Guarantees in with-profit and unitised with profit life insurance contracts: fair valuation problem in presence of the default option*

Laura Ballotta, Steven Haberman and Nan Wang
Faculty of Actuarial Science and Statistics, Cass Business School
City University London

Abstract

The purpose of the paper is to apply contingent claim theory to the valuation of the type of participating life insurance policies commonly sold in the UK. The paper extends the techniques developed by Haberman et al. (2003) to allow for the default option. The default option is a feature of the design of these policies, which recognizes that the insurance company's liability is limited by the market value of the reference portfolio of assets underlying the policies that have been sold. The valuation approach is based on the classical contingent claim pricing "machinery", underpinned by Monte Carlo techniques for the computation of fair values. The paper addresses in particular the issue of a fair contract design for a complex type of participating policy and analyzes in detail the feasible set of policy design parameters that would lead to a fair contract and the trade-offs between these parameters.

1 Introduction

Participating contracts of various types make up a significant part of the life insurance market of many countries including the US, Japan, Australia, Canada and several members of the European Union. The modelling, valuation and pricing of these contracts are important subjects for scientific

*The financial support from the Actuarial Research Club at Cass Business School and its corporate members is gratefully acknowledged.

analysis. This is because of the need by actuaries for appropriate and robust methods of internal financial risk management; the need by insurance companies to demonstrate solvency and the ability to pay claims (and hence benefits); the need to offer customers a "fair price" and be able to demonstrate this; and the need to measure profitability. The task, however, is made difficult by the nature of these life insurance contracts which incorporate a wide range of guarantees and option-like features.

In recent years, a series of studies have applied classical contingent claim theory, building on the pioneering work of Brennan and Schwartz (1976) on unit-linked policies, to different types of participating contracts: see, for example, Bacinello (2001, 2003), Grosen and Jørgensen (2000, 2001), Jensen et al. (2001), Miltersen and Persson (1999) and Persson and Aase (1997).

In this paper, our approach is to consider the most common policy design used in the UK for unitised with profit contracts and use classical contingent claim methodology to solve the valuation problem related to this contract. In particular, the policy design under examination incorporates both a reversionary-type bonus and a terminal bonus and hence differs from that considered by other authors. The paper is a companion to Haberman et al. (2003) but with the added feature of the default option which is discussed in more detail in section 3.

The paper is organised as follows. Section 2 describes the model and section 3 describes the valuation framework and the equilibrium condition for fair valuation, considering the separate claims of the policyholders and the equityholders. Section 4 provides the numerical results and section 5 provides some concluding comments.

2 Model of a life insurance company with participating contracts

The aim of this section is to set up a simple valuation model for the liabilities of a life insurance company implied by participating policies. Assume that at time $t = 0$ the insurance company acquires an asset portfolio A and finances this portfolio with the (single) premium, P_0, received from a policyholder and with paid-in capital, E_0, as illustrated in the company's balance sheet in Table 1. In return for the payment of the single premium, the policyholder is entitled to a fixed guaranteed benefit, together with the so-called reversionary bonus which is added periodically (i.e. a variable component reflecting the individual policyholder's smoothed share of the insurance company's profits), and a second variable component, which is based on the final surplus

	Assets	Liabilities	
At time 0			
	A_0	$L_0 = P_0 = \theta A_0$	single premium
		$E_0 = (1-\theta) A_0$	equity capital
	A_0	A_0	

	Assets	Liabilities	
At time t			
	$A(t)$	$L(t)$	policyholder claim
		$E(t)$	equity capital
	$A(t)$	$A(t)$	

Table 1: Balance sheet for the policy contract.

earned by the insurance company. We describe the guaranteed payoff and the reversionary bonus as constituting the policy reserve P, whilst the second variable part represents the so-called terminal bonus, R. All three components are payable at maturity or prior death of the policyholder. Depending on the policy design, elements of the three components could be payable if the policyholder surrenders the policy, prior to maturity. However, in this analysis we ignore both the risk from surrenders and from mortality.

An inspection of the "building blocks" of such a participating policy reveals that we can regard these contracts as a sequence of embedded options written on the asset portfolio A (which, from now, we call the reference portfolio), whose market value can be determined within the classical contingent claim pricing framework.

Consider a competitive market with continuous trading, and assume that the market is frictionless, i.e. that there are no taxes, no transaction costs, no restrictions on borrowing and short sales, and all securities are perfectly divisible. In such a context, let

A be the reference portfolio backing the policy; this is assumed to be 100% composed by equity and to follow the traditional geometric Brownian motion:

$$dA(t) = rA(t)\, dt + \sigma A(t)\, d\hat{W}(t),$$

under the risk-neutral probability measure $\hat{\mathbb{P}}$, where $r \in \mathbb{R}^+$ is the risk-free rate of interest and $\sigma \in \mathbb{R}^+$ is the portfolio volatility;

θ be the proportion of the initial reference portfolio financed by the policyholder, so that $(1-\theta)$ is the proportion of the portfolio financed by the equityholders; hence, θ is a cost allocation parameter. As we will note later, θ may be also interpreted as the leverage of the company, or a parameter governing the distribution of the financial risk among the policyholder and the equityholders;

r_G be the fixed guaranteed rate;

β be the participation rate of the policyholder in the returns generated by the reference portfolio;

$r_P(t)$ be the interest rate credited to the policyholder's account, including both the guaranteed benefit and the reversionary bonus;

γ be the terminal bonus rate (participation rate in the company's surplus at maturity);

T be the maturity date of the contract.

We now consider the participating contract in more detail. The policyholder enters the contract paying an initial single premium $P_0 = \theta A_0$, where A_0 is the initial value of the reference portfolio. In return, the policyholder receives a benefit, P, which accumulates at rate $r_P(t)$, so that

$$\begin{aligned} P(t) &= P(t-1)(1 + r_P(t)) \quad t = 1, 2, ...T, \\ P(0) &= P_0. \end{aligned} \tag{1}$$

As the reversionary bonus rate is usually determined by a smoothing adjustment to the rate of return on the policy's reference portfolio, we consider a scheme for the accumulation of the policy reserve based on the arithmetic average of the last τ year returns on the portfolio A, so that

$$\begin{aligned} r_P(t) &= \max\left\{ r_G, \frac{\beta}{n} \left(\frac{A(t)}{A(t-1)} + ... + \frac{A(t-n+1)}{A(t-n)} - n \right) \right\}, \\ n &= \min(t, \tau), \end{aligned} \tag{2}$$

where τ is the length of the averaging period (taken to be 3 years throughout this investigation)[1]. Thus, the rate credited to the policyholder account is the greater of the guaranteed rate, r_G, and the arithmetic average of the last τ period returns on the reference portfolio, multiplied by the participation rate β.

At the claim date of the contract, a terminal bonus is also paid based on the final surplus earned by the insurance company; this is defined as $\gamma R(T)$, where

$$R(T) = (\theta A(T) - P(T))^+ .$$

[1]Evidence from Needleman and Roff (1995), Chadburn (1998), and the recent Asset Share Survey by Tillinghast-Towers Perrin (2001) shows that the scheme under consideration is one of the smoothing mechanisms commonly used by insurance companies in the UK. Other possibilities include the geometric average of the last τ period returns on A, and schemes based on the concept of a smoothed asset share (see also Haberman et al., 2003, for further details).

If, at maturity, the insurance company is not capable of paying the policyholder's account, $P(T)$, then the policyholder takes those assets that are available, while the equityholders "walk away" empty-handed (i.e. the equityholders have limited liability).

3 Equilibrium condition for fair valuation

Given the contract specifications described in the previous section, we now turn our attention to the issue of determining the value of the life insurance policy in a market consistent manner, and in such a way that the contributions from the stakeholders of the life insurance company (i.e. the premium P_0 paid by the policyholder, and the equity capital E_0 provided by the equityholders) are fair with respect to the value of the benefits that they entitle the stakeholders to receive. By fair, we mean that these contributions do not leave room for arbitrage opportunities to arise. In other words, the policyholder receives value for the money paid in the form of the actual premium, but at the same time, the insurer is not offering the benefits too cheaply, compromising in this way the position of the equityholders.

These considerations imply that the two sides of the contract, i.e. the policyholder and the equityholders, need to be considered simultaneously for the market to be in equilibrium. Consequently, in the following section we describe the details of these two claims on the company's assets and derive a condition under which the market equilibrium is achieved with respect to both the policyholder and the equityholders.

3.1 Stakeholders' claims and no arbitrage in the market

This section analyzes the structure of the claims of the policyholder and the equityholders implied by the specification of the participating policy described in section 2 (in particular by equations 1 and 2), beginning with the specification of the liability of the company towards the policyholder.

- Policyholder's claim:

Based on the model introduced in the previous section, and referring in particular to Table 1, we consider the overall payoff that the policyholder is entitled to receive at the expiration of the contract. This is the liability, L, of the insurance company at maturity, which can be described as follows:

$$L(T) = \begin{cases} A(T) & \text{if } A(T) < P(T) \\ P(T) & \text{if } P(T) < A(T) < \frac{P(T)}{\theta} \\ P(T) + \gamma R(T) & \text{if } A(T) > \frac{P(T)}{\theta}. \end{cases}$$

In a more compact way, we can write this as:

$$L(T) = P(T) + \gamma R(T) - D(T),$$

where

$$D(T) = (P(T) - A(T))^+$$

represents the payoff of the so-called default option.

Applying risk-neutral valuation, it follows that the premium to be charged to the policyholder should satisfy the following condition in order to be fair, i.e. in order to clear any arbitrage opportunities from the market:

$$P_0 = e^{-rT}\hat{\mathbb{E}}\left[P(T) + \gamma R(T) - D(T)\right], \tag{3}$$

where $\hat{\mathbb{E}}$ denotes the expectation under the risk-neutral probability measure $\hat{\mathbb{P}}$. Alternatively

$$P_0 = e^{-rT}\hat{\mathbb{E}}\left[P(T)\right] + \gamma e^{-rT}\hat{\mathbb{E}}\left[R(T)\right] - e^{-rT}\hat{\mathbb{E}}\left[D(T)\right],$$

which we can write, in an obvious notation, as

$$P_0 = V_P(0) + \gamma V_R(0) - V_D(0). \tag{4}$$

- <u>Equityholders' claim</u>:

As shown in the balance sheet in Table 1, the equityholders at maturity can claim

$$E(T) = \begin{cases} 0 & \text{if } A(T) < P(T) \text{ (limited liability)} \\ A(T) - P(T) & \text{if } P(T) < A(T) < \frac{P(T)}{\theta} \\ A(T) - P(T) - \gamma R(T) & \text{if } A(T) > \frac{P(T)}{\theta}, \end{cases}$$

or

$$E(T) = (A(T) - P(T))^+ - \gamma R(T).$$

This implies that the fair contribution to the company's capital should satisfy the following:

$$\begin{aligned} E_0 &= e^{-rT}\hat{\mathbb{E}}\left[(A(T) - P(T))^+ - \gamma R(T)\right] \\ &= e^{-rT}\hat{\mathbb{E}}\left[(A(T) - P(T))^+\right] - \gamma V_R(0). \end{aligned}$$

Note that

$$P(T) - D(T) = A(T) - (A(T) - P(T))^{+}.$$

Hence,

$$E_0 = e^{-rT}\hat{\mathbb{E}}\left[A(T) - P(T) + D(T)\right] - \gamma V_R(0). \tag{5}$$

It follows that

$$(1-\theta)A(0) = E_0 = e^{-rT}\hat{\mathbb{E}}\left[A(T) - P(T) + D(T)\right] - \gamma V_R(0),$$

and therefore

$$P_0 = e^{-rT}\hat{\mathbb{E}}\left[P(T) - D(T)\right] + \gamma V_R(0).$$

In other words, it is sufficient that equation (3) is satisfied for the policy contract to be fair to both policyholders and equityholders.

3.2 The default option

In the previous section we introduced the default option with payoff

$$D(T) = (P(T) - A(T))^{+},$$

whose no-arbitrage value is given by:

$$V_D(0) = e^{-rT}\hat{\mathbb{E}}\left[(P(T) - A(T))^{+}\right], \tag{6}$$

where $\hat{\mathbb{E}}$ denotes the expectation under the risk neutral probability measure $\hat{\mathbb{P}}$.

Equation (6) highlights the point that the value of the default option corresponds to the so-called unconditional shortfall expectation (see, for example, Wirch and Hardy, 1999), the only difference being the probability measure which is used to quantify the probability of default. In fact, as equation (6) shows, in the case of the default option, calculations are carried out in the risk neutral world. However, the choice between risk neutral and real world probability of default depends on the purpose of the analysis: real world default probabilities should be used for scenario analyses to calculate potential future losses from defaults; whilst for the purpose of estimating the impact of default risk on the pricing of instruments, as in the case treated in this paper, risk neutral probabilities are the proper tool in order to carry out consistent valuations.

In the light of these considerations, the value V_D of the default option can be regarded as the price of an insurance contract which would cover the event of default. Maurer and Schlag (2002) observe that, if the life insurance company had the possibility of transferring the default risk to a reinsurance company, the shortfall expectation could be seen as an important element of the appropriate reinsurance premium. Equations (3) and (5) show that this premium should be funded by the equityholders and not charged to the policyholder.

Equations (6) also shows that the parameters affecting the value of the default option, V_D, are the contract parameters $(\theta, r_G, \beta, \tau)$, to the extent that these parameters affect the policy reserve, P, and the market parameters (σ, r). Given the structure of the contract under examination, the effect of the market parameters is partially moderated by the length of the averaging period, τ, the participation rate, β, but mainly by the parameter θ. As mentioned in section 2, θ determines the allocation of the premium cost among the two parties involved, the policyholder and the equityholders, and in this sense it could be considered as a wealth distribution coefficient. On the other hand, θ also determines the size of the debt in the company's balance sheet. The contribution coming from the policyholder could be regarded as a long term loan, on which the company pays interest by crediting each year the rate $r_P(t)$ to the policy reserve. The bigger the size of the loan, the more resources the company has available to fund its investment strategies. However, a bigger loan also means that more interest needs to be paid and a larger sum needs to be repaid at maturity. This implies a higher chance of default. In this sense, the parameter θ can be regarded as a leverage coefficient.

These considerations suggest that the default option value, V_D, captures the riskiness of the contract in terms of both market risk and credit risk. By market risk, we mean the risk arising from the possibility that market variables will move in such a way that the value of a contract to the financial institution, i.e. the insurance company, becomes negative. By credit risk, instead, we mean the risk that a loss will be experienced (by the policyholder) because of a default by the counterparty (the life insurance company) in a derivatives (the sequence of options embedded in the policy) transaction.

3.3 Further analysis of the policy components

From equation (1), it follows that the policy reserve can be expressed as

$$\begin{aligned} P(T) &= P_0 \prod_{t=1}^{T} (1 + r_P(t)) \\ &= \theta A_0 \prod_{t=1}^{T} (1 + r_P(t)) = \theta P^U(T), \end{aligned} \tag{7}$$

where $P^U(T)$ is the "unlevered" policy reserve. By analogy, if $R^U(T)$ denotes the "unlevered" terminal bonus, then

$$\begin{aligned} R(T) &= (\theta A(T) - P(T))^+ \\ &= \left(\theta A(T) - \theta A_0 \prod_{t=1}^{T} (1 + r_P(t))\right)^+ = \theta \left(A(T) - P^U(T)\right)^+; \\ &= \theta R^U(T); \end{aligned} \tag{8}$$

whilst the default option payoff can be rewritten as

$$\begin{aligned} D(T) &= (P(T) - A(T))^+ \\ &= \left(\theta A_0 \prod_{t=1}^{T} (1 + r_P(t)) - A(T)\right)^+ \\ &= \theta \left(P^U(T) - \frac{A(T)}{\theta}\right)^+. \end{aligned} \tag{9}$$

The three payoff equations $(7)-(9)$ show that adding the leverage feature to the model affects the structure of the default option, while the policy reserve and the terminal bonus are unaffected, as the leverage coefficient acts only as a rescaling factor.

This implies that the equilibrium condition (4) can be rewritten as:

$$\begin{aligned} P_0 &= e^{-rT}\hat{\mathbb{E}}[P(T)] + \gamma e^{-rT}\hat{\mathbb{E}}[R(T)] - e^{-rT}\hat{\mathbb{E}}[D(T)] \\ &= \theta e^{-rT}\hat{\mathbb{E}}\left[P^U(T)\right] + \gamma\theta e^{-rT}\hat{\mathbb{E}}\left[R^U(T)\right] - \theta e^{-rT}\hat{\mathbb{E}}\left[\left(P^U(T) - \frac{A(T)}{\theta}\right)^+\right] \\ &= \theta\left[V_P^U(0) + \gamma V_R^U(0) - e^{-rT}\hat{\mathbb{E}}\left[\left(P^U(T) - \frac{A(T)}{\theta}\right)^+\right]\right], \end{aligned}$$

where $V_P^U(0)$ is the fair value of the "unlevered" policy reserve and $V_R^U(0)$ is the fair value of the "unlevered" terminal bonus.

It follows that

$$A_0 = V_P^U(0) + \gamma V_R^U(0) - e^{-rT}\hat{\mathbb{E}}\left[\left(P^U(T) - \frac{A(T)}{\theta}\right)^+\right].$$

It is clear that P_0 increases as each of β, γ, θ and r_G increase. Moreover

$$\gamma = \frac{A_0 - V_P^U(0) + e^{-rT}\hat{\mathbb{E}}\left[\left(P^U(T) - \frac{A(T)}{\theta}\right)^+\right]}{V_R^U(0)}. \tag{10}$$

Note that, if $\theta = 1$ (so that there is no contribution from the shareholders), then

$$\begin{aligned}\gamma &= \frac{A_0 - V_P(0) + e^{-rT}\hat{\mathbb{E}}\left[(P(T) - A(T))^+\right]}{V_R(0)} \\ &= 1, \end{aligned} \tag{11}$$

since, as we observed before,

$$P(T) - D(T) = A(T) - (A(T) - P(T))^+ .$$

The result expressed in equation (11) can be interpreted as a "wealth distribution effect": if the policyholders are the only group contributing to the financing of the portfolio backing the policy (i.e. if $\theta = 100\%$), they have the right to receive the entire surplus of the company, and therefore they will fix the terminal bonus rate at its maximum value ($\gamma = 100\%$). The same feature has been observed in a similar context by Grosen and Jørgensen (2002). Equation (10) also shows that γ increases as θ increases.

Finally we note that any insurer's action on the leverage coefficient θ affects only the way in which the financial risk is redistributed between the company's stakeholders. As the market becomes more volatile, in fact, the insurance company might decide to change the exposure of new (and/or existing) policyholders to the higher volatility of the reference portfolio by changing the value of the parameter θ. This can be considered as a sort of asset substitution effect which, however, does not affect the riskiness of the portfolio backing the participating contract.

4 Numerical results

The path dependency implied by the structure of the reversionary bonus, as described by equations (1) and (2), and the complexity of the components of equation (3) mean that the investigation of the combination of parameters satisfying the equilibrium condition (3) needs to be carried out making use of numerical procedures. In particular, we use Monte Carlo techniques for the estimation of the values of the policy components, i.e. $V_P(0)$, $V_R(0)$ and $V_D(0)$. We implement the antithetic variates method for variance reduction purposes. The solutions to the equilibrium condition are then sought using the bisection method.

The Monte Carlo experiment is carried out by simulating 500,000 paths, with each path composed by 20 steps, equivalent to one observation per year over the life span of the contract (this is justified by the fact that the rate of return on the policy reserve, $r_P(t)$, is credited once a year). Unless otherwise stated, the base set of parameters is:

$$A_0 = 100; \quad r_G = 4\%; \quad \theta = 0.75; \quad \gamma = 0.7; \quad r = 6\%; \quad \sigma = 0.15; \quad T = 20 \text{ years}.$$

In the following, we adopt a comparative statics approach and consider the feasible fair combinations of parameters, where these are grouped two by two, leaving all of the others fixed at their base values as stated above. We first analyze how the contract parameters γ, r_G and θ need to be readjusted for the equilibrium condition to hold when the volatility of the reference portfolio is allowed to change. Then, we study the impact on the terminal bonus rate, γ, and on the minimum guarantee, r_G, of a different allocation of the financing of the reference portfolio between the policyholder and equityholders. Finally, we look at the trade-off between the guaranteed rate and the terminal bonus rate.

We recognize that the parameter θ would be fixed at the start of an individual contract. Also there would be an expectation (in the UK), among policyholders, that the design parameters β and r_G would not be altered often and, if so, only gradually during the lifetime of a contract. However, it would be common for the value of γ not to be guaranteed and indeed for its value only to become known near to the maturity of an individual contract.

Finally, we observe that part of the analyses we present in the next sections (precisely, sections 4.1, 4.2 and 4.4) have been already considered by other authors like Briys and De Varenne (1994) and Grosen and Jørgensen (2002). However, our results extend their findings as these authors do not include the variable part of the guaranteed benefit in their model of the policy reserve (i.e. they assume $\beta = 0$).

4.1 Impact of fund volatility on terminal bonus rates

Figure 1 and Figure 2 show the possible combinations of the fund's volatility, σ, and terminal bonus rate, γ, for which the equilibrium condition (3) is satisfied. In particular, Figure 1 depicts isopremium curves corresponding to different levels of the cost allocation coefficient, θ; Figure 2, instead, considers the case of isopremium curves corresponding to different levels of the guaranteed rate, r_G.

Both figures show that, if $\theta < 1$, as the fund volatility increases, the participation rate for the terminal bonus, γ, has to be reduced for the equilibrium condition to hold. However, for low values of the participation rate β, we observe that the curves have a U-shape, which disappears in the plots corresponding to higher values of β. From the equilibrium condition (4), it

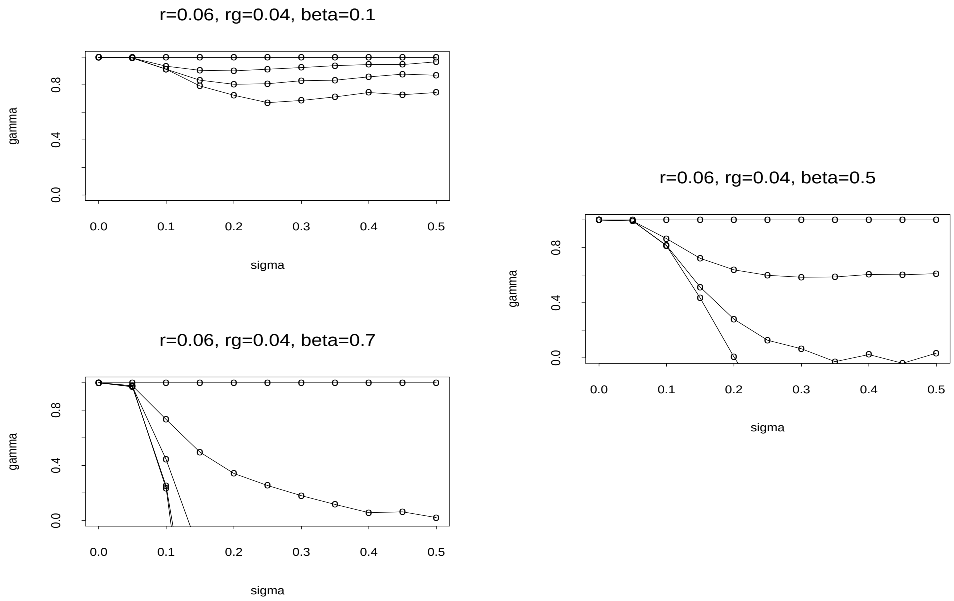

Figure 1: Isopremium curves over the range of possible values of σ and γ. The curves in each panel correspond to the values of θ equal to 0.25, 0.50, 0.75 and 1 from bottom to top.

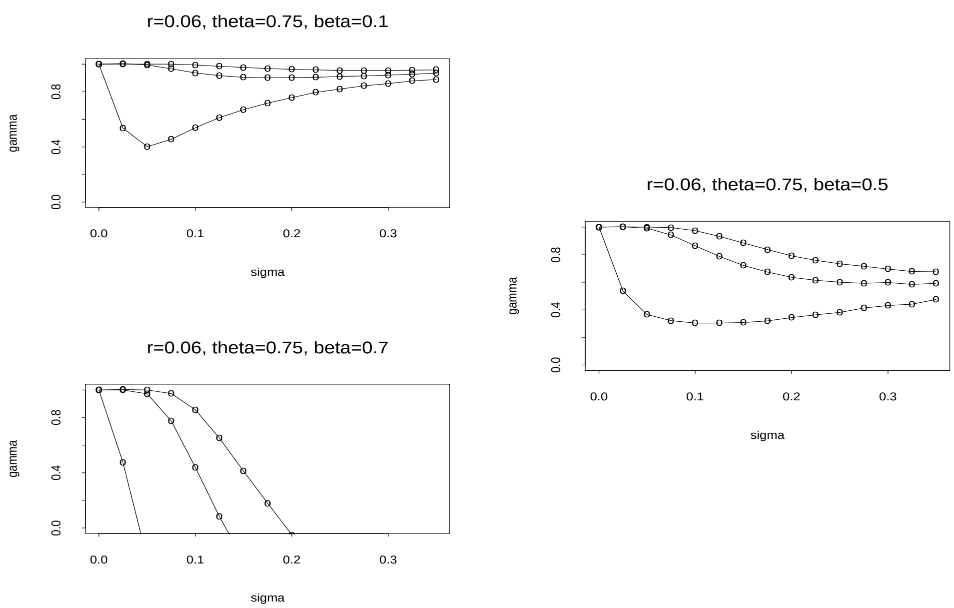

Figure 2: Isopremium curves over the range of possible values of σ and γ. The curves in each panel correspond to the values of r_G equal to 2%, 4% and 6% from top to bottom.

follows that:

$$\gamma = \frac{P_0 - V_P(0) + V_D(0)}{V_R(0)} \tag{12}$$

When σ increases, the value of both the policy reserve and the default option increases. The first one tends to reduce the numerator in equation (12) and hence γ, while the second tends to increase it. We note that the requirement that $\gamma > 0$ implies that $P_0 + V_D(0) > V_P(0)$. Hence, $V_P(0)$ cannot increase too much more than $V_D(0)$ does. For low β, the value at inception of the terminal bonus is an increasing function of σ (see Haberman et al., 2003); in this case, the default option looks like a vanilla put option as the policy reserve is not very sensitive to the fund volatility and its size is not very large, relatively. This mix determines the U-shape observed in the top panel of both Figure 1 and Figure 2. For high β, the value of the terminal bonus at inception decreases as the fund becomes riskier, as observed in Haberman et al. (2003); despite this effect, the bottom panels of Figures 1 and 2 show that γ decreases. This suggests that, as β is increased, the policy reserve becomes the predominant term in equation (12). It is as if the policy reserve "consumes" more and more from the reference portfolio; the default risk increases as well but the participation in the surplus (the terminal bonus rate) has to be cut in order to keep the contract feasible. This explanation is confirmed by the detailed plots in Figure 3, which depict the behaviour of both the fair values of the policy reserve and the default option as σ varies.

From Figure 1, we can also observe that the rate of change in γ increases as θ is reduced (the curves become steeper). This effect is due to a "wealth distribution effect", in the sense that when θ becomes smaller, the equity-holders become more dominant, and so they might need to reduce γ even more in order to preserve the value of their share of the portfolio, i.e. to maintain the equilibrium as described by equation (5). In fact, reductions in θ make the conditions for redistributing the terminal bonus very difficult to achieve. According to our model, the terminal bonus can be distributed when $A_T > P_T/\theta$. As θ decreases, this condition becomes very strong. This is particularly true when β is high, as A and P become similar in terms of their respective risk profiles. We recall from Haberman et al. (2003) that the value at inception of the terminal bonus may be considered to be the premium for the probability mass in the right tail. Clearly, when θ decreases, the barrier to be hit moves to the right, thereby reducing the probability mass.

Figure 2 highlights the peculiar behaviour of the isopremium curves corresponding to a guaranteed rate equal to 6%, i.e. the assumed risk-free rate

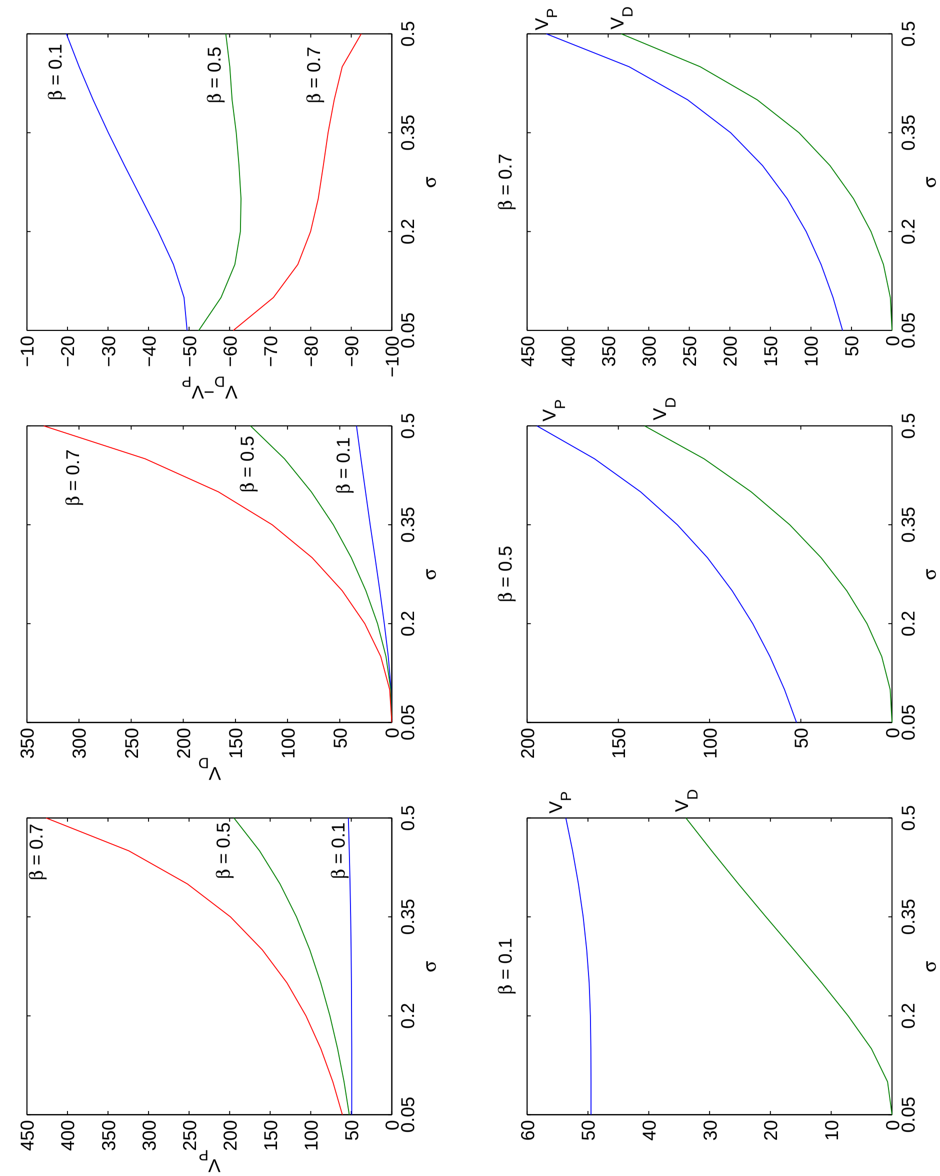

Figure 3: Sensitivity analysis for the fair value of the policy reserve, V_P, and the default option V_D, with respect to the underlying asset's volatility. Design parameters set: $r_G = 4\%$, $\theta = 0.75$, $r = 6\%$.

of interest[2]. We observe, in fact, that for this level of r_G, the above mentioned U-shape of the curves for $\beta = 0.1$ and $\beta = 0.5$ is even more accentuated. For $\beta = 0.7$, again it disappears. The detailed plots show that, for low values of the participation rate β, combined with high guaranteed rates, r_G, there exists a critical value of σ, say $\bar{\sigma}$, such that, for $\sigma < \bar{\sigma}$, any marginal increase in the volatility has an impact on the value of the terminal bonus V_R (see also Figure 8 in section 4.3), such that the terminal bonus rate γ needs to be reduced in order to preserve the fairness of the contract. For $\sigma > \bar{\sigma}$, instead, the risk of default, as measured by V_D, has a magnitude such that the life insurance company has to increase the participation rate in the final surplus, γ, in order to justify the same level of the premium (note that in this case θ is fixed at 75%), for a contract in which the policy reserve accumulates almost at the risk-free rate but that has a non-zero probability of default attached. It should be observed, in fact, that for a low participation rate β, very little of the fund volatility is inherited by the policy reserve which, consequently, accumulates almost at the fixed rate r_G (see Figure 8 in section 4.3).

4.2 Impact of fund volatility on the guaranteed rate

In Figure 4, we represent isopremium curves corresponding to different combinations of σ and r_G for fixed values of the cost allocation coefficient θ. Figure 5 depicts the same scenario but for different choices of the terminal bonus rate γ.

As both Figures show, when β is fixed at low values, there exists a $\hat{\sigma}$, such that, for $\sigma < \hat{\sigma}$, a decrease in σ requires a higher r_G for the premium to be fair compared to the benefits being promised. For $\sigma > \hat{\sigma}$, instead, the pattern is reversed, in the sense that an increase in σ requires a higher r_G. On the other hand, for high values of β, we observe that in both figures there is a change of concavity and an inversion of the trend, in the sense that now r_G is a monotone decreasing function of σ (actually, if $\beta = 0.5$ this happens for $\theta \leq 0.75$; if $\beta = 0.7$ it happens for $\theta \leq 0.90$, as can be observed in Figure 4). The observed decreasing pattern of the guaranteed rate, r_G, is as expected. As σ increases, in fact, the life insurance company faces a higher default risk as the value of its assets is more volatile and the value of the policyholder's claim increases. This is even more accentuated if the participation rate β is fixed at a high level. Hence, the insurer is forced to reduce r_G in order to

[2]We note that newly issued policies generally offer a minimum guaranteed rate below the market prevailing interest rate. However, the scenario of guarantees which lie above the current market rate of interest is well known in practice for the case of policies that have been issued in the past, in the light of the recent occurrence of separate periods of inflation and periods of falling interest rates in many industrialized economies.

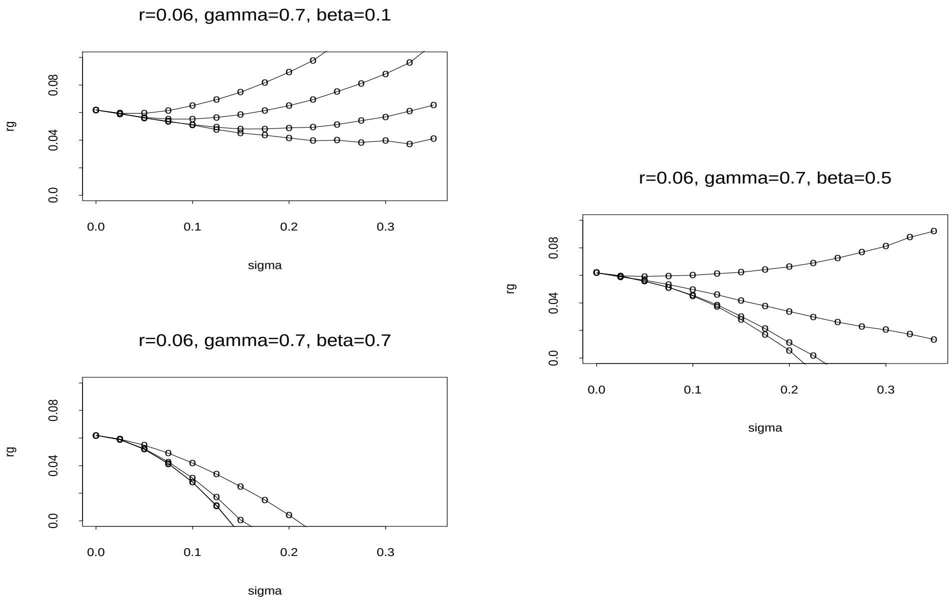

Figure 4: Feasible set for the asset's volatility and the minimum guarantee. The curves in each panel correspond to the values of θ equal to 0.20, 0.50, 0.75 and 1 from bottom to top.

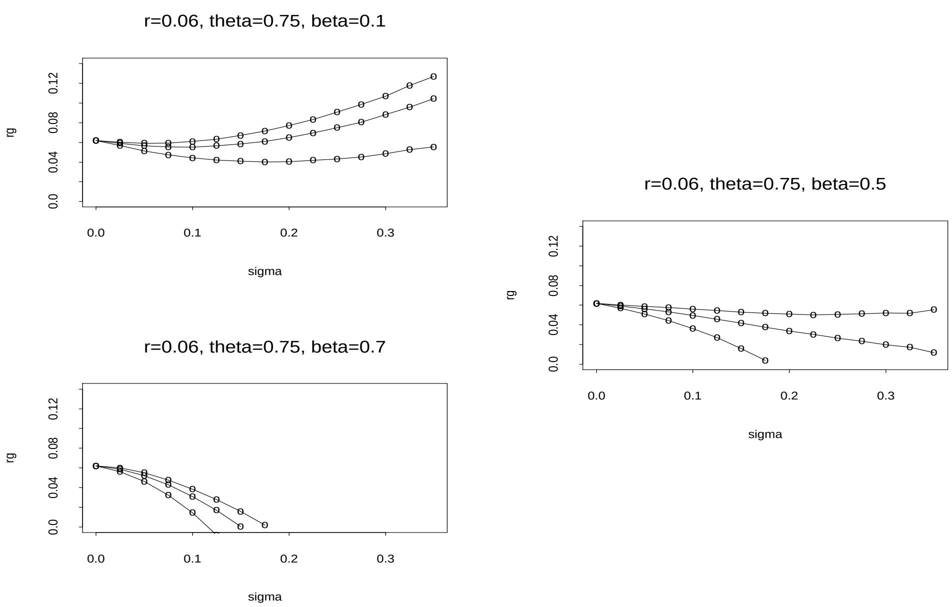

Figure 5: Feasible set for the asset's volatility and the minimum guarantee. The curves in each panel correspond to the values of γ equal to 0.50, 0.70, and 0.90 from top to bottom.

preserve its solvency. Further, as the plots show, this situation is the same for different capital structures, i.e. different θ, which is consistent with the Modigliani and Miller theorem (Modigliani and Miller, 1958). The U shape which we observe for low values of β, instead, is explained by the fact that $V_P(0)$ is not very sensitive to changes in the market volatility because of the low participation rate. Hence, the policyholder will buy the contract only if the guaranteed rate is attractive enough (and he/she receives some compensation for the low participation in asset returns implicit in the policy design). As β is low, the risk coming from the variable part of the policy reserve, $P(t)$, is limited.

We also note from Figure 4 that the rate of change in r_G increases as θ decreases (steeper curves). This again is a "wealth distribution effect": as θ decreases, the shareholders become the main group (contributing to the balance sheet) and therefore they try to preserve their share of the value of the portfolio. Since, in this case, γ is fixed, as well as β, the only design parameter on which they can act is the guaranteed rate, r_G. On the other hand, this decrease in θ corresponds to a reduction in the premium paid by the policyholder and, consequently, an increase in σ is accompanied by a corresponding fall in the guaranteed rate offered, r_G.

4.3 Impact of fund volatility on the cost allocation coefficient

Figures 6 and 7 represent the feasible set of fair combinations of the reference portfolio's volatility and the leverage coefficient, θ. From both Figures, we observe that, in general, θ is an increasing function of σ.

More precisely, both figures show that this is the case for $\beta = 0.5$, $\beta = 0.7$ and $r_G = 4\%$. The reason for the observed pattern follows from the arbitrage principle: when σ increases, the value of the option embedded inside the structure of the policy reserve increases; consequently, the premium paid by the policyholder to purchase such an option has to increase.

Figure 6 presents a first exception to this trend in the top panel, corresponding to the case of the participation rate $\beta = 0.1$, especially for $\gamma = 0.9$. Observing this particular isopremium curve, we note that for $\sigma > 0.15$, θ decreases as σ increases. This suggests that for values of the volatility higher than 15%, the contract is too risky. For this reason, the life insurance office needs to reduce the leverage in its liability structure.

Another exception to the general trend is presented in Figure 7, which shows that when the guaranteed rate is above or equal to the risk-free rate

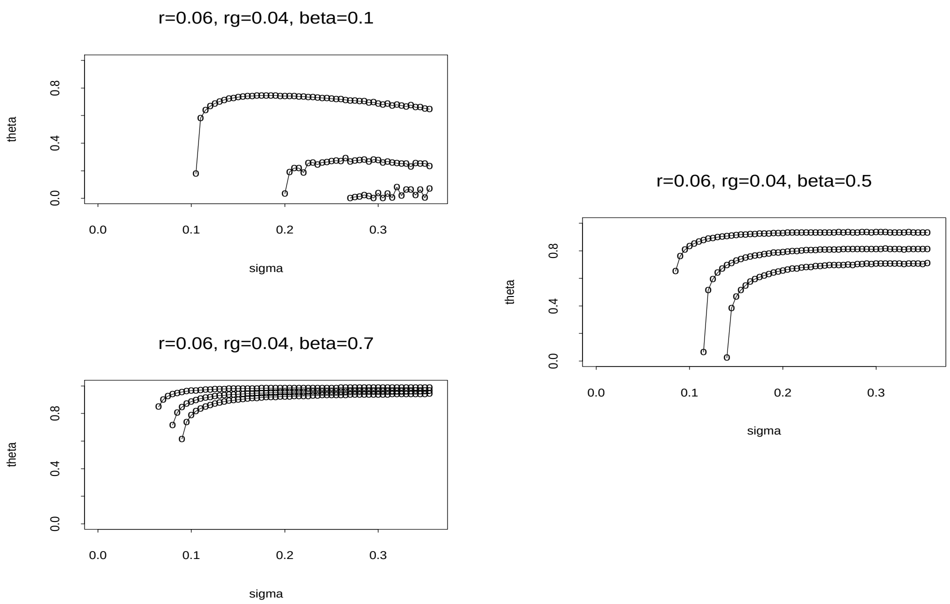

Figure 6: Isopremium curves over the range of possible values of σ and θ. The curves in each panel correspond to the values of γ equal to 0.5, 0.7 and 0.9 from bottom to top.

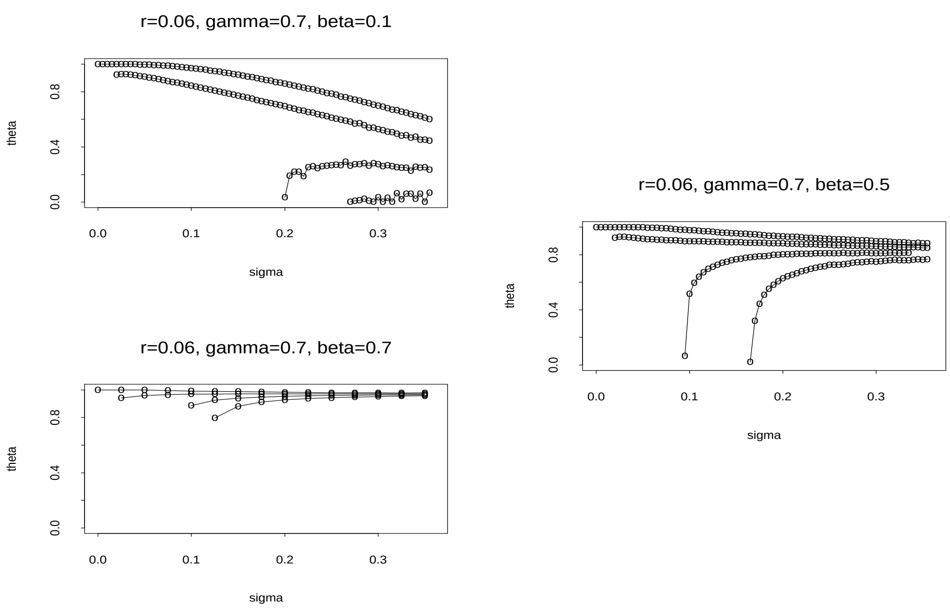

Figure 7: Isopremium curves over the range of possible values of σ and θ. The curves in each panel correspond to the values of r_G equal to 2%, 4%, 6% and 8% from bottom to top.

of interest (i.e. $r_G \geq r$)[3], θ is clearly a decreasing function of σ, when the participation rate β is fixed at a low value. In fact, as already observed, in such a situation the policy reserve is not very sensitive to changes in the market volatility. Figure 8 in fact shows that for $\beta = 0.1$, the policy reserve accumulates almost at the guaranteed rate. However, if σ increases, the default risk, as measured by the value of the default option, increases as well. This turns the contract into a product that earns a fixed rate of interest, which is higher than the rate of interest offered by the market, with a non-zero probability of default attached. In such a scenario, the life insurance company may find that the risk which it is bearing is too high and it therefore decides to reduce the leverage. Although, according to the Modigliani and Miller theorem, this does not change the value of the firm's assets, it does reduce the amount of the policyholder's claim.

Further evidence is presented in Figure 9, in which we plot the market value of the entire policy, i.e.

$$V_P(0) + \gamma V_R(0) - V_D(0), \tag{13}$$

against the volatility of the reference portfolio. For $\beta = 0.1$, we observe that, as r_G increases above the 6% level, the market value of the contract becomes a decreasing function of σ, implying that the value of the default option is predominant. Since equation (13) is the right hand side of the equilibrium condition (3) and $P_0 = \theta A_0$, the leverage coefficient has to decrease as well. As β increases, this feature is observed only for $r_G = 8\%$ in the panel corresponding to $\beta = 0.5$, while it disappears in the panel corresponding to $\beta = 0.7$. As Figure 8 shows in more detail, when β increases, the policy reserve is positively affected by increases in σ, compensating in this way for the increased risk of default.

4.4 Impact of the cost allocation parameter on the terminal bonus rate

From Figures 10 and 11, we observe that γ is an increasing function of θ. This effect has also been observed by Grosen and Jørgensen (2002) and it is due to the fact that, if policyholders are the only contributors to the establishment of the reference portfolio, they are the sole group entitled to the surplus of the company. Thus, as observed in section 3.3, when $\theta = 1$, $\gamma = 1$. These results confirm the monotonic relationship between γ and θ mentioned in the discussion of equation (12).

[3]See footnote 2 for a comment on the relevance of the case where $r_G \geq r$.

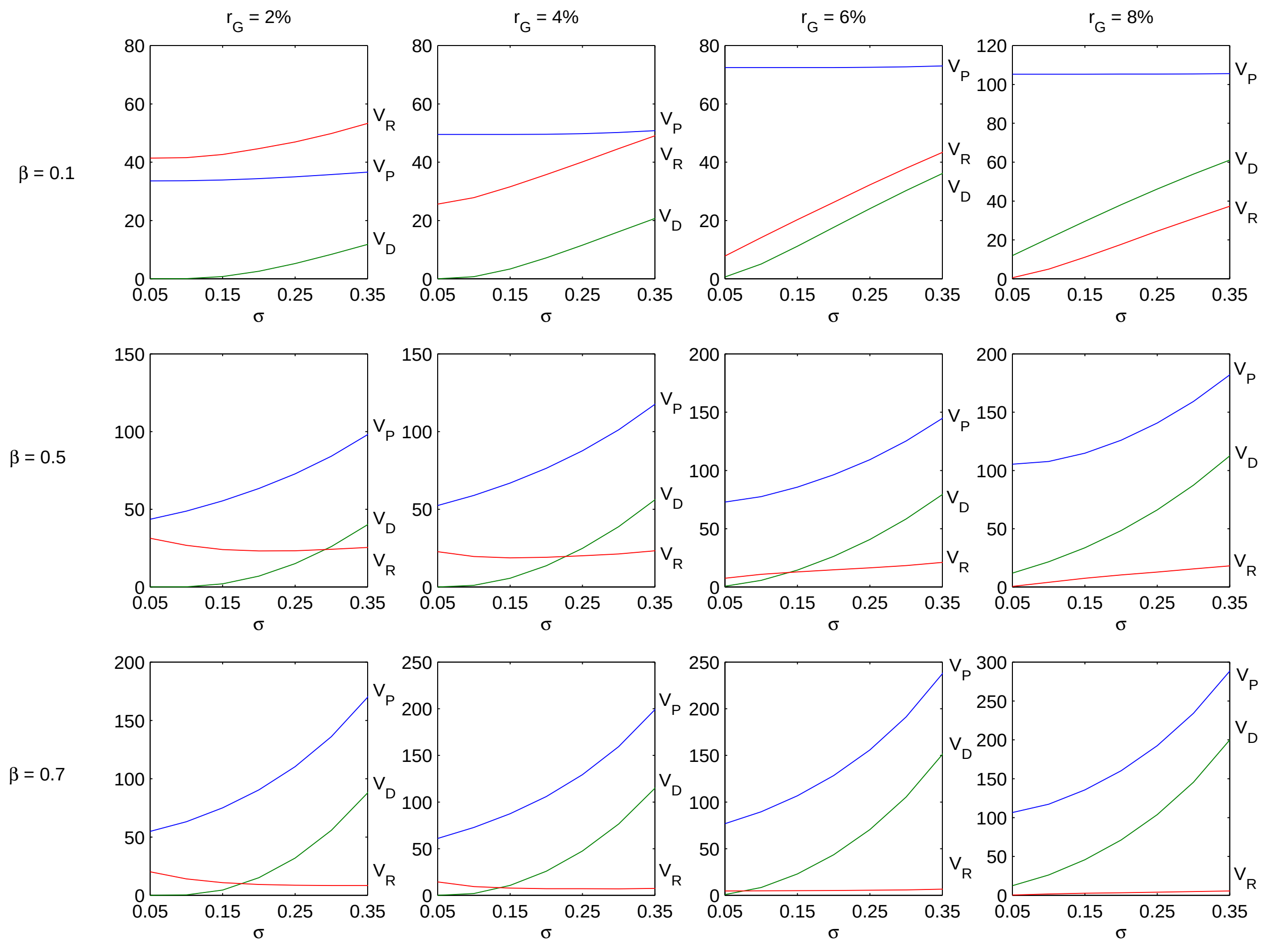

Figure 8: The components of the with profit policy. Comparison of sensitivities to changes in the fund volatility in different guarantee scenarios. Set of design parameters: $\theta = 0.75$, $r = 6\%$.

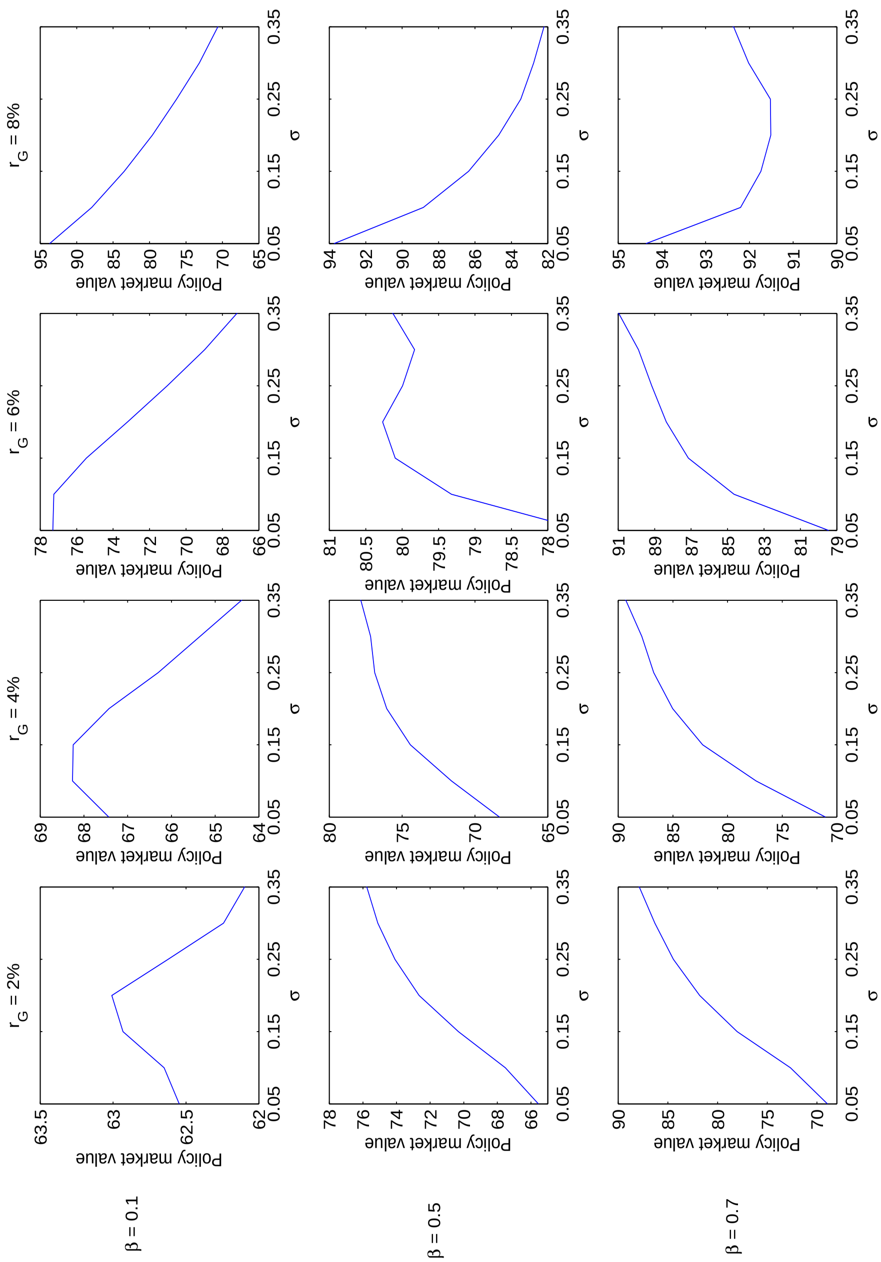

Figure 9: Sensitivity analysis for the market value of the contract (equation 13) with respect to the volatility of the reference portfolio. Design parameters set: $\theta = 0.75$, $\gamma = 0.7$, $r = 6\%$.

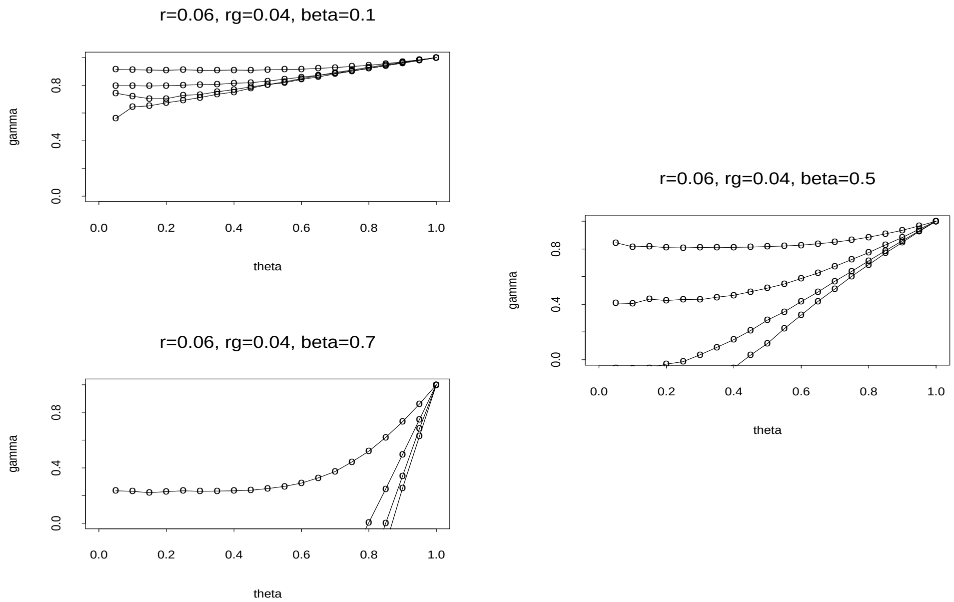

Figure 10: Isopremium curves over the range of possible values θ and γ. The curves in each panel correspond to the values of σ equal to 0.10, 0.15, 0.20 and 0.25 from top to bottom.

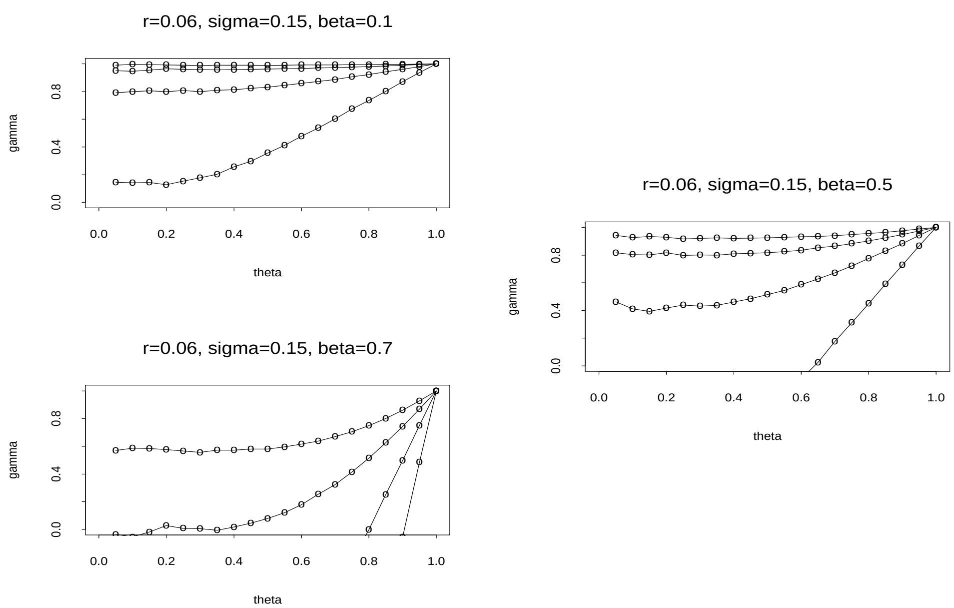

Figure 11: Feasible set for the cost allocation parameter, θ, and the terminal bonus rate, γ. The curves in each panel correspond to the values of r_G equal to 0%, 2%, 4% and 6% from top to bottom.

The detailed plots of Figure 10 show that if σ increases, the curves move towards the bottom-right corner and the spread between each volatility scenario becomes more evident as the participation rate β increases (which is consistent with what we have seen in section 4.1).

Further, in the top panel of Figure 10, we note that a lower bound is implied for γ for each choice of σ, so that a fair contract does not exist for some choices of γ. The lower bound depends on the volatility and is lower for higher values of the volatility. In the last panel of Figure 10, we note that the range of values, which the wealth distribution parameter θ can assume, narrows in high volatility scenarios.

The same consideration arises from Figure 11 in the case of high guaranteed rates: the curves, in fact, move towards the bottom part of the panels, or, in other words, the isopremium curves corresponding to higher levels of r_G are feasible only for lower values of the terminal bonus rate (this feature is explored in more details in section 4.6). We observe once again that increases in the participation rate, β, produce the effect of moving the isopremium curves towards the bottom right corner of the axes: ceteris paribus, either the policyholder pays a higher premium in return for the value of the benefit which he/she receives, or the policyholder accepts a lower guarantee.

4.5 Impact of the cost allocation parameter on the guaranteed rate

In Figures 12 and 13, we represent the feasible fair combinations of θ and r_G for different choices of σ and γ respectively. We note that in each plot, r_G is an increasing function of θ. This suggests that a high guaranteed rate is chosen when the contribution from the policyholder predominates over that from the equityholders.

In particular, in Figure 12 there is a cross-over feature that disappears for higher values of the participation rate, β. For $\beta = 0.1$, there is a value of θ, say $\bar{\theta}$, approximately equal to 0.5 such that for $\theta < \bar{\theta}$, higher guaranteed rates are chosen in lower volatility conditions than in higher volatility conditions. For $\theta > \bar{\theta}$, the situation is reversed and high guaranteed rates are chosen in high volatility scenarios. For $\beta = 0.5$, the cross-over appears at the higher level of $\bar{\theta} \cong 0.9$; while for $\beta = 0.7$ the cross-over disappears so that higher guaranteed rates are chosen for lower values of volatility than for higher values. This effect is due to the inversion of concavity that we have observed in Figure 4, discussed in section 4.2.

Further, both figures show that, as β increases, the guaranteed rate is lowered for any value of the market volatility (the curves move towards the

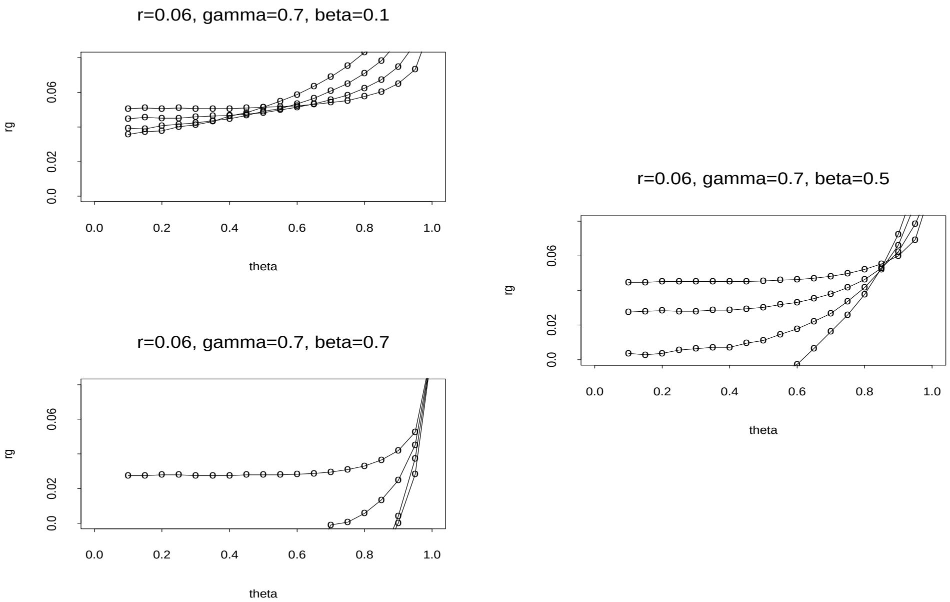

Figure 12: Feasible set for the cost allocation parameter θ and the minimum guarantee. The curves in each panel correspond to the values of σ equal to 0.10, 0.15, 0.20 and 0.25 from top to bottom at the left end of the θ scale.

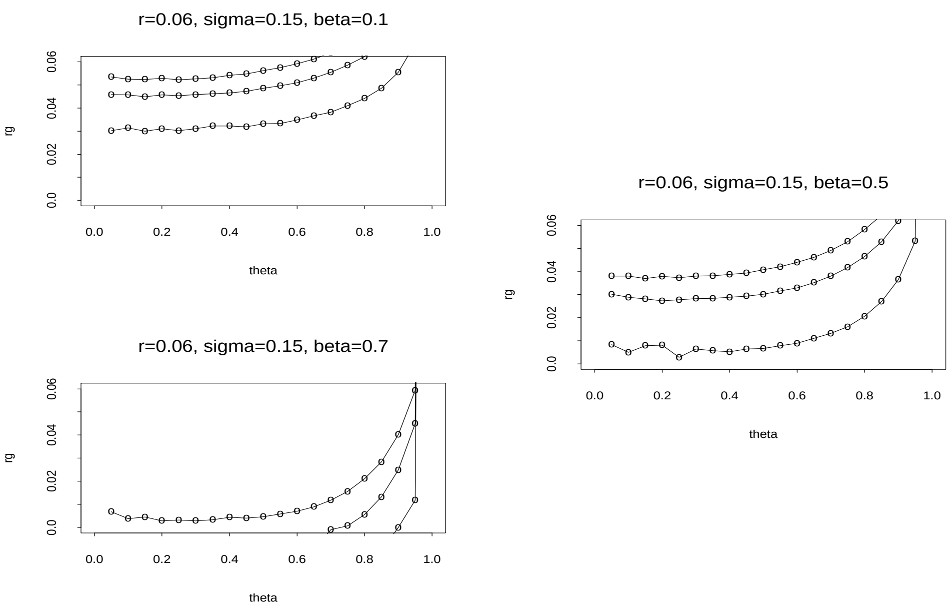

Figure 13: Isopremium curves over the range of possible values of θ and r_G. The curves in each panel correspond to the values of γ equal to 0.50, 0.75, and 0.90 from top to bottom.

bottom-right corner); for $\beta = 0.7$ and for $\sigma > 0.15$, the leverage coefficient needs to be approximately 1 in order for a positive guaranteed rate to be feasible. When β increases, both the size of the policy reserve and its sensitivity to the volatility increase; which means that the value of the default option increases under this double effect. If we regard the value of the default option as a measure of the default risk, then the last plot depicts a riskier situation and compensation is sought through reductions of the guaranteed rate (as shown in the bottom panels of both Figure 12 and Figure 13).

In Figure 13, we observe that when the terminal bonus rate increases, the curves move towards the bottom right corner of each panel. This shows that, in order to keep the same level of the guaranteed rate, the contribution from the policyholder has to increase, as for higher γ they receive a higher final benefit (which is consistent with what we have observed in section 4.2). Alternatively, if we want to keep the wealth distribution rate, θ, unchanged as γ increases, we need to reduce the level of the guarantee so that the value of the overall benefit, received by the policyholder, is fair compared to the premium paid.

4.6 Impact of the guaranteed rate on the terminal bonus rate

The feasible set of fair combinations for the minimum guarantee and the terminal bonus rate are shown in Figure 14 for different choices of θ, and in Figure 15 for different volatility scenarios.

Both figures show the existence of a trade-off between the two parameters, in the sense that low terminal bonus rates are associated with a high minimum guarantee for the contract to be fair to both sides. In other words, the benefit provided to the policyholder by a high guaranteed rate has to be compensated by a low terminal bonus rate in order to preserve the allocation of the financing of the reference portfolio between the two parties involved. On the other hand, it is worth noting that, when the policyholder receives a high guaranteed rate, it is unlikely that the conditions for the distribution of the terminal bonus are met.

Figure 14 also shows that higher terminal bonus rates for fixed r_G, or higher guarantees for fixed γ are feasible only if the contribution from the policyholder increases (as observed also in sections 4.4 and 4.5).

In Figure 15, we observe a cross-over feature in the panels corresponding to $\beta = 0.1$ and $\beta = 0.5$, in the sense that there is a critical value of r_G, $\bar{r}_G$ say, such that for $r_G < \bar{r}_G$ and fixed γ, reductions of the fund volatility cause the guarantee to be increased in order to maintain the same premium. For

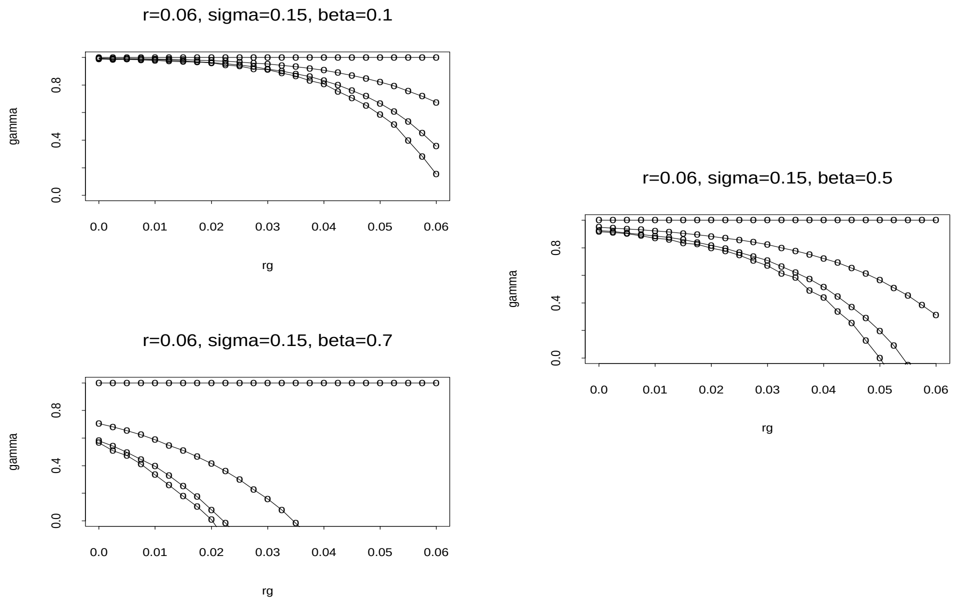

Figure 14: Isopremium curves over the range of possible values of r_G and γ. The curves in each panel correspond to the values of θ equal to 0.25, 0.5, 0.75 and 1 from bottom to top.

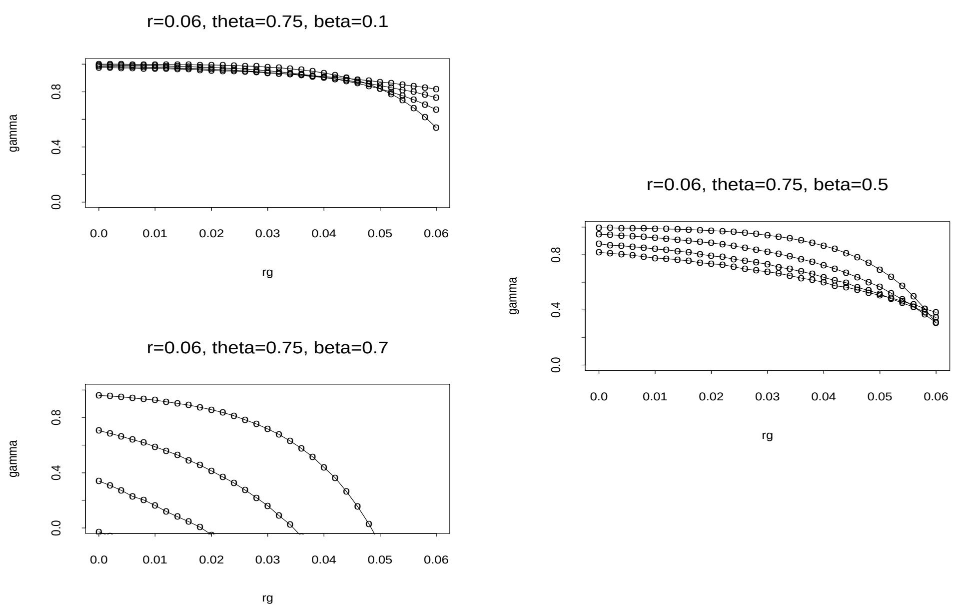

Figure 15: Isopremium curves over the range of possible values of r_G and γ. The curves in each panel correspond to the values of σ equal to 0.10, 0.15, 0.20 and 0.25 from top to bottom at the left end of the r_G scale.

$r_G > \bar{r}_G$ (and fixed γ), the situation is reversed: higher volatility scenarios require a higher guaranteed rate. This cross-over feature is related to the change of concavity that we have observed in Figure 5.

5 Concluding remarks

In this paper, we have applied a market-based valuation methodology for the most common unitised with-profits life insurance contracts sold in the UK. The contract design is also common in other European countries and Japan. These contracts contain complex guarantees and option-like features, which have not been analyzed yet in the literature. Specifically, we decompose these contracts into a policy reserve, comprising the guaranteed benefit and a periodically added reversionary bonus, a terminal bonus and a default option.

As mentioned earlier in the paper, the focus is on the fair design of these contracts when surrender opportunities are ignored. The option to surrender would give the policyholder the possibility to leave the policy scheme before maturity if this suits his/her needs; therefore, as shown by the theory of American options, a participating contract offering such an opportunity should be more valuable. However, it is common practice for life insurance companies to apply penalties when the policyholder decides to leave the scheme before the end of its term. Such charges could compensate for the added value generated by the American-type feature.

The analysis presented in this paper does not also take account of the mortality risk. Each benefit offered by the participating contract is in fact paid to the policyholder provided that he/she survives till each payment date. We, instead, assume that the policyholder will survive till the end of the contract's term with probability 1. In this respect, our paper provides a more prudential pricing rule, in the sense that we provide a possible upper bound for the life insurance liabilities.

Further, the model presented in this paper relies on a reference portfolio composed only of equities. We recognize, however, that the common practice amongst life insurance companies is to fund participating contracts with portfolios composed also of gilts, corporate bonds and instruments traded in foreign markets, whose proportions are not fixed over the lifetime of the contract, but change as the policy approaches maturity. A first attempt to incorporate part of these features is currently in progress.

References

[1] Bacinello, A. R., 2001, Fair Pricing of Life Insurance Participating Contracts with a Minimum Interest Rate Guaranteed, *Astin Bulletin*, 31: 275-297.

[2] Bacinello, A. R., 2003, Pricing Guaranteed Life Insurance Participating Policies with Annual Premiums and Surrender Option, *North American Actuarial Journal*, 7 (3): 1-17.

[3] Brennan, M. J. and E. S. Schwartz, 1976, The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee, *Journal of Financial Economics*, 3: 195-213.

[4] Briys, E. and F. De Varenne, 1994, Life Insurance in a Contingent Claim Framework: Pricing and Regulatory Implications, *The Geneva Papers on Risk and Insurance Theory,* 19: 53-72.

[5] Chadburn, R. G., 1998, Controlling Solvency and Maximizing Policyholders' Returns: a Comparison of Management Strategies for Accumulating With-Profits Long-Term Insurance Business, *Actuarial Research Paper* N^0 115, Department of Actuarial Science and Statistics, City University.

[6] Grosen, A. and P. L. Jørgensen, 2000, Fair Valuation of Life Insurance Liabilities: the Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies, *Insurance: Mathematics and Economics*, 26: 37-57.

[7] Grosen, A. and P. L. Jørgensen, 2002, Life Insurance Liabilities at Market Value: an Analysis of Investment Risk, Bonus Policy and Regulatory Intervention Rules in a Barrier Option Framework, *Journal of Risk and Insurance*, 69: 63-91.

[8] Haberman, S., L. Ballotta and N. Wang, 2003, Modelling and Valuation of Guarantees in With-Profit and Unitised With-Profit Life Insurance Contracts, *Actuarial Research Paper* N^0 146, City University London, under review.

[9] Jensen, B. P. L. Jørgensen and A. Grosen, 2001, A Finite Difference Approach to the Valuation of Path Dependent Life Insurance Liabilities, *Geneva Papers on Risk and Insurance Theory*, 26: 57-84.

[10] Maurer, R. and C. Schlag, 2002, Money-Back Guarantees in Individual Pension Accounts: Evidence from the German Pension Reform, *Centre for Financial Studies Working Paper* N^0 2002/03, Johan Wolfgang Goethe-Universität, Frankfurt am Main.

[11] Miltersen, K., R. and S. A. Persson, 1999, Pricing Rate of Return Guarantees in a Heath-Jarrow-Morton Framework, *Insurance: Mathematics and Economics*, 25: 307-325.

[12] Modigliani, F and M. H. Miller, 1958, The Cost of Capital, Corporation Finance and the Theory of Investment, *American Economic Review*, 48: 261-297.

[13] Needleman, P. D. and T. A. Roff, 1995, Asset Shares and their Use in the Financial Management of a With-Profits Fund, *British Actuarial Journal*, 1: 603-688.

[14] Persson, S. A. and K. K. Aase, 1997, Valuation of the Minimum Guaranteed Return Embedded in Life Insurance Products, *Journal of Risk and Insurance*, 64: 599-617.

[15] Tillinghast-Towers Perrin, 2001, Asset Share Survey 2000. London.

[16] Wirch, J. L. and M. R. Hardy, 1999, A Synthesis of Risk Measures for Capital Adequacy, *Insurance: Mathematics and Economics*, 25: 337-347.