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Symmetries, Groups, Groupoids and Systems of Systems

Eduardo Alonso, Nicos Karcanias
City University London
London, UK
{E.Alonso, N.Karcanias}@city.ac.uk

Ali G. Hessami
Vega Systems Ltd.
London, UK
hessami@vegaglobalsystems.com

***Abstract*–In this paper we propose an algebraic model of systems based on the concept of symmetry that can be instrumental in representing Systems of Systems two main characteristics, namely complexity and (hierarchical) emergence.**

Keywords–systems of systems; symmetries; conservative systems; emergence; groups and groupoids

I. INTRODUCTION

Many socio-technical systems consist of a large number of partly autonomous local systems with local self-management and possibly intelligence that may be strongly or weakly interact and where physical interactions may be expressed through physical streams of energy (electricity, steam) or material (water, gas, intermediates) and general relationships may be defined in terms of operational rules stemming from a central goal which may be explicitly or implicitly defined. Such systems belong to the general class of cyber-physical systems of systems. Examples of such systems are the electrical grid, buildings and building complexes, petrochemical and chemical production plants, water distribution systems, and gas networks. In such systems, there is frequently a conflict between the local optimization defined on individual units and the goals of the overall system. This class of systems have frequently an evolving nature and are highly dynamic.

The term *System of Systems* (SoS) has been used to describe a collection of subsystems /components, which are brought together to achieve a common goal. A commonly used characterisation of this notion of SoS was proposed in terms of features such as [1][2]:

- Operational independence of the subsystems (can operate independently).
- Managerial independence of the subsystems (separately acquired by different managerial entities).
- Evolutionary development (evolution over time with the addition, of elimination of parts)
- Emergent behaviour (exhibit unexpected behaviour that cannot be easily referred to constituent parts).

Defining SoS in terms of such features does not explain the fundamental difference between this new family and the traditional family of *Composite Systems* (CoS). It has been identified in [3]-[8] that the fundamental differentiation between the SoS and CoS classes is the nature of the interconnection rules between the subsystems. In CoS such relations may be described in terms of certain graph topologies, whereas in SoS we deal with a multitude of relations which are not necessarily well defined. They may be variable, possibly evolving and are frequently deduced by the rules of a game defined on the subsystems, which are assumed to have intelligence characteristics.

The current paper intends to contribute towards the definition of a formal framework to characterise such relations.

II. TOWARDS A GENERAL THEORY OF SYSTEMS OF SYSTEMS

A. Groups, Symmetries and (Conservative) Systems

A.1 Systems as Groups

In the study of SoS, the fundamental primitive is the concept of "system" itself. Traditionally, a system is considered simply as a set, a collection of elements defined precisely (and circularly) as being members of the set. For instance, the set of integers is Z={…, -3, -2 -1, 0, 1, 2, 3, …}. It is however clear that sets are insufficient to represent the underlying *structure* that characterizes systems, be it abstract systems like the integers or mundane systems like a traffic network. The collection of the elements a traffic network consists of, say vehicles, radars and traffic lights, doesn't make up the "traffic network system". What make a set of objects a system are their interactions, that is, their structure. The mathematical notion of group includes such structure: a group is a set together with an *operation* that combines any two elements to form a third element. For example, the set of integers together with the addition operation (Z,+) forms a group. Formally, a group is a set, G, together with an operation •, $(G, \bullet)$, that satisfies four axioms:

- Closure: For all a, b in G, the result of the operation, $a \bullet b$, is also in G.
- Associativity: For all a, b and c in G, $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.
- Identity element: There exists an element e in G, such that for every element a in G, the equation $e \bullet a = a \bullet e = a$ holds.

- Inverse element: For each a in G, there exists an element b in G such that $a \bullet b = b \bullet a = e$.

Following the integer example, (Z,+) is a group since

- For any two integers a and b, the sum $a + b$ is also an integer.
- For all integers a, b and c, $(a + b) + c = a + (b + c)$.
- If a is any integer, then $0 + a = a + 0 = a$.
- For every integer a, there is an integer b such that $a + b = b + a = 0$.

Another example may clarify the concept of group: translation of the plane is a rigid movement of every point of the plane for a certain distance in a certain direction. For instance "move in the North-East direction for 2 miles" is a translation of the plane. Two such translations a and b can be composed to form a new translation $a \circ b$ as follows: first follow the prescription of b, then that of a. For instance, if a = "move North-East for 3 miles", and b = "move South-East for 4 miles" then $a \circ b$ = "move East for 5 miles". The set of all translations of the plane with composition as operation forms a group:

- If a and b are translations, then $a \circ b$ is also a translation.
- Composition of translations is associative: $(a \circ b) \circ c = a \circ (b \circ c)$.
- The identity element for this group is the translation with prescription "move zero miles in whatever direction you like".
- The inverse of a translation is given by walking in the opposite direction for the same distance.

In fact, groups are formalisms that apply to any structured set of objects: elements do not need to be integers or points of the plane, they can be can be any abstraction –shapes, phrases, mathematical equations and even theories. And the operations of the group can be any transformation –from a rotation over an axis to the interaction of vehicles in a traffic network. Crucially, groups act on operations not on elements.

A.2 Symmetries and Symmetry Groups

There is one more characteristic that makes the concept of group an ideal candidate to formalize the idea of "system": the definition of group embeds the ontological principle that the whole (the system) remains invariant under a set of transformations. If it changes, then we are talking about a different system. In other words, a system can be identified with its symmetry group.

The study of symmetries flourished in the XIX century, originally as an instrument to solve algebraic equations: it was the young Évariste Galois who first understood that groups opened a new general way of finding the (invariant) structures that underlie the number and form of the solutions for equations of arbitrary degrees. This had an immediate effect in Physics: C. G. J. Jacobi developed a procedure for transforming step by step the Hamiltonian formulation of the dynamical equations of mechanics into new ones that are simpler but perfectly equivalent. In geometry, Felix Klein proposed the *Erlangen Program* to classify various geometries (Euclidean, affine, and projective) with respect to geometrical properties that are left invariant under rotations and reflections [9].

In fact, we can view theories in Physics in terms of their symmetries and groups. Newtonian classical mechanics is based on Galilei transformations formalized in the Galilei group; the special theory of relativity unified seemingly contradictory mechanical and electromagnetic phenomena of the hand of Lorentz transformations and their corresponding Lorentz groups; and the general theory of relativity explained gravity, the most symmetrical of field theories so far, under the group of all diffeomorphisms of a space-time. It has been, however, with quantum mechanics when symmetry groups have become an indispensable tool in Physics (see [10]): internal symmetries (*i.e.*, those which act on fields while at the nuclear level and cannot be reduced to "classical" spatiotemporal symmetries), both global and gauge, can only be fully understood when studied through the groups their representations form. In particular, the Standard Model classifies all elementary particles and their interactions according to their flavor, charge and color symmetries (the SU(3) ⊗ SU(2) ⊗ U(1) group), and, in so doing, unifies electromagnetism, QED and QCD and explains electroweak interactions through spontaneous symmetry breaking.

Summarizing, symmetry groups provide us with a formal tool to characterize and analyze systems. Indeed, the use of symmetries in the study of systems is two-fold: as argued in [11], we attribute symmetry properties to *theories* and *laws* (symmetry principles) from which we derive and test the validity of the laws of Nature; at the same time, we may derive specific consequences with regard to particular *phenomena* on the basis of their symmetry properties (symmetry arguments). Pierre Curie himself postulated a necessary condition for a given phenomenon to happen, namely, that it is compatible with the symmetry conditions established by a principle [12].

More specifically, symmetries play several inter-related roles that we illustrate with (point) groups in molecular biology (see *e.g.*, [13]):

Normative role: one the one hand, symmetries furnish a kind of selection rule. Given an initial situation with a specified symmetry, only certain phenomena are allowed to happen; on the other hand, it offers a falsification criterion: a violation of Curie's principle may indicate that something is wrong with the systems' description. That is, symmetries can be viewed as normative tools, as constraints on theories –the requirement of invariance with respect to a transformation group imposes several restrictions on the form the theory may take, limiting the types of quantities that may appear in the theory as well as the form of its fundamental equations. For instance, the rule that determines whether or not two atomic orbitals can form a chemical bond (*i.e.*, a molecule) is that they must belong to the same symmetry species within the point group of the molecule. The same applies to bonding in polyatomics;

Unification role: symmetries can be used as a heuristic to compare and unify theories, resulting from the possibility of

combining different types of symmetries by means of a unification of the corresponding transformation groups. Likewise, we can use symmetries to analyze whether or not different theories are, in fact, equivalent –or incomparable. Following our example in molecular biology, the analysis of symmetries and their corresponding groups provides us with a unifying approach to complex molecular behavior such as molecular vibrations and vibrational spectroscopy;

Classificatory role: classifications can be used to identify gaps in the theories but also to predict the existence of new phenomena. This applies when new phenomena can be predicted exclusively in terms of symmetry and when the predictions so postulated are coherent with those of existing models. For instance, all possible molecules can be classified according to symmetry operations on five symmetry elements: the identity operation (doing nothing) on the identity element (the entire molecule); rotation on the proper rotation axis; rotation on the improper rotation axis; reflection in the plane of symmetry; and inversion on the center of symmetry. We can group together molecules that posses the same symmetry elements and classify molecules according to their symmetry: for example, water belongs to the C_{2v} group which contains the identity, a 2-fold axis of rotation and 2 vertical mirror planes. Interestingly, Dymethyl ether also belongs to such group no matter how different its composition and that of water's may look – $O(CH_3)_2$ and H_2O respectively;

Explanatory role: symmetries are also explanatory in that phenomena can be explained as consequences of symmetry arguments. We know that the symmetry elements of the causes must be found in their effects and that the converse is not true. That is, the effects can be (and often are) more symmetric than their causes. In group-theoretic terms this means that the initial symmetry conditions are lowered into (more constrained) groups: the symmetry has been broken. In biology we know that for a molecule to have a permanent dipole moment it must have an asymmetric charge distribution. The point group of the molecule not only determines whether a molecule may have a dipole moment but also in which direction(s) it may point. The only groups compatible with a dipole moment are C_n, C_{nv} and C_s. Besides, in molecules belonging to C_n or C_{nv} the dipole must lie along the axis of rotation. Now, we can explain and predict, at least partially, how a molecule of water behaves.

A.3 Symmetries, Principle of Least Action and Conservation

Let's recapitulate, groups formalize the notion of system, as a structure that remains invariant, that is, as a symmetric structure. In addition, formalizing the idea of system using symmetry groups allows us to define the system's *dynamics*. In Physics, the true dynamical trajectories of a system are found by imagining all possible trajectories that the system could conceivably take, computing the *action* (a functional of the trajectory) for each of these trajectories, and selecting the one that makes the action "least" (actually stationary). Formally, the action to be minimized is the integral of a function, the Lagrangian, over time. That is, we try to minimize $S = \int_{\substack{entire \\ path}} L\,dt$. The Lagrangian itself describes completely the dynamics of the system under consideration as the difference between its kinetic energy (the energy due to the motion, how much is "happening") and its potential energy (the energy due to the position or configuration, how much could happen). In short, Nature is as lazy (thrifty, as de Maupertuis put it) as possible. Consider the path followed by a ball thrown into the air: on the one hand, the ball wants to spend a lot much time near the top of its trajectory since this is where the kinetic energy is least and the potential energy is greater. On the other hand, if it spends too much time near the top of its trajectory, it will really need to rush to get up there and get back down and this will take a lot of action. The perfect compromise is a parabolic path. Now, we know that this "least action condition" is equivalent to Euler-Lagrange's equation, $\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial v}$ of motion. This formulation is more elegant (works with energy-scalars rather than force-vectors) and universal than Newton's (applies to any framework or generalized co-ordinates), encapsulates the universal Principle of Least Action, and when transformed into its Hamiltonian form reflects the symmetries of Nature [14]. In addition, Emmy Noether's theorems state that there is a one-to-one correspondence between conservation laws and differentiable symmetries of (physical) systems [15]: for example, the conservation of energy follows from the time-invariance of physical systems, and the fact that physical systems behave the same regardless of how they are oriented in space gives rise to the conservation of angular momentum.

A.4 Control Systems

To summarize, the variational principles that shape the dynamics of systems reflect the symmetries of Nature. Groups formalize such symmetries. It is thus not a coincidence that systems are defined as structures that are preserved (conserved) under transformations. The same principles apply to control systems. The Hamiltonian of a control system is a function of four variables, $H(s,p,u,t) = L(s,u,t) + p_t^T f(s,u,t)$, where $p_t^T = -\frac{\partial H}{\partial s}$ is a costate interpreted as a Lagrange multiplier: if the state given by the function *f* represents constraints in the optimization problem, the costate represents the cost of violating those constraints. In other words, *p* is the rate of change of the Hamiltonian as a function of the constraint. For example, in Lagrangian mechanics, the force on a particle, $F = -\nabla V$, can be interpreted as *p* determining the change in action (transfer of potential to kinetic) following a variation in the particle's constrained trajectory. In economics, the optimal profit is calculated according to a constrained space of actions, where *p* is the increase in the value of the objective function due to the relaxation of a given constraint –the marginal cost of a constraint, called the shadow price. Intuitively, the constraint *f* can be thought of as competing with the desired function to pull the system to its minimum or maximum (or to a steady state). And the Lagrange multiplier *p* can be thought of as measure of how hard *f* has to pull in order to make those forces balance out in the constraint surface. The extremal principle used in such cases is Pontryagin's Minimum Principle [16], which states that $H(s_t^*, u_t^*, p_t^*, t) \leq H(s_t^*, u_t, p_t^*, t)$ with the associated conditions

for a maximum, $p_t = -\frac{\partial H}{\partial s}$, $s_t = -\frac{\partial H}{\partial p}$, and $\frac{\partial H}{\partial u} = 0$.

At this stage, we need to consider whether the concept of symmetry group is the right tool to formalize SoS. We argue that it is not: a SoS is characterized by emergence and evolution, and symmetry groups represent closed, conservative systems. To formalize open systems we need more sophisticated structures.

B. Groupoids, Partial Symmetries and Systems of Systems

B.1 Systems of Systems: Complexity and Emergence

Not all systems fit in the idea of "system" presented in the previous section. In fact, most natural systems are not conservative. Typically, systems are open and interact with the environment and other systems to form large complex systems. It is precisely such interactions what makes emergence and evolution possible –the defining characteristics of SoS. It is thus important that we ascertain which parts of our previous analysis apply to SoS. In order to do so, we follow Sharma and Annila in redefining evolutionary processes as probable motions [17]. This allows us to "recover" the Principle of Least Action. The main idea is that evolutionary processes direct along the steepest descent of and energy landscape by equalizing differences in energy via interactions (transformation processes), for instance, diffusion, heat flows, electric currents, chemical reactions, but also associative learning (as the formation of links between the representations of two stimuli), economic behavior and the evolution of populations. In large and complex systems, that is, in SoS, the flows are viewed to explore diverse evolutionary paths, and those that lead to a faster entropy increase, equivalent to a more rapid decrease in the free energy, become selected. This fitness criterion, "take the steepest gradient in energy", and Principle of Least Action are equivalent.

In particular the second law of thermodynamics can be expressed as a differential equation of motion for the probability P, $\frac{dP}{dt} = LP$, where the propagator, $L = \frac{1}{k_B T} = \sum_{j,k} \frac{dx_k}{dt}\left(\frac{\partial \mu_k}{\partial x_j} - \frac{\partial Q_k}{\partial x_j}\right)$ drives the transport $dx_k/dt = -\sum dx_j/dt$ between general coordinates, e.g. as diffusion and currents, by draining the potential energy gradients $\partial \mu_k / \partial x_j$ and the fields $\partial Q_k / \partial x_j$ that couple to the *jk*-transport process. After each dissipative event, i.e. an emission or absorption, the system settles via interactions to a new partition corresponding to a new average energy per particle, $k_B T$, the common reference. The system evolves by dissipation, i.e. by energy efflux or influx, in the quest to reduce the gradients and to attain a stationary state in its surroundings.

Similarly to transport processes, diverse transformation processes, e.g. chemical reactions converting N_k substrates to N_j products or vice versa, are driven by the propagator [18] $L = \frac{1}{RT} = \sum_{j,k} \frac{N_k}{dt}\left(\frac{\partial \mu_k}{\partial N_j} - \frac{\partial Q_k}{\partial N_j}\right)$, where the chemical potential is denoted as the gradient $\partial \mu_k / \partial N_j$. When the surrounding density-in-energy couples to the *jk*-transformation, it contributes by $\partial Q_k / \partial N_j$. For chemical reactions, the average energy $RT = N_A k_B T$ is, as usual, given per mole via Avogadro's number N_A and Boltzmann's constant k_B. Evolution as given by this equation, is essentially a restatement of the Gibbs–Duhem equation that relates a decrease in the chemical potential of one substance to an increase in the chemical potentials of the other substances. In accordance with Le Chatelier's principle, the system will evolve towards a stationary state by interacting with its surroundings –by emitting or acquiring quanta. In equilibrium, gradients vanish but energy is conserved.

Let's use a less technical example to illustrate this point: at the end of the day, animals are behavior *systems* –sets of behaviors that are organized around biological functions and goals like feeding, defense, or sex. When such systems are free to act as they please, their preferred or optimal distribution of activities defines a behavioral bliss point (BBP) or baseline level of activity. In dynamic terms the BBP is a natural, steady and stable, attractor.

This view encapsulates the behavioral regulation theory and generalizes the concept of homeostasis and negative feedback from physiology to psychology. Physiological homeostasis keeps physiological parameters such as body temperature close to an optimal or ideal level. This level is "defended" in that deviations from the target temperature trigger compensatory physiological mechanisms that return the system to its homeostatic levels. In behavioral systems, what is defended is the organism's BBP against instrumental contingencies that create disturbances to which the system adapts.

More specifically, John E. R. Staddon's model explains operant behavior in terms of time constraints and feedback constraints, the reinforcement schedule to which the animal is subjected [19]. Starting from BBP, the animal finds the optimal equilibrium between instrumental and contingent responses –the one that minimizes the cost involved. Instrumental conditioning procedures are considered as response constraints that disrupt the free behavior and interfere with how an organism makes choices among the available responses. Instrumental conditioning procedures do not allow the organism to return to the BBP, yet the organism achieves a contingent optimization by approaching its bliss point under the constraints of the instrumental conditioning procedure. Put it this way, the analysis of operant behavior is an optimal control problem and thus we should be able to express it in terms of the corresponding Hamiltonian: L, the Lagrangian, is defined as the cost to be minimized, f are the time and feedback constraints, and p, the multiplier or conjugate momentum, represents the gradient of the cost associated with a departure from a given distribution of actions. The dynamics of the system is defined according to

$\frac{Dr}{Ds}$ and $\frac{Df}{Ds}$. The former tells us how the rate of contingent responses changes as the distribution of responses changes and the latter how the constraints themselves change. These two quantities define the change of cost that we minimize and give us the (new) optimal distribution [20].

Crucially, this approach to evolutionary and adaptive systems does not only compute a solution to a standard optimization problem –in this case, the optimal distribution of responses. Of course, it does if we assume that the functions that describe the dynamics of the system are known; yet, in real systems, organisms adapt to the optimal distribution when the constraints are a moving target. This reflects the tension between optimality, taking advantage of changes in the environment, and stability, the persistence of existing populations. Considering the animal's behavior as an evolving SoS solves this "teleological conundrum": of course, animals do not know what the reinforcement schedule would be or the corresponding optimal response ratio –and yet they adapt to the optimal solution and they do so in an optimal way. Perhaps an analogy may clarify this point: Physicists found it puzzling that particles behaved as if they knew what the future would be; traditionally, the movement of particles was interpreted in terms of global symmetries and thus it was difficult to explain how particles abided by the Principle of Least Action locally, when constraints appeared and disappeared as the system interacted with "unexpected" forces. Surely, the symmetries were broken in such cases; and yet, Nature seemed to account for them so as to comply with global symmetries –"as if nothing had happened", symmetry was restored. We know that the answer lies in gauge symmetries: indeed, at each step, deviations are counter-balanced so as to bring the system back (or as close as possible) to the original symmetry. It is not coincidence that ODEs for optimal control problems share the same form as Lyapunov functions [21] and Nash equilibria [22].

Although this approach to psychology has not been formalized as equations of motion or in terms of transformations of a complex system of behaviors, it clearly follows Sharma and Annila's analysis of "evolutionary" systems, which in turn, can be understood as a special case of Ilya Prigogine's work on self-organizations, which embed the idea of "order though fluctuations" [23], and E. T. Jaynes' Minimum Entropy Production Principle [24]. The applications of this analysis are over-reaching: behavioral regulation as theorized in the bliss point approach is at the core of behavioral economics, in particular in the study of resource allocation and consumer demand. It is also widely used in studies of optimal foraging and population dynamics.

In some sense, the relation of extremal principles for conservative systems and for open systems should not be surprising: following the intuition of the Basque Pierre de Fermat and his principle of least time, de Maupertuis formulated the Principle of Least Action for any system, conservative or not [25]. That, given the difficulties that its formulation for dissipative systems and non-equilibrium systems entailed, it has been historically applied to conservative systems, for which clear-cut equations and symmetry groups can de defined, does not mean that open systems don't abide by it. Likewise, that different versions of the same principle seem to be far apart from each other still today, is due mainly to the fact that the formalisms of Physics and the language of more complex systems differ from each other (as Boltzmann himself stated, [26]).

B.2 Partial Symmetries, Symmetry Breaking and Groupoids

Now that we have made a connection between systems and SoS in terms of fundamental principles, we need to consider whether SoS can be formalized the same way as systems were, using symmetry groups. Symmetries, at least global symmetries, are broken as systems interact and evolve. From dissipation through friction to spontaneous decay or the creation and annihilation of particles in QFT, systems are not conservative or symmetrical. One way to deal with such systems is to "close" them, to consider the system and their surroundings as a (conservative) whole. The trick is to add the corresponding entropy to the Lagrangian of the system and to the equations of motion, in short, to reduce SoS to CoS. However interesting this approach might be, it avoids the main problem: since interactions are considered as "debris" there is not guarantee that a unique solution minimizes the system's trajectory. Instead, we take the view that interactions are systems in themselves –like the creation of a photon that results from the collision of an electron and a positron. As Sir Arthur Eddington expressed it: "We often think that when we have completed our study of one, we know all about two, because 'two' is 'one and one.' We forget that we still have to make a study of 'and' —which is the photon."

In terms of group theory, let's assume two closed systems, each with its corresponding symmetry group, G_1 and G_2, and "bring them together": the new system, a SoS, results from the interaction of the two systems. We know that, according to Curie's principle, the SoS cannot have more symmetry than its constituents. Energy cannot be created. In addition, the SoS cannot have less total symmetry. Energy cannot be destroyed either. We also know that interactions must be computed and thus that the SoS symmetry group, G_3, is not reducible to the addition of its constituents' symmetry groups, that is, $G_3 \neq G_1 \otimes G_2$. Hence, the only solution is that the symmetries of the two sub-systems are broken, evolving into new symmetries, G'_1 and G'_2; in addition, part of the original symmetry emerges as the symmetry of the interaction, G_I. Thus, $G_3 = G'_1 \otimes G'_2 \otimes G_I$. Energy can be transformed. Crucially, this analysis is recursive, that is, systems and SoS form hierarchies: at the top level, as SoS, symmetry groups are preserved where as the constituents interact and evolve, old symmetries being broken and new ones emerging in the process. In turn SoS can interact forming supra-SoS. Clearly, that, since the constitutive systems evolve, a rigid notion of global symmetry and the groups that formalize it is not longer useful.

We have seen that mathematicians (and physicists) tend to think of the notion of symmetry as being virtually synonymous with the theory of groups (symmetry groups). In fact, though groups are indeed sufficient to characterize homogeneous structures, there are plenty of objects which exhibit what we clearly recognize as symmetry, but which admit few or no nontrivial automorphisms. It turns out that the

symmetry, and hence much of the structure, of such objects can be characterized algebraically (and categorically) if we use groupoids and not just groups (see [27][28], for two formal introductions to groupoids).

Let's take the circle as an example: the circle is a highly symmetric object –every line through the center forms a line of reflection symmetry and it has rotational symmetry around the center for every angle. Its symmetry group is the orthogonal group O(2,R). What about a 2-D bowling ball, that is, a circle with three "holes"? Obviously, the bowling ball is not as symmetric as a plain circle. However, the ball still shows a lot of symmetry. Perhaps the ball cannot be formalized as a group, but it can as a structure with "less" symmetry, as a groupoid. The same would apply to a SoS: if we only allow transformations that leave the whole system unaltered then the evolution of a complex system or indeed the emergence of new properties is not possible.

Intuitively, a groupoid should be thought of as a group with many objects, or with many identities. A groupoid with an object is essentially just a group. So the notion of groupoid is an extension of that of group. This apparently innocuous distinction between one-object structures (groups) and many-objects structures (groupoids) is actually crucial. The homomorphisms defined in groups are always automorphisms (homomorphisms of the object to itself). In other words, as groups are one-object categories, *all* morphisms can be composed with all other morphisms. From this, the algebraic conditions for the formation of groups (closure, unique identity, unique total inverse and *total* associativity) follow directly. On the other hand, groupoids, can only compose morphisms (isomorphisms in their case) with the appropriate domains and co-domains. Algebraically, a groupoid is a set with a *partially* defined binary operation (that *is* associative *where* defined) and a total inverse function. Formally, a groupoid is a set G with a unary operation $^{-1}$ and a partial function *. Here * is not a binary operation because it is not necessarily defined for all possible pairs of G-elements. The operations, * and $^{-1}$, have the following axiomatic properties. Let a, b, and c be elements of G. Then:

- Associativity: If $a * b$ and $b * c$ are defined, then $(a * b) * c$ and $a * (b * c)$ are defined and equal.
- Inverse: $a^{-1} * a$ and $a * a^{-1}$ are always defined.
- Identity: If $a * b$ is defined, then $a * b * b^{-1} = a$, and $a^{-1} * a * b = b$.

What is important to get from this mathematical mumbo-jumbo is (a) that in groupoids associativity is partially defined, allowing us to investigate *variable symmetries* (symmetry groupoids) and (b) that in groupoids isomorphisms are defined over sets of base points (fundamental groupoids), permitting us to study *more symmetries*. Indeed, groupoids show new structures that do not show at a group level –more specifically, in groupoids, the inverse relation, although total, is defined over paths; besides, groupoids lead to higher dimensional algebras and help us move between n-categories through natural transformations, limits and co-limits.

Summarizing, groupoids present three very useful properties: (1) partial associativity, (2) path reversibility, and (3) hierarchism. SoS properties, precisely.

Admittedly, the debate over whether groupoids are useful or unmotivated abstractions is still going on [29]. Nevertheless, since they were introduced by H. Brandt in 1926 groupoids have been used in a wide area of mathematics as well as in theoretical physics, neurosciences, biodynamics and networks and logic and computer science (see, *e.g.*, [30]). From a purely mathematical point of view, the analysis of SoS in terms of groupoids opens up the possibility of studying their topology formally (see [31]).

More generally, the theory of groupoids does not differ widely in spirit and aims from the theory of groups. The recognition of the utility of groupoids gives gains over the corresponding groups without any consequent loss. Our contention is that the above-described characteristics make groupoids an ideal candidate to fill in the symmetry roles that, we have argued, would help define SoS and their characteristics: groupoids provide us with a multi-object language defined over paths along with rules of variance and rules of transformation with which to study both internal and external symmetries. In other words, the language of groupoids gives us the required expressiveness and flexibility to represent hierarchies, emergence and evolution in SoS. In short, symmetry groupoids and symmetry groups follow the same principles. Depending on the system under scrutiny one or the other applies but there are not fundamental differences between them. That is, symmetries and the algebraic structures in which they are formalized provide us with a general, abstract framework as well as with a wide range of tools that fit any system (or SoS).

III. CONCLUSIONS

In this paper a formal approach to systems and SoS has been introduced. The underlying idea is that any system, simple or complex, small or large, conservative or dissipative, follows some version of the Principle of Least Action. As such, systems show symmetries that can we formalized using algebraic structures, which, in turn, shape their constitutive equations and their dynamics. The most important contribution of the paper is the realization that where as systems can be represented and analyzed in terms of symmetry groups, SoS, their inherent emergence and evolution, require more flexible tools –namely, groupoids and n-categories. We have illustrated these concepts using examples from various fields, physics, control, biology, chemistry and psychology. In the future, we plan to apply them rigorously to cyber-physical systems and socio-technical systems. Needles to say, the work presented is preliminary. Nevertheless, it is important to appreciate that without an analysis such as the one proposed in this paper, attempts to specify, design and implement SoS are futile.

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