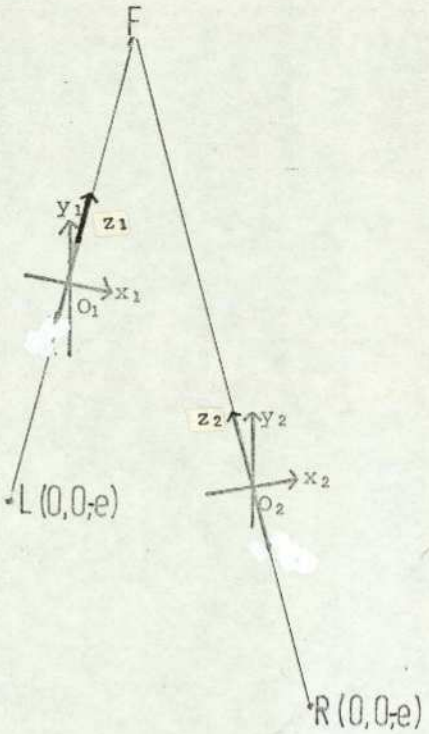


Fig. 7.1 The two co-ordinate systems $\underline{x}_1$ and $\underline{x}_2$.

$$\underline{x}_1 \equiv (x_1, y_1, z_1)$$

$$\underline{x}_2 \equiv (x_2, y_2, z_2)$$

the planes (7.5) intersect the planes Π_i in corresponding lines. They are thus defined as *corresponding planes*.

The straight lines joining the nodal points $(0,0,e)$, to corresponding points in the planes Π_i are called *corresponding lines of sight*.

A line of sight can be defined as the intersection of any pair of planes passing through it. (see fig. 7.2)

If we have two such planes intersecting in a line of sight, then the two corresponding planes intersect in a corresponding line of sight.

That is, each of the pair of planes:

$$\ell X_i + mY_i + nZ_i = 0 \qquad 7.7$$

$$\ell' X_i + m' Y_i + n' Z_i = 0 \qquad 7.8$$

intersect when

$$\frac{X_i}{n'm-nm'} = \frac{Y_i}{\ell'n-\ell n'} = \frac{Z_i}{m'\ell-m\ell'} \qquad 7.9$$

Hence, substituting from (7.2), (7.3), (7.4), for X_i, Y_i, Z_i, and putting $i = 1,2$ successively, gives the equations for two corresponding lines of sight.

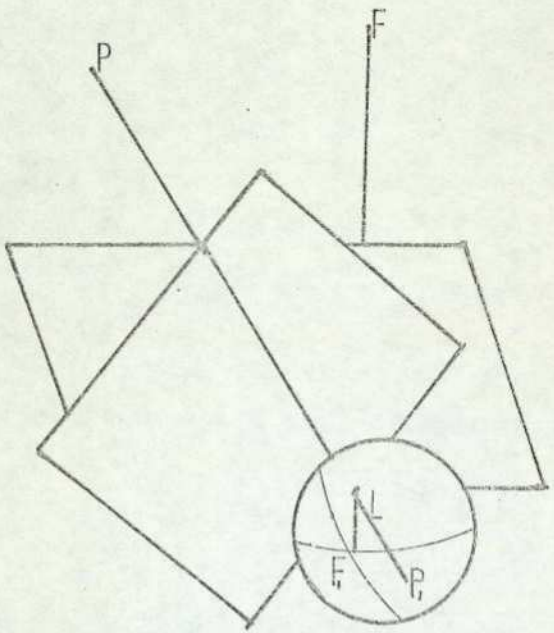


Fig. 7.2 A line of sight

The line of sight P_1LP
can be defined by any two planes
passing through it.

While the two corresponding planes:

$$\ell X_i + mY_i + nZ_i = 0 \qquad (i=1,2) \qquad 8.1$$

are defined in terms of co-ordinates relating to each eye separately, they exist in the same three-dimensional space. Therefore, they are either parallel or they must intersect in a line (see fig. 8.1).

This line lies in both corresponding planes, and is thus a self-corresponding line. That is to say, points on this line will stimulate corresponding retinal lines. They would thus give rise to the perception of lines of identical visual direction, which we can call *nonius-lines*. These lines would be perceived as single even when the stimulus to fusion was inoperative.

Note, that a self-corresponding line does not consist exclusively of self-corresponding points, that is, points in space that will stimulate corresponding retinal points. It is simply that each point on a nonius-line will stimulate in each retina, a point, whose corresponding point in the other retina, will also lie on the image of the line.

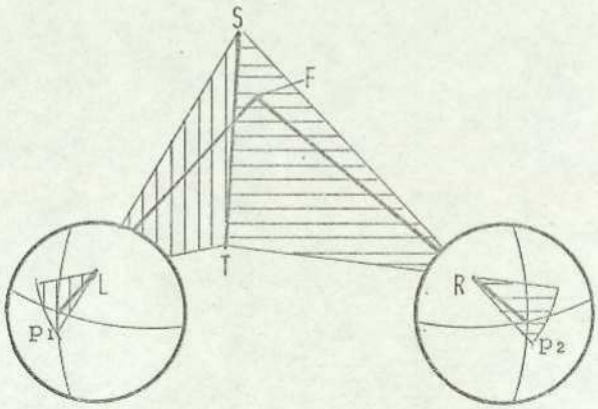


Fig. 8.1 The Nonius-Line

If p_1 and p_2 are corresponding retinal lines, then p_1L and p_2R are corresponding planes. These two planes intersect in ST, the self-corresponding nonius line.

That is to say, any point P' on the nonius line will stimulate a point P_1 (say) in the left eye. If P_2 is the point corresponding to P_1 in the right eye, then the line P_2R will meet the nonius-line in a point (say) P". (see fig. 8.2)

In other words, the distribution of corresponding retinal points induces a one-dimensional homography, ω, on the points of the nonius-line, such that

$$P'' = \omega P' \qquad 8.2$$

Further, as the point P_1 is also the point corresponding to P_2, we can also write

$$P' = \omega P'' \qquad 8.3$$

(8.2), (8.3) =>

$$\omega^2 = \varepsilon \quad \text{(identity)} \qquad 8.4$$

I.e. ω is a cyclic homography of period 2, that is, an involution.

Now, it is a property of an involution, [324] that there exists two "united-points", i.e. two points such that

$$P' = \omega P' \qquad 8.5$$

that is to say, there are two points on the nonius-line, where the route

$$P' \to P_1 \to P_2 \to P''$$

leads back to the original point P'.

Thus, there are two points on any nonius-line that will stimulate corresponding retinal points; i.e. there are two self-corresponding points on each nonius-line.

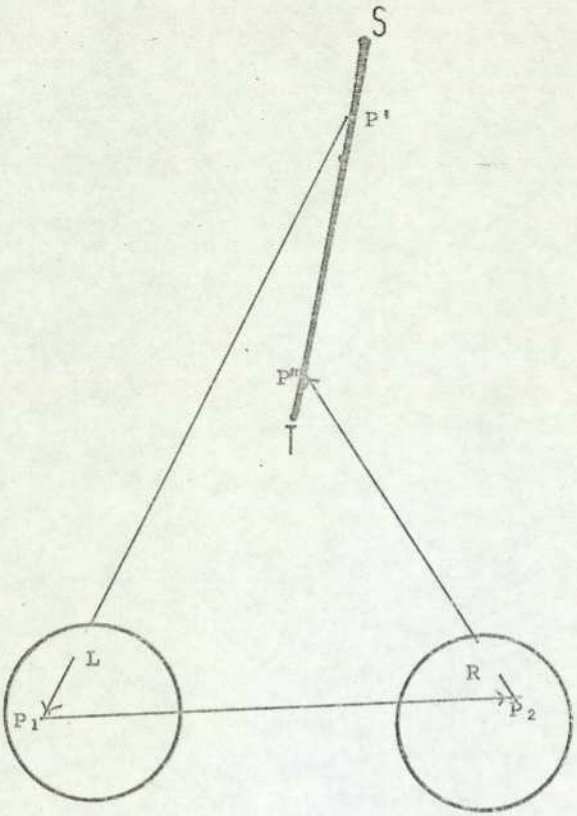


Fig. 8.2 The involution, ω

Illustrating the reciprocal relationship

$$P'' = \omega P'$$

$$P' = \omega P''$$

on the nonius line ST.

However, we can see that if two nonius-lines intersect at a point P', then the image points of P' in each retina must be corresponding retinal points, since each image must lie at the point of intersection of two corresponding retinal lines. Thus, the point of intersection of two nonius-lines is a self-corresponding point.

But, as we have just seen, there are two and only two united-points of an involution (not the identity involution), and thus only two self-corresponding points on a nonius-line.

Hence, every nonius-line intersects two, and only two, other nonius lines.

The surface in space that comprises the totality of all the nonius-lines, for a given fixation point, F, we will call the *rectilinear horopter*.

The corresponding planes (8.1), can be written

$$\underline{u}.\underline{X}_i = 0 \qquad (i=1,2) \qquad 8.6$$

where

$$\underline{u} = (\ell,m,n)$$

$$\underline{X}_i = (X_i,Y_i,Z_i) \quad (i=1,2)$$

Now (8.6), implies that

$$\underline{u}.\underline{X}_i = \rho(\underline{X}_1 x \underline{X}_2).\underline{X}_i$$
$$= 0 \qquad 8.7$$

since, for i=1 or 2, this is a triple scalar product with a repeated factor, and therefore vanishes for any ρ.

Hence

$$\underline{u} = \rho(\underline{X}_1 x \underline{X}_2) \qquad 8.8$$

i.e. $\underline{u}$ is a vector parallel to $\underline{X}_1 x \underline{X}_2$.

That is to say,

$\underline{u}$ has direction cosines

$$(\ell,m,n) = (Y_1Z_2 - Z_1Y_2,\ Z_1X_2 - X_1Z_2,\ X_1Y_2 - Y_1X_2) \qquad 8.9$$

Thus, we have that the equation of the nonius-line, which is the line of intersection of the two planes (8.6), is given by

$$\underline{u}.\underline{X}_1 = 0 = \underline{u}.\underline{X}_2 \qquad 8.10$$

where $\underline{u}$ is given by (8.8).

Thus, $\underline{u}$ is uniquely determined, unless:

$$\underline{X}_1 x \underline{X}_2 = 0 \qquad 8.11$$

that is, unless the vectors, $\underline{X}_1$ and $\underline{X}_2$ are parallel. In which case:

$$X_1Y_2 - Y_1X_2 = 0 \qquad 8.12$$
$$Y_1Z_2 - Z_1Y_2 = 0 \qquad 8.13$$
$$Z_1X_2 - X_1Z_2 = 0 \qquad 8.14$$

Note: any point in space, $\underline{x}$, will determine unique values of $\underline{X}_1$ and $\underline{X}_2$.

These, will, in general, determine a unique line

$$\underline{u}.\underline{X}_i = 0 \quad (i=1,2) \qquad 8.15$$

where $\underline{u} = \rho\underline{X}_1 x \underline{X}_2$

which is a nonius-line.

Thus there is a unique nonius-line that can be drawn through any general point of space.

However, if $\underline{x}$ is such that,

$$\underline{X}_1 x \underline{X}_2 = 0 \qquad 8.16$$

then any number of lines can be so drawn, parallel to $\underline{X}_1 x \underline{X}_2$.

That is to say, there is either, one or an infinite number of lines of the rectilinear horopter through every point of space.

The rectilinear horopter, then, is the envelope of nonius-lines, or, the surface on which lines are seen singly without fusion. Consider the two planes:

$$\underline{u}_1 . \underline{X}_1 = 0 \qquad 8.17$$

$$\underline{u}_2 . \underline{X}_1 = 0$$

These planes intersect in a "line of sight". Any plane through this line can be represented:

$$(\underline{u}_1 + k\underline{u}_2) . \underline{X}_1 = 0 \qquad 8.18$$

where k is a variable parameter.

The plane corresponding to (8.18) is

$$(\underline{u}_1 + k\underline{u}_2) . X_2 = 0 \qquad 8.19$$

and the planes (8.18),(8.19), intersect in a nonius-line.

As k varies, the family of lines represented by (8.18),(8.19), taken together, will generate a surface, whose equation may be found by eliminating k.

(8.18),(8.19) imply:

$$-k = \frac{\underline{u}_1 . \underline{X}_1}{\underline{u}_2 . \underline{X}_1} = \frac{\underline{u}_1 . \underline{X}_2}{\underline{u}_2 . \underline{X}_2} \qquad 8.20$$

which implies:

$$(\underline{u}_1 . \underline{X}_1)(\underline{u}_2 . \underline{X}_2) - (\underline{u}_1 . \underline{X}_2)(\underline{u}_2 . \underline{X}_1) = 0 \qquad 8.21$$

which, from the well known vector formula is equivalent to:

$$(\underline{u}_1 \times \underline{u}_2) . (\underline{X}_1 \times X_2) = 0 \qquad 8.22$$

If we take the components of the vector $\underline{u}_1 x \underline{u}_2$ as (L,M,N), we can write (8.2), as:

$$L(X_1Y_2-X_2Y_1) + M(Y_1Z_2-Y_2Z_1) + N(Z_1X_2-Z_2X_1) = 0 \quad 8.23$$

For any particular values of L,M,N, this expression represents a quadric surface. That is, the surface generated by co-ordinates satisfying a quadratic equation.

This is evident, as each term from (7.2),(7.3),(7.4) can be seen to be a product of two linear factors in x_i, y_i, z_i. Further, it is a particular type of quadric surface called a ruled surface, which is a surface that can be generated entirely by straight lines.

As L,M,N vary, so a whole family of different quadric surfaces are represented. So we can see that (8.23) is the equation of a family of such surfaces. If we put L,M,N equal to zero in pairs, we can see that the following quadrics are included in the family:

$$X_1Y_2 - X_2Y_1 = 0 \quad 8.24$$

$$Y_1Z_2 - Y_2Z_1 = 0 \quad 8.25$$

$$Z_1X_2 - Z_2X_1 = 0 \quad 8.26$$

The family of quadrics represented by (8.23) then, consists of all the quadrics through the points common to (8.24), (8.25), (8.26).

In general, three quadrics intersect in eight points only, but, inspection of these three equations reveals that any two implies the third, so that there are really only two linearly independent surfaces. Two quadrics intersect, in general, in a quartic curve.

Thus, the rectilinear horopter is given by (8.23), and consists of the pencil of quadrics through the common curve of intersection of any two of the surfaces (8.24),(8.25),(8.26).

Further, we have shown that at any point of this common curve, an infinite number of nonius-lines intersect, and thus, every point of this common curve is self-corresponding.

Thus, the common curve of intersection of the surfaces (8.24),(8.25),(8.26), consists entirely of self-corresponding points, and is thus, by definition, the horopter.

9 Co-ordinates relative to the visual plane

If we wish to explore the properties of the quadric surfaces (8.24), (8.25), (8.26), the form of the equations of §8, are not very useful. We need to represent the vectors defined there with respect to the two eyes in the same co-ordinate system. Consequently, we will now set up a system of co-ordinates (x', y', z') relative to the visual plane.

The visual plane (fig. 9.1) is defined by the points RLF.

Take the point of fixation, F, as the origin.

Take the visual plane, itself, as the plane

$$z' = 0$$

so that the z'-axis is directed upwards.

Let the angle RFL, i.e. the angle of convergence be given by 2γ.

Let the x'-axis bisect the angle of convergence, directed towards the eyes.

Let the y'-axis be such that the three axes form a right-handed orthogonal set.

Let

$$FL = a_1 \qquad 9.1$$

$$FR = a_2 \qquad 9.2$$

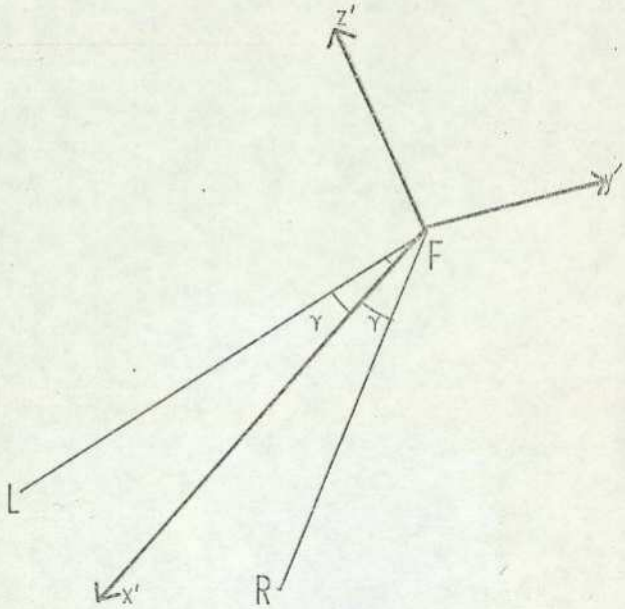


Fig. 9.1 The co-ordinate-system x' defined with respect to the visual plane.

Thus, we have,

$$L \equiv (a_1\cos\gamma,\ a_1\sin\gamma, 0) \qquad 9.3$$

$$R \equiv (a_2\cos\gamma, -a_2\sin\gamma, 0) \qquad 9.4$$

Now, let us consider two further sets of co-ordinate axes, with co-ordinates (x_i', y_i', z_i'), $(i=1,2)$ which are obtained from the (x', y', z') axes, by a rotation about the z'-axis, through an angle γ, and $-\gamma$ respectively, such that each x_i'-axis passes through R and L respectively.

If we write the vector $\underline{x}$ as $(x,y,z,1)$, then we can express these rotations by

$$\underline{x}_i' = R_i\underline{x}' \qquad 9.5$$

where

$$R_1 = \begin{pmatrix} \cos\gamma & \sin\gamma & 0 & 0 \\ -\sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad 9.6$$

and

$$R_2 = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad 9.7$$

Now the x_i'-axis $(i=1,2)$ is the line from the fixation point to each nodal point. This is the same line as the z_i-axis of §7. (see: fig. 9.2)

So, we can take

$$x_i' = a_i - (z_i + e) \qquad 9.8$$

Further, let us say that, with respect to $\underline{x}_i'$, the $\underline{x}_i$-co-ordinates (see fig. 7.1) are rotated through an angle ψ_i about this line as axis.

That is, ψ_i is the angle between the visual plane, $z_i' = 0$, and the plane $y_i = 0$, of §5.

And so we can put

$$x_i = \cos\psi_i y_i' - \sin\psi_i z_i' \qquad 9.9$$

$$y_i = \sin\psi_i y_i' + \cos\psi_i z_i' \qquad 9.10$$

and (9.8) can be written:

$$z_i = -x_i' + (a_i - e) \qquad 9.11$$

and, we can express (9.9),(9.10),(9.11) by

$$\underline{x}_i = \mathcal{B}_i \underline{x}_i' \qquad 9.12$$

where

$$\mathcal{B}_i = \begin{pmatrix} 0 & \cos\psi_i & -\sin\psi_i & 0 \\ 0 & \sin\psi_i & \cos\psi_i & 0 \\ -1 & 0 & 0 & a_i - e \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (i=1,2) \qquad 9.13$$

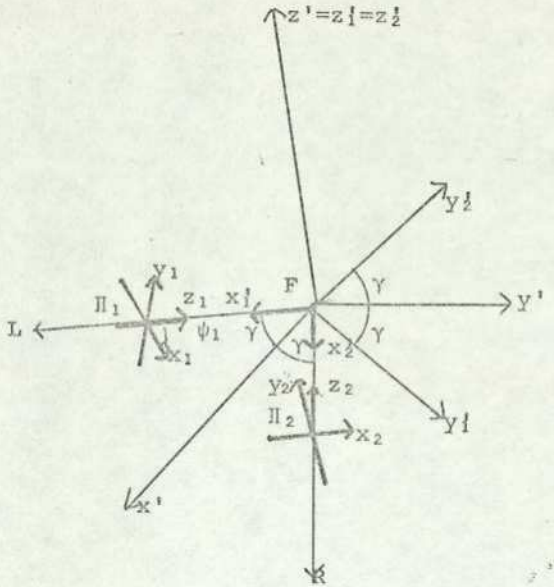


Fig. 9.2 Relation between the co-ordinates $\underline{x}'$, $\underline{x}'_i, \underline{x}_i$

$\underline{x}'$ are the co-ordinates fixed in the visual plane $z' = 0$

$\underline{x}'_i$ $(i=1,2)$ are obtained from $\underline{x}'$ by rotations of $\pm\gamma$ about the z'-axis.

$\underline{x}_i$ $(i=1,2)$ are the co-ordinates of §7, defined in the planes Π_i and the fixation lines; where the x_i-axis is inclined at an angle ψ_i to the visual plane.

Equations (7.2),(7.3),(7.4)

express the relationship

$$\underline{X}_i = A_i \underline{x}_i \qquad 9.14$$

where

$$\underline{X}_i \equiv (X_i, Y_i, Z_i, 1) \qquad 9.15$$

and

$$A_i = \begin{pmatrix} \sin\theta_i & \cos\theta_i & 0 & 0 \\ \sin\phi_i & \cos\phi_i & 0 & 0 \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad 9.16$$

However, without loss of generality, we can take the x_i-axis in §5, as the retinal horizon, i.e., we can put

$$\theta_i = 0 \quad (i=1,2) \qquad 9.17$$

Further, if we suppose that the apparently vertical meridians in each eye are symmetrically disposed with respect to each other, we can take

$$\phi_1 = \tfrac{1}{2}\pi + \varepsilon \qquad 9.18$$

$$\phi_2 = \tfrac{1}{2}\pi - \varepsilon \qquad 9.19$$

Substituting from (9.17),(9.18),(9.19) in (9.16), we have

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \cos\varepsilon & -\sin\varepsilon & 0 & 0 \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad 9.20$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \cos\varepsilon & \sin\varepsilon & 0 & 0 \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad 9.21$$

We also have the following relationships:

The angle between the visual plane and the x_i-axis is ψ_i.

The angle between the retinal horizon and the x_i-axis is θ_i. (from 5.15)

The angle between the visual plane and the retinal horizon is, by definition τ_i, where τ_i is the angle of torsion.

Thus, we can write

$$\psi_i = \theta_i + \tau_i \qquad 9.22$$

which (9.17) $\Rightarrow$ $\psi_i = \tau_i$ 9.23

Thus, (9.13) becomes

$$B_i = \begin{bmatrix} 0 & \cos\tau_i & -\sin\tau_i & 0 \\ 0 & \sin\tau_i & \cos\tau_i & 0 \\ -1 & 0 & 0 & a_i - e \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (i=1,2) \qquad 9.24$$

Thus, we have:

from (9.5) $\underline{x}_i' = R_i \underline{x}'$ 9.25

from (9.12) $\underline{x}_i = B_i \underline{x}_i'$ 9.26

$= B_i R_i \underline{x}'$ (from 9.25) 9.27

from (9.14) $\underline{X}_i = A_i \underline{x}_i$

$= A_i B_i R_i \underline{x}'$ (from 9.27) 9.28

So $\underline{X}_i = C_i \underline{x}'$ (say) 9.29

where $C_i = A_i B_i R_i$ 9.30

Thus, the equation (9.29), expresses the linear factors of the quadric surfaces (8.23) in terms of $\underline{x}'$, the co-ordinates relative to the visual plane.

Multiplying out the matrices in (9.30), we find:

$$C_1 = \begin{pmatrix} -\sin\gamma\sin\tau_1 & \cos\gamma\sin\tau_1 & \cos\tau_1 & 0 \\ -\sin\gamma\cos(\tau_1+\varepsilon) & \cos\gamma\cos(\tau_1+\varepsilon) & -\sin(\tau_1+\varepsilon) & 0 \\ -\cos\gamma & -\sin\gamma & 0 & a_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad 9.31$$

and

$$C_2 = \begin{pmatrix} \sin\gamma\sin\tau_2 & \cos\gamma\cos\tau_2 & \cos\tau_2 & 0 \\ \sin\gamma\cos(\tau_2-\varepsilon) & \cos\gamma\cos(\tau_2-\varepsilon) & -\sin(\tau_2-\varepsilon) & 0 \\ -\cos\gamma & \sin\gamma & 0 & a_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad 9.32$$

10 THE BASE QUADRICS

Consider the three quadrics that are the basis of the pencil (8.23)

Put

$$S_1 \equiv Y_1Z_2 - Z_1Y_2 = 0 \qquad 10.1$$

$$S_2 \equiv Z_1X_2 - X_1Z_2 = 0 \qquad 10.2$$

$$S_3 \equiv X_1Y_2 - Y_1X_2 = 0 \qquad 10.3$$

If we substitute from (9.29) into these equations, we find:

$$\begin{aligned} S_1 \equiv\ & x'^2\{\cos(\tau_1+\varepsilon) + \cos(\tau_2-\varepsilon)\}\sin\gamma\cos\gamma \\ & + x'y'\{\cos(\tau_2-\varepsilon) - \cos(\tau_1+\varepsilon)\} \\ & + y'^2\{\cos(\tau_1+\varepsilon) + \cos(\tau_2-\varepsilon)\}\sin\gamma\cos\gamma \\ & + x'z'\{\sin(\tau_1+\varepsilon) - \sin(\tau_2-\varepsilon)\}\cos\gamma \\ & - y'z'\{\sin(\tau_1+\varepsilon) + \sin(\tau_2-\varepsilon)\}\sin\gamma \\ & - x'\{a_2\cos(\tau_1+\varepsilon) + a_1\cos(\tau_2-\varepsilon)\}\sin\gamma \\ & + y'\{a_2\cos(\tau_1+\varepsilon) - a_1\cos(\tau_2-\varepsilon)\}\cos\gamma \\ & - z'\{a_2\sin(\tau_1+\varepsilon) - a_1\sin(\tau_2-\varepsilon)\} \\ & = 0 \end{aligned} \qquad 10.4$$

$$\begin{aligned} S_2 \equiv\ & -x'^2(\sin\tau_2 + \sin\tau_1)\sin\gamma\cos\gamma \\ & -x'y'(\sin\tau_2 - \sin\tau_1) - y'^2(\sin\tau_2 + \sin\tau_1)\sin\gamma\cos\gamma \\ & - x'z'(\cos\tau_2 - \cos\tau_1)\cos\gamma \\ & - y'z'(\cos\tau_2 + \cos\tau_1)\sin\gamma \\ & + x'(a_1\sin\tau_2 + a_2\sin\tau_1)\sin\gamma \\ & + y'(a_1\sin\tau_2 - a_2\sin\tau_1)\cos\gamma \\ & + z'(a_1\cos\tau_2 - a_2\cos\tau_1) \\ & = 0 \end{aligned} \qquad 10.5$$

$$S_3 \equiv x'^2\{\sin\tau_1\cos(\tau_2-\varepsilon) - \sin\tau_2\cos(\tau_1+\varepsilon)\}\sin^2\gamma$$
$$- y'^2\{\sin\tau_1\cos(\tau_2-\varepsilon) - \sin\tau_2\cos(\tau_1+\varepsilon)\}\cos^2\gamma$$
$$+ z'^2\{\cos\tau_1\sin(\tau_2-\varepsilon) - \cos\tau_2\sin(\tau_1+\varepsilon)\}$$
$$- x'z'\{\sin\tau_1\sin(\tau_2-\varepsilon) + \cos\tau_1\cos(\tau_2-\varepsilon)$$
$$+ \cos\tau_2\cos(\tau_1+\varepsilon) + \sin\tau_2\sin(\tau_1+\varepsilon)\}\sin\gamma$$
$$+ y'z'\{\sin\tau_1\sin(\tau_2-\varepsilon) - \cos\tau_1\cos(\tau_2-\varepsilon)$$
$$- \sin\tau_2\sin(\tau_1+\varepsilon) + \cos\tau_2\cos(\tau_1+\varepsilon)\}\cos\gamma$$
$$= 0 \qquad 10.6$$

(10.4),(10.5),(10.6), then, are the equations of three surfaces composed entirely of nonius-lines for the case when the fixation point is at (0,0,0). And, any quadric surface through their common curve of intersection , is also composed of such lines.

S_1 is a hyperboloid of one sheet, S_2 is a hyperbolic paraboloid, and S_3 is a cone, with a vertex at the origin (for certain non-zero values of the angle of torsion) see: fig. 10.1.

These equations can be simplified by making the substitutions

$$\sin(\tau_2-\varepsilon) = k_1\sin(\tau_1+\varepsilon) \qquad 10.7$$
$$\cos(\tau_2-\varepsilon) = k_2\cos(\tau_1+\varepsilon) \qquad 10.8$$
$$\sin\tau_2 = j_1\sin\tau_1 \qquad 10.9$$
$$\cos\tau_2 = j_2\cos\tau_1 \qquad 10.10$$

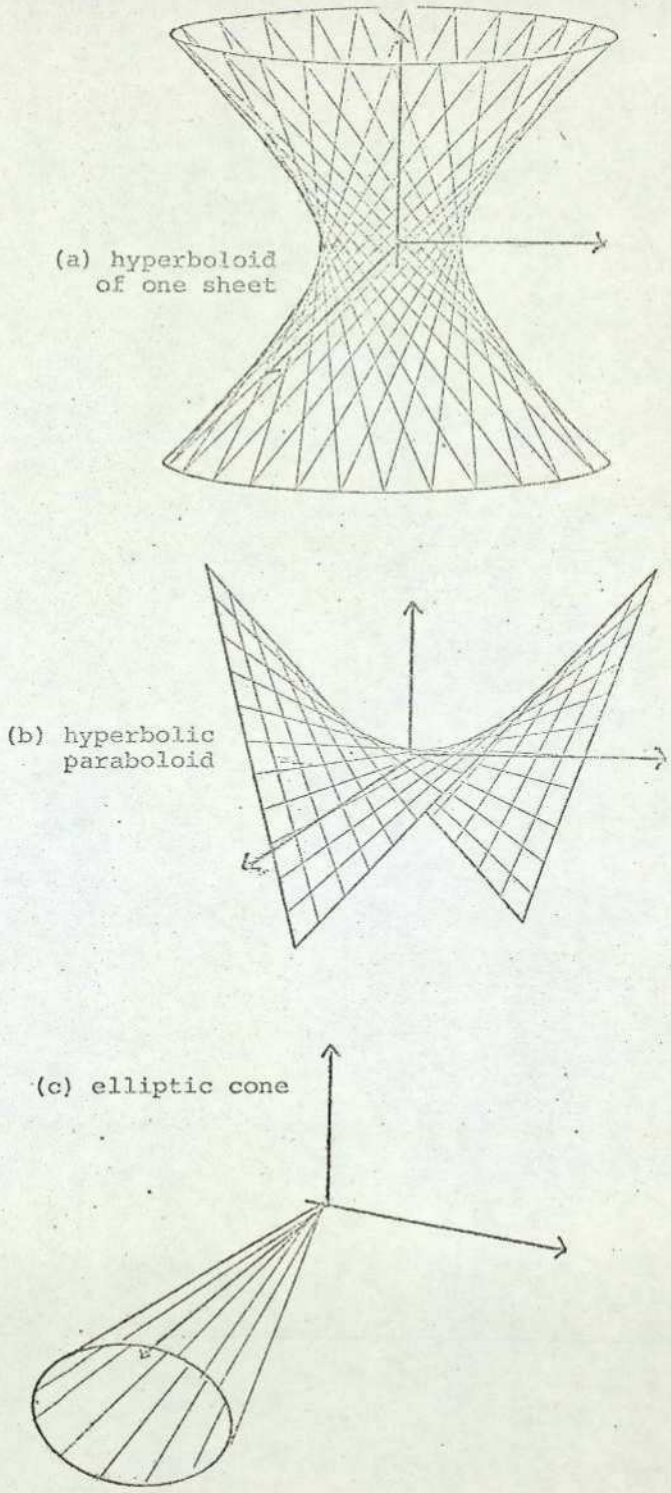


Fig. 10.1 Types of ruled quadric surfaces

that the base quadrics can take

When the basic rectilinear horopter surfaces, become

$$S_1 \equiv x'^2(1+k_2)SC + y'^2(1+k_2)SC - x'y'(1-k_2) + x'z'(1-k_1)CT - y'z'(1+k_1)ST - x'(a_2+a_1k_2)S + y'(a_2-a_1k_2)C - z'(a_2-a_1k_1)T = 0 \qquad 10.11$$

$$S_2 \equiv x'^2(1+j_1)SC + y'^2(1+j_1)SC - x'y'(1-j_1) - x'z'(1-j_2)CK + y'z'(1+j_2)SK - x'(a_2+a_1j_1)S y'(a_2-a_1j_1)C + z'(a_2-a_1j_2)K = 0 \qquad 10.12$$

$$S_3 \equiv x'^2(k_2-j_1)S^2 - y'^2(k_2-j_1)C^2 + z'^2(k_1-j_2)KT - x'z'\{(k_1+j_1)T + (k_2+j_2)K\}S + y'z'\{(k_1-j_1)T - (k_2-j_2)K\}C = 0 \qquad 10.13$$

where

$$S = \sin\gamma \qquad 10.14$$

$$C = \cos\gamma \qquad 10.14$$

$$T = \tan(\tau_1+\varepsilon) \qquad 10.15$$

$$K = \cot\tau_1 \qquad 10.16$$

And the rectilinear horopter is the family,

$$\lambda_1S_1 + \lambda_2S_2 + \lambda_3S_3 = 0 \qquad 10.17$$

The form that these three surfaces can take changes with the fixation point. Various numerical particular cases are considered below, (chapter 18).

However, we can notice the following properties now.

From (10.11), we can see that S_1, meets the visual plane, $z' = 0$, when

$$x'^2(1+k_2)SC + y'^2(1+k_2)SC - x'y'(1-k_2) - x'(a_2+a_1k_2)S + y'(a_2-a_1k_2)C = 0 \quad 10.18$$

and, we can note, that if

$$k_2 = 1 \quad 10.19$$

this reduces to

$$x'^2 + y'^2 - \frac{x'(a_2+a_1)}{2C} + \frac{y'(a_2-a_1)}{2S} = 0 \quad 10.20$$

which, substitution from (9.3),(9.4), will reveal is the circle through F,L and R;

i.e., it is the Vieth-Müller circle.

This occurs when, from (10.8)

$$\tau_2 = \tau_1 + 2\varepsilon \quad 10.21$$

or

$$\tau_2 = -\tau_1 \quad 10.22$$

From (10.12), it can be seen that S_2, will meet $z' = 0$, when

$$x'^2(1+j_1)SC + y'^2(1+j_1)SC - x'y'(1-j_1) - x'(a_2+a_1j_1)S + y'(a_2-a_1j_1)C = 0 \quad 10.23$$

and if

$$j_1 = 1 \quad 10.24$$

this becomes (10.20), the Vieth-Müller circle again.

This occurs, when, from (10.9)

$$\tau_2 = \tau_1 \quad 10.25$$

From (10.13), we can see that S_3, will meet the plane $z' = 0$, when

$$x'^2S^2 - y'^2C^2 = 0 \qquad 10.26$$

i.e., when

$$(x'S-y'C)(x'S+y'C) = 0 \qquad 10.27$$

that is, S_3 meets the visual plane in the two lines

$$y' = \tan\gamma x' \qquad 10.28$$

$$y' = -\tan\gamma x' \qquad 10.29$$

i.e., the two lines FL and FR.

Further, since S_3 is a homogeneous equation of the second degree, with no constant term, we know immediately that it is the equation of a cone, with its vertex at the origin. (see: fig. 10.1c)

11 DERIVATION OF THE HOROPTER

The horopter is the common curve of intersection of the three quadric surfaces, S_1, S_2, and S_3.

Any plane meets a quadric in a conic, thus, a general plane will meet two quadrics in the four points common to two conics. A curve that meets a general plane in four points is called a quartic curve. Thus, the curve of intersection of two quadrics is, in general, a quartic curve, Γ_4.

If the quadric is a ruled surface, comprising straight lines, then these lines are called generators of the quadric. If two ruled quadrics have a generator in common, then a general plane will meet Γ_4, their curve of intersection, in one point of this generator and three other points. Thus, Γ_4 breaks down into this generator and a twisted cubic curve Γ.

While a general plane will cut any quadric in a conic, if this plane is a tangent plane to the quadric at somepoint P, then the conic will degenerate into two straight lines. These two lines are the generators through P.

The equation of the tangent plane to a quadric

$$S = 0$$

at a point (x_1,y_1,z_1), is given by

$$(x-x_1)\frac{\partial S}{\partial x_1} + (y-y_1)\frac{\partial S}{\partial y_1} + (z-z_1)\frac{\partial S}{\partial z_1} = 0 \qquad 11.1$$

where $\frac{\partial S}{\partial x_1}$ is the partial derivative of S with respect to x at the point (x_1,y_1,z_1).

From (10.11),(11.1), we see that, Λ_1, the tangent plane to S_1 at the point (0,0,0) is

$$\Lambda_1 \equiv x'(a_2+a_1k_2)S - y'(a_2-a_1k_2)C + z'(a_2-a_1k_1)T = 0 \qquad 11.2$$

Substituting for z' from (11.2) into (10.11), we see that Λ_1 meets S_1, when

$$x'^2(k_1+k_2)(a_2-a_1)SC + y'^2(k_2-k_1)(a_1+a_2)SC + x'y'\{(a_1k_1+a_2k_2)+(S^2-C^2)(a_1k_2+a_2k_1)\} = 0 \qquad 11.3$$

which factorizes into the linear factors

$$\{x'(k_2+k_1)S+y'(k_2-k_1)C\}\{x'(a_2-a_1)C+y'(a_2+a_1)S\} = 0 \qquad 11.4$$

Now S_3 is a cone whose vertex is at the point (0,0,0). It is the property of the vertex of a cone that any plane through it cuts the cone in two generators. We can find the generators that Λ_1 meets S_3 by substituting for z' from (11.2) into (10.13), when we find, that

$$S^2(k_1+k_2)Lx'^2 - C^2(k_1-k_2)My'^2 + 2SCNx'y' = 0 \qquad 11.5$$

where

$$L = (a_2-a_1k_1)(a_2+a_1j_1)T + (a_2+a_1k_2)(a_2-a_1j_2)K \quad 11.6$$

$$M = (a_2-a_1k_1)(a_2-a_1j_1)T + (a_2-a_1k_2)(a_2-a_1j_2)K \quad 11.7$$

$$N = (a_1k_2^2-a_2k_1)(a_2-a_1j_2)K-(a_2-a_1k_1)(a_2k_1-a_1j_1k_2)T \quad 11.8$$

and (11.5) factorizes to

$$\{S(k_2+k_1)x' + C(k_2-k_1)y'\}\{SLx' - CMy'\} = 0 \quad 11.9$$

Thus from (11.2),(11.4),(11.9) we can see that the line

$$\left.\begin{aligned} x'(a_2+a_1k_2)S - y'(a_2-a_1k_2)C + z'(a_2-a_1k_1)T &= 0 \\ x'(k_2+k_1)S + y'(k_2-k_1)C &= 0 \end{aligned}\right\} \quad 11.10$$

which is equivalent to

$$\frac{-Sx'}{k_2-k_1} = \frac{Cy'}{k_2+k_1} = \frac{Tz'}{2k_2} \quad 11.11$$

is a generator common to both S_1 and S_3.

Hence as the surfaces S_1 and S_3 have a generator in common their residual curve of intersection is a twisted cubic curve Γ. So S_1 intersects Λ_1 in two generators intersecting at F(0,0,0). Let these be given by ℓ and m where ℓ is the common generator of S_1 and S_3 (fig. 11.1).

Now ℓ is a generator of the rectilinear horopter and is therefore a nonius line. We have seen in §8 that each nonius line contains two self-corresponding points. That is, each nonius line must meet Γ in two points.

Further since F is the vertex of the cone S_3 an infinite number of nonius lines pass through F, i.e. F must lie on Γ.

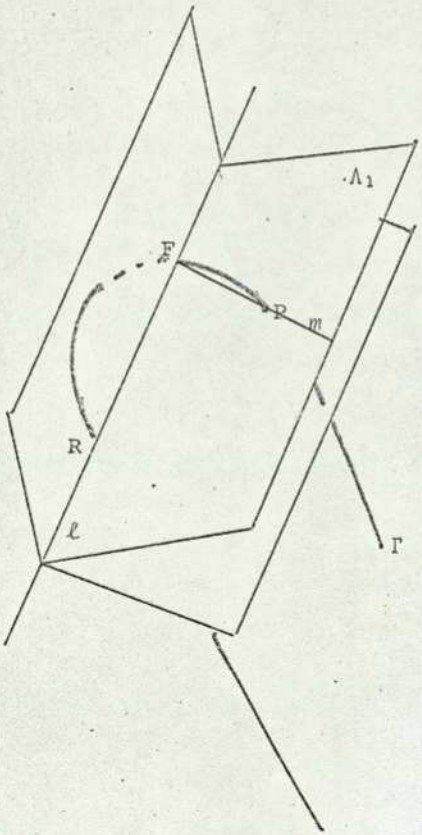


Fig. 11.1 The generation of Γ

Λ_1 is the tangent plane to S_1 at F.

ℓ and m are the generators of S_1 in Λ_1.

ℓ is the common generator of the base quadrics that meets Γ in two points F and R.

Any plane through ℓ therefore, meets Γ in Q and R and one further point.

Thus by considering every plane through ℓ we will generate every point of Γ.

So the line ℓ meets Γ in two points, one of which is F. Let the other point of intersection be R (fig. 11.1).

Now since the space horopter Γ is a twisted cubic curve it will intersect any plane in three points. Consider a general plane through the line ℓ. This plane will meet Γ in the points F and R and one further point (say) P. If the plane is then considered to rotate about the line ℓ, then the residual point of intersection P will generate the whole curve Γ.

From (11.11) we can write the equation of the common generator as

$$\begin{aligned} 2k_2Sx' + (k_2-k_1)Tz' &= 0 \\ 2k_2Cy' - (k_2+k_1)Tz' &= 0 \end{aligned} \qquad 11.12$$

and thus any plane through this line as

$$\{2k_2Sx'+(k_2-k_1)Tz'\} + t\{2k_2Cy'-(k_2+k_1)Tz'\} = 0 \qquad 11.13$$

where t is a variable parameter.

i.e.

$$2k_2Sx' + 2k_2tCy' + \{(k_2-k_1)-(k_2+k_1)t\}Tz' = 0 \qquad 11.14$$

As t varies, (11.14) will generate the whole family of planes through the common generator to S_1 and S_3. For each value of t, then, the plane (11.14) will meet Γ in one further point, and, as t varies, the whole curve Γ will be traced out. This point will occur where the plane (11.14) meets either S_1 or S_3. (fig. 11.1)

If we substitute for x' in (11.14) into either (10.11) or (10.13) we will obtain a quadratic expression in y' and z'. However, as we know that the generator (11.12) is common to the two quadrics and to the plane (11.14). We know one of the factors in the expression must be a linear term from (11.12). Hence, the other factor must also be linear, so we can write the result of the substitution as:

$$\{2k_2Cy'-(k_2+k_1)Tz'\}\{D_iy'+E_iz'+F_i\} = 0 \qquad 11.15$$

$$(i=1 \text{ or } 3)$$

and, substituting into (10.11), we find

$$D_1 = \frac{1}{2Sk_2}\{(1+k_2)C^2t^2 + (1-k_2)t + (1+k_2)S^2\} \qquad 11.16$$

$$E_1 = \frac{-CT}{4Sk_2^2}\{(1+k_2)(k_1+k_2)t^2-2(k_2^2-k_1)t+(k_2-k_1)(k_2-1) \qquad 11.17$$

$$F_1 = \frac{1}{2k_2}\{(a_2+a_1k_2)t + (a_2-a_1k_2)\} \qquad 11.18$$

and, substitution in (10.13) gives:

$$D_3 = \frac{C}{2k_2}(k_2-j_1)(t^2-1) \qquad 11.19$$

$$E_3 = -\frac{1}{4k_2^2}\{(k_2+k_1)(k_2-j_1)Tt^2-2\{(k_2^2+j_1k_1)T+k_2(k_2+j_2)K\}t + \{(k_2+j_1)(k_2-k_1)T + 2k_2(k_2-j_2)K\}\} \qquad 11.20$$

$$F_3 = 0 \qquad 11.21$$

Thus we have, that the residual point of intersection of the surfaces S_1 and S_3 in the plane (11.14) is at the intersection in this plane of the lines

$$D_1y' + E_1z' + F_1 = 0 \qquad 11.22$$

$$D_3y' + E_3z' = 0 \qquad 11.23$$

We can now change our co-ordinate representation to homogeneous cartesian co-ordinates. That is, we can let the point P, with co-ordinates (x,y,z) be represented by a four-dimensional vector (x_1,x_2,x_3,x_4), where

$$x = x_1/x_4 \qquad 11.24$$

$$y = x_2/x_4 \qquad 11.25$$

$$z = x_3/x_4 \qquad 11.26$$

These co-ordinates are determined only by their ratios, so the following values for the vector $\underline{x}$ can be taken without ambiguity to represent the same point.

$$\underline{x} = (x, y, z, 1) \qquad 11.27$$

$$= (x_1, x_2, x_3, x_4) \qquad 11.28$$

$$= (\rho x_1, \rho x_2, \rho x_3, \rho x_4) \qquad 11.29$$

where ρ has any value $\neq 0$.

The point $(x_1, x_2, x_3, 0)$ is taken to be the point at infinity with direction ratios x_1, x_2, x_3; and for any finite point we must have

$$x_4 \neq 0 \qquad 11.30$$

These co-ordinates are restricted in that they cannot all be simultaneously zero.

In these co-ordinates, we can solve the equations (11.14), (11.22), (11.23) simultaneously to give the co-ordinates of the residual point of intersection of the plane (11.14) with both S_1 and S_3, for any value of t.

Which gives

$$x_1' = -\frac{C}{S}t x_2' - \frac{T}{2Sk_2}\{(k_2-k_1)-(k_2+k_1)t\}x_3' \qquad 11.31$$

$$x_2' = -E_3F_1 \qquad 11.32$$

$$x_3' = D_3F_1 \qquad 11.33$$

$$x_4' = D_1E_3 - D_3E_1 \qquad 11.34$$

In which, substituting from (11.15)-(11.20) we find that we can write the co-ordinates of Γ as:

$$x_1' = q_{11}'t^3 + q_{12}'t^2 + q_{13}'t + q_{14}' \qquad 11.35$$

$$x_2' = q_{21}'t^3 + q_{22}'t^2 + q_{23}'t + q_{24}' \qquad 11.36$$

$$x_3' = q_{31}'t^3 + q_{32}'t^2 + q_{33}'t + q_{34}' \qquad 11.37$$

$$x_4' = q_{41}'t^3 + q_{42}'t^2 + q_{43}'t + q_{44}' \qquad 11.38$$

i.e $$\underline{x}' = Q'\underline{t} \qquad 11.39$$

where $$\underline{t} = (t^3, t^2, t, 1) \qquad 11.40$$

and, writing

$$A_1' = \tfrac{1}{2}(a_2 - a_1k_2) \qquad 11.41$$

$$A_2' = \tfrac{1}{2}(a_2 + a_1k_2) \qquad 11.42$$

we have

$$q_{11}' = 2CA_1'(k_2-k_1)(k_2-j_1)T \qquad 11.43$$

$$q_{12}' = -2C\{\{2(k_2^2-j_1k_1)A_1' - (k_2-j_1)(k_2-k_1)A_2'\}T + 2k_2(k_2-j_2)A_1'K\} \qquad 11.44$$

$$q_{13}' = -2C\{\{2(k_2^2-j_1k_1)A_2' - (k_2+j_1)(k_2+k_1)A_1'\}T + 2k_2\{(k_2-j_2)A_2' - (k_2+j_2)A_1'\}K\} \qquad 11.45$$

$$q_{14}' = 2CA_2'\{(k_2+j_1)(k_2+k_1)T + 2k_2(k_2+j_2)K\} \qquad 11.46$$

$$q_{21}' = 2S\{(k_2+j_1)(k_2-k_1)T + 2k_2(k_2-j_2)K\}A_1' \qquad 11.47$$

$$q_{22}' = 2S\{\{(k_2+j_1)(k_2-k_1)A_2' - 2(k_2^2+j_1k_1)A_1'\}T + 2k_2\{(k_2-j_2)A_2' - (k_2+j_2)A_1'\}K\} \qquad 11.48$$

$$q_{23}' = 2S\{\{(k_2+k_1)(k_2-j_1)A_1' - 2(k_2^2+j_1k_1)A_2'\}T - 2k_2(k_2+j_2)A_2'K\} \qquad 11.49$$

$$q_{24}' = 2SA_2'(k_2+k_1)(k_2-j_1)T \qquad 11.50$$

$$q'_{31} = -4SCk_2(k_2-j_1)A'_1 \qquad 11.51$$

$$q'_{32} = -4SCk_2(k_2-j_1)A'_2 \qquad 11.52$$

$$q'_{33} = 4SCk_2(k_2-j_1)A'_1 \qquad 11.53$$

$$q'_{34} = 4SCk_2(k_2-j_1)A'_2 \qquad 11.54$$

$$q'_{41} = -(k_2-k_1)\{(k_2+j_1)(1+k_2)S^2 - (k_2-j_1)(1-k_2)C^2\}T$$
$$-2k_2(1+k_2)(k_2-j_2)S^2K \qquad 11.55$$

$$q'_{42} = 2C^2(k_2-j_1)(k_2^2-k_1)T + 2S^2(1+k_2)(k_2^2+j_1k_1)T$$
$$-(1-k_2)(k_2+j_1)(k_2-k_1)T - 2k_2(1-k_2)(k_2-j_2)K$$
$$+2S^2k_2(1+k_2)(k_2+j_2)K \qquad 11.56$$

$$q'_{43} = -2k_2(k_2-k_1)(1+j_1)C^2T - (1+k_2)(k_2+k_1)(k_2-j_1)T$$
$$+2(1-k_2)(k_2^2+j_1k_1)T - 2C^2k_2(1+k_2)(k_2-j_2)K$$
$$+2k_2(1-k_2)(k_2+j_2)K \qquad 11.57$$

$$q'_{44} = -(1-k_2)(k_2+k_1)(k_2-j_1)T + 2k_2(1+k_2)(k_2+j_2)C^2K$$
$$+2k_2(k_2+k_1)(1+j_1)C^2T \qquad 11.58$$

Thus, we have, the equation of the horopter, that is, the locus in space of those points which give rise to corresponding retinal points is given by

$$\Gamma \equiv \underline{x}' = Q'\underline{t} \qquad 11.59$$

where the elements of Q' are given by (11.43)-(11.58), and $\underline{x}'$ are the co-ordinates with respect to the fixation point.

As t varies from $-\infty$ to $+\infty$ the whole curve Γ is traced out.

Note: For convenience the parameter, t, of (11.35)-(11.40) and (11.59) has been taken as the reciprocal of the t of (11.14) etc.

From (11.59), we can see, that Γ meets, the visual plane, i.e. $x_3^1 = 0$, when,

$$x_3^1 = q_{31}^1 t^3 + q_{32}^1 t^2 + q_{33}^1 t + q_{34}^1 = 0 \qquad 11.60$$

from which (11.51)-(11.54) =>

$$-A_1^1 t^3 - A_2^1 t^2 + A_1^1 t + A_2^1 = 0 \qquad 11.61$$

whose roots are

$$t = 1, -1, -A_2^1/A_1^1 \qquad 11.62$$

Substituting for these values in (11.59), we find:

$$t = 1 \Rightarrow \underline{x}' = \{(A_2^1 - A_1^1)C, (A_2^1 - A_1^1)S, 0, 1\}$$
$$\equiv R \qquad 11.63$$

$$t = -1 \Rightarrow \underline{x}' = \{(A_2^1 + A_1^1)C, -(A_2^1 + A_1^1)S, 0, 1\}$$
$$\equiv L \qquad 11.64$$

$$t = -A_2^1/A_1^1 \Rightarrow$$
$$\underline{x}' = (0,0,0,1)$$
$$\equiv F \qquad 11.65$$

Thus, the horopter curve, Γ, meets the visual plane in the three points R,L and F.

12 TRANSFER OF ORIGIN

Equation (11.59) gives the co-ordinates of Γ referred to the fixation point as origin. We now wish to transform to co-ordinates fixed in space, or rather, fixed in relation to the head. If the head is assumed to be erect and fixed these two systems are identical.

Let the distance between R and L be 2b, and let the mid-point of RL be O.

We can then take O as origin, and take the z-axis as the vertical through O (positive upwards), so that the horizontal plane is

$$z = 0 \qquad 12.1$$

Then the x-axis is where z = O meets the median plane (positive forward).
And the y-axis is the base-line RL, such that x,y,z form a right-handed orthogonal set.

Let the fixation point be at F, and let the distance OF be a. Let the line OF make an angle α with RL, and, finally, let the visual plane be inclined at an angle λ to the horizontal.

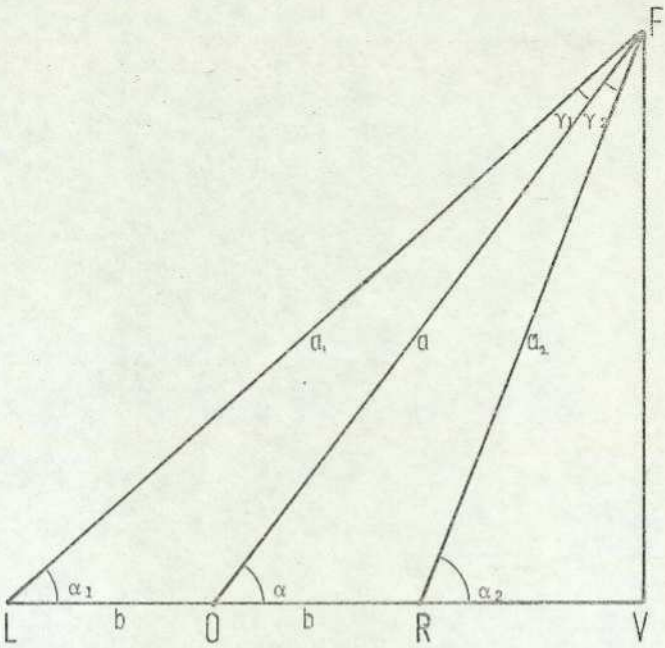


Fig. 12.1 Relationships in the visual plane

$VF = a\sin\alpha = a_1\sin\alpha_1 = a_2\sin\alpha_2$

$OV = a\cos\alpha \quad RV = a\cos\alpha - b \quad LV = a\cos\alpha + b$

$$\tan\alpha_1 = \frac{a\sin\alpha}{a\cos\alpha+b} \qquad \tan\alpha_2 = \frac{a\sin\alpha}{a\cos\alpha-b}$$

$$a_1^2 = a^2 + b^2 + 2ab\cos\alpha$$

$$a_2^2 = a^2 + b^2 - 2ab\cos\alpha$$

$$a_1 = \frac{a\sin\alpha}{\sin\alpha_1} \qquad a_2 = \frac{a\sin\alpha}{\sin\alpha_2}$$

$$\gamma = \tfrac{1}{2}(\gamma_1 + \gamma_2) = \tfrac{1}{2}(\alpha_2 - \alpha_1)$$

$$\tan 2\gamma = \tan(\alpha_2-\alpha_1) = \frac{2ab\sin\alpha}{a^2-b^2}$$

In the visual plane $z' = 0$, the following relationships hold (fig. 12.1)

$$a_1^2 = a^2 + b^2 - 2ab\cos\alpha \qquad 12.2$$

$$a_2^2 = a^2 + b^2 + 2ab\cos\alpha \qquad 12.3$$

$$\tan 2\gamma = \frac{2ab\sin\alpha}{a^2 - b^2} \qquad 12.4$$

If we put

$$A_1 = \tfrac{1}{2}(a_2 - a_1) \qquad 12.5$$

$$A_2 = \tfrac{1}{2}(a_2 + a_1) \qquad 12.6$$

then, it can be shown that the following relationships also hold using (10.14), (10.15)

$$b^2 = A_1^2C^2 + A_2^2S^2 \qquad 12.7$$

$$ab\sin\alpha = SC(A_2^2 - A_1^2) \qquad 12.8$$

$$ab\cos\alpha = A_1A_2$$

Now, in the co-ordinates related to the fixation point, $\underline{x}'$ (see fig. 9.1), we can write the co-ordinates of the nodal points, as

$$R \equiv \{(A_2 - A_1)C, (A_2 - A_1)S, 0, 1\} \qquad 12.9$$

$$L \equiv \{(A_2 + A_1)C, -(A_2 + A_1)S, 0, 1\} \qquad 12.10$$

and, if the new origin, O, is to be the mid-point of R and L, then, with respect to $\underline{x}'$

$$O \equiv (A_2C, -A_1S, 0, 1) \qquad 12.11$$

then we can transfer the origin to this point leaving the axes parallel to their original direction, giving a new co-ordinate system $\underline{x}_1$, say, by the transformation

$$\underline{x}_1 = \mathbb{P}_1\underline{x}' \qquad 12.12$$

(See: fig. 12.2)

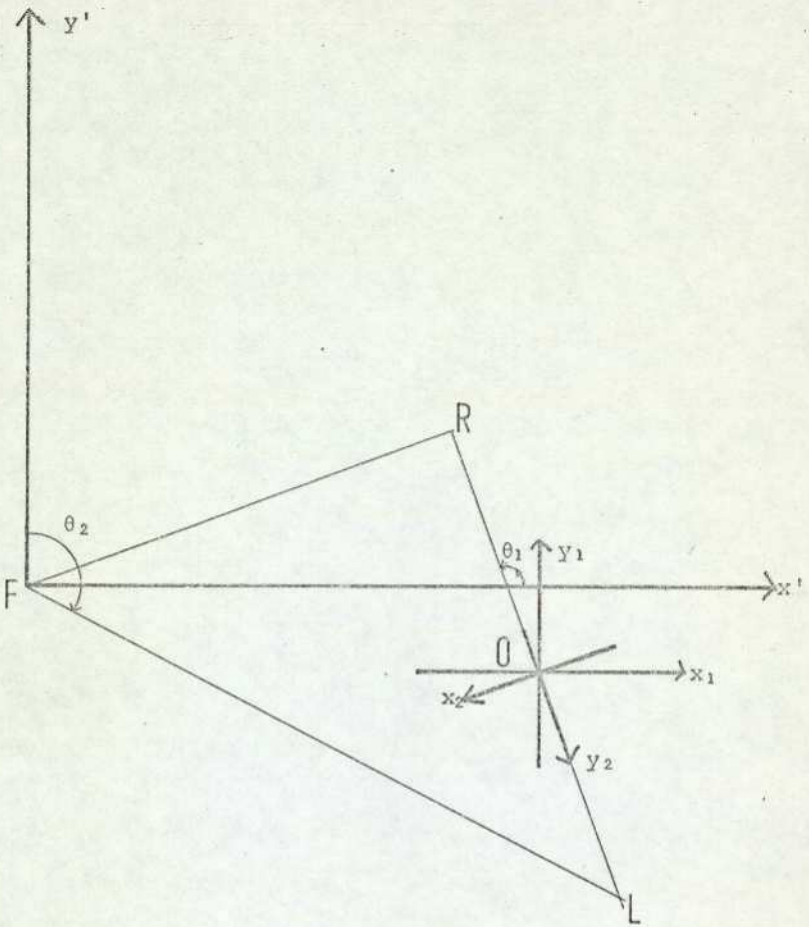


Fig 12.2 Transfer of origin

The translation P_1 moves the origin from F to O, with parallel axes $\underline{x}' \rightarrow \underline{x}_1$.

The rotation P_2 rotates the co-ordinates about the z_1-axis so that y_2-axis is along RL $\quad \underline{x}_1 \rightarrow \underline{x}_2$

The rotation P_3 rotates the co-ordinates about the base line RL until the z-axis is vertical, i.e. until the xy-plane is horizontal $\quad \underline{x}_2 \rightarrow \underline{x}$.

where

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & -A_2C \\ 0 & 1 & 0 & A_1S \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad 12.13$$

From (12.9),(12.10), we can deduce that with respect to the $\underline{x}'$ system, and thus, also to the $\underline{x}_1$ system , the line RL, in the visual plane makes an angle θ_1 with the x' axis where

$$\tan\theta_1 = \frac{-A_2S}{A_1C} \qquad 12.14$$

To rotate the axes, such that the line RL is the y_2-axis, and the x_2-axis is perpendicular to the $y_2$axis in the plane $z_1 = 0$, and x_2 is positive towards the fixation point, we need to rotate the axes about the z_1-axis, through an angle θ_2 , where θ_2 is $-LOy_1$.

i.e. $\theta_2 = -(\frac{1}{2}\pi+\theta_1)$

therefore $\tan\theta_2 = \cot\theta_1 = -\frac{A_1C}{A_2S}$ 12.15

Thus, we can put

$$\underline{x}_2 = P_2\underline{x}_1 \qquad 12.16$$

where

$$P_2 = \begin{pmatrix} -A_2S & -A_1C & 0 & 0 \\ A_1C & -A_2S & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \qquad 12.17$$

simplifying from (12.7)

If we finally, rotate the $\underline{x}_2$-axes to the $\underline{x}$-axes by a rotation through λ, about the y_2-axis, so that the z-axis is vertical, then, we have,

$$\underline{x} = P_3 \underline{x}_2 \quad \text{(say)} \qquad 12.18$$

where

$$P_3 = \begin{pmatrix} \cos\lambda & 0 & \sin\lambda & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\lambda & 0 & \cos\lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad 12.19$$

Thus, we have

from (12.12) $\underline{x}_1 = P_1 \underline{x}'$ 12.20

from (12.16) $\underline{x}_2 = P_2 \underline{x}_1$ 12.21

$= P_2 P_1 \underline{x}'$ (from 12.20) 12.22

from (12.18) $\underline{x} = P_3 \underline{x}_2$ 12.23

$= P_3 P_2 P_1 \underline{x}'$ (from 12.22) 12.24

$= P \underline{x}'$ (say) 12.25

where $P = P_3 P_2 P_1$ 12.26

Thus (12.25) gives us the transformation of co-ordinates $\underline{x}'$, defined with respect to the fixation point, to $\underline{x}$, those fixed in relation to the head.

If we evaluate the matrix P from (12.26), by substituting from (12.13), (12.17) (12.19) and multiplying, we find utilizing the relationships (12.7), (12.8)

$$P = \begin{pmatrix} -A_2 S\cos\lambda & -A_1 C\cos\lambda & b\sin\lambda & ab\sin\alpha\cos\lambda \\ A_1 C & -A_2 S & 0 & -abc\cos\alpha \\ A_2 S\sin\lambda & A_1 C\sin\lambda & b\cos\lambda & -ab\sin\alpha\sin\lambda \\ 0 & 0 & 0 & b \end{pmatrix} \qquad 12.27$$

If we put $\underline{x}' = (0,0,0,1)$ in (12.25), we find the co-ordinates of the fixation point in this system is

$$\underline{x}_f = (a\sin\alpha\cos\lambda, -a\cos\alpha, -a\sin\alpha\sin\lambda, 1) \equiv F \qquad 12.28$$

We can also see

$$R \equiv (0,b,0,1) \qquad 12.29$$

$$L \equiv (0,-b,0,1) \qquad 12.30$$

and the visual plane, i.e., the plane through R,L,F is

$$\Lambda \equiv \sin\lambda x_1 + \cos\lambda x_3 = 0 \qquad 12.31$$

From (11.59), we have that the equation of the horopter with respect to the $\underline{x}'$ co-ordinates is given by

$$\Gamma \equiv \underline{x}' = Q'\underline{\xi} \qquad 12.32$$

but, from (12.25), we have

$$\underline{x} = P\underline{x}'$$

and so, with respect to the new co-ordinates, we see, by multiplying both sides of 12.32 by P.

$$\Gamma \equiv \underline{x} = PQ'\underline{\xi} \qquad 12.33$$

$$= Q\underline{\xi} \quad \text{(say)} \qquad 12.34$$

where

$$Q = PQ' \qquad 12.35$$

So, evaluating (12.35) from (11.43)-(11.54) and (12.27) will give us (12.34) explicitly, which is the equation of the three dimensional space horopter.

Evaluating (12.35), we find, where q_{ij} are the elements of Q:
and, noting

$$2A_1 = A_2(1-k_2) + A_1(1+k_2) \qquad 12.36$$

$$2A_2 = A_2(1+k_2) + A_1(1-k_2) \qquad 12.37$$

$$\begin{aligned} q_{11} = \; & -2SCk_2\cos\lambda(k_2-k_1)\{(1+j_1)b^2+A_1A_2(1-j_1)\}T \\ & -2SCk_2\cos\lambda(k_2-j_2)\{(1+k_2)b^2+A_1A_2(1-k_2)\}K \\ & -2SCk_2\sin\lambda(k_2-j_1)b\{A_2(1-k_2)+A_1(1+k_2) \end{aligned} \qquad 12.38$$

$$\begin{aligned} q_{12} = \; & 2SCk_2\cos\lambda(k_2+k_1)\{(1+j_1)b^2+A_1A_2(1-j_1)\}T \\ & +2SCk_2\cos\lambda(k_2+j_2)\{(1+k_2)b^2+A_1A_2(1-k_2)\}K \\ & -2SCk_2\sin\lambda(k_2-j_1)b\{A_2(1+k_2)+A_1(1-k_2)\} \end{aligned} \qquad 12.39$$

$$\begin{aligned} q_{13} = \; & 2SCk_2\cos\lambda(k_2-k_1)\{(1+j_1)b^2+A_1A_2(1-j_1)\}T \\ & 2SCk_2\cos\lambda(k_2-j_2)\{(1+k_2)b^2+A_1A_2(1-k_2)\}K \\ & 2SCk_2\sin\lambda(k_2-j_1)b\{A_2(1-k_2)+A_1(1+k_2)\} \end{aligned} \qquad 12.40$$

$$\begin{aligned} q_{14} = \; & -2SCk_2\cos\lambda(k_2+k_1)\{(1+j_1)b^2+A_1A_2(1-j_1)\}T \\ & -2SCk_2\cos\lambda(k_2+j_2)\{(1+k_2)b^2+A_1A_2(1-k_2)\}K \\ & +2SCk_2\sin\lambda(k_2-j_1)b\{A_2(1+k_2)+A_1(1-k_2)\} \end{aligned} \qquad 12.41$$

$$q_{21} = (k_2-k_1)(1+k_2)(k_2-j_1)A_1^2C^2T$$
$$-(k_2-k_1)(1-k_2)(k_2+j_1)A_2^2S^2T$$
$$-2k_2(1-k_2)(k_2-j_2)A_2^2S^2K \qquad 12.42$$

$$q_{22} = (k_2-k_1)(k_2-j_1)(1-k_2)A_1^2C^2T$$
$$-2(k_2^2-j_1k_1)(1+k_2)A_1^2C^2T$$
$$+2(k_2^2+j_1k_1)(1-k_2)A_2^2S^2T$$
$$-(k_2-k_1)(k_2+j_1)(1+k_2)A_2^2S^2T$$
$$+2k_2(k_2+j_2)(1-k_2)A_2^2S^2K$$
$$-2k_2(k_2-j_2)(1+k_2)b^2K \qquad 12.43$$

$$q_{23} = (k_2+k_1)(k_2+j_1)(1+k_2)A_1^2C^2T$$
$$-2(k_2^2-j_1k_1)(1-k_2)A_1^2C^2T$$
$$+2(k_2^2+j_1k_1)(1+k_2)A_2^2S^2T$$
$$-(k_2+k_1)(k_2-j_1)(1-k_2)A_2^2S^2T$$
$$-2k_2(k_2-j_2)(1-k_2)A_1^2C^2K$$
$$+2k_2(k_2+j_2)(1+k_2)b^2K \qquad 12.44$$

$$q_{24} = (k_2+k_1)(1-k_2)(k_2+j_1)A_1^2C^2T$$
$$-(k_2+k_1)(1+k_2)(k_2-j_1)A_2^2S^2T$$
$$+2k_2(1-k_2)(k_2+j_2)A_1^2C^2K \qquad 12.45$$

$$q_{31} = 2SCk_2\sin\lambda(k_2-k_1)\{(1+j_1)b^2+A_1A_2(1-j_1)\}T$$
$$+2SCk_2\sin\lambda(k_2-j_2)\{(1+k_2)b^2+A_1A_2(1-k_2)\}K$$
$$-2SCk_2\cos\lambda(k_2-j_1)b\{A_2(1-k_2)+A_1(1+k_2)\} \quad 12.46$$

$$q_{32} = -2SCk_2\sin\lambda(k_2+k_1)\{(1+j_1)b^2+A_1A_2(1-j_1)\}T$$
$$-2SCk_2\sin\lambda(k_2+j_2)\{(1+k_2)b^2+A_1A_2(1-k_2)\}K$$
$$-2SCk_2\cos\lambda(k_2-j_1)b\{A_2(1+k_2)+A_1(1-k_2)\} \quad 12.47$$

$$q_{33} = -2SCk_2\sin\lambda(k_2-k_1)\{(1+j_1)b^2+ A_1A_2(1-j_1)\}T$$
$$-2SCk_2\sin\lambda(k_2-j_2)\{(1+k_2)b^2+ A_1A_2(1-k_2)\}K$$
$$+2SCk_2\cos\lambda(k_2-j_1)b\{A_2(1-k_2)+A_1(1+k_2)\} \quad 12.48$$

$$q_{34} = 2SCk_2\sin\lambda(k_2+k_1)\{(1+j_1)b^2+A_1A_2(1-j_1)\}T$$
$$+2SCk_2\sin\lambda(k_2+j_2)\{(1+k_2)b^2+A_1A_2(1-k_2)\}K$$
$$+2SCk_2\cos\lambda(k_2-j_1)b\{A_2(1+k_2)+A_1(1-k_2)\} \quad 12.49$$

$$q_{41} = -(k_2-k_1)(k_2+j_1)(1+k_2)bS^2T$$
$$+(k_2-k_1)(k_2-j_1)(1-k_2)bC^2T$$
$$-2k_2(1+k_2)(k_2-j_2)bS^2K \qquad 12.50$$

$$q_{42} = 2(k_2^2-k_1)(k_2-j_1)bC^2T$$
$$+2(k_2^2+j_1k_1)(1+k_2)bS^2T$$
$$-(k_2-k_1)(k_2+j_1)(1-k_2)bT$$
$$-2k_2(k_2-j_2)(1-k_2)bK$$
$$+2k_2(k_2+j_2)(1+k_2)bS^2K \qquad 12.51$$

$$q_{43} = -2k_2(k_2-k_1)(1+j_1)bC^2T$$
$$-(k_1+k_2)(k_2-j_1)(1+k_2)bT$$
$$+2(k_2^2+j_1k_1)(1-k_2)bT$$
$$-2k_2(k_2-j_2)(1+k_2)bC^2K$$
$$+2k_2(k_2+j_2)(1-k_2)bK \qquad 12.52$$

$$q_{44} = 2k_2(k_2+k_1)(1+j_1)bC^2T$$
$$-(k_2+k_1)(k_2-j_1)(1-k_2)bT$$
$$+2k_2(k_2+j_2)(1+k_2)bC^2K \qquad 12.53$$

13 Properties of the horopter curve, Γ

We have now determined that the locus of those points in space that will stimulate corresponding retinal points is given by (12.34) i.e.

$$\underline{x} = Q\underline{t} \qquad 13.1$$

where the elements of the matrix, Q, are given by the equations (12.38-12.53).

These elements, q_{ij}, are functions of (a,α,λ), i.e., the fixation point; b, the semi inter-nodal distance; c, the subjective deviation of the vertical in the primary position; and τ_1 and τ_2, the angles of torsion in the left and right eyes respectively.

Each row of (13.1) can be written

$$x_i = \Sigma q_{ij} t^{(4-j)} \quad (j=1,4) \qquad 13.2$$

and is a cubic expression in t.

Thus, each co-ordinate is given by a cubic expression of the parameter, and hence Γ is a twisted cubic curve.

(13.1) =>

$$\underline{t} = Q^{-1}\underline{x} \qquad 13.5$$

where Q^{-1} is the inverse of the matrix Q.

If we transform to a co-ordinate system, where

$$\underline{X} = \rho Q^{-1}\underline{x} \qquad 13.6$$

then (13.1) => that

$$\underline{X} = \rho Q^{-1}\underline{x} = \rho Q^{-1}(Q\underline{t})$$

$$= \rho\,\underline{t} \qquad 13.7$$

i.e. Γ is in canonical form

$$X_1 : X_2 : X_3 : X_4 = t^3 : t^2 : t : 1 \qquad 13.8$$

In this representation, let

$$\underline{u}.\underline{X} = 0 \qquad 13.9$$

be the plane containing the three points on Γ defined by $(t_1),(t_2),(t_3)$

Then, t_1, t_2, t_3 are the three roots of the equation

$$u_1 t^3 + u_2 t^2 + u_3 t + u_4 = 0 \qquad 13.10$$

$$\Rightarrow \frac{u_1}{1} = \frac{-u_2}{t_1+t_2+t_3} = \frac{u_3}{t_2t_3+t_3t_1+t_1t_2} = \frac{-u_4}{t_1t_2t_3} \qquad 13.11$$

Thus, the equation of the plane (13.9) is

$$X_1 - (t_1+t_2+t_3)X_2 + (t_2t_3+t_3t_1+t_1t_2)X_3 - t_1t_2t_3X_4 = 0 \qquad 13.12$$

If the three points t_i co-incide, then the plane defined by (13.12) is called the OSCULATING PLANE at the point t, and is given by

$$X_1 - 3tX_2 + 3t^2X_3 - t^3X_4 = 0 \qquad 13.13$$

Further, it is evident that every nonius-line is a chord of Γ, and is defined by the two points $(t_1),(t_2)$ at which it meets Γ. And the chord $(t_1),(t_2)$ must lie in the plane (t_1,t_2,t) for all values of t.

So, (13.12) =>

$$X_1-(t_1+t_2)X_2+t_1t_2X_3-t\{X_2-(t_1+t_2)X_3+t_1t_2X_4\} = 0 \qquad 13.14$$

for all values of t.

So, the chord $(t_1),(t_2)$ has the equation:

$$\left.\begin{aligned} X_1 - (t_1+t_2)X_2 + t_1t_2X_3 &= 0 \\ X_2 - (t_1+t_2)X_3 + t_1t_2X_4 &= 0 \end{aligned}\right\} \qquad 13.15$$

And, if we put $t_2 = t_1$, we find that the tangent at the point (t_1) is

$$\left.\begin{aligned} X_1 - 2t_1X_2 + t_1^2X_3 &= 0 \\ X_2 - 2t_1X_3 + t_1^2X_4 &= 0 \end{aligned}\right\} \qquad 13.16$$

Now, Γ meets the plane at infinity,

when $\qquad x_4 = 0 \qquad 13.17$

(13.17) being a cubic equation in t, it will always have at least one real root.

Let this root be t_∞.

Then, the asymptote plane to the cubic curve Γ, is the osculating plane at $t = t_\infty$

i.e. $\qquad X_1 - 3t_\infty X_2 + 3t_\infty^2X_3 - t_\infty^3X_4 = 0 \qquad 13.18$

Now, from (13.6), we have that for i = 1,4

$$X_i = \Sigma q^*_{ij}x_j \quad (j=1,4) \qquad 13.19$$

where (q^*_{ij}) are the coefficients of Q^{-1}.

So, the asymptote plane is

$$\Sigma\,(q^*_{1i} - 3t_\infty q^*_{2i} + 3t_\infty^2 q^*_{3i} - t_\infty^3 q^*_{4i})x_i = 0 \qquad (i=1,4) \qquad 13.20$$

Similarly, the asymptote line, i.e., the tangent at the point $t = t_\infty$, is given by:

$$\left.\begin{aligned}\Sigma(q^*_{1i} - 2t_\infty q^*_{2i} + t_\infty^2 q^*_{3i})x_i &= 0\\ \Sigma(q^*_{2i} - 2t_\infty q^*_{3i} + t_\infty^2 q^*_{4i})x_i &= 0\end{aligned}\right\} \quad (i=1,4) \qquad 13.21$$

If the cubic equation (13.17) has only one real root, then the entire curve Γ, lies on one side of the plane (13.20), and approaches the line (13.21) as an asymptote at either end.

The matrices Q and Q^{-1}, have been evaluated for various fixation points on The City University I.C.L 1905 Computer, and various projections of the curve Γ plotted. (see, below, chapter 18)

14 The quasi-symmetric case: the space horopter

We have, from §3, that, if the direction cosines of the line of fixation of an eye are given by

$$\underline{u} = (\sin\alpha'\cos\lambda, -\cos\alpha', -\sin\alpha'\sin\lambda) \qquad 14.1$$

then, if the eye moves in accordance with Listing's Law, the angle of torsion of the eye is a well-defined function of the co-ordinates of the point of fixation, viz. (from 3.58, 3.59)

$$\sin\tau = \frac{\cos\alpha'\sin\lambda}{1 + \sin\alpha'\cos\lambda} \qquad 14.2$$

$$\cos\tau = \frac{\sin\alpha' + \cos\lambda}{1 + \sin\alpha'\cos\lambda} \qquad 14.3$$

Now, in the co-ordinate system of §12, we have from (12.28) that the co-ordinates of the fixation point are:

$$\underline{x}_f = (a\sin\alpha\cos\lambda, -a\cos\alpha, -a\sin\alpha\sin\lambda) \qquad 14.4$$

which is equal to the vector $\underline{OF}$

Hence it is evident from the symmetry of fig. 12.1, that the vectors LF and RF are given by $\underline{y}_i$ (say) (i=1,2), where

$$\underline{y}_i = (a_i\sin\alpha_i\cos\lambda, -a_i\cos\alpha_i, -a_i\sin\alpha_i\sin\lambda) \qquad 14.5$$

$$(i=1,2)$$

i.e., the direction cosines of the two fixation lines LF and RF are $\underline{u}_i$ (i=1,2), respectively, where

$$\underline{u}_i = (\sin\alpha_i\cos\lambda, -\cos\alpha_i, -\sin\alpha_i\sin\lambda) \qquad 14.6$$

Thus, we have, from (14.2),(14.3)

$$\sin\tau_i = \frac{\cos\alpha_i \sin\lambda}{1 + \sin\alpha_i \cos\lambda} \qquad (i=1,2) \qquad 14.7$$

$$\cos\tau_i = \frac{\sin\alpha_i + \cos\lambda}{1 + \sin\alpha_i \cos\lambda} \qquad (i=1,2) \qquad 14.8$$

Also, from fig. 12.1, we can see that the following relationships hold:

$$\sin\alpha_i = \frac{a\sin\alpha}{a_i} \qquad (i=1,2) \qquad 14.9$$

$$\cos\alpha_1 = \frac{a\cos\alpha + b}{a_1} \qquad 14.10$$

$$\cos\alpha_2 = \frac{a\cos\alpha - b}{a_2} \qquad 14.11$$

So, if we now substitute for $\sin\alpha_i$, $\cos\alpha_i$ $(i=1,2)$ from (14.9),(14.10),(14.11) into (14.7) and (14.8), we find

$$\sin\tau_1 = \frac{(a\cos\alpha + b)\sin\lambda}{a_1 + a\sin\alpha\cos\lambda} \qquad 14.12$$

$$\sin\tau_2 = \frac{(a\cos\alpha - b)\sin\lambda}{a_2 + a\sin\alpha\cos\lambda} \qquad 14.13$$

$$\cos\tau_i = \frac{a\sin\alpha \pm a_i\cos\lambda}{a_i + a\sin\alpha\cos\lambda} \qquad (i=1,2) \qquad 14.14$$

Whence, the matrix Q, and thus the horopter curve, Γ, is completely determined, when a, b, α, λ and ε, are known.

If

$$\alpha = \tfrac{1}{2}\pi \qquad 14.15$$

i.e., the fixation point, F, lies in the median plane y = 0, then we have

$$a_1 = a_2 \qquad 14.16$$

in which case, (14.12)-(14.14), reveal

$$\sin\tau_1 = \frac{b\sin\lambda}{a_1 + a\cos\lambda} \qquad 14.17$$

$$\sin\tau_2 = \frac{-b\sin\lambda}{a_1 + a\cos\lambda}$$

$$= -\sin\tau_1 \qquad 14.18$$

and

$$\cos\tau_2 = \frac{a + a_1\cos\lambda}{a_1 + a\cos\lambda}$$

$$= \cos\tau_1 \qquad 14.19$$

(14.18),(14.19), together, =>

$$\tau_2 = -\tau_1 \qquad 14.20$$

So, if fixation is in the median plane, and the eye moves in accordance with Listing's Law, then the relationship (14.20) holds between the angles of torsion of the two eyes.

However, it has been shown[325], that Listing's Law is not generally true, particularly so when the eyes are converged to a near point.

However, considerations of symmetry indicate that it is not unreasonable to assume

that a relationship such as (14.20) holds for a certain region of fixation space in the vicinity of the median plane.

We will call the situation when (14.20) is assumed to hold the quasi-symmetric case.

If we substitute from (14.20) into (10.7), we find

$$k_1 = \frac{\sin(-\tau_1-\varepsilon)}{\sin(\tau_1+\varepsilon)} = -1 \qquad 14.21$$

Similarly

(10.8)=> $k_2 = 1$ 14.22

(10.9)=> $j_1 = -1$ 14.23

(10.10)=> $j_2 = 1$ 14.24

And, if these values are substituted into (12.38)-(12.53), the elements of Q become:

$$q_{11} = -SCA_1(A_2T\cos\lambda + b\sin\lambda) \qquad 14.25$$

$$q_{12} = SC(Kb^2\cos\lambda - A_2b\sin\lambda) \qquad 14.26$$

$$q_{13} = SCA_1(A_2T\cos\lambda + b\sin\lambda) \qquad 14.27$$

$$q_{14} = -SC(Kb^2\cos\lambda - A_2b\sin\lambda) \qquad 14.28$$

$$q_{21} = A_1^2C^2T \qquad 14.29$$

$$q_{22} = 0 \qquad 14.30$$

$$q_{23} = Kb^2 + A_2^2S^2T \qquad 14.31$$

$$q_{24} = 0 \qquad 14.32$$

$$q_{31} = SCA_1(A_2T\sin\lambda - b\cos\lambda) \qquad 14.33$$

$$q_{32} = -SC(Kb^2\sin\lambda + A_2b\cos\lambda) \qquad 14.34$$

$$q_{33} = -SCA_1(A_2T\sin\lambda - b\cos\lambda) \qquad 14.35$$

$$q_{34} = SC(Kb^2\sin\lambda + A_2b\cos\lambda) \qquad 14.36$$

$$q_{41} = 0 \qquad 14.37$$

$$q_{42} = (T + KS^2)b \qquad 14.38$$

$$q_{43} = 0 \qquad 14.39$$

$$q_{44} = KbC^2 \qquad 14.40$$

If we write

$$\Gamma(t) = (x_1, x_2, x_3, x_4) \qquad 14.41$$

from (13.2), then we find using (12.7)

$$\Gamma(1) = (0,\ b^2(K+T), 0, b(K+T)) \qquad 14.42$$

$$= (0,b,0,1) \quad \text{from (11.29)} \qquad 14.43$$

$$\equiv L$$

similarly

$$\Gamma(-1) = (0,-b,0,1) \qquad 14.44$$

$$\equiv R$$

and for the particular value $t = -A_2/A_1$ we find

$$\Gamma\left(-\frac{A_2}{A_1}\right)$$

$$x_1 = \frac{SC(A_2^2-A_1^2)(Kb^2+A_2^2T)\cos\lambda}{A_1^2} \qquad 14.45$$

$$x_2 = -\frac{A_2(Kb^2+A_2^2T)}{A_1} \qquad 14.46$$

$$x_3 = \frac{SC(A_1^2-A_2^2)(Kb^2+A_2^2T)\sin\lambda}{A_1^2} \qquad 14.47$$

$$x_4 = \frac{b(Kb^2+A_2^2T)}{A_1^2} \qquad 14.48$$

Simplifying with the aid of (12.8), we see that

$$\Gamma(-\frac{A_2}{A_1}) = (a\sin\alpha\cos\lambda, -a\cos\alpha, -a\sin\alpha\sin\lambda, 1) \equiv F \qquad 14.49$$

So, the quasi-symmetric horopter meets the visual plane in the two nodal points and the fixation point.

If we invert the matrix Q, with elements (14.25)-(14.40), we find that the inverse matrix can be written

$$Q^{-1} = -\frac{1}{SCA_1b^2(K+T)(Kb^2+A_2^2T)}Q^* \qquad 14.50$$

where the elements of Q^* are given by

$$q^*_{11} = (Kb\sin\lambda+A_2\cos\lambda)(Kb^2+A_2^2S^2T) \qquad 14.51$$

$$q^*_{12} = -SCA_1(Kb^2+A_2^2T) \qquad 14.52$$

$$q^*_{13} = (Kb\cos\lambda-A_2\sin\lambda)(Kb^2+A_2^2S^2T) \qquad 14.53$$

$$q^*_{14} = 0 \qquad 14.54$$

$$q^*_{21} = A_1C^2Kb(A_2T\sin\lambda-b\cos\lambda) \qquad 14.55$$

$$q^*_{22} = 0 \qquad 14.56$$

$$q^*_{23} = A_1C^2Kb(A_2T\cos\lambda+b\sin\lambda) \qquad 14.57$$

$$q^*_{24} = SCA_1(Kb^2+A_1^2T) \qquad 14.58$$

$$q^*_{31} = -A_1^2C^2T(Kb\sin\lambda+A_2\cos\lambda) \qquad 14.59$$

$$q^*_{32} = -SCA_1(Kb^2+A_2^2T) \qquad 14.60$$

$$q^*_{33} = -A_1^2C^2T(Kb\cos\lambda-A_2\sin\lambda) \qquad 14.61$$

$$q^*_{34} = 0 \qquad 14.62$$

$$q^{\sharp}_{1} = -A_1 b(T+KS^2)(A_2 T\sin\lambda - b\cos\lambda) \quad 14.63$$

$$q^{\sharp}_{2} = 0 \quad 14.64$$

$$q^{\sharp}_{3} = -A_1 b(T+KS^2)(A_2 T\cos\lambda + b\sin\lambda) \quad 14.65$$

$$q^{\sharp}_{4} = -SCA_1(Kb^2 + A_2^2 T) \quad 14.66$$

Now, we have from (13.2)

$$x_4 = q_{41}t^3 + q_{42}t^2 + q_{43}t + q_{44} \quad 14.67$$

and so from (14.37)-(14.40), we see

$$x_4 = (T+KS^2)bt^2 + KbC^2 \quad 14.68$$

Hence, the only real root of

$$x_4 = 0 \quad 14.69$$

is when[¶]

$$t = \infty \quad 14.70$$

Thus Γ is a twisted cubic with only one asymptote plane at the point given by (14.70).

From (13.20), we have, for this value of t, the equation of this plane is

$$\Sigma q^{*}_{4i} x_i = 0 \quad 14.71$$

and, therefore, if we substitute from (14.63)-(14.66), we find that the equation of the asymptote plane, is

---

¶ There is a theoretical possibility for (14.69) to have three real roots; this would require ε to be much greater than τ_1 and of opposite sign.

$$(A_2T\sin\lambda - b\cos\lambda)x_1 + (A_2T\cos\lambda + b\sin\lambda)x_3 + \frac{SC(Kb^2+A_2^2T)}{b(T+KS^2)}x_4 = 0 \qquad 14.72$$

and the asymptote line is found from (13.21) where this plane intersects the plane

$$\Sigma q^*_{3i}x_i = 0 \qquad 14.73$$

i.e., the plane

$$A_1CT(Kb\sin\lambda + A_2\cos\lambda)x_1 + S(Kb^2+A_2^2T)x_2 + A_1CT(Kb\cos\lambda - A_2\sin\lambda)x_3 = 0 \qquad 14.74$$

i.e., the line given by

$$\frac{x - \frac{SC(Kb^2+A_2^2T)}{bd(T+KS^2)}}{\sin\lambda + A_2T\cos\lambda} = \frac{Sy - \frac{A_1SC^2T(Kb\sin\lambda + A_2\cos\lambda)}{bd(T+KS^2)}}{A_1CT} = \frac{z}{d} \qquad 14.75$$

where

$$d = \cos\lambda - A_2T\sin\lambda \qquad 14.76$$

from which we can see that the quasi-symmetric horopter, Γ, can take the form given by fig. 14.1.

. If fixation is in the horizontal plane, then $\lambda = 0$, and the matrix for Q becomes:

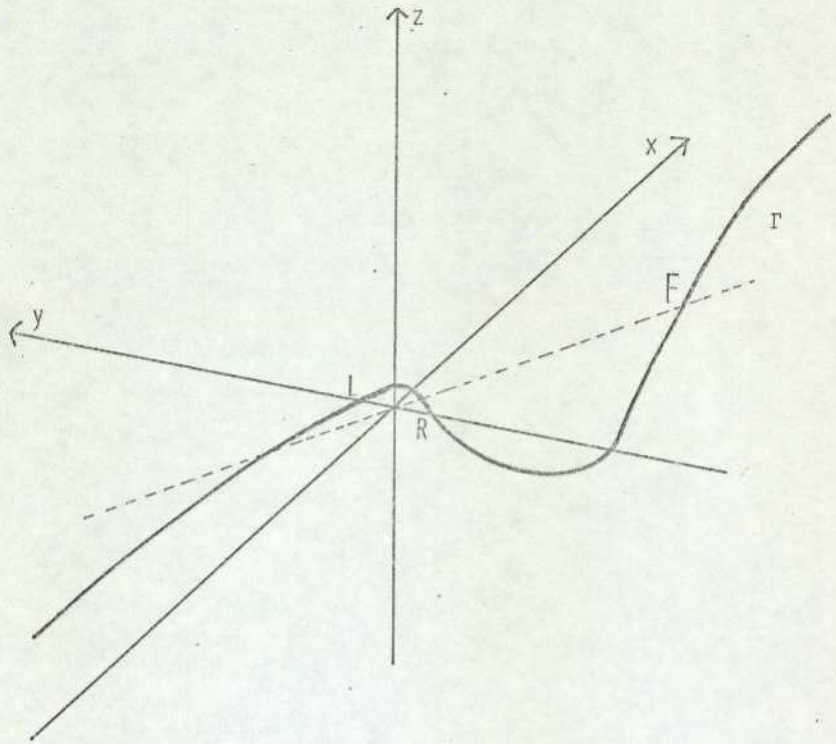


Fig. 14.1 The space horopter:
quasi-symmetric case

$$Q = \begin{pmatrix} -SCA_1A_2T & SCKb^2 & SCA_1A_2T & -SCKb^2 \\ A_1^2C^2T & 0 & Kb^2+A_2^2S^2T & 0 \\ -SCA_1b & -SCA_2b & SCA_1b & SCA_2b \\ 0 & (T+KS^2)b & 0 & KbC^2 \end{pmatrix} \qquad 14.77$$

and, the asymptote plane becomes

$$bx - A_2Tz = \frac{SC(Kb^2+A_2^2T)}{b(T+KS^2)} \qquad 14.78$$

and the asymptote line

$$\frac{x - \frac{SC(Kb^2+A_2^2T)}{b(T+KS^2)}}{A_2T} = \frac{Sy - \frac{A_1A_2SC^2T}{b(T+KS^2)}}{A_1CT} = z \qquad 14.79$$

15 The quasi-symmetric case: the rectilinear horopter

From (10.11)-(10.13), and (14.21)-(14.24), we can see that in the quasi-symmetric case the basic quadrics of the rectilinear horopter are given by

$$S_1 \equiv (x'^2 + y'^2)SC + CTx'z' - A_2Sx' + A_1Cy' - A_2Tz' = 0 \qquad 15.1$$

$$S_2 \equiv x'y' - SKy'z' + A_1Sx' - A_2Cy' - A_1Kz' = 0 \qquad 15.2$$

$$S_3 \equiv S^2x'^2 - C^2y'^2 - KTz'^2 - (K-T)Sx'z' = 0 \qquad 15.3$$

Let these three quadrics each be represented by a symmetric quadratic form such that

$$S_i \equiv \underline{x}'^T S_i' \underline{x}' \qquad 15.4$$

so that

$$S_1' = \begin{pmatrix} SC & 0 & \frac{1}{2}CT & -\frac{1}{2}A_2S \\ 0 & SC & 0 & \frac{1}{2}A_1C \\ \frac{1}{2}CT & 0 & 0 & -\frac{1}{2}A_2T \\ -\frac{1}{2}A_2S & \frac{1}{2}A_1C & -\frac{1}{2}A_2T & 0 \end{pmatrix} \qquad 15.5$$

$$S_2' = \begin{pmatrix} 0 & 1 & 0 & \frac{1}{2}A_1S \\ 1 & 0 & -\frac{1}{2}SK & -\frac{1}{2}A_2C \\ 0 & -\frac{1}{2}SK & 0 & -\frac{1}{2}A_1K \\ \frac{1}{2}A_1S & -\frac{1}{2}A_2C & -\frac{1}{2}A_1K & 0 \end{pmatrix} \qquad 15.6$$

$$S_3' = \begin{pmatrix} S^2 & 0 & -\tfrac{1}{2}(K-T)S & 0 \\ 0 & -C^2 & 0 & 0 \\ -\tfrac{1}{2}(K-T)S & 0 & -KT & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad 15.7$$

The form that a quadric surface can take can be determined by considering the determinant $|S_i'|$, and, the eigenvalues of the matrix $\mathcal{D}_i$, formed from the first three rows and columns.

Now

$$|S_1'| = A^2C^4T^2/16 \qquad 15.8$$
$$\neq 0$$

and

$$|\mathcal{D}_1| = -\tfrac{1}{4}SC^3T^2 \qquad 15.9$$
$$\neq 0$$

Put $$\mathcal{D}_1\underline{u} = \mu\underline{u} \qquad 15.10$$

=>

$$\begin{vmatrix} SC-\mu & 0 & \tfrac{1}{2}CT \\ 0 & SC-\mu & 0 \\ \tfrac{1}{2}CT & 0 & -\mu \end{vmatrix} = 0 \qquad 15.11$$

which gives

$$\mu = SC, \text{ or } \tfrac{1}{2}(S \pm \sqrt{S^2+T^2}) \qquad 15.12$$

Thus, since S is of rank 4, and $\mathcal{D}$ is of rank 3, and the eigenvalues are not all of the same sign, S_1 is a hyperboloid of one sheet. (see: fig. 10.1a)

S_1 meets the visual plane $z' = 0$, (fig. 15.1), in the circle,

$$x'^2 + y'^2 - \frac{A_2}{C}x' + \frac{A_1}{S}y' = 0 \qquad 15.13$$

which we have seen in §10 is the Vieth-Müller circle.

It meets the plane $y' = 0$ (fig. 15.2) in the two lines given by

$$(Sx' + Tz')(Cx' - A_2) = 0 \qquad 15.14$$

and the plane $x' = 0$ (fig. 15.3) in the parabola

$$SCy'^2 + A_1Cy' - A_2Tz' = 0 \qquad 15.15$$

We have also

$$|S_{\frac{1}{2}}| = A_1^2K^2C^4 \neq 0 \qquad 15.16$$

but

$$|\mathcal{D}_2| = 0 \qquad 15.17$$

i.e., there is at least one zero eigenvalue, the characteristic equation is

$$\begin{vmatrix} -\mu & 1 & 0 \\ 1 & -\mu & -\frac{1}{2}SK \\ 0 & -\frac{1}{2}SK & -\mu \end{vmatrix} = 0 \qquad 15.18$$

whose roots are

$$\mu = 0 \text{ or } \pm\sqrt{1-\tfrac{1}{4}S^2K^2} \qquad 15.19$$

Since S is of rank 4, and $\mathcal{D}$ of rank 2 and the non-zero eigenvalues are of opposite sign S_2 is a hyperbolic paraboloid. (see: fig. 10.1b)

S_2 meets the visual plane, $z' = 0$, (fig. 15.1), in the hyperbola

$$x'y' + A_1Sx' - A_2Cy' = 0 \qquad 15.20$$

and meets the plane, $y' = 0$ (fig. 15.2) in the single line

$$Sx' - Kz' = 0 \qquad 15.21$$

and meets the plane, $x' = 0$ (fig. 15.3) in the hyperbola

$$SKy'z' + A_2Cy' + A_1Kz' = 0 \qquad 15.22$$

For the third quadric, we have

$$|S_3^1| = 0 \qquad 15.23$$

$$|\mathcal{D}_3| = SC^2(KT + \tfrac{1}{2}S(K-T)^2) \qquad 15.24$$

$$\neq 0$$

therefore S_3 is a cone.

This cone meets $z' = 0$ (fig. 15.1) in the two lines

$$(Cy'+Sx')(Cy'-Sx') = 0 \qquad 15.25$$

and, meets the plane, $y' = 0$ (fig. 15.2) in the two lines

$$(Sx'-Kz')(Sx'+Tz') = 0 \qquad 15.26$$

and, meets the plane $x' = 0$ (fig. 15.3) in the single point $(0,0,0)$, i.e. the vertex of the cone.

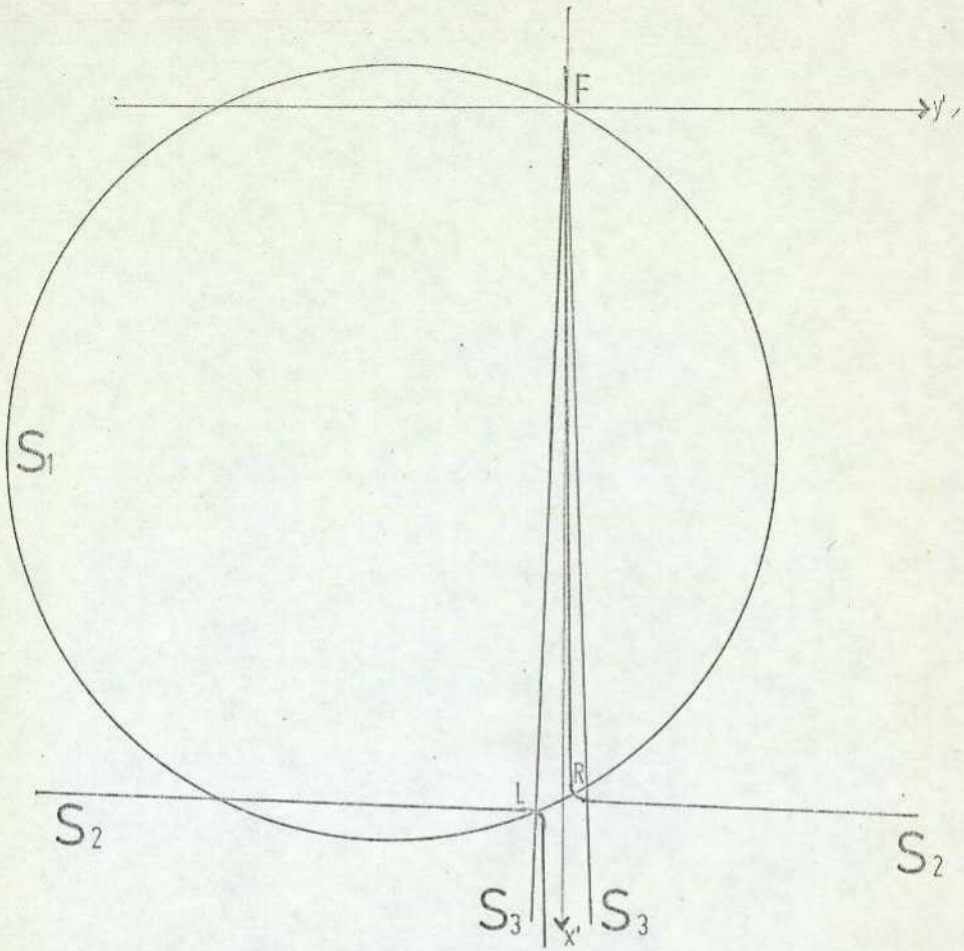


Fig. 15.1 The quasi-symmetric case: Intersection of the base quadrics, S_1, S_2, S_3, with the plane: $z' = 0$

$$S_1 \equiv x'^2 + y'^2 - \frac{A_2}{C}x' + \frac{A_1}{S}y' = 0$$

$$S_2 \equiv x'y' + A_1Sx' - A_2Cy' = 0$$

$$S_3 \equiv (Cy' + Sx')(Cy' - Sx') = 0$$

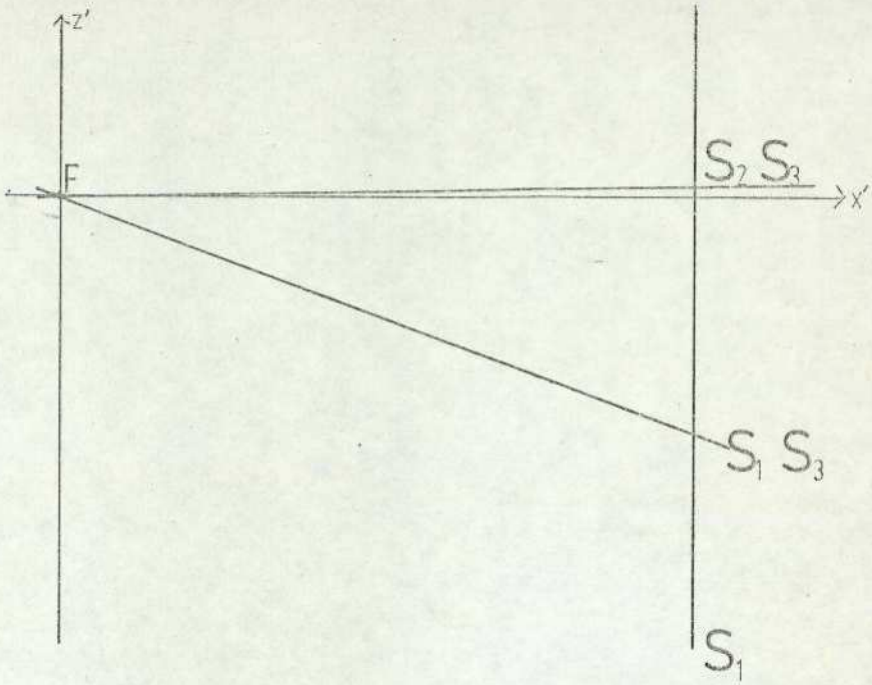


Fig. 15.2 The quasi-symmetric case:
Intersection of the base quadrics, S_1, S_2, S_3,
with the plane: $y' = 0$

$$S_1 \equiv (Sx' + Tz')(Cx' - A_2) = 0$$

$$S_2 \equiv Sx' - Kz' = 0$$

$$S_3 \equiv (Sx' - Kz')(Sx' + Tz') = 0$$

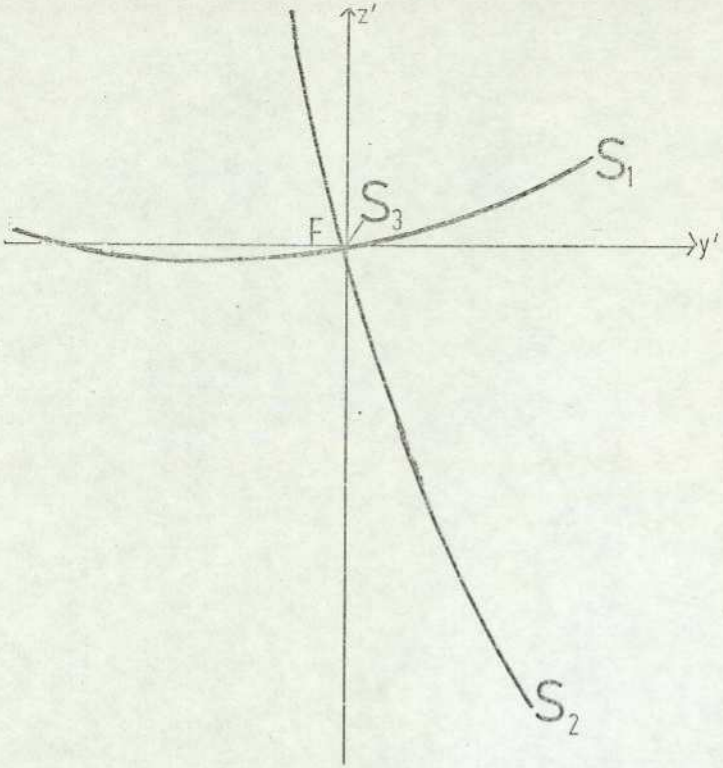


Fig. 15.3 The quasi-symmetric case: Intersection of the base quadrics, S_1, S_2, S_3, with the plane: $x' = 0$

$S_1 \equiv SCy'^2 + A_1Cy' - A_2Tz' = 0$

$S_2 \equiv SKy'z' + A_2Cy' + A_2Kz' = 0$

$S_3 \equiv$ the single point $(0,0)$

From (12.25), we have

$$\underline{x} = P\underline{x}' \qquad 15.27$$

where is defined by (12.27), and this can be written

$$\underline{x}' = P^{-1}\underline{x} \qquad 15.28$$

if we substitute for $\underline{x}'$ in (15.4), we can write

$$S_i \equiv (P^{-1}\underline{x})^T S_i' (P^{-1}\underline{x}) \qquad 15.29$$

$$= \underline{x}^T P^{-1T} S_i' P^{-1} \underline{x}$$

Thus, S_i, can be written with respect to the fixed co-ordinates $\underline{x}$, as

$$S_i \equiv \underline{x}^T S \underline{x} = 0 \qquad 15.30$$

where

$$S_i = P^{-1T} S_i' P^{-1} \qquad 15.31$$

16 The symmetric case: the space horopter

If the fixation point, F, lies in the median plane

$$x_2 = 0 \tag{16.1}$$

that is to say, if

$$\alpha = \tfrac{1}{2}\pi \tag{16.2}$$

then we can see from fig. 12.1, that the following relationships hold

$$a_1 = a_2 \tag{16.3}$$

$$a_1^2 = a^2 + b^2 \tag{16.4}$$

$$S = b/a_1 \tag{16.5}$$

$$C = a/a_1 \tag{16.6}$$

and, from (12.5),(12.6), we have

$$A_1 = 0 \tag{16.7}$$

$$A_2 = a_1 \tag{16.8}$$

and, from the discussion in 14, we can see that

$$\tau_2 = -\tau_1 \tag{16.9}$$

Thus, we can substitute these values in (14,25)-(14.40), when, the elements of $\mathcal{Q}$, become

$$q_{11} = 0 \qquad 16.10$$

$$q_{12} = \frac{ab}{a_1^2}(Kb^2\cos\lambda - a_1 b\sin\lambda) \qquad 16.11$$

$$q_{13} = 0 \qquad 16.12$$

$$q_{14} = -\frac{ab}{a_1^2}(Kb^2\cos\lambda - a_1 b\sin\lambda) \qquad 16.13$$

$$q_{21} = 0 \qquad 16.14$$

$$q_{22} = 0 \qquad 16.15$$

$$q_{23} = b^2(K+T) \qquad 16.16$$

$$q_{24} = 0 \qquad 16.17$$

$$q_{31} = 0 \qquad 16.18$$

$$q_{32} = -\frac{ab}{a_1^2}(Kb^2\sin\lambda + a_1 b\cos\lambda) \qquad 16.19$$

$$q_{33} = 0 \qquad 16.20$$

$$q_{34} = \frac{ab}{a_1^2}(Kb^2\sin\lambda + a_1 b\cos\lambda) \qquad 16.21$$

$$q_{41} = 0 \qquad 16.22$$

$$q_{42} = \frac{b}{a_1^2}(Kb^2 + a_1^2 T) \qquad 16.23$$

$$q_{43} = 0 \qquad 16.24$$

$$q_{44} = \frac{Kba^2}{a_1^2} \qquad 16.25$$

So, in the symmetric case

$$Q = \begin{Vmatrix} 0 & a(Kb^2\cos\lambda - a_1 b\sin\lambda) & 0 & -a(Kb^2\cos\lambda - a_1 b\sin\lambda) \\ 0 & 0 & a_1^2 b(K+T) & 0 \\ 0 & -a(Kb^2\sin\lambda + a_1 b\cos\lambda) & 0 & a(kb^2\sin\lambda + a_1 b\cos\lambda) \\ 0 & Kb^2 + a_1^2 T & 0 & Ka^2 \end{Vmatrix} \quad 16.26$$

Evidently, in this case, Γ has degenerated into the conic.

$$x_1 = a(Kb^2\cos\lambda - a_1 b\sin\lambda)(t^2-1) \quad 16.27$$

$$x_2 = a_1^2 b(K+T)t \quad 16.28$$

$$x_3 = -a(Kb^2\sin\lambda + a_1 b\cos\lambda)(t^2-1) \quad 16.29$$

$$x_4 = (Kb^2 + a_1^2 T)t^2 + Ka^2 \quad 16.30$$

This conic has been plotted by the computer, see Appendix .

If now we take $\lambda = 0$, so that fixation is in the horizontal plane, then Q reduces to

$$x_1 = Kb^2 a(t^2-1) \quad 16.31$$

$$x_2 = a_1^2 b(K+T)t \quad 16.32$$

$$x_3 = -a_1 ba(t^2-1) \quad 16.33$$

$$x_4 = (Kb^2 + a_1^2 T)t^2 + Ka^2 \quad 16.34$$

The considerations in §11, show, that in the symmetric case, all the quadrics of the family (10.17) have a common generator given by (11.11), which, with the simplifications of (16.3)-(16.9) become

$$\left.\begin{aligned} x_2^1 &= 0 \\ bx_1^1 + a_1 Tx_3^1 &= 0 \end{aligned}\right\} \quad 16.35$$

Hence, the line (16.35) is part of the common curve of intersection of the family of quadrics (10.17).

That is, it is part of Γ. Thus, we have, that in the symmetric case, Γ degenerates into the conic (16.27)-(16.30), and the common generator (16.35). (see: fig. 16.1)

Dividing (16.31) by (16.33) we find

$$\frac{x_1}{x_3} = -\frac{Kb}{a_1} \quad \text{for all values of } t$$

i.e. the horopter conic lies in the plane

$$a_1 x_1 + Kbx_3 = 0 \qquad 16.36$$

This is a plane through the y-axis. If it is inclined at an angle θ to the horizontal plane then

$$\tan\theta = -\frac{a_1}{Kb} \qquad 16.37$$

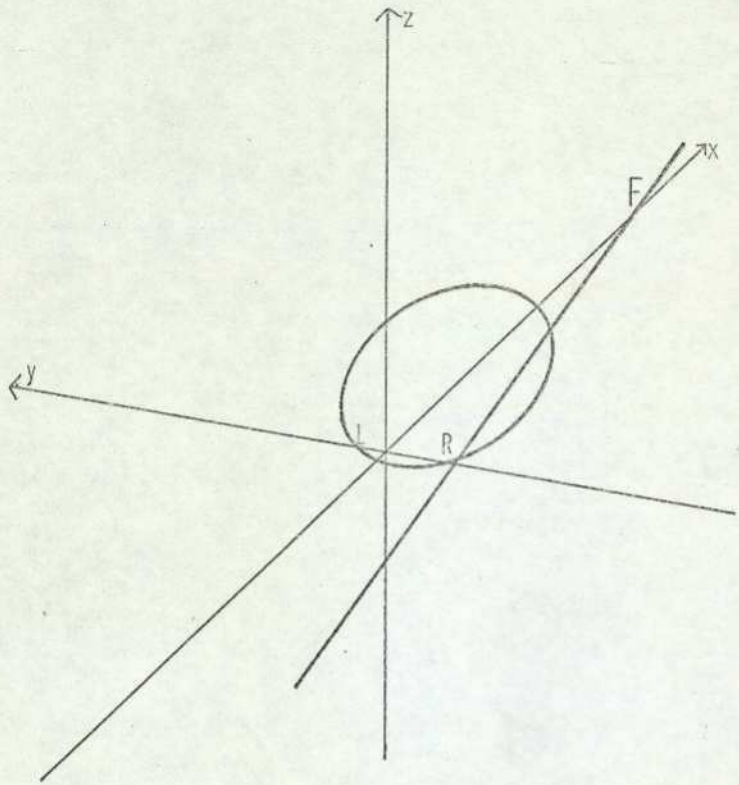


Fig. 16.1 The space horopter

symmetric case

17 The symmetric case: the rectilinear horopter

With the relationships (16.3)-(16.8) the base quadric equations become:

from (15.1) $$S_1 \equiv (x'^2+y'^2)ab + aa_1Tx'z' - a_1^2bx' - a_1^3Tz' = 0 \qquad 17.1$$

from (15.2) $$S_2 \equiv y'(a_1\dot{x}' - bKz' - aa_1) = 0 \qquad 17.2$$

from (15.3) $$S_3 \equiv b^2x'^2 - a^2y'^2 - a_1^2KTz'^2 - (K-T)a_1bx'z' = 0 \qquad 17.3$$

Now, from (17.1) we find

$$|S_1^1| = 0 \qquad 17.4$$

and

$$|\mathcal{D}_1| = -\tfrac{1}{4}a^3a_1^2bT^2 \neq 0 \qquad 17.5$$

and thus we can see that the hyperboloid of one sheet (15.1), reduces to a cone, with vertex

$$(\frac{a_1^2}{a}, 0, -\frac{a_1b}{aT}) \qquad 17.6$$

S_2 can be seen directly from (17.2) to have degenerated into the two intersecting planes;

$$\left.\begin{aligned} y' &= 0 \\ a_1x' - bKz' - aa_1 &= 0 \end{aligned}\right\} \qquad 17.7$$

And, we have from (17.3)

$$|S_3^1| = 0 \qquad 17.8$$

$$|\mathcal{D}_3| \neq 0 \qquad 17.9$$

so that S_3 is still a cone with its vertex at the fixation point.

S_1 meets the plane (fig. 17.4)

$$z' = z_0 \qquad 17.10$$

in the circle

$$(x'^2 + y'^2)ab + a_1(a\,Tz_0 - a_1b)x' - a_1^3Tz_0 = 0 \qquad 17.11$$

which, in the co-ordinate plane (fig. 17.1)

$$z_0 = 0 \qquad 17.12$$

becomes the Vieth-Müller circle:

$$x'^2 + y'^2 - \frac{a_1^2}{a}x' = 0 \qquad 17.13$$

S_1 meets the plane (fig. 17.5)

$$y' = y_0 \qquad 17.14$$

in the hyperbola

$$x'^2ab + aa_1Tx'z' - a_1^2bx' - a_1^3Tz' + aby_0^2 = 0 \qquad 17.15$$

which, in the co-ordinate plane (fig. 17.2)

$$y_0 = 0 \qquad 17.16$$

degenerates into the two lines

$$(ax' - a_1^2)(bx' + a_1Tz') = 0 \qquad 17.17$$

S_1 meets the plane (fig. 17.6)

$$x' = x_0 \qquad 17.18$$

in the hyperbola

$$abx_0^2 + y'^2ab + a_1T(ax_0 - a_1^2)z' - a_1^2bx_0 = 0 \qquad 17.19$$

which, in the co-ordinate plane (fig. 17.3)

$$x_0 = 0 \qquad 17.20$$

becomes

$$y'^2ab = a_1^3Tz' \qquad 17.21$$

S_2 meets the plane (fig. 17.4)

$$z' = z_0 \qquad 17.22$$

in the two lines

$$y' = 0 \qquad 17.23$$

$$a_1x' - bKz_0 - aa_1 = 0 \qquad 17.24$$

which, in the co-ordinate plane (fig. 17.1)

$$z_0 = 0 \qquad 17.25$$

become

$$y' = 0 \qquad 17.26$$

$$x' = a \qquad 17.27$$

S_2 meets the plane (fig. 17.5)

$$y' = y_0 \qquad 17.28$$

in the single line

$$a_1x' - bKz' - a_1a = 0 \qquad 17.29$$

but, in the co-ordinate plane (fig. 17.2)

$$y_0 = 0 \qquad 17.30$$

S_2 is the whole plane.

S_2 meets the plane (fig. 17.6)

$$x' = x_0 \qquad 17.31$$

in the two lines

$$y' = 0 \qquad 17.32$$

$$bKz' = a_1(x_0 - a) \qquad 17.33$$

which, in the co-ordinate plane (fig. 17.3)

$$x_0 = 0 \qquad 17.34$$

become the lines

$$y' = 0 \qquad 17.35$$

$$z' = -\frac{aa_1}{bK} \qquad 17.36$$

S_3 meets the plane (fig. 17.4)

$$z' = z_0 \tag{17.37}$$

in the hyperbola

$$b^2x'^2 - a^2y'^2 - (K-T)a_1bz_0x' - a_1^2KTz_0^2 = 0 \tag{17.38}$$

which, in the co-ordinate plane (fig. 17.1)

$$z_0 = 0 \tag{17.39}$$

degenerates into the pair of lines

$$(bx' + ay')(bx' - ay') = 0 \tag{17.40}$$

S_3 meets the plane (fig. 17.5)

$$y' = y_0 \tag{17.41}$$

in the hyperbola

$$b^2x'^2 - a_1^2KTz'^2 - (K-T)a_1bx'z' - a^2y_0^2 = 0 \tag{17.42}$$

which, in the co-ordinate plane (fig. 17.2)

$$y_0 = 0 \tag{17.43}$$

becomes the pair of lines

$$(bx' + a_1Tz')(bx' - a_1Kz') = 0 \tag{17.44}$$

S_3 meets the plane (fig. 17.6)

$$x' = x_0 \tag{17.45}$$

in the ellipse

$$a^2y'^2 + a_1^2KTz'^2 + (K-T)a_1bx_0z' - b^2x_0^2 = 0 \tag{17.46}$$

which, in the co-ordinate plane (fig. 17.3)

$$x_0 = 0 \tag{17.47}$$

becomes the single point (0,0).

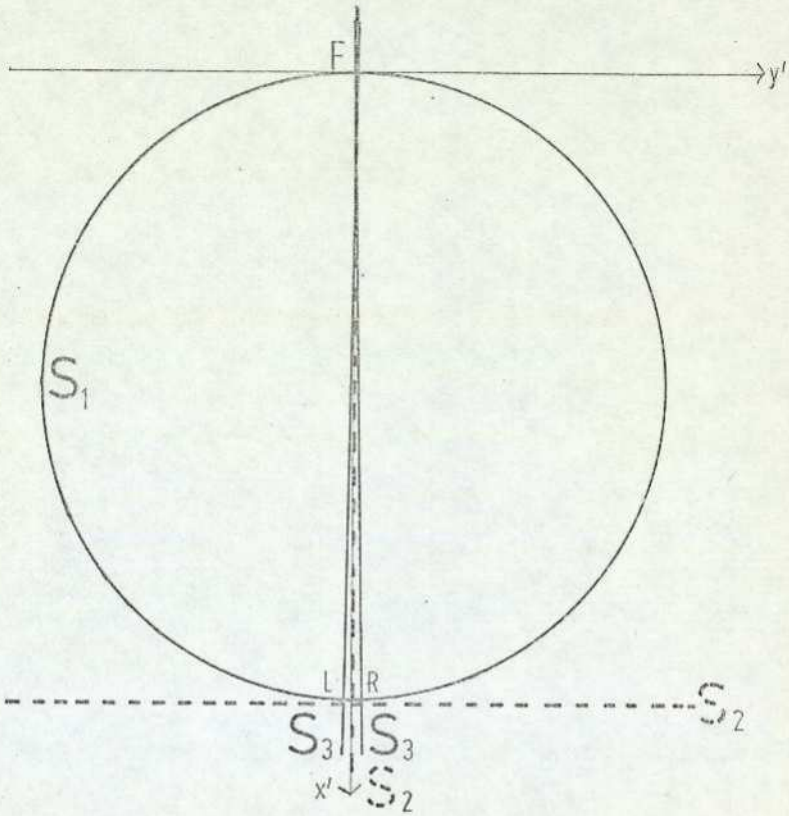


Fig. 17.1 The symmetric case:

Intersection of the base quadrics, S_1, S_2, S_3, with the plane : $z' = 0$

$$S_1 \equiv x'^2 + y'^2 - \frac{a_1^2}{a}x' = 0$$

$$S_2 \equiv y'(x' - a) = 0$$

$$S_3 \equiv (bx' + ay')(bx' - ay') = 0$$

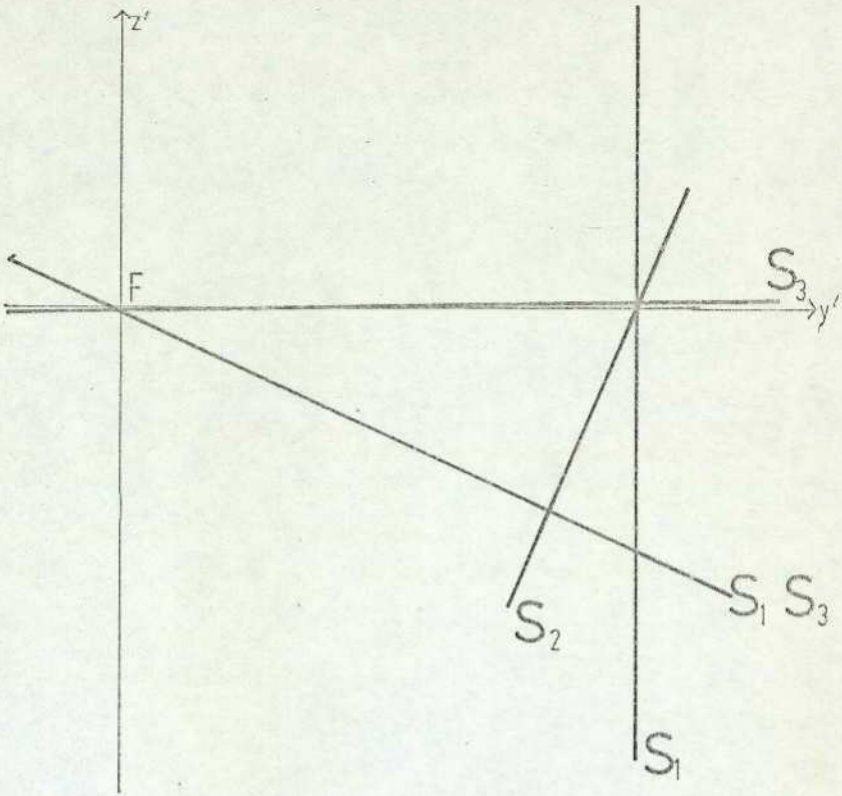


Fig. 17.2 The symmetric case:

Intersection of the base quadrics, S_1, S_2, S_3, with the plane: $y' = 0$

$S_1 \equiv (ax' - a_1^2)(bx' + a_1Tz') = 0$

$S_2 \equiv$ the whole plane;
the line:

$a_1x' - bKz' - a_1a = 0$

is the line of intersection of the two planes of S_2.

$S_3 \equiv (bx' + a_1Tz')(bx' - a_1Kz') = 0$

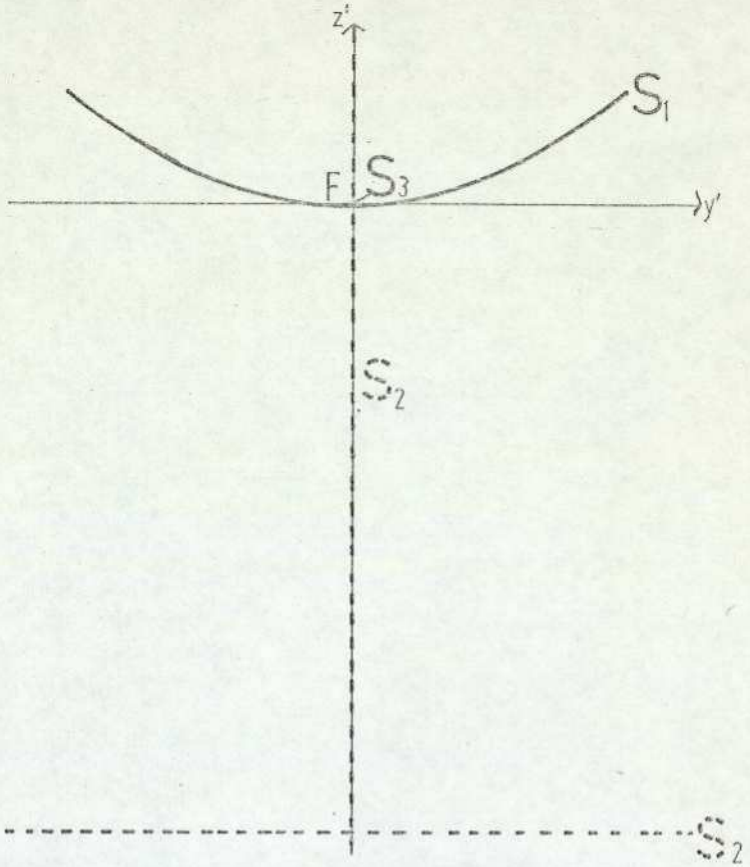


Fig. 17.3 The symmetric case:

Intersection of the base quadrics, S_1, S_2, S_3, with the plane: $x' = 0$

$S_1 \equiv y'^2 ab = a_1^3 Tz'$

$S_2 \equiv y'(bKz' + a_1 a) = 0$

$S_3 \equiv$ the single point (0,0)

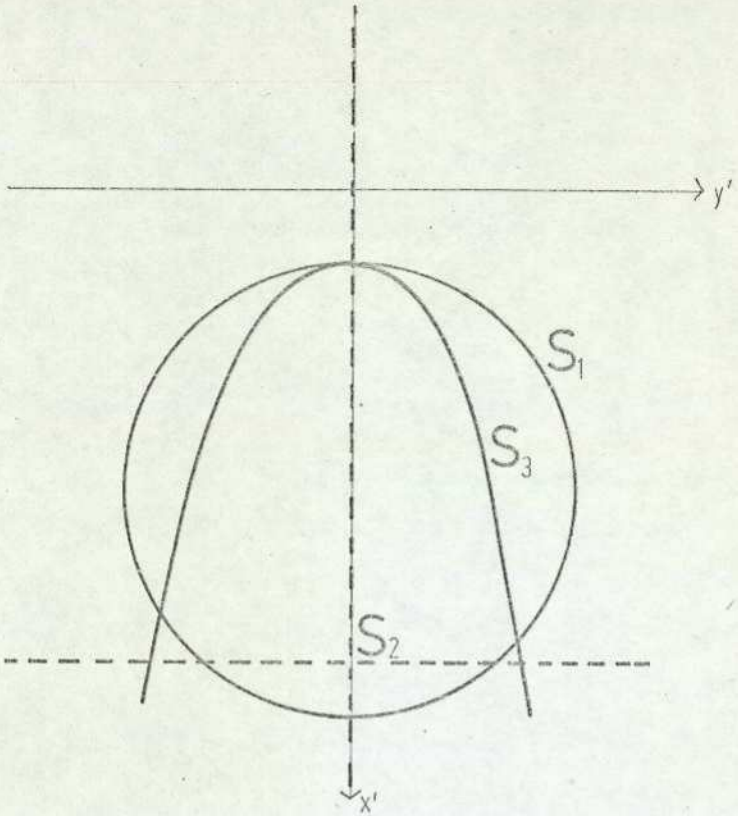


Fig. 17.4 The symmetric case:

Intersection of the base quadrics S_1, S_2, S_2, with the plane: $z' = z_0$

$$S_1 \equiv (x'^2+y'^2)ab + a_1(a_1Tz_0 - a_1b)x' - a_1^3Tz_0 = 0$$

$$S_2 \equiv y'(a_1x' - bKz_0 - aa_1) = 0$$

$$S_3 \quad b^2x' - a^2y'^2 - (K-T)a_1bz_0x' - a_1^2KTz_0^2 = 0$$

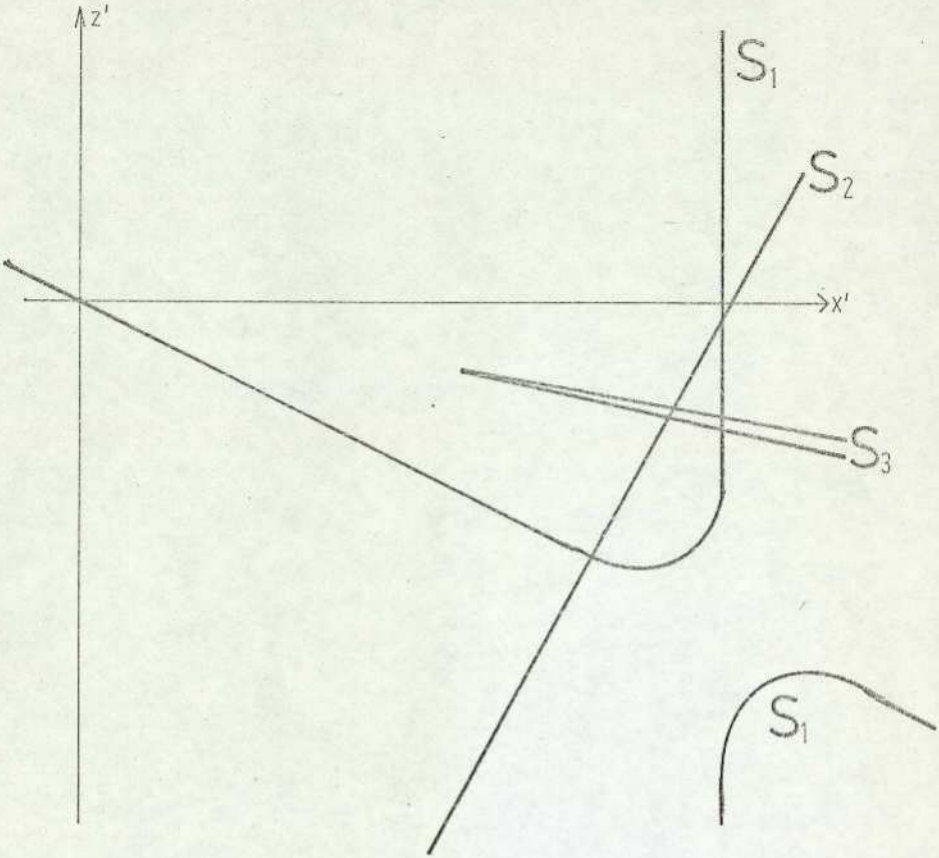


Fig. 17.5 The symmetric case:

Intersection of the base quadrics, S_1, S_2, S_3, with the plane: $y' = y_0$

$$S_1 \equiv x'^2 ab + aa_1 Tx'z' - a_1^2 bx' - a_1^3 Tz' + aby_0^2 = 0$$

$$S_2 \equiv a_1 x' - bKz' - a_1 a = 0$$

$$S_3 \equiv b^2 x'^2 - a_1^2 KTz'^2 - (K-T)a_1 bx'z' - a^2 y_0^2 = 0$$

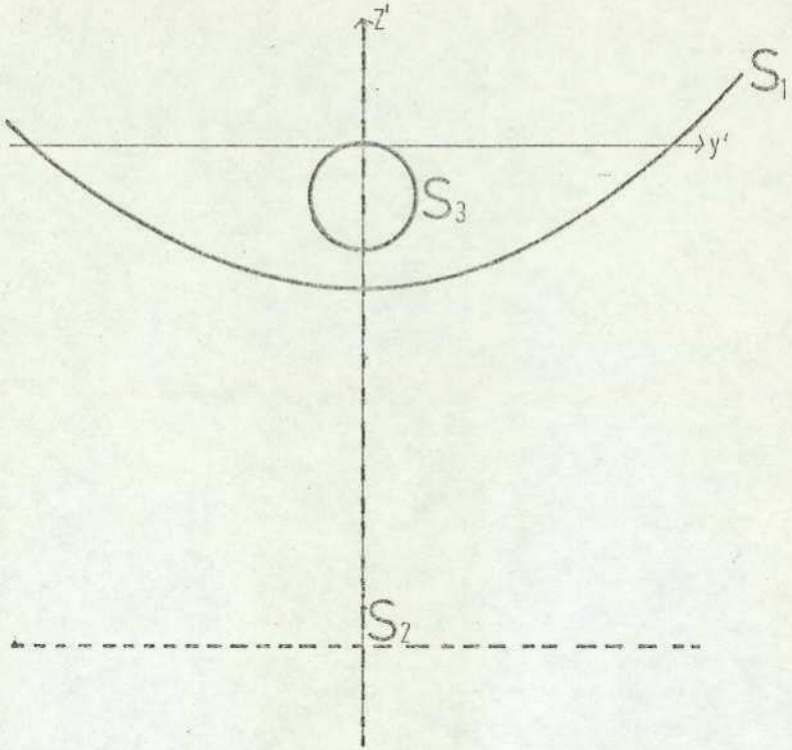


Fig. 17.6 The symmetric case:

Intersection of the base quadrics, S_1, S_2, S_3, with the plane: $x' = x_0$

$$S_1 \equiv y'^2 ab + a_1 T(ax_0 - a_1^2)z' - bx_0(ax_0 - a_1^2) = 0$$

$$S_2 \equiv y'\{bKz' - a_1(x_0 - a)\} = 0$$

$$S_3 \equiv a^2 y'^2 + a_1^2 KTz'^2 + (K-T)a_1 bx_0 z' - b^2 x_0^2 = 0$$

i) The quasi-symmetric case

We will take

$$a = 300 \quad (cm)$$

$$b = 3$$

$$\alpha = \tfrac{1}{4}\pi$$

$$\lambda = 0$$

$$\varepsilon = 0$$

and assume

$$\tau_1 = -\tau_2 = 0.02$$

The various functions have been evaluated by the computer programs shown in the Appendix. We find:

$$S_1 \equiv x'^2 + y'^2 + 2.828x'z' - 300.015x' + 299.985y' - 848.549x'z' = 0$$

$$|S_1| = 0.0018$$

$$|\mathcal{D}_1| = -5.57 \times 10^{-6}$$

centre $\equiv$ (300.15,-149.992,-106.074)

eigenvalues of $\mathcal{D}_1$ 0.02828, 0.01414, -0.01414

corresponding principal directions (i.e. eigenvectors)

$$\begin{matrix} 0.8165 & 0 & 0.5773 \\ 0 & -1 & 0 \\ 0.5773 & 0 & -0.8165 \end{matrix}$$

Hence, S_1 is a hyperboloid of one sheet.

A perspective view of the generating lines of this surface is shown in fig. 18.1 as output by the computer program

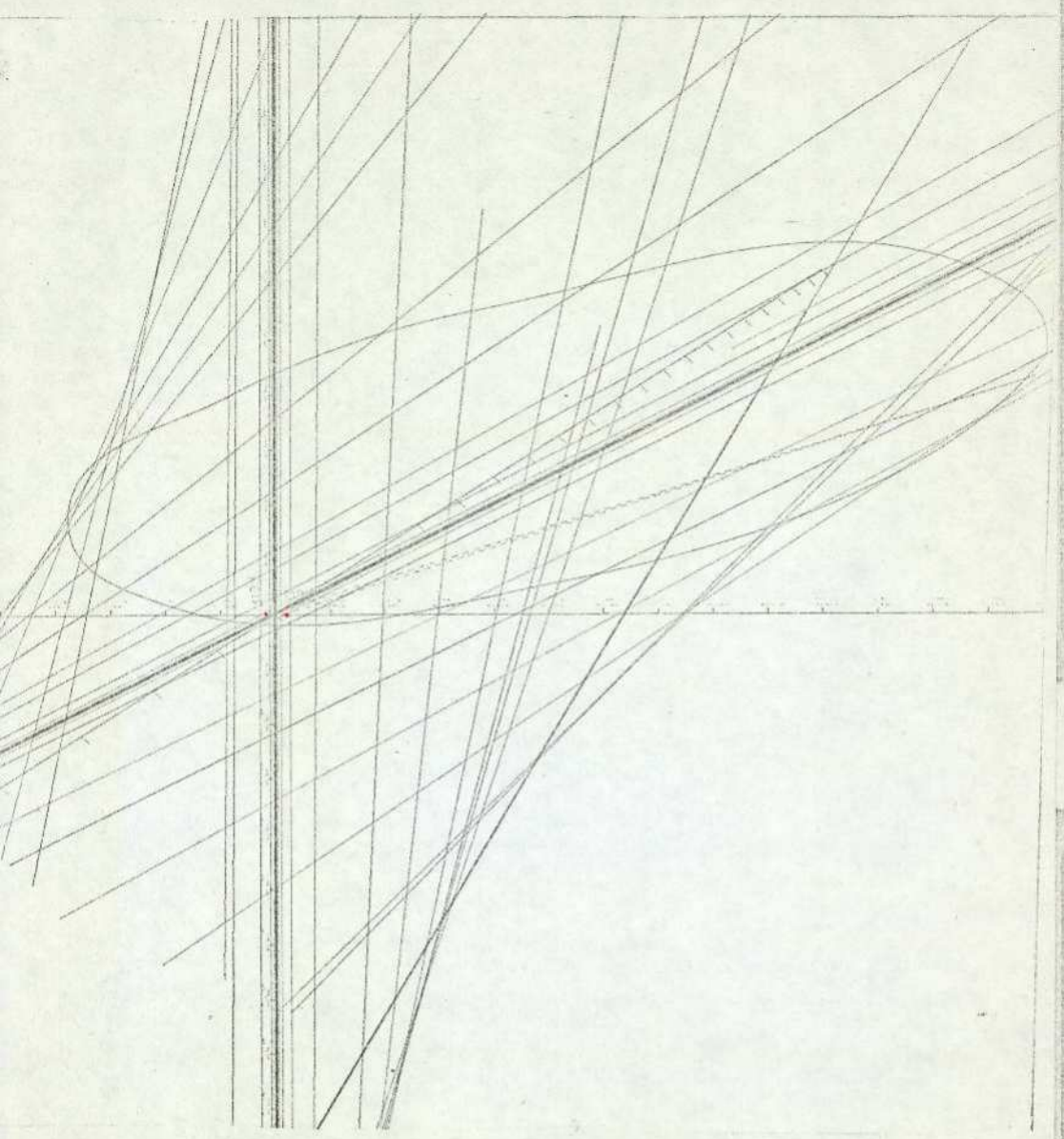


Fig. 18.1 Computer generation of S_1

quasi-symmetric case

perspective view

Note:

Throughout this chapter the nodal points are indicated by red dots.

$S_2 \equiv x'y' - 0.3535y'z' + 0.015x'$

$- 300y' - 106.06z' = 0$

$|S_2'| = 0.0018$

$|\mathcal{D}_2| = 0$

Thus there is no inverse of $\mathcal{D}_2$ and hence no centre to the quadric

eigenvalues of $\mathcal{D}_2$ 0.02121, 0, -0.02121

principal directions:

0.6666	-0.7071	-0.2357
0.3333	0	0.9428
0.6666	0.7071	-0.2357

quadric is a hyperbolic paraboloid

$S_3 \equiv x'^2 - 19998y'^2 - 19999z'^2 - 7068x'z' = 0$

$|S_3'| = 0$

$|\mathcal{D}_3| = 2.0 \times 10^{-6}$

centre $\equiv$ (0,0,0)

eigenvalues of $\mathcal{D}_3$ 0.0012,-0.04,-0.0412

principal directions:

0.986	0	-0.169
0	1	0
0.169	0	0.986

thus, the quadric S_3 is a cone.

The fixation point is at (212.132,-212.132,0)

The horopter is given by

$$x_1 = -0.0144t^3 + 0.5091t^2 + 0.0144t - 0.5091$$

$$x_2 = 0.0144t^3 + 72.0144t$$

$$x_3 = -0.0036t^3 - 0.5091t^2 + 0.0036t + 0.5091$$

$$x_4 = 0.0108t^2 + 24.0000$$

The real roots of

$x_1 = 0$ are $t = 35.3553, 1, -1$

$x_2 = 0$ $t = 0$

$x_3 = 0$ $t = -141.4284, 1, -1$

$x_4 = 0$ $t = \infty$

$\Gamma(1) = (0, 72.0288, 0, 24.0096)$

$= (0, 3, 0, 1)$

$= L$

$\Gamma(-1) = (0, -72.0288, 0, 24.0096)$

$= (0, -3, 0, 1)$

$= R$

$\Gamma(-141.4284) = (50916.27, -50916.27, 0.00, 240.0216)$

$= (212.132, -212.132, 0, 1)$

$= F$

$\Gamma(35.3553) = (0, 3182.4260, -794.8348, 37.4989)$

$= (0, 84.867, -21.196, 1)$

This is where the horopter crosses the vertical plane through the nodal points and is lost to sight of the eyes.

$\Gamma(0) = (-0.5091, 0, 0.5091, 24)$

$= (-0.0212, 0, 0.021, 1)$

So the horopter crosses the median plane .2mm behind the eyes and .2mm above the horizontal plane.

$$\Gamma(\infty) = (-0.0144, 0.0144, -0.0036, 0)$$
$$= (1, -1, 0.25, 0)$$

This is the direction of the horopter as it tends to infinity; the symmetry between the x and y directions is, of course, because we have chosen $\alpha = \frac{1}{4}\pi$.

$$|Q| = -15.85 .$$

$$Q^{-1} = \begin{pmatrix} -55.5450 & 0.0139 & -55.5436 & 0 \\ 0.3926 & 0 & -1.5706 & 0.0416 \\ 0.0111 & 0.0139 & 0.0111 & 0 \\ -0.0002 & 0 & 0.0007 & 0.0416 \end{pmatrix}$$

The asymptote plane is

$$x - 1.25y = 235.71$$

and the asymptote line

$$\frac{x - 235.71}{4} = \frac{y + 188.57}{-5} = z$$

This curve has been plotted by the computer (see the Appendix), both as projections onto the various co-ordinate planes, and also in two perspective views. (see: figs. 18.2, 18.3, 18.4, 18.5, 18.6)

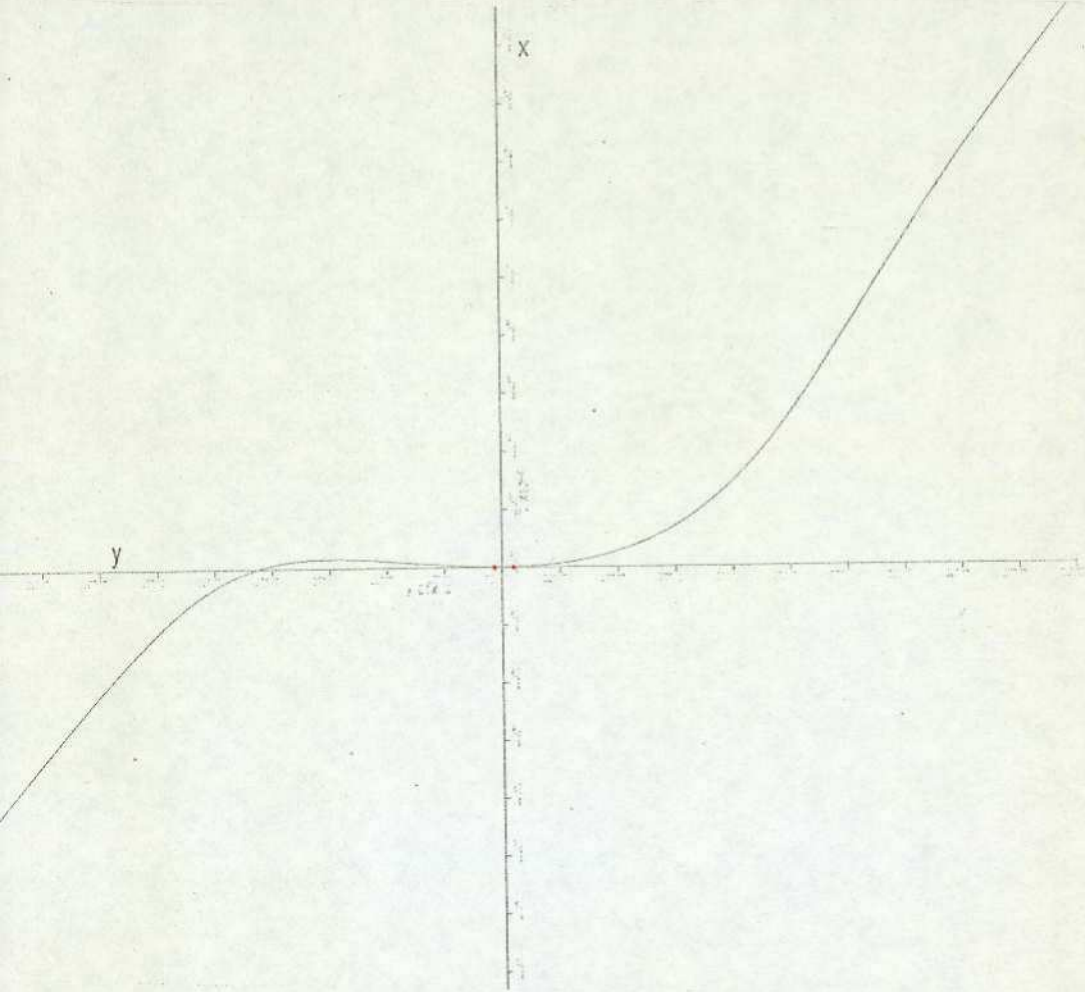


Fig. 18.2 Computer generation of Γ

quasi-symmetric case

projection onto the plane

$z = 0$

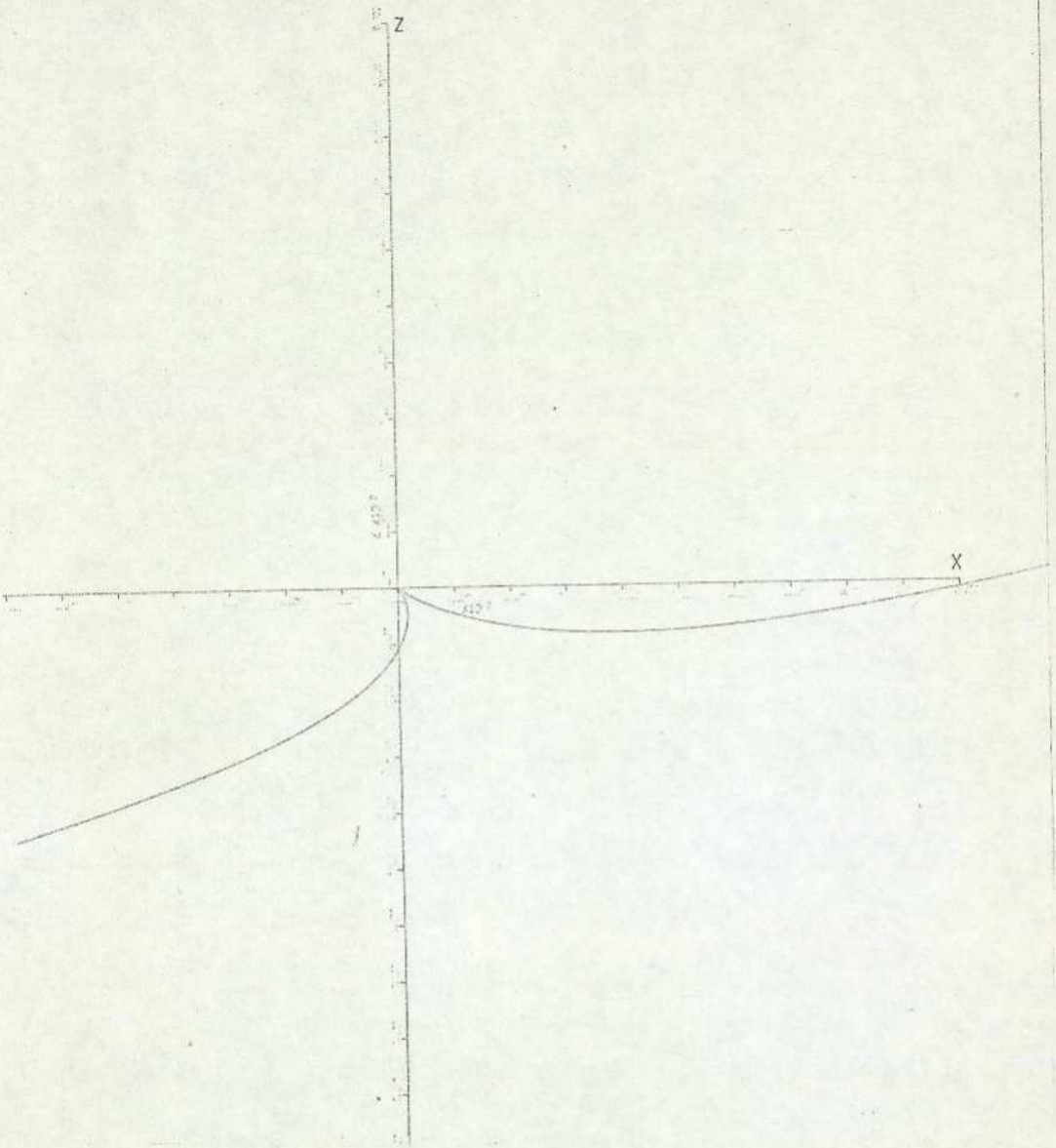


Fig. 18.3 Computer generation of Γ

quasi-symmetric case

projection onto the plane

$y = 0$

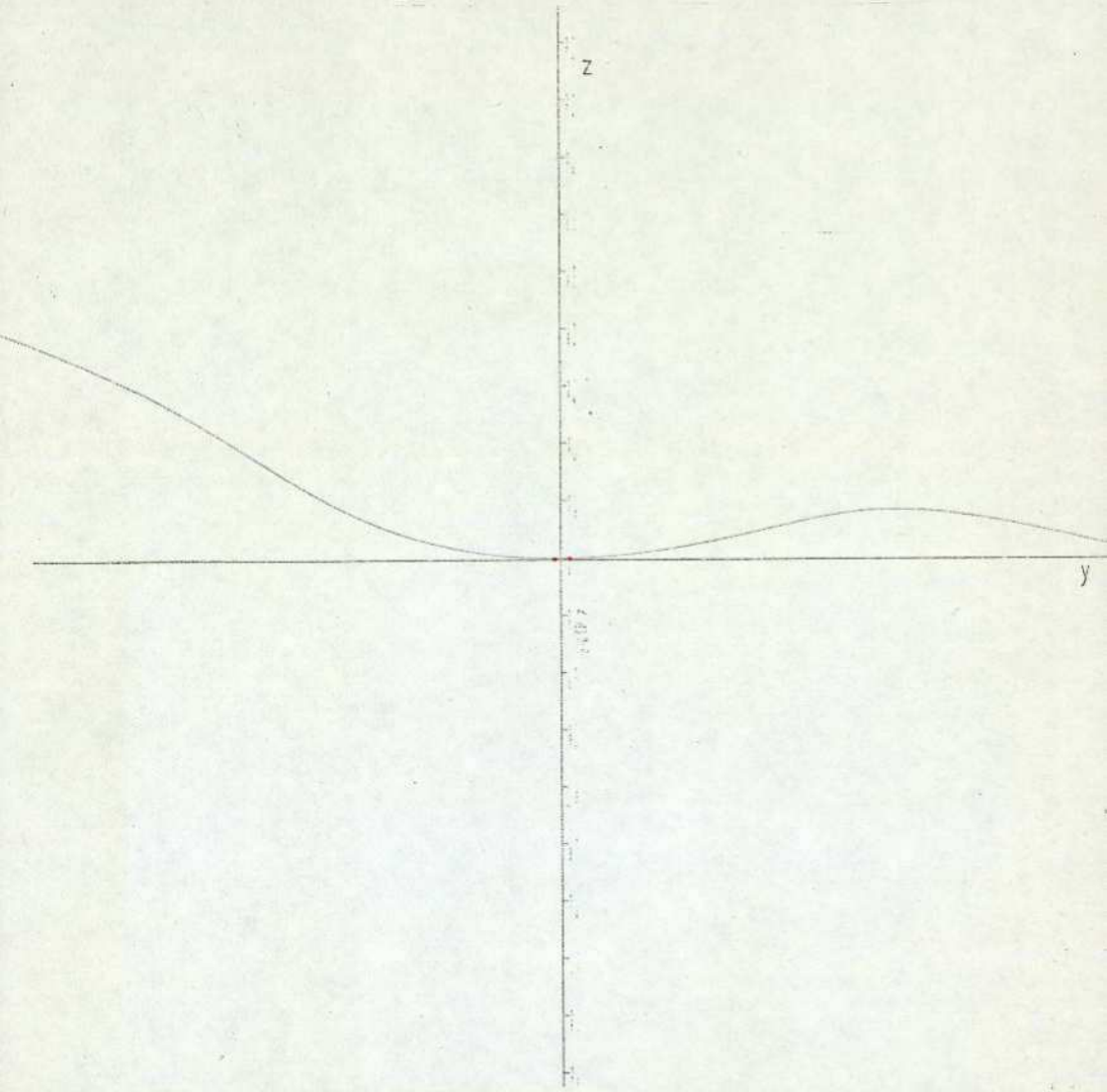


Fig. 18.4 Computer generation of Γ

quasi-symmetric case

projection onto the plane

$x = 0$

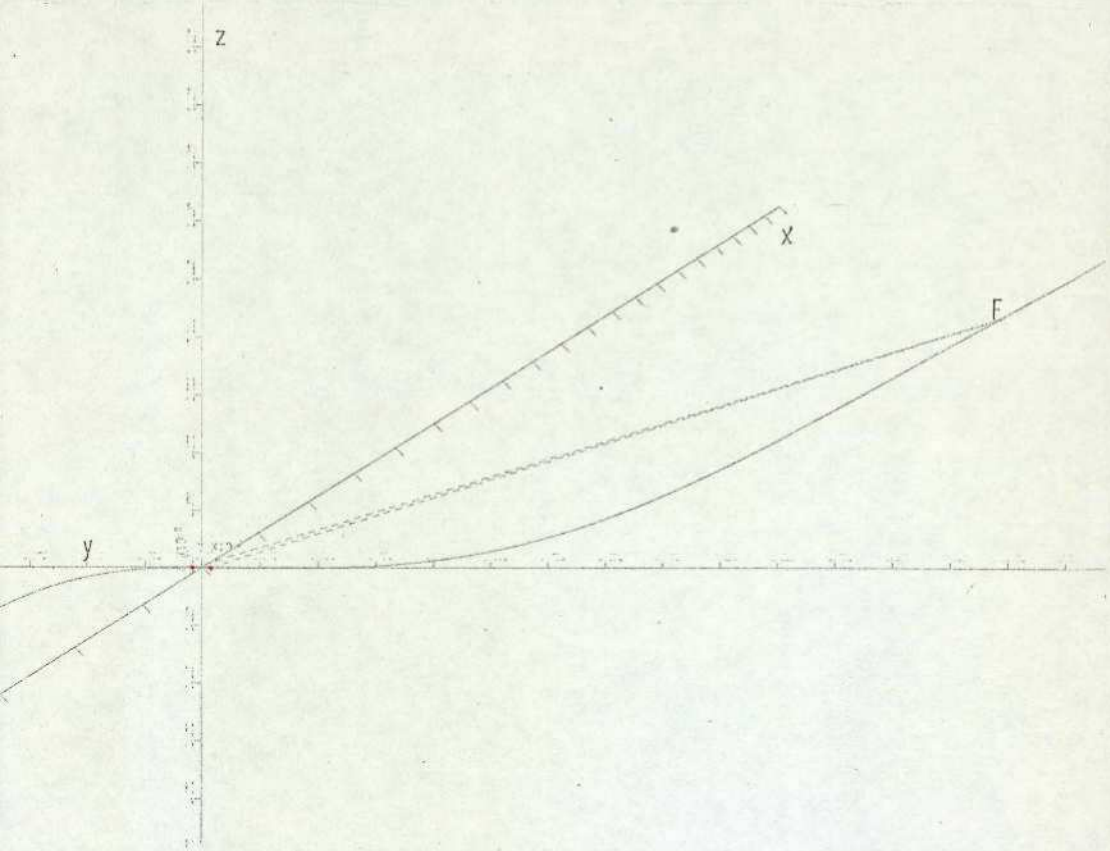


Fig. 18.5 Computer generation of Γ

quasi-symmetric case

perspective view

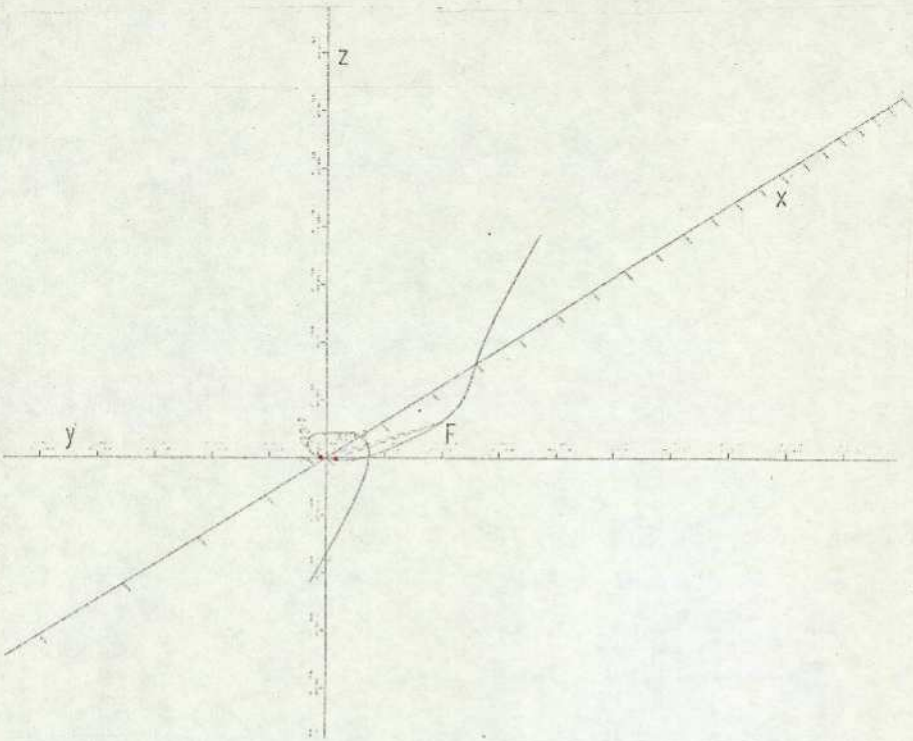


Fig. 18.6 Computer generation of Γ

quasi-symmetric case

perspective view

different scale and aspect to fig. 18.5

ii) The symmetric case

We will take

$$a = 300$$

$$b = 3$$

$$\alpha = \tfrac{1}{2}\pi$$

$$\lambda = 0$$

$$\epsilon = 0$$

and assume

$$\tau_1 = -\tau_2 = 0.02$$

when, we find:

$$S_1 \equiv x'^2 + y'^2 + 2x'z' - 300.015x' - 600.03z' = 0$$

$$|S_1^i| = 0$$

$$|\mathcal{D}_1| = -8\times10^{-6}$$

centre: (300.015, 0, -150.007)

eigenvalues of $\mathcal{D}_1$ 0.0324, 0.02, -0.0124

corresponding principal directions

$$\begin{matrix} 0.8506 & 0 & 0.5257 \\ 0 & -1 & 0 \\ 0.5257 & 0 & -0.8506 \end{matrix}$$

Hence the quadric S_1 is a cone.

But this quadric will meet any plane parallel to the visual plane (i.e. the horizontal plane), say

$$z' = z_0'$$

in the circle

$$x'^2 + y'^2 + 2(z_0' - 150.007)x' - 600.03z_0' = 0$$

That is, the circle whose centre is at the point

$$(-z_0'+150.007,\ 0)$$

and whose radius, R, is

$$z_0'+150.007$$

So, in the visual plane

$z_0' = 0$, this is the circle

centre: (150.007,0); radius: 150.007

i.e., the Vieth-Müller circle.

and, in the plane

$z_0' = -150.007$, this is the circle

centre: (300.015,0); radius: 0

Hence, the circle of intersection of S_1 with the horizontal plane reduces to a point, and thus the hyperboloid of one sheet reduces to a cone through this point.¶ (see: fig. 18.7)

$$S_2 \equiv 2x'y' - y'z' - 600.03y' = 0$$

$$|S_2| = |\mathcal{D}_2| = 0$$

which implies, and which can be seen directly, that S_2 degenerates into the two intersecting planes:

$$y'(2x' - z' - 600.03) = 0$$

(see; fig. 18.8)

¶ The computer (see Appendix) showed S_1 to be a hyperboloid of one sheet, due to rounding errors in the determination of $|S_1'|$; in spite of using double precision arithmetic to over twenty significant figures.

$$S_3 \equiv x'^2 - 10000y'^2 - 10000z'^2 - 4998x'z' = 0$$

$$|S_3^4| = 0$$

$$|\mathcal{D}_3| = 4 \times 10^{-6}$$

centre: (0,0,0)

eigenvalues of $\mathcal{D}_3$ 0.0024, -0.040, - 0.0424

principal directions

$$\begin{matrix} 0.9733 & 0 & -0.2300 \\ 0 & 1 & 0 \\ 0.2300 & 0 & 0.9733 \end{matrix}$$

Thus, S_3 is a cone with vertex at the origin. (fig. 18.9)

Now, S_1 can be written

$$(x'-300.03)(x'+2z') + y'^2 = 0$$

and S_3, as

$$(x'-5000z')(x'+2z') - 10000y'^2 = 0$$

Thus, it is evident that the line

$$y' = 0$$

$$x' + 2z' = 0$$

is common to all three surfaces, and is thus the common generator of these surfaces.

And, it can easily be shown, that the locus of the residual points of intersection is the ellipse:

$$5x'^2 + y'^2 - 2700x' + 360000 = 0$$

$$2x' - z' - 600.03 = 0$$

i.e. the explicit (non-parametric) form of the symmetric horopter.

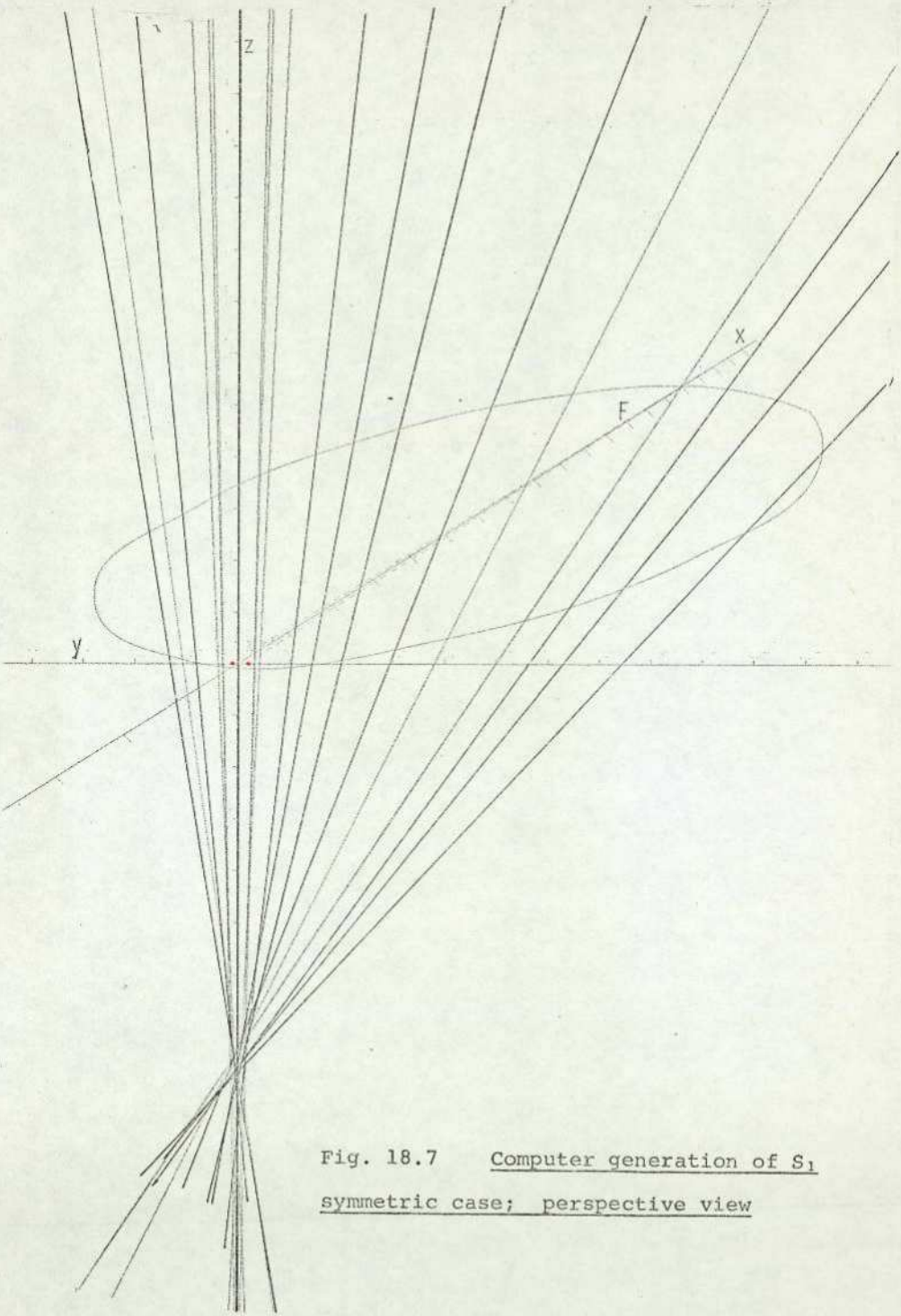


Fig. 18.7 Computer generation of S_1 symmetric case; perspective view

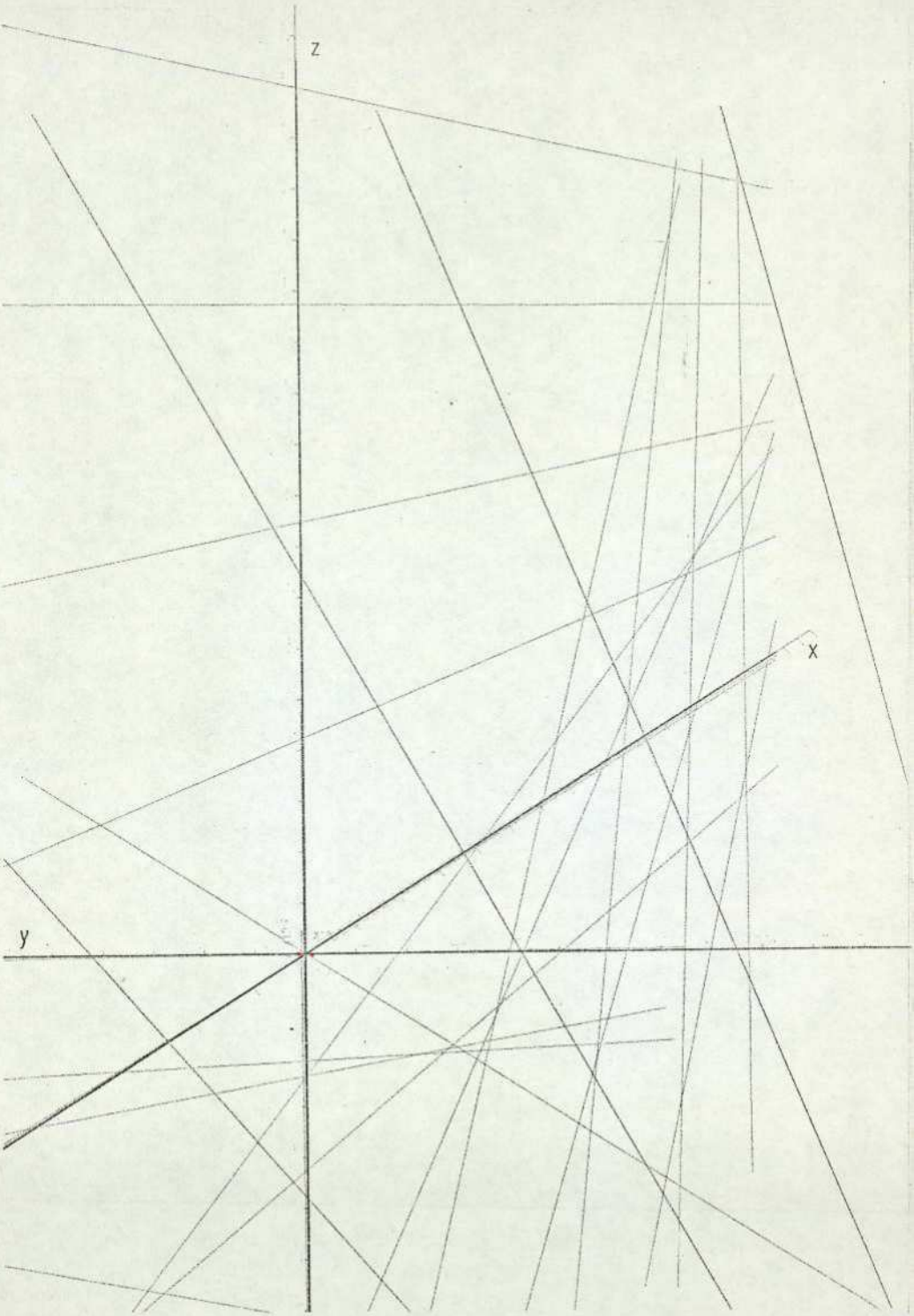


Fig. 18.8 Computer generation of S_2 symmetric case; perspective view two intersecting planes.

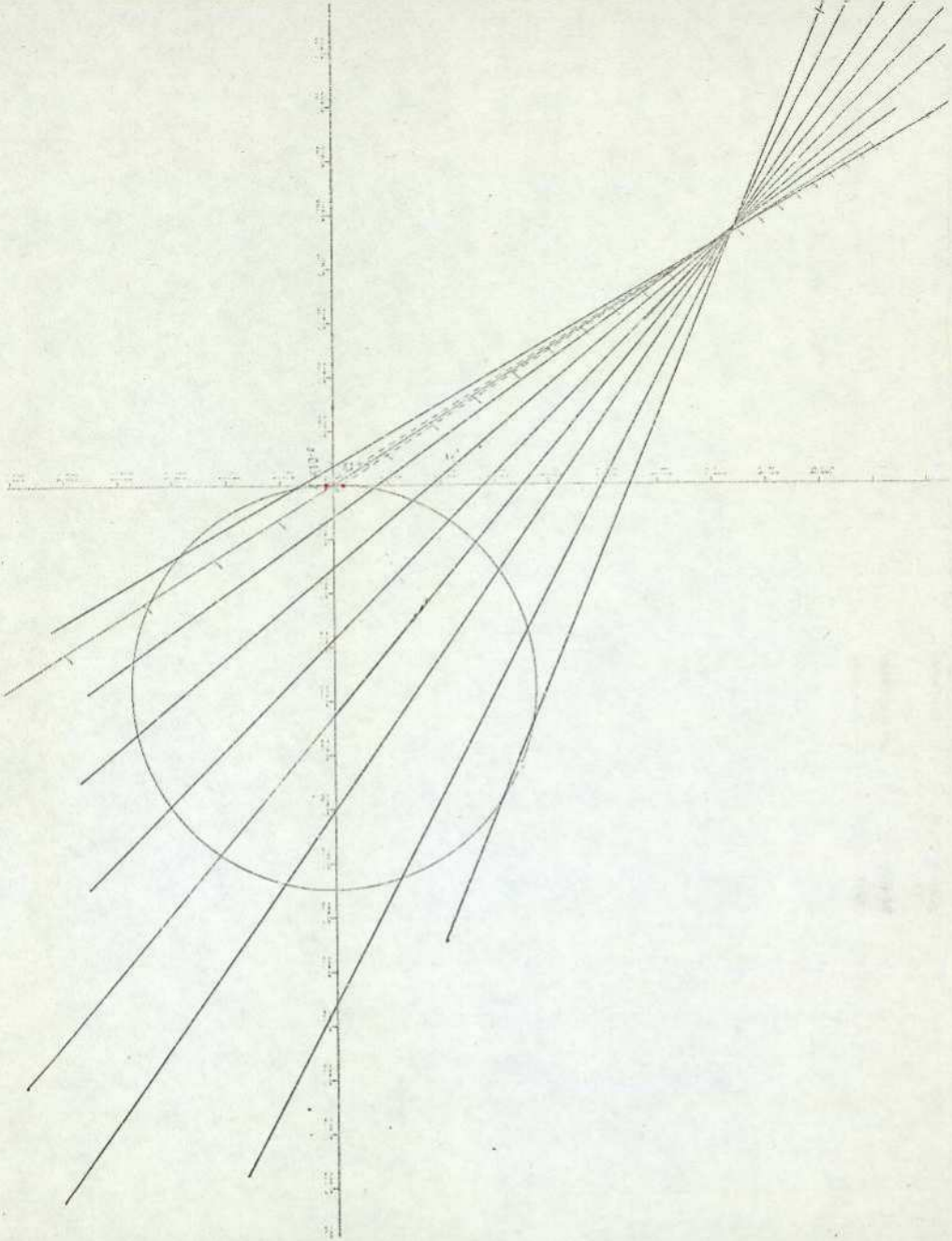


Fig. 18.9 Computer generation of S_3 symmetric case; perspective view

The fixation point is at (300,0,0)

The horopter in parametric form is given by

$$x_1 = 0.72(t^2-1)$$

$$x_2 = 72.00t$$

$$x_3 = -1.44(t^2-1)$$

$$x_4 = 0.012t^2 + 24.0$$

which, as we have already seen, and is indicated by the fact that each expression is a quadratic in the parameter, is a conic. (see: figs. (18.10-13))

The roots of each equation can be seen immediately, and we have:

$\Gamma(1) = (0,72,0,24) = (0,3,0,1)$

$\Gamma(-1) = (0,-3,0,1)$

$\Gamma(0) = (-0.03,0,0.06,1)$

$\Gamma(\infty) = (60,0,-120,1)$

Note, the point (∞) is on the conic, not at infinity.

$$|Q| = 0$$

so there is no inverse matrix.

The asymptote plane is

$$x - 2z - 300 = 0$$

and, the asymptote line is

$$\left.\begin{aligned} y &= 0 \\ x - 2z - 300 &= 0 \end{aligned}\right\}$$

We have from (12.25), that

$$\underline{x} = P\,\underline{x}'$$

Where, substituting the numerical values into (12.27),

$$P = \begin{pmatrix} -1 & 0 & 0 & 300 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

i.e., we have

$$x = -x' + 300$$

$$y = -y'$$

$$z = z'$$

if we substitute for x,y and z in the equations of the asymptotic line, they become

$$y' = 0$$

$$x' + 2z' = 0$$

so, the asymptote to the degenerate cubic represented by Q, is the common generator.

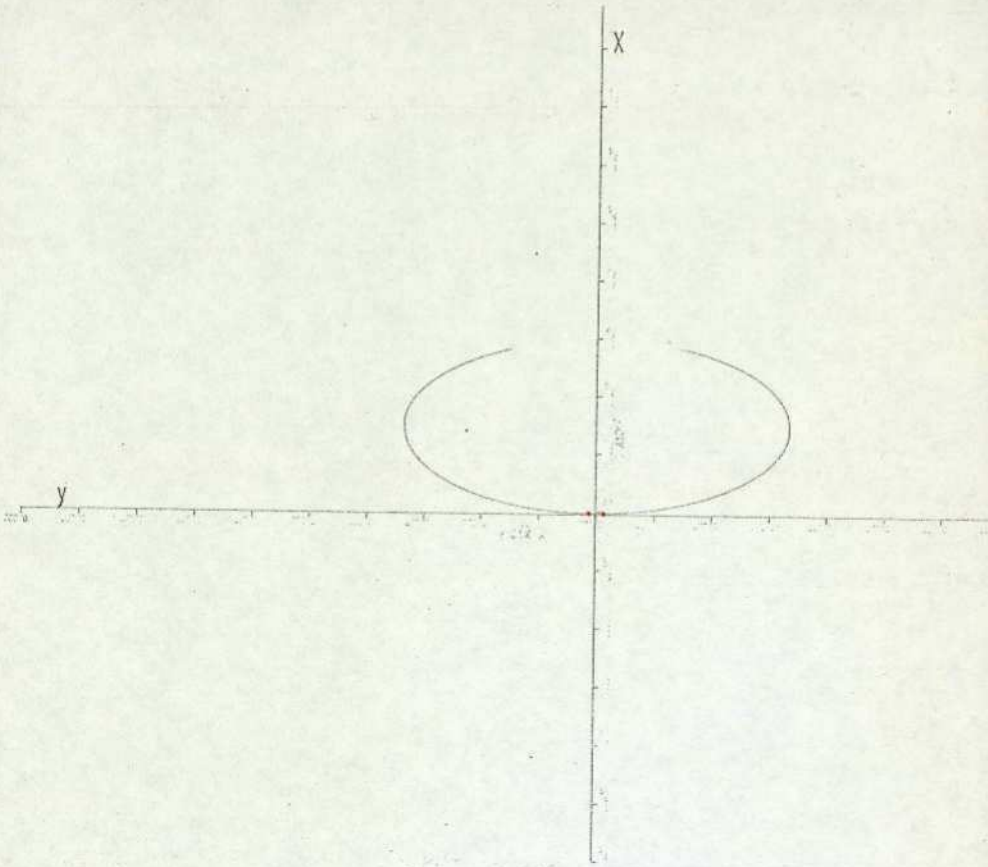


Fig. 18.10 Computer generation of Γ

symmetric case: projection onto the plane

$z = 0$

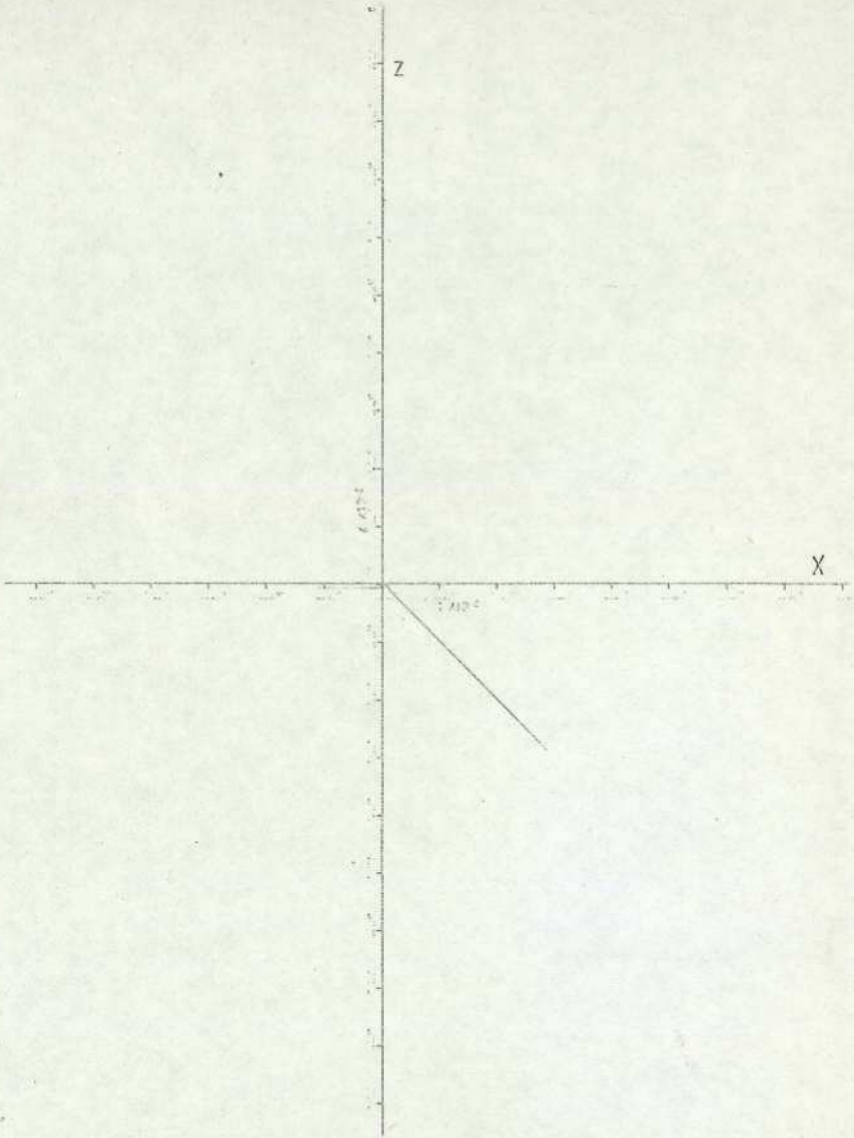


Fig. 18.11 Computer generation of Γ

symmetric case: projection onto the plane

$y = 0$

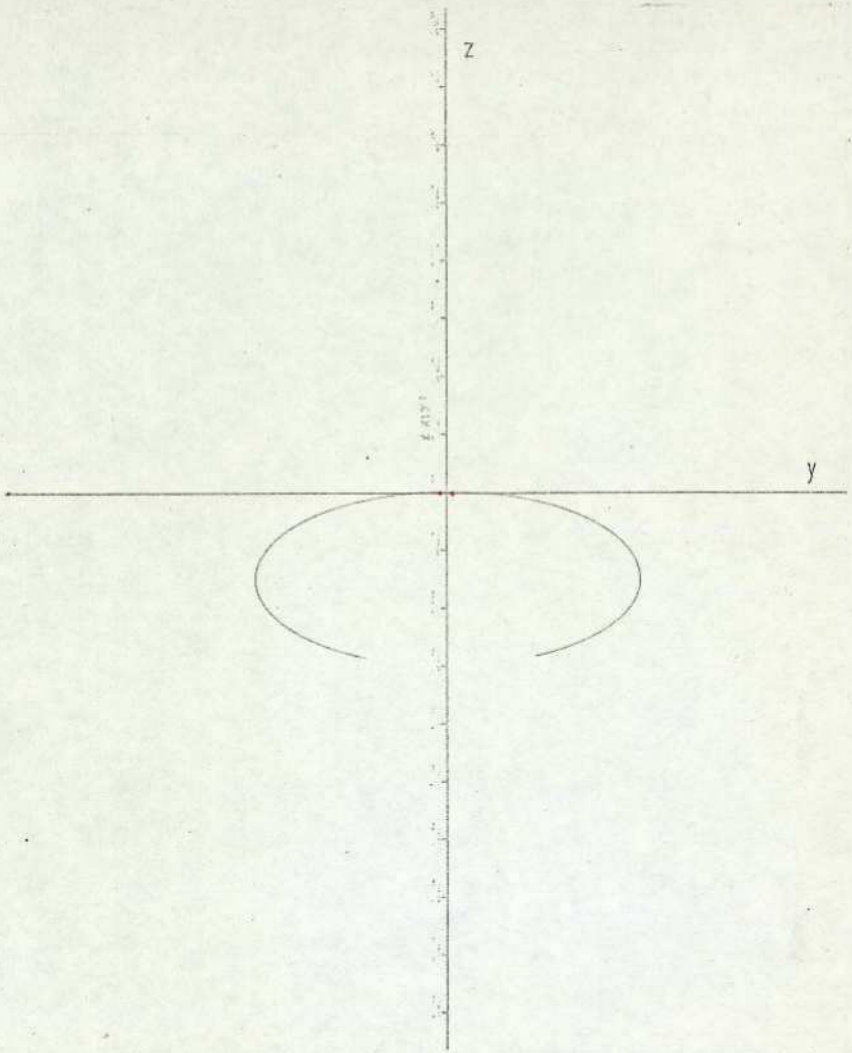


Fig. 18.12 Computer generation of Γ

symmetric case: projection onto the plane

$x = 0$.

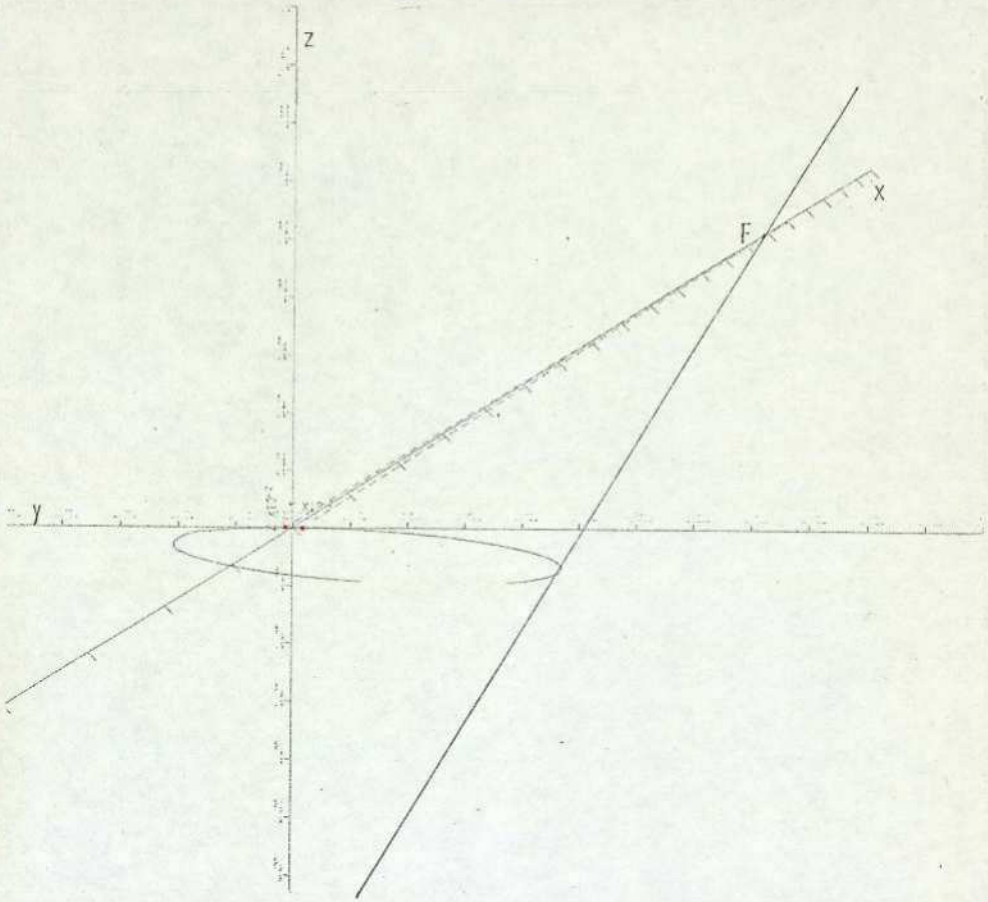


Fig. 18.13 Computer generation of Γ

symmetric case: perspective view

iii) The general case

We will take

$a = 300$

$b = 3$

$\alpha = 70$

$\lambda = 30$

$\varepsilon = 1$

and assume that the angles of torsion are determined by Listing's Law, when, we find:

$$S_1 \equiv x'^2 + y'^2 + 0.15x'y' + 1.58x'z' - 0.095y'z' - 300.025x' + 64.25y' - 484.94z' = 0$$

$$|S_1^4| = 1.66\times10^{-4}$$

$$|\mathcal{D}_1| = -4.22\times10^{-6}$$

centre: (302.75,-63.66,-187.09)

eigenvalues of $\mathcal{D}_1$ 0.027, 0.019, -0.008

corresponding principal directions

0.8737	0.0974	0.4767
0.0595	-0.9938	0.0941
0.4829	-0.0539	-0.8740

Consequently, S_1 is a hyperboloid of one sheet.

$$S_2 \equiv x'^2 + y'^2 + 2.91x'y' - 0.28z'x' + 10.56y'z' - 300x' - 764.16y' + 1235.8z' = 0$$

$$|S_2^4| = 2.98\times10^{-8}$$

$$|\mathcal{D}_2| = -1.69\times10^{-7}$$

centre: (309.62, -108.94, 7.64)

eigenvalues of $\mathcal{D}_2$ 0.011, 0.002, -0.008

principal directions

0.1971 0.7383 0.6450

0.9637 -0.0252 -0.2657

0.1799 -0.6740 0.7165

So, in this case, S_2 is also a hyperboloid of one sheet.

$$S_3 \equiv x'^2 - 11324.48y'^2 - 68408.08z'^2 - 4370233x'z' = 0$$

$$|S_3'| = 0$$

$$|\mathcal{D}_3| = -4.4 \times 10^{-6}$$

centre: (0, 0, 0)

eigenvalues of $\mathcal{D}_3$ 0.005, 0.003, -0.033

principal directions

0 -1 0

0.9599 0 -0.2804

0.2904 0 0.9599

Thus S_3 is a cone with its vertex at the origin

The fixation point is at

(244.139, -102.606, -140.954)

$$x_1 = -25.27t^3 + 16990.6t^2 + 25.27t - 16990.6$$

$$x_2 = -t^3 - 8773t^2 + 2058534.6t - 338.9$$

$$x_3 = 20.06t^3 - 7089.84t^2 - 20.06t + 7089.84$$

$$x_4 = 0.15t^3 + 38.25t^2 - 2897.4t + 686139.48$$

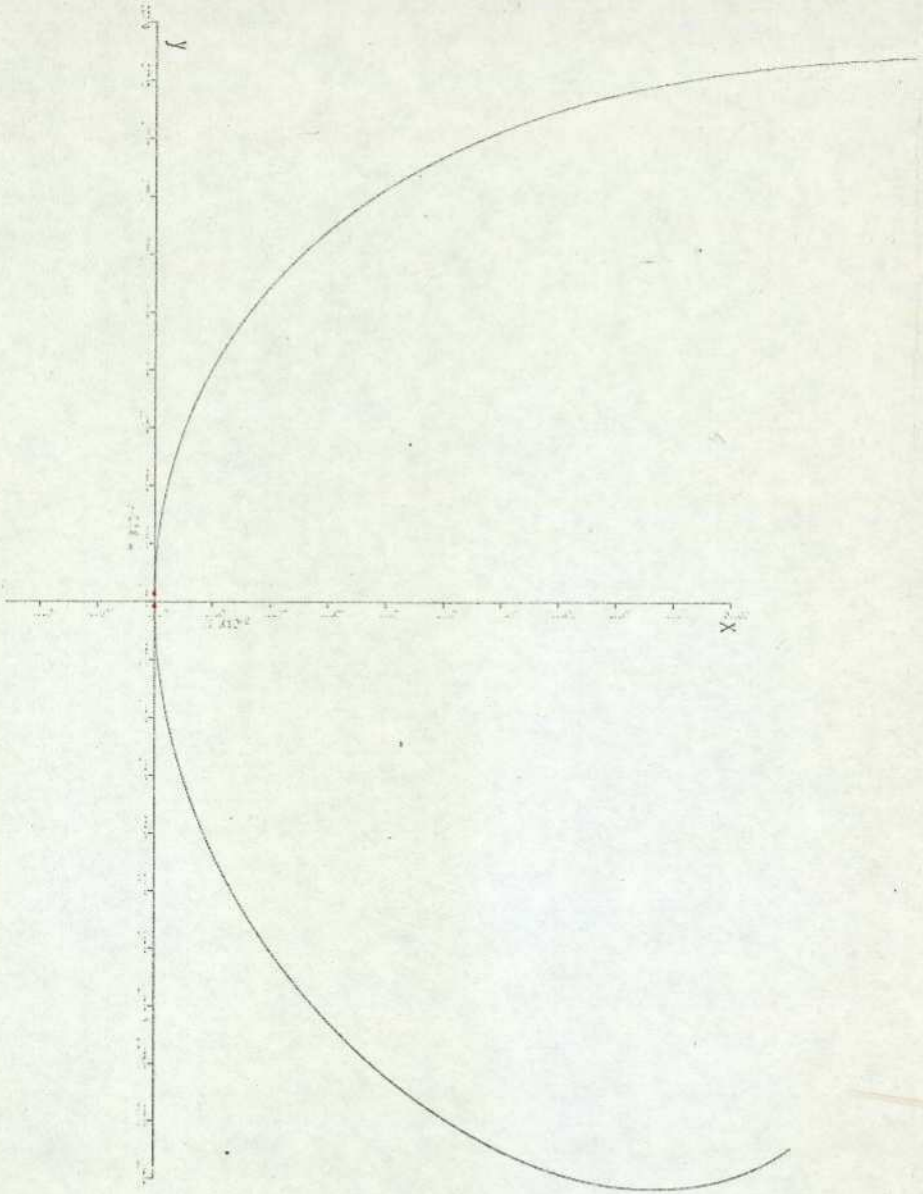


Fig. 18.14 Computer generation of Γ

The general case: projection onto the plane

$z = 0$

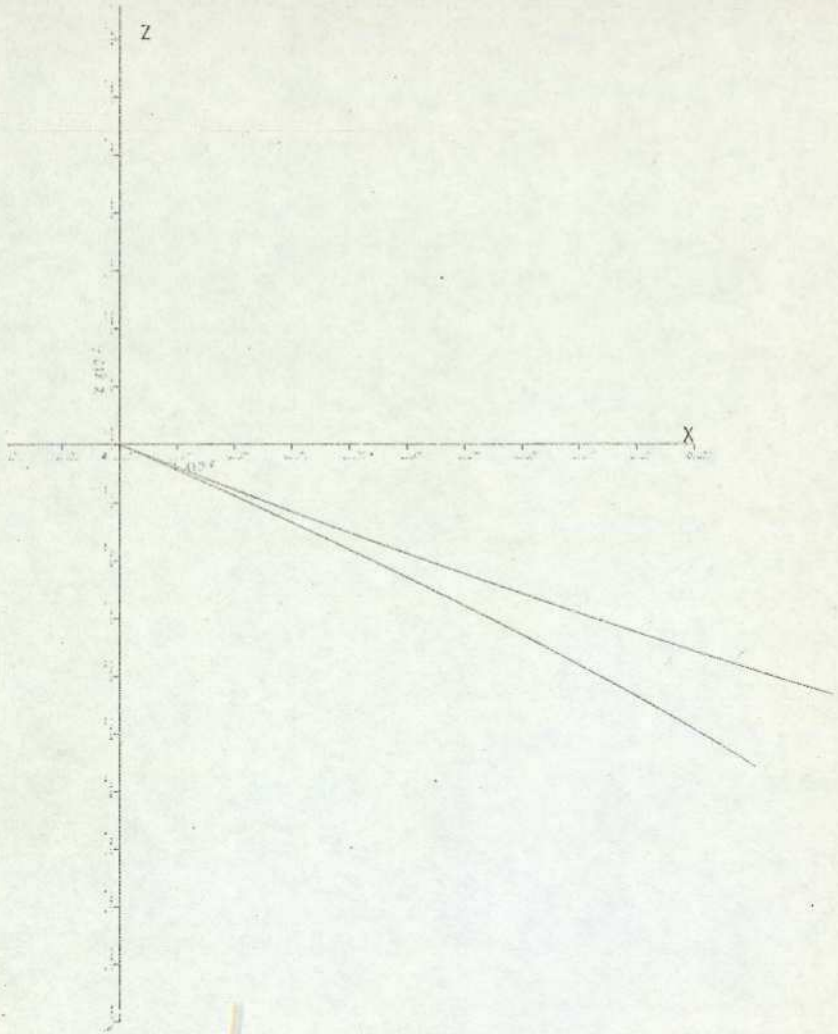


Fig. 18.15 Computer generation of Γ

The general case: projection onto the plane

$y = 0$

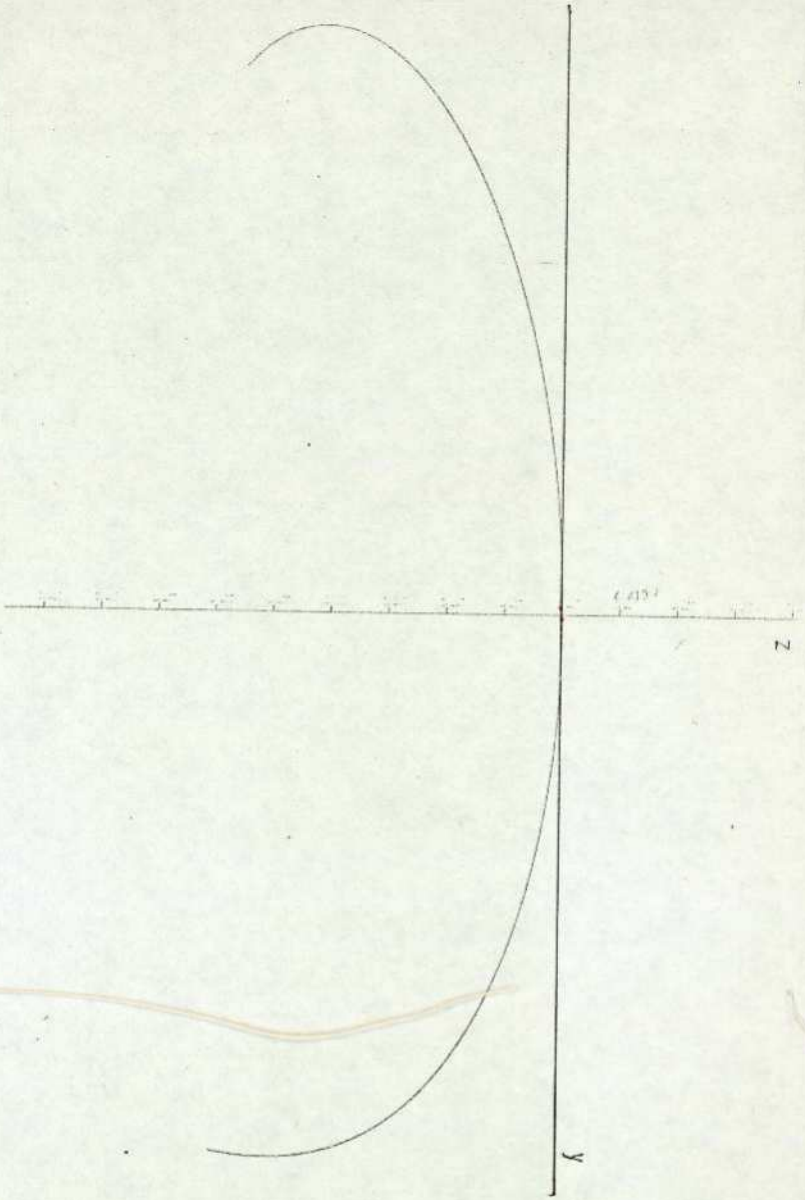


Fig. 18.16 Computer generation of Γ

The general case: projection onto the plane

$x = 0$

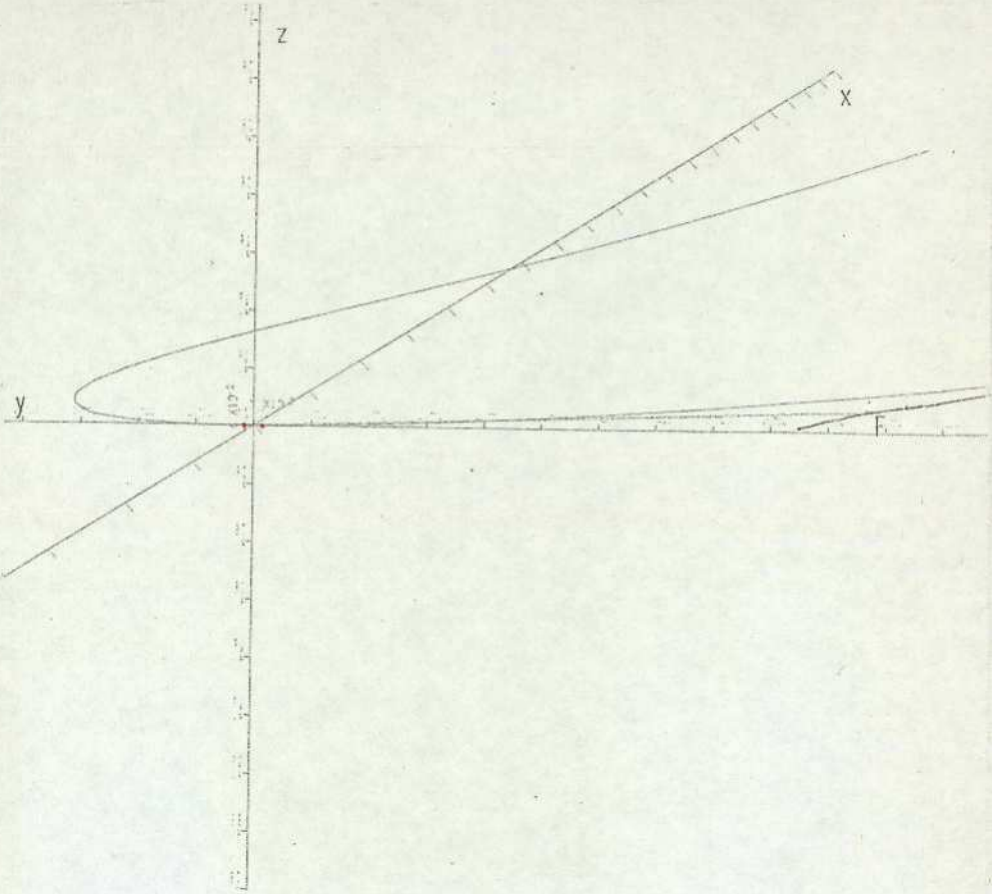


Fig. 18.17 Computer generation of Γ

The general case: perspective view

The real roots of

$x_1 = 0$ are $t = 672.342, 1, -1$

$x_2 = 0$ $t = -9001.76, 228.68, 0$

$x_3 = 0$ $t = 353.38, 1, -1$

$x_4 = 0$ $t = 251.677$

The fixation point corresponds to

$$t = -496.939$$

The point at infinity,

$$t_\infty = 251.677$$

$$Q^{-1} = \begin{pmatrix} -69.257 & 0.013 & -6.926 & 0 \\ 0.049 & 0 & -1.958 & 0.042 \\ 0.014 & 0.013 & 0.001 & 0 \\ 0 & 0 & 0 & 0.042 \end{pmatrix}$$

The asymptote plane is

$$x + 0.88y + 1.3z + 0.01 = 0$$

The asymptote line is

$$\frac{x + 0.04}{1.4} = \frac{y - 0.03}{-3.07} = z$$

The computer generation of Γ, as output to the graph plotter, are given in figs. (18.14)-(18.17)

19 The locus of parallel chords

It was shown in (8.15), that there is a unique nonius-line through every point of space. Each nonius-line, however, is a chord of the horopter curve, Γ. So, there is a chord of Γ passing through every point of space; and this chord is unique - except for points on Γ, which have an infinite number of such chords.

In view of the experimental methods used in plotting the nonius horopter[326] it is of interest to examine the curve in which the chords that are parallel to each other intersect the horizontal plane .

If the point at infinity on Γ is designated as in §13 as (t_∞), then the family of parallel chords are the lines joining the point (t_∞) to every other point (t) on Γ.

If, in (13.15), we write

$$p = t + t_\infty \qquad 19.1$$

$$q = tt_\infty \qquad 19.2$$

Then, the equations

$$\left.\begin{aligned} X_1 - pX_2 + qX_3 &= 0 \\ X_2 - pX_3 + qX_4 &= 0 \end{aligned}\right\} \qquad 19.3$$

will generate the family of parallel chords as t varies between $\pm\infty$.

From (13.6), we can deduce that, under the transformation

$$\underline{x} \rightarrow \underline{X}$$

the plane

$$\underline{u}.\underline{x} = 0 \qquad 19.4$$

will transform to a plane

$$\underline{U}.\underline{X} = 0 \text{ (say)} \qquad 19.5$$

where

$$\underline{U} = (Q^{-1})^{-1^T}\underline{u} \qquad 19.6$$

i.e.

$$\underline{U} = Q^T\underline{u} \qquad 19.7$$

where Q^T is the transpose of the matrix Q.

Now, in the fixed co-ordinate system $\underline{X}$, the horizontal plane is given by

$$x_3 = 0 \qquad 19.8$$

i.e.

$$\underline{u} = (0,0,1,0) \qquad 19.9$$

Thus, (19.7) =>

$$.\underline{U} = (q_{31},q_{32},q_{33},q_{34}) \qquad 19.10$$

i.e., the horizontal plane has the equation

$$q_{31}X_1 + q_{32}X_2 + q_{33}X_3 + q_{34}X_4 = 0 \qquad 19.11$$

in the representation $\underline{X}$.

Now, let the chord (19.3), meet the horizontal plane (19.11) at a point $\underline{Y}$ (say).

Then, it can be shown, that the co-ordinates of $\underline{Y}$ are given by

$$\underline{Y} = \begin{pmatrix} q_{32}q^2 + q_{33}pq + q_{34}(p^2-q) \\ -q_{31}q^2 + q_{33}q + q_{34}p \\ -q_{31}pq - q_{32}q + q_{34} \\ -q_{31}(p^2-q) - q_{32}p - q_{33} \end{pmatrix} \qquad 19.12$$

If this point corresponds to the point $\underline{y}$, in the original co-ordinates, then we have, from (13.6)

$$\underline{y} = \underline{Q}\underline{Y} \qquad 19.13$$

and, evaluation gives:-

$$\begin{aligned} y_i = {} & (q_{i1}q_{32}-q_{12}q_{31})q^2 + (q_{i1}q_{33}-q_{i3}q_{31})pq \\ & + (q_{i1}q_{34}-q_{14}q_{31})(p^2-q) + (q_{i2}q_{33}-q_{i3}q_{32})q \\ & + (q_{i2}q_{34}-q_{i4}q_{32})p + (q_{i3}q_{34}-q_{i4}q_{33}) \quad (i=1,4) \end{aligned} \qquad 19.14$$

If we put

$$A^i_{jk} = q_{ij}q_{3k} - q_{ik}q_{3j} \qquad 19.15$$

and substitute from (19.1), and (19.2), we find that the locus of the points where the parallel chords of Γ meet the horizontal plane $x_3 = 0$, is given by

$$y_i = (A^i_{12}t_\infty^2+A^i_{13}t_\infty+A^i_{14})t^2 + \{A^i_{13}t_\infty^2+(A^i_{14}+A^i_{23})t_\infty+A^i_{24}\}t + (A^i_{14}t_\infty^2+A^i_{24}t_\infty+A^i_{34}) \quad (i=1,4) \qquad 19.16$$

If we now limit our consideration to the quasi-symmetric case, with fixation in the horizontal plane, when Q is given by (14.77).

From which we can see that

$$x_4 = (T+KS^2)bt^2 + KbC^2 \qquad 19.17$$

which implies that x_4 will only vanish, when $t = \infty$

Hence

$$t_\infty = \infty \qquad 19.18$$

If we let $t_\infty \to \infty$ in (19.16), we find that the locus of parallel chords is given by

$$y_i = A^i_{12}t^2 + A^i_{13}t + A^i_{14} \quad (i=1,4) \qquad 19.19$$

and, if we substitute the values of q_{ij} from (14.77) into (19.15); and from (19.15) into (19.19), we find

$$y_1 = SC(Kb^2+A_2^2T)(t^2-1) \qquad 19.20$$

$$y_2 = -A_1A_2C^2Tt^2+b^2(T+K)t+A_1A_2C^2T \qquad 19.21$$

$$y_3 = 0 \qquad 19.22$$

$$y_4 = b(T+KS^2)t^2 + KbC^2 \qquad 19.23$$

Which we can see is a conic in the plane $x_3 = 0$

If now, we consider the symmetric case. I.e., we take a fixation point in the median line as most of the experimentalists have done. We put

$$\alpha = \tfrac{1}{2}\pi \tag{19.24}$$

and the simplifications (16.3)-(16.8) apply.

Then we have

$$y_1 = a(Kb^2 + a_1^2T)(t^2 - 1) \tag{19.25}$$

$$y_2 = a_1^2b(T + K)t \tag{19.26}$$

$$y_3 = 0 \tag{19.27}$$

$$y_4 = (Kb^2 + a_1^2T)t + Ka \tag{19.28}$$

If we eliminate t from these equations and revert to non-homogeneous co-ordinates, this conic can be written:

$$\left.\begin{aligned} ab^2Kx^2 + a(Kb^2+a_1^2T)y^2 + b^2(Kb^2+a_1^2T-Ka^2)x \\ - ab^2(Kb^2+a_1^2T) = 0 \\ z = 0 \end{aligned}\right\} \tag{19.29}$$

This conic is an ellipse, as can be seen by re-writing it in canonical form:

$$\frac{(x-C)^2}{A^2} + \frac{y^2}{B^2} = 1 \tag{19.30}$$

where:

$$A = \frac{a_1^2(K+T)}{2aK} \qquad 19.31$$

$$B = \frac{a_1^2(K+T)b}{2a\sqrt{K(Kb^2+a_1^2T}} \qquad 19.32$$

$$C = \frac{Ka^2-Kb^2-a_1^2T}{2aK} \qquad 19.33$$

Thus, the chords of Γ that are parallel to each other lie on the surface of an elliptic cylinder; (i.e. a cone with its vertex in the plane at infinity) which intersects the horizontal visual plane in the ellipse (19.30). That is, the ellipse with axes, A and B, and centre (C,O), where A,B and C are given by (19.31)-(19.33).

20 Comparison with the classical horopter

Experimental work on the horopter[329] has determined it to be a plane conic curve lying in the visual plane. The fact that a plane curve has been found, although, as the analysis in §11 has shown, the horopter is a cubic curve twisted in space, is implicit in the design of the nonius experiment.

Experimental work has been based, in the main, on two criteria, the apparent fronto-parallel plane and the identity of visual direction.

The former criterion seems to be so strongly influence by extraneous factors, such as the colour of the test-rods, the background, and other clues, that it is not amenable to analysis.

In fact, this technique is increasingly being discarded, as one authority puts it

> "The apparent frontal plane horopter ... has *no* direct relationship with corresponding retinal points."[330]

and perusal of the literature makes it puzzling that any-one should have thought that there was.

In the nonius technique[331], the subject fixates binocularly a point in the median line. He is then required to report the position in which the monocular images of

a rod, as seen by the right and left eyes, appear to be in line. The point where this rod meets the horizontal visual plane is taken to be a point of the horopter. The locus of such points is found to be a conic curve.

However, as the discussion in §8 has revealed, what this experiment is determining are not the self-corresponding points of the horopter, but what we have called the nonius-lines. That is, the nonius experiment is not plotting Γ, it is merely locating certain chords of Γ.

Now there is a unique nonius-line through every point of the visual plane. It is suggested here that the nonius experimental technique is selecting the nonius lines that are parallel to each other. Ogle[332] seemed to imply this possibility in his definition of the longitudinal horopter surface, but it is only mentioned in passing and is not developed.

Equation (2.11) is Ogle's explicit equation for the longitudinal horopter in terms of H, the Hering-Hillebrand deviation coefficient, which can be re-written (interchanging x and y to conform to our co-ordinates):

$$x^2\{1+\frac{Hb}{2a}\} + y^2\{1-\frac{Ha}{2b}\} - x\{\frac{a^2-b^2}{a} + Hb\} - b^2 + \frac{Hab}{2} = 0 \quad 20.1$$

which we can re-write in the form

$$\frac{(x-C')^2}{A'^2} + \frac{y^2}{B'^2} = 0 \quad 20.2$$

where

$$A' = \frac{a^2 + b^2}{2a + Hb} \quad 20.3$$

$$B'^2 = \frac{b(a^2+b^2)^2}{a(2a+Hb)(2b-Ha)} \quad 20.4$$

$$C' = \frac{(a^2-b^2+Hab)}{2a + Hb} \quad 20.5$$

If this curve does, in fact, represent the locus of parallel chords (19.30), then we must have

$$A = A' \quad 20.6$$

$$B = B' \quad 20.7$$

$$C = C' \quad 20.8$$

where A, B and C are given by (19.31)-(19.33).

Solving these equations simultaneously, we find

$$H = -\frac{2aT}{b(K+T)} \quad 20.9$$

Thus, since b, the semi-inter-nodal distance is (almost) fixed for a given observer, H is found to be a function of the fixation distance, and the mutual angle of torsion of the two eyes.

In Part II, above, we found an expression for the angle of torsion of an eye in terms of its azimuth angle and angle of elevation on the assumption that the eye moves in accordance with Listing's Law.

However, if the eyes do so move, and fixation is in the horizontal plane, we have from (3.65) that

$$\tau = 0 \qquad 20.10$$

If we put this value of the angle of torsion into the equations for the symmetric horopter (16.31)-(16.34), they reduce to

$$x_1 = ab^2(t^2-1) \qquad 20.11$$

$$x_2 = a\hat{1}bt \qquad 20.12$$

$$x_3 = 0 \qquad 20.13$$

$$x_4 = b^2t^2 + a^2 \qquad 20.14$$

whatever the value of the vertical deviation ε.

Eliminating t, can be written

$$ax^2 + ay^2 - (a^2-b^2)x - ab^2 = 0 \qquad 20.15$$

which is the equation of the Vieth-Müller circle.

Similarly, if (20.10) holds, then, from (20.9), H is zero too, and Ogle's equation (20.1) also reduces to the Vieth-Müller circle (20.15).

For this reason, we have not assumed Listing's Law to hold in the symmetric case, but have followed the results of other investigators[333] in assuming it to break down in an individual way when the eyes are converged; and have taken the quasi-symmetric approximation

$$\tau_2 = -\tau_1 \qquad 20.16$$

to apply.

Since the time of Hillebrand[334], the Hering-Hillebrand deviation has been held to be a measure of the relative asymmetry of the corresponding points of the two retinas. However, this leads to the problem, discussed above[335], of explaining the apparently linear variation of the coefficient H, with the fixation distance (see: fig. 2.22).

However, equation (20.9) reveals that on the basis of the preceding arguments, H is indeed a linear function of the fixation distance. Further, the asymmetry is introduced into the configuration not as a result of a structural retinal asymmetry, but as a consequence of the relative cyclo-torsional movements of the eye. Indeed, as we have just seen, if there is no angle of torsion there is no deviation. Hence, any arguments against the cortical linkage of corresponding retinal points on the basis of the horopter deviation are rendered redundant.

21 Some Experimental Considerations

(i) Inclination of the symmetric horopter

For symmetric horizontal fixation we have found that the horopter degenerates into a straight line through the fixation point intersecting a conic that passes through the nodal points of the eyes. This conic lies in a plane inclined at an angle θ to the horizontal plane, where θ (from 16.37) is given by

$$\tan\theta = -\frac{a_1}{b}\tan\tau_1 \qquad 21.1$$

So if the angle of torsion is zero Γ will lie in the horizontal plane.

The ability of the eye to resolve two points in the vicinity of the fixation point is said[336] to be 1' of arc. Hence a shift in the inclination of the plane of the horopter greater than this would be observable.

If we put $\theta = 1'$ and $b = 3$ cm. in (21.1), we find

$$\tan\tau_1 = -\frac{0.0009}{\sqrt{a^2 + 3}}$$

$$\Rightarrow \qquad |\tau| \simeq \frac{0.05}{a} \text{ degrees} \qquad 21.2$$

Thus for (say) $a = 300$ cm. τ would be of the order of 0.6" of arc which is beyond the precision of any current technique for measuring this angle.

Hence we can say that the presence of a demonstrable amount of torsion in the eye will produce an observable effect on the inclination of the horopter.

(ii) Stability of the fixation point

We have seen (in §16) that the twisted cubic curve Γ degenerates into a conic when fixation is in the median plane. However the work of Ditchburn[337] and Millidot[338] indicate that under prolonged fixation the fixation of the eyes shifts through about 30' of arc.

It is therefore worth considering the effect of small changes of fixation on the shape of the space horopter.

In the equations of §14 we can put $\lambda = 0$ so that fixation is in the horizontal plane and

$$\alpha = \tfrac{1}{2}\pi - \delta\alpha \qquad 21.3$$

where $\delta\alpha$ is small enough so that second order terms can be neglected.

The equations for the horopter curve then become

$$x_1 = -ab\{abT\delta\alpha t - Kb\}(t^2-1) \qquad 21.4$$

$$x_2 = a'^2b(K+T) \qquad 21.5$$

$$x_3 = -\frac{ab}{a'}\{ab\delta\alpha t + a'^2\}(t^2-1) \qquad 21.6$$

$$x_4 = (Kb^2 + a'^2T)t^2 + Ka^2 \qquad 21.7$$

where

$$a' = \sqrt{a^2+b^2} \qquad 21.8$$

i.e. $$a_1 = a' - \frac{ab\delta\alpha}{a'} \qquad 21.9$$

If we put $\delta\alpha=0$ then (21.4)-(21.7) reduce to (16.31)-(16.34), the symmetric case.

We can study the appearance of this curve by considering its projection onto the plane $y = 0$ from a viewing point with co-ordinates (d_1, d_2, d_3).

We can show that the co-ordinates in the plane of projection (X,Y) are given by

$$X = \frac{x_1 d_2 - x_2 d_1}{x_4 d_2 - x_2} \qquad 21.10$$

$$Y = \frac{x_3 d_2 - x_2 d_3}{x_4 d_2 - x_2} \qquad 21.11$$

If we substitute the parameters of the symmetric horopter from (16.31)-(16.34) into these expressions we find that the points (X_1, Y_1) of its projection are given by

$$X_1(t) = \frac{d_2 K b^2 a(t^2-1) - d_1 a_1^2 b(K+T)t}{d_2\{(Kb^2+a_1^2T)t^2 + Ka^2\} - a_1^2 b(K+T)t} \qquad 21.12$$

$$Y_1(t) = -\frac{d_2 a_1 ba(t^2-1) + d_3 a_1^2 b(K+T)t}{d_2\{(Kb^2+a_1^2T)t^2 + Ka^2\} - a_1^2 b(K+T)t} \qquad 21.13$$

This curve is shown as the part dotted conic LRPCQ in fig. 21.1 for $a = 300\text{cm}$ $b = 3\text{cm}$ $T = 0.02$ from a viewing point $(30,100,100)$.

If we substitute from (21.4)-(21.7) into (21.10), (21.11) we find that the projection of the horopter with the fixation point shifted through an angle $\delta\alpha$ is given by

$$X_2(t) = \frac{-a^2 bT\delta\alpha d_2 t^3}{d_2\{(Kb^2+a_1^2T)t^2 + Ka^2\} - a_1^2 b(K+T)t} + X_1(t) \qquad 21.14$$

$$Y_2(t) = \frac{-a^2 b^2 \delta\alpha d_2 t^3/a'}{d_2\{(Kb^2+a_1^2T)t^2 + Ka^2\} - a_1^2 b(K+T)t} + Y_1(t) \qquad 21.15$$

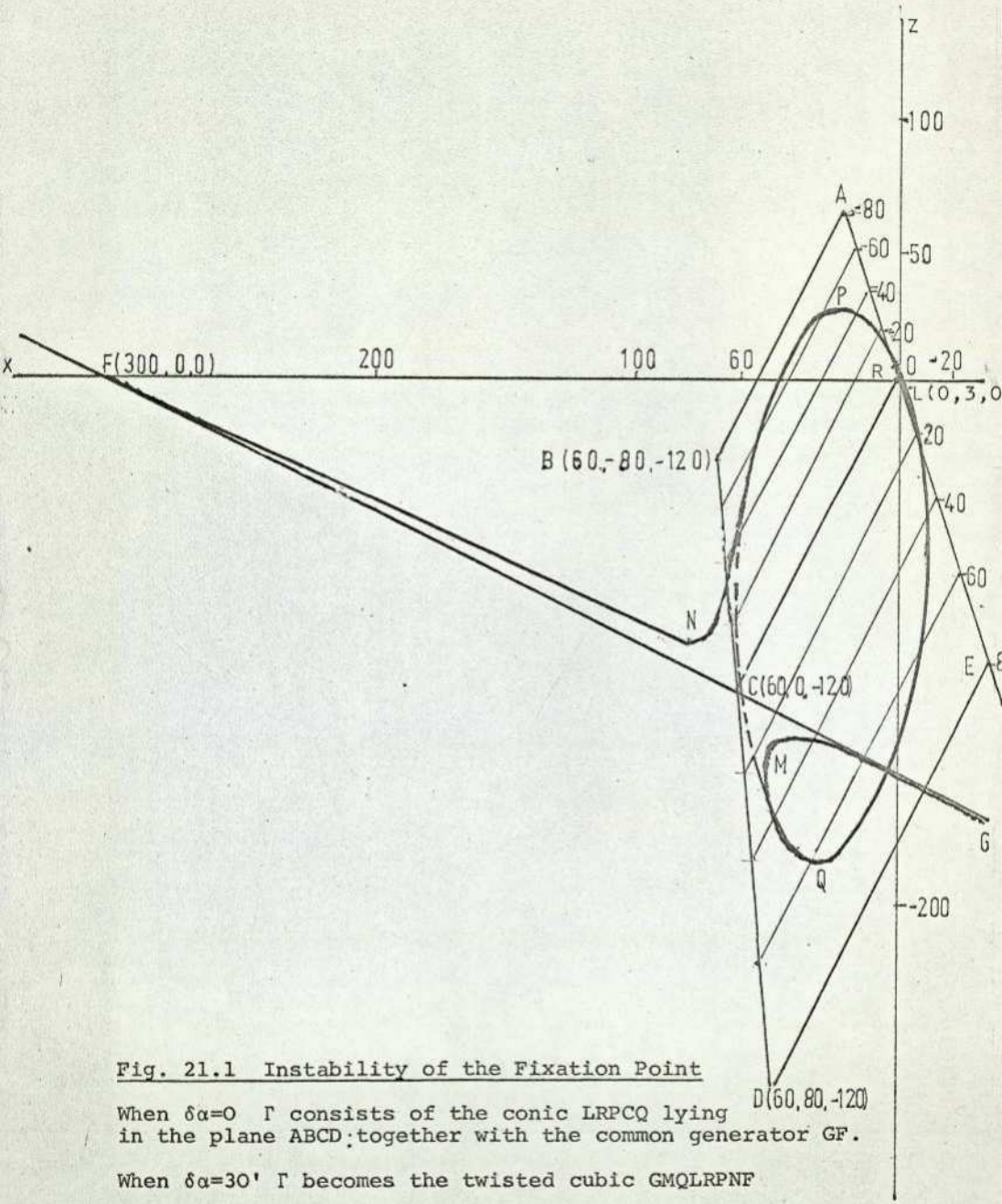


Fig. 21.1 Instability of the Fixation Point

When $\delta\alpha=0$ Γ consists of the conic LRPCQ lying in the plane ABCD together with the common generator GF.

When $\delta\alpha=30'$ Γ becomes the twisted cubic GMQLRPNF

$$ABCDE \equiv 2x + z = 0$$

$$GF \equiv \left.\begin{array}{l} x - 2z = 300 \\ y = 0 \end{array}\right\}$$

If we put $\delta\alpha = 30'$, we find that the horopter becomes the twisted cubic curve GMQLRPNF. The gap between M and N is found to subtend an angle of about 13° at the eye.

From this it would seem that as a consequence of the fluctuation of fixation the part of the conic shown dotted in fig. 21.1 would be indeterminate.

(iii) Comparison with experiment

There is surprisingly little actual experimental data on the horopter in the literature. However we have the following observations made by Ogle[339], for which a = 40.4cm b = 3.2cm.

lane	12°	8°	4°	2°	1°
mean reading (cm)	-5.5	-3.8	-2.3	-1.9	-1.7
deviation (cm)	±2.8	±1.6	±0.9	±0.6	±0.4

These figures are the mean positions and total deviation of 15 readings. Unfortunately we do not have the raw data nor the standard deviations. These points have been plotted in fig. 21.2.

The wide deviation in the 12° lane is an indication of the accuracy of the experimental method and is a measure of both the resolving power of the eye at this point of the visual field and of the instability of fixation.

The locus of parallel chords to the horopter in the horizontal plane for a small displacement of fixation through a small angle $\delta\alpha$ is given by (from 19.20-19.23)

$$y_1 = a(Kb^2+a'^2T)(t^2-1) \qquad 21.16$$

$$y_2 = -a^3\delta\alpha T(t^2-1) + a'^2b(T+K)t \qquad 21.17$$

$$y_3 = 0 \qquad 21.18$$

$$y_4 = (Kb^2+a'^2T)t^2 + Ka^2 \qquad 21.19$$

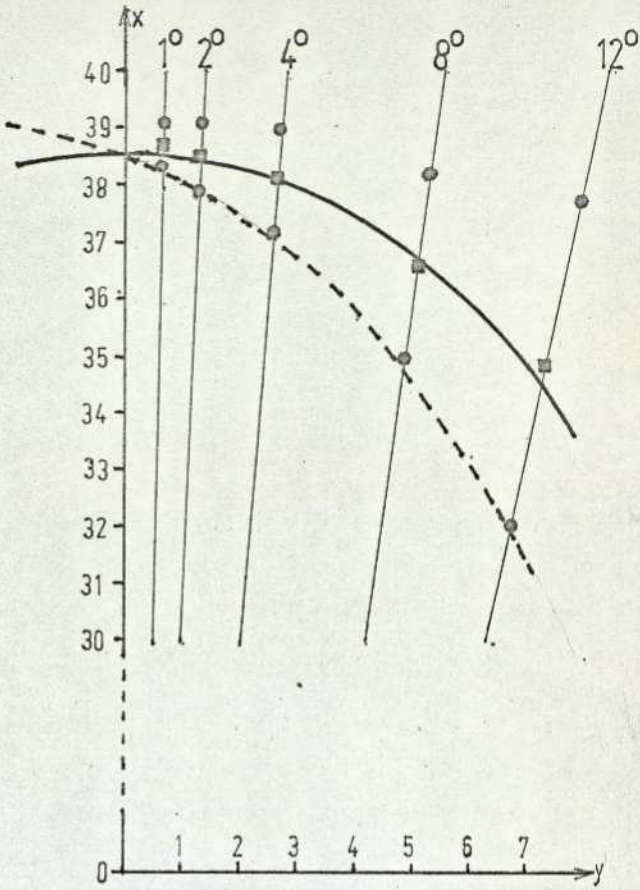


Fig. 21.2 The classical nonius experimental data

● experimental mean positions

■ experimental extreme positions

——— best conic through the experimental points

– – – conic rotated through 7.7°

which eliminating t can be written

$$ab^2Kx^2 + a(Kb^2+a'^2T)y^2 - 2a^3T\delta\alpha xy + b^2(Kb^2+a'^2T-Ka^2)x - ab^2(Kb^2+a'^2T) = 0 \qquad z = 0 \qquad 21.20$$

If we compare this equation with (19.29) the equation for the symmetric case, we see that the effect of the small angular displacement of the fixation direction by $\delta\alpha$ adds the term $-2a^3T\delta\alpha xy$ to the symmetric equation.

That is the conic (21.20) is the conic (19.29) rotated through an angle θ where

$$\tan 2\theta = \frac{2a^2\delta\alpha}{a'^2} \qquad 21.21$$

i.e. θ, the rotation of the conic is approximately equal to the displacement $\delta\alpha$.

Using a least squares method of curve fitting, the mean points of fig. 2.2 can be best fitted by the equation

$$x^2 + 2.62y^2 - 37.83x - 27.97 = 0 \qquad 21.22$$

this is plotted as the solid curve in fig. 21.2

Comparison of (21.20) and (21.22) suggest that the experimental curve is satisfied when

$$\delta\alpha = 0$$

$$\tau = 6.1^{\circ}$$

This angle of torsion is high for this fixation distance but is not impossibly high, see Allen and Carter (1967)[340].

If we put this value of the angle of torsion into 21.20 it becomes

$$x^2 + 2.62y^2 - 3.2\delta\alpha xy - 37.83x - 27.97 = 0 \qquad 21.23$$

The lower limit of the experimental curve can be seen from fig. 21.2 to be the point (32,6.8).

If (21.23) passes through this extreme point, we find

$$\delta\alpha = -0.134^r = -7.7^\circ$$

In this case (21.23) becomes

$$x^2 + 2.62y^2 + 0.43xy - 37.83x - 27.97 = 0 \qquad 21.24$$

which is plotted in fig. 2.2 as the dotted line.

It is unfortunate that the available experimental data does not permit a statistical analysis of the significance of the departure of a reading from the mean position. It can be seen however that a single set of readings could occur with the fixation shifted through 7° without that fact being revealed and it would take a large number of readings before any rotation of the fixation point could be demonstrated.

(iv) A proposed experiment

Fig. 21.3 illustrates a proposed experiment to determine the projections of corresponding retinal points.

ABCD is a screen, located about 1 metre from the two eyes R and L, and perpendicular to the median plane. On this screen there are placed small lamps one of which (F) is identified as the fixation target. One other lamp on the screen L_1 is illuminated and a polaroid filter is placed in front of it.

Another lamp (L_2) is placed half a metre away from the eyes and is freely moveable by the observer. This lamp is also shielded by a polaroid filter.

The polaroid filters are crossed with a pair worn by the observer who will see one lamp with the right eye only and one with the left.

The experiment will consist of the observer whose head must be rigidly fixed adjusting the position of L_2 until it appears to lie in the same direction as L_1. In this way we will be able to map the positions of corresponding retinal points and also to calculate the co-ordinates of the horopter points which will be located at the points (P) whenever the two lines RL_1 LL_2 intersect.

Care must be taken to ensure that fixation is maintained, and that fusion is not encouraged. The sensitivity could be increased by sequentially flashing the stimuli.

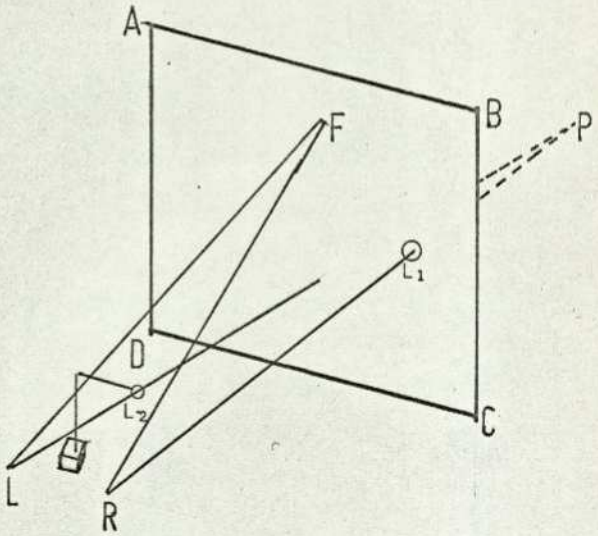


Fig. 21.3 Proposed Experimental Layout

Polaroid filters are so arranged that the light L_1 is seen by the right eye only and the light L_2 the left. While the eyes fixate the point F. The lamp L_2 is adjusted until it appears to lie in the same direction as L_1. In this way corresponding retinal points can be mapped on the plane ABCD. If the lines RL_1 and LL_2 intersect at P, then P is a point of the space horopter.

22 Conclusion

The horopter is the mathematical idealization of the psycho-physiological concept of corresponding retinal points.

The idea that there are points within the two eyes that must in some way correspond with each other is at least as old as Aristotle, and the concept of corresponding points and the horopter has been traced in outline from their early beginnings to the present day.

The original concept of the single perception of an external object as the criterion for the definition of the horopter was invalidated by Wheatstone's discovery of the ability of the eye to fuse disparate images.

The criterion that the horopter can be identified with the position of the apparent fronto-parallel plane appears to be based on no more than an assertion of Hering's, and there seems to be neither logical nor mathematical reasons for its consideration.

One must therefore agree with the current view that the only valid criterion for describing the horopter is that of the identity of visual direction in the sense defined in §4, above.

Except for Helmholtz, all previous workers on the horopter have limited their discussion to two dimensions. As well as leading to a misunderstanding of the true nature of the horopter curve, this restriction limits the interpretation of the horopter deviation in a most significant way.

If the discussion of the horopter is limited to two dimensions and corresponding points are identified by identical positions in the two eyes (see fig. 2.6), then as Vieth and Müller correctly surmized, the horopter must be identical with the Vieth-Müller circle. If, as was found, the horopter was not identical with this circle, then it followed that the corresponding points could not have identical co-ordinates in the two eyes; i.e. the relative retinal asymmetry (see: fig. 2.8).

It is only by extending the investigation to three dimensions that it is possible to preserve the relative structural symmetry of the two eyes, since, as the eyes rotate about their anterio-posterior axes, the positions of corresponding points can vary with respect to each other. This view is completely confirmed by equation (20.15) where we see that when the angle of torsion is zero, i.e. there is no asymmetry in the configuration, then the horopter curve does indeed reduce to the circle that Vieth and Müller predicted.

For this reason, we derived an expression for the angle of torsion of an eye in terms of its angles of azimuth and elevation, which could then be applied to the study of the horopter in three dimensions.

We introduced the notion of a nonius-line, and found that the concept of identity of visual direction induced an involution on such a line. The totality of all nonius-lines was called the rectilinear horopter. It was found to consist of a family of quadric surfaces with a common curve of intersection.

This curve, consisting exclusively of self-corresponding points, was the extension of the classical two-dimensional horopter curve to three dimensions. It was found to be in general a twisted curve in space and was named the space horopter.

Explicit expressions were found to describe both the rectilinear horopter and the space horopter, and their properties were explored in a variety of situations, both analytically and, by means of specially written computer programs, numerically.

By giving an explicit formula for the Hering-Hillebrand deviation coefficient, we have solved the two outstanding problems of horopter studies, namely, the linear variation of this coefficient with fixation distance, and the implied variation of retinal asymmetry with fixation. This analysis therefore confirms

retinal correspondence as being a cortical phenomenon.

The horopter has always been of interest to those who wished to understand how the two eyes functioned together. It has also been found to be a useful clinical tool for the study of such anomalies of binocular vision as aniseikonia, fixation disparity and anomalous retinal correspondence.

However, a firm conclusion from the preceding arguments is that previous workers have not been determining the horopter at all, but merely selected, experimentally, the points of intersection of certain nonius-lines with the horizontal plane, a locus that bears only a very indirect relationship to the true horopter.

It is hoped that as a result of the present work that a greater understanding of the horopter may prevail, and its utility extended further.

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APPENDIX

There are five specially written computer programs included in this appendix, that were used to determine the numerical results, and the graph plotter out-put included in §18.

For each of these programs, the input data was:

$$a,\ b,\ \alpha,\ \lambda,\ \varepsilon.$$

For

$$\lambda = 0$$

the quasi-symmetric situation was presumed to hold.

A description of each program precedes the listing.

found

1) the co-ordinates of the fixation point.

2) the matrix of the horopter coefficients

$$Q = (q_{ij})$$

3) the roots of the equations

$$x_i = 0$$

where $x_i = \Sigma q_{ij}^{(4-j)}$ $(i=1,4)$

4) the co-ordinates (x_i) for various values of t, converted them to non-homogeneous co-ordinates, and thus plotted the projections of Γ onto each of the co-ordinate planes.

see: figs. 18.2, 18.3, 18.4
18.10, 18.11, 18.12
18.14, 18.15, 18.16

```
      IMPLICIT REAL*8
      REAL LAMDA
      DIMENSION Q(4,4),T(40),XH(4),X(40),Y(40),Z(40),URUN(4),XR(3),P(3)
      COMMON/C1/PI,RM2,A,B,ALPHA,LAMDA,E1,KK,PIT
      CALL PROJECT(27HPROJECTIONS OF THE HONOPTER,27)
      PIT = 0.0174532925
      B = 3.0
   99 READ(1,901) A,ALPHA,LAMDA,E1,LAST
  901 FORMAT(4F5.2,I2)
      IF(LAST.EQ.1) GO TO 3'0
      KK = 1
      IF(LAMDA..EQ.0.0) KK = 0
      AL = ALPHA*PI
      BL = LAMDA *PIT
      P(1) = A*SIN(AL)*COS(BL)
      P(2) = - .*COS(AL)
      P(3) = - .*SIN(AL)*SIN(BL)
      WRITE(2,903) A,B,ALPHA,LAMDA,E1
  903 FORMAT(1H0,10X,4HA = ,F ,1,3X,4HB = ,F5.1,5X,8HALPHA = ,F .,1,5X,
     1    8HLAMDA = ,F5.1,5X,7HEPSILON,F5.1,1/)
      WRITE(2,904) P
  904 FORMAT(1H ,14HFIXATION POINT,3F15.5)
      CALL HMTQ
      WRITE(2,915)
  915 FORMAT(1H ,10X,21HHONOPTER COEFFICIENTS)
      DO 100 I = 1,4
      WRITE(2,920) (Q(I,J),J=1,4)
  920 FORMAT(1H ,4E20.7)
  100 CONTINUE
      WRITE(2,105)
  105 FORMAT(1H0,30X,6HHOUTS)
      DO 200 I = 1,4
      DO 150 J = 1,4
  150 XHOM(J) = Q(I,J)
      CALL HOUT(XHOM,XR,XI)
      WRITE(2,151) I,XR,XI
  151 FORMAT(1H0,1HA,I1,4E20.8)
  200 CONTINUE
      T(1) = 1.
      T(2) = -1.0
      IF(PIT.EQ.1.5/5) GO TO  25
      T(3) = PI
      GO TO 126
  125 T(3) = 0.7
  126 NX = 3
  127 DO 140 I = 1,NX
      DO 130 J = 1,4
  130 XH(J) = X(J)*(I)
      WRITE(2,930) T(I),XH
  930 FORMAT( X , 6.1,4E2 ,0)
      IF(XH(4)..EQ.0.0) GO TO 14
      X(I) = XH(1)/XH(4)
      Y(I) = XH(2)/XH(4)
      Z(I) = Z(3)/XH(4)
      IF(NX.NE.3) GO TO 140
      WRITE(2,934) T(I),X(I),Y(I),Z(I)
  934 FORMAT(1H ,4HE***,4F2 ,4)
  140 CONTINUE
      IF(NX.NE.3) GO TO 206
      T(1) = -2.0.3
      DO 205 I = 2,40
  205 T(I) = T(I-1) + 1.
      NX = 40
      GO TO 127
```

```
      930 FORMAT(1H1,28HNON-HOMOGENEOUS CO-ORDINATES)
          DO 210 I = 1,40
      210 WRITE(2 940) I,X(I),Y(I),Z(I)
      940 FORMAT(I4 ,4F20.6)

          N1 = 0
          DO 215    = 1,40
          IF(ABS(X(I)).GT.4.0.0) GO TO 215
          IF(ABS(Y(I)).GT.4.0.0) GO TO 215
          IF(ABS(Z(I)).GT.4.0.0) GO TO 215
          N1 = N1 + 1
          X(N1) = X(I)/20.0
          Y(N1) = Y(I)/20.0
          Z(N1) = Z(I)/20.0
      215 CONTINUE
          CALL H3PORIGIN(10.0,35.0)
          DO 300 I = 1,3
          IF(I.EQ.1) GO TO 22
          CALL H3PORIGIN(60.0,0.0)
      220 CALL H3PSYMBL(-10.0,25.,.1,79,19HHODROPT=K CURVE,0.0,19)
          CALL H3PSYMBL(-9.,-22.,.1,9,4HA = ,0.0,4)
          CALL H3PNUMBER(-5.0,-22.0,0.0,A,0.0,7.0)
          CALL H3PSYMBL(-9.,-24.,.1,9,5HALPHA = ,0.0,8)
          CALL H3PNUMBER(-1.0,-24.0,0.0,ALPHA,0.0,0.0,2.0)
          CALL H3PSYMBL(-9.,-26.,.1,9,5H-AMDA = ,0.0,8)
          CALL H3PNUMBER(-1.0,-26.0,0.0,LAMDA,0.0,0.0,2.0)
          GO TO(231,232,233),I
      231 CALL H3PAXIS (-10.0,0.0,24H                              X,-24,20.0,0.0,
         1-400.0,-10.0)
          CALL H3PAXIS (0.0,-10.0,24H                              Y,24,20.0,90.0,
         1-400.0,20.0)
          CALL H3PSYMBL(-10.0,-20.0,0.5,34HPROJECTION OF GAMMA ON PLANE Z =
         1 0.0,34)
          CALL H3PSCURV(X,Y,N1,0.0,0.0,0.0)
          GO TO 3.0
      232 CALL H3PAXIS (-10.0,-10.0,24H                            Y,24,20.0, 0.0,
         1-400.0,-10.0)
          CALL H3PAXIS (0.0,-10.0,24H                              Z,24,20.0,90.0,
         1-400.0,-10.0)
          CALL H3PSYMBL(-10.0,-20.0,0.5,34HPROJECTION OF GAMMA ON PLANE X =
         1 0.0,34)
          CALL H3PSCURV(Y,Z,N1,0.0,0.0,0.0)
          GO TO 3.0
      233 CALL H3PAXIS (-10.0,0.0,24H                              X,-24,20.0,0.0,
         1-400.0,-10.0)
          CALL H3PAXIS (0.0,-10.0,24H                              Z,24,20.0,90.0,
         1-400.0,-10.0)
          CALL H3PSYMBL(-10.0,-20.0,0.5,34HPROJECTION OF GAMMA ON PLANE Y =
         1 0.0,34)
          CALL H3PSCURV(X,Z,N1,0.0,0.0,0.0)
      300 CONTINUE
          CALL H3PORIGIN(20.0,-35.0)
          GO TO 95
      310 CALL H3PFINISH
          STOP
          END

END OF SEGMENT, LENGTH  1139, NAME  BV03
```

```
      FUNCTION X(I,T)
      DIMENSION Q(4,4)
      COMMON/CM1/Q
      X = Q(I,1)*T**3 + Q(I,2)*T**2 + Q(I,3)*T + Q(I,4)
      RETURN
      END
```

END OF SEGMENT, LENGTH 61, NAME X

```
      SUBROUTINE ROOT(A,XR,XI)
      DIMENSION A(4),XR(3),F(3)
      XI = 0.
      IF(ABS(A(1)).LT.1.E-9) GO TO 100
      CALL CUBIC(A,XR,XI)
      GO TO 200
  100 XR(1) = 1.E75
      IF(ABS(A(2)).LT.1.E-9) GO TO 150
      DO 110 I = 1,3
  110 F(I) = A(I+1)
      CALL QUAD(F,XR(2),XR(3),XI)
      GO TO 200
  150 XR(2) = 1.E75
      IF(ABS(A(3)).LT.1.E-9) GO TO 160
      XR(3) = -A(4)/A(3)
      GO TO 200
  160 XR(3) = 1.E75
  200 RETURN
      END
```

END OF SEGMENT, LENGTH 174, NAME ROOT

```
      READ FROM(MT,USERS EDIT.CUBE)
SUBFILE CUBE
SUBFILE CUBIC
      SUBROUTINE CUBIC(A,XR,XI)
      DIMENSION A(4),XR(3),AQ(3)
C     TO SOLVE CUBIC
C     A(1)*X**3+A(2)*X**2+A(3)*X+A(4)=0
      IPATH=2
      EX=1./3
      IF(A(1)) 106,104,106
  104 XR(1)=0
      GO TO 134
  106 A2=A(1)*A(1)
      Q=(27.*A2*A(4)-9.*A(1)*A(2)*A(3)+2.*A(2)**3)/(54.*A2*A(1))
      IF(Q) 110,108,114
  108 Z=0.
      GO TO 132
  110 Q=-Q
      IPATH=1
  114 P=(3.*A(1)*A(3)-A(2)*A(2))/(9.*A2)
      ARG=P*P*P-Q*Q
      IF(ARG) 116,118,120
  116 Z=-2.*SQRT(-P)*COS(ATAN(SQRT(-ARG)/Q)/3.)
      GO TO 128
  118 Z = -2.*Q**EX
      GO TO 128
  120 SARG = SQRT(ARG)
      IF(P) 122,124,126
  122 Z=-(Q+SARG)**EX-(Q-SARG)**EX
      GO TO 128
  124 Z=-(2.*Q)**EX
      GO TO 128
  126 Z=(SARG-Q)**EX-(SARG+Q)**EX
  128 GO TO (130,132),IPATH
  130 Z=-Z
  132 XR(1)=(-A(2)+Z)/(3.*A(1))
  134 AQ(1)=A(1)
      AQ(2)=A(2)+XR(1)*A(1)
      AQ(3)=A(3)+XR(1)*AQ(2)
      CALL QUAD(AQ,XR(2),XR(3),XI)
      RETURN
      END

END OF SEGMENT, LENGTH   399, NAME  CUBIC
```

```
      END OF SUBFILE
      SUBFILE QUAD
            SUBROUTINE QUAD(A,XR1,XR2,XI)
      C     TO SOLVE QUADRATIC
      C     A(1)XSQ+A(2)X+A(3)=0
            DIMENSION A(3)
            X1 = -A(2)/(2.*A(1))
            DISC=X1*X1-A(3)/A(1)
            IF(DISC) 210,220,220
        210 X2 = SQRT(-DISC)
            XR1 = X1
            XR2 = X1
            XI  = X2
            GO TO 230
        220 X2 = SQRT(DISC)
            XR1 = X1+X2
            XR2 = X1-X2
            XI  = 0.
        230 RETURN
            END

END OF SEGMENT, LENGTH  112, NAME  QUAD
```

```
      END OF SUBFILE
      END OF SUBFILE
      READ FROM(MT,USERS EDIT.QMATRIX)
SUBFILE QMATRIX
      SUBROUTINE MAIN
      REAL LAMDA
      DOUBLE PRECISION A,B,AL,EL,E,AB,A1(2),AA(2),BAR,S,C,B1(2),L(2),
     1M(2),ST(2),CT(2),J1,J2,K1,K2,TK,TO,M1,P2,R1,R2,AD1,AD2,A1S,A2S,T1,
     2X1,X2
      DIMENSION D(4,4)
      COMMON/M1/D/CM2/AS,BS,ALPHA,LAMDA,EPS,K,PPT
      PIT = 0.3174532925
      A = AS
      B = BS
      AL = ALPHA*PIT
      EL = LAMDA*PIT
      E0 = EPS*PIT
      E = TAN(E0)
      AB = A**2 + B**2
      D = 2.0*A*B*DCOS(AL)
      A1(1) = DSQRT(AB + D)
      A1(2) = DSQRT(AB - D)
      A(1) = (A1(2)+A1(1))/2.0
      A(2) = (A1(2)-A1(1))/2.0
      BAR = 0.5*DATAN2(2. *A*B*DSIN(AL),A*A-B*B)
      S = DSIN(BAR)
      C = DCOS(BAR)
      B1(1) = A*D.OS(AL) + B
      B1(2) = A*D.OS(AL) - B
      DO 100 I = 1,2
      L(I) = A1(I) + A*DSIN(AL)*DCOS(EL)
      M(I) = A*DSIN(AL) + A1(I)*DCOS(EL)
      B(I) = B1(I)*DSIN(EL)/L(I)
  100 ST(I) = M(I)/L(I)
C     A .NE. 1 SYMMETRIC APPROXIMATION
      IF(A.EQ.1) GO TO 10
      J1 = -1.0
      J2 = 1.
      K1 = -1.0
      K2 = 1.
      TK = 0.2
      TO = 0. 2
      GO TO 12
  101 TK = ST(1)/CT(1)
      TO = (TK+E)/(1.0-TK*E)
      J1 = ST(2)/ST(1)
      J2 = CT(2)/CT(1)
      K1 = (J1*TK-J2*E)/(TK+E)
      K2 = (J2+J1*TK*E)/(1.0+TK*E)
  102 P1 = B*A1(2) - B*B1(1)*J1
      P2 = B*A1(2) - B*B1(1)*K2
      R1 = 2. *S*C*K2*DCOS(EL)
      R2 = 2. *S*C*K2*DSIN(EL)
      AD1 = (A1(2) - A1(1)*K2)/2.0
      AD2 = (A1(2) + A1(1)*K2)/2.0
      IF(ABS( D1) LT.1.E-3) GO TO 105
      PPT = P1*D2/AD1
      GO TO 1 6
  105 PPT = 1.E75
  106 A1S = (AA(1)*L)**2
      A2S = (AA(2)*S)**2
      T1 = TO**2
```

```
      A(1,1) = -R1*(K2-K1)*M1*T1 + K1*(K2-J2)*P2 - K2*(K2-J1)*B*ADI*TK
      A(1,2) =  R1*(K2-K1)*P1*T1 + R1*(K2+J2)*P2 - K2*(K2-J1)*B*AD2*TK
      A(1,3) = - A(1,1)
      A(1,4) = -A(1,2)
      A(2,1) =  (K2-K1)*(1.+K2)*(K2-J1)*A1C*T1
     1         -(K2-K1)*(1.-K2)*(K2+J1)*A2S*T1
     2         -2.0*K2*(1.+K2)*(K2*J2)*A2S
      A(2,2) =((K2-K1)*(1.+K2)*(K2-J1)-2.0*KM*(1.+K2))*A1C*T1
     1         -(2. *KP*(1.-K2) + (K2-K1)*(K2+J1)*(1.+K2))*A2S*T1
     2         -2.0*K2*(K2+J2)*(1.+K2)*A2S - 2.0*K2*(K2-J2)*(1.-K2)*B**2
      A(2,3) = ((K2+K1)*(K2-J1)*(1.+K2) + 2.0*KM*(1.-K2))*A1C*T1
     1         +(2.0*KP*(1.-K2) - (K2+K1)*(K2-J1)*(1.-K2))*A2S*T1
     2        -2.0*K2*(K2-J2)*(1.+K2)*A1C + 2.0*K2*(K2-J2)*(1.-K2)*B**2
      A(2,4) = (K2+K1)*(1.+K2)*(K2+J1)*A1C*T1
     1         -(K2+K1)*(1.+K2)*(K2-J1)*A2S*T1
     2         +2.0*K2*(1.-K2)*(K2*J2)*A1C
      A(3,1) = R2*(K2-K1)*P1*T1 + R2*(K2-J2)*P2 - R1*(K2-J1)*B*ADI*TK
      A(3,2) = -R2*(K2+K1)*P1*T1 + K2*(K2+J2)*P2 - R1*(K2-J1)*B*AD2*TK
      A(3,3) = -A(3,1)
      A(3,4) = -A(3,2)
      A(4,1) = -B*(K2-K1)*(K2-J1)*(1.+K2)*S*S*T1
     1         +B*(K2-K1)*(K2-J1)*(1.-K2)*C*C*T1
     2         -2.0*B*K2*(1.+K2)*(K2*J2)*S*S
      A(4,2) =  2.0*B*(K2-J1)*(K2*K2+K1)*C*C*T1
     1         +2.0*B*(1.+K2)*KP*S*S*(1. + B*(1.-K2)*(K2+J1)*(K2-K1)*T1
     2         -2.0*B*K2*(1.+K2)*(K2*J2) + 2.0*B*K2*(1.+K2)*(K2.J2)*S*S
      A(4,3) = -2.0*B*K2*(K2-K1)*(1.+J1)*C*C*T1
     1         -B*(1.+K2)*(K2-K1)*(K2-J1)*T1 + 2.0*B*(1.-K2)*KP*T1
     2         -2.0*B*K2*(1.+K2)*(K2*J2)*C*C + 2.0*B*K2*(1.-K2)*(K2*J2)
      A(4,4) =  2.0*B*K2*(K2+K1)*(1.+J1)*C*C*T1
     1         +2.0*B*K2*(K2+J2)*(1.+K2)*C*C
     2         -B*(1.-K2)*(K2-K1)*(K2-J1)*T1
      RETURN
      END

END OF SEGMENT, LENGTH  2320, NAME  MATQ
```

A = 3 0.0 B = 3.0 ALPHA = 70.0 LAMDA = 50.0 EPSILON 1.

FIXATION POINT 244.13 334 -102.686843 -140.953893

HURWITZ COEFFICIENTS

-0.6965964E-03	0.8046193E 00	0.6965964E-03	-0.8028193E 00
-0.3547435E-04	0.1126846E 00	0.7535569E 02	-0.1202412E-01
0.7118261E-03	0.2519443E 00	-0.7118261E-03	0.2515443E 00
-0.5295853E-05	0.3169796E-02	-0.1027967E 00	0.2434392E 02

ROOTS

X1	0.6728418YE 03	0.1 00000-E 01	-0.100.0000E 01	0.000000000E 00
X2	-1.900.757/E 04	0.2869812/E 03	0.18085791E-03	0.000000000E 00
X3	0.35337c6 E 03	0.1 0000097E 01	-0.1000000VE 01	0.000000000E 00
X4	0.2516777nE 01	0.2977099YE 01	0.2277895OE 01	0.13512721E 03

	0.0 0 0000E 0	0.7272594E 02	0.0000000E 00	0.2424277E 02
****	1.0000 00	0.000000	2.999384	0.00000
	0.0 0 0000E 0	-0.7307913E 02	0.0000000E 00	0.2444885E 02
****	-1.000000	0.000000	-3.000000	0.00000
	0.2588933IE 00	-0.1000000E 00	-0.1496721E 04	0.1186043E 04
****	-4.6 6 939277	244.139304	-102.686857	-140.953893
	0.3124474E 05	-0.2677439E 05	-0.1572594E 05	0.1414869E 05
	0.2994/599E 05	-0.2487/6133E 05	-0.1396277GE 05	0.1291865E 05
	0.2475955E 05	-0.2304453E 05	-0.1239101E 05	0.1176992E 05
	0.2189574E 05	-0.2123973E 05	-0.1076645E 05	0.1078549E 05
	0.191 3887E 05	-0.1994888E 05	-0.9354487E 04	0.9722218E 02
	0.1656871E 05	-0.1793910E 05	-0.8061804E 04	0.8618927E 02
	0.1427479E 05	-0.1622846E 05	-0.6885167E 04	0.7966432E 02
	0.1216764E 05	-0.1467212E 05	-0.5811465E 04	0.7427559E 02
	0.1 2292E 05	-0.1319522E 05	-0.4851935E 04	0.6937195E 02
	0.9486782E 04	-0.1177303E 05	-0.3999796E 04	0.6512201E 02
	0.6924897E 04	-0.1048076E 05	-0.3228459E 04	0.6148943E 02
	0.5535717E 04	-0.9366211E 04	-0.2550111E 04	0.5844801E 02
	0.4316426E 04	-0.8311112E 04	-0.1974029E 04	0.5596394E 02
	0.326 6015E 04	-0.7382514E 04	-0.1478421E 04	0.5400559E 02
	0.2663197E 04	-0.6493054E 04	-0.1059619E 04	0.5254067E 02
	0.1618475E 04	-0.5642532E 04	-0.7179517E 03	0.5155826E 02
	0.1 212044E 04	-0.4831201E 04	-0.4477478E 03	0.5099597E 02
	0.566 1577E 03	-0.4059258/E 04	-0.2453023E 03	0.5078215E 02
	0.247479?5E 03	-0.1361951?E 04	-0.1064653E 03	0.5085057E 02
	0.6 5n6743E 02	-0.7815181E 03	-0.2560758E 02	0.5112904E 02
	-5.6 7819326 0	-0.1204120E-01	0.2515442E 00	0.2434392E 02
	0.5a7 6480E 02	0.9991848IE 03	-0.2419817E 02	0.2344833E 02
	0.2335? 37E 03	0.1582915E 04	-0.9460578E 02	0.2278837E 02
	0.5177583E 03	0.1907981E 04	-0.2085493E 03	0.2233825E 02
	0.9 A9617E 03	0.2221128E 04	-0.3566938E 03	0.2206420E 02
	0.1394417E 04	0.2566191E 04	-0.5396868E 03	0.2193449E 02
	0.1979935?E 04	0.2929924E 04	-0.7913980E 03	0.2191721E 02
	0.2465742E 04	0.3273132?E 04	-0.9884607E 03	0.2198072E 02
	0.3A9549?E 04	0.3634598E 04	-0.1249238?E 04	0.2209320E 02
	0.4228695E 04	0.4020107?E 04	-0.1519397E 04	0.2222873E 02
	0.5?31?e36E 04	0.4159446?E 04	-0.1804436?E 04	0.2233795E 02
	0.6 . 239?E 04	0.4224078E 04	-0.2096071?E 04	0.2240678 E 02
	0.713 7643E 04	0.4284?7?2E 04	-0.2394?56E 04	0.2243978E 02
	0.8217337E 04	0.4159329?E 04	-0.2685?059E 04	0.2227800?E 02
	0.9354?208E 04	0.4029855?E 04	-0.2978?647E 04	0.2201710?E 02
	0.1 536953E 05	0.3853?1692E 04	-0.3257?179E 04	0.2158257?E 02
	0.1175925?E 05	0.3572?02?5E 04	-0.3526?755E 04	0.2094?983E 02
	0.1 1?1?E 05	0.3?82?34?E 04	-0.3?7?3?65E 04	0.2?1?6?3?E 02

NON-HOMOGENEOUS CO-ORDINATES

-200.000000	221.616169	-189.145714	+111.310628
-190.000000	216.949436	-192.512792	+108.081972
-180.000000	210.352994	-195.621732	+104.512336
-170.000000	203.873025	-198.378057	+100.569460
-160.000000	196.497148	-200.662831	-96.220865
-150.000000	188.146157	-202.327937	-91.445488
-140.000000	178.737431	-203.255077	-86.195713
-130.000000	168.199983	-203.371059	-80.450773
-120.000000	156.478853	-202.696767	-74.220958
-110.000000	143.551767	-198.799674	-67.593309
-100.000000	129.447928	-194.071452	-60.328854
-90.000000	114.652494	-187.182948	-52.754933
-80.000000	98.191912	-177.888421	-44.941869
-70.000000	81.205916	-165.615976	-36.905463
-60.000000	64.871924	-150.381025	-28.081777
-50.000000	48.257336	-131.954043	-21.395002
-40.000000	32.979990	-110.393359	-14.459344
-30.000000	19.862154	-85.796591	+8.520940
-20.000000	9.179194	-58.734420	+3.929824
-10.000000	2.373765	-29.846143	+1.003711
0.000000	0.024763	-0.000494	0.010333
10.000000	2.507491	29.820839	+1.032066
20.000000	10.240451	58.622593	+4.105004
30.000000	23.177900	85.502123	+9.263944
40.000000	41.067440	109.731027	-16.166043
50.000000	63.971754	130.807256	-24.643601
60.000000	90.194213	148.464312	-34.292505
70.000000	120.366473	162.648958	-44.957975
80.000000	153.828977	173.474147	-56.362755
90.000000	190.485723	181.169547	-68.325988
100.000000	229.792974	186.026287	-80.734125
110.000000	272.290092	188.334719	-93.546647
120.000000	317.576899	188.458008	+106.800157
130.000000	366.254248	186.566916	+120.614809
140.000000	424.876935	182.897879	+135.247444
150.000000	488.215735	177.558468	+150.917471
160.000000	561.491421	170.560235	+168.259352
170.000000	648.854706	161.775979	+187.992372
180.000000	755.868924	150.868746	+211.340871
190.000000	893.380449	137.157979	+240.244852

A = 3.0.0 B = 3.0 ALPHA = 45.0 LAMDA = 0.0 EPSILON 0.

FIXATION POINT 212.132034 -212.132035 0.000000

HURWITZIAN DEFINITENTS

-3.1469.6.8E-33	0.5091169E 00	0.1440000E-01	-0.5091169E 00
0.1439.5.6E-01	0.0000000E 00	0.7201440E 02	0.0000000E 00
-0.359941.E-32	-0.5091296E 00	0.3599910E-02	0.5091296E 00
0.0000.0.E 0	0.1650006E 01	0.0000000E 00	0.2399880E 02

ROOTS

X1	0.3933333E 02	0.1100000E 01	-0.1000000E 01	0.0000000E 0
X2	0.0000000E 0	0.0 0000000E 0	0.0000000E 00	0.70721286E 02
X3	-0.1414284E 03	0.1100000E 01	-0.1000000E 01	0.0000000E 0
X4	0.1000000E 76	0.0 0000000E 0	0.0000000E 00	0.4713914E 02
1,0	0.0000000E 0	0.7204000E 02	0.0000000E 00	0.2400960E 02
****	1.0.0.0	0.000000	3.00000-	0.00000
-1,0	0.0.0.0E 0	-0.7204000E 02	0.0000000E 00	0.2400960E 02
****	-1.0.0000	0.000000	-3.00000-	0.00000
-141,4	0.5.91627.E 05	-0.5091627E 05	0.1152120E-0A	0.2400215E 03
****	-141.428424	212.132034	-212.132034	0.00000
-400,0	0.1853612E 05	-0.1292916E 05	0.5430864E 04	0.4568012E 03
-190,0	0.1171454E 04	-0.1129424E 05	0.6314289E 04	0.4138859E 03
-180,0	0.1.047309E 05	-0.9594499E 05	0.4496736E 04	0.3739207E 03
-170,0	0.8548572 E 05	-0.8295257E 05	0.2972489E 04	0.3361285E 03
-160,0	0.7201.2979E 05	-0.7049886E 05	0.1711446E 04	0.3004803E 03
-150,0	0.5.52460E 05	-0.5934730E 05	0.6944915E 03	0.2670001E 03
-140,0	0.4041.766E 05	-0.4951665E 05	-0.1007821E 03	0.2356799E 03
-130,0	0.4 2386.4E 05	-0.4099560E 05	-0.6952470E 03	0.2065195E 03
-120,0	0.3202.2845E 05	-0.3354244E 05	-0.1111744E 04	0.1795196E 03
-110,0	0.2632.621E 05	-0.2705.069E 05	-0.1366874E 04	0.1546795E 03
-100,0	0.1946922.E 05	-0.2148000E 05	-0.1491257E 04	0.1319994E 03
-90,0	0.1461.642E 05	-0.1697.846E 05	-0.1499433E 04	0.1114792E 03
-80,0	0.1062.487E 05	-0.1343215E 05	-0.1410644E 04	0.9311918E 02
-70,0	0.7433.356E 04	-0.9977142E 04	-0.1259708E 04	0.7691909E 02
-60,0	0.4941.675E 04	-0.7435931E 04	-0.1054992E 04	0.6287901E 02
-50,0	0.3.715631E 04	-0.5405401E 04	-0.8225614E 03	0.5099595E 02
-40,0	0.17391.19E 04	-0.3806839E 04	-0.5838408E 03	0.4127889E 02
-30,0	0.846.6427E 03	-0.2549193E 04	-0.3606179E 03	0.3371885E 02
-20,0	0.318.4963E 03	-0.1555476E 04	-0.1744143E 03	0.2831682E 02
-10,0	0.6465587E 02	-0.7343425E 03	-0.4683492E 02	0.2507886E 02
0,0	0.5.911.68E 01	0.0000000E 00	0.5891296E 00	0.2399880E 02
10,0	0.3414.857E 02	0.7334925E 03	-0.5390774E 02	0.2507886E 02
20,0	0.8525.539E 02	0.1555476E 04	-0.2315764E 03	0.2831682E 02
30,0	0.1.4328.7E 03	0.2549193E 04	-0.5547975E 03	0.3371885E 02
40,0	-0.1.496611E 03	0.3806839E 04	-0.1048485E 04	0.4128896E 02
50,0	-0.5264991E 03	0.5405401E 04	-0.1722123E 04	0.5099895E 02
60,0	-0.1277.243E 04	0.7435931E 04	-0.2607922E 04	0.6287901E 02
70,0	-0.2644.204E 04	0.9977142E 04	-0.3728743E 04	0.7691904E 02
80,0	-0.4113.191E 04	0.1343215E 05	-0.5155785E 04	0.9311918E 02
90,0	-0.5837.966E 04	0.1697.846E 05	-0.6747451E 04	0.1114729E 03
100,0	-0.7837.033E 04	0.2148000E 05	-0.8690338E 04	0.1319994E 03
110,0	-0.1011.5.1E 05	0.2705.069E 05	-0.1092104E 05	0.1546795E 03
120,0	-0.1255.698E 05	0.3354244E 05	-0.1353117E 05	0.1795196E 03
130,0	-0.2301.362E 05	0.4099559E 05	-0.1651235E 05	0.2065195E 03
140,0	-0.2958.402E 05	0.4951665E 05	-0.1985.685E 05	0.2356799E 03
150,0	-0.3714.321E 05	0.5934730E 05	-0.2356485E 05	0.2670001E 03
160,0	-0.4594.721E 05	0.7044880E 05	-0.2777756E 05	0.3004834E 03
170,0	-0.5621.215E 05	0.8295257E 05	-0.3234980E 05	0.3361285E 03

NON-HOMOGENEOUS CO-ORDINATES

-200.0	297.282739	-264.190832	18.495312
-190.0	283.041962	-271.678257	15.254851
-180.0	268.781395	-259.239414	12.081256
-170.0	254.247247	-246.883081	8.843284
-160.0	239.659939	-234.620465	5.695702
-150.0	224.919956	-222.461674	2.690183
-140.0	209.987146	-210.419910	-0.427623
-130.0	194.848340	-198.508416	-3.356491
-120.0	179.435752	-186.734464	-6.187319
-110.0	163.723162	-175.118878	-8.849749
-100.0	147.644474	-163.637109	-11.297301
-90.0	131.182225	-152.295974	-13.450304
-80.0	114.169470	-141.086019	-15.196165
-70.0	96.629926	-129.743462	-16.377064
-60.0	78.992794	-118.178584	-16.778139
-50.0	60.227987	-105.895123	-16.127904
-40.0	42.033831	-92.187269	-14.143983
-30.0	25.091721	-75.601419	-10.694846
-20.0	11.231933	-54.927499	-6.158993
-10.0	2.278416	-29.289376	-1.867709
0.0	-0.021414	0.000000	0.021213
10.0	1.441319	29.289376	2.151926
20.0	8.118941	54.927499	6.187840
30.0	20.050483	75.601419	10.453616
40.0	-2.290918	92.187269	25.299818
50.0	-10.338989	105.895123	33.787826
60.0	-20.812818	118.178584	41.503862
70.0	-31.774914	129.743462	48.476179
80.0	-44.177898	141.086019	54.776965
90.0	-57.187269	152.295974	60.526581
100.0	-70.914716	163.637109	65.836189
110.0	-84.077131	175.118878	70.796273
120.0	-97.764766	186.734464	75.485712
130.0	-111.921917	198.508416	79.955115
140.0	-126.411961	210.419910	84.250179
150.0	-141.113102	222.461674	88.404582
160.0	-156.913945	234.620465	92.444865
170.0	-168.701458	246.883081	96.391262
180.0	-190.474756	259.239414	100.260063
190.0	-174.230370	271.678257	104.064144

A = 300.0 B = 0.0 ALPHA = 90.0 LAMDA = 0.0 EPSILON 0.

FIXATION POINT 600.0 000 -0.000000 0.000000

AUROPTEN COEFFICIENTS

-0.21817936E-11	0.11994000E 00	-0.71992000E 00
0.92912248E-19	0.00000000E 00	0.00000000E 00
-0.11990371E-26	-0.71996400E 00	0.71996400E 00
0.00000 0 E 00	0.11997600E 01	0.23997600E 02

ROOTS

X1	0.100000 0 E 76	0.1 00000E 01	-0.100 0000E 01	0.00000000E 00
X2	0.100000 0 E 76	0.1 00000E 76	0.000 0000E 00	0.0000 000E 00
X3	0.100000 0 E 76	0.1 00000E 01	-0.100 0000E 01	0.00000000E 00
X4	0.100000 0 E 76	0.0 00000E 01	0.000 0000E 00	0.44719971E 02

1.0	0.000000 0E 0	0.72026880E 02	0.00000000E 00	0.24000960E 02
*****	1.000 00	0.000000	3.00000	0.000000
-1.	0.000000 0E 0	-0.72026880E 02	0.00000000E 00	0.24000960E 02
*****	-1.000 00	0.000000	-3.00000	0.00000
*-557931744 4.5	0.11206941E 20	-0.20087870E 14	0.17179689E 11	0.3736c471E 20
*****	55795174434.493333	900.000000	-0.000001	0.00000
*400.0	0.2877640 E 05	-0.1440 5760E 05	-0.2877 640E 05	0.5039680E 03
*390.0	0.2391 881E 05	-0.1360 5472E 05	-0.2595 936E 05	0.4571609E 03
*380.0	0.2332 498E 05	-0.1290 5184E 05	-0.2332 114E 05	0.4127689E 03
*370.0	0.2 5 .820 E 05	-0.1221 4896E 05	-0.2070 240E 05	0.3707900E 03
*360.0	0.1842943E 05	-0.1152 4608E 05	-0.1843 035E 05	0.3311914E 03
*350.0	0.1619766E 05	-0.1083 4320E 05	-0.1619 647E 05	0.2939922E 03
*340.0	0.141 9869E 05	-0.1014 4032E 05	-0.1411 057E 05	0.2591929E 03
*330.0	0.12166 64E 05	-0.9387 7440E 04	-0.1216 672E 05	0.2267935E 03
*320.0	0.1038 268E 05	-0.8648 3600E 04	-0.1038 762E 05	0.1967941E 03
*310.0	0.8717 4 9E 04	-0.7921 6960E 04	-0.8711 845E 04	0.1691947E 03
*300.0	0.7198 3 0E 04	-0.7205 0880E 04	-0.7199 201E 04	0.1439952E 03
-290.0	0.583 697 E 04	-0.6498 9920E 04	-0.5833 984E 04	0.1211956E 03
-280.0	0.46 08 93E 04	-0.5764 3040E 04	-0.4607 496E 04	0.1007960E 03
-270.0	0.3926927E 04	-0.5044 0160E 04	-0.3527 136E 04	0.8279624E 02
-260.0	0.2591 20E 04	-0.4327 7280E 04	-0.2591 504E 04	0.6719673E 02
-250.0	0.17991 0 E 04	-0.3601 4400E 04	-0.1799 1900E 04	0.5599700E 02
-240.0	0.1151 1 4E 04	-0.2881 1520E 04	-0.1151 2224E 04	0.4419721E 02
-230.0	0.6472 129E 03	-0.2161 8640E 04	-0.6472 4704E 03	0.3479736E 02
-220.0	0.2872 5127E 03	-0.1440 5760E 04	-0.2872 6504E 03	0.2879750E 02
-210.0	0.7127 2873E 02	-0.7205 8880E 03	0.7127 6436E 02	0.2519757E 02
*0.0	*0.7199 2801E 0	0.000 0000E 00	0.7199 6400E 00	0.2399760E 02
10.0	0.7127 2873E 02	0.7205 8880E 03	-0.7127 6436E 02	0.2519757E 02
20.0	0.2872 5127E 03	0.1440 5760E 04	-0.2872 6504E 03	0.2879750E 02
30.0	0.6472 1528E 03	0.2161 8640E 04	-0.6472 4704E 03	0.3479736E 02
40.0	0.1151 1 4E 04	0.2881 1520E 04	-0.1151 2224E 04	0.4419721E 02
50.0	0.1799 1 0E 04	0.3601 4400E 04	-0.1799 1900E 04	0.5599700E 02
60.0	0.2591 20E 04	0.4327 7280E 04	-0.2591 504E 04	0.6719673E 02
70.0	0.3526 927E 04	0.5044 0160E 04	-0.3527 136E 04	0.8279624E 02
80.0	0.4607 893E 04	0.5764 3040E 04	-0.4607 496E 04	0.1007960E 03
90.0	0.583 697E 04	0.6498 9920E 04	-0.5833 984E 04	0.1211956E 03
100.0	0.7198 3 0E 04	0.7205 0880E 04	-0.7199 201E 04	0.1439952E 03
110.0	0.8717 4 9E 04	0.7921 6960E 04	-0.8711 845E 04	0.1691947E 03
120.0	0.1 366 43E 05	0.8648 3600E 04	-0.1036 762E 05	0.1967941E 03
130.0	0.1216 6.63E 05	0.9387 7440E 04	-0.1216 672E 05	0.2267935E 03
140.0	0.1419 869E 05	0.1008 4032E 05	-0.1411 057E 05	0.2591929E 03
150.0	0.1619 766E 05	0.1083 4320E 05	-0.1619 647E 05	0.2939922E 03
160.0	0.1842943E 05	0.1154 4608E 05	-0.1843 035E 05	0.3311914E 03

NON-HOMOGENEOUS CO-ORDINATES

-200.0 0.00	57.157476	-28.583738	-57.199932
-190.0 0.00	56.544211	-29.933953	-56.847352
-180.0 0.00	56.939827	-31.408981	-56.508451
-170.0 0.00	56.110966	-33.023744	-56.113170
-160.0 0.00	55.048872	-34.797419	-55.648655
-150.0 0.00	55.699265	-36.750363	-55.098299
-140.0 0.00	54.457715	-38.905911	-54.440437
-130.0 0.00	53.043782	-41.287524	-53.646464
-120.0 0.00	52.675969	-43.921367	-52.670202
-110.0 0.00	51.481972	-46.828702	-51.484146
-100.0 0.00	49.991968	-50.021967	-49.994167
-90.0 0.00	48.139785	-53.488850	-48.112190
-80.0 0.00	45.784857	-57.167948	-45.798642
-70.0 0.00	42.997380	-60.896944	-42.599710
-60.0 0.00	38.958731	-64.314952	-38.560659
-50.0 0.00	33.618219	-66.697938	-33.320165
-40.0 0.00	26.649453	-66.697931	-26.650385
-30.0 0.00	18.999038	-62.098461	-18.600468
-20.0 0.00	9.974967	-50.024339	-9.975368
-10.0 0.00	2.828961	-28.585607	-2.828702
0.0 0.00	-0.000000	0.000000	0.000001
10.0 0.00	2.828961	28.585607	-2.828702
20.0 0.00	9.974967	50.024339	-9.975368
30.0 0.00	18.999038	62.098461	-18.600468
40.0 0.00	26.649453	66.697931	-26.650385
50.0 0.00	33.618219	66.697938	-33.320165
60.0 0.00	38.958731	64.314952	-38.560659
70.0 0.00	42.997380	60.896944	-42.599710
80.0 0.00	45.784857	57.167948	-45.798642
90.0 0.00	48.139784	53.488850	-48.112190
100.0 0.00	49.991967	50.021967	-49.994167
110.0 0.00	51.481972	46.828702	-51.484146
120.0 0.00	52.675969	43.921367	-52.670202
130.0 0.00	53.643781	41.287524	-53.646464
140.0 0.00	54.457714	38.905911	-54.440437
150.0 0.00	55.099243	36.750363	-55.098299
160.0 0.00	55.648970	34.797419	-55.648655
170.0 0.00	56.110964	33.023744	-56.113170
180.0 0.00	56.508925	31.408981	-56.508451
190.0 0.00	56.844209	29.933953	-56.847352

EXECUTION ERROR 100 PROGRAM TERMINATED

found

1) the matrix Q
2) the parameter t_∞
3) $|Q|$
4) Q^{-1}
5) the coefficients of the equation of the asymptote plane.
6) the equation of the asymptote
7) the coefficients of the equation of the locus of parallel chords

```
      MASTER GW.4
      REAL LAMDA
      DIMENSION Q(4,4),QROW(4),TR(3),W2(3),U(4),T(3),ID(2),UU(2,4),
     1TR(4,2)
      COMMON/CM1/W2,U,A,B,ALPHA,LAMDA,E1,KK,PI
      PI = 3.0
   98 READ(1,900) A,ALPHA,LAMDA,E1,LAST
  900 FORMAT(4F6.2,12)
      IF(LAST EQ.1) GO TO 3:0
      KK = 1
      IF(LAMDA..EQ.0.0) KK = 0
      WRITE(2,905) A,U,ALPHA,LAMDA,E1
  905 FORMAT(1H1,6HA = ,F7.1,5X,4HB = ,F5.1,5X,8HALPHA = ,F5.1,5X,
     1   8HLAMDA = ,F5.1,5X,7HEPSILON,F5.1)
      CALL MATD
      WRITE(2,915)
  915 FORMAT(1H0,13X,21HHOROPTER COEFFICIENTS)
      DO 120 N1 = 1,4
      WRITE(2,920) (Q(N,N1),N1=1,4)
  920 FORMAT(1H ,4E20.7)
  120 CONTINUE
      DO 150 J = 1,4
  150 QROW(J) = Q(4,J)
      CALL MOUT(QROW,TR,TI)
  154 NOR = 3
      IF(TI.NE.0.0) NOR = 1
      IF(NOR.EQ.3) GO TO 156
      WRITE(2,155) TR(1)
  155 FORMAT(1H0,33HPOINT AT INFINITY, GIVEN BY: T = ,E20.8)
      GO TO 158
  156 WRITE(2,157) TR
  157 FORMAT(1H0,34HPOINTS AT INFINITY, GIVEN BY: T = ,3E20.8)
  158 CALL INVERT(Q,4,QROW,1,DET,ID)
      GO TO(161,159),ID
  159 WRITE(2,160)
  160 FORMAT(1H ,44HQ IS SINGULAR, I.E, GAMMA IS A CONIC)
      GO TO 175
  161 WRITE(2,162) DET
  162 FORMAT(1H ,17X,19HTHE DETERMINANT IS ,E15.8)
      WRITE(2,163)
  163 FORMAT(1H0,15X,12HINVERSE OF Q/)
      WRITE(2,920) ((U(I,J),J=1,4),I=1,4)
      DO 170 I = 1,NOR
      T(1) = TR(I)
      T(2) = TR(I)
      T(3) = TR(I)
      CALL TPLANE(T,C)
      WRITE(2,165) TR(I),C
  165 FORMAT(1H0,4HT = ,E 5.8,31HCOEFFICIENTS OF ASYMPTOTE PLANE,4E15,8)
      TR(1) = TR(I)
      T(2) = TR(I)
      CALL TCHORD(T,U)
      WRITE(2,166) ((UU(J,K),K=1,4),J=1,2)
  166 FORMAT(1H ,9X,32HCOEFFICIENTS OF ASYMPTOTIC LINES,4E15.8/
     1   1H ,9X,4E15.8)
  170 CONTINUE
  175 CALL FLUCUS(TR(1),R)
      WRITE(2,180) ((R(I,J),J=1,3),I=1,4)
  180 FORMAT(1H0,24HLUCUS OF PARALLEL CHORDS/1H ,3E20.7)
      GO TO 98
  310 STOP
```

```
      SUBROUTINE ROOT(A,XR,XI)
      DIMENSION A(4),XR(3),F(5)
      XI = 0.
      IF(ABS(A(1)).LT.1.E-9) GO TO 100
      CALL GUBIC(A,XR,XI)
      GO TO 200
  100 XR(1) = 1.E75
      IF(ABS(A(2)).LT.1.E-9) GO TO 150
      DO 110 I = 1,3
  110 F(I) = A(I+1)
      CALL GUBIC(F,XR(2),XR(3),XI)
      GO TO 200
  150 XR(2) = 1.E75
      IF(ABS(A(3)).LT.1.E-9) GO TO 160
      XR(3) = -A(4)/A(3)
      GO TO 200
  160 XR(3) = 1.E75
  200 RETURN
      END

END OF SEGMENT, LENGTH    79, NAME  ROOT
```

```
      SUBROUTINE TPLANE(T,Q)
      DIMENSION Q(4,4),T(3),Q(4)
      COMMON/CM1/Q
      BIG = 1.E70
      N = 0
      DO 50 K = 1,3
      IF(T(K).LT.BIG) GO TO 50
      N = N + 1
      T(K) = 0.
   50 CONTINUE
      T1 = T(1) + T(2) + T(3)
      T2 = T(1)*T(2) + T(1)*T(3) + T(2)*T(3)
      IF(N.NE.0) GO TO 20
      T3 = T(1)*T(2)*T(3)
      DO 100 I = 1,4
  100 Q(I) = Q(1,I) - Q(2,I)*T1 + Q(3,I)*T2 - Q(4,I)*T3
      RETURN
  200 GO TO(210,220,230),N
  210 DO 215 I = 1,4
  215 Q(I) = - Q(2,I) + Q(3,I)*T1 - Q(4,I)*T2
      RETURN
  220 DO 225 I = 1,4
  225 Q(I) = Q(3,I) - Q(4,I)*T1
      RETURN
  230 DO 235 I = 1,4
  235 Q(I) = Q(4,I)
      RETURN
      END

END OF SEGMENT, LENGTH   321, NAME  TPLANE
```

```
      SUBROUTINE TCHORD(T,G)
      SUBROUTINE TCHORD(T,G)
      DIMENSION Q(4,4),T(2),G(2,4)
      COMMON/CM1/Q
      BIG = 1.E70
      N = 0
      DO 50 K = 1,2
      IF(T(K).LT.BIG) GO TO 50
      N = N + 1
      T(K) = 0.0
   50 CONTINUE
      T1 = T(1) + T(2)
      IF(N.NE.0) GO TO 200
      T2 = T(1)*T(2)
      DO 100 I = 1,4
      G(1,I) = Q(1,I) - Q(2,I)*T1 + Q(3,I)*T2
  100 G(2,I) = Q(2,I) - Q(3,I)*T1 + Q(4,I)*T2
      RETURN
  200 GO TO(210,220),N
  210 DO 215 I = 1,4
      G(1,I) = - Q(2,I) + Q(3,I)*T1
  215 G(2,I) = - Q(3,I) + Q(4,I)*T1
      RETURN
  220 DO 225 I = 1,4
      G(1,I) = Q(3,I)
  225 G(2,I) = Q(4,I)
      RETURN
      END


END OF SEGMENT, LENGTH  293, NAME  TCHORD
```

```
      SUBROUTINE FLATCOCUS(T,R)
      DIMENSION B(4,4),R(3,3),A(4,4)
      COMMON/ M1/B
      DO 200 I = 1,3
      DO 100 J = 1,4
      DO 100 K = 1,4
  100 A(J,K) = B(1,J)*B(3,K) - B(1,K)*B(3,J)
      IF(T.GT.1-E/0) GO TO 15
      R(I,1) = A(1,2)*T**2 + A(2,3)*T + A(1,4)
      R(I,2) = A(1,3)*T**2 - (A(1,4)+A(2,3))*T + A(2,4)
      R(I,3) = A(1,4)*T**2 + A(2,4)*T + A(3,4)
      GO TO 2 0
   15 DO 160 J = 1,3
  160 R(I,J) = A(1,J+1)
  200 CONTINUE
      RETURN
      END

END OF SEGMENT, LENGTH   251, NAME  FLATLOCUS
```

```
      SUBROUTINE INVERT(A,N1,N,M1,DETERM,ID)
C     A SUBROUTINE TO INVERT A MATRIX OR TO SOLVE SIMULTANEOUS EQUATION
      DIMENSION A(N1,M1),B(N1,1),INDEX(11,3)
      EQUIVALENCE (IROW,JROW),(ICOLUM,JCOLUM),(AMAX,T,SWAP)
C     INITIALIZATION
C
      M = M1
      V = M1
   10 DETERM = 1.
   15 DO 20 J = 1,N
   20 INDEX(J,3) =
   30 DO 550 I =1, N
C
C     SEARCH FOR PIVOT ELEMENT
C
   40 AMAX = 0.0
   45 DO 105 J=1,N
      IF(INDEX(J,3)-1) 60,105,60
   60 DO 100 K =1,N
      IF(INDEX(K,3) -1) 80,100,710
   80 IF(AMAX -ABS(A(J,K))) 85,100,100
   85 IROW = J
   90 ICOLUM = K
      AMAX = ABS(A(J,K))
  100 CONTINUE
  105 CONTINUE
      INDEX(ICOLUM,3) = INDEX(ICOLUM,3) + 1
  260 INDEX(I,1) = IROW
  270 INDEX(I,2) = ICOLUM
C
C     INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
C
  130 IF(IROW -ICOLUM) 140,310,140
  140 DETERM = -DETERM
  150 DO 200 L =1,M
  160 SWAP = A(IROW,L)
  170 A(IROW,L) = A(ICOLUM,L)
  200 A(ICOLUM,L) = SWAP
      IF(M) 310,310,210
  210 DO 250 L = 1,M
  220 SWAP = B(IROW,L)
  230 B(IROW,L) = B(ICOLUM,L)
  250 B(ICOLUM,L) = SWAP
C
C     DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
  310 PIVOT = A(ICOLUM,ICOLUM)
      DETERM = DETERM*PIVOT
  330 A(ICOLUM,ICOLUM) = 1.0
  340 DO 350 L =1,N
  350 A(ICOLUM,L) = A(ICOLUM,L)/PIVOT
  355 IF(M) 380,380,360
  360 DO 370 L = 1,M
  370 B(ICOLUM,L) = B(ICOLUM,L)/PIVOT
C
C     REDUCE NON-PIVOT ROWS
C
  380 DO 550 L1 =1,N
  390 IF(L1-ICOLUM) 400,550,400
  400 T=A(L1,ICOLUM)
  420 A(L1,ICOLUM) = .0
  430 DO 450 L = 1,N
  450 A(L1,L)=A(L1,L) - A(ICOLUM,L)*T
  455 IF(M) 550,550,460
```

```
      DO 550 L = 1,N
      B(L,L) = B(L,L) - B(ICOLUM,L)*T
  550 CONTINUE
C
C     INTERCHANGE COLUMNS
C
  600 DO 710 I = 1,N
  610 L = N + 1 - I
  620 IF(INDEX(L,1)-INDEX(L,2)) 630,710,630
  630 JROW=INDEX(L,1)
  640 JCOLUM=INDEX(L,2)
  650 DO 705 K=1,N
  660 SWAP=A(K,JROW)
  670 A(K,JROW) = A(K,JCOLUM)
  700 A(K,JCOLUM) = SWAP
  705 CONTINUE
  710 CONTINUE
      DO 730 K = 1,N
      IF(INDEX(K,3)-1) 715,730,715
  715 IQ = 2
      GO TO 740
  720 CONTINUE
  730 CONTINUE
      IQ = 1
  740 RETURN
      END

END OF SEGMENT, LENGTH  544, NAME  INVERT
```

```
      READ FROM(MT,USER5 EDIT.CUBIC)
SUBFILE CUBIC
SUBFILE CUBIC
      SUBROUTINE CUBIC(A,XR,XI)
      DIMENSION A(4),XR(3),AQ(3)
C     TO SOLVE CUBIC
C     A(1)*X**3+A(2)*X**2+A(3)*X+A(4)=0
      IPATH=2
      EX=1./3.
      IF(A(4)) 100,104,106
  104 XR(1)=0.
      GO TO 134
  106 A2=A(1)*A(1)
      Q=(27.*A2*A(4)-9.*A(1)*A(2)*A(3)+2.*A(2)**3)/(54.*A2*A(1))
      IF(Q) 110,108,114
  108 Z=0.
      GO TO 132
  110 Q=-Q
      IPATH=1
  114 P=(3.*A(1)*A(3)-A(2)*A(2))/(9.*A2)
      ARG=P*P*P+Q*Q
      IF(ARG) 116,118,120
  116 Z=-2.*SQRT(-P)*COS(ATAN(SQRT(-ARG)/Q)/3.)
      GO TO 128
  118 Z = -2.*Q**EX
      GO TO 128
  120 SARG = SQRT(ARG)
      IF(P) 122,124,126
  122 Z=-(Q+SARG)**EX-(Q-SARG)**EX
      GO TO 128
  124 Z=-(2.*Q)**EX
      GO TO 128
  126 Z=(SARG-Q)**EX-(SARG+Q)**EX
  128 GO TO (130,132),IPATH
  130 Z=-Z
  132 XR(1)=(3.*A(1)*Z-A(2))/(3.*A(1))
  134 AQ(1)=A(1)
      AQ(2)=A(2)+XR(1)*A(1)
      AQ(3)=A(3)+XR(1)*AQ(2)
      CALL QUAD(AQ,XR(2),XR(3),XI)
      RETURN
      END

END OF SEGMENT, LENGTH  299, NAME  CUBIC
```

```
 END OF SUBFILE
SUBFILE QUAD
      SUBROUTINE QUAD(A,XR1,XR2,XI)
C     TO SOLVE QUADRATIC
C     A(1)X**2+A(2)X+A(3)=0
      DIMENSION A(3)
      X1 = -A(2)/(2.*A(1))
      DISC=X1*X1-A(3)/A(1)
      IF(DISC) 210,220,220
  210 X2 = SQRT(-DISC)
      XR1 = X1
      XR2 = X1
      XI  = X2
      GO TO 230
  220 X2 = SQRT(DISC)
      XR1 = X1+X2
      XR2 = X1-X2
      XI  = 0.
  230 RETURN
      END

END OF SEGMENT, LENGTH   112, NAME  QUAD
```

```
      END OF SUBFILE
      END OF SUBFILE
      READ FROM(MAT,USERS EDIT.QMATRIX)
SUBFILE QMATR X
      SUBROUT NO MAIN
      REAL LAND
      DOUBLE PRECISION A,B,AL,EL,E,AB,A1(2),AA(2),GAM,S,C,B1(2),L(2),
     1M(2),ST(2),  T(2),J1  J2,K1,K2,TK,TD,M1,P2,M1,R2,AD1,AD2,A10,A25,T1,
     2AP,KM
      DIMENSION D(4,4)
      COMMON/CM1/D/CM2/AB,BS,ALPHA,LAMDA,EPS,A,PI
      PIT = 0.01745329D
      A = AN
      B = BS
      AL = ALPHA*PIT
      EL = LAMDA*PIT
      EU T EPS*PIT
      E = TAN(EU)
      AB = A**2 + B**2
      D = 2.0*A*B*DCOS(AL)
      A1(1) = DSQRT(AB - D)
      A1(2) = DSQRT(AB + D)
      AA(1) = (A1(2)-A1(1))/2.0
      AA(2) = (A1(2)+A1(1))/2.0
      GAM = 0.5*DATAN2(2. *A*B*DSIN(AL),A**2-B**2)
      S = DSIN(GAM)
      C = DCOS(GAM)
      B1(1) = A*DCOS(AL) - B
      B1(2) = A*DCOS(AL) - B
      DO 100    = 1,2
      L(I) = A1(I) + A*DSIN(AL)*DCOS(EL)
      M(I) =  *DSIN(AL) + A1(I)*DCOS(EL)
      BT(I) = B1(I)*DSIN(EL)/L(I)
  100 ST(I) = M(I)/L(I)
C     K .NE. 1 SYMMETRIC APPROXIMATION
      IF(K.EQ.1) GO TO 101
      J1 = -1.0
      J2 = 1.
      K1 = -1.0
      K2 = 1.
      TK = 0.2
      TD = 0.2
      GO TO 102
  101 TK = ST(1)/ST(1)
      TD = (TK*E)/(1. -TK*E)
      J1 = ST(2)/ST(1)
      J2 = ST(2)/ST(1)
      K1 = (J1*TK-J2*E)/(TK-E)
      K2 = (J2+J1*TK*E)/(1.0+TK*E)
  102 P1 = B*B1(2) - B*B1(1)*J1
      P2 = B*B1(2) - B*B1(1)*K2
      R1 = B. *SEC*K2*DCOS(EL)
      R2 = B. *SEC*K2*DSIN(EL)
      AD1 = (A1(2) - A1(1)*K2)/2.0
      AD2 = (A1(2) + A1(1)*K2)/2.0
      IF(ABS(AD1).LT.1.E-5) GO TO 105
      FPT = AD2/AD1
      GO TO 106
  105 FPT = 1.E75
  106 A10 = (AA(1)*D)**2
      A25 = (AA(2)**2)**2
```

```
      W(1,1) = -R1*(K2-K1)*P1*T1  + M1*(K2-J2)*P2 - M2*(K2-J1)*B*A01*TK
      W(1,2) =  R1*(K2+K1)*P1*T1  + M1*(K2+J2)*P2 - M2*(K2-J1)*B*A02*TK
      W(1,3) = -D(1,1)
      W(1,4) = -D(1,2)
      W(2,1) =  (K2-K1)*(1.+K2)*(K2-J1)*A1C*T1
     1         -(K2-K1)*(1.+K2)*(K2+J1)*A2S*T1
     2         -2.0*K2*(1.+K2)*(K2-J2)*A2S
      W(2,2) =((K2-K1)*(1.+K2)*(K2*J1+K2,0*KM*(1.+K2))*A1C*T1
     1         +(2.0*KP*(1.+K2) + (K2*K1)*(K2+J1)*(1.+K2))*A2S*T1
     2         -2.0*K2*(K2+J2)*(1.+K2)*A2S - 2.0*K2*(K2-J2)*(1.+K2)*B**2
      W(2,3) = ((K2+K1)*(K2-J1)*(1.+K2) - 2.0*KM*(1.+K2))*A1C*T1
     1         +(2.0*KP*(1.+K2) - (K2+K1)*(K2+J1)*(1.+K2))*A2S*T1
     2        -2.0*K2*(K2-J2)*(1.+K2)*A1C + 2.0*K2*(K2+J2)*(1.+K2)*B**2
      W(2,4) = (K2-K1)*(1.+K2)*(K2+J1)*A1C*T1
     1        -(K2+K1)*(1.+K2)*(K2-J1)*A2S*T1
     2         +2.0*K2*(1.+K2)*(K2+J2)*A1C
      W(3,1) = RP*(K2-K1)*P1*T1 + M2*(K2-J2)*P2 - R1*(K2-J1)*B*AD1*TK
      W(3,2) = -R2*(K2+K1)*P1*T1  + M2*(K2+J2)*P2 - M1*(K2-J1)*B*A02*TK
      W(3,3) = -D(3,1)
      W(3,4) = -D(3,2)
      W(4,1) = -B*(K2-K1)*(K2+J1)*(1.+K2)*S*S*T1
     1         -B*(K2-K1)*(K2-J1)*(1.+K2)*L*C*T1
     2         -2.0*B*K2*(1.+K2)*(K2+J2)*S*S
      W(4,2) =  2.0*B*(K2-J1)*(K2**2+1)*C*C*T1
     1         +2.0*B*(1.+K2)*KP*S*S*T1 - B*(1.+K2)*(K2+J1)*(K2-K1)*T1
     2         -2.0*B*K*(1.+K2)*(K2+J2) + 2.0*B*K2*(1.+K2)*(K2-J2)*S*S
      W(4,3) = -2.0*B*K2*(K2-K1)*(1.+J1)*C*C*T1
     1         -B*(1.+K2)*(K2+K1)*(K2-J1)*T1 + 2.0*B*(1.-K2)*KP*T1
     2         -2.0*B*K2*(1.+K2)*(K2+J2)*C*C + 2.0*B*K2*(1.-K2)*(K2+J2)
      W(4,4) =  2.0*B*K2*(K2+K1)*(1.+J1)*C*C*T1
     1         +2.0*B*K2*(K2+J2)*(1.+K2)*C*C
     2         -B*(1.-K2)*(K2+K1)*(K2-J1)*T1
      RETURN
      END

END OF SEGMENT, LENGTH  2320, NAME  MATQ
```

A = 300.0 B = 3.0 ALPHA = 70.0 LAMDA = 90.0 EPSILON 1.0

HOMOGENEOUS COEFFICIENTS

-0.9565+54E 10	,80781743E 00	0.9565+64E-03	-0.0028190E 00
-0.3547+55E-04	-,31729480E 00	0.7[illegible]5E 02	-0.1202412E-01
0.7118261E-03	-,25[illegible]9443E 00	-0.7118261E-03	0.2515440E 00
-0.2295[illegible]E-02	,3[illegible]79E-02	-0.1027967E 00	0.2434342E 02

POINT AT INFINITY GIVEN BY: T = 0.25167778E 03

THE DETERMINANT IS 0.30145455E 06

INVERSE OF Q

,1283687E 04	,13[illegible]E-01	0.2961275E 04	0.1618219E-03
,8495875E 01	0,17[illegible]E-04	0.44[illegible]E 01	0.4107649E-01
,1[illegible]E-[illegible]	,13[illegible]E-01	0.2821135E-01	0.1618227E-03
,1393[illegible]E-03	0,57[illegible]E-04	0.4840178E-03	0.4107649E-01

T = 0.25167778E 03 COEFFICIENTS OF ASYMPTOTE PLANE=-0.24560740E 06+0.21567452E 06-0.3[illegible]E 06-0.28950294E 04

COEFFICIENTS OF ASYMPTOTIC LINES 0.4572216[illegible]E 03 0.8[illegible]E 03 0.2 2 2762E 04+0.515893[illegible]E 01
0.45209[illegible]E 01+0.[illegible]E 01 0.24[illegible]E 02 0.26016044E 04

LOCUS 1: PARALLEL CHORDS

,1[illegible]E 07	-,13[illegible]E 07	0.1423172E 05

LOCUS 2: PARALLEL CHORDS

,3[illegible]E 04	,13[illegible]E 03	-0.2[illegible]E 00

LOCUS 3: PARALLEL CHORDS

,[illegible]E 0[illegible]	,[illegible]E 00	[illegible]E 00

LOCUS 4: PARALLEL CHORDS

,6227 2[illegible]E-[illegible]	-,45[illegible]E 00	-0.4847[illegible]E 02

```
A =  3.00E 0     B =  3.0     ALPHA =  45.0     LAMDA =  0.0     EPSILON  0.0

        HURWITZ COEFFICIENTS
 -0.1449E-01          -0.5911E 00          0.1449E-01          -0.5911E 00
  0.1449E-01           0.0000E 00          0.7251E-02           0.0000E 00
 -0.3599E-02          -0.1296E 00          0.3599E-02           0.1296E 00
  0.0000E 00          -0.6000E-01          0.0000E 00           0.2399E-02

POINT AT INFINITY, GIVEN BY: T =       0,1000000E 00

              THE DETERMINANT IS =-0,1504E-02

              INVERSE OF Q

 -0.3554E 02           0.3858E-01          -0.3554E 02          -0.1142E-10
  0.3925E 02           0.0000E 00          -0.1570E 01           0.4165E-01
  0.1110E-01           0.3858E-01           0.1110E-01           0.2925E-14
 -0.1766E-03           0.0000E 00           0.7068E-03           0.4165E-01

T =  0.1000000E 00  ASYMPTOTE COEFFICIENTS OF ASYMPTOTE PLANE=-0.1766E-03  0.0000E 00  0.7068E-03  0.4165E-01

              COEFFICIENTS OF ASYMPTOTIC LINES  0.1110E-01  0.1585E-01  0.1110E-01  0.2925E-14
                                                -0.1766E-03  0.0000E 00  0.7068E-03  0.4165E-01

LOCUS OF PARALLEL CHORDS
 -0.7713E 00           0.0000E 00          0.0000E 00

LOCUS OF PARALLEL CHORDS
  0.9451E-02           0.2100E-01          -0.4625E-03

LOCUS OF PARALLEL CHORDS
 -0.3000E 00           0.0000E 00          0.0000E 00

LOCUS OF PARALLEL CHORDS
 -0.2453E-03          -0.9820E-05          -0.4625E-03
```

```
A =   300.0     B =   3.0     ALPHA =  90.0     LAMDA =   0.0     EPSILON  0.0

          PERTURBATION COEFFICIENTS
 0.5167372E-19          0.7199284E 00          0.5167372E-19         -0.7199284E 00
 0.9251224E-19          0.0000000E 00          0.9251224E 00          0.0000000E 00
-0.1290937 E-1          0.7199284E 00          0.1290937E-10          0.7199284E 00
 0.0000 0 E 0           0.1199761E-01          0.2399761E 00          0.2399761E 02

POINT AT INFINITY, GIVEN BY: T =        0.1000000E 76

                THE DETERMINANT IS =0.8033033E-02

                INVERSE OF Q

-0.1549097E 11          0.3403348E-01         -0.1549419E 11         -0.3524059E-02
 0.2775448E 00          0.3383722E-23         -0.1110634E 01          0.4165021E-01
 0.1990654E-10          0.3403348E-01          0.1990659E-10          0.4526224E-23
-0.1380334E-03          0.0000000E 00          0.8553612E-03          0.4165021E-01

T =  0.100000 OR ACOEFFICIENTS OF ASYMPTOTE PLANE-0.1380334E-03 0.0000000E 00 0.8553612E-03 0.4165021E-01

                COEFFICIENTS OF ASYMPTOTIC LINES 0.1990654E-10 0.1380334E-01 0.1990659E-10 0.4526224E-23
                                                -0.1380334E-03 0.0000000E 00 0.8553612E-03 0.4165021E-01

LOCUS OF PARALLEL CHORDS
-0.2181277E 09          0.0000000E 00          0.0000000E 00

LOCUS OF PARALLEL CHORDS
 0.3834632E-02          0.2763594E-13         -0.8291077E-12

LOCUS OF PARALLEL CHORDS
 0.0000 3E 0            0.0000000E 00          0.0000000E 00

LOCUS OF PARALLEL CHORDS
-0.1927473E-05          0.1381668E-13         -0.8291077E-12
```

```
A =   3.0.0     B =   3.0     ALPHA =  45.0     LAMDA =   1.0     EPSILON  0.0

        HOROPTER COEFFICIENTS
      0.8C8D544E-06        0.50909976 00       -0.8C8D544E-06       -0.5090976E 00
      0.1981727E-06        0.75/8995E-04        0.7280657E 02        0.1881456E-04
      0.1301125E-04       -0.7045999E-02       -0.1301125E-04        0.7045539E-02
      0.6273 57E-04        0.2002498E-02        0.3763196E-04        0.2999999E 02

POINT AT INFINITY, GIVEN BY: T =       0,1000000E 76

                THE DETERMINANT IS  0,1145107E-01

              INVERSE OF Q

      0.1063208E 04        0,1388820E-01        0.7683164E 05       -0.5431920E-07
      0.1983988E 01       -0,21 79928E-07      -0.4623377E-01        0.4166460E-01
     -0.4831237E-05        0,1388820E-01       -0.2867421E-03       -0.5443871E-07
     -0.9822426E-04       -0,2177592E-07        0.2944684E-06        0.4166460E-01

T =  0,1000000E 76COEFFICIENTS OF ASYMPTOTE PLANE-0.98224262E-04,0.2177591QE-07 0.2944/844E-06 0.41664598E-01

                  COEFFICIENTS OF ASYMPTOTIC LINES-0.4831236E-05 0.1388819E-01-0.2867421E-03-0.5443870E-07
                                                  -0.98224262E-04,0.2177591QE-07 0.2944/844E-06 0.41664598E-01

LOCUS OF PARALLEL CHORDS
      0.1473728E 02        0,1577497E 00       -0.5788444E-04

LOCUS OF PARALLEL CHORDS
      0.2725987E-01       -0,2943891E-03        0.9439962E-07

LOCUS OF PARALLEL CHORDS
      0.0000000E 00        0,7000000E 00        0.0000000E 00

LOCUS OF PARALLEL CHORDS
     -0.1364158E-05        0,1911917E-07        0.2012x69E-08
```

A = 540.0 B = 3.0 ALPHA = 45.0 LAMDA = 0.0 EPSILON 1.0

HOMOGENEOUS COEFFICIENTS

-0.1440[illegible]E-01	0.10011092 00	0.1440[illegible]00E-01	-0.5091164E 00
0.1439856E-01	0.0000000E 00	0.7201440E 02	0.0000000E 00
-0.3599911E-02	-0.5091296E 00	0.3599910E-02	0.5091295E 00
0.0000000E 00	0.10000006E-01	0.0000000E 00	0.2399880E 02

POINT AT INFINITY, GIVEN BY: T = 0.10000000E 00

THE DETERMINANT IS -0.15040480E 02

INVERSE OF Q

-0.3654601E 02	0.1338334E-01	-0.5554362E 02	-0.1202947E-10
0.3926447E 00	0.10000000E 00	-0.1578016E 01	0.4165001E-01
0.1110567E-01	0.1338334E-01	0.1110539E-01	0.2525137E-14
-0.1766999E-03	0.00000000E 00	0.7068174E-03	0.4165001E-01

T = 0.10000000E 00 /COEFFICIENTS OF ASYMPTOTE PLANE=-0.1766999E-03 0.00000000E 00 0.7068 738E-03 0.41650007E-01

COEFFICIENTS OF ASYMPTOTIC LINES 0.11105669E-01 0.13883336E-01 0.111 5391E-01 0.2525136BE-14
-0.1766999E-03 0.00000000E 00 0.7063 738E-03 0.41650007E-01

LOCUS OF PARALLEL CHORDS

-0.7713 [illegible]	0.10000000 00	0.1000000E 00

LOCUS OF PARALLEL CHORDS

0.9451216E-02	0.21000324E-01	-0.4625512E-03

LOCUS OF PARALLEL CHORDS

[illegible]	0.10000006 00	0.0000 00E 00

LOCUS OF PARALLEL CHORDS

-0.2493 [illegible]E-05	-0.98120028-05	-0.4625512E-03

A = 30.0 B = 3.0 ALPHA = 45.0 LAMDA = 0.0 EPSILON 0.0

HORUPTER COEFFICIENTS

-0.1439928E-01	0.5090914E 01	0.1439928E-01	-0.5090914E 01
0.1425401E-01	0.0	0.7261454E 02	0.0
-0.3593869E-01	-0.5135699E 00	0.3594869E-01	0.5135699E 00
0.0	0.0	0.0	0.2387941E 02

POINT AT INFINITY GIVEN BY: T = 0.1000000E 76

THE DETERMINANT IS =0.3288409E 03

INVERSE OF C

-0.2883446E 01	0.1393334E-01	-0.2878729E 02	-0.6086177E-11
0.1878139E 01	-0.4408992E-15	-0.7531370E-01	0.4165001E-01
0.5812535E-03	0.1393334E-01	0.5298056E-02	0.1204820E-14
-0.1023993E-02	0.1734723E-17	0.4166211E-03	0.4165001E-01

T = 0.1000000E 76 COEFFICIENTS OF ASYMPTOTE PLANE=-0.1023993E-02 0.1734723E-17 0.4166211E-03 0.4165001E-01

COEFFICIENTS OF ASYMPTOTIC LINES 0.5312535E-03 0.1393334E-01 0.5298056E-02 0.1204820E-14
-0.1023993E-02 0.1734723E-17 0.4166211E-03 0.4165001E-01

LOCUS OF PARALLEL CHORDS

-0.3725256E-01	0.1000000E 00	0.0000000E 00

LOCUS OF PARALLEL CHORDS

0.2607478E-02	0.1035282E-02	-0.2212512E-04

LOCUS OF PARALLEL CHORDS

0.0000000E 0	0.2000000E 00	0.0000000E 00

LOCUS OF PARALLEL CHORDS

-0.1421630E-04	-0.5644088E-05	-0.2212512E-04

A = 3000.0 B = 3.0 ALPHA = 45.0 LAMDA = 0.0 EPSILON 0.0

NORMALIZED COEFFICIENTS

-0.1440000E-01	0.5911169E-01	0.1440000E-01	-0.5911169E-01
0.1439999E-01	0.0000000E 00	0.7201440E 02	0.0000000E 00
-0.3599999E-03	-0.5911170E 00	0.3599999E-03	0.5911170E 00
0.0000 0.E 0	0.36-2000E-02	0.5000000E 00	0.2399999E 02

POINT AT INFINITY. GIVEN BY: T = 0.10000000E 76

THE DETERMINANT IS =0.12710294E 02

INVERSE OF Q

-0.6925242E 02	0.1388384E-01	-0.6925240E 01	0.0000000E 00
0.4895255E-01	0.1102238E-15	-0.1958502E 01	0.4165001E-01
0.1384870E-01	0.1388384E-01	0.1384870E-02	0.0000000E 00
-0.1960951E-04	-0.5421011E-19	0.7843881E-03	0.4165001E-01

T = 0.15000000E 76 COEFFICIENTS OF ASYMPTOTE PLANE-0.19609510E-04 0.54210109E-19 0.7843881E-03 0.41650007E-01

COEFFICIENTS OF ASYMPTOTIC LINES 0.13848700E-01 0.13883336E-01 0.1384-8697E-02 0.00000000E 00
-0.19609510E-04 0.54210109E-19 0.7843881E-03 0.41650007E-01

LOCUS OF PARALLEL CHORDS
-0.9617164E 01 0.00000000E 00 0.00000000E 00

LOCUS OF PARALLEL CHORDS
0.6797635E-03 0.27190926E-01 -0.5767984E-03

LOCUS OF PARALLEL CHORDS
0.0000000E 0 0.0000000E 00 0.00000000E 00

LOCUS OF PARALLEL CHORDS
-0.2722454E-06 -0.1088981E-04 -0.5767984E-03

found

1) the matrix Q

2) the homogeneous co-ordinates (x_i) for various values of t and converted them into co-ordinates (x,y) that represented a perspective view from a particular point.

see: figs. 18.5, 18.6, 18.13, 18.17

(no line-printer output has been included for this program)

```
      MASTER BV05
      REAL LAMDA
      DIMENSION Q(4,4),T(40),XH(4),X(40),Y(40),Z(40),GROW(4),XR(3),F(4
      COMMON/CM1/Q/CM2/A,B,ALPHA,LAMDA,E1,KK,PIT
      CALL HSPSTART(14HHYPOCYCLOID,14)
      PIT = 0.0174532925
      B = 3.0
   95 READ(1,900) A,ALPHA,LAMDA,E1,LAST
  900 FORMAT(4F6.2,I2)
      IF(LAST.EQ.1) GO TO 310
      KK = 1
      IF(LAMDA.EQ.0.0) KK = 0
      AL = ALPHA*PIT
      EL = LAMDA *PIT
      F(1) = A*SIN(AL)*COS(EL)
      F(2) = - A*COS(AL)
      F(3) = - A*SIN(AL)*SIN(EL)
      F(4) = 1.0
      WRITE(2,905) A,B,ALPHA,LAMDA,E1
  905 FORMAT(1H0,10X,4HA = ,F7.1,2X,4HB = ,F5.1,5X,8HALPHA = ,F5.1,5X,
     1       8HLAMDA = ,F5.1,5X,7HEPSILON,F5.1/)
      CALL MATQ
      T(1) = -200.0
      DO 205 I = 2,40
  205 T(I) = T(I-1) + 10.0
      N1 = 0
      DO 140 I = 1,40
      DO 130 N = 1,4
  130 XH(N) = XQ(N,T(I))
      CALL PROJECTION(XH,X(I),Y(I))
      WRITE(2,930) T(I),XH,X(I),Y(I)
  930 FORMAT(1H ,F6.1,6F12.3)
      IF(ABS(X(I)).GT.400.0) GO TO 140
      IF(ABS(Y(I)).GT.400.0) GO TO 140
      N1 = N1 + 1
      X(N1) = X(I)/20.0
      Y(N1) = Y(I)/20.0
  140 CONTINUE
      CALL PERSP(X,Y,N1,F)
      GO TO 95
  310 CALL HSPFINISH
      STOP
      END

END OF SEGMENT, LENGTH   343, NAME  BV05
```

```
      FUNCTION XB(I,T)
      DIMENSION Q(4,4)
      COMMON/CM1/Q
      XB = Q(I,1)*T**3 + Q(I,2)*T**2 + Q(I,3)*T + Q(I,4)
      RETURN
      END
```

END OF SEGMENT, LENGTH 61, NAME XB

```
      SUBROUTINE PROJECTION(XH,X,Y)
      DIMENSION XH(4),D(3)
      D(1) = -400.0
      D(2) = -400.0
      D(3) = 10.0*25
      XD = XH(4)*D(1) - XH(1)
      IF(XD.EQ.0.0) GO TO 100
      X = -(D(1)*XH(2) - D(2)*XH(1))/XD
      Y =  (D(1)*XH(3) - D(3)*XH(1))/XD
      GO TO 105
  100 X = 500.0
      Y = 500.0
  105 RETURN
      END

END OF SEGMENT, LENGTH   152, NAME  PROJECTION
```

```
      READ FROM(MT,USERS EDIT.QMATRIX)
SUBFILE QMATRIX
      SUBROUTINE MATQ
      REAL LAMDA
      DOUBLE PRECISION A,B,AL,EL,E,AB,A1(2),AA(2),GAM,S,C,B1(2),L(2),
     1M(2),ST(2),CT(2),J1,J2,K1,K2,TK,TD,P1,P2,R1,R2,AD1,AD2,A1C,A2S,T1,
     2KP,KM
      DIMENSION Q(4,4)
      COMMON/CM1/Q/CM2/AS,BS,ALPHA,LAMDA,EPS,K,FPT
      PIT = 0.0174532925
      A = AS
      B = BS
      AL = ALPHA*PIT
      EL = LAMDA*PIT
      ED = EPS*PIT
      E = TAN(ED)
      AB = A*A + B*B
      D = 2.0*A*B*DCOS(AL)
      A1(1) = DSQRT(AB - D)
      A1(2) = DSQRT(AB + D)
      AA(1) = (A1(2)-A1(1))/2.0
      AA(2) = (A1(2)+A1(1))/2.0
      GAM = 0.5*DATAN2(2.0*A*B*DSIN(AL),A*A-B*B)
      S = DSIN(GAM)
      C = DCOS(GAM)
      B1(1) = A*DCOS(AL) - B
      B1(2) = A*DCOS(AL) + B
      DO 100 I = 1,2
      L(I) = A1(I) + A*DSIN(AL)*DCOS(EL)
      M(I) = A*DSIN(AL) + A1(I)*DCOS(EL)
      ST(I) = B1(I)*DSIN(EL)/L(I)
  100 CT(I) = M(I)/L(I)
C     K .NE. 1 SYMMETRIC APPROXIMATION
      IF(K.EQ.1) GO TO 101
      J1 = -1.0
      J2 = 1.0
      K1 = -1.0
      K2 = 1.0
      TK = 0.02
      TD = 0.02
      GO TO 102
  101 TK = ST(1)/CT(1)
      TD = (TK+E)/(1.0-TK*E)
      J1 = ST(2)/ST(1)
      J2 = CT(2)/CT(1)
      K1 = (J1*TK-J2*E)/(TK+E)
      K2 = (J2+J1*TK*E)/(1.0-TK*E)
  102 P1 = B*B1(2) - B*B1(1)*J1
      P2 = B*B1(2) - B*B1(1)*K2
      R1 = 2.0*S*C*K2*DCOS(EL)
      R2 = 2.0*S*C*K2*DSIN(EL)
      AD1 = (A1(2) - A1(1)*K2)/2.0
      AD2 = (A1(2) + A1(1)*K2)/2.0
      IF(ABS(AD1).LT.1.E-9) GO TO 105
      FPT = -AD2/AD1
      GO TO 106
  105 FPT = 1.E75
  106 A1C = (AA(1)*C)**2
      A2S = (AA(2)*S)**2
      T1 = TD*TK
      KP = K2*K2 + J1*K1
      KM = K2*K2 - J1*K1
      Q(1,1) = -R1*(K2-K1)*P1*T1 - R1*(K2-J2)*P2 - R2*(K2-J1)*B*AD1*TK
      Q(1,2) =  R1*(K2+K1)*P1*T1 + R1*(K2+J2)*P2 - R2*(K2-J1)*B*AD2*TK
```

```
      Q(1,3) = - Q(1,1)
      Q(1,4) = -Q(1,2)
      Q(2,1) =  (K2-K1)*(1.+K2)*(K2-J1)*A1C*T1
     1         -(K2-K1)*(1.-K2)*(K2+J1)*A2S*T1
     2         -2.0*K2*(1.+K2)*(K2-J2)*A2S
      Q(2,2) =((K2-K1)*(1.-K2)*(K2-J1)-2.0*KM*(1.+K2))*A1C*T1
     1         +(2.0*KP*(1.-K2) - (K2-K1)*(K2+J1)*(1.+K2))*A2S*T1
     2         +2.0*K2*(K2+J2)*(1.-K2)*A2S - 2.0*K2*(K2-J2)*(1.+K2)*B**2
      Q(2,3) = ((K2+K1)*(K2+J1)*(1.+K2) + 2.0*KM*(1.-K2))*A1C*T1
     1        +(2.0*KP*(1.+K2) - (K2+K1)*(K2-J1)*(1.-K2))*A2S*T1
     2        -2.0*K2*(K2-J2)*(1.-K2)*A1C + 2.0*K2*(K2+J2)*(1.+K2)*B**2
      Q(2,4) = (K2+K1)*(1.-K2)*(K2+J1)*A1C*T1
     1         -(K2+K1)*(1.+K2)*(K2-J1)*A2S*T1
     2         +2.0*K2*(1.-K2)*(K2+J2)*A1C
      Q(3,1) = R2*(K2-K1)*P1*T1 + R2*(K2-J2)*P2 - R1*(K2-J1)*B*AD1*TK
      Q(3,2) = -R2*(K2+K1)*P1*T1 - R2*(K2+J2)*P2 - R1*(K2-J1)*B*AD2*TK
      Q(3,3) = -Q(3,1)
      Q(3,4) = -Q(3,2)
      Q(4,1) = -B*(K2-K1)*(K2+J1)*(1.+K2)*S*S*T1
     1         +B*(K2-K1)*(K2-J1)*(1.-K2)*C*C*T1
     2         -2.0*B*K2*(1.+K2)*(K2-J2)*S*S
      Q(4,2) =  2.0*B*(K2-J1)*(K2*K2-K1)*C*C*T1
     1         +2.0*B*(1.+K2)*KP*S*S*T1 + B*(1.-K2)*(K2+J1)*(K2-K1)*T1
     2         -2.0*B*K2*(1.-K2)*(K2-J2) + 2.0*B*K2*(1.+K2)*(K2+J2)*S*S
      Q(4,3) = -2.0*B*K2*(K2-K1)*(1.+J1)*C*C*T1
     1         -B*(1.+K2)*(K2-K1)*(K2-J1)*T1 + 2.0*B*(1.-K2)*KP*T1
     2         -2.0*B*K2*(1.+K2)*(K2-J2)*C*C + 2.0*B*K2*(1.-K2)*(K2+J2)
      Q(4,4) =  2.0*B*K2*(K2+K1)*(1.+J1)*C*C*T1
     1         +2.0*B*K2*(K2+J2)*(1.+K2)*C*C
     2         -B*(1.-K2)*(K2+K1)*(K2-J1)*T1
      RETURN
      END

END OF SEGMENT, LENGTH  2320, NAME  MATQ
```

```
      SUBROUTINE PERSP(X,Y,N,F)
      REAL LAMDA
      DIMENSION X(40),Y(40),F(4),XM(4),XT(31,2),YT(31,2)
      COMMON/COM2/A,B,ALPHA,LAMDA,E1,KA,FPT
      CALL HGPORIGIN(20.0,22.0)
      CALL HGPAXIS(-20.0,0.0,8H        Y,8,40,0,0.0,-400.0,20.0)
      CALL HGPAXIS(0.0,-20.0,8H        Z,8,40,0,90.,-400.0,20.0)
      XM(2) = 0.0
      XM(3) = 0.0
      XM(4) = 1.0
      DO 100 I = 1,31
      XI = I
      XM(1) = -220.0 + 20.0*XI
      CALL PROJECTION(XM,XA,YA)
      XT(I,1) = XA/20.0
      YT(I,1) = YA/20.0
      XT(I,2) = XT(I,1) + 0.14
  100 YT(I,2) = YT(I,1) - 0.14
      CALL HGPDASHLN(XT(1,1),YT(1,1),XT(31,1),YT(31,1),0.0)
      DO 105 I = 1,31
  105 CALL HGPDASHLN(XT(I,1),YT(I,1),XT(I,2),YT(I,2),0.0)
      CALL PROJECTION(F,F1,F2)
      F1 = F1/20.0
      F2 = F2/20.0
      B1 = B/20.0
      CALL HGPSYMBL(10.0,-15.0,0.25,4HA = ,0.0,4)
      CALL HGPSYMBL(10.0,-15.5,0.25,4HB = ,0.0,4)
      CALL HGPSYMBL(10.0,-16.0,0.25,8HALPHA = ,0.0,8)
      CALL HGPSYMBL(10.0,-16.5,0.25,8HLAMDA = ,0.0,8)
      CALL HGPSYMBL(10.0,-17.0,0.25,8HEPS   = ,0.0,8)
      CALL HGPSYMBL(10.0,-18.0,0.30,19HHORDPTER CURVE,0.0,19)
      CALL HGPNUMBER(12.0,-15.0,0.25,A,0,0.0,4,0)
      CALL HGPNUMBER(12.0,-15.5,0.25,B,0,0.0,4,0)
      CALL HGPNUMBER(12.0,-16.0,0.25,ALPHA,0,0,0,4,0)
      CALL HGPNUMBER(12.0,-16.5,0.25,LAMDA,0,0,0,4,0)
      CALL HGPNUMBER(12.0,-17.0,0.25,E1.0.0,0,4,0)
      CALL HGPDASHLN(B1,0.0,F1,F2,0.5)
      CALL HGPDASHLN(-B1,0.0,F1,F2,0.5)
      CALL HGPSCURVE(X,Y,N,0,0,0,0,0)
      CALL HGPORIGIN(30.0,-22.0)
      RETURN
      END

END OF SEGMENT, LENGTH   467, NAME  PERSP
```

found

1) the matrices S_1^1, S_2^1, S_3^1
2) $|S_1^1|$, $|S_2^1|$, $|S_3^1|$
3) $|\mathcal{D}_1|$, $|\mathcal{D}_2|$, $|\mathcal{D}_3|$
4) the centre of each quadric
5) the eigenvalues of $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$
6) the corresponding eigenvectors
7) analysed the above to determine the form of each quadric

```
      MASTER BV06
      REAL LAMDA
      DIMENSION S(4,4),SS(4,4),D(6),W(3),Z1(9),X1(12),NAME(15,3),IW(3)
      COMMON/CM1/S/CM2/A,B,ALPHA,LAMDA,E1,KK
      B = 3.0
      DO 90 I = 1,15
   90 READ(1,899)(NAME(I,J),J=1,3)
  899 FORMAT(3A8)
   95 READ(1,900) A,ALPHA,LAMDA,E1,LAST
  900 FORMAT(4F6.2,I2)
      IF(LAST.EQ.1) GO TO 310
      KK = 1
      IF(LAMDA.EQ.0.0) KK = 0
      WRITE(2,905) A,B,ALPHA,LAMDA,E1
  905 FORMAT(1H1,10X,4HA = ,F7.1,5X,4HB = ,F5.1,5X,8HALPHA = ,F5.1,5X
     1       8HLAMDA = ,F5.1,5X,7HEPSILON,F5.1/)
      DO 800 IL = 1,3
      WRITE(2,906) IL
  906 FORMAT(1H0,25HCOEFFICIENTS OF QUADRIC S,I1,35H WITH RESPECT TO THE
     1 FIXATION POINT)
      CALL QUADRIC(IL)
      WRITE(2,910)((S(I,J),J=1,4),I=1,4)
  910 FORMAT(1H ,4E20.7)
      DO 96 I = 1,4
      DO 96 J = 1,4
   96 SS(I,J) = S(I,J)
      CALL INVERT(SS,4,D,3,DETS,ID)
      IF(ID.EQ.2) DETS = 0.0
      WRITE(2,914) DETS
  914 FORMAT(1H0,7HDETS = ,E20.7)
      ST = 1.E-10
      DD = 0.0
      DO 101 I = 1,3
  101 DD = DD + COFD(1,I)*S(4,I)
      WRITE(2,915) IL,DD
  915 FORMAT(1H0,1HD,I1,3H = ,E20.7)
      DO 103 I = 1,3
      P = 0.0
      DO 102 J = 1,3
  102 P = P + COFD(I,J)*S(J,4)
      W(I) = P
      IF(ABS(DD).LE.ST) GO TO 104
  103 W(I) = -P/DD
      WRITE(2,916) W
  916 FORMAT(1H0,21HCOORDINATES OF CENTRE,3F12.6)
      ITA = 1
      GO TO 110
  104 IF(ABS(W(1)).LE.ST) GO TO 105
      ITA = 2
      GO TO 110
  105 IF(ABS(COFD(1,1)).LE.ST) GO TO 106
      ITA = 3
      GO TO 110
  106 AST = S(2,3)*S(1,4) - S(1,3)*S(1,2)
      ITA = 4
      IF(ABS(AST).LE.ST) ITA = 5
  110 WRITE(2,917) ITA
  917 FORMAT(1H0,6HITA = ,I2)
      DO 115 I = 1,3
      DO 115 J = 1,4
      K = I*(I-1)/2 + J
  115 B(K) = S(I,J)
      N = 3
      IVS = 3
```

```
      EIGVEN = 1
      N1 = 3
      N2 = 1
      R1 = 0.0
      R2 = 0.0
      CALL FPROOT1(N,ST,IVS,D(1),W(1),Z1(1),X1(1),NE,IGIVEN,N1,N2,R1,R2)
      WRITE(2,923) NE,IL,(I,W(I),I=1,NE)
920   FORMAT(1H0,I2,1X,16HEIGENVALUES OF D,I1,7H FOUND:/
     1       (1H ,4(1X,I2,1H:,F8.5)))
      WRITE(2,925)
925   FORMAT(1H0,31HTHE CORRESPONDING EIGENVECTORS:)
      DO 120 J = 1,NE
      K = (J-1)*N
120   WRITE(2,930) J,(Z1(I),I=K+1,K+N)
930   FORMAT(1H ,I2,1H:,3(F9.5,3X))
      DO 150 I = 1,3
150   IW( ) = 0
      DO 190 I = 1,6
      IM = 2
      IF(W(I).GT.ST) IM = 1
      IF(W(I).LT.-ST) IM = 3
180   IW( M) = IW(IM) + 1
190   CONTINUE
      WRITE(2,931) IW
931   FORMAT(1H0,3HIW:,3I2)
      IW1 = IW(1)
      IW2 = IW(2)
      IF(IW(2).NE.0) GO TO 25
      IF( ETS.EQ.0.0) GO TO 230
      ITYPE = 4 - IW(1)
      IF(ITYPE.LT.1) ITYPE = 15
      IF(ITYPE.GT.3) ITYPE = 15
      GO TO 300
230   GO TO(232,232,231),IW1
231   ITYPE = 4
      GO TO 300
232   ITYPE = 5
      GO TO 300
250   IF( TA.EQ.0.4) GO TO 275
      GO TO(251,261,271),IW2
251   IF(W(1).EQ.0.0) GO TO 252
      IF(W(3).EQ.0.0) GO TO 253
      W(2) = W(3)
      GO TO 253
252   W(1) = W(3)
253   WW = W(1)*W(2)
      IF( TA.EQ.0.2) GO TO 28
      IF( TA.EQ.0.4) GO TO 256
      IF( TA.EQ.0.5) GO TO 256
      IF(WW) 254,254,255
254   ITYPE = 7
      GO TO 300
255   ITYPE = 6
      GO TO 300
256   IF(WW) 257,257,258
257   ITYPE = 9
      GO TO 300
258   ITYPE = 8
      GO TO 300
261   ITYPE = 10
      GO TO 300
271   ITYPE = 11
      GO TO 300
275   ITYPE = 14
      GO TO 300
28    IF(WW) 281,282,282
```

```
      291 ITYPE = 13
          GO TO 300
      292 ITYPE = 12
      300 WRITE(2,940) ITYPE,(NAME(ITYPE,J),J=1,6)
      940 FORMAT(1H0,I2,5X,14HSQUARERIG IS ,5X,6A8)
      800 CONTINUE
          GO TO 95
      810 STOP
          END

END OF SEGMENT, LENGTH  1742, NAME  BV06
```

```
      SUBROUTINE QUADRIC(IQ)
      REAL LAMDA
      DOUBLE PRECISION A,B,AL,EL,E,AB,A1(2),AA(2),GAM,S,C,B1(2),L(
     1),M(2),ST(2),CT(2),J1,J2,K1,K2,TK,TD,T1,AD1,AD2
      DIMENSION Z(4,4)
      COMMON/CM1/Z/CM2/AS,BS,ALPHA,LAMDA,EPS,K
      PIT = 0.0174532925
      A = AS
      B = BS
      AL = ALPHA*PIT
      EL = LAMDA*PIT
      ED = EPS*PIT
      E = TAN(ED)
      A1 = A*A + B*B
      D = 2.0*A*B*DCOS(AL)
      A1(1) = DSQRT(A1 - D)
      A1(2) = DSQRT(A1 + D)
      AA(1) = (A1(2)-A1(1))/2.0
      AA(2) = (A1(2)+A1(1))/2.0
      GAM = 0.5*DATAN2(2.0*A*B*DSIN(AL),A*A-B*B)
      S = DSIN(GAM)
      C = DCOS(GAM)
      B1(1) = A*DCOS(AL) - B
      B1(2) = A*DCOS(AL) + B
      DO 100 I = 1,2
      L(I) = A1(I) + A*DSIN(AL)*DCOS(EL)
      M(I) = A*DSIN(AL) + A1(I)*DCOS(EL)
      ST(I) = B1(I)*DSIN(EL)/L(I)
  100 CT(I) = M(I)/L(I)
C     K .EQ. 1 SYMMETRIC APPROXIMATION
      IF(K.EQ.1) GO TO 101
      J1 = -1.0
      J2 = 1.0
      K1 = -1.0
      K2 = 1.0
      TK = 0.2
      TD = 0.2
      GO TO 102
  101 TK = ST(1)/CT(1)
      TD = (TK+E)/(1.0 - TK*E)
      J1 = ST(2)/ST(1)
      J2 = CT(2)/CT(1)
      K1 = (J1*TK-J2*E)/(TK+E)
      K2 = (J2+J1*TK*E)/(1.0 +TK*E)
  102 AD1 = (A1(2) - A1(1)*K2)/2.0
      AD2 = (A1(2) - A1(1)*K2)/2.0
      T1 = TD*TK
      GO TO(200,300),IQ
  200 Z(1,1) =  (1.0+K2)*S*C
      Z(1,2) = -(1.0-K2)/2.0
      Z(1,3) = ((1.0-K1)*C*TD)/2.0
      Z(1,4) = -AD2*S
      Z(2,2) =  Z(1,1)
      Z(2,3) =-((1.0+K1)*S*TD)/2.0
      Z(2,4) =  AD1*C
      Z(3,3) =  0.0
      Z(3,4) = -(A1(2) - A1(1)*K1)*TD/2.0
      GO TO 500
  300 Z(1,1) =  (1.0+J1)*S*C*TK
      Z(1,2) =  -(1.0-J1)*TK/2.0
      Z(1,3) = -(1.0-J2)*C/2.0
      Z(1,4) = -(A1(2) + A1(1)*J1)*S*TK/2.0
      Z(2,2) = Z(1,1)
      Z(2,3) =  (1.0+J2)*S/2.0
```

```
      Z(2,4) =  (A1(2) - A1(1)*J1)*C*(K/2.0
      Z(3,3) = 0.0
      Z(3,4) =  (A1(2) - A1(1)*J2)/2.0
      GO TO 500
  400 Z(1,1) =  (K2-J1)*TK*S**2
      Z(1,2) =  0.0
      Z(1,3) = -((K1+J1)*T1 + (K2+J2))*S/2.0
      Z(1,4) =  0.0
      Z(2,2) = -(K2-J1)*TK*C**2
      Z(2,3) =  ((K1-J )*T1 + (K2-J2))*C/2.0
      Z(2,4) =  0.0
      Z(3,3) =  (K1-J2)*TD
      Z(3,4) =  0.0
  500 Z(2,1) =  Z(1,2)
      Z(3,1) =  Z(1,3)
      Z(3,2) =  Z(2,3)
      Z(4,1) =  Z(1,4)
      Z(4,2) = Z(2,4)
      Z(4,3) = Z(3,4)
      Z(4,4) = 0.0
      RETURN
      END

END OF SEGMENT, LENGTH  1271, NAME  QUADRIC
```

```
      FUNCTION CFD(IL,JL)
      DIMENSION S(4,4),C(2,2)
      COMMON/CM1/S
      IK = 0
      DO 200 I = 1,4
      IF(I.EQ.IL) GO TO 200
      IK = IK + 1
      JK = 0
      DO 100 J = 1,4
      IF(J.EQ.JL) GO TO 100
      JK = JK + 1
      C(IK,JK) = S(I,J)
  100 CONTINUE
  200 CONTINUE
      SIGN = (-1)**(IL+JL)
      CFD = SIGN*(C(1,1)*C(2,2) - C(1,2)*C(2,1))
      RETURN
      END

END OF SEGMENT, LENGTH   145, NAME  CFD
```

```
      SUBROUTINE INVERT(A,N1,B,M1,DETERM,ID)
C     A SUBROUTINE TO INVERT A MATRIX OR TO SOLVE SIMULTANEOUS EQUATION
      DIMENSION A(N1,N1),B(N1,1),INDEX(11,3)
      EQUIVALENCE (IROW,JROW),(ICOLUM,JCOLUM),(AMAX,T,SWAP)
C     INITIALIZATION
C
      N = N1
      M = M1
   10 DETERM = 1.0
   15 DO 20 J = 1,N
   20 INDEX(J,3) = 0
   30 DO 550 I =1, N
C
C     SEARCH FOR PIVOT ELEMENT
C
   40 AMAX = 0.0
   45 DO 105 J=1,N
      IF(INDEX(J,3)-1) 60,105,60
   60 DO 100 K =1,N
      IF(INDEX(K,3) -1) 80,100,715
   80 IF(AMAX -ABS(A(J,K))) 85,100,100
   85 IROW = J
   90 ICOLUM = K
      AMAX = ABS(A(J,K))
  100 CONTINUE
  105 CONTINUE
      INDEX(ICOLUM,3) = INDEX(ICOLUM,3) + 1
  260 INDEX(I,1) = IROW
  270 INDEX(I,2) = ICOLUM
C
C     INTERCHANGE ROWS TO PUT PIVAT ELEMENT ON DIAGONAL
C
  130 IF(IROW -ICOLUM) 140,310,140
  140 DETERM = -DETERM
  150 DO 200 L =1,N
  160 SWAP = A(IROW,L)
  170 A(IROW,L) = A(ICOLUM,L)
  200 A(ICOLUM,L) = SWAP
      IF(M) 310,310,210
  210 DO 260 L = 1,M
  220 SWAP = B(IROW,L)
  230 B(IROW,L) = B(ICOLUM,L)
  250 B(ICOLUM,L) = SWAP
C
C     DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
  310 PIVOT = A(ICOLUM,ICOLUM)
      DETERM = DETERM*PIVOT
  330 A(ICOLUM,ICOLUM) = 1.0
  340 DO 350 L =1,N
  350 A(ICOLUM,L) = A(ICOLUM,L)/PIVOT
  355 IF(M) 360,360,380
  360 DO 370 L = 1,M
  370 B(ICOLUM,L) = B(ICOLUM,L)/PIVOT
C
C     REDUCE NON-PIVOT ROWS
C
  380 DO 550 L1 =1,N
  390 IF(L1-ICOLUM) 400,550,400
  400 T=A(L1,ICOLUM)
  420 A(L1,ICOLUM) = 0.0
  430 DO 450 L = 1,N
  450 A(L1,L)=A(L1,L) - A(ICOLUM,L)*T
  455 IF(M) 550,550,460
```

```
      460 DO 560 L = 1,M
      550 B(I,L) = B(L1,L) - B(ICOLUM,L)*T
      560 CONTINUE
C
C     INTERCHANGE COLUMNS
C
      600 DO 710 I = 1,N
      610 L = N + 1 - I
      620 IF( INDEX(L,1)-INDEX(L,2)) 630,710,630
      630 JROW=INDEX(L,1)
      640 JCOLUM=INDEX(L,2)
      650 DO 705 K=1,M
      660 SWAP=A(K,JROW)
      670 A(K,JROW) = A(K,JCOLUM)
      700 A(K,JCOLUM) = SWAP
      705 CONTINUE
      710 CONTINUE
          DO 730 K = 1,N
          IF(INDEX(K,3)-1) 715,725,715
      715 IO = 2
          GO TO 740
      725 CONTINUE
      730 CONTINUE
          IO = 1
      740 RETURN
          END

END OF SEGMENT, LENGTH   648, NAME=  INVERT
```

```
        A =   3.0.0     B =   3.0     ALPHA =  45.0     LAMBDA =    0.0     EPSILON  1.0

 COEFFICIENTS OF QUADRIC S: WITH RESPECT TO THE FIXATION POINT
        0.1414214E-01          .00 0000E 00          0.1999950E-01          -0.2121426E 01
        .000 10E 00            0.1414214E-01         0.0000000E 00          0.21212146 01
        .1999950E-01           .00 0000E 00          0.0000000E 00          -7.. 0001 E 01
      -0.21214266 01           0.21212146 01        -0.6000150E 01          -0.0000000E 00

 DETS =          0.179973 E-12

 D1 =         -0.5656571E-16

 COORDINATES OF CENTRE  30..215001 -14.,992501 -100,073972

 ITA =  1

  3 EIGENVALUES OF D1 FOUND:
   1:  0,12 0   2:  0. 1414   3:-  , 1414

 THE CORRESPONDING EIGENVECTORS:
  1;  0,4105E          .0000          .37735
  2;  0,  00          -1  0000         0,00000
  3;  0,37735         -.,0000        -0,31990

 IWI 2 0 1

   2     QUADRIC IS;     HYPERBOLOID OF ONE SHEET

 COEFFICIENTS OF QUADRIC S2 WITH RESPECT TO THE FIXATION POINT
          .  00 00E          -  ,00 0000E-01          0.0000000E 0          -2.0000 01E-05
        -0.2000 606E-01          ,0000000E 00          0.7071245E-02          0.0000000E 01
          .  0000E 0             .7071245E-02          0.0000000E 0           0.21212676 01
        -0.0000010E-03           3.5000000E 01         0.21212676 01          3.1000000E 00

 DETS =          0.1299736E-12

 D2 =          0.3041 03E 00

 ITA =  2

  3 EIGENVALUES OF D2 FOUND:
   1:  ,2 21   2:  0.00   3:-  ,02121

 THE CORRESPONDING EIGENVECTORS:
  1;  0,46166          -.70711          -,23571
  2;  0,33334          -.00000           0,74281
  3;  0,56466           .70711          -0.,23571

 IWI 1 1 1

  13     QUADRIC IS;     HYPERBOLIC PARABOLOID

 COEFFICIENTS OF QUADRIC S2 WITH RESPECT TO THE FIXATION POINT
          .2 001 0E-05           .5000000E 01         -0.7046416E-02           .7070 00E 0
          .  0000E 00          -0.3999900E-01          0.5000000E 0          -2.0000 00E 00
        -0.0684 6E-12            .0 0000E 00          -0.4000000E-01          0.000 00E 0
          .  0000E 0             .0010000E 00          0.0000000E 0           3.1000 0E 0

 DETS =          0.0 1 0000E 0
```

```
IFA = 1

 3 EIGENVALUES OF DA FOUND:
  1: 0.00121   2: -0.04000   3: -0.24121

THE CORRESPONDING EIGENVECTORS:
 1:  0.98561     0.00000     -0.16904
 2:  0.00000     1.00000      0.00000
 3:  0.16904    -0.00000      0.98561

I=: 1 0 2

 >      QUADRIC IS:      CONE
```

```
A =  3.0     B =  3.0     ALPHA =  70.0     LAMBDA =  30.0     EPSILON =

COEFFICIENTS OF QUADRIC S1 WITH RESPECT TO THE FIXATION POINT
 1.7818908E-01     -.4.9375E-02      0.1486432E-01     -1.2823471E 01
  .1699758E-02      -.8.1898E-01     -0.8910938E-03      0.1045435E 00
  .4834329E-        -.1.9 0346E-01     0.0000000E 00     -0.5629689E 01
 -0.2823471E 01      .0094059E 01     -0.4562989E 01      0.0000000E 01

DET1 =        0.1387342E-03

D1 =       -0.4221531E-05

COORDINATES OF CENTRE  382,752279  -83,859799  -167.386680

ITA =

 3 EIGENVALUES OF D1 FOUND:
  1: -.2410   2:  .1862   3: -.00625

THE CORRESPONDING EIGENVECTORS:
 1:  0.97526     0.09745     0.07559
 2:  0.1580     -.99378      .9411
 3:  0.43289    -0.65366    -0.17342

IMI = 0 1

  2    QUADRIC IS:    HYPERBOLOID OF ONE SHEET

COEFFICIENTS OF QUADRIC S2 WITH RESPECT TO THE FIXATION POINT
 -.1771564E-02      0.25 9731E-02     -0.2452310E-03     -0.1648732E 01
  .2585731E-02      0.1771914E-02      0.9394317E-02     -0.1797860E 01
 -.2492410E-03      .1394317E-02      -0.8888888E 01      0.1099339E 01
 -0.1648732E 01     -.1797860E 01      0.1099339E 01      0.0000000E 01

DET2 =        0.2582344E-07

D2 =       -0.1693562E-08

COORDINATES OF CENTRE  384,522975  -108,939285    ,639370

ITA =

 3 EIGENVALUES OF D2 FOUND:
  1:  .1 68   2:  .0125   3: -.0993

THE CORRESPONDING EIGENVECTORS:
 1:  0.1 09      .3831      0.84903
 2:  0.5 73     -.2516     -0.46867
 3:  0.7902     -.8741      0.74849

IMI = 0 1

  2    QUADRIC IS:    HYPERBOLOID OF ONE SHEET

COEFFICIENTS OF QUADRIC S2 WITH RESPECT TO THE FIXATION POINT
 -.1 44 65E- 5      .3 0 0 E 00      -0.9492288E-02      0.0000000E 00
  .1 08 0 3E 02     0.9494927E-02      0.1786631E-21      0.0000000E 00
 -0.9492288E-02      .7865316E-21     -0.2971614E-01      0.0000000E 00
  .0 9 0 0 0E 1      .0 0 0 0 E 00      0.0000000E 00      0.0000000E 01

DET3 =
```

```
COORDINATES OF CENTRE      0.000000     0.000000     0.000000

ITA =  1

 3 EIGENVALUES OF B3 FOUND:
  1:  0.00192   2:  0.0277   3: 0.28299

THE CORRESPONDING EIGENVECTORS:
 1:  0.96880     -1.00000      0.00000
 2:  0.39687      0.00000      0.28044
 3:  0.28044      1.00000      0.95967

IW: 2 3 1

  5      QUADRIC IS:      CONE
```

```
                F =   3 . 3        B =   3.0      ALPHA =   ..0      LAMBDA =    . 0      EPSILON =

     COEFFICIENTS OF QUADRIC S  WITH RESPECT TO THE FIXATION POINT
          .992E-02 .            . 5 .0E-0 .          0.1999X00E-01          . .E 0 0 E
          . .00 .0E             .19 9900E-01         0.-.E0 0 00E 0         .1376A00E-0A
          .1000 .0E- 1          . 3 0.0AE 0.         -. .0 00E 0            . .0 0A0 E
        -.3.90 .0E- 1          .53.6000E-02         -0.A00.03E0E 01          . 000.00E

     DETS =          0.1.562.75e- 9

     D1 =          -0.79934 3E-09

     COORDINATES OF CENTRE  A0 .333000   -0. 01000 -150.70/500

     IN: 2 0 1

      3 EIGENVALUES OF D1 FOUND:
       1: 1.03/30  2: . 2 00  3:-0.31236

     THE CORRESPONDING EIGENVECTORS:
      1:  0.95165        .00000        0.52574
      2:  .  .0          -1. 0.0       0.00000
      3:  0.52574        - .00000      -0. 5065

     IN: 2 0 1

       2     QUADRIC IS:      HYPERBOLOID OF ONE SHELT

     COEFFICIENTS OF QUADRIC S2 WITH RESPECT TO THE FIXATION POINT
          1. .000.00E 0          - .2000.00E-01          0. .000.00E 0          -0.1075362E-11
        - .2000.00E-01           1.000000E 00            0.9999500E-02          0.0000000E 00
          . .000.00E 0           1.999950E-02            0.00.00E0E 0           0.5377.78E-08
        -0.1075362E-11           1.00.00000E 01          0.5377E78E-08          1.0000.00E 00

     DETS =          0.1.562.78-19

     D2 =          0.000 000E 00

     IN: 3

      3 EIGENVALUES OF D2 FOUND:
       1: 0.00236  2:  .00000  3:-0. 2236

     THE CORRESPONDING EIGENVECTORS:
      1:  0.63266      - .70711      -0.31662
      2:  0.44720      - .70710      1.99944
      3:  0.63246        .70711      -0.31622

     IN: 1 1 1

       3     QUADRIC IS:      HYPERBOLIC CYLINDER

     COEFFICIENTS OF QUADRIC S3 WITH RESPECT TO THE FIXATION POINT
          1.1999X00E- 5          0.00 0000E 00          -0.9995500E-02          . 000.00E 0
          1.0000.00E 00          -0.39995.0E-01          0. .0.0.00E 00          . 000.00E 0
        -1.4995000E- 2           .00.00.0E 0.           -0.40000.00E-01         .1000.00E 0
          1.1000.00E 01          0.0000000E 00          0.00000000E 00          . 000.00E .

     DETS =          0.000 0000E 00

     D3 =          0.40024.0E- 5

     COORDINATES OF CENTRE     0.000000      0.10.000      0.000000
```

```
IIA =  1

 3 EIGENVALUES OF DA FOUND:
  1:  0,00236  2:-0.04000  3:-0,04236

THE CORRESPONDING EIGENVECTORS:
 1:  0,97527      0.00000     -0,22965
 2:  0,00000      1.00000      0,00000
 3:  0,22965      0.00000      0,97527

IM: 1 0 2

  3      QUADRIC IS:      CONE
```

PROGRAM BVO7

found

1) the inverse transformation matrix P^{-1}
2) S_1', S_2', S_3'
3) S_1, S_2, S_3
4) the equation of the conic in which each of the quadrics met each of the co-ordinate planes.
5) the nature of this conic
6) the tangent plane to this quadric at various points of the conic
7) the direction cosines of the generating lines of the quadric at each of these points
8) the points at which each of these lines met the limiting planes.
9) converted the co-ordinates of these points to their perspective equivalents and thus plotted the quadrics with respect to the fixed co-ordinates in terms of their generating lines.

see: figs: 18.1, 18.7, 18.8, 18.9

(only a sample line printer output has been included)

```
      MASTER 8V07
      REAL LAMDA
      DIMENSION NAME(4,2),S(4,4),P(4,4),PT(4,4),F(4),XM(4),XT(31,4),
     1YT(31,2),S1(4,4),X0(2),B1(2),X1(2),TH(60,4),XX(11),YY(11),SL(6),
     2A0(6),C(3,3),BR(2,3),TMIN(2)
      COMMON/CM1/S/CM2/A,B,ALPHA,LAMDA,E1,KK/CM3/P
      PIT = 0.0174532925
      B = 3.0
      SI = 1.E-10
      DO 85 I = 1,4
   89 READ(1,86) (NAME(I,J),J=1,2)
   86 FORMAT(2A8)
      CALL H3PSTART(2 HRECTILINEAR HORDPT=R,20)
   98 READ(1,900 ) A,ALPHA,LAMDA,E1,LAST
  900 FORMAT(4F6.2,I2)
      IF(LAST.EQ.1) GO TO 510
      KK = 1
      IF(LAMDA.EQ.0.0) KK = 0
      WRITE(2,905) A,B,ALPHA,LAMDA,E1
  905 FORMAT(1H1,10X,4HA = ,F7.1,5X,4HB = ,F5.1,5X,8HALPHA = ,F5.1,5X,
     1       8HLAMDA = ,F5.1,5X,7HEPSILON,F5.1/)
      AL = ALPHA*PIT
      EL = LAMDA *PIT
      F(1) = A*SIN(AL)*COS(EL)
      F(2) = - A*COS(AL)
      F(3) = - A*SIN(AL)*SIN(EL)
      F(4) = 1.0
      CALL QUADRIC(1)
      CALL PMAT(-1)
      WRITE(2,909)
  909 FORMAT(1H0,45HCOEFFICIENTS OF INVERSE TRANSFORMATION MATRIX)
      DO 80 I = 1,4
      DO 80 J = 1,4
   80 PT(I,J) = P(J,I)
      WRITE(2,910) ((P(I,J),J=1,4),I=1,4)
  910 FORMAT(1H ,4E20.7)
      DO 500 IL = 1,3
      CALL QUADRIC(IL)
      WRITE(2,914) IL
  914 FORMAT(1H ,25HCOEFFICIENTS OF QUADRIC S,I1,35H WITH RESPECT TO THE
     1 FIXATION POINT)
      WRITE(2,910) ((S(I,J),J=1,4),I=1,4)
      CALL H3PORIGIN(20.0,22.0)
      CALL H3PAXIS(-20.0, ,0,8H        Y,8,40,0,0.0,-400.0,20.0)
      CALL H3PAXIS( 0.0,-20.0,8H       Z,8,40,0,90.,-400.0,20.0)
      XM(2) = 0.0
      XM(3) = 0.0
      XM(4) = 1.0
      DO 100 I = 1,31
      XI = I
      XM(1) = -22.0 + 20.0*XI
      CALL PROJECTION(XM,XA,YA)
      XT(I,1) = XA/20.0
      YT(I,1) = YA/20.0
      XT(I,2) = XT(I,1) + 0.1
  100 YT(I,2) = YT(I,1) - 0.1
      CALL H3PDASHLN(XT(1,1),YT(1,1),XT(31,1),YT(31,1),0.0)
      DO 105 I = 1,31
  105 CALL H3PDASHLN(XT(I,1),YT(I,1),XT(I,2),YT(I,2),0.0)
      CALL PROJECTION(F(1),F(2),F1,F2)
      F1 = F1/20.0
      F2 = F2/20.0
      B2 = B/20.0
      IL = IL
```

```
      CALL HGPSYMBL(10.0,-14.5,0.25,31HRECTILINEAR HOROPTER SURFACE S ,0
     1.0,31)
      CALL HGPNUMBER(35.7,-14.5,0.15,IL,0.0,0,1,0)
      CALL HGPSYMBL(10.0,-15.0,0.25,4HA = ,0.0,4)
      CALL HGPSYMBL(10.0,-15.5,0.25,4HB = ,0.0,4)
      CALL HGPSYMBL(10.0,-16.0,0.25,6HALPHA = ,0.0,6)
      CALL HGPSYMBL(10.0,-16.5,0.25,6HLAMDA = ,0.0,6)
      CALL HGPSYMBL(10.0,-17.0,0.25,6HEPS   = ,0.0,6)
      CALL HGPNUMBER(12.0,-15.0,0.25,A,0.0,2,4,0)
      CALL HGPNUMBER(12.0,-15.5,0.25,B,0.0,2,4,0)
      CALL HGPNUMBER(12.0,-16.0,0.25,ALPHA,0.0,0,4,0)
      CALL HGPNUMBER(12.0,-16.5,0.25,LAMDA,0.0,0,4,0)
      CALL HGPNUMBER(12.0,-17.0,0.25,E1,0.0,0,4,0)
      CALL HGPDASHLN(82.0,0.0,F1,F2,0.5)
      CALL HGPDASHLN(-84.,0.0,F1,F2,0.2)
      CALL PROD(4.4,4.S,P,S1)
      CALL PROD(4.4,4,PT,S1,S)
      WRITE(2,915) IL
915 FORMAT(1H0,25HCOEFFICIENTS OF QUADRIC S,I1,39H WITH RESPECT TO THE
     1 FIXED CO-ORDINATES )
      WRITE(2,916) ((S(I,J),J=1,4),I=1,4)
      M1 = 0
      DO 300 I = 1,3
      J = I + 1
      IF(J.GE.4) J = J - 3
      K = I + 2
      IF(K.GE.4) K = K - 3
      C(1,1) = S(J,J)
      C(1,2) = S(J,K)
      C(1,3) = S(J,4)
      C(2,2) = S(K,K)
      C(2,3) = S(K,4)
      C(3,3) = S(4,4)
      C(2,1) = C(1,2)
      C(3,1) = C(1,3)
      C(3,2) = C(2,3)
      WRITE(2,919) I,IL
919 FORMAT(1H0,46HCOEFFICIENTS OF THE CONIC IN WHICH THE PLANE y,I1,
     1 5H = 0 ,19HMEETS THE QUADRIC S,I1)
      WRITE(2,920) ((C(I1,J1),J1=1,3),I1=1,3)
920 FORMAT(1H ,3E20.7)
      R = DET3(C)
      WRITE(2,921) R
921 FORMAT(1H ,23HDETERMINANT OF CONIC IS,5X,E20.7)
      IF(ABS(DET3(C)).LT.ST) GO TO 113
      T2 = S(J,K)**2 - S(J,J)*S(K,K)
      IF(T2) 110,111,112
110   IK = 3
      GO TO 114
111   IK = 2
      GO TO 114
112   IK = 1
      GO TO 114
113   IK = 4
114   WRITE(2,924) (NAME(IK,I1),I1=1,6)
924 FORMAT(1H0,11HCONIC IS  ,2A8)
      T1 = 0.5*ATAN2(2.*S(J,K),S(J,J)-S(K,K))
      CO = COS(T1)
      SO = SIN(T1)
      C(1,1) = S(J,J)*CO**2 + S(K,K)*SO**2 + 2.0*S(J,K)*SO*CO
      C(2,2) = S(J,J)*SO**2 + S(K,K)*CO**2 - 2.0*S(J,K)*SO*CO
      C(1,2) = 0.
      C(1,3) = S(J,4)*CO + S(K,4)*SO
      C(2,3) = -S(J,4)*SO + S(K,4)*CO
      C(3,3) = S(4,4)
      C(2,1) = C(1,2)
```

```
      C(3,2) = C(2,3)
      T = T0/PIT
      WRITE(2,929) I,T
  929 FORMAT(1H0,40HCOEFFICIENTS OF THE CONIC IN THE PLANE X,I1,5H = 0 A
     1X,26HROTATED THROUGH THE ANGLE ,F7.2,8H DEGREES)
      WRITE(2,920) ((C(I1,J1),J1=1,3),I1=1,3)
      R = DET3(C)
      DO 119 J1 = 1,2
  119 IF(ABS(C(J1,J1)).LT.ST) GO TO 125
      DO 120 J1 = 1,2
      X0(J1) = -C(J1,3)/C(J1,J1)
  120 B1(J1) = -R/(C(J1,J1)*C(1,1)*C(2,2))
      WRITE(2,925) X0,B1
  925 FORMAT(1H0,7HCENTRE:,2F10.4,10X,5HAXES:,2F10.4)
  125 IPATH = 0
      IF(IK.EQ.1) IPATH = 1
      IF(IK.EQ.4) IPATH = 1
  126 N1 = 0
      DO 200 IT = 1,11
      TI = IT
      GO TO(130,140,150,160),IK
  130 T = -6.0+ TI
      IF(B1(1).LT.0.0) GO TO 131
      X1(1) = X0(1) + SQRT(B1(1))*COSH(T)
      X1(2) = X0(2) + SQRT(-B1(2))*SINH(T)
      IF(IPATH.EQ.2) X1(1) = -X1(1)
      GO TO 170
  131 X1(1) = X0(1) + SQRT(-B1(1))*SINH(T)
      X1(2) = X0(2) + SQRT(B1(2))*COSH(T)
      IF(IPATH.EQ.2) X1(2) = -X1(2)
      GO TO 170
  140 L1 = 1
      L2 = 2
      IF(ABS(C(1,1)).GT.ST) GO TO 141
      L1 = 2
      L2 = 1
  141 X0(L1) = -C(L1,3)/C(L1,L1)
      TOP = C(L1,3)**2 - C(L1,L1)*C(3,3)
      BOT = 2.0*C(L1,L1)*C(L2,3)
      X0(L2) = TOP/BOT
      B1(1) = -0.5*C(L2,3)/C(L1,L1)
      WRITE(2,926) X0,B1(1)
  926 FORMAT(1H0,7HCENTRE:,2F10.4,10X,5HAXIS:,F10.4)
      TMAX = (350.0-X0(L2))/B1(1)
      TMAX = SQRT(TMAX)
      T = TMAX*(TI - 5.0)/5.0
      X1(L1) = X0(L1) + 2.0*B1(1)*T
      X1(L2) = X0(L2) + B1(1)*T**2
      GO TO 170
  150 IF(B1(1).LT.0.0) GO TO 151
      T = (TI-1.0)* .6283185306
      X1(1) = X0(1) + SQRT(B1(1))*COS(T)
      X1(2) = X0(2) + SQRT(B1(2))*SIN(T)
      GO TO 170
  151 WRITE(2,930) I
  930 FORMAT(1H0,7HPLANE X,I1,15H = 0:  NO LOCUS)
      GO TO 300
  160 T = 300.0*(TI-6.0)/5.0
      IF(ABS(C(1,1)).LT.ST) GO TO 166
      CM = -C(2,2)/C(1,1)
      IF(CM) 161,162,165
  161 WRITE(2,935)
  935 FORMAT(1H0,12HSINGLE POINT)
      X1(1) = X0(1)
      X1(2) = X0(2)
```

```
      IPATH = 4
      GO TO 170
  162 WRITE(2,936)
  936 FORMAT(1H0,5HERROR)
      GO TO 300
  163 CM = SQRT(CM)
      IF(IPATH.EQ.2) GO TO 164
      BM = X0(1) - X0(2)*CM
      GO TO 165
  164 BM = X0(1) + X0(2)*CM
      CM = -CM
  165 X1(1) = CM*T + BM
      X1(2) = T
      GO TO 170
  166 IF(ABS(C(2,2)).GT.ST) GO TO 162
      IPATH = 0
      WRITE(2,940)
  940 FORMAT(1H0,11HSINGLE LINE)
      IF(ABS(C(1,3)).GT.ST) GO TO 167
      X1(1) = T
      X1(2) = -C(3,3)/(2.0*C(2,3))
      GO TO 170
  167 CM = -C(2,3)/C(1,3)
      BM = -C(3,3)/(2.0*C(1,3))
      GO TO 165
  170 XM(J) = X1(1)*COS(T0) - X1(2)*SIN(T0)
      XM(K) = X1(1)*SIN(T0) + X1(2)*COS(T0)
      XM(1) = 0.0
      XM(4) = 1.0
      DO 175 I1 = 1,3
  175 IF(ABS(XM(I1)).GT.400.0) GO TO 200
      IF(XM(1).LT.-150.0) GO TO 200
      CALL PROJECTION(XM,XA,YA)
      IF(ABS(XA).GT.400.0) GO TO 200
      IF(ABS(YA).GT.400.0) GO TO 200
      N1 = N1 + 1
      M1 = M1 + 1
      DO 180 I1 = 1,4
  180 YM(M1,I1) = XM(I1)
      IF(IPATH.EQ.4) GO TO 300
      XX(N1) = XA/20.0
      YY(N1) = YA/20.0
  200 CONTINUE

      CALL HGPSCURVE(XX,YY,N1,0.0,0.0,0.0,0.0)
      IPATH = IPATH + 1
      IF(IPATH.EQ.2) GO TO 120
  300 CONTINUE

      DO 400 IT = 1,M1
      DO 310 I = 1,5
      SL(I) = 0.0
      DO 305 J = 1,4
  305 SL(I) = SL(I) + S(I,J)*YM(IT,J)
  310 CONTINUE
      WRITE(2,950) IT,(YM(IT,J),J=1,4),SL
  950 FORMAT(1H ,I2,5X,4F10.4,5X,5F10.4)
      DO 315 I1 = 1,3
      IF(ABS(SL(I1)).GT.ST) GO TO 320
  315 CONTINUE
      WRITE(2,936)
      GO TO 400
  320 I = I1
      J = I1 + 1
      K = I1 + 2
      IF(J.GE.4) J = J - 3
```

```
      IF(K.GE.4) K = K - 3
      AC(1) = S(I,I)*SL(J)**2 + S(J,J)*SL(I)**2 - 2.0*S(I,J)*SL(I)*SL(J)
      AC(2) = 2.0*(S(I,I)*SL(J)*SL(K) + S(J,K)*SL(I)**2
     1       -S(I,K)*SL(I)*SL(J) - S(I,J)*SL(I)*SL(K))
      AC(3) = S(I,I)*SL(K)**2 + S(K,K)*SL(I)**2 - 2.0*S(I,K)*SL(I)*SL(K)
      IF(ABS(AC(1)).GT.ST) GO TO 330
      IF(ABS(AC(3)).GT.ST) GO TO 325
      DR(1,I) = -SL(K)
      DR(1,J) = 0.0
      DR(1,K) = SL(I)
      DR(2,I) = SL(J)
      DR(2,J) = -SL(I)
      DR(2,K) = 0.0
      GO TO 335
  325 I1 = J
      J = K
      K = I1
      BM = AC(1)
      AC(1) = AC(3)
      AC(3) = BM
  330 CALL QUAD(AC,DR(1,J),DR(2,J),YI)
      IF(YI.EQ.0.0) GO TO 331
      WRITE(2,965) AC,YI
  965 FORMAT(1H ,13HCOMPLEX ERROR,4F10.4)
  331 DO 332 I1 = 1,2
      DR(I1,K) = 1.0
  332 DR(I1,I) = -(DR(I1,J)*SL(J) + DR(I1,K)*SL(K))/SL(I)
  335 DO 340 I1 = 1,2
      DSQ = 0.0
      DO 336 J1 = 1,3
  336 DSQ = DSQ + DR(I1,J1)**2
      DSQ = SQRT(DSQ)
      DO 337 J1 = 1,3
  337 DR(I1,J1) = DR(I1,J1)/DSQ
  340 CONTINUE
      WRITE(2,960) ((DR(I1,J1),J1=1,3),I1=1,2)
  960 FORMAT(1H0,17HDIRECTION COSINES,10X,3F10.4/1H ,27X,3F10.4)
      DO 390 I1 = 1,2
      TMIN(1) = 1.E10
      TMIN(2) = -1.E10
      DO 350 J1 = 1,2
      DO 350 J2 = 1,3
      TS = 350.0
      IF(J1.EQ.1) GO TO 343
      TS = -350.0
      IF(J2.EQ.1) TS = -450.0
  343 IF(ABS(DR(I1,J2)).LT.ST) GO TO 344
      T = (TS-YN(I1,J2))/DR(I1,J2)
      GO TO 345
  344 T = TS
  345 IF(T.GT.0.0) GO TO 346
      IF(T.LE.TMIN(2)) GO TO 350
      TMIN(2) = T
      GO TO 350
  346 IF(T.GE.TMIN(1)) GO TO 350
      TMIN(1) = T
      GO TO 350
  349 TMIN(J1) = TS
  350 CONTINUE
      WRITE(2,965) TMIN
  965 FORMAT(1H0,49X,1HI,2F10.4)
      XM(4) = 1.0
      DO 370 J1 = 1,2
      DO 360 J2 = 1,3
  360 XM(J2) = YN(I1,J2) + DR(I1,J2)*TMIN(J1)
      WRITE(2,970) XM
```

```
  370 FORMAT(1H ,4X,2HX=,F10.4,
     CALL PROJECTION(XA,YA)
     IF(ABS(XA).GT.400.0) GO TO 390
     IF(ABS(YA).GT.400.0) GO TO 390
     X1(J1) = XA/20.0
  380 X1(J1) = YA/20.0
     CALL HGPOLYLN(X0(1),X1(1),X0(2),X1(2),0,0)
  390 CONTINUE
  400 CONTINUE
     CALL HGPORIGIN(90.0,-22.0)
  500 CONTINUE
     GO TO 95
  510 CALL HGPFINISH
     STOP
     END

END OF SEGMENT, LENGTH  3010, NAME  BV07
```

```
      SUBROUTINE PROJECTION(XH,X,Y)
      DIMENSION XH(4),D(3)
      D(1) = -400.0
      D(2) = -400.0
      D(3) = 10.0*20
      XD = XH(4)*D(1) - XH(1)
      IF(XD.EQ.0.0) GO TO 100
      X = -(D(1)*XH(2) - D(2)*XH(1))/XD
      Y =  (D(1)*XH(3) - D(3)*XH(1))/XD
      GO TO 105
  100 X = 500.0
      Y = 500.0
  105 RETURN
      END
```

END OF SEGMENT, LENGTH 152, NAME PROJECTION

```
      FUNCTION DET3(C)
      DIMENSION C(3,3)
      DET3 = 0.0
      DO 100 I = 1,3
  100 DET3 = DET3 + C(1,I)*CFD(C,1,I)
      RETURN
      END
```

END OF SEGMENT, LENGTH   51, NAME DET3

```
      FUNCTION CFD(D,IL,JL)
      DIMENSION D(3,3),C(2,2)
      IK = 0
      DO 200 I = 1,3
      IF(I.EQ.IL) GO TO 200
      IK = IK + 1
      JK = 0
      DO 100 J = 1,3
      IF(J.EQ.JL) GO TO 100
      JK = JK + 1
      C(IK,JK) = D(I,J)
  100 CONTINUE
  200 CONTINUE
      SIGN = (-1)**(IL+JL)
      CFD = SIGN*(C(1,1)*C(2,2) - C(1,2)*C(2,1))
      RETURN
      END

END OF SEGMENT, LENGTH   151, NAME  CFD
```

```
      SUBROUTINE PROD(LC,MC,NC,A,B,C)
      DIMENSION A(NC,LC),B(LC,MC),C(NC,MC)
C     TO CALCULATE THE MATRIX PRODUCT C = AB
      DO 120 I = 1,NC
      DO 120 J = 1,MC
      C(I,J) = 0.0
      DO 100 K = 1,LC
  100 C(I,J) = C(I,J) + A(I,K)*B(K,J)
  120 CONTINUE
      RETURN
      END
```

END OF SEGMENT, LENGTH 117, NAME PROD

```
      SUBROUTINE PMAT(I)
      REAL LAMDA
      DIMENSION P(4,4)
      COMMON/CM2/A,B,ALPHA,LAMDA,E1,AA/CM3/P/CM4/A1,A2,S,C
      IF(I.EQ.-1) GO TO 100
      P(1,1) = A2*S*COS(LAMDA)
      P(1,2) = -A1*C*COS(LAMDA)
      P(1,3) = B*SIN(LAMDA)
      P(1,4) = -A*B*SIN(ALPHA)*COS(LAMDA)
      P(2,1) = A1*C
      P(2,2) = -A2*S
      P(2,3) = 0.
      P(2,4) = -A*B*COS(ALPHA)
      P(3,1) = A2*S*SIN(LAMDA)
      P(3,2) = -A1*C*SIN(LAMDA)
      P(3,3) = B*COS(LAMDA)
      P(3,4) = -A*B*SIN(ALPHA)*SIN(LAMDA)
      P(4,1) = 0.0
      P(4,2) = 0.0
      P(4,3) = 0.0
      P(4,4) = B
      GO TO 105
  100 AS = A2*S
      AC = A1*C
      C1 = COS(LAMDA)
      S1 = SIN(LAMDA)
      B1 = B*B
      P(1,1) = -AS*C1/B1
      P(1,2) = AC/B1
      P(1,3) = AS*S1/B1
      P(1,4) = A2*C/B
      P(2,1) = -AC*C1/B1
      P(2,2) = -AS/B1
      P(2,3) = AC*S1/B1
      P(2,4) = -A1*S/B
      P(3,1) = S1/B
      P(3,2) = 0.
      P(3,3) = C1/B
      P(3,4) = 0.0
      P(4,1) = 0.
      P(4,2) = 0.0
      P(4,3) = 0.
      P(4,4) = 1./B
  105 RETURN
      END

END OF SEGMENT, LENGTH   475, NAME  PMAT
```

```
      SUBROUTINE QUADRIC(IQ)
      REAL LAMDA
      DOUBLE PRECISION A,B,AL,EL,E,AB,A1(2),AA(2),GAM,S,C,B1(2),L(2),
     1M(2),ST(2),CT(2),J1,J2,K1,K2,TK,TD,T1,AU1,AU2
      DIMENSION Z(4,4)
      COMMON/CM1/Z/CM2/AS,BS,ALPHA,LAMDA,EPS,K/CM4/AX1,AX2,SX,CX
      PIT = 0.01745329252
      A = AS
      B = BS
      AL = ALPHA*PIT
      EL = LAMDA*PIT
      EO = EPS*PIT
      E = TAN(EO)
      AB = A*A + B*B
      D = 2.0*A*B*DCOS(AL)
      A1(1) = DSQRT(AB - D)
      A1(2) = DSQRT(AB + D)
      AA(1) = (A1(2)-A1(1))/2.0
      AA(2) = (A1(2)+A1(1))/2.0
      GAM = 0.5*DATAN2(2.0*A*B*DSIN(AL),A*A-B*B)
      S = DSIN(GAM)
      C = DCOS(GAM)
      AX1 = AA(1)
      AX2 = AA(2)
      SX = S
      CX = C
      B1(1) = A*DCOS(AL) - B
      B1(2) = A*DCOS(AL) + B
      DO 100 I = 1,2
      L(I) = A1(I) - A*DSIN(AL)*DCOS(EL)
      M(I) = A*DSIN(AL) + A1(I)*DCOS(EL)
      ST(I) = B1(I)*DSIN(EL)/L(I)
  100 CT(I) = M(I)/L(I)
C     K .NE. 1 SYMMETRIC APPROXIMATION
      IF(K.EQ.1) GO TO 101
      J1 = -1.0
      J2 = 1.0
      K1 = -1.0
      K2 = 1.0
      TK = 0.02
      TD = 0.02
      GO TO 102
  101 TK = ST(1)/CT(1)
      TD = (TK+E)/(1.0-TK*E)
      J1 = ST(2)/ST(1)
      J2 = CT(2)/CT(1)
      K1 = (J1*TK-J2*E)/(TK+E)
      K2 = (J2+J1*TK*E)/(1.0-TK*E)
  102 AU1 = (A1(2) - A1(1)*K2)/2.0
      AU2 = (A1(2) + A1(1)*K2)/2.0
      T1 = TD*TK
      GO TO(200,300,400),IQ
  200 Z(1,1) =  (1.0+K2)*S*C
      Z(1,2) = -(1.0-K2)/2.0
      Z(1,3) = ((1.0-K1)*C*TD)/2.0
      Z(1,4) = -AU2*S
      Z(2,2) =  Z(1,1)
      Z(2,3) =-((1.0+K1)*S*TD)/2.0
      Z(2,4) =  AU1*C
      Z(3,3) =  0.0
      Z(3,4) = -(A1(2) - A1(1)*K1)*TD/2.0
      GO TO 500
  300 Z(1,1) =  (1.0+J1)*S*C*TK
      Z(1,2) =  -(1.0-J1)*TK/2.
```

```
      Z(1,4) = -(A1(2) + A1(1)*J1)*S*TK/2.0
      Z(2,2) = Z(1,1)
      Z(2,3) =  (T1 +J2)*S/2.0
      Z(2,4) =  (A1(2) - A1(1)*J1)*C*TK/2.0
      Z(3,3) = 0.
      Z(3,4) =  (A1(2) - A1(1)*J2)/2.0
      GO TO 500
  400 Z(1,1) =  (K2-J1)*TK*S**2
      Z(1,2) =  0.0
      Z(1,3) = -((K1+J1)*T1 + (K2+J2))*S/2.0
      Z(1,4) =  0.0
      Z(2,2) = -(K2-J1)*TK*C**2
      Z(2,3) = ((K1-J1)*T1 + (K2+J2))*C/2.0
      Z(2,4) =  0.0
      Z(3,3) =  (K1-J2)*TD
      Z(3,4) =  0.0
  500 Z(2,1) =  Z(1,2)
      Z(3,1) =  Z(1,3)
      Z(3,2) =  Z(2,3)
      Z(4,1) =  Z(1,4)
      Z(4,2) = Z(2,4)
      Z(4,3) = Z(3,4)
      Z(4,4) = 0.0
      RETURN
      END

END OF SEGMENT, LENGTH  1305, NAME  QUADRIC
```

```
      READ FROM(1,USERS EDIT.QUAD)
SUBFILE QUAD
      SUBROUTINE QUAD(A,XR1,XR2,X1)
C     TO SOLVE QUADRATIC
C     A(1)XSQ+A(2)X+A(3)=
      DIMENSION A(3)
      X1 = -A(2)/(2.*A(1))
      DISC=X1*X1-A(3)/A(1)
      IF(DISC) 210,220,220
  210 X2 = SQRT(-DISC)
      XR1 = X1
      XR2 = X1
      X1  = X2
      GO TO 230
  220 X2 = SQRT(DISC)
      XR1 = X1+X2
      XR2 = X1-X2
      X1  = 0.
  230 RETURN
      END

END OF SEGMENT, LENGTH   112, NAME  QUAD
```

```
          A =   5.0.0     B =   0.0     ALPHA =  90.0     LAMDA =   0.0     EPSILON  0.1

 COEFFICIENTS OF INVERSE TRANSFORMATION MATRIX
      -0.33333336E 00      0.59742328-09       0.00000000E 00       0.10000000E 05
      -0.59742328E-09     -0.33333336E 00       0.00000000E 00      -0.17922798E-10
       0.10000000E 01      0.00000000E 00       0.33333336E 00       0.00000000E 00
       0.00000000E 00      0.00000000E 00       0.00000000E 00       0.33333336E 00

 COEFFICIENTS OF QUADRIC S1 WITH RESPECT TO THE FIXATION POINT
       0.19999000E-01      0.00000000E 00       0.19999000E-01      -0.30000000E 01
       0.00000000E 00      0.19999000E-01       0.00000000E 00       0.53760000E-08
       0.19999000E-01      0.00000000E 00       0.00000000E 00      -0.60000000E 01
      -0.30000000E 01      0.53760000E-08      -0.60000000E 01       0.00000000E 00

 COEFFICIENTS OF QUADRIC S1 WITH RESPECT TO THE FIXED CO-ORDINATES
       0.22222000E-02     -0.52939560E-22      -0.22222111E-02      -0.33326670E 00
      -0.52939560E-22      0.22220000E-02       0.39826220E-11       0.00000000E 00
      -0.22222111E-02      0.39826220E-11       0.00000000E 00      -0.66666340E-04
      -0.33326670E 00      0.00000000E 00      -0.66666340E-04      -0.19998000E-01

 COEFFICIENTS OF THE CONIC IN WHICH THE PLANE X1 = 0 MEETS THE QUADRIC S1
       0.22220000E-02      0.39826220E-11       0.00000000E 00
       0.39826220E-11      0.00000000E 00      -0.66666340E-04
       0.00000000E 00     -0.66666340E-04      -0.19998000E-01

 DETERMINANT OF CONIC IS          -0.7874571E-11

 CONIC IS  STRAIGHT LINES

 COEFFICIENTS OF THE CONIC IN THE PLANE X1 = 0      ROTATED THROUGH THE ANGLE     0.00 DEGREES
       0.22220000E-02      0.00000000E 00      -0.11946470E-12
       0.00000000E 00     -0.71382980E-20      -0.66666340E-04
      -0.11946470E-12     -0.66666340E-04      -0.19998000E-01

 COEFFICIENTS OF THE CONIC IN WHICH THE PLANE X2 = 0 MEETS THE QUADRIC S1
       1.00000000E 00     -0.22221110E-02       -0.66666340E-04
      -0.22221110E-02      0.22220000E-02       -0.33326670E 00
      -0.66666340E-04     -0.33326670E 00       -0.19998000E-01

 DETERMINANT OF CONIC IS          -0.9276667E-14

 CONIC IS  STRAIGHT LINES

 COEFFICIENTS OF THE CONIC IN THE PLANE X2 = 0      ROTATED THROUGH THE ANGLE  -89.28 DEGREES
       0.3995371E-02      0.00000000E 00       0.28345660E 00
       0.00000000E 00     -0.1873371E-02       -0.17528920E 00
       0.28345660E 00     -0.17528920E 00       -0.19999000E-01

 CENTRE  -78.0394 -127.6490          AXES:  -0.0000     0.0000

 COEFFICIENTS OF THE CONIC IN WHICH THE PLANE X3 = 0 MEETS THE QUADRIC S1
       0.22220000E-02      -0.52939560E-22       -0.33326670E 00
      -0.52939560E-22       0.22220000E-02        0.00000000E 00
      -0.33326670E 00       0.00000000E 00       -0.19998000E-01

 DETERMINANT OF CONIC IS          -0.4468589E-03

 CONIC IS  ELLIPSE

 COEFFICIENTS OF THE CONIC IN THE PLANE X3 = 0      ROTATED THROUGH THE ANGLE  -45.00 DEGREES
       0.22220000E-02      0.00000000E 00       -0.23568951E 00
       0.00000000E 00      0.22220000E-02       -0.23568951E 00
      -0.23568951E 00     -0.23568951E 00       -0.19998000E-01
```

```
     0.0000    0.0000 -300.0000    1.0000            0.3334  -0.0000   -0.0001

DIRECTION COSINES              0.0002    0.0200    0.9998
                               0.0002   -0.0200    0.9998

                                             T  650.1400  -50.0100
                                             XM    0.1400   12.9987  350.0000    1.0000
                                             XM   -0.0100   -0.9999 -350.0000    1.0000

                                             T  650.1400  -50.0100
                                             XM    0.1400  -12.9987  350.0000    1.0000
 2       0.0000   -0.0000 -240.0000    1.0000 XM  -0.0100    0.9999 -350.0000    1.0000
                                                   0.2000  -0.0000   -0.0001

DIRECTION COSINES              0.0003    0.0255    0.9997
                               0.0003   -0.0255    0.9997

                                             T  590.1966 -110.0067
                                             XM    0.1966   15.2409  350.0000    1.0000
                                             XM   -0.0067   -2.0397 -350.0000    1.0000

                                             T  590.1966 -110.0067
                                             XM    0.1966  -15.2409  350.0000    1.0000
 3       0.0000   -0.0000 -180.0000    1.0000 XM  -0.0067    2.0397 -350.0000    1.0000
                                                   0.0667  -0.0000   -0.0001

DIRECTION COSINES              0.0010    0.0446    0.9990
                               0.0010   -0.0446    0.9990

                                             T  530.5294 -170.1698
                                             XM    0.5296   23.6881  350.0000    1.0000
                                             XM   -0.1699   -7.5981 -350.0000    1.0000

                                             T  530.5294 -170.1698
                                             XM    0.5296  -23.6881  350.0000    1.0000
 4       0.0000   -0.0000 -120.0000    1.0000 XM  -0.1699    7.5981 -350.0000    1.0000
                                                  -0.0666  -0.0000   -0.0001
COMPLEX ERROR    0.0000    0.0000    0.0000    0.0448

DIRECTION COSINES             -0.0010   -0.0000    1.0000
                              -0.0010   -0.0000    1.0000

                                             T  470.0002 -230.0001
                                             XM   -0.4704   -0.0000  350.0000    1.0000
                                             XM    0.2302    0.0000 -350.0000    1.0000

                                             T  470.0002 -230.0001
                                             XM   -0.4704   -0.0000  350.0000    1.0000
 5       0.0000   -0.0000  -60.0000    1.0000 XM   0.2302    0.0000 -350.0000    1.0000
                                                  -0.1999  -0.0000   -0.0001
COMPLEX ERROR    0.0001    0.0000    0.0000    0.0258

DIRECTION COSINES             -0.0003   -0.0000    1.0000
                              -0.0003   -0.0000    1.0000

                                             T  410.0000 -200.0000
                                             XM   -0.1467   -0.0000  350.0000    1.0000
                                             XM    0.0967    0.0000 -350.0000    1.0000

                                             T  410.0000 -200.0000
                                             XM   -0.1467   -0.0000  350.0000    1.0000
 6       0.1500    0.0000     0000    1.0000  XM   0.0967    0.0000 -350.0000    1.0000
                                                  -0.3433    0.0010   -0.0001
```