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Valuation of guaranteed annuity conversion options*

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Abstract

In this note we introduce a theoretical model for the pricing and valuation of guaranteed annuity conversion options associated with certain deferred annuity pension-type contracts in the UK. The valuation approach is based on the similarity between the payoff structure of the contract and a call option written on a coupon-bearing bond. The model makes use of a one-factor Heath-Jarrow-Morton framework for the term structure of interest rates. Numerical results are investigated and the sensitivity of the price of the option to changes in the key parameters is also analyzed.

Keywords: guaranteed annuity option; risk neutral valuation; Heath-Jarrow-Morton model.

JEL Classification: G13, G23

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Subject codes: E50, E51

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1 Introduction

A guaranteed annuity option provides the holder of the contract the right to either receive at retirement a cash payment or receive an annuity which would be payable throughout his/her remaining lifetime and which is calculated at a guaranteed rate, depending on which has the greater value. This guarantee of the conversion rate between cash and pension income was a common feature of pension policies sold in the UK during the 1970s and 1980s. Thus, in a survey conducted by Bolton et al. (1997), annuity conversion guarantees were found to apply to just over 10% of the long term liabilities of the responding insurance companies.

Until recently, the cash benefit was more valuable than the guaranteed annuity payment since a higher pension could be obtained by applying the cash on the best annuity rates available in the market (the so-called "open market option"). After the reduction in the level of market interest rates over recent years, and particularly since 1998, the position has become reversed and the guaranteed annuity is now usually worth more than the cash benefit; improvements in mortality rates since these policies were issued have also made them more valuable to policyholders. As a result of these two combined effects, many insurance companies have experienced solvency problems requiring the setting up of extra reserves (using ad hoc methods) and leading one large life insurer (Equitable Life, the world's oldest life insurance company) to be closed to new business.

In this paper, we concentrate on unit-linked deferred annuity contracts purchased originally by a single premium. The pricing of options embedded in insurance contracts with guarantees has been addressed in the literature over the past 25 years. Thus, Brennan and Schwartz (1976) and Boyle and Schwartz (1977) analyzed unit-linked life insurance contracts with maturity guarantees using an approach centered on financial economics theory while the MGWP (1980) used a simulation-based methodology. More recently, Grosen and Jørgensen (2000) have analyzed with profit policies, allowing for the bonus guarantee and surrender option.

The approach advocated in this paper follows the above-cited literature and exploits the traditional option valuation procedure in order to provide indications in terms of pricing, reserving and hedging of the guaranteed annuity option contract. In this regard, our methodology differs from that proposed by Yang et al. (2001), who use a simulation-based asset model which is not arbitrage free. The approach also differs from that of Bezooyen et al. (1998), Pelsser (2002) and Yang et al. (2002) who each model the dynamics of the annuity price rather than modelling the term structure of interest rates. We believe that a methodology that begins with the term structure

of interest rates is more sound in that it allows the effect of changes in the term structure on the value of the guaranteed annuity option contract to be explored. At the time of writing, we have become aware of the work of Boyle and Hardy (2002) which follows a similar methodology to the one described below. Other work in this area has compared the guaranteed annuity option to swaption contracts: Lee (2001).

The option pricing approach we propose to valuation of these guarantees is based on the similarity between the payoff structure of the contract under consideration and a call option written on a coupon-bearing bond. The model makes use of a one-factor Heath-Jarrow-Morton framework for the term structure of interest rates. This choice is justified by the need to avoid dependence of the model on the market price of interest rate risk, which usually implies an arbitrary specification of the model's parameters leading to arbitrage opportunities (Heath, Jarrow and Morton, 1992). Also, single-factor models allow a mathematically tractable solution to the coupon bond option pricing problem (Jamshidian, 1989). We present two alternative formulations of the HJM framework based on different specifications for the forward rate volatility. The first relies on the assumption of constant volatility, while the second uses an exponentially decaying volatility structure, typical of the Vasicek (1977) class of models. Under the additional assumption of an unsystematic mortality risk, independent of the financial risk, a general pricing framework is proposed and closed analytical formulae for the value of the guaranteed annuity option are obtained. In both models, the pricing formulae derived implicitly contain the dynamic investment strategy that replicates the contract. Numerical results for both models are investigated and the sensitivity of the price of the option to changes in the key parameters is also analyzed.

The paper is organized as follows. Section 2 describes the financial model. Section 3 presents the guaranteed annuity option as a contingent claim. Section 4 considers in details two models for the dynamic of the term structure (in the HJM framework) and obtains closed form solutions for the value of the guaranteed annuity option at inception of the contract. In section 5, we present some numerical examples and sensitivity analysis results.

2 The financial model

Assume a frictionless market with continuous trading, no taxes, no transaction costs, no restrictions on borrowing or short sales and perfectly divisible securities. The insurance company invests the single premium paid by each policyholder in an equity fund, S, whose dynamic under the risk-neutral

equivalent martingale measure $\hat{\mathbb{P}}$ is described by the following.

$$dS_t = r_t S_t dt + \sigma_S S_t d\hat{Z}_t,$$

where $\left(\hat{Z}_t : t \geq 0\right)$ is a standard one-dimensional $\hat{\mathbb{P}}$-Brownian motion and $\sigma_S \in \mathbb{R}^+$. Assume also that the evolution of the forward rate is modeled in a single-factor Heath, Jarrow and Morton (1992) framework, that is

$$df\left(t,T\right) = \left(\sigma_f\left(t,T\right)\int_t^T \sigma_f\left(t,u\right)du\right)dt + \sigma_f\left(t,T\right)d\hat{W}_t, \tag{1}$$

where the volatility function $\sigma_f\left(t,T\right)$ is $\mathcal{F}_t$-adapted satisfying

$$\int_0^T \sigma_f^2\left(s,T\right)ds < \infty \qquad a.s.,$$

and $\left(\hat{W}_t : t \geq 0\right)$ is a standard one-dimensional $\hat{\mathbb{P}}$-Brownian motion correlated with $\hat{Z}$, so that

$$d\hat{W}_t d\hat{Z}_t = \rho dt,$$

for any $\rho \neq 0$. Hence

$$\hat{Z}_t = \rho\hat{W}_t + \sqrt{1-\rho^2}\hat{W}'_t,$$

where $\left(\hat{W}'_t : t \geq 0\right)$ is a $\hat{\mathbb{P}}$-Brownian motion independent of $\hat{W}_t$. Under these assumptions, the price of a zero-coupon bond with redemption date at T is

$$P_t\left(T\right) = e^{-\int_t^T f(t,u)du},$$

while the money market account is given by

$$B_t = e^{\int_0^t r_u du},$$

where $r_t := \lim_{T\to t} f\left(t,T\right)$ is the short rate.

Assume further that the mortality risk is independent of the financial risk and is unsystematic. Finally, let τ_x be a random variable which represents the remaining lifetime of the policyholder and which depends on the age, x, of the policyholder at the time of issue. The survival function of the random variable τ_x is given by

$${}_tp_x = \mathbb{P}\left(\tau_x > t\right).$$

3 The guaranteed annuity option

We consider now a guaranteed annuity option, which is a contract giving the holder the right to receive at retirement the greater of (a) a cash payment equal to the current value of the investment in the equity fund, S, and (b) the expected present value of the life annuity obtained by converting this investment at the guaranteed rate. In other words, if at inception the policyholder is aged x, and if N is the normal retirement age, then the guaranteed annuity option payoff at maturity is

$$C_T = \left(gS_T \sum_{t=0}^{w-(T+x)} {}_tp_{T+x} P_T\left(T+t\right) - S_T \right)^+,$$

where $T = N - x$ is the option lifetime, w is the largest survival age and g is the guaranteed annuity rate. Note that

$$C_T = gS_T \left(\sum_{t=0}^{w-(T+x)} {}_tp_{T+x} P_T\left(T+t\right) - K \right)^+,$$

where $K = 1/g$. This last equality shows the similarity between the payoff of the guaranteed annuity option and a call option written on a coupon bond, with "coupon dates" $T < T+1 < ... < w-x$. Applying the risk-neutral valuation procedure and bearing in mind that the mortality risk is assumed to be unsystematic and independent of the financial risk, the value of the contract entered at time $t = 0$ by a policyholder aged x is

$$\begin{aligned} V_x\left(x, t=0, T=N-x\right) &= \hat{\mathbb{E}}\left[B_T^{-1} C_T 1_{(\tau_x > T)}\right] \\ &= \hat{\mathbb{E}}\left[B_T^{-1} C_T\right] \hat{\mathbb{E}}\left[1_{(\tau_x > T)}\right] \\ &= \hat{\mathbb{E}}\left[B_T^{-1} C_T\right] \mathbb{E}\left[1_{(\tau_x > T)}\right] \\ &= {}_Tp_x C_0\,, \end{aligned}$$

where

$$C_0 = \hat{\mathbb{E}}\left[B_T^{-1} C_T\right].$$

Define a probability measure $\tilde{\mathbb{P}} \sim \hat{\mathbb{P}}$ by the density process (Geman, El Karoui, Rochet, 1995)

$$\eta_T := \left.\frac{d\tilde{\mathbb{P}}}{d\hat{\mathbb{P}}}\right|_{\mathcal{F}_T} = \frac{S_T}{S_0 B_T}. \tag{2}$$

Then

$$\begin{aligned} C_0 &= \hat{\mathbb{E}}\left[B_T^{-1} C_T\right] \\ &= \hat{\mathbb{E}}\left[B_T^{-1} g S_T \left(\sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T\left(T+t\right) - K\right)^+\right] \\ &= g\hat{\mathbb{E}}\left[\eta_T S_0 \left(\sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T\left(T+t\right) - K\right)^+\right] \\ &= g S_0 \hat{\mathbb{E}}\left[\eta_T\right] \tilde{\mathbb{E}}\left[\left(\sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T\left(T+t\right) - K\right)^+\right] \\ &= g S_0 \tilde{\mathbb{E}}\left[\left(\sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T\left(T+t\right) - K\right)^+\right], \end{aligned}$$

where $\tilde{\mathbb{E}}$ denotes the expectation under the probability measure $\tilde{\mathbb{P}}$ which takes the asset S as numeraire.

Since the proposed framework makes use of a single-factor model of interest rates and the mortality risk is assumed independent of the financial risk, the correspondence previously observed between the guaranteed annuity option and an option contract on a coupon bond suggests the possibility of following the approach introduced by Jamshidian (1989), and rewriting the annuity option payoff as the payoff generated by a portfolio of zero-coupon bond options with appropriate strike prices, K_t, and weights equal to the survival probabilities, ${}_t p_{T+x}$, for $t = 0, 1, ..., w-(T+x)$. In fact, since the bond price is a monotonic (decreasing) function of the interest rate, it is possible to find that critical value such that

$$\sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T\left(T+t\right) = K,$$

and define a new "artificial" strike price K_t as the bond price which is calculated to correspond to this critical interest rate level, that is

$$K_t = P_T^*\left(T+t\right).$$

From the relationship between interest rates and bond prices, it follows that

$$\left(\sum_{t=0}^{w-(T+x)} {}_t p_{T+x} P_T\left(T+t\right) - K\right)^+ = \sum_{t=0}^{w-(T+x)} {}_t p_{T+x} \left(P_T\left(T+t\right) - K_t\right)^+,$$

which implies that

$$\begin{aligned} C_0 &= gS_0\tilde{\mathbb{E}}\left[\left(\sum_{t=0}^{w-(T+x)} {}_tp_{T+x}P_T\left(T+t\right)-K\right)^+\right] \\ &= gS_0\sum_{t=0}^{w-(T+x)} {}_tp_{T+x}\tilde{\mathbb{E}}\left[\left(P_T\left(T+t\right)-K_t\right)^+\right]. \end{aligned} \tag{3}$$

To value this contingent claim, expression (3) demonstrates that we need to define the dynamic of the forward rate under the stock-risk-adjusted probability measure $\tilde{\mathbb{P}}$. We specify this dynamic and also illustrate how the abstract pricing procedure presented in this section works in practice with two concrete examples in the next section.

4 Term structure movements and option pricing

This section presents two examples to illustrate in details the valuation procedure introduced in section 3.

In the first example, we assume that the volatility of the forward rate process (1) is a positive constant, $\sigma_f\left(t,T\right)=\sigma_f\in\mathbb{R}^+$. This is a continuous time limit of Ho and Lee's (1986) model which may prove useful in practical applications due to its computational simplicity. However, according to this model, all rates fluctuate in the same way. Another related disadvantage is that this model has no mean-reversion. Therefore, in the second example, we use an exponentially decaying structure for the forward rate volatility. This leads to a governing process for the short rate resembling the Vasicek (1977) model.

4.1 Contingent claim valuation: constant volatility

If $\sigma_f\left(t,T\right)=\sigma_f>0$, the stochastic process for the forward rate under the risk-neutral equivalent martingale measure $\hat{\mathbb{P}}$ is described by

$$df\left(t,T\right)=\sigma_f^2\left(T-t\right)dt+\sigma_f d\hat{W}_t.$$

Equation (2) defines the equivalent martingale measure $\tilde{\mathbb{P}}$ through the following density process

$$\begin{aligned}\eta_T &= \frac{S_T}{S_0 B_T} = e^{-\frac{\sigma_S^2}{2}T+\sigma_S \hat{Z}_T} \\ &= e^{-\rho^2\frac{\sigma_S^2}{2}T-\left(1-\rho^2\right)\frac{\sigma_S^2}{2}T+\sigma_S\rho\hat{W}_T+\sigma_S\sqrt{1-\rho^2}\hat{W}'_t}.\end{aligned}$$

The multidimensional version of the Girsanov theorem implies that

$$\begin{aligned}\tilde{W}_t &: = \hat{W}_t - \rho\sigma_S t \\ \tilde{W}'_t &: = \hat{W}'_t - \sigma_S\sqrt{1-\rho^2}t\end{aligned}$$

are $\tilde{\mathbb{P}}$-standard Brownian motions. The new dynamic of the forward rate is then

$$df\left(t,T\right) = \left(\sigma_f^2\left(T-t\right)+\rho\sigma_f\sigma_S\right)dt + \sigma_f d\tilde{W}_t;$$

which implies that the price of a zero-coupon bond maturing at T is

$$P_t\left(T\right) = \frac{P_0\left(T\right)}{P_0\left(t\right)}e^{-\frac{\sigma_f^2}{2}Tt(T-t)-\rho\sigma_f\sigma_S(T-t)t-\sigma_f(T-t)\tilde{W}_t}.$$

The corresponding stochastic process for the short rate r_t is

$$r_t = f\left(0,t\right)+\frac{\sigma_f^2}{2}t^2+\rho\sigma_f\sigma_S t+\sigma_f\tilde{W}_t. \tag{4}$$

Therefore

$$\begin{aligned}P_t\left(T\right) &= \frac{P_0\left(T\right)}{P_0\left(t\right)}e^{-\frac{\sigma_f^2}{2}Tt(T-t)-\rho\sigma_f\sigma_S(T-t)t-\sigma_f(T-t)\tilde{W}_t} \\ &= \frac{P_0\left(T\right)}{P_0\left(t\right)}e^{-\frac{\sigma_f^2}{2}t(T-t)^2-(T-t)\left(\frac{\sigma_f^2}{2}t^2+\rho\sigma_f\sigma_S t+\sigma_f\tilde{W}_t\right)} \\ &= \frac{P_0\left(T\right)}{P_0\left(t\right)}e^{-\frac{\sigma_f^2}{2}t(T-t)^2-(T-t)(r_t-f(0,t))}.\end{aligned}$$

Let $\sigma_r^2\left(t\right) = \sigma_f^2 t$ be the variance of the short rate, then

$$P_t\left(T\right) = \frac{P_0\left(T\right)}{P_0\left(t\right)}e^{-\frac{1}{2}(T-t)^2\sigma_r^2(t)-(T-t)(r_t-f(0,t))}. \tag{5}$$

As equation (5) shows, the bond price is a monotonic function of the current short rate. Therefore it is possible to find that level r^* such that

$$\sum_{t=0}^{w-(T+x)} {}_tp_{T+x}\frac{P_0\left(T+t\right)}{P_0\left(T\right)}e^{-\frac{1}{2}t^2\sigma_r^2(T)-t(r_T-f(0,T))} = K, \tag{6}$$

and define the artificial strike price K_t as

$$K_t = \frac{P_0(T+t)}{P_0(T)} e^{-\frac{1}{2}t^2\sigma_r^2(T) - t\left(r_T^* - f(0,T)\right)}. \tag{7}$$

It is now possible to evaluate the expectation in equation (3) :

$$\tilde{\mathbb{E}}\left[\left(P_T(T+t) - K_t\right)^+\right].$$

Equation (5) implies that

$$\begin{aligned} & \tilde{\mathbb{E}}\left[\left(P_T(T+t) - K_t\right)^+\right] \\ = \; & \tilde{\mathbb{E}}\left[\left(\frac{P_0(T+t)}{P_0(T)} e^{-\frac{1}{2}t^2\sigma_r^2(T) - t(r_T - f(0,T))} - K_t\right)^+\right]. \end{aligned}$$

Since $(r_T - f(0,T)) \sim N\left(\frac{\sigma_f^2}{2}T^2 + \rho\sigma_f\sigma_S T, \sigma_f^2 T\right)$ under $\tilde{\mathbb{P}}$, if we set $m_r(T) = \frac{\sigma_f^2}{2}T^2 + \rho\sigma_f\sigma_S T$, then

$$\begin{aligned} & \tilde{\mathbb{E}}\left[\left(P_T(T+t) - K_t\right)^+\right] \\ = \; & \int_{\mathbb{R}} \left(\frac{P_0(T+t)}{P_0(T)} e^{-\frac{1}{2}t^2\sigma_r^2(T) - t(m_r(T) + \sigma_r(T)y)} - K_t\right)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy, \end{aligned}$$

where $y \sim N(0,1)$. The last equality is equivalent to

$$\begin{aligned} & \tilde{\mathbb{E}}\left[\left(P_T(T+t) - K_t\right)^+\right] \\ = \; & \int_{-\infty}^{d_t} \left(\frac{P_0(T+t)}{P_0(T)} e^{-\frac{1}{2}t^2\sigma_r^2(T) - t(m_r(T) + \sigma_r(T)y)} - K_t\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy, \end{aligned}$$

where

$$d_t = \frac{1}{t\sigma_r(T)}\left[\ln\frac{P_0(T+t)}{K_t P_0(T)} - \frac{1}{2}\sigma_r^2(T)\, t^2 - m_r(T)\, t\right]. \tag{8}$$

Hence

$$\begin{aligned} & \tilde{\mathbb{E}}\left[\left(P_T(T+t) - K_t\right)^+\right] \\ = \; & \frac{P_0(T+t)}{P_0(T)} e^{-\frac{1}{2}t^2\sigma_r^2(T) - t m_r(T)} \int_{-\infty}^{d_t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y^2 + 2t\sigma_r(T)y\right)} dy \\ & - K_t \int_{-\infty}^{d_t} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\ = \; & \frac{P_0(T+t)}{P_0(T)} e^{-\frac{1}{2}t^2\sigma_r^2(T) - t m_r(T)} e^{\frac{1}{2}t^2\sigma_r^2(T)} \int_{-\infty}^{d_t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y + t\sigma_r(T))^2} dy \\ & - K_t N(d_t) \\ = \; & \frac{P_0(T+t)}{P_0(T)} e^{-t m_r(T)} N(d_t') - K_t N(d_t), \end{aligned} \tag{9}$$

where $N(\cdot)$ is the cumulative distribution function of a standard Normal random variable and

$$\begin{aligned} d'_t &= d_t + t\sigma_r(T) \\ &= \frac{1}{t\sigma_r(T)}\left[\ln\frac{P_0(T+t)}{K_t P_0(T)} + \frac{1}{2}\sigma_r^2(T)\,t^2 - m_r(T)\,t\right]. \end{aligned}$$

Equations (3) and (9) imply that the value of C_0 is given by

$$\frac{gS_0}{P_0(T)}\sum_{t=0}^{w-(T+x)} {}_tp_{T+x}\left[P_0(T+t)\,e^{-tm_r(T)}N(d'_t) - P_0(T)\,K_t N(d_t)\right].$$

Therefore the guaranteed annuity option value at inception[1] is

$$\begin{aligned} & V_x(x, t=0, T=N-x) \\ = \; & \frac{{}_Tp_x gS_0}{P_0(T)}\sum_{t=0}^{w-(T+x)} {}_tp_{T+x}\left[P_0(T+t)\,e^{-tm_r(T)}N(d'_t) - P_0(T)\,K_t N(d_t)\right] \end{aligned} \tag{10}$$

with

$$d_t = \frac{1}{t\sigma_r(T)}\left[\ln\frac{P_0(T+t)}{K_t P_0(T)} - \frac{1}{2}\sigma_r^2(T)\,t^2 - m_r(T)\,t\right]$$

and

$$d'_t = d_t + t\sigma_r(T).$$

[1]Equation (10) can be simplified further substituting for K_t as in (7) into both (8) and (10). In fact

$$\begin{aligned} d_t &= \frac{1}{t\sigma_r(T)}\left[\ln\frac{P_0(T+t)}{K_t P_0(T)} - \frac{1}{2}\sigma_r^2(T)\,t^2 - m_r(T)\,t\right] \\ &= \frac{1}{t\sigma_r(T)}\left[t\left(r_T^* - f(0,T)\right) - m_r(T)\,t\right]. \end{aligned}$$

Since $\left(r_T^* - f(0,T)\right) \sim N\left(m_r(T), \sigma_r^2(T)\right)$, then

$$d_t = y^*,$$

where y^* is the value of the standard Normal random variable which solves (6). Hence

$$\begin{aligned} & V_x(x, t=0, T=N-x) \\ = \; & \frac{{}_Tp_x gS_0}{P_0(T)}\sum_{t=0}^{w-(T+x)} {}_tp_{T+x}P_0(T+t)\,e^{-tm_r(T)}\left[N(d'_t) - e^{-\frac{1}{2}\sigma_r^2(T)t^2 - t\sigma_r(T)y^*}N(y^*)\right] \end{aligned}$$

with

$$d'_t = y^* + t\sigma_r(T).$$

4.2 Contingent claim valuation: exponentially decaying volatility

In this second part, we assume that the volatility of the forward rate follows an exponentially decaying structure, that is

$$\sigma_f(t,T) = \sigma e^{-\lambda(T-t)},$$

where $\sigma > 0, \lambda > 0$. Hence, the forward rate dynamic is given by

$$df(t,T) = \left(\sigma^2 e^{-\lambda(T-t)} \int_t^T e^{-\lambda(u-t)} du\right) dt + \sigma e^{-\lambda(T-t)} d\hat{W}_t,$$

under $\hat{\mathbb{P}}$, while under $\tilde{\mathbb{P}}$ is

$$df(t,T) = \left(\sigma e^{-\lambda(T-t)} \left(\sigma \int_t^T e^{-\lambda(u-t)} du + \rho\sigma_S\right)\right) dt + \sigma e^{-\lambda(T-t)} d\tilde{W}_t. \quad (11)$$

Under these assumptions, it follows that the short rate process is

$$r_t = f(0,t) + \int_0^t \tilde{\mu}_f(v,t)\, dv + \sigma \int_0^t e^{-\lambda(t-v)} d\tilde{W}_v, \quad (12)$$

where

$$\begin{aligned} \tilde{\mu}_f(v,t) &= \sigma e^{-\lambda(t-v)} \left(\sigma \int_v^t e^{-\lambda(x-v)} dx + \rho\sigma_S\right) \\ &= \sigma e^{-\lambda(t-v)} \left[\frac{\sigma}{\lambda}\left(1 - e^{-\lambda(t-v)}\right) + \rho\sigma_S\right]. \end{aligned} \quad (13)$$

Therefore

$$r_t = f(0,t) + \left(1 - e^{-\lambda t}\right)\left[\frac{\sigma^2}{2\lambda^2}\left(1 - e^{-\lambda t}\right) + \frac{\rho\sigma\sigma_S}{\lambda}\right] + \sigma \int_0^t e^{-\lambda(t-v)} d\tilde{W}_v. \quad (14)$$

As the last equation shows, the exponentially decaying structure of the forward rate volatility leads to a mean-reverting form of the short rate that closely resembles an extended version of the Vasicek (1977) model.

Since

$$P_t(T) = e^{-\int_t^T f(t,u)du},$$

equations (11) and (13) imply that

$$P_t(T) = \frac{P_0(T)}{P_0(t)} e^{-\int_t^T \left(\int_0^t \tilde{\mu}_f(v,u)dv + \sigma \int_0^t e^{-\lambda(u-v)} d\tilde{W}_v\right) du}.$$

In particular

$$\begin{aligned}
&\int_t^T \left(\int_0^t \tilde{\mu}_f(v,u)\, dv + \sigma \int_0^t e^{-\lambda(u-v)} d\tilde{W}_v \right) du \\
= &\int_0^t dv \int_t^T \tilde{\mu}_f(v,u)\, du + \sigma \int_0^t d\tilde{W}_v \int_t^T e^{-\lambda(u-v)} du.
\end{aligned}$$

Notice that since $v \in [0,t]$,

$$\begin{aligned}
\int_t^T e^{-\lambda(u-v)} du &= \int_t^T e^{-\lambda(u-t)-\lambda(t-v)} du \\
&= e^{-\lambda(t-v)} \int_t^T e^{-\lambda(u-t)} du \\
&= e^{-\lambda(t-v)} \left(\frac{1 - e^{-\lambda(T-t)}}{\lambda} \right) \\
&= e^{-\lambda(t-v)} \gamma(t,T), \qquad (15)
\end{aligned}$$

where

$$\gamma(t,T) = \left(\frac{1 - e^{-\lambda(T-t)}}{\lambda} \right). \qquad (16)$$

Analogously, using equation (16) and following for example Jarrow and Turnbull (1994) and Chiarella and Kwon (2001), we get

$$\begin{aligned}
&\int_t^T e^{-\lambda(u-v)} \int_v^u e^{-\lambda(x-v)} dx du \\
= &\; e^{-\lambda(t-v)} \int_t^T e^{-\lambda(u-t)} \left(\int_v^t e^{-\lambda(x-v)} dx + \int_t^u e^{-\lambda(x-t)-\lambda(t-v)} dx \right) du \\
= &\; e^{-\lambda(t-v)} \int_t^T e^{-\lambda(u-t)} du \int_v^t e^{-\lambda(x-v)} dx + e^{-2\lambda(t-v)} \int_t^T e^{-\lambda(u-t)} \int_t^u e^{-\lambda(x-t)} dx du \\
= &\; \gamma(t,T)\, e^{-\lambda(t-v)} \int_v^t e^{-\lambda(x-v)} dx + e^{-2\lambda(t-v)} \int_t^T \gamma(t,u) \left(\frac{d}{du} \gamma(t,u) \right) du \\
= &\; \gamma(t,T)\, e^{-\lambda(t-v)} \int_v^t e^{-\lambda(x-v)} dx + \frac{1}{2} \gamma^2(t,T)\, e^{-2\lambda(t-v)}. \qquad (17)
\end{aligned}$$

Equations (13), (15) and (17) imply that

$$\begin{aligned}
&\int_t^T \left(\int_0^t \tilde{\mu}_f(v,u)\, dv + \sigma \int_0^t e^{-\lambda(u-v)} d\tilde{W}_v \right) du \\
= &\int_0^t dv \int_t^T \tilde{\mu}_f(v,u)\, du + \sigma \int_0^t d\tilde{W}_v \int_t^T e^{-\lambda(u-v)} du \\
= &\sigma \int_0^t dv \int_t^T e^{-\lambda(u-v)} \left(\sigma \int_v^u e^{-\lambda(x-v)} dx + \rho\sigma_S \right) du + \sigma\gamma(t,T) \int_0^t e^{-\lambda(t-v)} d\tilde{W}_v \\
= &\gamma(t,T) \int_0^t \tilde{\mu}_f(v,t)\, dv + \frac{1}{2}\gamma^2(t,T)\,\sigma^2 \int_0^t e^{-2\lambda(t-v)} dv + \sigma\gamma(t,T) \int_0^t e^{-\lambda(t-v)} d\tilde{W}_v \\
= &\gamma(t,T)\,(r_t - f(0,t)) + \frac{1}{2}\gamma^2(t,T)\,\sigma^2 \left(\frac{1-e^{-2\lambda t}}{2\lambda} \right),
\end{aligned}$$

where the last equality follows in virtue of (12). According to (14), under $\tilde{\mathbb{P}}$

$$(r_t - f(0,t)) \sim N\left(m_r(t), \sigma_r^2(t) \right)$$

where

$$\begin{aligned}
m_r(t) &= \left(1 - e^{-\lambda t}\right) \left[\frac{\sigma^2}{2\lambda^2} \left(1 - e^{-\lambda t}\right) + \frac{\rho\sigma\sigma_S}{\lambda} \right], \\
\sigma_r^2(t) &= \sigma^2 \left(\frac{1-e^{-2\lambda t}}{2\lambda} \right).
\end{aligned}$$

Therefore

$$P_t(T) = \frac{P_0(T)}{P_0(t)} e^{-\frac{1}{2}\gamma^2(t,T)\sigma_r^2(t) - \gamma(t,T)(r_t - f(0,t))}. \tag{18}$$

Although equations (14) and (18) are similar to the expressions derived by Vasicek (1977), they differ in the fact that they are obtained taking the initial term structure as exogenous, while for the Vasicek model the initial term structure is endogenous.

As in section 4.1, it is possible to find the critical value r^* such that

$$\sum_{t=0}^{w-(T+x)} {}_tp_{T+x} \frac{P_0(T+t)}{P_0(T)} e^{-\frac{1}{2}\gamma^2(T,T+t)\sigma_r^2(T) - \gamma(T,T+t)(r_T - f(0,T))} = K, \tag{19}$$

so that

$$K_t = \frac{P_0(T+t)}{P_0(T)} e^{-\frac{1}{2}\gamma^2(T,T+t)\sigma_r^2(T) - \gamma(T,T+t)\left(r_T^* - f(0,T)\right)}. \tag{20}$$

Hence, the expectation in equation (3) can be solved as follows.

$$\begin{aligned}
& \tilde{\mathbb{E}}\left[\left(P_T\left(T+t\right)-K_t\right)^+\right] \\
= & \tilde{\mathbb{E}}\left[\left(\frac{P_0\left(T+t\right)}{P_0\left(T\right)}e^{-\frac{1}{2}\gamma^2(T,T+t)\sigma_r^2(T)-\gamma(T,T+t)(r_T-f(0,T))}-K_t\right)^+\right] \\
= & \int_{-\infty}^{d_t}\left(\frac{P_0\left(T+t\right)}{P_0\left(T\right)}e^{-\frac{1}{2}\gamma^2(T,T+t)\sigma_r^2(T)-\gamma(T,T+t)(m_r(T)+\sigma_r(T)y)}-K_t\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy
\end{aligned}$$

where $y\sim N\left(0,1\right)$ and

$$d_t=\frac{\ln\frac{P_0(T+t)}{K_tP_0(T)}-\frac{1}{2}\sigma_r^2\left(T\right)\gamma^2\left(T,T+t\right)-m_r\left(T\right)\gamma\left(T,T+t\right)}{\gamma\left(T,T+t\right)\sigma_r\left(T\right)}.$$

Therefore

$$\begin{aligned}
& \tilde{\mathbb{E}}\left[\left(P_T\left(T+t\right)-K_t\right)^+\right] \\
= & \frac{P_0\left(T+t\right)}{P_0\left(T\right)}e^{-\frac{1}{2}\gamma^2(T,T+t)\sigma_r^2(T)-\gamma(T,T+t)m_r(T)}\int_{-\infty}^{d_t}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(y^2+2\gamma(T,T+t)\sigma_r(T)y\right)}dy \\
& -K_t\int_{-\infty}^{d_t}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\
= & \frac{P_0\left(T+t\right)}{P_0\left(T\right)}e^{-\gamma(T,T+t)m_r(T)}\int_{-\infty}^{d_t}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y+\gamma(T,T+t)\sigma_r(T))^2}dy \\
& -K_tN\left(d_t\right) \\
= & \frac{P_0\left(T+t\right)}{P_0\left(T\right)}e^{-\gamma(T,T+t)m_r(T)}N\left(d_t'\right)-K_tN\left(d_t\right),
\end{aligned}$$

where

$$\begin{aligned}
d_t' &= d_t+\gamma\left(T,T+t\right)\sigma_r\left(T\right) \\
&= \frac{\ln\frac{P_0(T+t)}{K_tP_0(T)}+\frac{1}{2}\sigma_r^2\left(T\right)\gamma^2\left(T,T+t\right)-\gamma\left(T,T+t\right)m_r\left(T\right)}{\gamma\left(T,T+t\right)\sigma_r\left(T\right)}.
\end{aligned}$$

Hence,

$$C_0=\frac{gS_0}{P_0\left(T\right)}\sum_{t=0}^{w-(T+x)}{}_tp_{T+x}\left[P_0\left(T+t\right)e^{-\gamma(T,T+t)m_r(T)}N\left(d_t'\right)-P_0\left(T\right)K_tN\left(d_t\right)\right],$$

while[2]

$$\begin{aligned} & V_x\left(x, t=0, T=N-x\right) \\ = \quad & \frac{{}_Tp_x g S_0}{P_0\left(T\right)} \sum_{t=0}^{w-(T+x)} {}_tp_{T+x} \left[P_0\left(T+t\right) e^{-\gamma(T,T+t) m_r(T)} N\left(d_t'\right) - P_0\left(T\right) K_t N\left(d_t\right)\right] \end{aligned} \tag{21}$$

with

$$d_t = \frac{\ln \frac{P_0(T+t)}{K_t P_0(T)} - \frac{1}{2}\sigma_r^2\left(T\right)\gamma^2\left(T, T+t\right) - \gamma\left(T, T+t\right) m_r\left(T\right)}{\gamma\left(T, T+t\right)\sigma_r\left(T\right)}$$

and

$$d_t' = d_t + \gamma\left(T, T+t\right)\sigma_r\left(T\right).$$

Equation (21) shows that the price of the guaranteed annuity option closely resembles the price of a bond option as in the standard Black-Scholes framework. However, the payoff of the guaranteed annuity option depends not only on the interest rate, likewise the bond option, but also on the dynamics of the equity fund. This last aspect is captured by the "correction factor" $m_r\left(T\right)\gamma\left(T, T+t\right)$. Analogous considerations hold for equation (10), which expresses the value of the contract for the constant volatility case and which can be derived also as a particular case of equation (21) for the limiting case $\lambda \to 0$.

The two pricing equations (10) and (21) also contain a first indication in terms of hedging strategy. In fact, according to the valuation formula, the guaranteed annuity option can be seen as a portfolio consisting of a long position in the $(T+t)$-zero coupon bond which has to be funded by a short position in the T-zero coupon bond.

[2]As in section 4.1, the pricing equation can be simplified further. If y^* is the value of the standard Normal random variable which solves (19), then equation (20) implies:

$$\begin{aligned} & V_x\left(x, t=0, T=N-x\right) \\ = \quad & \frac{{}_Tp_x g S_0}{P_0\left(T\right)} \sum_{t=0}^{w-(T+x)} {}_tp_{T+x} P_0\left(T+t\right) e^{-\gamma(T,T+t) m_r(T)} \left[N\left(d_t'\right) - e^{-\frac{1}{2}\sigma_r^2(T)\gamma(T,T+t)^2 - \gamma(T,T+t)\sigma_r(T) y^*} N\left(y^*\right)\right] \end{aligned}$$

with

$$d_t' = y^* + \gamma\left(T, T+t\right)\sigma_r\left(T\right).$$

4.3 Uncorrelated markets: the case of $\rho = 0$

As shown in section 3, the fair value of the guaranteed annuity option contract is

$$\begin{aligned} V_x(x, t=0, T=N-x) &= {}_Tp_x \hat{\mathbb{E}}\left[B_T^{-1} C_T\right] \\ &= {}_Tp_x g \hat{\mathbb{E}}\left[B_T^{-1} S_T \left(\sum_{t=0}^{w-(T+x)} {}_tp_{T+x} P_T(T+t) - K\right)^+\right], \end{aligned}$$

where $B_T = e^{\int_0^T r_t dt}$ is the money market account. Since

$$S_T = S_0 e^{\int_0^T r_t dt - \frac{\sigma_S^2}{2}T + \sigma_S \hat{Z}_T},$$

$$V_x(x, t=0, T=N-x) = {}_Tp_x g S_0 \hat{\mathbb{E}}\left[e^{-\frac{\sigma_S^2}{2}T + \sigma_S \hat{Z}_T} \left(\sum_{t=0}^{w-(T+x)} {}_tp_{T+x} P_T(T+t) - K\right)^+\right].$$

But $P_t(T) = e^{-\int_t^T f(t,u)du}$, i.e. the bond price depends only on the process W independent of Z when $\rho = 0$. Therefore

$$\begin{aligned} V_x(x, t=0, T=N-x) &= {}_Tp_x g S_0 \hat{\mathbb{E}}\left[e^{-\frac{\sigma_S^2}{2}T + \sigma_S \hat{Z}_T}\right] \hat{\mathbb{E}}\left[\left(\sum_{t=0}^{w-(T+x)} {}_tp_{T+x} P_T(T+t) - K\right)^+\right] \\ &= {}_Tp_x g S_0 \hat{\mathbb{E}}\left[\left(\sum_{t=0}^{w-(T+x)} {}_tp_{T+x} P_T(T+t) - K\right)^+\right] \\ &= {}_Tp_x g S_0 \sum_{t=0}^{w-(T+x)} {}_tp_{T+x} \hat{\mathbb{E}}\left[\left(P_T(T+t) - K_t\right)^+\right], \end{aligned}$$

where the last equality follows in virtue of Jamshidian's decomposition.

In other words, when $\rho = 0$, only the initial value of equity fund affects the contract price whilst its dynamics becomes irrelevant. For this reason the second change of measure is now unnecessary. Hence, the value of the guaranteed annuity option contract is given by

$$\begin{aligned} & V_x(x, t=0, T=N-x) \\ = \; & \frac{{}_Tp_x g S_0}{P_0(T)} \sum_{t=0}^{w-(T+x)} {}_tp_{T+x} \left[P_0(T+t) e^{-t m_r(T)} N(d_t') - P_0(T) K_t N(d_t)\right] \end{aligned} \tag{22}$$

with

$$\begin{aligned} d_t &= \frac{1}{t\sigma_r(T)}\left[\ln\frac{P_0(T+t)}{K_t P_0(T)} - \frac{1}{2}\sigma_r^2(T)\,t^2 - m_r(T)\,t\right], \\ d_t' &= d_t + t\sigma_r(T), \\ m_r(T) &= \frac{\sigma_f^2}{2}T^2, \end{aligned}$$

when we assume constant forward rate volatility, and

$$\begin{aligned} & V_x(x, t=0, T=N-x) \\ = \; & \frac{{}_Tp_x g S_0}{P_0(T)} \sum_{t=0}^{w-(T+x)} {}_tp_{T+x}\left[P_0(T+t)\,e^{-\gamma(T,T+t)m_r(T)}N(d_t') - P_0(T)\,K_t N(d_t)\right] \end{aligned} \tag{23}$$

with

$$\begin{aligned} d_t &= \frac{\ln\frac{P_0(T+t)}{K_t P_0(T)} - \frac{1}{2}\sigma_r^2(T)\,\gamma^2(T,T+t) - \gamma(T,T+t)\,m_r(T)}{\gamma(T,T+t)\,\sigma_r(T)}, \\ d_t' &= d_t + \gamma(T,T+t)\,\sigma_r(T), \\ m_r(T) &= \left(1-e^{-\lambda T}\right)\left[\frac{\sigma^2}{2\lambda^2}\left(1-e^{-\lambda T}\right)\right], \end{aligned}$$

when we assume instead an exponentially decaying structure for the volatility of the forward rate.

5 Numerical results and sensitivity analysis

The results obtained in the previous section have been used to study the behaviour of the guaranteed annuity option under different scenarios. Throughout the following analysis, unless otherwise stated, the basic set of parameters is

$$S_0 = 100; \quad \sigma_S = 0.2; \quad \rho = 1; \quad g = 0.111; \quad x = 50; \quad T + x = N = 65.$$

In particular, the choice of the parameter g follows the indication of Bolton et al. (1997) as the most common parameter value in the UK. As far as the volatility function of the forward rate is concerned, we fix $\sigma_f = 0.001$ for the constant volatility model (section 4.1), and $\sigma = 0.01$ and $\lambda = 0.15$ for the exponentially decaying volatility model (section 4.2). In order to compute the initial bond prices $P_0(T)$ and $P_0(T+t)$, $t = 0, ..., w-(T+x)$, a flat initial

term structure is assumed and fixed at 4%, i.e. $f(0,\cdot) = f_0 = 0.04$. Although the valuation procedure presented in section 3 makes allowance for stochastic mortality trends (provided that mortality risk is still assumed independent of the financial risk), we use UK mortality tables for the calculation of the survival probabilities. In particular, results are obtained for the PMA92-C20 mortality table and then extended also to the PA90 and PMA80-C10 mortality tables[3]. Finally, we assume that the annuity has a 5-year guarantee period (so that the first five annual payments of the annuity scheme would be definitely payable, providing that the policyholder survives to age 65).

Table 1 contains the extra premia that the insurer should charge at inception for the option embedded in the policy. In order to perform a sensible comparison between the two models, the table contains also the values corresponding to the exponentially decaying volatility parameters $\sigma = 0.001$ and $\lambda = 0.001$. In fact, as outlined in section 4.2, the constant volatility model can be retrieved from the exponentially decaying volatility one as the limiting case for $\lambda \to 0$. As we can see, for the most recent mortality table, the initial cost of the guaranteed annuity option is about 45% of the original single premium, S_0, paid by the policyholder at inception.

The sensitivity of the annuity option to the correlation coefficient between the equity fund and interest rates is shown in Figure 1 for the constant volatility case and in Figure 2 for the exponentially decaying model. An inspection of equations (4) and (5) (for the constant volatility model), and equations (14) and (18) (for the exponentially decaying volatility model) reveals that the expected present value of the annuity payment is a decreasing function of ρ. In fact, the so-called open market option, i.e. the opportunity to invest the cash amount at the current market rates, becomes more and more attractive the more that equity and interest rates move in the same direction. Hence, the patterns shown in Figures 1 and 2 of the value of the option decreasing as the correlation parameter, ρ, moves from -1 to 1 are as expected.

The same equations also show that the expected present value of the annuity payments is an increasing or decreasing function of the equity fund

[3]These mortality tables have been produced by the Continuous Mortality Investigation Bureau of the Institute and Faculty of Actuaries for insurance company data on male pensioner mortality. They are extensively used for the calculation of premiums and reserves. The PA90 table is based on data for the period 1967-70 projected to 1990, PMA80-C10 is based on data for the period 1979-82 projected to 2010 and PMA92-C20 is based on data for the period 1991-94 projected to 2020. Because of the declining trend in mortality rates over time, and hence the increasing trend in survival probabilities ${}_tp_{T+x}$, the expected present value of the life annuity increases as we move the assumption from PA90 to PMA80-C10 to PMA92-C20.

volatility according to whether ρ is negative or positive. When $\rho < 0$ and the volatility in the equity market is high, a decreasing trend in the interest rates might induce the policyholder to expect a corresponding increase in the equity value. This would make the fraction locked in by the guarantee and the guaranteed rate itself appear very attractive. On the other hand, if interest rates are increasing, the policyholder might prefer not to exercise the option. When ρ is positive, instead, the expectation of an increase in the equity value induced by increasing interest rates makes the open market option appear very competitive, and therefore the value of the guaranteed annuity option drops. This kind of behaviour is reflected in the patterns shown in Figure 3 and Figure 4.

The behaviour of the annuity option value for different values of the forward rate volatility for the constant volatility model, with all the other parameters left unchanged, is represented in Figure 5. The observed decreasing pattern finds a first explanation in equation (5), which shows that if the rates of interest are very volatile, the present value of the annuity payment falls. When ρ is close to its minimum value, this effect is offset by the attractiveness of the guarantee. Thus, as the forward rate volatility increases, the chance that interest rates will perform very well or very badly increases. The policyholder may benefit from falls in interest rates since the guaranteed payment appears more competitive; but the policyholder faces a limited downside risk in the event of a rise in interest rates because of the option scheme embedded in the contract. However, as the correlation between the equity market and interest rates becomes less and less negative, the decreasing effect induced by the drop in the expected present values of the annuity payments prevails. In fact, as Figure 5 shows, as ρ increases, the policyholder seems to prefer not to exercise the option, but to take the cash payment instead and reinvest it at more favorable conditions. Similar patterns apply for the other mortality tables; details are not presented here but are available from the authors. The changes in the value of the guaranteed annuity option under the exponentially decaying volatility model are summarized in Figure 6 and Figure 7. The sensitivity to the parameter σ (Figure 6) can be justified using the same argument as before. In fact, as equation (18) shows, the short rate volatility, σ_r, is an increasing function of σ, while the bond price, and therefore the expected present value of the annuity payments, is a decreasing function of σ_r. However, the volatility of the short rate is a monotonic decreasing function of the speed of adjustment, λ, and hence the pattern is inverted relative to Figure 6, as shown in Figure 7.

The behaviour of the contract for different ages of the policyholder at inception of the contract (for the PMA92-C20 mortality table) is analyzed

in Figure 8 for the constant volatility model and in Figure 9 for the exponentially decaying volatility model. We note that the observed patterns represent the dynamics over time of the guaranteed annuity option. In fact, by virtue of the no-arbitrage principle, if the contract had been tradable in the secondary market, its price should have been such that

$$V_x\left(x+t, t\in(0,T], \tau = T-t\right) = V_{x+t}\left(x+t, t\in(0,T], \tau = T-t\right).$$

In other words, the value at time t of the policy entered at age x by a policyholder now aged $x+t$ and with time to maturity $(T-t)$ would have been the same as the value of a policy entered at time t by a policyholder aged $x+t$ and with expiration date $\tau = T-t$. Both Figure 7 and Figure 9 show an increasing time evolution for the value of the guaranteed annuity option, which is mainly due to the time value of money: the later the policyholder enters the contract, the shorter is the time horizon over which the value of the annuity is discounted.

Figure 10 and Figure 11, instead, show a negative correlation between the annuity option value and the initial redemption yield: higher current interest rates make the guaranteed annuity payment locked in by the contract less attractive than the current rates available in the market.

This last analysis suggests the idea of tracking guaranteed annuity options values for contracts entered in 1970 using historical data of interest rates, and following their evolution over time up to the present day in order to analyze in more detail the effects produced on these contracts by the drop in both interest rates and mortality rates. In particular, for the initial term structure values, we used the annual average of retail bank's base rates over the past 32 years (Bank of England, September 2001 updated to February 2002). The T-year survival probability for an individual aged x (i.e. pre-retirement), ${}_Tp_x$, has been computed using a fixed mortality table, namely the AM92 mortality table. However, in order to take into account the improvements in mortality rates, post-retirement, the survival probabilities linked to each annuity payment due after maturity, ${}_tp_{T+x}$, are instead computed using the most up-to-date mortality table available for practical use at the moment at which the valuation is performed. If the option contract is evaluated during the period from 1970 to 1990, ${}_tp_{T+x}$ is calculated using the PA90 mortality table. The PMA80-C10 table was introduced in 1991 and is used here over the 1991-1999 valuation period, while the PMA92-C20 is used from year 2000 onward. In order to separate the "mortality effect" from the "interest rate effect", values have been computed also for the case of post-retirement survival probabilities obtained using the PA90 table only. The pattern of this contract for a policyholder aged 20 at inception in 1970 is shown in Figures 12

and 13 for the two volatility models considered. (The patterns for other ages at inception show similar results and are available from the authors). As we can see from these plots, the guaranteed annuity option contract had zero-value for most of its life, precisely from 1973 to 1992 when the level of interest rates was oscillating between 9.5% and 15%. As the rates dropped in 1993 to 5.50%, the option price rose to about 11.5%. We also note the effect induced on the contract's value by neglecting improvements in mortality rates: the results based on the PA90 table only, in fact, produce a relative underpricing of the guaranteed annuity option of about 60% in year 2002, with a peak of about 90% in year 1998. These results indicate the relative contributions made to the value of the guaranteed annuity option from falling interest rates and falling mortality rates (and anticipated future falls).

6 Conclusions

In this paper we have introduced a theoretical model, based on the one-factor Heath-Jarrow-Morton term structure framework, for the valuation of guaranteed annuity conversion options attached to single premium deferred annuity contracts. The approach depends on the correspondence between the contingent claim under consideration and an option contract written on a coupon paying bond. Two set of results are derived for the cases of (a) constant volatility and (b) of exponentially decaying volatility of the forward rate. Insurance company expenses, tax, profits and pre-retirement death benefits are ignored.

The model has been illustrated with numerical results and a sensitivity analysis. This indicates how the value of the annuity option varies with the key parameters, including the forward rate volatility, the equity fund volatility, the correlation coefficient between the equity and bond markets, mortality tables used in the calculation of the expected present value of the annuity payments, age at inception and initial term structure. In particular, we note the estimated value of the guarantee in relation to the single premium, $S_0 = 100$, and the effects of lower mortality rates on this estimated value. This evidence strongly suggests the need for a stochastic model for hazard rates; a first attempt based on the example of Milevsky and Promislow (2001) is currently in progress. We notice that the valuation framework proposed in this paper relies on a single-factor HJM model for the term structure of interest rates, and on the assumption of independence between the financial risk and the mortality risk for the Jamishidian (1989) decomposition to work. We also highlight that the model presented in this paper is based on a geometric Brownian motion-driven uncertainty in the equity

market. Limitations of this assumption are well known in literature since Black (1975); given the durations involved in the contract specification, a stochastic volatility model would be a more suitable choice.

Although pension contracts with guaranteed annuity conversion options may no longer be being issued (eg. in the UK), there remains a significant practical problem of estimating appropriate reserves for those contracts sold in the past and where the option has not yet been exercised (Bolton et al., 1997). Thus, we believe that results (10) and (21) will be of considerable assistance to insurance companies for estimating such reserves, and for reporting and regulatory purposes.

As we note above, equations (10) and (21) provide some guidance as to the theoretical hedging strategy which should be employed. We acknowledge that there are practical considerations to take into account, for example, the question of the availability of $(T+t)$-zero coupon bonds for such long maturities as the ones implied by the contract.

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	Constant volatility $\sigma_f = 0.001$	Exponentially decaying volatility (the benchmark case) $\sigma = 0.01, \lambda = 0.15$	Exponentially decaying volatility (the limiting case) $\sigma = 0.001, \lambda = 0.001$
PMA92-C20	45.8083	42.0175	45.8667
PMA80-C10	29.7541	26.5497	29.7971
PA90	16.3342	13.7925	16.3651

Table 1: Price of the guaranteed annuity option. Parameter set: $S_0 = 100; \sigma_S = 0.2; \rho = 1; g = 11.1\%; f_0 = 0.04; x = 50; N = 65$.

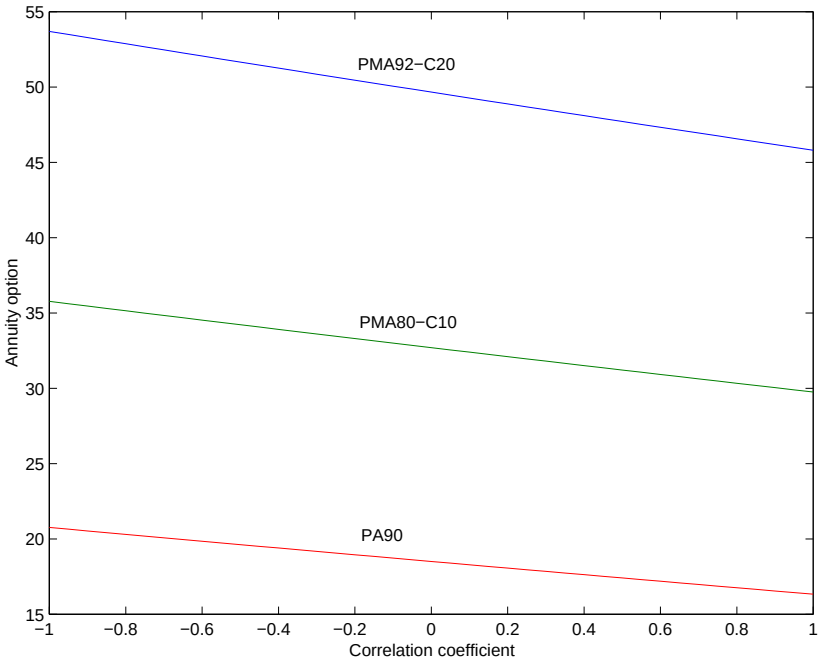

Figure 1: Sensitivity to the correlation coefficient: the constant volatility case.

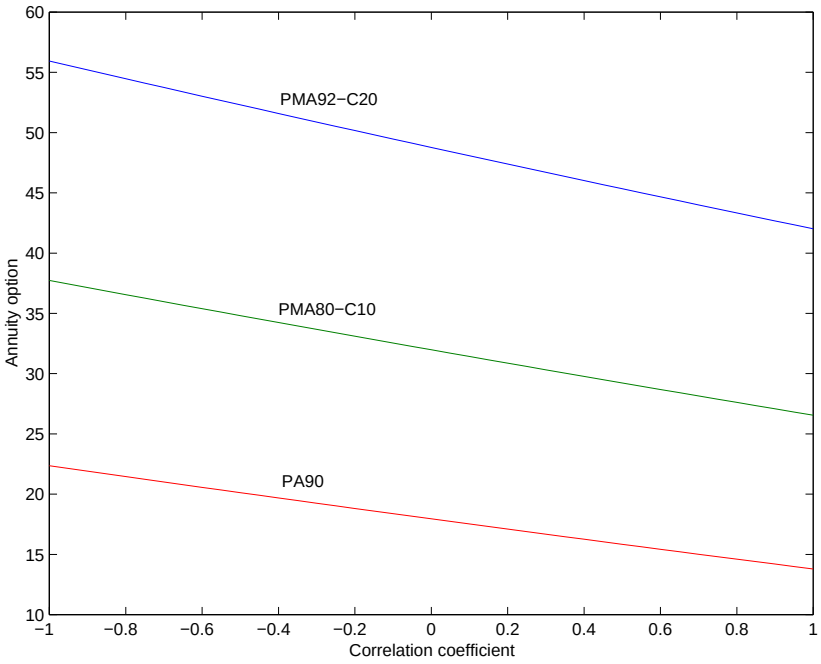

Figure 2: Sensitivity to the correlation coefficient: the exponentially decaying volatility model.

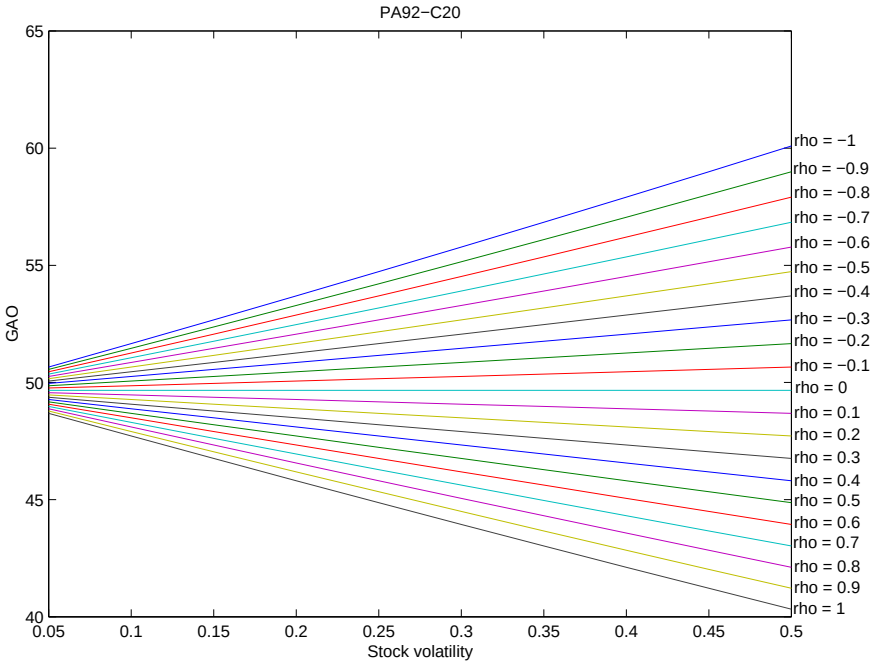

Figure 3: Sensitivity to the equity fund volatility: the constant volatility model.

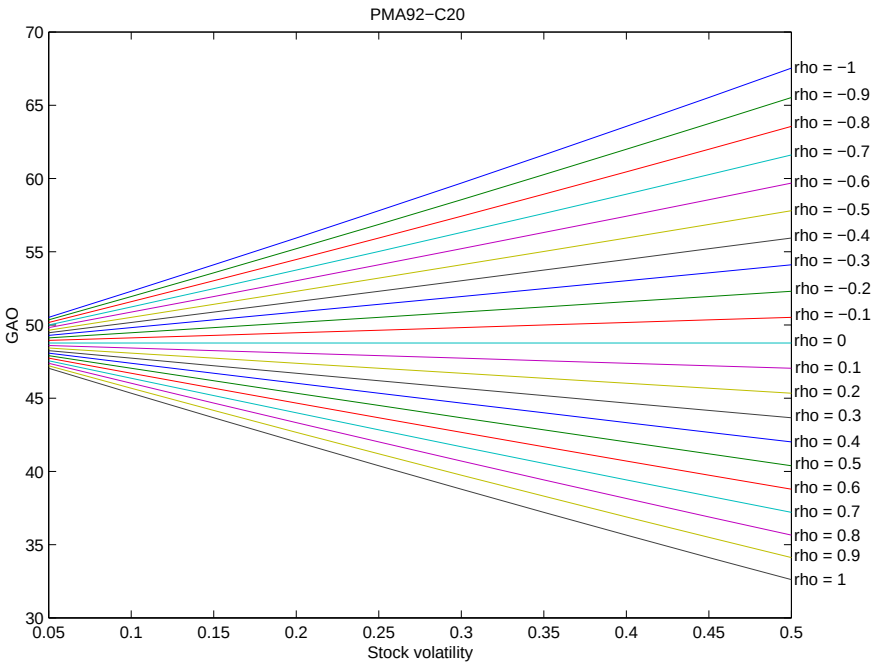

Figure 4: Sensitivity to the equity fund volatility: the exponentially decaying volatility model.

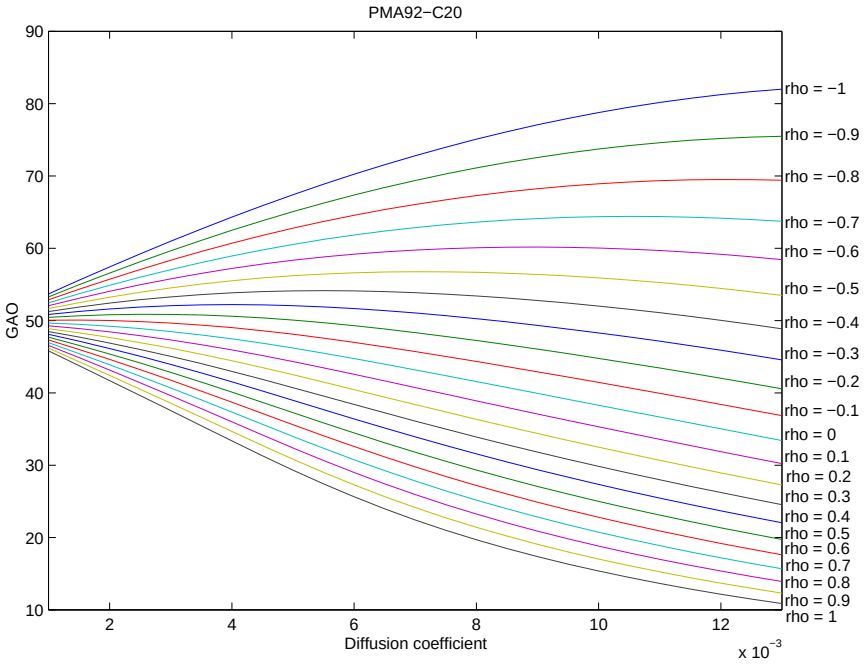

Figure 5: Sensitivity to the forward rate volatility: the constant volatility case.

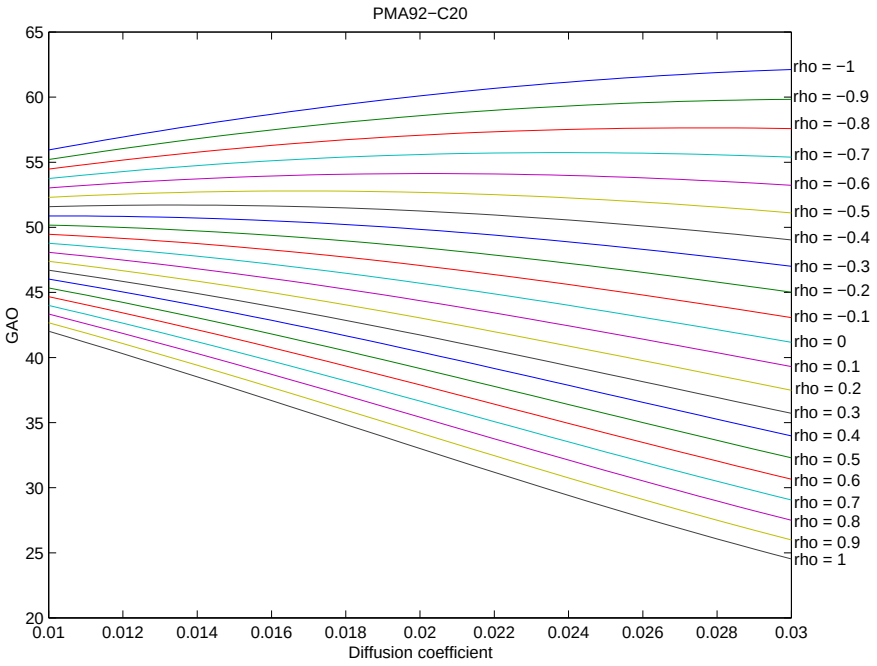

Figure 6: Exponentially decaying volatility model: guaranteed annuity option sensitivity to the diffusion term.

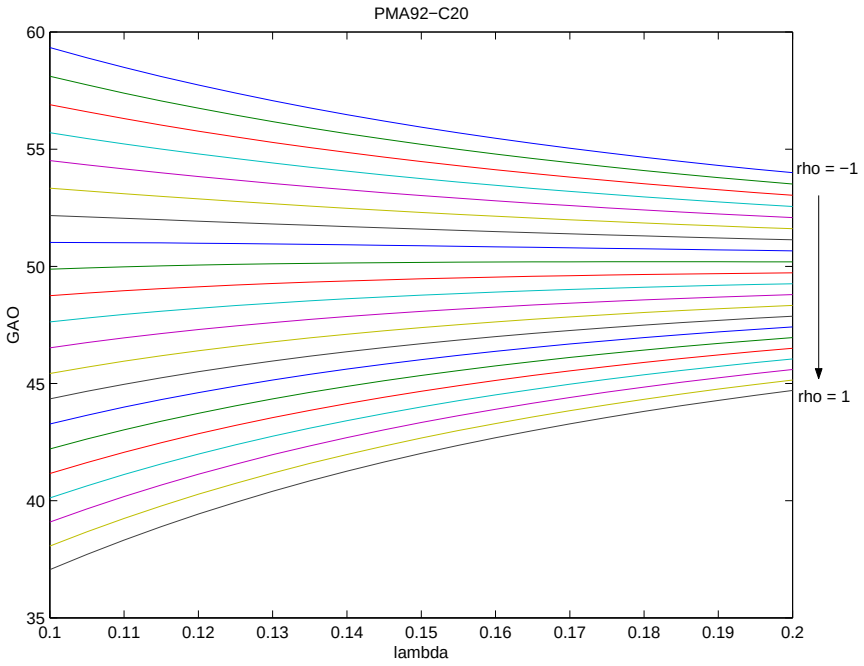

Figure 7: Exponentially decaying volatility model: guaranteed annuity option sensitivity to the speed of adjustment.

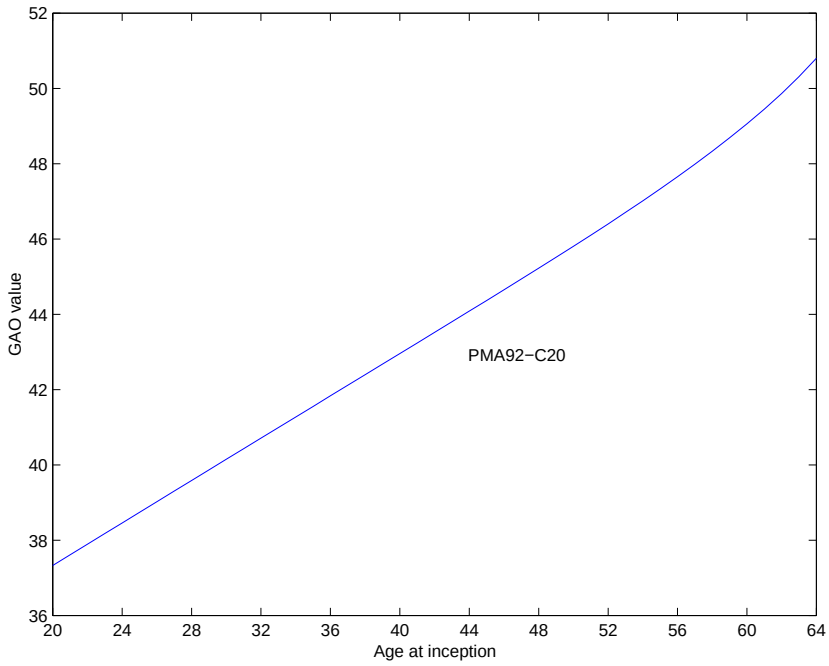

Figure 8: Time evolution of the guaranteed annuity option: the constant volatility model.

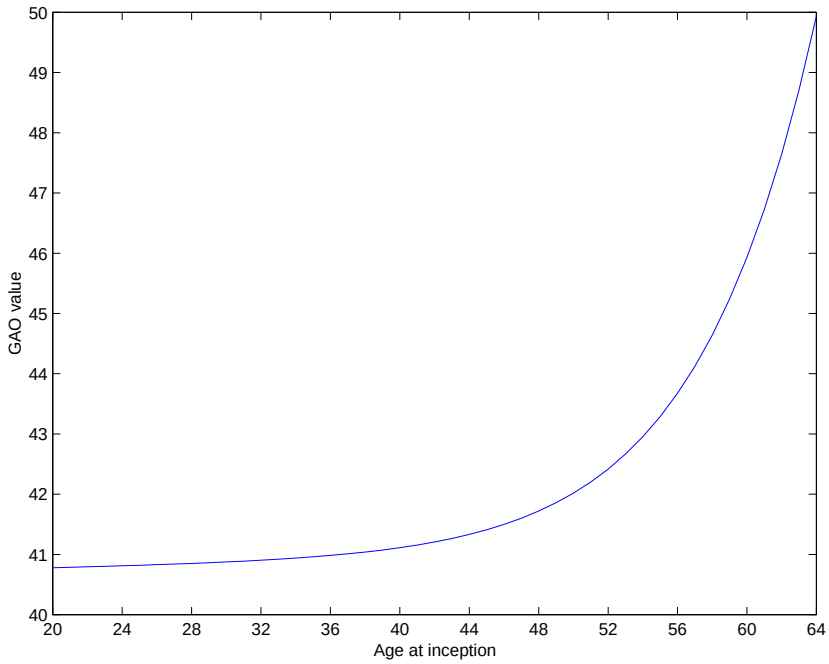

Figure 9: Time evolution of the guaranteed annuity option: the exponentially decaying volatility model.

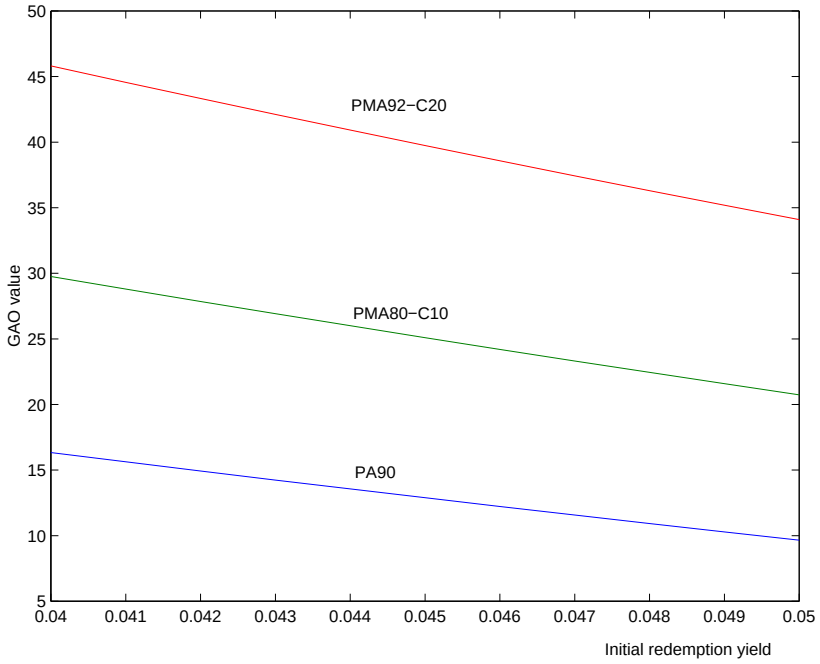

Figure 10: Sensitivity to the initial term structure: the constant volatility model.

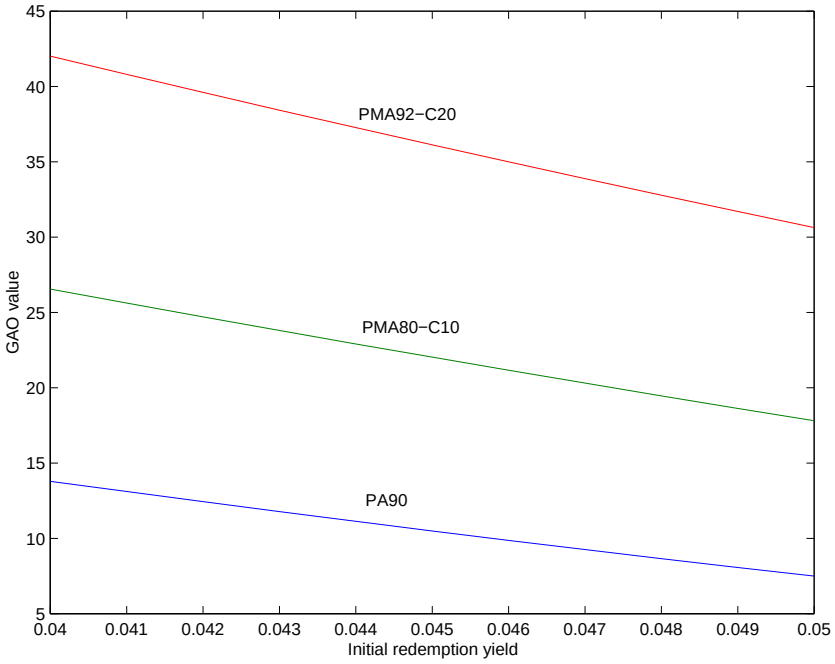

Figure 11: Sensitivity to the initial term structure: the exponentially decaying volatility model.

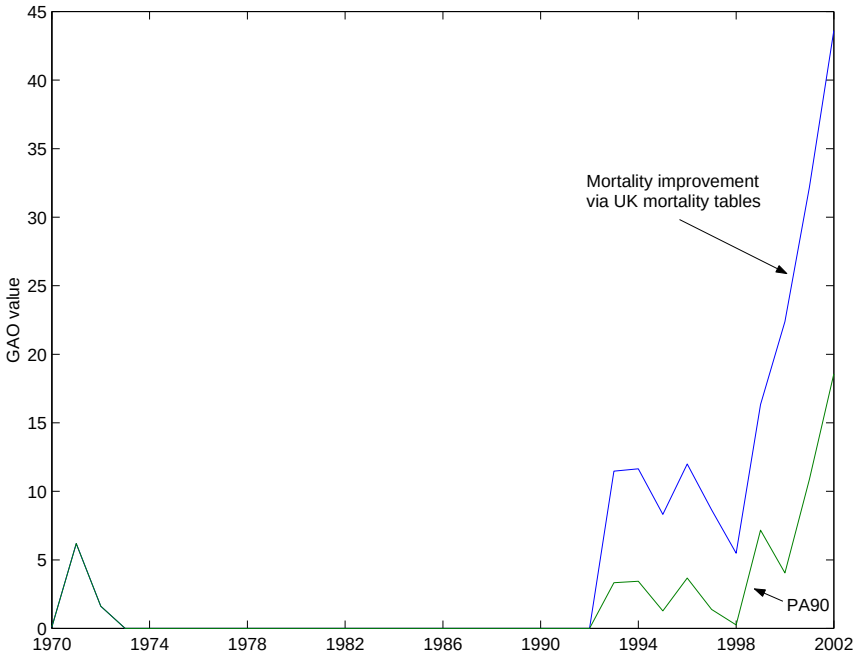

Figure 12: "Historical evolution" of the guaranteed annuity option from 1970 to the present day: the constant volatility model.

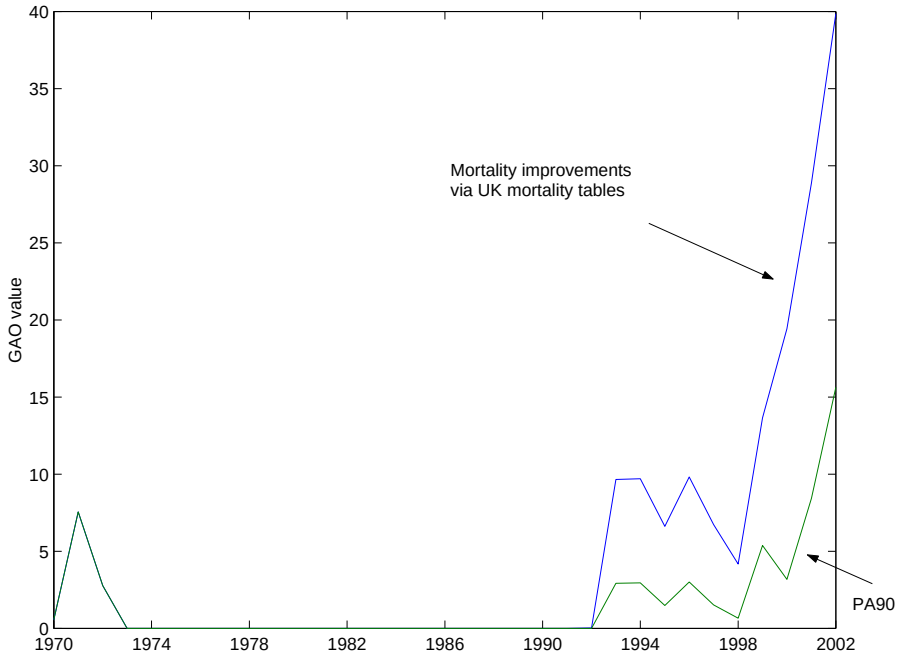

Figure 13: "Historical evolution" of the guaranteed annuity option from 1970 to the present day: the exponentially decaying volatility model.