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Convertible bonds valuation in a jump diffusion setting with stochastic interest rates

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This paper proposes an integrated pricing framework for convertible bonds, which comprises firm value evolving as an exponential jump diffusion, correlated stochastic interest rates movements and an efficient numerical pricing scheme. By construction, the proposed stochastic model fits in the framework of affine jump diffusion processes of Duffie, Pan and Singleton [Duffie, D., Pan, J. and Singleton, K., Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 2000, **68**, 1343–1376] with tractable behaviour. We define the firm's optimal call policy and investigate its impact on the computed convertible bond prices. We illustrate the performance of the numerical scheme and highlight the effects originated by the inclusion of jumps, stochastic interest rates and a non-zero correlation structure between firm value and interest rates.

Keywords: Convertible bonds pricing; Stochastic interest rates; Affine jump diffusion model; Optimal call strategy

JEL Classification: G12; G13; C63

1 Introduction

In this communication, we introduce a valuation framework for convertible bonds (CBs) based on a structural default model including jumps in the firm value dynamics and correlated stochastic interest rates which fits in the class of affine jump diffusions of Duffie *et al.* (2000). The proposed framework also includes a pricing algorithm based on a backward convolution approach, which is facilitated by the fact that the characteristic function of the relevant driving risk factors is known in analytical form.

CBs are corporate bonds that can be converted into shares of the issuing firm at the bondholder's discretion; these instruments represent a pricing challenge because of their hybrid nature and their complex design. Firstly, they depend on variables related to the underlying firm value (or stock), the fixed income part, which includes both interest rates and default risk, and the interaction between these components. Secondly, CBs usually carry call provisions giving the issuer the right to demand premature redemption in exchange for the current call price. A put provision, which allows the investor to force the issuing firm to prematurely repurchase the CB for a pre-specified price, is also sometimes met. Due to these early-exercise features, CBs valuation requires an accurate numerical scheme capable of handling efficiently the complexity of the contract.

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Closed form solutions for the price of the CB in the standard Black–Scholes–Merton economy have been obtained by Ingersoll (1977a) for the case of non-callable/callable products; however, the introduction in the valuation model of a more realistic specification including, e.g., discretely payable coupons, dividends on the underlying stock, hard call constraints (which protect the CB holder from the issuer calling the bond soon after issue) and soft call provisions (which vary from a notice period before the CB can be called, to stock price trigger and/or make-whole features) prevent the derivation of explicit pricing formulae. For these reasons, various techniques have been considered in the literature such as numerical solutions to partial differential equations/inequalities adopted by, e.g., Brennan and Schwartz (1977, 1980), Carayannopoulos (1996), Tsiveriotis and Fernandes (1998), Takahashi *et al.* (2001), Barone-Adesi *et al.* (2003) and Kovalov and Linetsky (2008); lattice methods developed, amongst others, by Derman (1994) and Takahashi *et al.* (2001); and Monte Carlo simulation, see Lvov *et al.* (2004) and Ammann *et al.* (2008).

Early contributions by Ingersoll (1977a), Brennan and Schwartz (1977, 1980), and Carayannopoulos (1996) fall in the structural class of default risk where the relevant state variable is the firm value. More recently the focus has been on reduced form equity default models with/out stochastic interest rates as in Tsiveriotis and Fernandes (1998), Takahashi *et al.* (2001), Barone-Adesi *et al.* (2003), Carayannopoulos and Kalimipalli (2003), Yiğitbaşioğlu and Alexander (2006), and Kovalov and Linetsky (2008) (who also include stochastic stock price volatility and default intensity factor).[1] The shift away from structural models had been mainly due to the fact that the spreads generated by these models were not in line with the observed ones (see, e.g., Eom *et al.* 2004). However, recent evidence provided by Brigo and Morini (2006), Cariboni and Schoutens (2007), Lipton and Sepp (2009), Fang *et al.* (2010), Fiorani *et al.* (2010) and Dao and Jeanblanc (2012), amongst others, shows that enhanced structural models, which incorporate, for example, time varying volatility and/or jumps, exhibit high fit to market quotes and, under suitable constructions, high tractability as well. This implies that many of the documented shortcomings are to be attributed to the pure diffusion form of the traditional models.

In light of the previous discussion, in this paper we adopt a jump diffusion model for the firm value with correlated stochastic interest rates, hence comprising four risk factors: the Brownian motion governing the diffusion part of the firm value, the compound Poisson process with random jump sizes modelling the shocks of the firm value, and the Brownian motion driving the short rate of interest. Correlation between interest rates and firm value is imposed on the diffusion components of the two processes. We consider both the cases of Merton's jump diffusion and Kou's double exponential jump diffusion for the log-increments of the underlying. In this respect, our setup generalizes the firm value approach of Bermúdez and Webber (2004), in which the default triggering event is represented by a single jump of fixed severity. Further, because of the hybrid nature of CBs, the flexibility to correlate the underlying firm value with the term structure of interest rates yields a realistic model of practical importance for pricing this contract. Finally, we propose a pricing algorithm which belongs to the class of backward price recursion techniques for contracts with early exercise and/or path-dependence (see, e.g., Lord *et al.* 2008, Fang and Oosterlee 2009, Černý and Kyriakou 2011 and Ruijter and Oosterlee 2012), aimed at handling effectively both the number of risk factors and real-world CB specifications, including discrete cash flows and conversion which either takes place voluntarily at the holders' choice, or can be forced by a call on notice by the issuer.

As the proposed numerical approach is shown to be flexible enough to handle the dimensionality imposed by the proposed market model, while remaining smoothly convergent and precise, we study the impact of model error on CBs prices, where by model error we mean the error induced by misspecifying the dynamics of either the firm value or the interest rates or both. In particular, we examine the discrepancy between the prices generated by the two jump diffusion processes and the traditional lognormal framework as a function of the model parameter values

[1] For a more detailed review, we refer to Yiğitbaşioğlu and Alexander (2006) and references therein.

and the moneyness (measure of the likelihood of conversion) of the convertible bond. We also explore the effects of modelling explicitly the uncertainty in the term structure of interest rates and the effects of a non-zero correlation structure between the firm value and interest rate processes on CB pricing. Finally, we study the implications of coupons payable to the CB holders, dividends distributed to the current stock holders, and the impact of a varying call policy on the computed prices.

The remainder of the paper is organized as follows. Section 2 presents our assumptions for the firm value and interest rate processes. In Section 3, we introduce the basic notation and describe the CB design focusing on the optimal call strategy assumed for the issuing firm. In Section 4, we develop the theoretical ground for the backward pricing algorithm. The impact of model error and the effects of additional CBs features are analyzed in Section 5, whilst Section 6 concludes.

2 Market model

From a valuation perspective, a pricing model for CBs requires assumptions on the dynamics followed by the asset underlying the conversion, the firm's default-driving mechanism and the term structure of interest rates.

We start from the underlying of the contract and the default-triggering event. We adopt a structural approach to default and assume that the dynamics of the firm value is driven by a jump diffusion process, so that most of the time the process follows a Brownian motion portraying marginal changes due to temporary imbalances between demand and supply, whilst market shocks induced by the arrival of important new information are captured by a compound Poisson process. In details, let $\left(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t>0}, P\right)$ be a complete filtered probability space, where P is a risk neutral probability measure[1]. Under the measure P, we assume that the firm value V is given by

$$V(t) = e^{Y(t)},\ t \geq 0,$$

where Y follows a jump diffusion process of the form

$$Y(t) = \tilde{Y}(t) + \sum_{k=1}^{N(t)} X_k - \lambda E\left(e^X - 1\right) t$$

with

$$\tilde{Y}(t) = Y(0) + \int_0^t \left(r(s) - \frac{\sigma^2}{2}\right) \mathrm{d}s + \sigma W(t)$$

and $Y(0) = \ln V(0)$. r is the short rate process, W is a $\mathbb{F}$-adapted standard Brownian motion, $\sigma \geq 0$ is the diffusion coefficient, N is a time-homogeneous Poisson process with constant arrival rate $\lambda > 0$ and jumps of size X, which is modelled by a sequence of i.i.d. random variables with characteristic function $\phi_X(u) := E\left(e^{iuX}\right)$. The random factors W, N and X are assumed to be mutually independent. Two popular choices in the literature for the distribution of X are the normal distribution (see Merton 1976) with characteristic function

$$\phi_X(u) = e^{iu\mu_X - \frac{1}{2}u^2\sigma_X^2},\ \mu_X \in \mathbb{R}, \sigma_X \geq 0,$$

[1]Due to the presence of jumps, the market is incomplete in the sense that there are infinitely many risk neutral pricing measures. Incompleteness can be resolved by calibration of the model to market quotes of traded instruments and credit spreads.

and the double exponential distribution (see Kou 2002) with characteristic function

$$\phi_X(u) = \frac{p\eta_1}{\eta_1 - iu} + \frac{q\eta_2}{\eta_2 + iu},$$

where $\eta_1 > 1$, $\eta_2 > 0$, and $p, q \geq 0$ represent the probabilities of an upward and a downward jump respectively, such that $p + q = 1$, under the measure P. We note at this stage that both models generate leptokurtic distributions of the firm (log) value process $Y(t)$ (see Appendix A).

The proposed construction belongs to the family of the so-called (structural) credit barrier models inspired by the Merton (1976) original contribution and extensions proposed by Black and Cox (1976), Longstaff and Schwartz (1995) (with stochastic interest rates), Leland (1994) and Leland and Toft (1996), to mention a few. Structural models offer an economic rationale behind default in that this occurs if the firm value goes below an appropriate threshold, as opposed to reduced form models in which the default event has exogenous components that are independent of market information. Structural models equipped with unforeseeable jumps, which reflect external impacts on the firm value evolution, allow flexible fitting to the observed credit spreads, see, e.g., Lipton and Sepp (2009), Fiorani *et al.* (2010) and Dao and Jeanblanc (2012). This way, previous limitations due to a pure diffusion driving process, such as vanishing spreads at very short maturities and poor fit over longer maturities (see, e.g., Eom *et al.* 2004), are circumvented, as shown for example by Brigo and Morini (2006), Cariboni and Schoutens (2007), Brigo *et al.* (2010), Fang *et al.* (2010), Fiorani *et al.* (2010) and Marazzina *et al.* (2012). The empirical studies reported in these contributions show, in fact, that enhanced structural models can be calibrated exactly to credit spreads using the default probability formulae generated by the chosen models.

The short rate process r is assumed to evolve according to the Vašíček (1977) model

$$r(t) = r(0)e^{-\kappa t} + \mu_r \left(1 - e^{-\kappa t}\right) + \sigma_r \int_0^t e^{-\kappa(t-s)} \mathrm{d}W_r(s),$$

for $\kappa, \mu_r, \sigma_r > 0$, where W_r is a standard Brownian motion, such that W and W_r have constant correlation coefficient $\rho \in [-1, 1]$, whereas W_r is independent of N and X. (For ease of notation and to distinguish directly from quantities related to the firm (log) value, we will be using hereafter the subscript 'r' on quantities related to the stochastic short rate of interest.) The price $B(t, \upsilon)$ at $t \geq 0$ of a pure-discount bond maturing at $\upsilon \geq t$ is given by

$$B(t, \upsilon) = E\left[e^{-\int_t^\upsilon r(s)\mathrm{d}s} \middle| \mathcal{F}_t\right] = \exp(A_1(\upsilon - t) - A_2(\upsilon - t)\, r(t)), \tag{1}$$

where

$$A_1(\tau) = \frac{1}{\kappa^2}\left(A_2(\tau) - \tau\right)\left(\kappa^2 \mu_r - \frac{\sigma_r^2}{2}\right) - \frac{\sigma_r^2 A_2^2(\tau)}{4\kappa}, \tag{2}$$

$$A_2(\tau) = \frac{1}{\kappa}\left(1 - e^{-\kappa\tau}\right) \tag{3}$$

(see Vašíček 1977).

Under the given assumptions, the process $(r, \tilde{Y})^{\mathrm{T}}$ is affine as the drift and diffusion terms are linear (in the state variables) up to a constant, see Duffie *et al.* (2000), allowing us to derive a closed form expression for the characteristic function as we show next. We note that the Vašíček model adopted in this paper suffers from the possibility of negative interest rates, as these are Gaussian. However, the analytical tractability implied by the affine structure of $(r, \tilde{Y})^{\mathrm{T}}$ does not hold when assuming other sensible dynamics for the short rate process, such as the Cox–Ingersoll–Ross (CIR) model (see Cox *et al.* 1985), due to the non-zero correlation structure assumed between the processes in our setting[1]. Hence, the choice between the CIR and Vašíček models is a trade-off between non-negative rates and non-zero correlation. However, negative interest rates implied by the Vašíček model is a well-known limitation to practitioners which can be controlled, e.g., see Rogers (1995). In fact, the Vašíček model (or its generalized version with time-dependent drift term, e.g., see Hull and White 1996) is frequently met in the recent literature, e.g., see van Haastrecht *et al.* (2009), Grzelak and Oosterlee (2011), Grzelak *et al.* (2012) with applications in other hybrid products, and Grzelak *et al.* (2011) for a multi-factor extension.

By affinity of the process $(r, \tilde{Y})^{\mathrm{T}}$, it follows (e.g., see Filipović 2009) that

$$E\left[e^{-\int_t^v r(s)\mathrm{d}s + iu_1 r(v) + iu_2 \tilde{Y}(v)} \middle| \mathcal{F}_t\right] = \exp(\psi_0\left(v-t, u_1, u_2\right) + \psi_1\left(v-t, u_1, u_2\right) r(t) \\ + \psi_2\left(v-t, u_2\right) \tilde{Y}(t)), \tag{4}$$

where (ψ_0, ψ_1, ψ_2) are given by

$$\begin{aligned}
\psi_0\left(\tau, u_1, u_2\right) = {} & \frac{\sigma_r^2}{2}\left[\frac{\left(iu_1\right)^2}{2\kappa}\left(1-e^{-2\kappa\tau}\right) + \frac{iu_1\left(iu_2-1\right)}{\kappa^2}\left(1-2e^{-\kappa\tau}+e^{-2\kappa\tau}\right)\right. \\
& \left. + \frac{\left(iu_2-1\right)^2}{2\kappa^3}\left(2\kappa\tau - 3 + 4e^{-\kappa\tau} - e^{-2\kappa\tau}\right)\right] \\
& + \left(\sigma\sigma_r\rho iu_2 + \kappa\mu_r\right)\left[iu_1\frac{1-e^{-\kappa\tau}}{\kappa} + \frac{iu_2-1}{\kappa^2}\left(\kappa\tau - 1 + e^{-\kappa\tau}\right)\right] \\
& + \frac{\sigma^2}{2} iu_2\left(iu_2-1\right)\tau, \\
\psi_1\left(\tau, u_1, u_2\right) = {} & iu_1 e^{-\kappa\tau} + \left(iu_2-1\right)\frac{1-e^{-\kappa\tau}}{\kappa}, \\
\psi_2\left(\tau, u_2\right) = {} & iu_2.
\end{aligned}$$

Further, Eq. (4), the independence between the jump part of process Y, its continuous part $\tilde{Y}$ and the short rate process r, and the Lévy-Khintchine formula imply that

$$E\left[e^{-\int_t^v r(s)\mathrm{d}s + iu_1 r(v) + iu_2 Y(v)} \middle| \mathcal{F}_t\right] = \exp(\psi_0\left(v-t, u_1, u_2\right) + \psi_1\left(v-t, u_1, u_2\right) r(t) \\ + \psi_2\left(v-t, u_2\right) Y(t) + \psi_3\left(v-t, u_2\right)) \tag{5}$$

with

$$\psi_3\left(v-t, u_2\right) = \lambda\left(\phi_X\left(u_2\right) - 1 - iu_2\left(\phi_X\left(-i\right) - 1\right)\right)\left(v-t\right).$$

We conclude this section with few considerations regarding the calibration of the proposed model to market data. In a structural approach, default occurs as soon as the firm value falls below a preset threshold before or at maturity of the debt. Hence, default probabilities can be derived in a straightforward manner from the given model. Fast and efficient numerical schemes for this purpose have been proposed in similar settings by, for example, Fang *et al.* (2010) and Marazzina *et al.* (2012). Due to the link between default probabilities and spreads of credit default swaps (CDS), once the interest rate term structure is fitted to LIBOR and swap rates,

[1]It is worth noting that Grzelak and Oosterlee (2011) investigate affine approximations for the case of the CIR interest rate dynamics and derive approximate characteristic functions, the use of which is left to future research.

the default probability of the underlying reference entity can be calibrated to quoted market CDS spreads, using for example the bootstrapping procedure described in O'Kane and Turnbull (2003) to recover the default probability term structure, and the minimization procedure in Fang *et al.* (2010).

3 Convertible bonds

3.1 *Contract features*

A convertible bond is a corporate bond which offers the investors the option to exchange it for a predetermined number of shares of the issuing firm at certain points in time. Upon conversion each investor receives the conversion value $\gamma V(t)$, where $t \in (0, T]$, $T > 0$ is the maturity time of the CB, $\gamma \in (0, 1/m)$ denotes the dilution factor (Ingersoll 1977a), i.e., the fraction of common stock possessed by each CB holder post-conversion, and m the number of CBs outstanding. The CB issue usually offers regular aggregate coupon payments C_{t_j} at times $t_j \in (0, T]$, i.e., a payment of $c_{t_j} := C_{t_j}/m$ per bond. If the issue is kept alive to time T, then this is redeemed for a total face value mF. Instead, the firm's stock holders receive discrete aggregate dividends D_{t_i} at the dividend dates $t_i \in (0, T]$ (see Brennan and Schwartz 1977).

CBs also carry a call provision allowing the issuer to redeem it prematurely in exchange for the current call price (e.g., see Ingersoll 1977a, Ingersoll 1977b and Brennan and Schwartz 1977). In general, the issuer announces its decision to call the bond a certain period in advance (call notice period). Once the CB is called, the investor needs to consider if it is the case to exercise the conversion option at the end of the call notice period (so-called "forced conversion by call") instead of receiving the call price. The existence of a call provision implies that the CB payoff depends on the optimal exercise strategy adopted by the issuer. We discuss this in more details in the next section.

3.2 *The optimal call strategy*

Under the assumption of a market that is not subject to any imperfections, in which the Modigliani–Miller theorem holds and no call notice applies, Ingersoll (1977a) proves that the optimal call policy for a callable convertible issue is to call as soon as the firm value $V(t)$ reaches the critical level K_t/γ, for a call price K_t which is usually fixed by the firm at the issue of the contract or is time-dependent. This feature endows the CB with path-dependence, hence implies the need for frequent monitoring. However, based on the empirical findings of Ingersoll (1977b), firms tend to call when the conversion value is in excess of the call price.

Asquith and Mullins (1991) and Asquith (1995) demonstrate that, in the absence of call protection, existence of significant cash flow advantages causes delayed calls, i.e., the firm might be saving cash by delaying the call if, for example, the after-corporate tax coupons on the CB are less than the dividends payable post-conversion. A good portion of the observed delay is also explained by the existence of a call notice and a safety premium, i.e., the issuer delays announcing the call until the conversion value exceeds the call price enough, so that there is high chance of this to still exceed the call price at the end of the call notice period and, hence, avoid the bond redemption in cash.

Thus, abstracting from factors such as taxes, we formulate the optimal call policy for the CB as follows. The firm announces the decision to call as soon as the conversion value exceeds the call price increased by some safety premium $\vartheta \in [0, 1)$; consequently, the firm's optimal call announcement is given by the stopping time

$$\tau^c := \inf \left\{ t \in (0, T) : Y(t) \geq \ln \frac{(1 + \vartheta)\, K_t}{\gamma} \right\}.$$

In addition, denote by s^c the call notice period and define the accrued interest on coupons $\mathrm{AccIR} = ((\tau^c + s^c - t_j)/(t_{j+1} - t_j))c_{t_{j+1}}$, $t_j \le \tau^c + s^c < t_{j+1}$. At the end of the call notice period, the bondholder chooses whether to convert and receive the conversion value or redeem the bond for the call price accrued by any coupon due. The no-arbitrage value of this option as at the time of call announcement is

$$Q_{\tau^c}(r(\tau^c), Y(\tau^c)) := E\left[e^{-\int_{\tau^c}^{\tau^c+s^c} r(s)\mathrm{d}s} \max\left\{ \gamma e^{Y(\tau^c+s^c)}, K_{\tau^c} + \mathrm{AccIR} \right\} \middle| \mathcal{F}_{\tau^c} \right]. \tag{6}$$

As we show in the following section, the call payoff (6) is the amount the CB holders will receive when the issuer announces a call.

3.3 *The payoff function*

Because of the early-exercise rights embedded in the CB, we distinguish between the contract payoff function H_t at any possible decision time $t \in (0, T]$ and the no-arbitrage (continuation) value G_t. The payoff function is defined as follows.

At maturity, $t = T$, the investors can choose between converting to common stock (see Brennan and Schwartz 1977) and receiving the face value and the final coupon, provided that the firm can afford the total of this payment. Otherwise, they recover the outstanding firm value at that time. Hence,

$$H_T(r(T), Y(T)) = \begin{cases} \gamma e^{Y(T)}, & \text{if } Y(T) \ge \ln \frac{F+c_T}{\gamma} \\ F + c_T, & \text{if } \ln(mF + C_T) \le Y(T) < \ln \frac{F+c_T}{\gamma} \\ e^{Y(T)}/m, & \text{if } Y(T) < \ln(mF + C_T) \end{cases}. \tag{7}$$

At a date where no coupon or dividend payments are due, $t \ne t_j, t_i$, CB may be forced by a call to conversion ($t = \tau^c$) or continue to exist at least until the next decision date ($t \ne \tau^c$), i.e.,

$$H_t(r(t), Y(t)) = \begin{cases} Q_t(r(t), Y(t)), & \text{if } Y(t) \ge \ln \frac{(1+\vartheta)K_t}{\gamma} \\ G_t(r(t), Y(t)), & \text{if } Y(t) < \ln \frac{(1+\vartheta)K_t}{\gamma} \end{cases} \tag{8}$$

where Q_t is given by Eq. (6).

At a coupon date, $t = t_j \ne t_i$, the payoff of the CB depends on whether the firm has enough funding to meet the claim. If $V(t_{j\_}) \le C_{t_j}$, the CB defaults, its value is $G_{t_j} = 0$ as $0 \le G_{t_j} \le V(t_j)$ by limited liability and the Modigliani–Miller theorem, whilst the investors receive the available assets. If instead $V(t_{j\_}) > C_{t_j}$ and for as long as the CB is uncalled ($t_{j\_} \ne \tau^c$), the contract remains in force and the coupons are paid in full. On the other hand, if the CB is called ($t_{j\_} = \tau^c$), its holders receive both the call payoff (6) and the coupon. Hence,

$$H_{t\_}(r(t), Y(t\_)) = \begin{cases} Q_{t\_}(r(t), \ln(e^{Y(t\_)} - C_t)) + c_t, & \text{if } Y(t\_) \ge \ln \frac{(1+\vartheta)K_t}{\gamma} \\ G_{t\_}(r(t), \ln(e^{Y(t\_)} - C_t)) + c_t, & \text{if } \ln C_t < Y(t\_) < \ln \frac{(1+\vartheta)K_t}{\gamma} \\ e^{Y(t\_)}/m, & \text{if } Y(t\_) \le \ln C_t \end{cases}. \tag{9}$$

Finally, at a dividend date, $t = t_i \ne t_j$, the existing stock holders are entitled to receive dividends for as long as the firm can afford their payment, providing that it has already met all the other claims ranking above them. Hence, the investors may find optimal to convert prior to the dividend payment (so-called "voluntary conversion"). The following condition, which is proved in Brennan and Schwartz (1977), applies

$$H_{t\_}(r(t), Y(t\_)) = \max\left\{ G_{t\_}(r(t), \ln(e^{Y(t\_)} - D_t 1_{Y(t\_) > \ln D_t})), \gamma e^{Y(t\_)} \right\}. \tag{10}$$

If, in addition, the coupon and dividend dates coincide, $t = t_j = t_i$, then (10) is extended to

$$H_{t_-}(r(t), Y(t_-)) = \begin{cases} \max\left\{G_{t_-}(r(t), \ln(e^{Y(t_-)} - C_t - D_t 1_{Y(t_-)>\ln(D_t+C_t)})) + c_t, \gamma e^{Y(t_-)}\right\}, & \text{if } Y(t_-) > \ln C_t \\ e^{Y(t_-)}/m, & \text{if } Y(t_-) \le \ln C_t \end{cases}. \quad (11)$$

A call protection provision may further apply which will prohibit any call back by the issuer typically for 2 or 3 years after issue. Hence, when pricing the convertible, the call feature must be disabled at the dates falling within the period of call protection. In addition, we may take into account a put provision, i.e., a right which allows the holder of the convertible to sell it back to the issuer for some cash amount R (which can be time-dependent). Upon active put provision, we need to modify the payoff function as follows: in (8), use $\max\{G_t(r(t), Y(t)), R + \text{AccIR}\}$ for $Y(t) < \ln((1+\vartheta) K_t/\gamma)$; in (9), use $\max\{G_{t_-}(r(t), \ln(e^{Y(t_-)} - C_t)) + c_t, R + \text{AccIR}\}$ for $\ln C_t < Y(t_-) < \ln((1+\vartheta) K_t/\gamma)$; in (10), use $\max\{G_{t_-}(r(t), \ln(e^{Y(t_-)} - D_t 1_{Y(t_-)>\ln D_t})), \gamma e^{Y(t_-)}, R + \text{AccIR}\}$ (payoff 11 is modified accordingly). The terminal payoff (7) remains unaffected. In this paper, we only account for call notice features, while a call protection period can be very easily incorporated if necessary. We also do not account for put features which are less common than call features.

4 Price convolution for convertible bonds

Consider the time partition $\mathcal{T} := \{t_k\}_{k=0}^n$, $n \in \mathbb{N}$, of the contract's term $[0, T]$ denoting the set of decision dates. For ease of exposition, assume that these dates are equidistant so that $t_k - t_{k-1} =: \delta t$ for $0 < k \le n$, with $t_0 := 0$ and $t_n := T$. With these assumptions in mind, the price of the CB is the solution to the dynamic programming problem described next.

We write G_{k_-} (resp. H_{k_-}) for the continuation value $G_{t_{k_-}}$ (resp. payoff function $H_{t_{k_-}}$) of the CB as at t_k, $0 \le k \le n$. (We use the convention $0_- \equiv 0$ and $n_- \equiv n$ in the subscript.) Based on the fundamental theorem of asset pricing, we write for $0 < k \le n$ the iteration

$$\begin{aligned} G_{k-1_-}(r(t_{k-1}), Y(t_{k-1_-})) &= E\left[e^{-\int_{t_{k-1}}^{t_k} r(s)\mathrm{d}s} H_{k_-}(r(t_k), Y(t_{k_-})) \middle| \mathcal{F}_{t_{k-1}} \right] \\ &= e^{A_1(\delta t) - A_2(\delta t) r(t_{k-1})} E^* \left[H_{k_-}(r(t_k), Y(t_{k_-})) \middle| \mathcal{F}_{t_{k-1}} \right], \end{aligned} \quad (12)$$

where the second equality follows by a change to the t_k-forward measure $P^*_{t_k}$ induced by taking as numéraire the price $B(t, t_k)$ of a pure-discount bond maturing at t_k as at time $t \in [t_{k-1}, t_k]$, see Eq. (1). (For more details on probability measure changes, we refer to Geman *et al.* 1995.)

In order to analyze (12) further, it is essential that we introduce some key functions and random variables. First, we define the functions

$$l_k(y) = \begin{cases} y, & \text{if } k \ne i, j \\ \ln(e^y - C_{t_k}), & \text{if } k = j \ne i, \; y > \ln C_{t_k} \\ \ln\left(e^y - D_{t_k} 1_{y > \ln D_{t_k}}\right), & \text{if } k = i \ne j \\ \ln\left(e^y - C_{t_k} - D_{t_k} 1_{y > \ln(D_{t_k} + C_{t_k})}\right), & \text{if } k = i = j, \; y > \ln C_{t_k} \end{cases}, \quad (13)$$

$$g_k(y_r, y) = l_k(y) - A_1(\delta t) + A_2(\delta t) y_r, \quad (14)$$

$$g_{r,k}(y_r) = y_r e^{-\kappa \delta t}, \; \forall k, \quad (15)$$

where the functions A_1, A_2 are given by Eqs. (2)–(3). From Eqs. (13)–(15), we obtain $g_{r,k-1}(r(t_{k-1})) = r(t_{k-1}) e^{-\kappa\delta t}$ and $g_{k-1}(r(t_{k-1}), Y(t_{k-1_-})) = Y(t_{k-1}) - A_1(\delta t) + A_2(\delta t) r(t_{k-1})$ as, by definition, $l_{k-1}(Y(t_{k-1_-})) = Y(t_{k-1})$, that is, the firm log-value at the beginning of the sub-period $[t_{k-1}, t_k]$ after any discrete payment has taken place: no payment if $t_{k-1} \neq t_j, t_i$, coupon payment if $t_{k-1} = t_j \neq t_i$, dividend payment if $t_{k-1} = t_i \neq t_j$, both coupon and dividend payments if $t_{k-1} = t_j = t_i$. In addition, we define the pairs

$$(Z_{r,k}, Z_k) = \big(r(t_k) - g_{r,k-1}(r(t_{k-1})), Y(t_{k_-}) - g_{k-1}(r(t_{k-1}), Y(t_{k-1_-}))\big)\,,\ 0 < k \leq n. \tag{16}$$

This allows us to rewrite Eq. (12) as

$$\begin{aligned} G_{k-1_-}(r(t_{k-1}), Y(t_{k-1_-})) = e^{A_1(\delta t) - A_2(\delta t) r(t_{k-1})} \\ \times \tilde{G}_{k-1_-}(g_{r,k-1}(r(t_{k-1})), g_{k-1}(r(t_{k-1}), Y(t_{k-1_-}))), \end{aligned} \tag{17}$$

where

$$\begin{aligned} &\tilde{G}_{k-1_-}(g_{r,k-1}(r(t_{k-1})), g_{k-1}(r(t_{k-1}), Y(t_{k-1_-}))) && (18) \\ &:= E^*\left[H_{k_-}(r(t_k), Y(t_{k_-}))\middle|\, \mathcal{F}_{t_{k-1}}\right] \\ &= \int_{\mathbb{R}^2} H_{k_-}(g_{r,k-1}(r(t_{k-1})) + z_r, g_{k-1}(r(t_{k-1}), Y(t_{k-1_-})) + z) f_k^*(z_r, z)\, \mathrm{d}(z_r, z) && (19) \end{aligned}$$

with f_k^* denoting the $P_{t_k}^*$-density function of $(Z_{r,k}, Z_k)$. In fact, the distribution law of $(Z_{r,k}, Z_k)$ is known via its characteristic function which follows from Eqs. (5) and (16)

$$\begin{aligned} \varphi(\delta t, u_1, u_2) &:= E^*\left[e^{iu_1 Z_{r,k} + iu_2 Z_k}\middle|\, \mathcal{F}_{t_{k-1}}\right] = e^{-A_1(\delta t) + A_2(\delta t) r(t_{k-1})} E\left[e^{-\int_{t_{k-1}}^{t_k} r(s)\mathrm{d}s + iu_1 Z_{r,k} + iu_2 Z_k}\middle|\, \mathcal{F}_{t_{k-1}}\right] \\ &= \exp(\psi_0(\delta t, u_1, u_2) + (iu_2 - 1) A_1(\delta t) + \psi_3(\delta t, u_2)) \end{aligned} \tag{20}$$

for all k. Clearly, our choice of functions g_r and g ensures that (20) does not depend on the state variables r and Y, facilitating the numerical computation of the convolution (19) as described in Appendix B. Few comments are in order. Recursion (19) is initialized by (7). Evaluating then the function $\tilde{G}_{k-1_-}$ in (18) at $(g_{r,k-1}, g_{k-1})$ gives us access to G_{k-1_-} in (17), i.e., the continuation value of the CB at the beginning of the sub-period $[t_{k-1}, t_k]$, for all k, prior to any discrete payment taking place. For this reason the function g in Eq. (14) (see also Eq. 13) is defined for firm values higher than the coupon level over which the CB does not default (continues to exist). The continuation value G_{k-1_-} is used to compute the new payoff function H_{k-1_-} consistently with (8)–(11), which is applied in the subsequent iteration. Ultimately, the price of the CB at inception is G_0.

In addition, upon a call of the CB by the issuing firm, we need to compute the payoff to the CB holders (see Eqs. 8–9). From (6),

$$Q_{\tau^c}(r(\tau^c), Y(\tau^c)) = e^{A_1(s^c) - A_2(s^c) r(\tau^c)} E^*\left[\max\left\{\gamma e^{Y(\tau^c + s^c)}, K_{\tau^c} + \mathrm{AccIR}\right\}\middle|\, \mathcal{F}_{\tau^c}\right],$$

where the expectation is taken under the $P_{\tau^c + s^c}^*$ measure. Similarly to g in (14) we define the function

$$g_{\tau^c}^c(y_r, y) = \begin{cases} y - A_1(s^c) + A_2(s^c) y_r, & \text{if } \tau^c \neq t_j, t_i \\ \ln(e^y - C_{\tau^c}) - A_1(s^c) + A_2(s^c) y_r, & \begin{array}{l}\text{if } \tau^c = t_{j_-} \neq t_{i_-}, \\ y > \ln C_{\tau^c}\end{array} \end{cases}$$

and then the random variable

$$Z_{\tau^c + s^c}^c = Y(\tau^c + s^c) - g_{\tau^c}^c(r(\tau^c), Y(\tau^c))$$

whose probability distribution is assumed to have density function $q^*_{\tau^c+s^c}$ under the measure $P^*_{\tau^c+s^c}$. Finally, we obtain

$$\begin{aligned} &Q_{\tau^c}(r(\tau^c), Y(\tau^c)) \\ &= e^{A_1(s^c)-A_2(s^c)r(\tau^c)} \int_{\mathbb{R}} \max\left\{\gamma e^{g^c_{\tau^c}(r(\tau^c),Y(\tau^c))+z}, K_{\tau^c} + \mathrm{AccIR}\right\} q^*_{\tau^c+s^c}(z)\mathrm{d}z. \end{aligned} \tag{21}$$

The characteristic function of $Z^c_{\tau^c+s^c}$ under the measure $P^*_{\tau^c+s^c}$ is given by

$$\varphi^c(s^c, u) := E^*\left[e^{iuZ^c_{\tau^c+s^c}} \middle| \mathcal{F}_{\tau^c}\right] = \exp(\psi_0(s^c, 0, u) + (iu-1)A_1(s^c) + \psi_3(s^c, u)).$$

5 Numerical study

In this section, we analyze the performance of the proposed pricing algorithm and study the behaviour of the CB price under different modelling assumptions and for different payoff structures. In first place, we study the convergence of the numerical pricing scheme in the case of constant interest rates and lognormal firm value, as to benchmark our algorithm to the exact pricing formula of Ingersoll (1977a). Results are presented in Section 5.1. Secondly, we introduce jumps and stochastic interest rates; we compare the results with the CB prices obtained under the standard Gaussian economy and constant interest rates in an attempt to quantify model error, meant as the error originated by the misspecification of the relevant driving risk factors. The effect of jumps only in presence of constant interest rates is discussed in Section 5.2, whilst the impact of stochastic interest rates and the complex interaction with firm value and CB prices is shown in Section 5.3. Finally we analyze the impact of different contract specifications: the introduction of coupons and dividends is examined in Section 5.4, whilst alternative call policies (in terms of different call notice periods and safety premia) are presented in Section 5.5.

5.1 *Convergence of the pricing scheme*

In the following, we illustrate the numerical performance of the proposed pricing scheme under various CB payoff structures. In particular, we consider callable CBs with: (a) $K = 40$, $s^c = \vartheta = 0$, $C = D = 0$; (b) $K = 40$, $s^c = \vartheta = 0$, $C = 1$ (payable semiannually), $D = 2$ (payable at the first quarter and third quarter of the year); (c) $K = 40$, $s^c = 1/12$, $\vartheta = 0.2$, $C = D = 0$; (d) $K = 40$, $s^c = 1/12$, $\vartheta = 0.2$, $C = 1$, $D = 0$. In order to benchmark our numerical scheme, we compute prices for 5-year CBs with daily sampling, assuming constant interest rates and firm value driven by the lognormal model, as in this case we can compare against the prices obtained using the exact analytical formula of Ingersoll (1977a) for continuously (infinitely sampled) callable CBs.

Let α^n_N be the CB price calculated using the numerical scheme with N grid points (see Appendix B) and n sampling dates. For fixed n, we compute the absolute price differences $\beta^n_N := |\alpha^n_N - \alpha^n_{N/2}|$ for $N = 2^{12}, 2^{13}, \ldots, 2^{16}$ and plot these on a log-log scale in Fig. 1. The solid lines in Fig. 1 correspond to regular linear convergence in the number of grid points, which clearly remains unaffected under the various payoff structures. Smooth convergence further enables the use of Richardson extrapolation as a way to accelerate convergence (e.g., see Andricopoulos *et al.* 2003). To this end, we obtain CB prices $\alpha^n_{\mathrm{RE},N}$ from Richardson extrapolation of the pairs of prices $(\alpha^n_N, \alpha^n_{N/2})$ and calculate successive differences $\beta^n_{\mathrm{RE},N} := |\alpha^n_{\mathrm{RE},N} - \alpha^n_{\mathrm{RE},N/2}|$ for $N = 2^{13}, \ldots, 2^{16}$ which exhibit quadratic convergence in N (see dotted lines in Fig. 1).

Richardson extrapolation can also be applied on pairs of callable CB prices obtained for different number of sampling dates $(\alpha^n_N, \alpha^{n/2}_N)$, N held fixed and varying $n = 2^5, 2^6, \ldots, 2^{11}$, to price an otherwise equivalent continuously callable CB. Results are shown in Table 1 for a callable CB with $K = 40$, where we denote by $\alpha^{\mathrm{RE},n}_N$ the extrapolated continuously callable CB

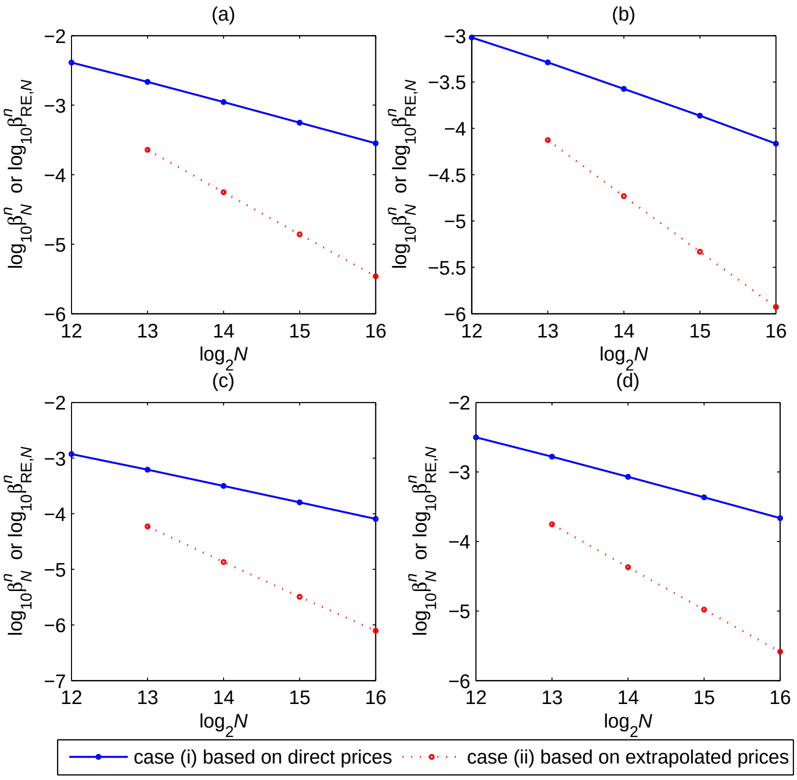

Figure 1. Convergence of CB pricing scheme in number of grid points N. Cases: (i) α^n_N denotes callable CB price calculated directly using the numerical scheme in Appendix B with N grid points and n sampling dates. $\beta^n_N = |\alpha^n_N - \alpha^n_{N/2}|$ denotes absolute price differences for $N = 2^{12}, 2^{13}, \ldots, 2^{16}$. (ii) $\alpha^n_{\mathrm{RE},N}$ denotes callable CB price obtained from Richardson extrapolation of $(\alpha^n_N, \alpha^n_{N/2})$. $\beta^n_{\mathrm{RE},N} = |\alpha^n_{\mathrm{RE},N} - \alpha^n_{\mathrm{RE},N/2}|$ denotes absolute differences of extrapolated prices for $N = 2^{13}, 2^{14}, \ldots, 2^{16}$. Solid lines correspond to case (i) indicating linear convergence in N, dotted lines correspond to case (ii) indicating quadratic convergence in N. CB specifications: (a) $K = 40$, $s^c = \vartheta = 0$, $C = D = 0$; (b) $K = 40$, $s^c = \vartheta = 0$, $C = 1$ (payable semiannually), $D = 2$ (payable at the first and third quarters of the year); (c) $K = 40$, $s^c = 1/12$, $\vartheta = 0.2$, $C = D = 0$; (d) $K = 40$, $s^c = 1/12$, $\vartheta = 0.2$, $C = 1$, $D = 0$. Other CB parameters: $T = 5$, $n = 1250$, $F = 40$, $m = 1$, $\gamma = 0.2$. Firm value parameters: $V(0) = 100$, $\sigma = 0.25$. Constant interest rate: $r = 0.04$.

price based on pair $(\alpha^n_N, \alpha^{n/2}_N)$. Obviously, with increasing n, the numerical CB price approaches smoothly the one of the continuously callable CB computed using the exact price result of Ingersoll (1977a) which we denote by α^∞. Richardson extrapolation speeds up convergence to the exact price.

Note that the convergence of the numerical scheme is not affected when we introduce stochastic interest rates and/or jumps in the firm value dynamics (in the interest of space, results are omitted and available by the authors upon request). However, changing from a constant interest rates setting to a stochastic interest rates one increases the computing time (by an average factor of 14) due to the change from one to two state variables and double price integrals. The computing time also increases (by an average factor of 1.5) from the case of a simply callable CB to a callable CB with coupons (and a dividend-paying stock) because of the increasingly power-demanding surface-fitting procedure in the numerical implementation of the pricing algorithm (see *Step 2* in the description given in Appendix B). In the sections to follow, we produce and make use in our analysis CB price results which are precise to the fifth and third decimal place, respectively, in the case of constant and stochastic interest rates, aiming for the lowest possible computing time for given interest rate modelling assumptions, while achieving sufficient accuracy for practical applications.

Table 1. Callable CB prices.

$\log_2 n$	α_N^n	$\alpha_N^{\mathrm{RE},n}$	% error
5	32.99098	-	-
6	33.03468	33.14018	0.0127
7	33.06626	33.14250	0.0057
8	33.08890	33.14356	0.0025
9	33.10504	33.14400	0.0012
10	33.11652	33.14424	0.0005
11	33.12465	33.14428	0.0004

α_N^n denotes callable CB price calculated using N grid points and n sampling dates (5 decimal place accuracy achieved with $N = 2^{13}$). $\alpha_N^{\mathrm{RE},n}$ denotes price of continuously callable CB obtained from Richardson extrapolation of $(\alpha_N^n, \alpha_N^{n/2})$. "% error" given by $|\alpha_N^{\mathrm{RE},n} - \alpha^\infty|/\alpha^\infty$, where $\alpha^\infty = 33.1444004$ is the true price of the continuously callable CB from the formula of Ingersoll (1977a). Firm value parameters: $V(0) = 100$, $\sigma = 0.25$. Constant interest rate: $r = 0.04$. CB parameters: $T = 5$, $F = 40$, $K = 40$, $C = D = 0$, $m = 1$, $\gamma = 0.2$, $\vartheta = 0$, $s^c = 0$.

5.2 *Model error: jumps*

In this section, we examine the impact of introducing jumps; to this end, we ignore for convenience and without loss of generality the case of stochastic interest rates due to their independence of the jump component.

For illustration purposes, in this analysis we use the same values as in Dao and Jeanblanc (2012) for the diffusion coefficient, i.e., $\sigma = 0.2$, and the parameters of the jump part of the double exponential jump diffusion (DEJD) model. Therefore, we assume probability of an upward jump $p = 30\%$ (hence, probability of a downward jump $q = 70\%$); further, the jumps have upward mean size $1/\eta_1 = 0.02$ and downward mean size $1/\eta_2 = 0.03$, whilst the jump arrival rate is $\lambda = 3$. The emphasis, therefore, is on downward jump risk. The parameters of Merton's jump diffusion (MJD) model, i.e., μ_X, σ_X, are matched to the mean and standard deviation of the jump size distribution of the DEJD model (see Eqs. A5, A6, A7).[1] By assuming common jump arrival rate for both models, the mean and the volatility of the MJD and DEJD firm log-value processes (see Eqs. A1–A2) are also matched. The parameters are reported in Table 2 together with the resulting coefficients of skewness and excess kurtosis (see Eqs. A3–A4). In addition, Table 2 reports the parameters and resulting distribution features for the different cases I–VI used to analyze the impact of jumps on the CB prices. These cases are obtained by changing the parameters of the jump process in order to change the level of skewness and excess kurtosis in the distribution. In particular, we consider cases I–II of higher/lower frequency of occurrence of jumps; cases III–IV (cases V–VI) in which the mean jump size (standard deviation of the jump size) are either doubled or halved compared to the base case for both models. Similarly to the base case, assuming common jump arrival rate for both models, we obtain the remaining DEJD parameters by matching the mean and the volatility of the MJD and DEJD firm log-value processes.

As reported in Table 2, the DEJD distribution shows consistently higher left skewness and excess kurtosis than the MJD distribution. In order to highlight the effect of these features on the CB prices, in Table 3 we report the differences in the prices generated by the MJD, DEJD models and the Gaussian model (i.e., with the jump part ignored) in correspondence

[1] Note that in the MJD model the parameters μ_X, σ_X coincide with the mean and standard deviation of the jump size distribution (see Eqs. A5, A7), whilst this is not the case with the DEJD model (see Eqs. A6, A7).

Table 2. Model parameters.

case	MJD & DEJD			MJD				DEJD				
	λ	mean	vol	μ_X	σ_X	skew	exc kurt	η_1	η_2	p	skew	exc kurt
base	3	0.0178	0.2110	–0.0150	0.0357	–0.0194	0.0101	50	33.3333	0.3	–0.0316	0.0224
I	5	0.0163	0.2179	–0.0150	0.0357	–0.0293	0.0147	50	33.3333	0.3	–0.0478	0.0327
II	1	0.0193	0.2037	–0.0150	0.0357	–0.0072	0.0039	50	33.3333	0.3	–0.0117	0.0086
III	3	0.0168	0.2157	–0.0300	0.0357	–0.0424	0.0174	13.4374	31.4533	0.0169	–0.0442	0.0507
IV	3	0.0180	0.2097	–0.0075	0.0357	–0.0095	0.0082	38.8943	38.6981	0.3558	–0.0099	0.0165
V	3	0.0121	0.2366	–0.0150	0.0714	–0.0527	0.0813	19.6218	19.2543	0.3590	–0.0575	0.1629
VI	3	0.0192	0.2041	–0.0150	0.0179	–0.0063	0.0014	26.0657	62.9170	0.0165	–0.0064	0.0041

"mean", "vol", "skew" and "exc kurt" denote respectively mean, volatility, skewness coefficient and excess kurtosis of $\ln(V(1)/V(0))$ (see Eqs. A1, A2, A3, A4). Cases I–VI are consistent with departures of λ, mean jump size $E(X)$ (see Eqs. A5, A6) and standard deviation of jump sizes $\mathrm{Var}(X)^{1/2}$ (see Eq. A7) from base values, with corresponding (scale-free, where applicable) cumulants and fitted DEJD model parameters so that mean and standard deviation of jump sizes are matched. Other parameters (fixed in all cases): $r = 0.04$, $\sigma = 0.2$.

Table 3. Model error: MJD vs DEJD & Gaussian vs DEJD.

case	MJD vs DEJD price diff.			Gaussian vs DEJD price diff.		
	CB moneyness			CB moneyness		
	(0.6,1.0)	[1.0,1.2)	[1.2,1.4)	(0.6,1.0)	[1.0,1.2)	[1.2,1.4)
base	0.006	0.004	0.001	0.016	0.012	0.006
I	0.009	0.006	0.002	0.025	0.017	0.007
II	0.002	0.001	0.001	0.006	0.006	0.004
III	0.010	0.021	0.012	0.028	0.023	0.020
IV	0.003	0.006	0.008	0.010	0.016	0.019
V	0.023	0.050	0.026	0.101	0.130	0.090
VI	0.001	0.003	0.004	0.005	0.005	0.004

Estimated average (across a range of $V(0)$ values) price difference in correspondence of the parameters combinations in Table 2 (volatility of Gaussian model matched to that of MJD and DEJD models), and CB moneyness in the regions (0.6,1.0), [1.0,1.2), [1.2,1.4). Moneyness calculated as ratio between conversion and investment values; investment value defined as hypothetical bond value in the absence of conversion option and credit risk (definition follows Ammann *et al.* 2008). CB parameters: $T = 5$, $n = 1250$, $F = 40$, $K = 50$, $C = D = 0$, $m = 1$, $\gamma = 0.2$, $\vartheta = 0$, $s^c = 0$. Constant interest rate: $r = 0.04$. CB prices considered are for $\ln V(0) = \ln 100 + k\delta v \in [\ln 100, \ln(K/\gamma))$, for $k = 0, \ldots, 15534$, $\delta v = 5 \times 10^{-5}$. For deep in-the-money CBs (moneyness ≥ 1.4), average price difference tends practically to zero level due to the firm's highly likely call (excluded from the table).

of the alternative parameters combinations considered (the volatility of the Gaussian model is matched to that of the MJD and DEJD models) and the moneyness of the CB. In particular, we observe that the DEJD prices are lower than the ones generated by the MJD model (i.e. the price differences are positive) due to the distribution features described above, which jointly guarantee higher probability of default and lower likelihood of a call by the firm under the DEJD model specification. The reduction in value caused by the default effect is strong enough to overshadow the raise in the CB value caused by the call effect. The observed negative skewness and leptokurtosis of the two jump diffusions also explain the higher prices generated by the Gaussian model, as this underestimates the probability of default.

Further, we note from Table 2 that cases II, IV and VI are characterized by a firm log-value distribution with very low skewness and excess kurtosis when compared against cases I, III and V. In this respect, the distributions resulting from cases II, IV and VI are the closest to the Gaussian distribution and are expected to yield prices close to the ones generated by the Gaussian model. This is, in fact, the case as shown in Table 3; the prices originated by the two jump diffusion models and the Gaussian model coincide to penny accuracy in cases II, IV and VI. On the other hand, in cases I, III and V, the MJD versus DEJD and Gaussian versus DEJD price deviations increase noticeably: these can reach up to 5 and 13 pence respectively when the standard deviation of the jump size increases (i.e., in case V which features the highest left skewness and excess kurtosis).

Table 4. Model error: constant interest rates vs stochastic interest rates.

model prices (const. interest rates)			model prices (stoch. interest rates)			price differences (const. vs stoch. interest rates)	
r	MJD	DEJD	σ_r	MJD	DEJD	MJD	DEJD
3.90%	33.55010	33.54485	0.047	33.660	33.655	–0.110	–0.110
2.42%	35.83379	35.82866	0.188	36.096	36.093	–0.262	–0.264
0.45%	39.19397	39.18902	0.288	38.308	38.307	0.886	0.882

Prices and price differences under stochastic and constant interest rates in correspondence of the Vašíček model with varying interest rate volatility σ_r and constant interest rates r set equal to matched 5-year yields to maturity generated by the Vašíček model. Remaining Vašíček model parameters: $r(0) = \mu_r = 0.04$, $\kappa = 0.858$. Correlation coefficient: $\rho = 0.2$. Firm value parameters: $V(0) = 100$, base set (see Table 2). CB parameters: $T = 5$, $n = 1250$, $F = 40$, $K = 50$, $C = D = 0$, $m = 1$, $\gamma = 0.2$, $s^c = 0$, $\vartheta = 0$.

Moreover, in all cases, for a deep in-the-money CB, all the models' prices converge to the call price as the CB is then forced to conversion by call. From Table 3, in case III the MJD versus DEJD price difference peaks when the CB is close to the money because of high default probability, while in case IV of lower left skewness and excess kurtosis the peak occurs only when the CB moves in the money. A similar pattern is noted respectively in cases V and VI.

5.3 *Model error: interest rates*

The impact on the callable CB prices of alternative interest rates modelling assumptions is examined in Table 4 and Fig. 2. The parameters of the Vašíček model are $r(0) = \mu_r = 0.04$, $\kappa = 0.858$; the interest rate volatility σ_r is set to generate a 5-year yield to maturity which decreases from 3.9% to 2.42% and 0.45% (the latter being more in line with recent market quotes). These values of σ_r are reported in Table 4, together with CB prices obtained under the Vašíček model assuming correlation coefficient $\rho = 0.2$ between the Brownian motions driving the interest rate process and the continuous part of the firm value process. In Table 4, we also report the CB prices calculated for constant interest rates set equal to the different yields to maturity generated by the Vašíček model, and the resulting price differences. Prices are reported to highlight the effects of different assumptions on the yield to maturity; price differences instead emphasize the impact of model misspecification. In addition, in Fig. 2, we illustrate the variation in the CB prices[1] obtained under the Vašíček model when we vary the correlation level: in panel (a) we consider the benchmark value of the dilution factor assumed in the paper, i.e., $\gamma = 0.2$; in panel (b) we consider a higher value $\gamma = 0.45$ in order to assess the CB behaviour when the bond component has smaller weight in the contract structure.

From Table 4, we note that, in line with the results of Brennan and Schwartz (1980), the CB prices decrease as the yield to maturity increases. However, differently from Brennan and Schwartz (1980), we observe that the difference between the prices originated by the two interest rates modelling assumptions increases as the bond component of the contract becomes more valuable. In fact, the (absolute) price difference - expressed as percentage of the price in the case of stochastic interest rates - is only marginal (0.33%) when the 5-year yield to maturity is at its highest level, whilst it increases up to 2.31% in the case of a 0.45% 5-year yield to maturity, independently of the chosen firm value model. This price difference induced by 'interest rate model error' increases with the correlation level and can reach up to 5.09%. This is due to the fact that when ρ increases, the bond component becomes more sensitive to variations in the firm value and the firm's credit quality. The effects on the CB price of the interaction between interest rates and firm value can, in fact, be quite complex. The interest rate affects CB prices as it reflects the discounting applied to any (certain) future cash flows. Changes in the firm value, on the other hand, impact the value of the conversion option but also the probability of default and, consequently, the straight debt component. In presence of (positive) correlation between

[1]Prices are not reported in the interest of space and are available by the authors upon request.

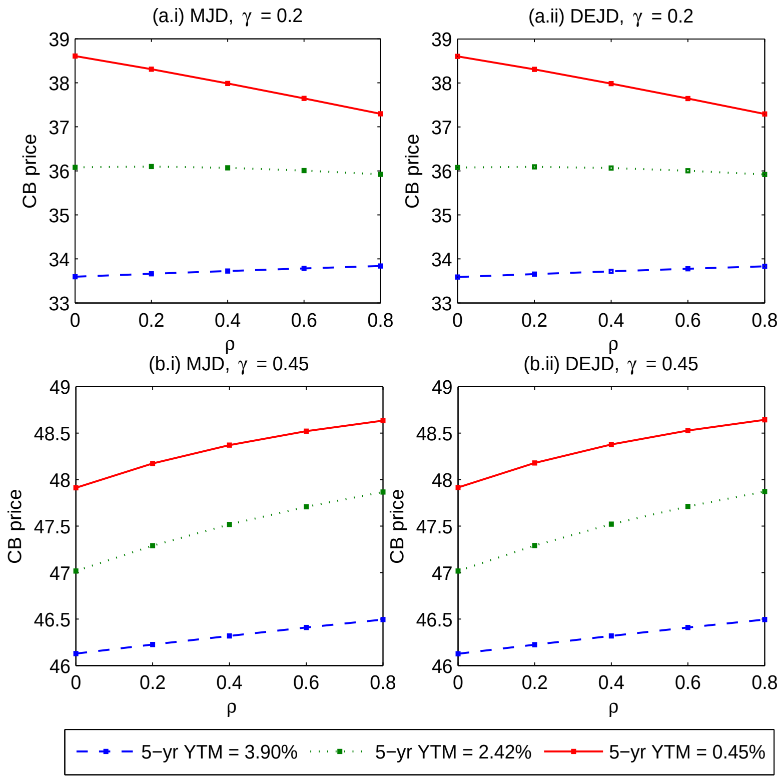

Figure 2. Impact of 5-year yield to maturity (5-yr YTM), correlation level ρ and dilution factor γ on callable CB prices. Stochastic interest rates parameters: $r(0) = \mu_r = 0.04$, $\kappa = 0.858$, $\sigma_r \in \{0.047, 0.188, 0.282\}$. 5-yr YTM $\in \{3.90\%, 2.42\%, 0.45\%\}$ generated by the Vašíček model in correspondence of the different σ_r levels. Firm value parameters: $V(0) = 100$, base set (see Table 2). CB parameters: $T = 5$, $n = 1250$, $F = 40$, $K = 50$, $C = D = 0$, $m = 1$, $s^c = 0$, $\vartheta = 0$.

the risk drivers, interest rates and firm value are expected to move in the same direction most of times, therefore their impact on the CB prices is amplified; a feature which would be missed if the rate of interest were to be assumed constant.

The ramifications of the complex interactions between interest rates, firm value and CB prices are illustrated in Fig. 2, where we plot CB prices for increasing levels of $\rho > 0$ for different yields to maturity and dilution factors. When $\rho > 0$, interest rates and firm value are more likely to either increase or decrease together. If they increase, the conversion option is more valuable and it offsets the loss in the bond value. If interest rates and firm value decrease, the increase in the bond value counteracts the price reduction of the conversion rights; however the probability of default deteriorates with a clear negative impact on the CB prices. Which of these effects prevails on the CB price depends, as shown in Fig. 2, on the current market conditions and the contract specification concerning the dilution factor. In presence of a low dilution factor (see Fig. 2, panels a.i–a.ii), the CB prices decrease with increasing ρ in a regime of low yields to maturity. This is due to the higher weight given to the bond position with respect to the conversion option and a probability of default high enough to offset any increase in the bond value when interest rates reduce. On the other hand, if the 5-year yield to maturity is set at a very high level, the bondholders will find the option to convert comparatively more attractive even when the interest rates fall, as by converting they can exit the now riskier contract and sell any acquired shares on the market, eliminating this way their exposure to the firm's default risk. In these circumstances the CB prices increase with ρ, although marginally. In presence of a higher dilution factor (see Fig. 2, panels b.i–b.ii), instead, the straight debt component has a relatively lower weight in the contract, hence the bondholders will always benefit of the downside protections provided by the

Table 5. CB prices for different contract specifications.

CB specification	model prices (const. inter. rate)		model prices (stoch. inter. rates)	
	MJD	DEJD	MJD	DEJD
call	33.40333	33.39807	33.660	33.655
call, coupons	41.91545	41.91100	42.089	42.085
call, coup., div.	40.76062	40.75578	40.702	40.697

Callable CB, callable CB with coupon payments, callable CB with coupon and dividend payments. Firm value parameters: $V(0) = 100$, base set (see Table 2). Case of constant interest rate: $r = 0.04$. Case of stochastic interest rates: $r(0) = \mu_r = 0.04$, $\kappa = 0.858$, $\sigma_r = 0.047$, $\rho = 0.2$. CB parameters: $T = 5$, $n = 1250$, $F = 40$, $K = 50$, $C = 1$ (where applicable), $D = 2$ (where applicable), $m = 1$, $\gamma = 0.2$, $s^c = 0$, $\vartheta = 0$.

CB, and its price increases as ρ increases regardless of the level of the yield to maturity.

5.4 *Effects of discrete coupon and dividend payments*

In the following, we explore the impact of adding discrete coupons and dividends to the valuation framework. Table 5 reports prices for 5-year callable CBs with daily sampling under constant and stochastic interest rates. In the case of stochastic interest rates, we choose parameter values $r(0) = \mu_r = 0.04$, $\kappa = 0.858$, $\sigma_r = 0.047$ and $\rho = 0.2$; for the constant interest rates assumption instead we set $r = r(0)$. We employ the base set of parameters as in Table 2 for the firm value process, and further assume $V(0) = 100$, $F = 40$, $K = 50$, $C = 1$ (payable semiannually), $D = 2$ (payable at the first and third quarters of the year), $m = 1$, $\gamma = 0.2$, $s^c = 0$, $\vartheta = 0$.

Few comments are in order. Adding coupons to the bond indenture raises substantially the payoff to the investors and, consequently, the CB value, while the default event becomes more likely negatively affecting the CB value. Nevertheless, the first effect offsets the second one, justifying the overall increase in the CB value observed. We also note that, for the case of a dividend-paying common stock, the contract's price increases less compared to the case of a non-dividend-paying stock. This occurs because the dividends are not payable to the CB holders pre-conversion, although they affect the market perception of the firm's liquidity and therefore the chances of a future default. Further, the MJD model prices remain higher than the DEJD model prices. The price discrepancies are smaller in the case of a coupon-bearing CB, as the coupons have a primary positive upshot on the value of the bond.

5.5 *Effects of call policy*

In this section, we investigate the impact of the adopted call strategy by studying how changing the call notice length and the safety premium affects the model prices produced under the base and case V sets of parameters, which corresponds to the more significant departure from the Gaussian model (see Section 5.2). We compute prices for 5-year callable CBs with daily sampling, $V(0) = 100$, $F = 40$, $K = 50$, $D = 0$, $m = 1$, $\gamma = 0.2$. For the interest rates model, we assume the same parameters as in Section 5.4.

We observe from Table 6 that introducing a call notice period and a safety premium affects the CBs prices. In details, under the base set with constant interest rates, the percentage price changes incurred by varying (s^c, ϑ) from $(0, 0)$ range from 0.25% to -0.76% in the absence of coupons, and 0.19% to -0.24% when non-zero coupon payments are assumed in the contract specification. Results are similar under the assumption of stochastic interest rates. Introducing a call notice increases the value of the CB as it yields a higher probability of a significant favourable movement in the firm value, and therefore the conversion value. The effect is particularly evident under the case V of high volatility of the firm log-value process, as the option to convert becomes

Table 6. Callable CB prices for varying call specification (s^c, ϑ).

(s^c, ϑ)	model prices (constant interest rate)				model prices (stochastic interest rates)			
	base parameters		case V parameters		base parameters		case V parameters	
$C = 0$	MJD	DEJD	MJD	DEJD	MJD	DEJD	MJD	DEJD
(0.00, 0.00)	33.40333	33.39807	33.56636	33.55806	33.660	33.655	33.797	33.788
(1/12,0.00)	33.48615	33.48020	33.69869	33.69552	33.751	33.745	33.936	33.932
(1/12,0.20)	33.20939	33.20351	33.40978	33.40891	33.451	33.445	33.634	33.633
(1/12,0.25)	33.14899	33.14321	33.33957	33.33896	33.385	33.380	33.561	33.560
(1/24,0.00)	33.46714	33.46131	33.67012	33.66780	33.729	33.724	33.905	33.902
(1/24,0.20)	33.20935	33.20345	33.40940	33.40846	33.451	33.445	33.634	33.633
(1/24,0.25)	33.14899	33.14319	33.33946	33.33877	33.385	33.380	33.561	33.560
$C = 1$								
(0.00, 0.00)	41.91545	41.91100	41.85611	41.84985	42.089	42.085	42.002	41.996
(1/12,0.00)	41.99472	41.98956	41.98329	41.98102	42.169	42.165	42.128	42.126
(1/12,0.20)	41.85176	41.84633	41.85319	41.85248	42.021	42.016	42.000	41.999
(1/12,0.25)	41.81551	41.81010	41.81250	41.81196	41.994	41.989	41.970	41.969
(1/24,0.00)	41.97622	41.97120	41.95501	41.95350	42.160	42.156	42.109	42.107
(1/24,0.20)	41.85170	41.84626	41.85276	41.85200	42.019	42.013	41.997	41.996
(1/24,0.25)	41.81550	41.81008	41.81237	41.81178	41.999	41.994	41.974	41.973

Firm value parameters: $V(0) = 100$, base and case V sets of parameters (see Table 2). Case of constant interest rate: $r = 0.04$. Case of stochastic interest rates: $r(0) = \mu_r = 0.04$, $\kappa = 0.858$, $\sigma_r = 0.047$, $\rho = 0.2$. CB parameters: $T = 5$, $n = 1250$, $F = 40$, $K = 50$, $D = 0$, $m = 1$, $\gamma = 0.2$.

more valuable. The price rise is smaller in the presence of coupons in the bond indenture. On the contrary, the presence of the safety premium, which relates to the firm's decision on the date of the call announcement, reduces the price of CBs as a successful forced conversion by call is more likely. The reduction is though slightly mitigated by the accrued interests on any coupon still due; this effect is more pronounced under the case V.

6 Conclusions

This paper examines the pricing of convertible bonds in a structural setting with jumps and stochastic interest rates correlated to the firm value process. To this aim, we have developed and implemented a pricing algorithm which has proved to be accurate and capable of flexibly handling a number of contract-design features, such as call provisions, discrete coupons and dividends.

As the proposed jump diffusion market setup for pricing CBs is new in the literature, we analyze the behaviour of the contract price under this more complex representation of the firm value. Our numerical analysis shows that misspecifying the underlying process driving the firm value, i.e., falsely specifying a jump diffusion process as a continuous Brownian motion, can substantially overstate the value of the CB due to the higher probability of default generated by the inclusion of market shocks in the model.

The results of our analysis also indicate the importance of an accurate modelling of the term structure of interest rates corroborating the results of Yiğitbaşioğlu and Alexander (2006); the mispricing, in fact, can be significant depending on prevailing market conditions, such as yield to maturity and correlation between risk drivers, and contract specifications, in particular the dilution factor. For practical purposes, we note that the additional complexity induced by a more sophisticated model for the term structure of interest rates can be handled efficiently by the proposed numerical scheme.

Finally, it is shown that ignoring the call notice period and the safety premium when valuing convertibles yields noticeable pricing biases.

References

Ammann, M., Kind, A. and Wilde, C., Simulation-based pricing of convertible bonds. *Journal of Empirical Finance*, 2008, **15**, 310–331.

Andricopoulos, A.D., Widdicks, M., Duck, P.W. and Newton, D.P., Universal option valuation using quadrature methods. *Journal of Financial Economics*, 2003, **67**, 447–471.

Asquith, P., Convertible bonds are not called late. *Journal of Finance*, 1995, **50**, 1275–1289.

Asquith, P. and Mullins, D.W., Convertible debt: corporate call policy and voluntary conversion. *Journal of Finance*, 1991, **46**, 1273–1289.

Barone-Adesi, G., Bermúdez, A. and Hatgioannides, J., Two-factor convertible bonds valuation using the method of characteristics/finite elements. *Journal of Economic Dynamics and Control*, 2003, **27**, 1801–1831.

Bermúdez, A. and Webber, N., An asset based model of defaultable convertible bonds with endogenized recovery. Working Paper, 2004.

Black, F. and Cox, J.C., Valuing corporate securities: some effects of bond indenture provisions. *Journal of Finance*, 1976, **31**, 351–367.

Brennan, M.J. and Schwartz, E.S., Convertible bonds: valuation and optimal strategies for call and conversion. *Journal of Finance*, 1977, **32**, 1699–1715.

Brennan, M.J. and Schwartz, E.S., Analyzing convertible bonds. *Journal of Financial and Quantitative Analysis*, 1980, **15**, 907–929.

Brigo, D. and Morini, M., Structural credit calibration. *Risk Magazine*, 2006, **April issue**.

Brigo, D., Morini, M. and Tarenghi, M., Credit calibration with structural models and equity return swap valuation under counterparty risk. In *Recent advances in theory and practice of credit derivatives*, edited by Bielecki, Brigo and Patras, 2010, Bloomberg Press.

Carayannopoulos, P., Valuing convertible bonds under the assumption of stochastic interest rates: an empirical investigation. *Quarterly Journal of Business and Economics*, 1996, **35**, 17–31.

Carayannopoulos, P. and Kalimipalli, M., Convertible bond prices and inherent biases. *The Journal of Fixed Income*, 2003, **13**, 64–73.

Cariboni, J. and Schoutens, W., Pricing credit default swaps under Lévy models. *The Journal of Computational Finance*, 2007, **10**, 71–91.

Černý, A., Introduction to fast Fourier transform in finance. *The Journal of Derivatives*, 2004, **12**, 73–88.

Černý, A. and Kyriakou, I., An improved convolution algorithm for discretely sampled Asian options. *Quantitative Finance*, 2011, **11**, 381–389.

Cox, J.C., Ingersoll, J.E. and Ross, S.A., A theory of the term structure of interest rates. *Econometrica*, 1985, **53**, 385–407.

Dao, B. and Jeanblanc, M., Double exponential jump diffusion process: a structural model of endogenous default barrier with roll-over debt structure. *Journal of Credit Risk*, 2012, **8**, 21–42.

Derman, E., Valuing convertible bonds as derivatives. Quantitative strategies research notes, Goldman Sachs, New York, 1994.

Duffie, D., Pan, J. and Singleton, K., Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 2000, **68**, 1343–1376.

Eom, Y.H., Helwege, J. and Huang, J.Z., Structural models of corporate bond pricing: an empirical analysis. *Review of Financial Studies*, 2004, **17**, 499–544.

Fang, F., Jönsson, H., Oosterlee, C.W. and Schoutens, W., Fast valuation and calibration of credit default swaps under Lévy dynamics. *The Journal of Computational Finance*, 2010, **14**, 57–86.

Fang, F. and Oosterlee, C.W., Pricing early-exercise and discrete barrier options by Fourier-cosine series expansions. *Numerische Mathematik*, 2009, **114**, 27–62.

Filipović, D., *Term-Structure Models: A Graduate Course*, 2009 (Springer: New York).

Fiorani, F., Luciano, E. and Semeraro, P., Single and joint default in a structural model with purely discountinuous asset prices. *Quantitative Finance*, 2010, **10**, 249–263.

Geman, H., El-Karoui, N. and Rochet, J.C., Changes of numéraire, changes of probability measure and option pricing. *Journal of Applied Probability*, 1995, **32**, 443–458.

Grzelak, L.A. and Oosterlee, C.W., On the Heston model with stochastic interest rates. *SIAM Journal on Financial Mathematics*, 2011, **2**, 255–286.

Grzelak, L.A., Oosterlee, C.W. and van Weeren, S., The affine Heston model with correlated Gaussian interest rates for pricing hybrid derivatives. *Quantitative Finance*, 2011, **11**, 1647–1663.

Grzelak, L.A., Oosterlee, C.W. and van Weeren, S., Extension of stochastic volatility equity models with the Hull–White interest rate process. *Quantitative Finance*, 2012, **12**, 89–105.

Hull, J.C. and White, A.D., Using Hull–White interest rate trees. *The Journal of Derivatives*, 1996, **3**, 26–36.

Ingersoll, J.E., A contingent-claims valuation of convertible securities. *Journal of Financial Economics*, 1977a, **4**, 289–321.

Ingersoll, J.E., An examination of corporate call policies on convertible securities. *Journal of Finance*, 1977b, **32**, 463–478.

Kou, S.G., A jump-diffusion model for option pricing. *Management Science*, 2002, **48**, 1086–1101.

Kovalov, P. and Linetsky, V., Valuing convertible bonds with stock price, volatility, interest rate, and default risk. Working paper, 2008.

Leland, H.E., Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance*, 1994, **49**, 1213–1252.

Leland, H.E. and Toft, K.B., Optimal capital structure, endogenous bankruptcy and the term structure of credit spreads. *Journal of Finance*, 1996, **51**, 987–1019.

Lipton, A. and Sepp, A., Credit value adjustment for credit default swaps via the structural default model. *Journal of Credit Risk*, 2009, **5**, 123–146.

Longstaff, F.A. and Schwartz, E.S., A simple approach to valuing risky fixed and floating rate debt. *Journal of Finance*, 1995, **50**, 789–819.

Lord, R., Fang, F., Bervoets, F. and Oosterlee, C.W., A fast and accurate FFT-based method for pricing early-exercise options under Lévy processes. *SIAM Journal on Scientific Computing*, 2008, **30**, 1678–1705.

Lvov, D., Yiğitbaşioğlu, A.B. and El-Bachir, N., Pricing convertible bonds by simulation. Working Paper, 2004.

Marazzina, D., Fusai, G. and Germano, G., Pricing credit derivatives in a Wiener–Hopf framework. In *Topics in Numerical Methods for Finance*, edited by Cummins et al., 2012 (Springer Science+Business Media: New York).

Merton, R.C., Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 1976, **3**, 125–144.

O'Kane, D. and Turnbull, S., Valuation of credit default swaps. Fixed income quantitative credit research, Lehman Brothers, 2003.

Rogers, L.C.G., Which model for term-structure of interest rates should one use?. In *Mathematical Finance, IMA Volumes in Mathematics and its Applications 65*, edited by Davis, Duffie, Fleming and Shreve, 1995 (Springer: New York).

Ruijter, M.J. and Oosterlee, C.W., Two-dimensional Fourier cosine series expansion method for pricing financial options. *SIAM Journal on Scientific Computing*, 2012, **34**, B642–B671.

Takahashi, A., Kobayashi, T. and Nakagawa, N., Pricing convertible bonds with default risk: a Duffie-Singleton approach. *The Journal of Fixed Income*, 2001, **11**, 20–29.

Tsiveriotis, K. and Fernandes, C., Valuing convertible bonds with credit risk. *The Journal of Fixed Income*, 1998, **8**, 95–102.

van Haastrecht, A., Lord, R., Pelsser, A. and Schrager, D., Pricing long-dated insurance contracts with stochastic interest rates and stochastic volatility. *Insurance: Mathematics and Economics*, 2009, **45**, 436–448.

Vašíček, O.A., An equilibrium characterization of the term structure. *Journal of Financial Economics*, 1977, **5**, 177–188.

Yiğitbaşioğlu, A.B. and Alexander, C., Pricing and hedging convertible bonds: delayed calls and uncertain volatility. *International Journal of Theoretical and Applied Finance*, 2006, **9**, 415–453.

Appendix A: General properties of a jump diffusion process

A jump diffusion process is a Lévy process, i.e., a process with stationary and independent increments, which can be decomposed as the sum of a Brownian motion and an independent compound (time-homogeneous) Poisson process. It follows from the construction proposed in Section 2 that under the assumption of constant interest rates, the cumulants of the process $\ln(V(t)/V(0))$ under the risk neutral measure are

$$c_1 := E\left[\ln\frac{V(t)}{V(0)}\right] = \left(r - \sigma^2/2 - \lambda E\left(e^X - 1\right) + \lambda E(X)\right)t, \tag{A1}$$

$$c_2 := \mathrm{Var}\left[\ln\frac{V(t)}{V(0)}\right] = \left(\sigma^2 + \lambda E\left(X^2\right)\right)t, \tag{A2}$$

$$c_3 := E\left[\left(\ln\frac{V(t)}{V(0)} - E\left(\ln\frac{V(t)}{V(0)}\right)\right)^3\right] = \lambda E\left(X^3\right)t,$$

$$c_4 := E\left[\left(\ln\frac{V(t)}{V(0)} - E\left(\ln\frac{V(t)}{V(0)}\right)\right)^4\right] - 3c_2 = \lambda E\left(X^4\right)t.$$

The skewness coefficient and excess kurtosis are given, respectively, by

$$\frac{c_3}{c_2^{3/2}} = \frac{\lambda E\left(X^3\right)t^{-1/2}}{\left(\sigma^2 + \lambda E\left(X^2\right)\right)^{3/2}}, \tag{A3}$$

$$\frac{c_4}{c_2^2} = \frac{\lambda E\left(X^4\right)t^{-1}}{\left(\sigma^2 + \lambda E\left(X^2\right)\right)^2}, \tag{A4}$$

where the formulae for $E\left(X^j\right)$, $j = 1, 2, 3, 4$, for the Merton and Kou processes are, respectively,

$$\begin{aligned} E(X) = \mu_X, \quad E\left(X^2\right) = \mu_X^2 + \sigma_X^2, \\ E\left(X^3\right) = \mu_X\left(\mu_X^2 + 3\sigma_X^2\right), \quad E\left(X^4\right) = \mu_X^4 + 6\mu_X^2\sigma_X^2 + 3\sigma_X^4, \end{aligned} \tag{A5}$$

and

$$\begin{aligned} E(X) = p/\eta_1 - q/\eta_2, \quad E\left(X^2\right) = 2\left(p/\eta_1^2 + q/\eta_2^2\right), \\ E\left(X^3\right) = 6\left(p/\eta_1^3 - q/\eta_2^3\right), \quad E\left(X^4\right) = 24\left(p/\eta_1^4 + q/\eta_2^4\right). \end{aligned} \tag{A6}$$

In addition, knowledge of $E(X)$ and $E\left(X^2\right)$ gives us access to

$$\mathrm{Var}(X) = E\left(X^2\right) - E(X)^2. \tag{A7}$$

We note from (A3) that the skewness of the process distribution is governed by the third (raw) moment of the jump size distribution, whilst it follows from (A4) that the process is leptokurtic. Finally, (A3) and (A4) show that both skewness and excess kurtosis vanish over very long time horizons.

Appendix B: Computation of convolutions

For a fast computation of convolution (19), we use the fast Fourier transform (FFT). We present a discrete approximation to (19) which allows computation by FFT directly (see Eqs. B1–B2); the derivation follows the one in Lord *et al.* (2008) for the 1-D case, hence it is omitted here. Further, we provide implementation details which are specific to our CB price application. We emphasize that the suggested FFT pricing approach requires knowledge of the distribution law of the pair of random variables $(Z_{r,k}, Z_k)$ defined in (16) only via the associated characteristic function which is given by (20).[1]

B.1 *Implementation of price recursion*

Based on backward recursion, we compute at each sampling date t_{k-1}, $k = n, \ldots, 1$, the continuation value of the CB and, subsequently, the CB payoff for use in the next iteration. In what follows, we provide detailed steps of the pricing procedure; a summarized version is given in Table B1.

Step 0.a. Choose grid size N to be a power of 2 for fast computation of the sums (B1) and (B2) using the FFT with radix 2 (see Černý 2004).

Step 0.b. For each of the state variables, i.e., the short rate of interest and the firm log-value, select equally spaced grids $\boldsymbol{x_r} := \{x_{r,0} + m_1\delta_r\}_{m_1=0}^{N-1}$ and $\boldsymbol{x} := \{x_0 + m_2\delta\}_{m_2=0}^{N-1}$ on which we will evaluate the continuation value of the CB. The grid ranges are chosen based on the cumulants of $Z_{r,k}$ and Z_k as in Lord *et al.* (2008): $x_{r,0} := c_{r,1} - \varrho\sqrt{c_{r,2}}$, $x_0 := c_1 - \varrho\sqrt{c_2 + \sqrt{c_4}}$ and grid spacings $\delta_r := 2\varrho\sqrt{c_{r,2}}/N$, $\delta := 2\varrho\sqrt{c_2 + \sqrt{c_4}}/N$, where ϱ is a user-defined proportionality constant, e.g., $\varrho = 5$, $c_{r,1} = -i\partial \ln \varphi(\delta t, 0, u_2)/\partial u_1$, $c_{r,2} = -\partial^2 \ln \varphi(\delta t, 0, u_2)/\partial u_1^2$, $c_1 = -i\partial \ln \varphi(\delta t, u_1, 0)/\partial u_2$, $c_2 = -\partial^2 \ln \varphi(\delta t, u_1, 0)/\partial u_2^2$ and $c_4 = \partial^4 \ln \varphi(\delta t, u_1, 0)/\partial u_2^4$. Given the closed form expression for the characteristic function $\varphi(\delta t, u_1, u_2)$ in (20), the cumulants can be obtained explicitly by straightforward differentiation using any symbolic computation package such as Mathematica. Cumulant c_4 is included to ensure that the tails of the distribution of Z_k are sufficiently captured.

Step 0.c. Select equally spaced grids $\boldsymbol{y_r}$ and $\boldsymbol{y}$ on which we will evaluate the CB payoff for the relevant decision date. For convenience, we choose $\boldsymbol{y_r} = \boldsymbol{x_r}$ and $\boldsymbol{y} = \boldsymbol{x}$.

Step 0.d. Select uniform, symmetric grids of size N with spacings δ_r^u and δ^u, respectively, $\boldsymbol{u_r} := \{(j_1 - N/2)\delta_r^u\}_{j_1=0}^{N-1}$ and $\boldsymbol{u} := \{(j_2 - N/2)\delta^u\}_{j_2=0}^{N-1}$. The range of values of the grids $(\boldsymbol{u_r}, \boldsymbol{u})$ is determined to ensure that the tail of the absolute value of the characteristic function $|\varphi(\delta t, u_1, u_2)|$ is sufficiently captured. More specifically, given N from *Step 0.a*, choose grid spacings δ_r^u, δ^u, hence grid end-points $(N/2)\delta_r^u$ and $(N/2)\delta^u$, so that $|\varphi(\delta t, (N/2)\delta_r^u, (N/2)\delta^u)| \leq 10^{-\varrho'}$ holds, where the characteristic function φ is given in (20) and the value $\varrho' \in \mathbb{N}$ is guided by the desired precision, e.g., choose $\varrho' = 5$ when computing results to 5 decimal places.

Step 0.e. As required for a FFT implementation, check that the Nyquist relations $\delta_r^u\delta_r = 2\pi/N$, $\delta^u\delta = 2\pi/N$ hold. If not, adjust accordingly the original grid spacings (δ_r, δ) and/or (δ_r^u, δ^u).

Step 0.f. Calculate the values of φ on the grid $(-\boldsymbol{u_r}, -\boldsymbol{u})$ and denote these by $\boldsymbol{\varphi}$. Store for use in all iterations.

[1] An alternative accurate methodology for 2-D pricing problems based on the use of characteristic functions has been recently proposed by Ruijter and Oosterlee (2012). However, as the authors show, the full applicability of their method is restricted in the case of the (non-Lévy) Heston model, even when pricing simpler Bermudan put options. Similarly, the jump diffusion model with stochastic interest rates assumed in this paper does not belong to the Lévy class. Given this and also the complex payoff structure of the CB, we anticipate similar limitations in our case. We leave the study of their methodology for future research.

Step 1. Denote by $\tilde{\mathbf{G}}_{k-1_-} := \{\tilde{G}_{k-1_-,m_1,m_2}\}_{m_1=0,m_2=0}^{N-1,N-1}$ the values approximating the function $\tilde{G}_{k-1_-}$ in (18) on the grid $(\boldsymbol{x_r}, \boldsymbol{x})$. Compute

$$\tilde{G}_{k-1_-,m_1,m_2} := \frac{1}{4\pi^2} e^{i\frac{N}{2}\delta_r^u(\boldsymbol{x}_{\boldsymbol{r}}^{\mathrm{T}}-x_{r,0})+i\frac{N}{2}\delta^u(\boldsymbol{x}^{\mathrm{T}}-x_0)}\delta_r^u\delta^u \times \sum_{j_1=0}^{N-1}\sum_{j_2=0}^{N-1} e^{-i\frac{2\pi}{N}j_1m_1-i\frac{2\pi}{N}j_2m_2}\left(e^{-i\boldsymbol{u}_{\boldsymbol{r}}^{\mathrm{T}}x_{r,0}}e^{-i\boldsymbol{u}x_0}\circ\mathbf{D}_{k_-}\circ\boldsymbol{\varphi}\right)_{j_1,j_2}, \quad \text{(B1)}$$

where we denote by $\mathbf{D}_{k_-} := \{D_{k_-,j_1,j_2}\}_{j_1=0,j_2=0}^{N-1,N-1}$ the values approximating the discrete Fourier transform of (19) on the grid $(\boldsymbol{u_r}, \boldsymbol{u})$ with

$$D_{k_-,j_1,j_2} := e^{i(\boldsymbol{u}_{\boldsymbol{r}}^{\mathrm{T}}+\frac{N}{2}\delta_r^u)y_{r,0}+i(\boldsymbol{u}^{\mathrm{T}}+\frac{N}{2}\delta^u)y_0}\delta_r\delta \times \sum_{m_1=0}^{N-1}\sum_{m_2=0}^{N-1} e^{i\frac{2\pi}{N}j_1m_1+i\frac{2\pi}{N}j_2m_2}\left(e^{-i\frac{N}{2}\delta_r^u\boldsymbol{y}_{\boldsymbol{r}}^{\mathrm{T}}}e^{-i\frac{N}{2}\delta^u\boldsymbol{y}}\circ\mathbf{H}_{k_-}\circ\boldsymbol{w}\right)_{m_1,m_2} \quad \text{(B2)}$$

and $\mathbf{H}_{k_-} := \{H_{k_-,m_1,m_2}\}_{m_1=0,m_2=0}^{N-1,N-1}$ the payoff function values H_{k_-} on the grid $(\boldsymbol{y_r}, \boldsymbol{y})$ initialized by (7) (see also *Step 4*). $\circ$ denotes the Hadamard element-wise product. Finally, $\boldsymbol{w} := \{w_{m_1,m_2}\}_{m_1=0,m_2=0}^{N-1,N-1}$ are the trapezoidal weights, where

$$w_{m_1,m_2} := \begin{cases} 1/4, & \text{if } (m_1,m_2) = (0,0),(0,N-1),(N-1,0),(N-1,N-1) \\ 1/2, & \text{if } m_1 = 1,\ldots,N-2, \quad m_2 = 0, N-1 \\ 1/2, & \text{if } m_1 = 0, N-1, \quad m_2 = 1,\ldots,N-2 \\ 1, & \text{if } m_1 = 1,\ldots,N-2, \quad m_2 = 1,\ldots,N-2 \end{cases}. \quad \text{(B3)}$$

Step 2. Consider the functions g and g_r given by (14)–(15). By fitting a surface to the nodes $(\boldsymbol{x_r}, \boldsymbol{x}, \tilde{\mathbf{G}}_{k-1_-})$, we evaluate $\tilde{G}_{k-1_-}$ at $g_{r,k-1}(\boldsymbol{y_r}) \subseteq \boldsymbol{x_r}$, $g_{k-1}(\boldsymbol{y_r}, \boldsymbol{y}) \subseteq \boldsymbol{x}$ using cubic interpolation. For $g_{k-1}(\boldsymbol{y_r}, \boldsymbol{y}) \nsubseteq \boldsymbol{x}$ we use linear extrapolation in e^x. See Remark *B1* for more details.

Step 3. Compute $\mathbf{G}_{k-1_-} := \{G_{k-1_-,m_1,m_2}\}_{m_1=0,m_2=0}^{N-1,N-1}$ the continuation values of the CB in (17), where

$$G_{k-1_-,m_1,m_2} := e^{A_1(\delta t)-A_2(\delta t)\boldsymbol{y}_{\boldsymbol{r},m_1}}\tilde{G}_{k-1_-}(g_{r,k-1}(\boldsymbol{y}_{\boldsymbol{r},m_1}), g_{k-1}(\boldsymbol{y}_{\boldsymbol{r},m_1}, \boldsymbol{y}_{m_2})).$$

Step 4. Compute the payoff function values $\mathbf{H}_{k-1_-}$ on the grid $(\boldsymbol{y_r}, \boldsymbol{y})$ from Eqs. (8)–(11) for the relevant decision date. Continue with *Step 1* until $k = 1$.

Remark *B1***:** From *Step 1*, we obtain the values $\tilde{\mathbf{G}}_{k-1_-}$ of the function $\tilde{G}_{k-1_-}$ on the grid $(\boldsymbol{x_r}, \boldsymbol{x})$. However to compute the CB continuation values G_{k-1_-}, as it is obvious from the right-hand side of (17), we require knowledge of $\tilde{G}_{k-1_-}$ on the non-uniformly spaced grid $(g_{r,k-1}(\boldsymbol{y_r}), g_{k-1}(\boldsymbol{y_r}, \boldsymbol{y}))$. As we have immediate access to $\tilde{G}_{k-1_-}(\boldsymbol{x_r}, \boldsymbol{x})$ only, we compute $\tilde{G}_{k-1_-}(g_{r,k-1}(\boldsymbol{y_r}), g_{k-1}(\boldsymbol{y_r}, \boldsymbol{y}))$ by cubic interpolation (using, for example, the built-in algorithm available in Matlab). In addition, to evaluate $\tilde{G}_{k-1_-}$ wherever $g_{k-1}(\boldsymbol{y_r}, \boldsymbol{y})$ lies outside the range of $\boldsymbol{x}$, we use linear extrapolation in the firm value dimension (i.e., in e^x).

Remark *B2***:** The 1-D convolution in (21) is computed separately by implementing *Steps 1–2* above using, however, a 1-D transform (similarly to a European vanilla option, see Lord *et al.* 2008). In the absence of accrued interest, this computation is performed once as a preliminary step and stored for use at the relevant decision dates. If we allow for accrued interest, then the

Table B1. CB pricing algorithm.

Inputs: grid parameters: $N, (\delta_r, \delta), (\delta_r^u, \delta^u)$ (see *Steps 0.a–e*)

1. Set grids in spatial domain $\boldsymbol{x_r} \leftarrow \{x_{r,0} + m_1\delta_r\}_{m_1=0}^{N-1}, \boldsymbol{x} \leftarrow \{x_0 + m_2\delta\}_{m_2=0}^{N-1}, \boldsymbol{y_r} \leftarrow \boldsymbol{x_r}, \boldsymbol{y} \leftarrow \boldsymbol{x}$
2. Set grids in Fourier domain $\boldsymbol{u_r} \leftarrow \{(j_1 - N/2)\delta_r^u\}_{j_1=0}^{N-1}, \boldsymbol{u} \leftarrow \{(j_2 - N/2)\delta^u\}_{j_2=0}^{N-1}$
3. Compute characteristic function (see Eq. 20) on grid $(-\boldsymbol{u_r}, -\boldsymbol{u})$ for use in all iterations: $\boldsymbol{\varphi} \leftarrow \varphi(\delta t, -\boldsymbol{u_r}, -\boldsymbol{u})$

Iterate for $k = n, \ldots, 1$:

4. Compute discr. Four. trans. of (19) on grid $(\boldsymbol{u_r}, \boldsymbol{u})$:

$$\mathbf{D}_{k_-} \leftarrow \left\{ \begin{array}{l} e^{i(\boldsymbol{u_r}^{\mathrm{T}} + \frac{N}{2}\delta_r^u)y_{r,0} + i(\boldsymbol{u}^{\mathrm{T}} + \frac{N}{2}\delta^u)y_0}\delta_r\delta \\ \times \sum_{m_1=0}^{N-1}\sum_{m_2=0}^{N-1} e^{i\frac{2\pi}{N}j_1m_1 + i\frac{2\pi}{N}j_2m_2}\left(e^{-i\frac{N}{2}\delta_r^u\boldsymbol{y_r}^{\mathrm{T}}}e^{-i\frac{N}{2}\delta^u\boldsymbol{y}} \circ \mathbf{H}_{k_-} \circ \boldsymbol{w}\right)_{m_1,m_2} \end{array} \right\}_{j_1=0,j_2=0}^{N-1,N-1}$$

5. Compute $\tilde{G}_{k-1_-}$ (see Eq. 18) on grid $(\boldsymbol{x_r}, \boldsymbol{x})$:

$$\tilde{\mathbf{G}}_{k-1_-} \leftarrow \left\{ \begin{array}{l} \frac{1}{4\pi^2}e^{i\frac{N}{2}\delta_r^u(\boldsymbol{x_r}^{\mathrm{T}} - x_{r,0}) + i\frac{N}{2}\delta^u(\boldsymbol{x}^{\mathrm{T}} - x_0)}\delta_r^u\delta^u \\ \times \sum_{j_1=0}^{N-1}\sum_{j_2=0}^{N-1} e^{-i\frac{2\pi}{N}j_1m_1 - i\frac{2\pi}{N}j_2m_2}\left(e^{-i\boldsymbol{u_r}^{\mathrm{T}}x_{r,0}}e^{-i\boldsymbol{u}x_0} \circ \mathbf{D}_{k_-} \circ \boldsymbol{\varphi}\right)_{j_1,j_2} \end{array} \right\}_{m_1=0,m_2=0}^{N-1,N-1}$$

6. Compute CB continuation values G_{k-1_-} (see Eq. 17):
   a. Evaluate $\tilde{G}_{k-1_-}(g_{r,k-1}(\boldsymbol{y_r}), g_{k-1}(\boldsymbol{y_r}, \boldsymbol{y}))$ using interpolation/extrapolation (see *Step 2*)
   b. $\mathbf{G}_{k-1_-} \leftarrow \left\{e^{A_1(\delta t) - A_2(\delta t)\boldsymbol{y}_{\boldsymbol{r},m_1}}\tilde{G}_{k-1_-}(g_{r,k-1}(\boldsymbol{y}_{\boldsymbol{r},m_1}), g_{k-1}(\boldsymbol{y}_{\boldsymbol{r},m_1}, \boldsymbol{y}_{m_2}))\right\}_{m_1=0,m_2=0}^{N-1,N-1}$
7. Compute CB payoff function values $\mathbf{H}_{k-1_-}$ on grid $(\boldsymbol{y_r}, \boldsymbol{y})$ (see Eqs. 8–11)
8. Continue with next iteration until $k = 1$

Return CB prices $\mathbf{G}_0$ on grid $(\boldsymbol{y_r}, \boldsymbol{y})$

Note. Inputs: grid spacings are determined as explained in preliminary *Steps 0.a–e*. Iteration part: initialized by CB payoff (7). Trapezoidal weights $\boldsymbol{w}$ given by (B3). $\circ$ denotes the Hadamard element-wise product. Output: CB prices $\mathbf{G}_0$ on uniform 2-D grid of short rate and firm log-values.

computation needs to be performed as many times as the number of sampling dates between two consecutive coupon dates; this is because the accrued interest varies at different sampling dates between two coupon dates.

Remark *B3***:** At each decision date, we need to identify the critical value of $Y(t_-) =: y^*$ at which the CB payout changes. These are obvious for the payoffs (7)–(9), e.g., in (8) at $y^* = \ln((1+\vartheta)K_t/\gamma)$ continuation of the CB changes to call by the issuer. However, the optimal conversion point in the payoff function (10) is not fixed and is unknown. We need to locate at each time $t = t_i$ the critical value $Y(t_{i_-}) =: y_i^*$ which satisfies $G_{i_-}(r(t_i), \ln(e^{y_i^*} - D_{t_i}1_{y_i^* > \ln D_{t_i}})) - \gamma e^{y_i^*} = 0$. This equation can be solved fast numerically using, for example, the Trust-Region Dogleg algorithm in Matlab with user-defined termination tolerance level and maximum number of iterations. The same procedure can be used for the payoff function (11).

Remark *B4***:** If we assume constant interest rates, 2-D transforms (B1)–(B2) reduce to 1-D transforms based on a single state variable, that is, the firm log-value. (From a numerical implementation perspective, the valuation problem in this case is similar to that of a Bermudan option, as in Lord *et al.* 2008, based instead on the relevant CB payoff at each decision date.)