Discrete-time volatility forecasting with persistent leverage effect and the link with continuous-time volatility modeling *

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Abstract

We first propose a reduced form model in *discrete time* for S&P500 volatility showing that the forecasting performance can be significantly improved by introducing a persistent leverage effect with a long-range dependence similar to that of volatility itself. We also find a strongly significant positive impact of lagged jumps on volatility, which however is absorbed more quickly. We then estimate *continuous-time* stochastic volatility models which are able to reproduce the statistical features captured by the discrete-time model. We show that a single-factor model driven by a fractional Brownian motion is unable to reproduce the volatility dynamics observed in the data, while a multi-factor Markovian model fully replicates the persistence of both volatility and leverage effect. The impact of jumps can be associated with a common jump component in price and volatility.

JEL classification: C13; C22; C51; C53

Keywords: Volatility Forecasting; Leverage Effect; Jumps; Fractional Brownian Motion; Multifactor Models.

*This paper supersedes the previously circulating versions *Volatility determinants: heterogeneity, leverage and jumps* and *HAR volatility modelling with heterogenous leverage and jumps*. The paper is complemented by a Web Appendix downloadable from our web pages and containining supplementary material. The daily variables used in this paper are available from the authors upon request. We would like to acknowledge Davide Pirino for research assistence, Federico Bandi, Tim Bollerslev, Alvaro Cartea, Giampiero Gallo, Roel Oomen, Eduardo Rossi, Paolo Santucci de Magistris and Fabio Trojani for useful suggestions, and the partecipants at the SITE Summer Segment in Stanford, June 2009 and at seminars in Madrid, Universidad Carlos III and Tokio, Bank of Japan. Università della Svizzera Italiana and Carey Business School at Johns Hopkins University are kindly acknowledged for support. All errors and omissions are our own.

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1 Introduction

The relevance of financial market volatility led to a very large literature trying to take into account its most salient dynamic features: clustering, slowly decaying auto-correlation, asymmetric responses. The advent of high-frequency data, allowing for specification and estimation of models for realized volatility, elicited a considerable advancement in this field. However, a considerable gap still exists in the literature between models devised for volatility forecasting, which are commonly specified in discrete time, and volatility modeling in continuous time which is used, among other things, for option pricing. This is particularly annoying for leverage effect, whose interpretation is completely different in discrete time, where it is typically interpreted as a negative correlation between lagged negative returns and volatility, and in continuous time, where the negative correlation between price and volatility shocks is contemporaneous.

This paper contributes to this literature in two directions and aims at filling this apparent gap. In the first part of the paper, we propose a new reduced-form model in *discrete time*, the LHAR-CJ model, which is able to provide a remarkable forecasting performance for volatility over a time horizon which ranges from one day to one month, along with a positive and significant risk-return trade-off. Our specification is extremely simple to implement and it is based on the incorporation of three effects. The first is the well know volatility persistence, which is modeled with the HAR specification of Corsi (2009). However, we do not restrict to lagged volatilities (at daily, weekly and monthly frequency) as possible sources of future volatility, but we also add jumps (as in Andersen et al., 2007) and, as a novel contribution of this paper, negative returns over the past day, week and month, thus imposing a common heterogeneous structure to the explanatory variables.

The empirical findings in the first part of the paper are also relevant because of the important implications they bear on the set of continuous-time models consistent with the empirical features of financial data. In the second part of the paper, we then estimate continuous-time models via indirect inference using the proposed discrete-time specifications as auxiliary models, thus reproducing the very same features captured by the discrete-time model. In particular, we show that a single-factor model is unable to reproduce the type of integrated volatility persistence which is displayed by the data, even when allowing shocks to be driven by a fractional Brownian motion (Comte and Renault, 1998), that is by a genuine long-memory component. This induces us to investigate multi-factor specifications. We find that a Markovian two-factor model is able not only to replicate the persistence of integrated volatility, but also the persistence in the leverage effect by correlating more than one volatility factor with the price shocks. The analyzed multi-factor models do not need long-memory shocks to achieve their goal. While not ruling out the possible presence of long memory in volatility, this shows that the autocorrelation function of realized volatility is not necessarily a signature of genuine long memory in the data generating process, and corroborates the framework according to which market volatility is generated by a superposition of different frequencies, as suggested in Corsi (2009) and Muller et al. (1997).

In the literature, it is well known that volatility tends to increase more after a negative shock than after a positive shock of the same magnitude see e.g. Christie (1982); Campbell and Hentschel (1992); Glosten et al. (1989) and more recently Bollerslev et al. (2006), Bollerslev et al. (2009) and Martens et al. (2009). We extend the heterogeneous structure to the standard leverage effect by including lagged negative returns at different frequencies as explanatory variables to forecast volatility. This idea traces back to Corsi (2005) and can also be found in concurrent work of Scharth and Medeiros (2009) and Allen and Scharth (2009) as well as in Fernandes et al. (2009) to forecast implied volatility. However, with this paper we are able to provide a novel and clear evidence on the fact that the impact of negative returns on future volatility of S&P500 is also highly persistent and extends for a period of at least one month, thus displaying a long-range dependence similar to that of volatility itself.

With respect to the literature on jumps, we follow the separation of the quadratic variation in "continuous" volatility and jumps proposed in Andersen et al. (2007) for the explanatory variables and extended in Busch et al. (2011) to the dependent variable. However, contrary to the above-mentioned studies, we do not use bipower variation (Barndorff-Nielsen and Shephard, 2004) to measure the jumps contribution to quadratic variation, but we follow Corsi et al. (2010) using *threshold bipower variation*, a measure of continuous quadratic variation which is able to crucially soften the small-sample issues of bipower variation. This provides a superior forecasting performance, and allows to reveal that volatility does increase after a jump (both positive or negative) but that this shock is absorbed quickly in the volatility dynamics. Jumps are instead found to be almost unpredictable. When modeling in continuous time, the transient jump impact is captured by co-jumps between price and a single volatility factor.

The paper is organized as follows. Section 2 presents the main reduced-form model in discrete time. Section 3 contains the estimates and various robustness checks. In Section 4, we estimate continuous-time models via indirect inference. Section 5 concludes.

2 The discrete-time model

This section is devoted to the specification of the reduced-form model in discrete time. We first define the data generating process, the variables of interest and their estimators: We then specify the LHAR-CJ model.

2.1 Construction of the variables of interest

We assume that the data generating process X_t (the log-price) is a real-valued process that can be put, in a standard probability space, in the form of an Ito's semimartingale:

$$dX_t = \mu_t dt + \sigma_t dW_t + dJ_t \tag{2.1}$$

where W_t is a standard Brownian motion, μ_t is predictable, σ_t is cádlág; $dJ_t = c_t dN_t$ is a jump process where N_t is a non-explosive Poisson process whose intensity is an adapted stochastic process λ_t and c_t is the adapted random variable measuring the size of the jump at time t and satisfying, $\forall t \in [0,T],\ \mathcal{P}(\{c_t = 0\}) = 0$.

While in Section 4 we propose possible specifications of model (2.1), including the possibility that σ_t is driven by a fractional Brownian motion with constant volatility (Comte and Renault, 1998), here we concentrate on reduced-form models for quadratic variation, which is defined by:

$$[X]_t^{t+T} = \int_t^{t+T} \sigma_s^2 ds + \sum_{j=N_t}^{N_{t+T}} c_{\tau_j}^2. \tag{2.2}$$

where we denote by τ_j the times in which jumps occurr.

These quantities are not directly observable and they have to be replaced with consistent realized estimators, which we denote by $\widehat{\mathsf{V}}_t$ (for $[X]_t^{t+T}$), $\widehat{\mathsf{C}}_t$ (for $\int_t^{t+T} \sigma_s^2 ds$), and $\widehat{\mathsf{J}}_t$ (for $\sum_{j=N_t}^{N_{t+T}} c_{\tau_j}^2$). We use $T = 1$ day and we denote the daily (close-to-close) return by r_t. We remark that realized volatility models need both the specification of the dynamics of quadratic variation and the choice of small-sample estimators. For example, two models can share the same dynamics (e.g. the HAR model for total quadratic variation) but be different just because quadratic variation estimators (e.g. realized volatility versus two-scale estimator) are different.

In order to mitigate the impact of microstructure effects on our estimates, $\widehat{\mathsf{V}}_t$ is the two-scale estimator (TSRV_t) proposed by Zhang et al. (2005), which is consistent also in the presence of jumps. Details on the construction of the estimator are provided in a related Web Appendix. Aït-Sahalia and Mancini (2008) show that using the two-scale estimator instead of standard realized volatility measures yields significant gains in volatility forecasting.

To define $\widehat{\mathsf{C}}_t$ and $\widehat{\mathsf{J}}_t$ we use the following approach. We first pre-test the data for jumps using the C-Tz statistics proposed in Corsi et al. (2010) and formally defined in the Web Appendix. The C-Tz test, which is distributed as a standard Normal in the absence of jumps, is computed daily. When the null is not rejected (namely, when $\mathsf{C\text{-}Tz} < 3.0902$, corresponding to the 99.9% significance level), we set $\widehat{\mathsf{C}}_t = \widehat{\mathsf{V}}_t = \mathsf{TSRV}_t$ and $\widehat{\mathsf{J}}_t = 0$. When instead the test rejects the null, we set $\widehat{\mathsf{C}}_t = \mathsf{TBPV}_t$ and $\widehat{\mathsf{J}}_t = \max(\mathsf{TSRV}_t - \mathsf{TBPV}_t, 0)$, where TBPV_t is the threshold bipower variation estimator introduced in Corsi et al. (2010):

$$\mathsf{TBPV}_t = \frac{\pi}{2}\frac{M}{M-2}\sum_{j=0}^{M-2} |\Delta_{t,j}X| \cdot |\Delta_{t,j+1}X| I_{\{|\Delta_{t,j}X|^2 \le \vartheta_{j-1}\}} I_{\{|\Delta_{t,j+1}X|^2 \le \vartheta_j\}} \tag{2.3}$$

where $\Delta_{t,j}X$ is the j-th intraday return of day t (we use 5-minutes returns here), with $j = 1, \ldots, M$; $I_{\{\cdot\}}$ is the indicator function and ϑ_j is a threshold function (see the Web Appendix for its precise definition) which is designed to remove jumps from the returns time series (Mancini, 2009). We have $\widehat{\mathsf{V}}_t = \widehat{\mathsf{C}}_t + \widehat{\mathsf{J}}_t$ provided $\mathsf{TSRV}_t > \mathsf{TBPV}_t$ in days with jumps, which is always the case empirically. Equation (2.3) shows that TBPV_t is very similar to the bipower variation (BPV_t) of Barndorff-Nielsen and Shephard (2004), with the difference being the two indicator functions which remove returns larger than the threshold ϑ_j. While this difference is irrelevant asymptotically, it has been shown by Corsi et al. (2010) to be crucial in small samples (for example, we often have $\mathsf{TSRV}_t < \mathsf{BPV}_t$ in days with jumps because of the bias of BPV_t).

Figure 1 shows, for the S&P 500 series studied in this paper, the lagged correlation function between the two-scale estimator TSRV_t and TSRV_{t-h} itself, negative returns, positive returns and $\widehat{\mathsf{J}}_{t-h}$. The autocorrelation of TSRV_t decays very slowly, as it is well known. The lagged correlation between TSRV_t and negative returns shows the leverage effect: volatility is correlated with lagged negative returns. Figure 1 also shows that the impact of negative returns on future volatility is slowly decaying as well. Also jumps have a positive and large impact (as large as the negative returns when $h = 1$), which however decays more rapidly. Finally, positive returns have a very small and negligible impact on future volatility.

The slowly decaying impact of negative returns might well be a by-product of the slowly decaying auto-correlation function of volatility. However, since the same phenomenon is not observed with the jump component, it can be also suggestive of the fact that leverage effect might be very

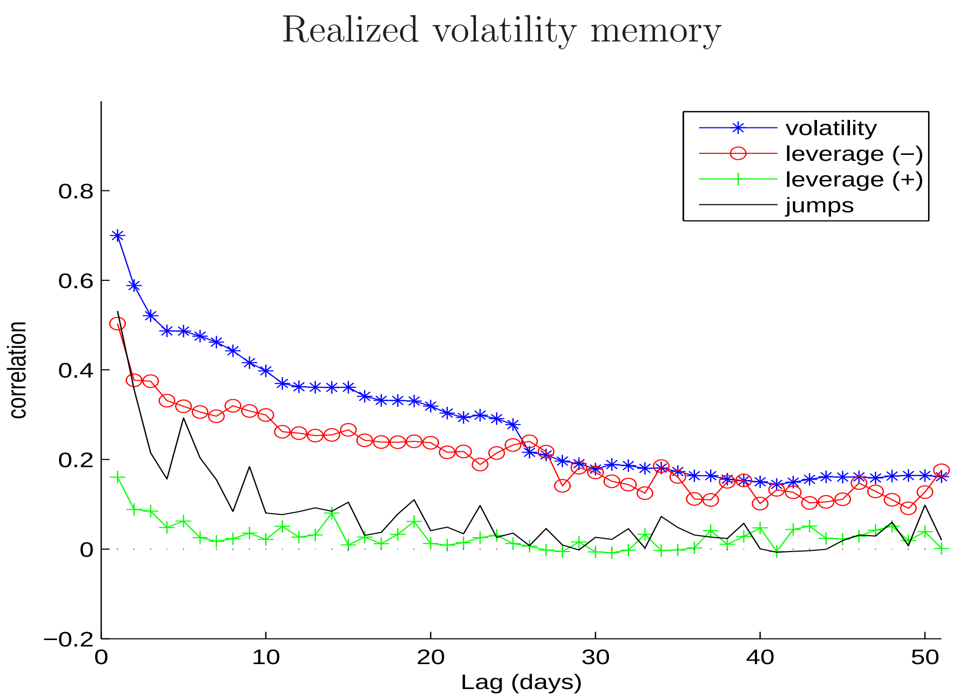

Figure 1: Lagged correlation function between past values Y_{t-h} and current, daily, integrated variance estimates TSRV_t as a function of h, with Y_{t-h} being TSRV_{t-h} itself, negative returns, positive returns and jumps quadratic variation $\widehat{\mathsf{J}}_{t-h}$ for the S&P500 futures from 28 April 1982 to 5 February 2009 (6,669 observations).

persistent, a possibility which has been seldom investigated so far.

Persistence in the leverage effect can be induced, in continuous time, by making the leverage effect an explicit function of volatility, as in Bandi and Renò (2011b). Our reduced-form model, which is in discrete time, explores an alternative possibility. We follow Corsi (2009) in modeling the slowly decaying auto-correlation function by means of a heterogeneous structure induced by a volatility cascade, and we extend this structure to negative returns and jumps.

2.2 The LHAR-CJ model

Combining heterogeneity in realized volatility, leverage, and jumps we construct the *Leverage Heterogeneous Auto-Regressive with Continuous volatility and Jumps* (LHAR-CJ) model. As it is common in practice, we use daily, weekly and monthly frequencies. Then, using variable specified in logs, we introduce averaged variables, which are defined over an integer number h of days as (jumps are aggregated instead of averaged):

$$\log \widehat{\mathsf{V}}_t^{(h)} = \frac{1}{h}\sum_{j=1}^{h} \log \widehat{\mathsf{V}}_{t-j+1}, \quad \log \widehat{\mathsf{C}}_t^{(h)} = \frac{1}{h}\sum_{j=1}^{h} \log \widehat{\mathsf{C}}_{t-j+1}, \quad r_t^{(h)} = \frac{1}{h}\sum_{j=1}^{h} r_{t-j+1}, \quad \widehat{\mathsf{J}}_t^{(h)} = \sum_{j=1}^{h} \widehat{\mathsf{J}}_{t-j+1}\,.$$

To model the leverage effect at different frequencies, we define $r_t^{(h)-} = \min(r_t^{(h)}, 0)$. The proposed model reads:

$$\begin{aligned} \log \widehat{\mathsf{V}}_{t+h}^{(h)} = c \quad &+ \quad \beta^{(d)} \log \widehat{\mathsf{C}}_t + \beta^{(w)} \log \widehat{\mathsf{C}}_t^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_t^{(22)} \\ &+ \quad \alpha^{(d)} \log(1+\widehat{\mathsf{J}}_t) + \alpha^{(w)} \log(1+\widehat{\mathsf{J}}_t^{(5)}) + \alpha^{(m)} \log(1+\widehat{\mathsf{J}}_t^{(22)}) \\ &+ \quad \gamma^{(d)} r_t^{-} + \gamma^{(w)} r_t^{(5)-} + \gamma^{(m)} r_t^{(22)-} + \varepsilon_{t+h}^{(h)}, \end{aligned} \tag{2.4}$$

with real parameters $\{c, \beta^{(d,w,m)}, \alpha^{(d,w,m)}, \gamma^{(d,w,m)}\}$ and where $\varepsilon_t^{(h)}$ is IID noise. Model (2.4) nests other models which have been successfully used for realized volatility. When $\alpha^{(d,w,m)} = \gamma^{(d,w,m)} = 0$ and $\widehat{\mathsf{C}}_t = \widehat{\mathsf{V}}_t$, the model becomes the HAR model of Corsi (2009). When $\gamma^{(d,w,m)} = 0$, we get the HAR-CJ model proposed by Andersen et al. (2007) which separately include continuous and discontinuous component as explanatory variables. When $\alpha^{(d,w,m)} = 0$ and $\widehat{\mathsf{C}}_t = \widehat{\mathsf{V}}_t$ the model is referred to as the LHAR model. The model can also be specified directly for $\widehat{\mathsf{V}}_t$ and for $\sqrt{\widehat{\mathsf{V}}_t}$, as in Andersen et al. (2007) and Corsi et al. (2010).

We estimate model (2.4) and its variants, with h ranging from 1 to 22 to make multiperiod predictions, by OLS with Newey-West covariance correction for serial correlation.

3 Empirical evidences

The purpose of this section is to empirically analyze the performance of the LHAR-CJ model (2.4) and related ones, both in-sample and out-of-sample. Our data set covers a long time

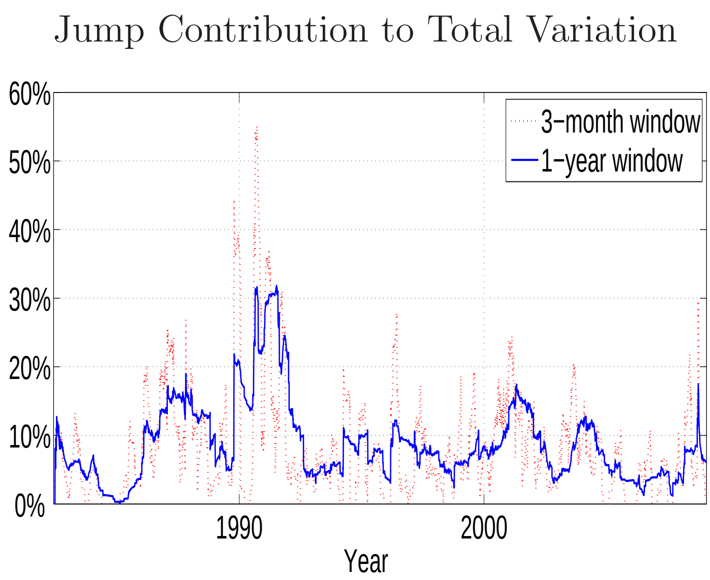

Figure 2: Percentage contribution of daily jump to total quadratic variation measured over a moving window of 3-month (dotted line) and 1-year (solid line) for the S&P500 futures from 28 April 1982 to 5 February 2009 ($6,669$ observations) excluding the October 1987 crash. The C-Tz statistics is computed with a confidence interval $\alpha = 99.9\%$.

span of almost 28 years of high frequency data for the S&P 500 futures from 28 April 1982 to 5 February 2009. We leave out from the sample the week of the 1987 October crash (when included, results are qualitatively very similar but less clear-cut) and days with less than 500 trades. We are left with 6,669 days. All the quantities of interest are computed on an annualized base. Figure 2 reports the relative contribution of the quadratic variation of jumps with respect to total quadratic variation, computed on a 3-month and 1-year moving window. In line with the results in Andersen et al. (2007) and Huang and Tauchen (2005) we find a jump contribution varying between 2% and 30% of total variation (with an overall sample mean of about 6%).

3.1 In-sample analysis

The results of the estimation of the LHAR-CJ on the S&P500 sample with $h = 1, 5, 10, 22$ are reported in Table 1, together with their statistical significance evaluated with the Newey-

S&P500 LHAR in-sample regression, period 1982–2009

Variable	One day	One week	Two weeks	One month
c	0.442*	0.549*	0.662*	0.858*
	(10.699)	(9.258)	(8.525)	(7.756)
$\widehat{\mathsf{C}}$	0.307*	0.201*	0.154*	0.116*
	(16.983)	(14.158)	(12.984)	(10.590)
$\widehat{\mathsf{C}}^{(5)}$	0.369*	0.359*	0.332*	0.286*
	(13.908)	(11.251)	(9.166)	(6.784)
$\widehat{\mathsf{C}}^{(22)}$	0.222*	0.319*	0.370*	0.415*
	(10.958)	(10.913)	(10.198)	(9.344)
$\widehat{\mathsf{J}}$	0.043*	0.020*	0.017*	0.012*
	(7.057)	(4.453)	(4.485)	(3.804)
$\widehat{\mathsf{J}}^{(5)}$	0.011*	0.013*	0.011*	0.010
	(3.373)	(3.112)	(2.256)	(1.913)
$\widehat{\mathsf{J}}^{(22)}$	0.005*	0.008*	0.010*	0.014*
	(2.199)	(2.106)	(2.205)	(2.336)
r^{-}	-0.007*	-0.005*	-0.004*	-0.003*
	(-9.669)	(-10.435)	(-8.298)	(-5.518)
$r^{(5)-}$	-0.008*	-0.006*	-0.008*	-0.007*
	(-4.412)	(-3.059)	(-4.012)	(-3.472)
$r^{(22)-}$	-0.009*	-0.012*	-0.009	-0.004
	(-2.845)	(-2.314)	(-1.481)	(-0.467)
$\overline{R}^2$	0.7664	0.8137	0.8030	0.7629
HRMSE	0.2168	0.1692	0.1699	0.1796

Table 1: OLS estimates of LHAR-CJ regressions, model (2.4), for the S&P500 futures from 28 April 1982 to 5 February 2009 ($6,669$ observations). The LHAR-CJ model is estimated with $h = 1$ (one day), $h = 5$ (one week), $h = 10$ (two weeks) and $h = 22$ (one month). The significant jumps are computed using a critical value of $\alpha = 99.9\%$. Reported in parenthesis are t-statistics based on Newey-West correction with $L = 2 + 2h$ number of lags and Bartlett kernel. A star denotes 95% significance.

West robust t-statistic. The forecasts of the different models are evaluated on the basis of the adjusted R^2 of the regressions, and the heteroskedasticity-adjusted root mean square error (HRMSE) proposed by Bollerslev and Ghysels (1996).

As usual, all the coefficients of the three continuous volatility components are positive and, in general, highly significant. The impact of daily and weekly volatility decreases with the forecasting horizon of future volatility, while the impact of monthly volatility increases. The coefficient which measures the impact of monthly volatility on future daily volatility is approximately double than that of daily volatility on future monthly volatility. This finding is consistent with Corsi (2009).

Estimation of model (2.4) also reveals the strong significance (with an economically sound negative sign) of the negative returns at all the daily, weekly and monthly aggregation frequency, which unveils a heterogeneous structure in the leverage effect as well. Not only daily negative returns affect the next day volatility (the well-know leverage effect) but, in addition, also the negative returns of the past week and past month have an impact on forthcoming volatility. This novel finding suggests that the market might aggregate daily, weekly and monthly memory, observing and reacting to price declines happened in the past week and month, revealing a persistent leverage effect.

A similar heterogeneous structure is present in the impact of jumps on future volatility. However, while the daily and weekly jumps are highly significant and positive, their impact decreases with the forecasting horizon at a fast rate. The monthly jump component is also slightly significant over all forecasting horizons, with its impact increasing with the horizon.

Figure 3 shows the Mincer-Zarnowitz R^2 for different models at various horizons, which obtaines its maximum at one week. Moreover, Figure 3 shows unambiguously that the inclusion of both the heterogeneous jumps and the heterogeneous leverage effects considerably improves the forecasting performance of the S&P 500 volatility at any forecasting horizon. In particular, the inclusion of heterogeneous leverage effect provides the most relevant overall benefit in the in-sample performance. We confirm this result out-of-sample in Section 3.6.

3.2 Forecasting jumps and continuous volatility

Following Busch et al. (2011), we can use the continuous and jumps component of total quadratic variation as dependent variables as well, and investigate the possibility of forecasting them separately. We then specify the LHAR-C-CJ model for forecasting the continuous quadratic

In-sample Mincer-Zarnowitz R^2

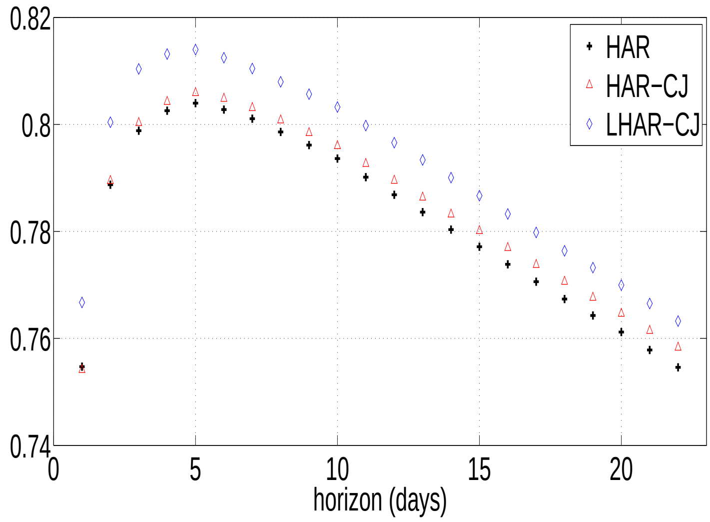

Figure 3: R^2 of Mincer-Zarnowitz regressions for static in sample one-step ahead forecasts for horizons ranging from 1 day to 1 month of the S&P500 futures from 28 April 1982 to 5 February 2009 (6,669 observations). The forecasting models are the standard HAR with only heterogeneous volatility, the HAR-CJ with heterogeneous jumps and the LHAR-CJ model.

variation as:

$$\begin{aligned}\log \widehat{\mathsf{C}}_{t+h}^{(h)} = c &+ \beta^{(d)} \log \widehat{\mathsf{C}}_t + \beta^{(w)} \log \widehat{\mathsf{C}}_t^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_t^{(22)} \\ &+ \alpha^{(d)} \log(1+\widehat{\mathsf{J}}_t) + \alpha^{(w)} \log(1+\widehat{\mathsf{J}}_t^{(5)}) + \alpha^{(m)} \log(1+\widehat{\mathsf{J}}_t^{(22)}) \\ &+ \gamma^{(d)} r_t^- + \gamma^{(w)} r_t^{(5)-} + \gamma^{(m)} r_t^{(22)-} + \varepsilon_{t+h}^{(h)},\end{aligned} \tag{3.1}$$

and the LHAR-J-CJ model for forecasting jumps as:

$$\begin{aligned}\log(1+\widehat{\mathsf{J}}_{t+h}^{(h)}) = c &+ \beta^{(d)} \log \widehat{\mathsf{C}}_t + \beta^{(w)} \log \widehat{\mathsf{C}}_t^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_t^{(22)} \\ &+ \alpha^{(d)} \log(1+\widehat{\mathsf{J}}_t) + \alpha^{(w)} \log(1+\widehat{\mathsf{J}}_t^{(5)}) + \alpha^{(m)} \log(1+\widehat{\mathsf{J}}_t^{(22)}) \\ &+ \gamma^{(d)} r_t^- + \gamma^{(w)} r_t^{(5)-} + \gamma^{(m)} r_t^{(22)-} + \varepsilon_{t+h}^{(h)},\end{aligned} \tag{3.2}$$

Corresponding models with $\alpha^{(d,w,m)} = 0$ and $\widehat{\mathsf{C}}_t = \widehat{\mathsf{V}}_t$ on the right-hand side are named LHAR-C and LHAR-J respectively. Estimation results for the daily horizon ($h = 1$) are presented in Table 2. We can see that, as already recognized in the literature, the jump component is essentially unpredictable, with an adjusted R^2 of just 1.52%. We find a strong significant impact on future jumps only for the monthly jump component, which is a clear indication of jump clustering. Also daily and weekly volatilities are significant, but with opposite signs. The impact of monthly jumps on $\widehat{\mathsf{C}}_t$ is instead not significant, confirming that the impact of jumps on volatility is quite transitory in nature.

While the inability of forecasting jumps has been signaled also by Busch et al. (2011), they find, contrary to our analysis, that the impact of daily jumps on the future daily continuous quadratic variation is significantly negative, a result which would imply, on average, a volatility decrease after a jump. Corsi et al. (2010) show that this result is induced by the small-sample bias of bipower variation measures. Building on their work, we use threshold bipower variation and uncover the positive (and transitory) impact of jumps on future volatility also in the presence of a persistent leverage effect.

3.3 Risk-return trade-off

Our volatility forecasts can also be evaluated in term of the implied risk-return trade-off, since economic theory posits that there should be a positive relation between returns and perceived risk. The literature on the risk-return trade-off is very large. Recent research on this topic includes Ghysels et al. (2005); Christensen and Nielsen (2007); Bandi and Perron (2008); Bollerslev et al. (2008). Here we use as a measure of risk the daily volatility forecast of i) the standard HAR model, ii) the HAR-CJ model and iii) the LHAR-CJ model. All models are specified in the logarithmic form. We regress the return on the variance forecasts $\widetilde{\widehat{\mathsf{V}}}_t$, that is on the exponential of the logarithmic forecasts $\log \widetilde{\widehat{\mathsf{V}}}_t$. Estimation results, obtained via OLS, are shown in Table 3.

The results are in agreement with those in Bali and Peng (2006), who analyze the same data

LHAR-CJ, LHAR-C-CJ and LHAR-J-CJ regression

	Dependent Variables		
Variable	$\widehat{\mathsf{V}}$	$\widehat{\mathsf{C}}$	$\widehat{\mathsf{J}}$
c	0.442*	0.440*	-0.287*
	(10.699)	(10.995)	(-2.770)
$\widehat{\mathsf{C}}$	0.307*	0.324*	-0.097*
	(16.983)	(17.803)	(-2.388)
$\widehat{\mathsf{C}}^{(5)}$	0.369*	0.354*	0.121*
	(13.908)	(13.406)	(2.089)
$\widehat{\mathsf{C}}^{(22)}$	0.222*	0.214*	0.065
	(10.958)	(10.822)	(1.261)
$\widehat{\mathsf{J}}$	0.043*	0.037*	0.040
	(7.057)	(5.941)	(1.829)
$\widehat{\mathsf{J}}^{(5)}$	0.011*	0.010*	0.000
	(3.373)	(3.051)	(0.029)
$\widehat{\mathsf{J}}^{(22)}$	0.005*	0.001	0.026*
	(2.199)	(0.461)	(4.313)
r^-	-0.007*	-0.007*	-0.002
	(-9.669)	(-10.173)	(-1.128)
$r^{(5)-}$	-0.008*	-0.007*	-0.006
	(-4.412)	(-4.151)	(-1.241)
$r^{(22)-}$	-0.009*	-0.010*	0.014
	(-2.845)	(-2.869)	(1.341)
$\overline{R}^2$	0.7664	0.7594	0.0152

Table 2: OLS estimate for the LHAR-CJ model using as dependent variable $\log \widehat{\mathsf{V}}_t$, $\log \widehat{\mathsf{C}}_t$, $\log(1 + \widehat{\mathsf{J}}_t)$, daily forecasting horizon, for the S&P500 futures from 28 April 1982 to 5 February 2009 (6,669 observations). The significant jumps are computed using a critical value of $\alpha = 99.9\%$. Reported in parenthesis are t-statistics based on Newey-West correction with $L = 2 + 2h$ number of lags and Bartlett kernel. A star denotes 95% significance.

until 2002. With all models, we find a significant impact of volatility forecasts on returns which is compatible with economic theory, even if we have a very low R^2, as it is common in this kind of applications. The inclusion of jumps is not benefitial to return forecasting. Instead, the inclusion of the leverage component increases the slope coefficient and almost doubles the significance of the effect. Similar results are obtained regressing r_t on $\sqrt{\widetilde{\widehat{\mathsf{V}}}_t}$, or replacing $\widetilde{\widehat{\mathsf{V}}}_t$ with the volatility forecasts of $\widetilde{\widehat{\mathsf{C}}}_t$ obtained with the LHAR-C-CJ model discussed in Section 3.2.

Risk-return regression:

$$r_t = c + \beta\widetilde{\widehat{\mathsf{V}}}_t + \varepsilon_t$$

Model	slope $\hat{\beta}$	t-stat	$\bar{R}^2$
HAR	0.0026018	3.4257	0.16%
HAR-CJ	0.0025964	3.4113	0.16%
LHAR-CJ	0.0033497	6.7266	0.66%

Table 3: OLS estimates of the regression of daily returns on daily variance forecasts $\widetilde{\widehat{\mathsf{V}}}_t$ obtained with different models

3.4 Is leverage effect induced by jumps?

An open research question is whether, and to which extent, the leverage effect is induced by jumps, see e.g. Bandi and Renò (2011a). In our setting, we investigate this issue by separating the daily jump contribution to quadratic variation in a positive and negative part. To this purpose, we define:

$$\widehat{\mathsf{J}}_t^+ = \widehat{\mathsf{J}}_t \cdot I_{\{r_t > 0\}}$$

$$\widehat{\mathsf{J}}_t^- = \widehat{\mathsf{J}}_t \cdot I_{\{r_t < 0\}}$$

and we insert $\widehat{\mathsf{J}}_t^+$ and $\widehat{\mathsf{J}}_t^-$ in the LHAR model in place of $\widehat{\mathsf{J}}_t$, denoting by LHAR-CJ$^+$ the newly obtained model. We also estimate the HAR-CJ$^+$ model, which is the same without leverage terms. Results are reported in Table 4. Given the evidence provided by Todorov and Tauchen (2011) and Bandi and Renò (2011a), with different statistical methods, of a strong negative correlation between price and volatility jumps, we expect the coefficient on $\widehat{\mathsf{J}}_t^-$ to be larger than that on $\widehat{\mathsf{J}}_t^+$.

When we estimate the HAR-CJ$^+$ model, this is exactly what we find: the coefficient on negative jumps is almost double than that of positive jumps, and this is true for all the considered forecasting horizons, ranging from one day to one month. However, when we estimate the full

	HAR-CJ$^+$ regression					LHAR-CJ$^+$ regression			
	1 day	1 week	2 weeks	1 month		1 day	1 week	2 weeks	1 month
c	0.232*	0.377*	0.505*	0.747*	c	0.442*	0.549*	0.661*	0.858*
	(5.774)	(6.217)	(6.418)	(6.736)		(10.724)	(9.277)	(8.531)	(7.778)
$\widehat{C}$	0.398*	0.265*	0.214*	0.165*	$\widehat{C}$	0.307*	0.201*	0.154*	0.116*
	(21.521)	(18.225)	(16.084)	(12.442)		(16.972)	(14.185)	(13.007)	(10.608)
$\widehat{C}^{(5)}$	0.366*	0.368*	0.346*	0.291*	$\widehat{C}^{(5)}$	0.369*	0.359*	0.332*	0.286*
	(13.889)	(11.697)	(9.750)	(7.327)		(13.885)	(11.237)	(9.144)	(6.777)
$\widehat{C}^{(22)}$	0.190*	0.291*	0.338*	0.390*	$\widehat{C}^{(22)}$	0.222*	0.319*	0.370*	0.415*
	(9.470)	(9.743)	(9.059)	(8.875)		(10.914)	(10.905)	(10.183)	(9.336)
$\widehat{J}^{+}$	0.044*	0.018*	0.016*	0.013*	$\widehat{J}^{+}$	0.044*	0.018*	0.015*	0.012*
	(6.099)	(3.264)	(3.000)	(2.538)		(6.176)	(3.182)	(2.819)	(2.395)
$\widehat{J}^{-}$	0.074*	0.040*	0.039*	0.027*	$\widehat{J}^{-}$	0.043*	0.019*	0.020*	0.011*
	(6.909)	(6.833)	(6.658)	(5.351)		(4.598)	(3.387)	(3.633)	(2.285)
$\widehat{J}^{(5)}$	0.009*	0.012*	0.010*	0.010	$\widehat{J}^{(5)}$	0.011*	0.013*	0.011*	0.010
	(2.645)	(2.724)	(2.028)	(1.799)		(3.372)	(3.110)	(2.254)	(1.914)
$\widehat{J}^{(22)}$	0.005	0.007	0.009*	0.014*	$\widehat{J}^{(22)}$	0.005*	0.008*	0.010*	0.014*
	(1.845)	(1.875)	(2.026)	(2.242)		(2.200)	(2.104)	(2.203)	(2.337)
-					r^{-}	-0.007*	-0.005*	-0.004*	-0.003*
						(-9.772)	(-10.057)	(-7.804)	(-5.341)
-					$r^{(5)-}$	-0.008*	-0.006*	-0.008*	-0.007*
						(-4.409)	(-3.068)	(-4.020)	(-5.341)
-					$r^{(22)-}$	-0.009*	-0.012*	-0.008	-0.004
						(-2.844)	(-2.315)	(-1.484)	(-0.467)
$\overline{R}^2$	0.7543	0.8060	0.7960	0.7582	$\overline{R}^2$	0.7664	0.8137	0.8030	0.7629
HRMSE	0.2201	0.1721	0.1722	0.1812	HRMSE	0.2168	0.1692	0.1698	0.1796

Table 4: OLS estimate for the LHAR-CJ$^+$ and HAR-CJ$^+$ model in which we separate daily jumps in positive and negative, for the S&P500 futures from 28 April 1982 to 5 February 2009 (6,669 observations). The models are estimated with $h = 1$ (one day), $h = 5$ (one week), $h = 10$ (two weeks) and $h = 22$ (one month). The significant jumps are computed using a critical value of $\alpha = 99.9\%$. Reported in parenthesis are t-statistics based on Newey-West correction with $L = 2 + 2h$ number of lags and Bartlett kernel. A star denotes 95% significance.

LHAR-CJ$^+$ model, which includes all the leverage terms (which are also affected by the jump component), the impact of positive and negative jumps is estimated to be roughly the same, again at all the considered horizons. Our interpretation of this result is that the number of co-jumps is likely too small to allow for the joint detection of the continuous leverage effect and the covariance part genuinely due to jumps.

3.5 Robustness to other volatility measures

In the literature many volatility measures have been proposed as explanatory variables for the volatility dynamics. Forsberg and Ghysels (2007) proposed the use of realized absolute variation (RAV) which shows a more persistent dynamics than realized volatility being more robust to microstructure noise and jumps. The range, i.e. the difference between the highest and the lowest price within a day, has also been found to be significant by many authors, see e.g. Brandt and Jones (2006) and Engle and Gallo (2006). Recently, Barndorff-Nielsen et al. (2010) proposed the realized semivariance as the sum of square negative returns to capture the impact on volatility of downward price pressures. Visser (2008) combines RAV and semivariance by taking the sum of negative absolute squared returns.

In the spirit of Forsberg and Ghysels (2007), we compare the relative explanatory power of different volatility measures by estimating a set of models (for space concerns we limit ourselves to the one day horizon) obtained by adding explanatory variables to model (2.4). Estimation results are reported in Table 5. In line with previous literature, we find that the realized absolute variation (RAV) computed at 5-minute frequency and the range have a significant impact on future volatility. However, they seem to be mainly substitutes for continuous volatility and jumps, which is not totally surprising since they are estimators (though noisy) of total quadratic variation. Indeed, for instance, when the range replaces the jumps (LHAR-Range model, not reported), the coefficients of daily continuous volatility almost halves. The adjusted R^2 of the two competing regressions (LHAR-Range and LHAR-CJ) is practically the same. When the range is inserted together with the jumps (LHAR-CJ-Range), both the coefficients of daily volatility and jumps decrease, although they remain highly significant. The significance of the heterogeneous leverage effect is instead unaffected by the presence of RAV and range. We thus conclude that the RAV and the range, while partially proxying for both volatility and jumps, are also able to capture some other (small) effect which is not captured by the other variables in the model. However, the adjusted R^2 of the encompassing regression increases only marginally. The realized semivariance (semiRV) of Barndorff-Nielsen et al. (2010) and the downward absolute power variation of Visser (2008) (semiRAV) have a weaker impact. Realized semivariance and semi-power-variation are significant in explaining future volatility, and, again, they are correlated with both the daily two-scale estimator and the jumps (typically depleting the significance of the corresponding coefficients without totally removing it), while unrelated with the leverage. However, their contribution to the model performance is not significant (as measured by the Diebold-Mariano test). Moreover, when they are included in the all-encompassing model they both remain insignificant.

Summarizing, the results of this section show that when the other volatility measures proposed in the literature are inserted in the baseline LHAR-CJ model they either do not contribute significantly or only marginally contribute to the performance of the model. Moreover, they mainly act as substitutes of continuous volatility and jumps. Hence, we conclude the LHAR-CJ model seems to capture the main determinants of volatility dynamics.

3.6 Out-of-sample analysis

In this section, we evaluate the performance of the LHAR-CJ model on the basis of a genuine out-of-sample analysis. For the out-of-sample forecast of $\widehat{\mathsf{V}}_t$ on the $[t, t+h]$ interval we keep the same forecasting horizons ranging from one day to one month and re-estimate the model at each day t on an increasing window of all the observation available up to time $t-1$. The out-of-sample forecasting performance for $\log \widehat{\mathsf{V}}_t$ in terms of Mincer-Zarnowitz R^2 is reported in Figure 4, together with the Diebold-Mariano test computed for the HRMSE loss function at all the considered horizons.

The superiority of the LHAR-CJ model at all horizons, with respect to the HAR and the HAR-CJ model, is statistically significant, validating the importance of including both the heterogeneous leverage effects and jumps in the forecasting model. The out-of-sample exercise confirms that the maximum R^2 is obtained at a forecasting horizon of one week.

S&P500 in-sample estimates, period 1982–2009

Variable	LHAR-CJ	LHAR-CJ-RAV	LHAR-CJ-Range	LHAR-CJ-SemiRV	LHAR-CJ-SemiRAV	LHAR-CJ-All
const	0.442*	0.593*	0.444*	0.470*	0.554*	0.616*
	(10.699)	(12.419)	(10.860)	(11.029)	(9.869)	(8.914)
$\widehat{\mathsf{C}}$	0.307*	0.116*	0.207*	0.249*	0.252*	0.085*
	(16.983)	(3.355)	(10.035)	(9.588)	(9.489)	(2.449)
$\widehat{\mathsf{C}}^{(5)}$	0.369*	0.374*	0.384*	0.372*	0.372*	0.389*
	(13.908)	(14.125)	(14.568)	(14.074)	(14.032)	(14.693)
$\widehat{\mathsf{C}}^{(22)}$	0.222*	0.221*	0.223*	0.223*	0.222*	0.223*
	(10.958)	(10.943)	(11.071)	(11.003)	(10.963)	(11.035)
$\widehat{\mathsf{J}}$	0.043*	0.018*	0.024*	0.033*	0.037*	0.010
	(7.057)	(2.519)	(3.893)	(4.850)	(5.620)	(1.331)
$\widehat{\mathsf{J}}^{(5)}$	0.011*	0.012*	0.012*	0.011*	0.011*	0.012*
	(3.373)	(3.717)	(3.789)	(3.409)	(3.413)	(3.959)
$\widehat{\mathsf{J}}^{(22)}$	0.005*	0.006*	0.005*	0.006*	0.006*	0.006*
	(2.199)	(2.334)	(2.207)	(2.266)	(2.277)	(2.329)
r^{-}	-0.007*	-0.007*	-0.006*	-0.006*	-0.006*	-0.005*
	(-9.669)	(-10.156)	(-8.193)	(-8.197)	(-7.792)	(-5.630)
$r^{(5)-}$	-0.008*	-0.007*	-0.008*	-0.008*	-0.008*	-0.008*
	(-4.412)	(-4.147)	(-4.847)	(-4.368)	(-4.278)	(-4.582)
$r^{(22)-}$	-0.009*	-0.009*	-0.009*	-0.010*	-0.010*	-0.010*
	(-2.845)	(-2.687)	(-2.868)	(-2.980)	(-2.951)	(-2.853)
RAV		0.185*				0.077
		(6.533)				(1.867)
Range			0.088*			0.086*
			(9.364)			(7.911)
SemiRV				0.058*		-0.011
				(3.305)		(-0.330)
SemiRAV					0.054*	0.056
					(3.141)	(1.455)
$\overline{R}^2$	0.7664	0.7681	0.7696	0.7668	0.7668	0.7704
HRMSE	0.2168	0.2158*	0.2148*	0.2165	0.2165	0.2142*
DM		(2.7060)	(4.6323)	(1.5020)	(1.8539)	(4.6778)

Table 5: Estimated parameters, adjusted R^2 and Heteroskedasticity adjusted RMSE (HRMSE), of alternative specifications of the baseline LHAR-CJ model for the S&P500 futures from 28 April 1982 to 5 February 2009 (6,669 observations); t-statistic and Diebold-Mariano (DM) test for HRMSE are in parenthesis. A star denotes 95% significance.

The superiority of the HAR-CJ model vs the HAR model is instead milder, but the reason is that the improvements appear only in days which follow a jump (368 out of $6,669$), and thus on a small subsample. However, it is important to note that the inclusion of the jump component helps also in forecasting longer horizon volatility, see the results in the web appendix and Andersen et al. (2007).

4 Continuous-time models

The main motivation of this section is to provide a continuous-time model which delivers the stylized facts documented in the previous sections and captured by the newly proposed discrete-time model. Such a continuous-time model would then, at the same time, not only provide a more accurate statistical representation of the data, but also bridge the gap between continuous and discrete time modelling. Since the inception of the GARCH literature, indeed, volatility forecasting is mostly set up in discrete time, and the LHAR-CJ model is no exception. However, models used in practice, e.g. for option pricing, are often specified in continuous time. In the literature, the link between GARCH-like models and continuous time models is well established, see e.g. Nelson (1990); Duan (1997); Corradi (2000). However, the link between continuous-time models and HAR-like models is unclear.

We achieve this goal by estimating continuous-time models via *indirect inference* (Gourieroux et al. 1993; see Bollerslev et al. 2006 for an application similar to ours) using the (L)HAR(-CJ) specification as auxiliary model. The idea of the indirect inference approach is to estimate the auxiliary model both on actual data and on data simulated from the structural model, and then to minimize the distance (labelled by χ^2), as a function of the structural model parameters, between the estimated coefficients weighted with the inverse variance-covariance matrix of the estimates. Details are provided in the Web Appendix.

It is usually suggested that the long-range autocorrelation function of realized volatility is

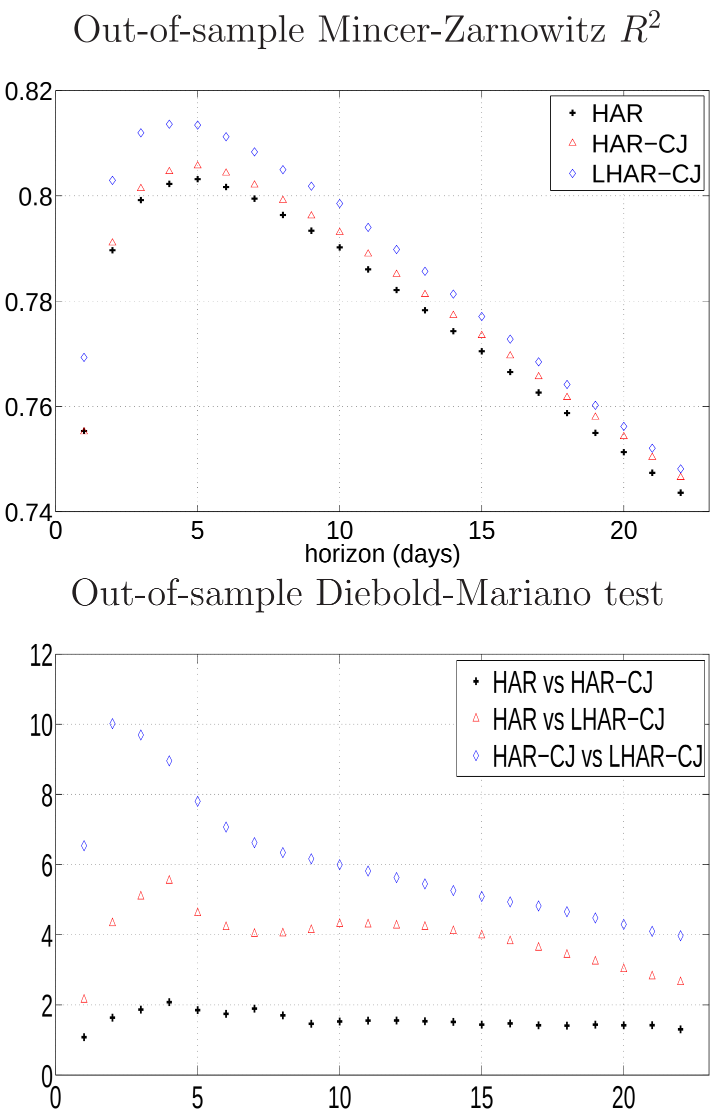

Figure 4: Top: R^2 of Mincer-Zarnowitz regressions for out-of-sample forecasts. Bottom: Diebold-Mariano test for the out-of-sample HRMSE. Horizons range from 1 day to 1 month of the S&P500 from 28 April 1982 to 5 February 2009 (6,669 observations, the first 2,500 observations are used to initialize the models). The forecasting models are the standard HAR with only heterogeneous volatility, the HAR-CJ with heterogeneous jumps and the LHAR-CJ model. The comparison of HAR-CJ and LHAR-CJ is made employing the Clark and West (2007) adjustment to the Diebold-Mariano test for nested models. In both cases, the test is asymptotically standard Normal under the null.

generated by a long-memory model. For this reason, we start by estimating the Comte and Renault (1998) continuous-time model:

$$\begin{aligned} dX_t &= \sigma_t dW_t, \\ d\log\sigma_t &= k(\omega - \log\sigma_t)dt + \eta dW_t^{(d)}, \end{aligned} \tag{4.1}$$

where W_t is a standard Brownian motion and $dW_t^{(d)}$ is an independent *fractional Brownian motion* with memory parameter $d \in [0, 0.5]$ (ensuring stationarity). The value $d = 0$ corresponds to the standard Brownian motion, while higher d corresponds to higher memory in the time series. Estimation of model (4.1) has only been performed, to the best of our knowledge, in Casas and Gao (2008) using spectral methods. For simulation studies, see Nielsen and Frederiksen (2008) and Rossi and Spazzini (2010). A discrete time specification of model (4.1) is instead estimated more routinely, see e.g. Comte and Renault (1996) and Christensen and Nielsen (2007). To assess the impact on the results of the fractional difference parameter d, we first estimate model (4.1) for different fixed values d, and then estimate the four parameters (k, ω, η, d) jointly.

The results, reported in Table 6, show that model (4.1) is substantially unable to reproduce the coefficients of the HAR model. The best fit is obtained with a value of $d = 0.491$ very close to non-stationarity. However, even for this fit, the implied daily coefficient of the HAR model is still too high, and the implied weekly coefficient is still too low. To understand the motivation of this failure, it is interesting to look at the estimates obtained for fixed and increasing values of d. When $d = 0$, the model is assimilable to an $AR(1)$ specification and thus is unsurprisingly unable to reproduce the HAR coefficients. As d increases, persistence comes from two terms: the mean-reverting term $k(\omega - \log\sigma_t^2)dt$ and the fractional Brownian motion $\eta dW_t^{(d)}$. However, these two components can vary only in a rigid fashion. For example, when d increases, the mean-reversion parameter k has to increase sharply because of the mean reversion observed in the volatility series which would not be reproduced by $\eta dW_t^{(d)}$ alone. Notice that the ratio

Structural model:

$$d \log \sigma_t = k(\omega - \sigma_t)dt + \eta dW_t^{(d)}$$

parameter	estimates						
	$d=0$	$d=0.1$	$d=0.2$	$d=0.3$	$d=0.4$	$d=0.49$	unconstrained d
k	0.144	0.325	0.750	1.917	7.706	145.081	144.853
ω	−0.166	−0.218	−0.268	−0.290	−0.313	−0.446	−0.447
η	0.248	0.272	0.352	0.625	2.512	118.350	123.746
d	0.000	0.100	0.200	0.300	0.400	0.490	0.491
χ^2	1109.4187	1137.1615	931.5612	527.4070	178.8777	43.2839	42.6094

Auxiliary model:

$$\log \widehat{\mathsf{C}}_{t+1} = c + \beta^{(d)} \log \widehat{\mathsf{C}}_t + \beta^{(w)} \log \widehat{\mathsf{C}}_t^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_t^{(22)} + \varepsilon_t$$

parameter	estimated	implied						
		$d=0$	$d=0.1$	$d=0.2$	$d=0.3$	$d=0.4$	$d=0.49$	unconstrained d
c	0.208	0.661	0.875	1.019	0.966	0.688	0.440	0.436
$\beta^{(d)}$	0.388	0.913	0.877	0.756	0.575	0.449	0.420	0.421
$\beta^{(w)}$	0.368	−0.044	−0.083	−0.059	0.057	0.203	0.281	0.281
$\beta^{(m)}$	0.203	0.004	0.035	0.099	0.173	0.209	0.211	0.211
σ_ε^2	0.198	0.188	0.191	0.195	0.198	0.199	0.198	0.197

Table 6: Estimates (daily units) via indirect inference of the long memory model (4.1) using the HAR model as auxiliary model, and implied HAR coefficients.

$k_\infty = \eta d/k$ remains approximately constant when changing d. This rigidity makes model (4.1) unable to reproduce the HAR model.

Given the failure of the single-factor long memory model, we get inspiration from the very nature of the HAR model, which reproduces a slowly decaying autocorrelation function via the aggregation of different frequencies, and estimate affine multi-factor models with jumps:

$$\begin{aligned} dX_t &= \sum_{i=1}^{N} \sqrt{V_t^i} dW_t^i + dJ_t^X \\ dV_t^i &= \kappa_i(\omega_i - V_t^i)dt + \eta_i \sqrt{V_t^i} dW_t^{i+N} + dJ_t^i \qquad i = 1, \ldots, N \\ corr(dW^i, dW^{i+N}) &= \rho_i \end{aligned} \tag{4.2}$$

where $W^1, \ldots, W^{2N}$ is a multivariate (possibily correlated) Brownian motion and $\mathbf{J} = \{J^X, J^1, \ldots, J^N\}$ is a multivariate (possibly correlated) Poisson process with constant intensities, Normal

Structural model:

$$dX_t = \sqrt{V_t^1}dW_t^1 + \sqrt{V_t^2}dW_t^2$$
$$dV_t^1 = \kappa_1(\omega - V_t^1)dt + \eta_1\sqrt{V_t^1}dW_t^3$$
$$dV_t^2 = \kappa_2(\omega - V_t^2)dt + \eta_2\sqrt{V_t^2}dW_t^4$$

parameter	estimates
κ_1	2.1461
κ_2	0.0042
ω	0.4497
η_1	0.8513
η_2	0.3110
χ^2	0.0000026

Auxiliary model:

$$\log \widehat{\mathsf{C}}_{t+1} = c + \beta^{(d)} \log \widehat{\mathsf{C}}_t + \beta^{(w)} \log \widehat{\mathsf{C}}_t^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_t^{(22)} + \varepsilon_t$$

parameter	estimated	implied
c	0.208	0.208
$\beta^{(d)}$	0.388	0.388
$\beta^{(w)}$	0.368	0.368
$\beta^{(m)}$	0.203	0.203
σ_ε^2	0.198	0.198

Table 7: Estimates (daily units) via indirect inference of model (4.2) with $N = 2$ and $\omega_1 = \omega_2 = \omega$, using the HAR model as auxiliary model, and implied HAR coefficients.

jump sizes in the prices and exponential jump sizes in volatility. In the case of no jumps, when $N = 1$, this is the well known Heston (1993) model; Duffie et al. (2000) and Pan (2002) include jumps as in the Eraker et al. (2003) model considered earlier. With $N = 2$, this model has been used for example in Bates (2000) and, more recently, by Christoffersen et al. (2009).

When using the HAR model as auxiliary model, we set $N = 2$, $\rho_i = 0$, $\mathbf{J} = 0$ and to achieve identification $\omega_1 = \omega_2 = \omega$. Corresponding estimates, together with the implied HAR coefficients, are reported in Table 7. Contrary to the single-factor model with fractional Brownian motion, the two-factor model is perfectly able to reproduce the HAR coefficients, obtaining an objective function χ^2 close to zero. Estimates are compatible with those typically encountered in the literature: the fit implies the presence of a fast mean-reverting factor with a half-life less than one day, and a slowly mean-reverting factor with a half-life of nearly 200 days. The superposition of these two frequencies produces the desired effect in terms of volatility persistence. Lieberman and Phillips (2008) suggest that also the usage of integrated volatility measures produces a longer memory than that implied in the dynamics of spot volatility. Our result also explains why multi-factor model works so well in describing the dynamics of options, see e.g. Bates (2000), since they are able to reproduce the volatility dynamics under the natural probability. It also suggests that two factors might be redundant if the volatility dynamics is specified directly with a model similar to HAR: an attempt in this direction is the paper of Corsi et al. (2011), which develops an option pricing model with HAR volatility dynamics providing remarkable pricing performance with a single volatility factor. Finally, two volatility factors have also been shown to be priced in the cross-section of expected returns, see Adrian and Rosenberg (2008).

When the auxiliary model is LHAR, the natural approach is to allow for nonzero correlation coefficients to introduce a leverage effect, again setting $\mathbf{J} = 0$. We report estimates of the two-factor model with leverage effect in Table 8. When fitting the two-factor model using the LHAR model as auxiliary model, we find $\rho_1 > 0$, that is a positive correlation coefficient between the fast mean-reverting volatility factor and returns, while ρ_2 is negative. This fact is not totally surprising since it echoes the results of Chernov et al. (2003) and Bollerslev et al. (2006), who also estimate (among other models) a two-factor affine model on S&P500 returns via efficient method of moments (using an auxiliary GARCH model) and find the correlation coefficient associated to the fast mean-reverting volatility factor to be positive.

In presence of two factors, the interpretation of the leverage effect is not trivial since, as also Chernov et al. (2003) explain, the average leverage can be negative even with a positive correlation coefficient. However, the reason why a positive correlation arises with the fastest volatility factor remained unclear. Figure 5 can help to provide a possible explanation for this occurrence. We simulate model (4.2) with the coefficients estimated in Table 7, and we vary ρ_1 (with $\rho_2 = 0$) and ρ_2 (with $\rho_1 = 0$) to evaluate the impact of the introduced correlations on the LHAR coefficients. When ρ_1 (the leverage effect of the fast mean-reverting factor) is different from zero, the impact of daily negative returns on future volatility follows the sign of ρ_1, but the opposite hold for the impact of weekly and monthly negative returns. For example, when ρ_1 is negative, we find $\gamma^{(d)} < 0$ but $\gamma^{(w)}, \gamma^{(m)} > 0$. This is due to an *overshooting* effect: a positive correlation at a higher frequency becomes negative at a slower one and viceversa. However, for the slowly mean-reverting factor, the sign of ρ_2 induces a leverage effect with the same sign on the daily,

Structural model:

$$dX_t = \sqrt{V_t^1}dW_t^1 + \sqrt{V_t^2}dW_t^2 + \sqrt{V_t^3}dW_t^3 + c_X dN_t$$
$$dV_t^1 = \kappa_1(\omega_1 - V_t^1)dt + \eta_1\sqrt{V_t^1}dW_t^4 + c_\sigma dN_t$$
$$dV_t^2 = \kappa_2(\omega_2 - V_t^2)dt + \eta_2\sqrt{V_t^2}dW_t^5$$
$$dV_t^3 = \kappa_3(\omega_3 - V_t^3)dt + \eta_3\sqrt{V_t^3}dW_t^6$$
$$corr(dW^1, dW^4) = \rho_1$$
$$corr(dW^2, dW^5) = \rho_2$$
$$corr(dW^3, dW^6) = \rho_3$$
$$N_t \sim Poisson(\lambda t), \quad c_X \sim \mathcal{N}(0,\sigma_J^2), \quad c_\sigma \sim exp(\mu_\sigma)$$

parameter	two factor	three factor	three factor with jumps
κ_1	8.1088	6.7647	5.7233
κ_2	0.0003	0.6556	0.8390
κ_3	–	0.0036	0.00004
ω_1	0.3030	0.2480	0.2468
ω_2	0.5165	0.1348	0.1740
ω_3	–	0.1894	0.1311
η_1	1.6348	2.0128	1.8970
η_2	0.3748	0.3880	0.4137
η_3	–	0.2849	0.3412
ρ_1	0.9847	0.3201	0.5040
ρ_2	−0.9807	−0.9949	−0.8947
ρ_3	–	−0.9173	−0.9714
λ	–	–	0.0129
σ_J	–	–	0.0254
μ_σ	–	–	0.1420
χ^2	123.301	0.221	28.369

Auxiliary model:

$$\log \widehat{\mathsf{C}}_{t+1} = c + \beta^{(d)} \log \widehat{\mathsf{C}}_t + \beta^{(w)} \log \widehat{\mathsf{C}}_t^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_t^{(22)} + \\ + \gamma^{(d)} r_t^- + \gamma^{(w)} r_t^{(5)-} + \gamma^{(m)} r_t^{(22)-} + \varepsilon_t$$

parameter	estimated	two factor	three factor
c	0.421	0.561	0.428
$\beta^{(d)}$	0.299	0.238	0.302
$\beta^{(w)}$	0.366	0.530	0.358
$\beta^{(m)}$	0.236	0.105	0.239
$\gamma^{(d)}$	−0.007	−0.004	−0.007
$\gamma^{(w)}$	−0.008	−0.014	−0.008
$\gamma^{(m)}$	−0.009	−0.010	−0.010
σ_ε^2	0.187	0.188	0.187

Auxiliary model:

$$\log \widehat{\mathsf{V}}_{t+1} = c + \beta^{(d)} \log \widehat{\mathsf{C}}_t + \beta^{(w)} \log \widehat{\mathsf{C}}_t^{(5)} + \beta^{(m)} \log \widehat{\mathsf{C}}_t^{(22)} + \\ \alpha^{(d)} \log(1 + \widehat{\mathsf{J}}_t) + \alpha^{(w)} \log(1 + \widehat{\mathsf{J}}_t^{(5)}) + \alpha^{(m)} \log(1 + \widehat{\mathsf{J}}_t^{(22)}) + \\ + \gamma^{(d)} r_t^- + \gamma^{(w)} r_t^{(5)-} + \gamma^{(m)} r_t^{(22)-} + \varepsilon_t$$

parameter	estimated	three factor with jumps
c	0.446	0.384
$\beta^{(d)}$	0.304	0.306
$\beta^{(w)}$	0.369	0.349
$\beta^{(m)}$	0.222	0.260
$\alpha^{(d)}$	0.042	0.017
$\alpha^{(w)}$	0.011	0.011
$\alpha^{(m)}$	0.005	−0.000
$\gamma^{(d)}$	−0.007	−0.006
$\gamma^{(w)}$	−0.008	−0.006
$\gamma^{(m)}$	−0.009	−0.014
σ_ε^2	0.183	0.182

Table 8: Estimates (daily units) via indirect inference of model (4.2) with two and three factors, using the LHAR model as auxiliary model, and implied LHAR coefficients.

weekly and monthly coefficients, with the impact increasing with the horizon. The effect of introducing correlations on volatility coefficients is instead marginal. Thus, with positive ρ_1 we get the right sign for the weekly and monthly coefficient, while the daily coefficient is adjusted with a negative ρ_2.

Using the two-factor model, this mechanism is able to reproduce the LHAR model only partially: the signs are correct but the coefficients estimated on the data can be reproduced only to a limited extent. In order to get a satisfactory agreement with the LHAR model, we need to introduce a third factor (in this case, the parameters of the structural model are not identified). Estimates of the three-factor model are again reported in Table 8, and they show that with $\rho_1 > 0$ and $\rho_2, \rho_3 < 0$ we can reproduce completely the LHAR model.

Finally, we include jumps in the structural model with the aim of reproducing the results of the LHAR-CJ model. Extensive Monte Carlo analysis, not reported here for brevity but available in the Web Appendix, shows that a possible mechanism explaining the significant impact of jumps on future volatility is given by the presence of contemporaneous jumps in price and volatility, a possibility which has been recently empirically confirmed by Todorov and Tauchen (2011) and Bandi and Renò (2011a). For this reason, in our last estimation we introduce a single Poisson process N_t with constant intensity λ, and we set $dJ^X = c_X dN_t$, $dJ^1 = c_V dN_t$, $J^2 = J^3 = 0$ with $c_X \sim \mathcal{N}(0, \sigma_J^2)$ and $c_V \sim exp(\mu_\sigma)$, that is we introduce co-jumps in price and in a single volatility factor, namely the less persistent (also in this case the structural model is not identified). Estimation results are reported in Table 8 and indicate that introducing co-jumps provides a reasonable fit of the LHAR-CJ model, since we are able to reproduce both the short-range persistence of jumps and the long-range persistence of leverage.

Concluding, we have seen that the LHAR-CJ model, and some of its relevant restrictions, are fully consistent with a multi-factor Markovian volatility model. While it is certainly outside the scope of this paper to provide a thorough interpretation on the mechanism which generated the stylized facts described by the estimated statistical models, both in discrete time and in continuous time, a possible interpretation of the empirical results goes as follows. Volatility is highly persistent, and this persistence can be generated by a superposition of factors with different frequencies. Negative returns and jumps are correlated with volatility. However, jumps

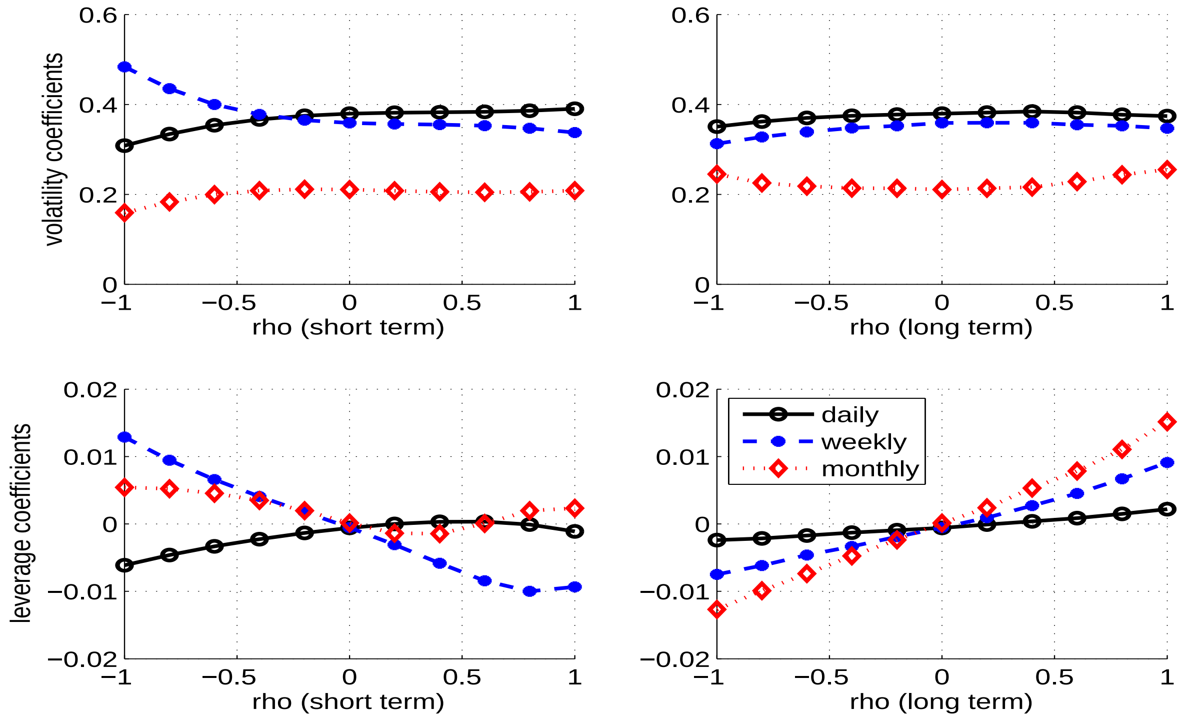

Figure 5: Sensitivity of the coefficients of the LHAR specification (top row: β coefficients of volatility; bottom row: γ coefficients on negative returns) the value of the leverage coefficients ρ_1 (correlation with the fastly mean reverting factor, left column), when $\rho_2 = 0$, and ρ_2 (correlation with the slowly mean reverting factor, right column), when $\rho_1 = 0$.

are only correlated to a fast-reverting volatility factor via the mechanism of co-jumps, so that their impact can only be short-lived. On the contrary, negative returns are correlated to *all* volatility factors through the correlation of the shocks. For this reason, negative returns can have a long-span impact on volatility, thus producing a persistent leverage effect.

5 Conclusions

In this paper, we uncover new stylized facts about volatility dynamics. While it is well known that past negative returns are correlated with current volatility (leverage effect), we show that the forecasting power of past negative returns remains significant even when considering them over long horizons. The data also suggest that past jumps are (positively) correlated with current volatility, but the forecasting power of aggregated jumps is milder when the aggregation horizon is large. We then specify, both in discrete and in continuous time, suitable models which are able to capture these novel stylized facts along with the well estabilished volatility features.

In the first stage, we propose a new discrete-time model for realized volatility measures, the LHAR-CJ model, which naturally identifies three main determinants of volatility dynamics, namely heterogeneous lagged continuous volatility, heterogeneous lagged negative returns and heterogeneous lagged jumps. We find that each of the components in the discrete-time model plays a different role at different forecasting horizons, but all the three are highly significant and neglecting each one of them is detrimental to the forecasting performance.

In the second stage, we look for continuous-time models which reproduce the very same stylized facts which are captured by the discrete-time specification. This is achieved by using the discrete-time model as a convenient statistical metric in an indirect inference framework. A multi-factor Markovian specification is found to be consistent with the empirical results and compatible with the LHAR-CJ. To reproduce the long-term impact of negative returns, all the volatility factors have to be correlated with the price shocks, while to reproduce the transient impact of jumps it is enough to correlate price jumps with the jumps of one volatility factor only (co-jumps).

We conclude by noting that our discrete-time model is very simple to implement, as it does not require sophisticated computational technique. The estimation of the model parameters can be performed through a simple OLS regression, and the computation of the explanatory variables is trivial. We think that, for all the aforementioned reasons, the LHAR-CJ model may be effectively used for risk management.

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