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Dynamic capital allocation with distortion risk measures

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Abstract

Tsanakas and Barnett (2002) employed concepts from cooperative game theory (Aumann and Shapley, 1974) for the allocation of risk capital to portfolios of pooled liabilities, when distortion risk measures (Wang et al., 1997) are used. In this paper we generalise previously obtained results in three directions. Firstly, we allow for the presence of non-linear portfolios. Secondly, based on the concept of correlation order (Dhaene and Goovaerts, 1996) we proceed with discussing the links between dependence structures, capital allocation and pricing, as well as dropping a restrictive assumption on the continuity of probability distributions. Finally, we generalise the capital allocation methodology to a dynamic setting and conclude with a numerical example.

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1 Introduction

One of the imperatives of risk management and regulation is the determination of capital requirements for portfolios of risky positions, that is, the calculation of the 'risk capital' that has to be safely invested to compensate for the risk of holding random assets and liabilities. For this purpose, risk measures, which are real-valued functions from a collection of risky positions to the real line, are used. A set of properties that a risk measure should satisfy has been proposed by Artzner et al. (1999) in their influential definition of coherent measures of risk. We note that investigations of alternative classes of risk measures and their respective sets of properties have been part of actuarial science for more than thirty years, with risk measures interpreted as principles of premium calculation. From such a perspective, coherent risk measures based on distorted probabilities have been proposed by Denneberg (1990) and Wang et al. (1997).

A financial entity that holds a number of (possibly dependent) risky portfolios, such as the liability portfolios of an insurance company, allocates risk capital to them. The aggregate allocated capital should be equal to the risk measure of the insurance company's aggregate liabilities. Capital allocation can be considered as a method of measuring the performance of a portfolio, in terms of the diversification that it contributes to the company, or as a tool for pricing insurance liabilities. Such considerations must be reflected in the properties that the capital allocation rule satisfies.

The use of concepts originating in cooperative game theory for allocating risk capital has been proposed by Denault (2001), who identified the Aumann-Shapley (1974) value and the fuzzy core (Aubin, 1981) as possible

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solutions. Allocation methodologies based on the fuzzy core are derived on the premise that no linear portfolio should be allocated more capital than its risk measure. Tsanakas and Barnett (2002) obtained an explicit allocation formula for the fuzzy core, in the case that distortion risk measures are used, and highlighted the relationship between capital allocation and the pricing of insurance liabilities.

In this paper we extend previously obtained results into several directions. Allocation methods based on the fuzzy core only consider the formation of linear portfolios. However, non-linear contracts are very common both in the insurance (e.g. stop-loss treaties) and the financial (e.g. options) markets. Hence we define the non-atomic core, as an extension of the fuzzy core allowing for the presence of non-linear portfolios. Subsequently we show that a result from the theory of non-additive integrals (Denneberg, 1994) guarantees the existence of an allocation methodology in the non-linear setting that belongs to the non-atomic core.

It is possible that two portfolios whose random payoffs are characterised by identical probability distributions are allocated different amounts of risk capital. This is due to the different degrees to which the portfolios stochastically depend on the aggregate, as such dependence determines the amount of diversification that a portfolio contributes to the aggregate position of its holder. We formalise these considerations using the correlation order on sets of random vectors with fixed marginals that was discussed by Dhaene and Goovaerts (1996) and Wang and Dhaene (1998). Using correlation order we also show that the capital allocation formula calculated by Tsanakas and Barnett (2002) holds in the more general setting considered in this paper.

Capital allocation in essence produces a system of prices, according to

which a financial entity values its portfolios. Invoking concepts from cooperative game theory ensures that these prices are internally consistent, from the perspective of the financial entity's risk management. The question then arises of whether these prices are also consistent with prices in a market where the portfolios can be traded. Based on an equilibrium model by Tsanakas and Christofides (2003), we show how the prices produced by capital allocation can be interpreted as market prices in an insurace/financial market, where competing entities determine their investment by minimising a distortion risk measure.

Risk measurement and capital allocation has been discussed so far in a static rather than a dynamic setting. The effect of time and the situation where additional information on the development of risks becomes gradually available have not been considered. The rest of the paper is devoted to a dynamic generalisation of the risk measurement and capital allocation methodologies discussed. Distortion risk measures are updated using the general conditioning rule for non-additive set functions (Denneberg, 1994b). Based on that rule, we show that distortion risk measures can be updated by simultaneously conditioning the original probability distribution and using a modified distortion function, which we call an updated distortion. Subsequently, we discuss the properties of updated distortion functions, also analysing the case that conditioning events have zero probability. Using the dynamic extension of risk measures, we then produce a dynamic generalisation of the capital allocation methodology.

Finally, the results presented in the paper are illustrated via a simple numerical example, where correlated Brownian motions with drift correspond to stochastic liability processes.

The structure of the paper is as follows. Risk measures based on distorted probabilities and the concept of correlation order are introduced in sections 2 and 3 respectively. Risk capital allocation with distortion measures is discussed in section 4, where we present our results on non-linear portfolios, the relationship between dependence and capital allocation, and the link between allocated capital and market prices. The extension of capital allocation to a dynamic setting is discussed in section 5, while the numerical example is presented in section 6.

2 Distortion risk measures

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$. A risk measure is a real-valued functional, ρ, defined on a set of random variables $\mathcal{X}$, standing for risky portfolios of assets and/or liabilities. For a portfolio $X \in \mathcal{X}$, its risk measure, $\rho(X)$, represents the amount of safely invested capital that a regulator would require the owner of X to hold. Specifically, $\rho(X)$ is interpreted as "the minimum extra cash that the agent has to add to the risky position X, and to invest 'prudently', to be allowed to proceed with his plans" (Artzner et al., 1999). For simplicity in this paper we take 'invest prudently' to mean 'with zero interest'. Also note that here and in the subsequent discussion, positive values of elements of $\mathcal{X}$ will be considered to represent losses, while negative values will represent gains.

A coherent measure of risk is defined by Artzner et al. (1999) as a risk measure that satisfies the following properties:

Monotonicity: If $X \leq Y$ a.s. then $\rho(X) \leq \rho(Y)$

Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$

Positive Homogeneity: If $a \in \mathbb{R}_+$ then $\rho(aX) = a\rho(X)$

Translation Invariance: If $a \in \mathbb{R}$ then $\rho(X + a) = \rho(X) + a$

It is furthermore shown in Artzner et al. (1999) that all functionals satisfying the above properties allow a representation:

$$\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} E_{\mathbb{P}}[X] \tag{1}$$

where $\mathcal{P}$ is a collection of probability measures, or 'generalised scenarios'.

An additional desirable property is defined via the concept of comonotonicity (e.g., Denneberg, 1994a):

Definition 1. *Two random variables X, Y are called comonotonic if there is a random variable U and non-decreasing real functions e, d such that $X = e(U), Y = d(U)$.*

As discussed in the next section, comonotonicity corresponds to the strongest form of positive dependence between random variables (e.g. Wang and Dhaene (1998), Embrechts et al. (2002)). In economic terms, comonotonic investment positions cannot used as hedges for each other and pooling comonotonic positions yields no gains from diversification (Yaari, 1987).

An additional desirable property of risk measures is additivity for comonotonic risks:

Comonotonic Additivity: If X, Y comonotonic then $\rho(X+Y) = \rho(X)+\rho(Y)$

It can be shown that, if and only if $\rho(X)$ is a coherent risk measure satisfying comonotonic additivity, it has a representation as a Choquet integral with respect to a set function (capacity), v (Denneberg, 1990):

$$\rho(X) = \int X dv = \int_{-\infty}^{0} (v(X > t) - 1)dt + \int_{0}^{\infty} v(X > t)dt, \tag{2}$$

where v is submodular ($v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$ for all $A, B \in \mathcal{F}$) and monotone with respect to set inclusion ($A \subset B \Rightarrow v(A) < v(B)$). Choquet (1953) integrals are defined with respect to (non-additive) monotone set functions instead of measures; a textbook on the subject is Denneberg (1994a).

Let $\mathbb{P}_0$ be the real-world (actuarial) probability measure. If $g : [0,1] \mapsto [0,1]$ is a continuous, increasing and concave function, with $g(0) = 0$ and $g(1) = 1$, then $v(A) = g(\mathbb{P}_0(A))$ is a submodular set function (Denneberg, 1994a). Thus, the following integral is a comonotonic additive coherent risk measure:

$$\rho(X) = \int_{-\infty}^{0} (g(\mathbb{P}_0(X > t)) - 1)dt + \int_{0}^{\infty} g(\mathbb{P}_0(X > t))dt, \tag{3}$$

We will call g a *distortion function*, $g(\mathbb{P}_0)$ a *distorted probability* and the risk measure (3) a *distortion risk measure*. It can be shown that, subject to a technical condition, any submodular set function can be represented by a distorted probability (Wang et al., 1997).

Distortion risk measures have been axiomatically defined in the context of insurance pricing by Denneberg (1990) and Wang et al. (1997). Choquet integrals have also found application as non-linear pricing functionals in financial markets with frictions (Chateauneuf et al., 1996).

Finally, note that since distortion risk measures are coherent, they will allow a representation through 'generalised scenarios'. Specifically, the set $\mathcal{P}$ of probability measures in the representation (1) corresponds to the core of the set function $v = g(\mathbb{P}_0)$ (Denneberg, 1994a):

$$\mathcal{P} = \{\mathbb{P} : \ \mathbb{P}(A) \leq g(\mathbb{P}_0(A)) \ \forall A \in \mathcal{F}\} \tag{4}$$

3 Dependence between risks and correlation order

The concept of comonotonicity, briefly discussed in section 2, is an interesting point of convergence between statistics and economics, as it represents both a form of extreme (positive) dependence between risks and a lack of portfolio diversification. In the sequel we will need notions of dependence weaker than comonotonicity, as well as the means of comparing the dependence structures between risks. For this purpose, we provide in this section a brief exposition of the concept of correlation orderings between pairs of risks, our main references here being Dhaene and Goovaerts (1996) and Wang and Dhaene (1998).

Consider cumulative probability distribution functions, $F_1,\ F_2$ and let $R(F_1, F_2)$ be the class of all pairs of random variables (X_1, X_2) with marginal distribution functions $F_1(x_1) = \mathbb{P}(X_1 \leq x_1)$ and $F_2(x_2) = \mathbb{P}(X_2 \leq x_2)$ respectively. Thus elements of $R(F_1, F_2)$ are random 2-vectors with fixed marginal behaviour but undetermined joint distribution $F_{X_1,X_2}(x_1, x_2) = \mathbb{P}(X_1 \leq x_1 \cap X_2 \leq x_2)$. Note that we can also write:

$$F_{X_1,X_2}(x_1, x_2) = C_{X_1,X_2}(F_1(x_1), F_2(x_2)), \tag{5}$$

where $C_{X_1,X_2} : [0, 1]^2 \mapsto [0, 1]^2$ is the copula function of (X_1, X_2), summarising their dependence structure (Embrechts et al., 2002).

The following result on the joint distribution of any $(X_1, X_2) \in R(F_1, F_2)$, gives the well-known Frechet-Hoeffding bounds on $F_{X_1,X_2}(x_1, x_2)$ (Dhaene and Goovaerts, 1996):

Proposition 1. *For any* $(X_1, X_2) \in R(F_1, F_2)$ *the following inequality holds:*

$$\max\{F_1(x_1) + F_2(x_2) - 1, 0\} \leq F_{X_1,X_2}(x_1, x_2) \leq \min\{F_1(x_1), F_2(x_2)\} \tag{6}$$

The upper and lower bounds are themselves bivariate distributions with marginals $F_1,\ F_2$.

The upper bound actually corresponds to the joint distribution function of X_1, X_2, when they are comonotonic (Wang and Dhaene, 1998):

Proposition 2. *Consider* $(X_1, X_2) \in R(F_1, F_2)$. *The following three statements are equivalent:*

a) X_1 *and* X_2 *are comonotonic.*

b) There exist increasing functions h_1, h_2 *and a random variable* U *such that* $X_1 = h_1(U), X_2 = h_2(U)$.

c) The joint distribution function of X_1 *and* X_2 *is* $\min\{F_1(x_1), F_2(x_2)\}$

Note that comonotonicity is the most dangerous dependence structure between risks, in terms of the stop-loss ordering of their sums (Dhaene and Goovaerts, 1996). Correspondingly, pairs of risks whose joint distribution function is the lower bound in (6) are called countermonotonic, and are characterised by the safest possible dependence structure.

Weaker notions of positive and negative dependence are Positive and Negative Quadrant Dependence, respectively (Wang and Dhaene (1998)):

Definition 2. *The random variables* X_1 *and* X_2 *are called Positive Quadrant Dependent (PQD), if:*

$$F_{X_1,X_2}(x_1, x_2) \geq F_1(x_1)F_2(x_2) \tag{7}$$

X_1 *and* X_2 *are called Negative Quadrant Dependent (NQD), if:*

$$F_{X_1,X_2}(x_1, x_2) \leq F_1(x_1)F_2(x_2) \tag{8}$$

It is desirable to compare elements of $R(F_1, F_2)$ in terms of their dependence. The correlation ordering discussed in Dhaene and Goovaerts (1996) provides a way of carrying out this comparison:

Definition 3. *Let (X_1, X_2) and (Y_1, Y_2) be elements of $R(F_1, F_2)$. We then say that (X_1, X_2) are less correlated than (Y_1, Y_2), and write $(X_1, X_2) \leq_{corr} (Y_1, Y_2)$, if:*

$$Cov(h_1(X_1), h_2(X_2)) \leq Cov(h_1(Y_1), h_2(Y_2)) \tag{9}$$

for all non-decreasing functions h_1, h_2 for which the covariances exist.

The following result is proved in Dhaene and Goovaerts (1996):

Proposition 3. *Let (X_1, X_2) and (Y_1, Y_2) be elements of $R(F_1, F_2)$. The following two statements are equivalent:*

a) $(X_1, X_2) \leq_{corr} (Y_1, Y_2)$

b) $F_{X_1,X_2}(x_1, x_2) \leq F_{Y_1,Y_2}(x_1, x_2)$

From the above proposition the following two lemmas are easily obtained:

Lemma 1. *Let $(Y_1, Y_2) \in R(F_1, F_2)$ be comonotonic (countermonotonic). Then:*

$$(Y_1, Y_2) \ \geq_{corr} (\leq_{corr}) \ (X_1, X_2) \quad \forall (X_1, X_2) \in R(F_1, F_2) \tag{10}$$

Lemma 2. *Let $(Y_1, Y_2) \in R(F_1, F_2)$ be independent and $(X_1, X_2) \in R(F_1, F_2)$ be PQD (NQD). Then:*

$$(Y_1, Y_2) \leq_{corr} \ (\geq_{corr}) \ (X_1, X_2) \tag{11}$$

For all non-decreasing functions h_1, h_2 for which the covariances exist:

$$Cov(h_1(X_1), h_1(X_2)) \ \geq (\leq) \ 0 \tag{12}$$

Remark: Dhaene and Goovaerts (1996) and Wang and Dhaene (1998) were concerned with pure insurance liabilities and thus only considered non-negative random variables. However, the results obtained in those papers are more general and hold with no assumptions on positivity of random variables.

4 Capital allocation with distortion risk measures

Denault (2001) proposed applying concepts from non-atomic (fuzzy) cooperative game theory to the problem of allocating the risk capital corresponding to portfolio to its constituents. The Aumann-Shapley (1974) value, which for positively homogenous and subadditive cost functions, such as coherent measures of risk, belongs to the fuzzy core (Aubin, 1981) of the game was shown to yield an appropriate risk capital allocation mechanism. Explicit formulae were obtained by Tasche (2000a), in the case that the risk measure is Expected Shortfall, and by Tsanakas and Barnett (2002), when distortion risk measures are used.

In the latter paper, results were obtained under the assumption of continuous conditional probability density functions. However, this assumption appears to be quite restrictive in practice, where discontinuous distribution functions are often encountered. Another restriction has been the consideration only of linear (sub)portfolios, in the definition of the fuzzy core. In this section we review the concept of the fuzzy core in the context of risk capital allocation. Using elements of non-additive integration theory (Denneberg, 1994) and correlation orderings (Dhaene and Goovaerts, 1996), we show the result of Tsanakas and Barnett (2002) holds in the more general case of dis-

continuous distributions and non-linear portfolios. The effect of dependence on the capital allocated to individual portfolios is also discussed. Finally we comment on the consistency of the resulting allocation functional with market prices.

4.1 The fuzzy core for distortion risk measures

Let $X_1, X_2, ..., X_n \in \mathcal{X}$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P}_0)$, representing insurance liabilities. Specifically we let $\mathcal{F}$ be the $\sigma-$algebra generated by $X_1, X_2, ..., X_n$. Let Z be the aggregate portfolio, consisting of the pooled liabilities X_i:

$$Z = \sum_{j=1}^{n} X_j \tag{13}$$

The holder of the portfolio, Z, will have to hold risk capital, $\rho(Z)$, in order to satisfy a regulator. We assume that ρ is a distortion risk measure, such as the ones discussed in section 2. The risk capital allocation problem then consists of determining a vector $d \in \mathbb{R}^n$, such that:

$$\sum_{j=1}^{n} d_j = \rho(Z), \tag{14}$$

where d_i is the capital allocated to liability X_i.

We need to impose additional conditions, in order to obtain a capital allocation that satisfies some desirable properties. Cooperative game theory provides a suitable framework for such problems. We consider the (holders of the) risks $X_1, X_2, ..., X_n$ as players in a cooperative game. Cooperation is understood as the pooling of their liabilities, which (due to the subadditivity of the risk measure) produces aggregate savings in risk capital. It is reasonable to require that the aggregate risk capital be allocated in such a way that no incentive is produced for a player to withdraw the whole or

part of its investment from the pool, since this would reduce the aggregate savings. This requirement is formalised via the concept of the *fuzzy core* (Aubin, 1981).

Define a subportfolio Z^u of Z as:

$$Z^u = \sum_{j=1}^{n} u_j X_j, \quad u \in [0,1]^n \tag{15}$$

The fuzzy core, $\mathcal{C}$, will consist of all allocations, $d^{\mathcal{C}}$, that satisfy (14) and do not allocate to any portfolio more capital than its individual risk assessment, were it not part of the pool:

$$\mathcal{C} = \{d \in \mathbb{R}^n | \sum_{j=1}^{n} d_j = \rho(Z) \quad \text{and} \quad \rho(Z^u) \geq \sum_{j=1}^{n} u_j d_j \ \forall u \in [0,1]^n\} \tag{16}$$

Thus, allocations in the fuzzy core do not produce an incentive for any subportfolio to leave the pool.

If the cost functional $\rho(Z^u)$ is subadditive and positively homogenous in u, i.e.:

$$\rho(Z^{\phi+\psi}) \leq \rho(Z^{\phi}) + \rho(Z^{\psi}) \quad \forall \phi, \psi \in [0,1]^n \tag{17}$$

and

$$\rho(Z^{\lambda u}) = \lambda \rho(Z^u) \quad \forall \lambda \geq 0, \tag{18}$$

then the fuzzy core is convex, compact and non empty (Aubin, 1981). Furthermore, if $\rho(Z^u)$ is differentiable at the n-vector of ones, $u = \mathbf{1}$, then the fuzzy core consists only of the gradient vector of $\rho(Z^u)$ at $u = \mathbf{1}$ (Aubin, 1981):

$$d_i^{\mathcal{C}} = \frac{\partial \rho(Z^u)}{\partial u_i}|_{u_j = 1 \forall j} \tag{19}$$

In the case of distortion risk measures (3), conditions (17) and (18) are clearly satisfied. Assuming that conditional densities are continuous, then

applying quantile derivatives (Tasche, 2000b) and using the quantile representation of the Choquet integral (Denneberg, 1994), Tsanakas and Barnett (2002) showed that $\rho(Z^u)$ is differentiable in u and, by direct calculation, obtained the following formula for the unique allocation in the fuzzy core:

$$d_i^{\mathcal{C}} = E[X_i g'(S_Z(Z))] \tag{20}$$

Thus, the resulting allocation mechanism can be represented by an expectation under a change of probability measure:

$$d_i^{\mathcal{C}} = E_{\mathbb{Q}}[X_i], \quad \frac{d\mathbb{Q}}{d\mathbb{P}_0} = g'(S_Z(Z)) \tag{21}$$

4.2 Non-linear portfolios

Note that the definition (16) of the fuzzy core is quite restrictive, as it only prevents linear subportfolios from leaving the pool. However, the formation of portfolios corresponding to nonlinear functions of $X_1, X_2, ..., X_n$ is conceivable in an insurance/financial market. The presence of non-linear insurance treaties (e.g. stop-loss) and financial derivatives (e.g. call options) is testimony to this. Thus, we would be interested in allocations that produce no incentive to leave the pool to any portfolio X, which is a, possibly non-linear, function of $X_1, X_2, ..., X_n$, i.e. any random variable that is measurable with respect to the σ-algebra, $\mathcal{F}$, generated by $X_1, X_2, ..., X_n$.

Let $\mathcal{X} \subseteq L_1(g(\mathbb{P}_0))$ be a class of integrable (for a rigorous definition of the linear space $L_1(g(\mathbb{P}_0))$, see Denneberg (1994a)), $\mathcal{F}$-measurable random variables. An allocation mechanism will then correspond to a linear functional on $\mathcal{X}$. Let Γ be the class of real-valued linear functionals defined on

$\mathcal{X}$. We define the *non-atomic core* as:

$$\begin{aligned}\mathcal{NA} = \quad & \{d \in R^n | \exists \gamma \in \Gamma \text{ for which } \gamma(X_i) = d_i \ \forall i, \quad \gamma(Z) = \rho(Z) \\ & \text{and} \quad \rho(X) \geq \gamma(X) \ \forall X \in \mathcal{X}\}\end{aligned} \tag{22}$$

It is obvious that $\mathcal{NA} \subseteq \mathcal{C}$, as the number of portfolios that could potentially have an incentive leave the pool increases. In the case that the risk measure is given by a distortion risk measure (with concave distortion function, g) the non-atomic core will be non-empty, by the following result from non-additive integration theory (Proposition 10.1 in Denneberg (1994a)):

Proposition 4. *Let v be monotone and submodular on 2^Ω. For any class $\mathcal{A} \subset L_1(v)$ of comonotonic functions, there is an additive set function $\mathbb{Q}$ on 2^Ω, such that:*

$$\mathbb{Q} \leq v, \quad \int X dv = \int X d\mathbb{Q}, \forall X \in \mathcal{A}$$

Let $v = g(\mathbb{P}_0)$. From the definition of the Choquet integral it immediately follows that:

$$\mathbb{Q} \leq v \Rightarrow E_{\mathbb{Q}}[X] = \int X d\mathbb{Q} \leq \int X dv = \rho(X), \ \forall X \in \mathcal{X} \tag{23}$$

Hence, the linear functional, $E_{\mathbb{Q}}[\cdot] \in \Gamma$ on $\mathcal{X}$, produces a risk capital allocation in the non-atomic core. Note that no assumption on the continuity of probability distributions was made. On the other hand, allocations in the non-atomic core are not necessarily unique. However, whenever conditional densities are continuous in the sense of Tasche (2000b), the non-atomic core coincides with the fuzzy core and contains only one allocation.

4.3 Capital allocation and correlation order

In section 4.2 it was shown that there exists a capital allocation mechanism in the non-atomic core, which takes into account the presence of non-linear

portfolios and its existence does not rely on the continuity of probability distributions. This capital allocation will not in general be unique. Here, using the concept of correlation order discussed in section 3, we show that the capital allocation given by equation (20) does actually belong to the non-atomic core. We furthermore comment on the relationship between correlation order, capital allocation and pricing.

As before, we denote by $R(F_1, F_2)$ the class of pairs of random variables (X_1, X_2) with marginal cumulative distributions F_1 and F_2, respectively. The following two lemmas will be used in the sequel:

Lemma 3. *Let (X_1, X_2) and (Y_1, Y_2) be elements of $R(F_1, F_2)$. Then:*

$$(X_1, X_2) \leq_{corr} (Y_1, Y_2) \Rightarrow E[h_1(X_1)h_2(X_2)] \leq E[h(Y_1)h_2(Y_2)], \qquad (24)$$

for any non-decreasing functions h_1, h_2, such that the covariances $Cov(h_1(X_1), h_2(X_2))$ and $Cov(h_1(Y_1), h_2(Y_2))$ exist.

Proof: For any pair of random variables $h_1(X_1), h_2(X_2)$, with finite covariance, it is:

$$E[h_1(X_1)h_2(X_2)] = E[h_1(X_1)]E[h_2(X_2)] + Cov(h_1(X_1), h_(X_2)).$$

Since $(X_1, X_2) \leq_{corr} (Y_1, Y_2)$, it will be:

$$Cov(h_1(X_1), h_2(X_2)) \leq Cov(h_1(Y_1), h_2(Y_2)).$$

Additionally, $E[h_1(X_1)] = E[h_1(Y_1)]$ and $E[h_2(X_2)] = E[h_2(Y_2)]$, because both (X_1, X_2) and (Y_1, Y_2) are elements of $R(F_1, F_2)$. Hence:

$$E[h_1(X_1)h_2(X_2)] = E[h_1(X_1)]E[h_2(X_2)] + Cov(h_1(X_1), h_2(X_2)) \leq$$
$$E[h_1(Y_1)]E[h_2(Y_2)] + Cov(h_1(Y_1), h_2(Y_2)) = E[h_1(Y_1), h_2(Y_2)].$$

The next result follows directly from Lemmas 1 and 3:

Lemma 4. *Let* $(Y_1, Y_2) \in R(F_1, F_2))$ *be comonotonic. Then:*

$$E[h_1(Y_1)h_2(Y_2)] \geq E[h_1(X_1)h_2(X_2)], \quad \forall (X_1, X_2) \in R(F_1, F_2),$$

for any non-decreasing functions h_1, h_2*, such that the covariances* $Cov(h_1(X_1), h_2(X_2))$ *and* $Cov(h_1(Y_1), h_2(Y_2))$ *exist.*

Let us return to the capital allocation problem. Consider the linear functional:

$$\gamma(X) = E[Xg'(S_Z(Z))], \ X \in \mathcal{X}$$

Now consider the risk measure (3). Integration by parts yields:

$$\rho(X) = E[Xg'(S_X(X))], \ X \in \mathcal{X}$$

Note that X and $g'(S_X(X))$ are comonotonic, since g is concave and S_X is decreasing. Also note that both $S_X(X)$ and $S_Z(Z)$ are uniformly distributed. Thus, Lemma 4 yields:

$$\gamma(X) = E[Xg'(S_Z(Z))] \leq E[Xg'(S_X(X))] = \rho(X), \ X \in \mathcal{X} \tag{25}$$

Summarising, we have just shown that:

Proposition 5. *The capital allocation mechanism:*

$$d_i = E[X_i g'(S_Z(Z))] \tag{26}$$

belongs to the non-atomic core.

In Tsanakas and Barnett (2002), it was argued that the amount of capital allocated to a liability X_i increases when X_i and Z are highly correlated. Using the correlation ordering on pairs of random variables, we can simply

show why this is the case. Assume that X_i and X_j have the same marginal probability distribution. Lemma 3, yields:

$$(X_i, Z) \leq_{corr} (X_j, Z) \Rightarrow d_i \leq d_j \tag{27}$$

The interpretation of this result is that high correlation of a risk to the aggregate portfolio induces a higher amount of allocated capital, since the risk contributes less to the diversification taking place by pooling.

Note that the capital allocation problem can also be viewed as a pricing exercise, where a reinsurer passes on to the cedents the savings from pooling liabilities arising from different contracts (Tsanakas and Barnett, 2002). Then, the term $g'(S_Z(Z))$ corresponds to a price density (or state-price deflator, Duffie (1988)) and the price $\pi(X)$ of $X \in \mathcal{X}$ is given by:

$$\pi(X) = E[Xg'(S_Z(Z))] = E[X] + Cov(X, g'(S_Z(Z))) \tag{28}$$

Thus, dependence between a traded risk and the price density can give us a clue as to the risk loading (or indeed discount), $\pi(X) - E[X]$, that will apply. Specifically, Lemma 2, yields the following result:

Proposition 6. *If (X, Z) are PQD, the risk loading, $\pi(X) - E[X]$, is positive. If they are NQD, it is negative.*

4.4 Consistency with market prices

The capital allocation functional (26) discussed here essentially produces a system of prices. Its game theoretical origins ensure that these prices are internally consistent, in the sense that they do not produce an incentive for pooled portfolio to split. If we assume that the portfolios of liabilities subject to capital requirements are traded in a risk (e.g. insurance) market it would

be desirable that the allocation mechanism is also externally consistent, in the sense that it is consistent with market prices.

Consider the case where n holders of random liabilities participate in an exchange, where their risky positions, $X_1, X_2, \ldots, X_n$, can be traded. We assume that the market is regulated and that the holder of liability X has to hold capital $\rho(X)$, where ρ is a distortion risk measure. Market prices are represented by a linear functional π, such that:

$$\pi(X) = E[\zeta X], \tag{29}$$

where $E[\zeta] = 1$.

Let the liability holders take investment decisions by minimising the risk that they are exposed to. Thus, the ith player determines the liability Y_i that it will hold after the exchange by solving:

$$\min_{Y_i \in \mathcal{X}} \rho(Y_i), \quad \text{such that} \quad \pi(X_i) \geq \pi(Y_i) \tag{30}$$

The budget constraint in (30) is equivalent to saying that the ith agent can afford the reinsurance, $Y_i - X_i$, that it buys.

Equilibrium is reached when each of the liability holders solves its minimisation problem and the market clears:

$$\sum_{j=1}^{n} Y_j = \sum_{j=1}^{n} X_j = Z \tag{31}$$

Risk exchange models with preference functionals and risk measures based on distorted probabilities are studied in Tsanakas and Christofides (2003). In that paper it is shown that equilibrium prices for risk exchange (30), (31) are given by:

$$\pi(X) = E[Xg'(S_Z(Z))] \quad X \in \mathcal{X} \tag{32}$$

The price functional (32) is of course identical to the capital allocation mechanism discussed in the previous sections. Note though that the random variable Z here represents the aggregate risk in the market and not the liability held by a market agent (insurance company), which is represented by Y_i. However, it can be shown that at equilibrium Z and Y_i are comonotonic (Tsanakas and Christofides, 2003) and thus $g'(S_Z(Z)) = g'(S_{Y_i}(Y_i))\forall i$. Hence, the prices used by liability holders in order to allocate capital to their portfolios are the same as those with which the risks are traded in the market. Subsequently, at equilibrium no reinsurance for a liability can be bought at a price lower than the capital allocated to it. If that was not the case, capital allocation would produce incentives for sub-optimal investment. On the other hand, the fact that (32) produces prices in the non-atomic core, has the implication that every player benefits from participating in the exchange and there are no incentives for any portfolio to leave the market.

5 Extension to the dynamic setting

Here we introduce a dynamic generalisation of the capital allocation methodology presented in the paper. This relies on a generalisation of the risk measures we use to a dynamic setting, which in turn goes through defining an updating rule for the distortion risk measure. In what way the updating should be carried out, depends on whether we view the distorted probability as a transformation of an objective probability measure (Yaari, 1987) representing preferences or as a submodular set function directly reflecting a subjective probability assessment (Schmeidler, 1989). In the former case a reasonable strategy would be to update the probability measure according

to Bayes' rule and distort it thereafter, whereas in the latter case an updating mechanism for non-additive set functions is called for, such as the ones discussed in Denneberg (1994b) (see also the discussion in Wang and Young (1998) and Young (1998)). In this paper we adopt the latter approach, as it is consistent with the 'worst-case-scenario' interpretation the risk measure. We then comment on classes of distortion functions that emerge as a result of updating distorted probabilities and finally extend the capital allocation methodology discussed in previous sections to the dynamic setting.

5.1 Updating submodular set functions and Choquet integrals

The distortion risk measure (3) can be viewed as an expectation with respect to the submodular set function $g(\mathbb{P}_0)$. In order to use the risk measure in a dynamic setting, we need to be able to condition the set function $v = g(\mathbb{P}_0)$, as well as the corresponding Choquet integral, on information becoming available. Recall the representation of the risk measure via a set of probability measures:

$$\rho(X) = \sup_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}[X], \quad \mathcal{P} = \{\mathbb{P} : \mathbb{P}(A) \leq v(A)\ \forall A \in \mathcal{F}\} \tag{33}$$

If each of the probability measures $\mathbb{P}$ is interpreted as a scenario, we can define the updated risk measure by conditioning each of those scenarios. Thus, the risk measure of an $\mathcal{F}$-measurable random variable, conditional upon an event $B \in \mathcal{F}$, will be defined by:

$$\rho_{|B}(X) = \sup_{\mathbb{P}\in\mathcal{P}} E_{\mathbb{P}}[X|B], \quad \forall B \in \mathcal{F} \tag{34}$$

For a construction of conditional expectation on a discrete probability space and a list of properties (many of them inherited from the conditional

expectations with respect to the additive probability measures in $\mathcal{P}$) see Denneberg (2000). Related approaches can be found in Walley (1991), who studied the problem in the context of quasi-Bayesian statistics and decision theory, and Lehrer (1996), who proposed a geometrical construction.

Owing to the good properties of submodular set functions we can actually move towards a concrete analytical formulation for the expression (34). This goes through defining an updating rule for the set function v, as a generalisation of Bayes' rule. A choice argued for in Denneberg (1994b), Walley (1991) is:

$$v_{|B}(A) = \frac{v(A \cap B)}{v(A \cap B) + \overline{v}(A^c \cap B)} \quad A, B \in \mathcal{F} \tag{35}$$

where $\overline{v}$ is the conjugate set function of v, $\overline{v}(A) = 1 - v(A^c)$ (note that if $v = g(\mathbb{P}_0)$, then $\overline{v} = h(\mathbb{P}_0)$, where $h(t) = 1 - g(1-t)$).

The set function $v_{|B}$ satisfies two properties that are crucial for our application. Firstly, if v is a submodular set function, so is $v_{|B}$ (Walley (1991), Denneberg (1994b)). Thus, if a Choquet integral with respect to $v_{|B}$ is defined, it will be a comonotonic-additive coherent risk measure as discussed in section 2. $v_{|B}$ can then be expressed as the upper envelope of all probability measures $\mathbb{P}$ such that $\mathbb{P} \leq v_{|B}$. Furthermore it is shown by Denneberg (1994b) that $v_{|B}$ can also be expressed as the upper envelope of all probability measures $\mathbb{P}$ such that $\mathbb{P} \leq v$, conditioned on the set B:

$$v_{|B}(A) = \sup_{\mathbb{P} \leq v_{|B}} \mathbb{P}(A) = \sup_{\mathbb{P} \leq v} \mathbb{P}(A|B) \tag{36}$$

It follows that for $B \in \mathcal{F}$ we can write:

$$\rho_{|B}(X) = \sup_{\mathbb{P} \leq v} E_{\mathbb{P}}[X|B] = \sup_{\mathbb{P} \leq v_{|B}} E_{\mathbb{P}}[X] = \int X dv_{|B} \tag{37}$$

And of course we can calculate the Choquet integral (37) explicitly in terms

of the real-world probability measure, $\mathbb{P}_0$, as:

$$\int X dv_{|B} = \int_{-\infty}^{0} (v_{|B}(X > x) - 1)dx + \int_{0}^{\infty} v_{|B}(X > x)dx, \tag{38}$$

where

$$v_{|B}(X > x) = \frac{g(\mathbb{P}_0(X > x \cap B))}{g(\mathbb{P}_0(X > x \cap B)) + h(\mathbb{P}_0(X \leq x \cap B))} \tag{39}$$

5.2 Updated distortion functions

Note that we can write (39) as:

$$v_{|B}(X > x) = \frac{g(S_{X|B}(x)\mathbb{P}_0(B))}{g(S_{X|B}(x)\mathbb{P}_0(B)) + h((1 - S_{X|B}(x))\mathbb{P}_0(B))}, \tag{40}$$

where $S_{X|B}(x) = \mathbb{P}_0(X > x|B)$. Thus we can represent $v_{|B}(X > x)$ by:

$$v_{|B}(X > x) = g_u(S_{X|B}(x); \mathbb{P}_0(B)) \tag{41}$$

where $g_u(s; p) : [0, 1] \mapsto [0, 1]$, $p \in (0, 1]$, is the *updated distortion function*, defined by:

$$g_u(s; p) = \frac{g(sp)}{g(sp) + h((1 - s)p)} \tag{42}$$

Thus, the updated distortion function $g_u(s; p)$, can be interpreted as a distortion function which is applied to the conditional survival function of X and parameterised by the probability of the conditioning event B.

The following result summarises some properties of $g_u(s; p)$.

Proposition 7. *Let $g(s) : [0, 1] \mapsto [0, 1]$ be a continuous, twice differentiable, increasing and concave (distortion) function, with $g(0) = 0$ and $g(1) = 1$. Then, the updated distortion function $g_u(s; p) : [0, 1] \mapsto [0, 1]$, with $p \in (0, 1]$, defined by equation (42), satisfies the following properties:*

a) $g_u(s; p)$ is increasing and concave in s with $g_u(0; p) = 0$, $g_u(1; p) = 1$ for all $s \in [0, 1]$, $p \in (0, 1]$.

b) $g_u'(0;p) \geq g'(0),\ g_u'(1;p) \leq g'(1)$ *for all* $p \in (0,1]$.

c) $g_u(s;p) \geq g(s)$ *for all* $s \in [0,1],\ p \in (0,1]$.

d) $g_u(s;1) = g(s)$ *for all* $s \in [0,1]$.

Proof:

a) The first derivative of $g_u(s;p)$ with respect to p is:

$$\frac{\partial}{\partial s} g_u(s;p) = p \frac{g'(sp)h((1-s)p) + g(sp)h'((1-s)p)}{(g(sp) + h((1-s)p))^2} \leq 0, \tag{43}$$

since $g(t)$ and $h(t) = 1 - g(1-t)$ are both positive and increasing. Hence $g_u(s;p)$ is increasing s. To show that it is concave, it is sufficient to show that the numerator of $\frac{\partial}{\partial s} g_u(s;p)$ is decreasing and its denominator increasing in s. We have:

$$\begin{gathered} \tfrac{\partial}{\partial s}(g'(sp)h((1-s)p) + g(sp)h'((1-s)p)) = \\ = pg''(sp)h((1-s)p) - pg(sp)h''((1-s)p) \leq 0, \end{gathered} \tag{44}$$

because g is concave and h convex, and

$$\begin{gathered} \tfrac{\partial}{\partial s}(g(sp) + h((1-s)p)) = pg'(sp) - ph'((1-s)p) = \\ = pg'(sp) - pg'(sp + 1 - p) \geq 0, \end{gathered} \tag{45}$$

because g' is decreasing and $p \leq 1$. Hence $g_u(s;p)$ is concave. $g_u(0;p) = 0$ and $g_u(1;p) = 1$ are obvious.

b) It is:

$$g_u'(0;p) = g'(0)\frac{p}{h(p)} \geq g'(0) \tag{46}$$

and

$$g_u'(1;p) = h'(0)\frac{p}{g(p)} = g'(1)\frac{p}{g(p)} \leq g'(1) \tag{47}$$

c) Follows from a) and b).

d) Obvious.

Proposition 7a) reflects the fact that distorting a probability measure with g_u as in (41) yields a submodular set function. Thus a risk measure constructed using this distortion function will be coherent and comonotonic-additive.

The second and third parts of the proposition have the interpretation that the updated distortion function is stricter than the original one, in the sense that it would yield a higher risk assessment in comparison to an approach where the real-world probability is first updated and then distorted:

$$v_{|B}(X > x) = g_u\left(S_{X|B}(x); \mathbb{P}_0(B)\right) \geq g(S_{X|B}(x)) \Rightarrow$$
$$\rho_{|B}(X) = \int X dv_{|B} \geq \int X d(g(P_0|B)) \tag{48}$$

Insofar, the general conditioning rule for set functions produces a more 'prudent' updated risk measure than the application of the distortion function to the updated probability (see also the discussion in Denneberg (1994b) where the general conditioning rule is shown to be more prudent than the Bayes and Dempster-Shafer conditioning rules).

Proposition 7d) has the interpretation that when conditioning on an event with probability one the updated distortion function reduces to the original one. This means that the risk measure is not going to be updated until some new, unknown until then information, becomes available.

We conclude this section with a note on the updating of distortion risk measures, when the event on which we seek to condition has measure zero. Such a situation frequently arises, for example in the case that the conditioning events are observations of a random variable Y with continuous probability distribution, i.e. $B = \{\omega : Y(\omega) = y\}$. This can cause problems

in updating the risk measure. Consider (40). If $\mathbb{P}_0(B) = 0$ it is obvious that the expression for the updated set function $v_{|B}(X > x)$ is indeterminate. Correspondingly, equation (42) does not yield a value for $p = 0$.

A way to address this problem is to approximate B with sets of measure greater than zero, as proposed by Walley (1991) and Wang and Young (1998). For $\delta > 0$ define the event $B^\delta = \{\omega : Y(\omega) \in [y, \delta)\}$. Updated distorted probabilities can then be calculated by taking the limit:

$$\lim_{\delta \downarrow 0} v_{|B^\delta}(X > x) = \lim_{\delta \downarrow 0} g_u \left(\mathbb{P}_0(X > x | B^\delta); \mathbb{P}_0(B^\delta) \right) \tag{49}$$

It is easy to calculate the above limit, provided that the conditional probabilities $\mathbb{P}_0(X > x|Y = y)$ are well defined. It will be:

$$\lim_{\delta \downarrow 0} v_{|B^\delta}(X > x) = \frac{g'(0)\mathbb{P}_0(X > x|B^\delta)}{g'(0)\mathbb{P}_0(X > x|B^\delta) + g'(1)\mathbb{P}_0(X \leq x|B^\delta)} \tag{50}$$

Thus, using the notation $S_{X|B}(x) = \mathbb{P}_0(X > x|B)$, we can write:

$$\lim_{\delta \downarrow 0} v_{|B^\delta}(X > x) = g_u \left(S_{X|B}(x); 0 \right), \tag{51}$$

where the updated distortion function g_u for $p = 0$ is given by:

$$g_u(s; 0) = \lim_{p \downarrow 0} g_u(s; p) = \frac{s}{s + \frac{g'(1)}{g'(0)}(1 - s)} \tag{52}$$

Equation (52) defines a new class of distortion functions, characterised by the parameter $g'(1)/g'(0) \leq 1$. It is remarkable that when conditioning a distorted probability on a zero-probability event, for any type of differentiable distortion function the updated distortion function will belong to the same class. Furthermore, the updated distortion function only depends on the values of the first derivative of the original distortion function at 0 and 1. We summarise the properties of the distortion functions (52) in the following proposition:

Proposition 8. *Let $g(s) : [0,1] \mapsto [0,1]$ be a continuous, twice differentiable, increasing and concave (distortion) function, with $g(0) = 0$ and $g(1) = 1$. Then, the updated distortion function $g_u(s;0) : [0,1] \mapsto [0,1]$, defined by equation (52), satisfies the following properties:*

a) $g_u(s;0)$ is increasing and concave in s with $g_u(0;0) = 0$, $g_u(1;0) = 1$ for all $s \in [0,1]$.

b) $g'_u(0;0) \geq g'(0)$, $g'_u(1;0) \leq g'(1)$.

c) $g_u(s;0) \geq g(s)$ for all $s \in [0,1]$.

Proof: Essentially same as proposition 7.

5.3 Dynamic risk measurement and capital allocation

We consider now the problem of risk measurement in a dynamic setting. Let the time of the initial risk assessment be 0 and the time horizon with respect to which capital is set be T. A random liability will now be a stochastic process, $X_{t\in[0,T]}$ on a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathbb{P})$. Then we can condition the risk measure of the terminal value X_T on any $\mathcal{F}_t$-measurable event B_t, using (38), (39).

As the updated risk measure will again be a Choquet integral with respect to a submodular set function, is will be a comonotonic-additive coherent risk measure. We can then proceed with the problem of dynamic risk capital allocation in a way similar to the method developed in section 4. Let $X_t^1, X_t^2, \ldots, X_t^n$ be $\mathcal{F}_t$-adapted stochastic processes corresponding to the different liabilities that are being pooled and the aggregate risk process be $Z_t = \sum_j X_t^j$. For simplicity assume that $X_t^1, X_t^2, \ldots, X_t^n$, as well as Z_t,

are Markov processes. The aggregate required capital at time t will now be represented by the stochastic process:

$$\rho_{|B_t}(Z_T) = \sup_{\mathbb{P} \leq v = g(\mathbb{P}_0)} E_{\mathbb{P}}[Z_T | B_t], \tag{53}$$

where the event $B_t \in \mathcal{F}_t$ is defined as:

$$B_t = \{\omega : X_t^1(\omega) = y_1, X_t^2(\omega) = y_2, \ldots, X_t^n(\omega) = y_n\} \tag{54}$$

As discussed in section 5.1, (53) can then be calculated by:

$$\rho_{|B_t}(Z_T) = \int_{-\infty}^{0} (v_{|B_t}(Z_T > z) - 1)dz + \int_{0}^{\infty} v_{|B_t}(Z_T > z)dz, \tag{55}$$

where

$$v_{|B_t}(Z_T > z) = \frac{g(\mathbb{P}_0(Z_T > z \cap B_t))}{g(\mathbb{P}_0(Z_T > z \cap B_t)) + h(\mathbb{P}_0(Z_T \leq z \cap B_t))} \tag{56}$$

As discussed in section 5.2 we can express (56) through an updated distortion function:

$$v_{|B_t}(Z_T > z) = g_u\left(S_{Z_T|B_t}(z); \mathbb{P}_0(B_t)\right)$$

and rewrite the updated risk measure (55) as:

$$\rho_{|B_t}(Z_T) =$$

$$= \int_{-\infty}^{0} \left(g_u\left(S_{Z_T|B_t}(z); \mathbb{P}_0(B_t)\right) - 1\right) dz + \int_{0}^{\infty} g_u\left(S_{Z_T|B_t}(z); \mathbb{P}_0(B_t)\right) dz \tag{57}$$

We can now proceed with determining the stochastic processes, d_t^i, that represent the amount of risk capital allocated to the portfolios, X_t^i. We assume that the allocation of capital to the portfolios is performed at each time $t \in [0, T]$. At each time t the allocation should belong to the non-atomic core of the game. Denote by $\mathcal{X}_t$ the set of random variables measurable with respect to the $\sigma-$algebra $\mathcal{F}_t$, generated by $X_t^1, X_t^2, \ldots, X_t^n$. Let Γ be the

class of real-valued linear functionals on $\mathcal{X}_T$. We require that the stochastic processes d_t^i are such that for all $t \in [0,T]$ there exists a linear functional $\gamma_t \in \Gamma$ such that:

$$\gamma_t(X_T^i) = d_t^i \ \forall i, \ \gamma_t(Z_T) = \rho_{|B_t}(Z_T) \text{ and } \rho_{|B_t}(Y) \geq \gamma_t(Y) \ \forall Y \in \mathcal{X}_T \quad (58)$$

As the set function $v_{|B_t}$ is submodular, all arguments on the existence of solutions in the fuzzy core made in section 4 apply to this case as well. Thus there will always exist a linear functional $\gamma_t \in \Gamma$ such that (58) is true. Furthermore, one such functional is:

$$\gamma_t(X_T) = E[X_T g_u' \left(S_{Z_T|B_t}(Z_T); \mathbb{P}_0(B_t)\right) |B_t] \quad (59)$$

Thus, the capital, d_t^i, allocated to the ith liability will follow the stochastic process:

$$d_t^i = E[X_T^i g_u' \left(S_{Z_T|B_t}(Z_T); \mathbb{P}_0(B_t)\right) |B_t] \quad (60)$$

6 Application with correlated Brownian motions

As an illustrative example, we apply the dynamic capital allocation methodology studied in the previous section to the case that the pooled instruments correspond to correlated Brownian motions with drift. By simulating paths of the liability processes, as well as of the processes representing allocated risk capital, the relationship between correlation order and capital allocation is demonstrated. Brownian motion and stochastic processes based on it are prominent in the financial mathematics literature. Furthermore, the fact that Brownian motion's increments are multi-normally distributed allows for an explicit calculation of the aggregate liability process, while the (Pearson) correlations give an accurate reflection of correlation order.

6.1 Liability processes

Let $W_t \in \mathbb{R}^n$ be an n-dimensional Brownian motion, starting at 0. Define the vector $X_t \in \mathbb{R}^m$ of individual liability processes via the stochastic differential equation:

$$dX_t = \alpha dt + \beta dW_t, \tag{61}$$

where $\alpha \in \mathbb{R}^m$ is a vector of drifts and $\beta \in \mathbb{R}^{m \times n}$ is a matrix of volatilities. It is then apparent that the individual liabilities X_t^i, as well as the aggregate liability Z_t, are themselves drifted Brownian motions. Specifically:

$$dX_t^i = \alpha_i dt + \sum_{j=1}^{n} \beta_{ij} dW_t^j = \alpha_i dt + \bar{\beta}_i d\bar{W}_t^i, \; X_0^i = 0, \tag{62}$$

where $\bar{\beta}_i = \sqrt{\sum_{j=1}^n \beta_{ij}^2}$ and $\bar{W}_t^i$ is a Brownian motion such that $\sum_{j=1}^n \beta_{ij} W_t^j = \bar{\beta}_i \bar{W}_t^i$. Also:

$$dZ_t = \sum_{k=1}^{m} dX_t^k = \bar{\alpha} dt + \bar{\beta} d\bar{W}_t, \; Z_0 = 0, \tag{63}$$

where $\bar{\alpha} = \sum_{k=1}^m \alpha_k$, $\bar{\beta} = \sqrt{\sum_{j=1}^n (\sum_{k=1}^m \beta_{kj})^2}$ and $\bar{W}_t$ is a Brownian motion such that $\sum_{k=1}^m \sum_{j=1}^n \beta_{kj} W_t^j = \bar{\beta} \bar{W}_t$

Define B_t as being the event $B_t = \{\omega \in \Omega : X_t^1(\omega) = x_t^1, ..., X_t^m(\omega) = x_t^m\}$. In order to apply the dynamic allocation methodology of the previous section, we have to determine the conditional probability distribution $F_{X_T^i, Z_T | B_t}(x_T, z_T) = \mathbb{P}_0(X_T^i \leq x_T \cap Z_T \leq z_T | B_t)$.

From the textbook properties of Brownian motion (e.g. Karatzas and Shreve, 1989), we know that the processes X_t^i have independent and normally distributed increments. Specifically we have that, given B_t, X_T is a

normally distributed vector:

$$\begin{pmatrix} X_T^1 \\ X_T^2 \\ \dots \\ X_T^m \end{pmatrix} = \begin{pmatrix} x_t^1 + \alpha_1(T-t) \\ x_t^2 + \alpha_2(T-t) \\ \dots \\ x_t^m + \alpha_m(T-t) \end{pmatrix} + \sqrt{T-t} \begin{pmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2n} \\ \dots & \dots & \dots & \dots \\ \beta_{m1} & \beta_{m2} & \dots & \beta_{mn} \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \\ \dots \\ N_n \end{pmatrix}, \tag{64}$$

where $N \in \mathbb{R}^n$ is a vector of independently distributed normal random variables.

In order to obtain the distribution of the random vector, $(X_T^i \; Z_T)'$ (we denote by A' the transpose of a matrix A), we apply to X_T the transformation:

$$\begin{pmatrix} X_T^i \\ Z_T \end{pmatrix} = D^i X_T, \tag{65}$$

where $D^i \in \mathbb{R}^{2\times m}$, such that $D^i_{1i} = 1$, $D^i_{1j} = 0$, $j \neq i$ and $D^i_{2j} = 1$, $\forall j$. From the properties of multivariate normal distributions (e.g. Embrechts et al., 2002), we know that the random vector $(X_T^i \; Z_T)'$ will be normally distributed, with mean $D^i E[X_T]$ and covariance matrix $(T-t)(D^i\beta)(D^i\beta)'$. Calculations yield that $(X_T^i \; Z_T)'$ will be normal with mean:

$$\begin{pmatrix} x_t^i + \alpha_i(T-t) \\ z_t + \sum_{k=1}^m \alpha_k(T-t) \end{pmatrix} \tag{66}$$

and covariance matrix:

$$(T-t) \begin{pmatrix} \sum_{j=1}^n \beta ij^2 & \sum_{j=1}^n \sum_{k=1}^m \beta_{ij}\beta_{kj} \\ \sum_{j=1}^n \sum_{k=1}^m \beta_{ij}\beta_{kj} & \sum_{j=1}^n \left(\sum_{k=1}^m \beta_{kj}\right)^2 \end{pmatrix} \tag{67}$$

Thus, for any $t \in [0,1]$, the correlation r_i between X_T^i and Z_T given B_t is:

$$r_i = \frac{\sum_{j=1}^n \sum_{k=1}^m \beta_{ij}\beta_{kj}}{\left(\sum_{j=1}^n \beta ij^2 \sum_{j=1}^n \left(\sum_{k=1}^m \beta_{kj}\right)^2\right)^{1/2}} \tag{68}$$

6.2 The distortion function

We use the exponential distortion function:

$$g(s) = \frac{1 - \exp(-hs)}{1 - \exp(-h)}, \quad h > 0 \tag{69}$$

This function has first derivative:

$$g'(s) = \frac{h \exp(-hs)}{1 - \exp(-h)} \tag{70}$$

Since the events on which the liability processes are conditioned have zero probability, the updated distortion function (52) becomes:

$$g_u(s; 0) = \frac{s}{s + \exp(-h)(1 - s)} \tag{71}$$

The functions g and g_u are shown in figure 1 for $h = 1$. It can be seen that the updated distortion function takes higher values than the original one.

6.3 Numerical example

We consider the 3-vector of liabilities $X_t = [X_t^1 \ X_t^2 \ X_t^3]'$:

$$X_t = \alpha dt + \beta W_t, \tag{72}$$

where $W_t = [W_t^1 \ W_t^2 \ W_t^3]^T$ is a 3-dimensional Brownian motion and

$$\alpha = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}, \quad \beta = \frac{1}{3} \begin{pmatrix} \sqrt{1.5} & -\sqrt{1.5} & 0 \\ 0 & \sqrt{1.5} & \sqrt{1.5} \\ 1 & 1 & 1 \end{pmatrix} \tag{73}$$

Note that each of the individual liability processes X_t^i is a Brownian motion with volatility $\frac{\sqrt{3}}{3}$ and drift 0.2. According to equation (68), given

B_t, the correlations between each individual liability and the aggregate at T are:

$$\begin{aligned} r_1 &= 0.26 \\ r_2 &= 0.69 \\ r_3 &= 0.95 \end{aligned} \tag{74}$$

Thus, while the individual liability processes have identical dynamics, they differ in their correlation to the aggregate liability, i.e. $(X_T^1, Z_T), (X_T^2, Z_T)$ and (X_T^3, Z_T) are all members of the same class $R(F_1, F_2)$. In fact, for bivariate normal distributions, correlation order can be completely determined through Pearson's correlation coefficient. Consider two random variables Y_1, Y_2 distributed according to the bivariate normal distribution, with correlation r and, for simplicity, unit means and standard deviations:

$$F_{Y_1,Y_2}(y_1, y_2) = \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} \frac{1}{2\pi(1-r^2)^{1/2}} \exp\left\{ -\frac{s^2 - 2rst + t^2}{2(1-r^2)} \right\} dsdt \tag{75}$$

It is apparent that F_{Y_1,Y_2} is increasing in r. Thus, given B_t, it is:

$$(X_T^1, Z_T) \leq_{corr} (X_T^2, Z_T) \leq_{corr} (X_T^3, Z_T) \tag{76}$$

Using an exponential distortion function with $h = 1$, we can determine the capital allocated to each liability X_t^i. It thus is:

$$d_t^i = E[X_T^i g_u' \left(S_{Z_T|B_t}(Z_T); 0 \right) | B_t], \tag{77}$$

where joint distribution function of $(X_T^i, Z_T)'$, conditional upon B_t, is normal with mean vector and covariance matrix as determined by equations (66) and (67) respectively.

Paths of the liability processes, as well as of the processes representing dynamic risk measures and allocated capital, are simulated with time horizon $T = 5$. In figure 2 paths of the individual liability processes X_t^1, X_t^2, X_t^3

are shown. In figure 3 the risk measure of the aggregate liability, $\rho_{|B_t}(Z_t)$ is compared to the sum of the risks of the individual liabilities, $\sum \rho_{|B_t}(X_t^j)$. The difference between the two lines represents aggregate savings from pooling the liabilities. Finally, in figures 4, 5 and 6, the risk measure of each liability, $\rho_{|B_t}(X_t^i)$, is compared to the capital, d_t^2, allocated to it. The difference between the lines represents the savings from pooling that the holder of X_t^i makes. It can be seen that the highest savings are made by the first player, lower savings made by the second, while the third one almost makes no savings at all. This is consistent with the correlation ordering (76) of the pairs of random variables, $(X_T^1, Z_T), (X_T^2, Z_T)$ and (X_T^3, Z_T).

7 Conclusions

The problem of allocating capital to pooled portfolios of risky positions was studied for the case of risk measures based on distorted probabilities and previously obtained results were extended. The non-atomic core was defined as a generalisation of the fuzzy core in order to account for the potential formation of non-linear portfolios. Using the correlation order discussed in Dhaene and Goovaerts (1996) it was then shown that the capital allocation methodology derived by Tsanakas and Barnett (2002) is consistent with the non-atomic core property defined in this paper. Furthermore, correlation order gave us the means to formulate explicitly the effect of dependence on the capital allocated to a portfolio. Specifically, for two portfolios whose payoffs are equal in distribution, more capital is allocated to the one which is more correlated to the aggregate risk that its holder is exposed to.

The requirement for consistency between market prices and prices orig-

inating in capital allocation was discussed, drawing from an equilibrium model by Tsanakas and Christofides (2003). If market prices are determined in a risk exchange by agents' minimising their (distortion) risk measure, then market prices are consistent with the capital allocation methodology proposed in this paper.

Next, the need to extend the capital allocation methodology to a dynamic setting was addressed. Such an extension goes through defining a dynamic version of the distortion risk measures used in the paper. The general conditioning rule for set functions (Denneberg, 1994b) provided a suitable mechanism for updating distortion risk measures. It turns out that the updated risk measure is again a distortion risk measure with respect to a modified distortion function. This function, which we call an updated distortion, dominates the original distortion function and thus yields more prudent risk assessments. Finally, a numerical example of dynamic risk measurement and capital allocation was presented, with correlated Brownian motions standing for liability processes. The example demonstrated the effect of correlation on the capital allocation.

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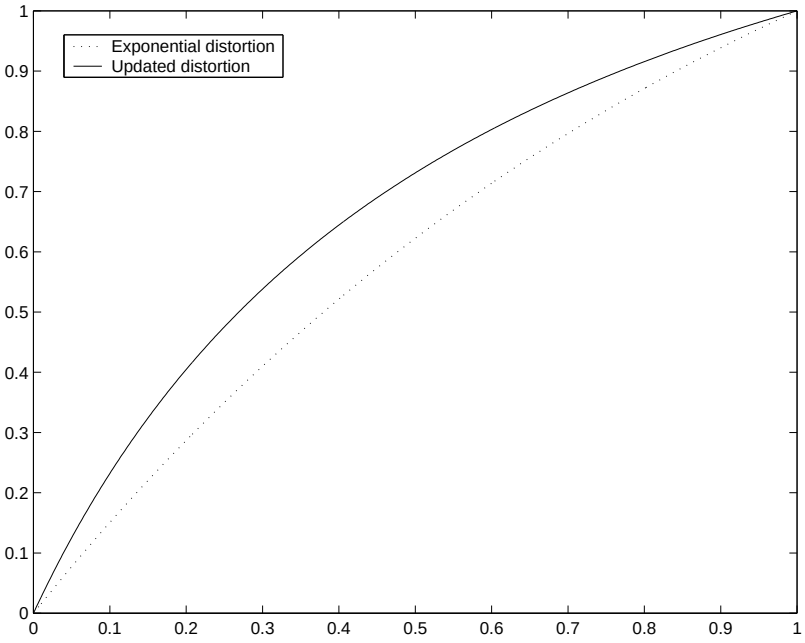

Figure 1: Exponential and updated exponential distortion functions (h=1).

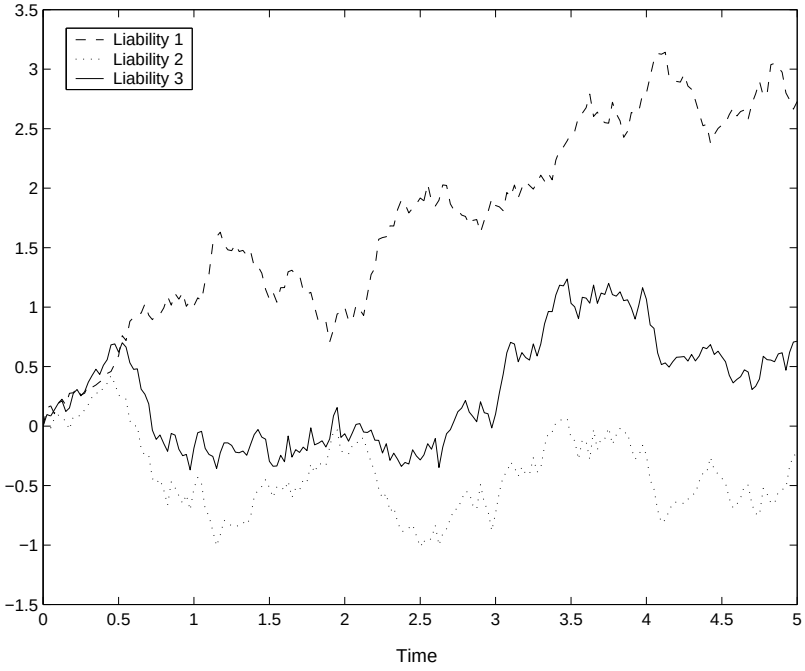

Figure 2: Simulated path of individual liabilities, X_t^1, X_t^2, X_t^3.

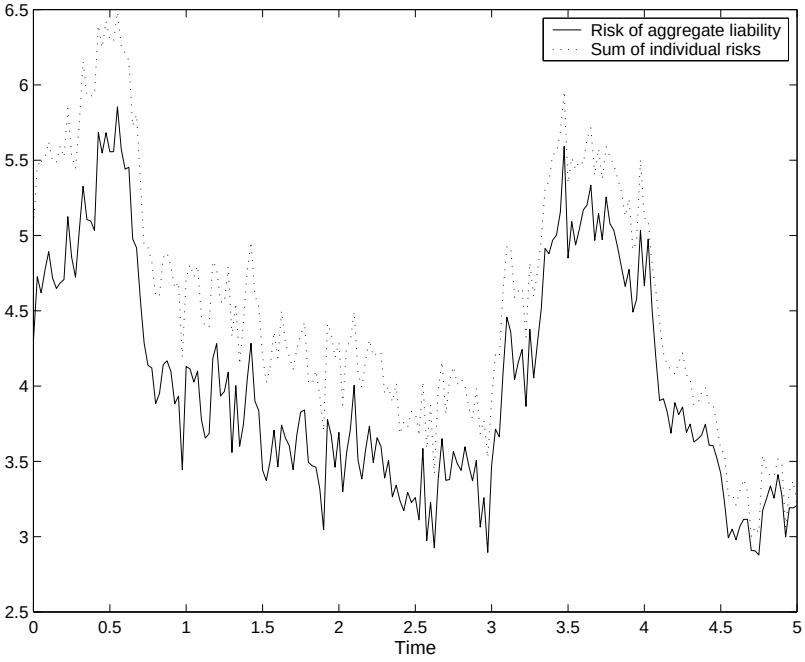

Figure 3: Risk measure of aggregate liability, $\rho_{|B_t}(Z_t)$, versus sum of risks of individual liabilities, $\sum \rho_{|B_t}(X_t^j)$.

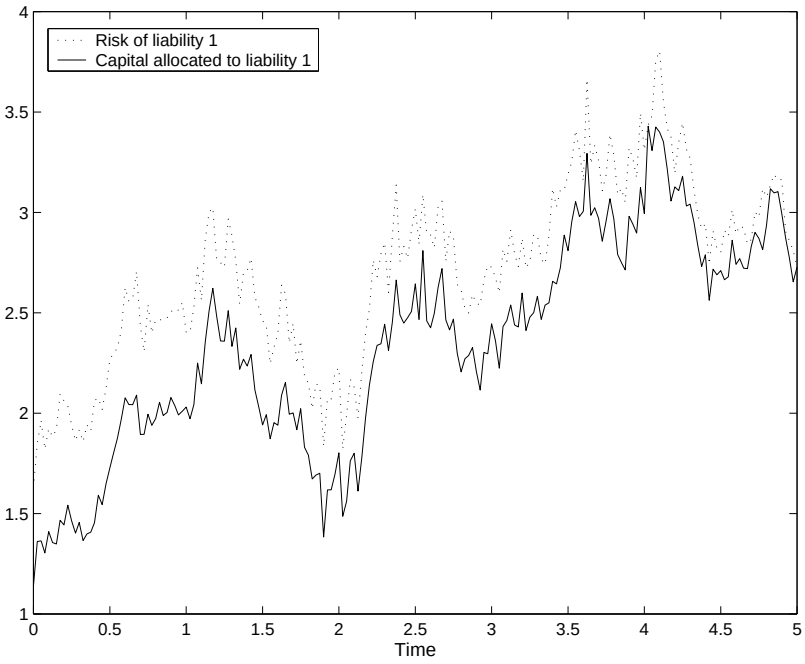

Figure 4: Risk measure, $\rho_{|B_t}(X_t^1)$, and capital, d_t^1, allocated to the first liability, X_t^1.

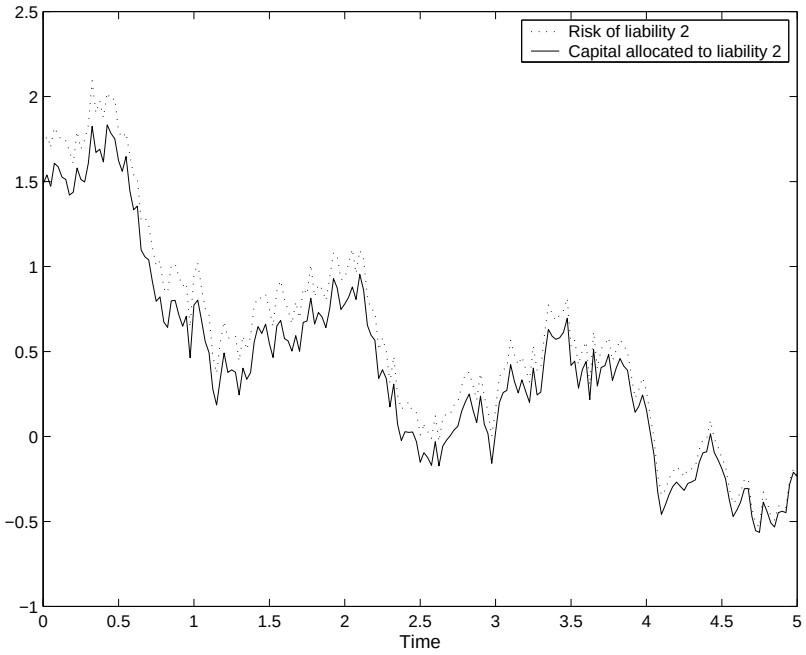

Figure 5: Risk measure, $\rho_{|B_t}(X_t^2)$, and capital, d_t^2, allocated to the second liability, X_t^2.

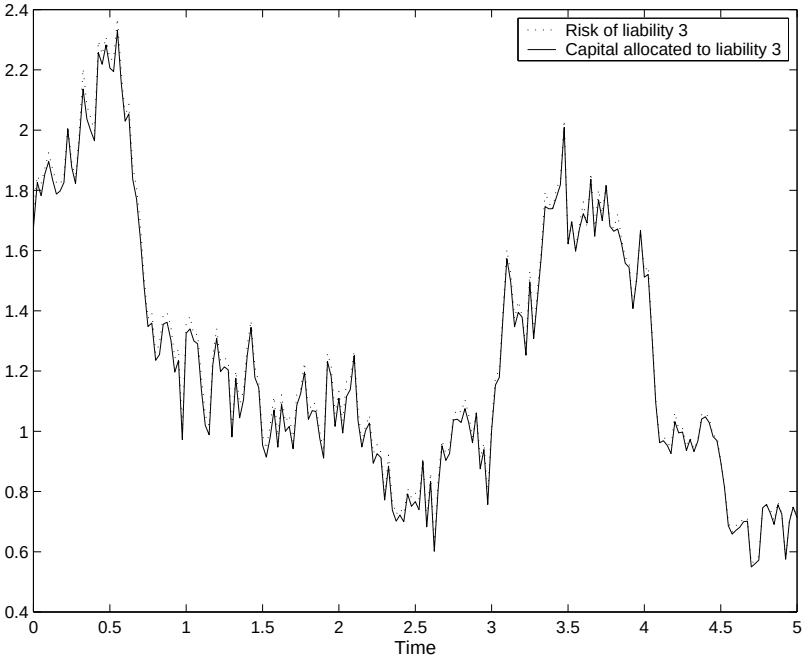

Figure 6: Risk measure, $\rho_{|B_t}(X_t^3)$, and capital, d_t^3, allocated to the third liability, X_t^3.