First-Differenced Inference for Panel Factor Series

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Abstract

We complement existing inferential theory for panel factor models by deriving the asymptotics for the first differences of the estimated factors and common components obtained from a non-stationary panel factor model. As an application, we propose an estimator for the long run variance of the common components.

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1 Introduction

Consider the non-stationary panel factor series

$$X_{it} = \lambda_i' F_t + e_{it}, \tag{1}$$

where $i = 1, ..., n$, $t = 1, ..., T$, F_t is a k-dimensional vector with DGP $F_t = F_{t-1} + \varepsilon_t$, and e_{it} is stationary. Bai (2004) develops the inferential theory for (1) - specifically, for F_t, λ_i, and for the non-stationary common component $C_{it} \equiv \lambda_i' F_t$. Alternatively, one may also consider the stationary, first-differenced model

$$x_{it} = \lambda_i' f_t + u_{it}, \tag{2}$$

where $x_{it} = \Delta X_{it}$ and $f_t = \Delta F_t$. In this case, estimators for λ_i, f_t and $c_{it} \equiv \lambda_i' f_t$ ($\hat{\lambda}_i$, $\hat{f}_t$ and $\hat{c}_{it}$ respectively) are provided by Bai (2003).

This note complements the existing inferential theory on (1) and (2), by studying estimation based on the first difference of the estimator of F_t, say $\hat{F}_t$, computed from (1). Indeed, instead of estimating f_t from (2), one could use $\tilde{f}_t = \hat{F}_t - \hat{F}_{t-1}$. Thence, using the either the estimated λ_i from (1), say $\hat{\lambda}_i$, or estimating λ_i from (2) using $\tilde{f}_t$, one can compute the first differenced estimator of c_{it} as $\tilde{c}_{it} \equiv \tilde{\lambda}_i' \tilde{f}_t$. Estimating f_t and c_{it} is useful for various purposes; in this paper we consider the estimation of the long run covariance matrices (henceforth, LRV) of F_t and C_{it}.

Some results have already been developed by Trapani (2012) in the context of bootstrapping nonstationary factor models. This note completes the inferential theory for the first-differenced estimators, reporting rates of convergence for: $\tilde{f}_t$; for the estimator of λ_i based on $\tilde{f}_t$, say $\tilde{\lambda}_i$; and for a weighted-sum-of-covariances estimator of the LRV of C_{it} based on $\tilde{f}_t$.

2 Results

All results are derived under the same assumptions as in Bai (2003, 2004), omitted for brevity. Henceforth, we define the $r \times r$ rotation matrix $H \equiv \left(\frac{\hat{F}'F}{T^2}\right)\left(\frac{\Lambda'\Lambda}{n}\right)$, where $F = [F_1, ..., F_T]'$ ($\hat{F}$ is defined similarly) and $\Lambda = [\lambda_1, ..., \lambda_n]'$. The number of factors, r, is assumed known.

We firstly report a Lemma containing rates of convergence for $\tilde{f}_t = \hat{F}_t - \hat{F}_{t-1}$.

Lemma 1 *As $(n, T) \to \infty$, it holds that*

$$\tilde{f}_t - H'f_t = O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{T^{3/2}}\right), \quad (3)$$

$$\max_{1 \le t \le T} \left\| \tilde{f}_t - H'f_t \right\| = O_p\left(\frac{1}{T}\right) + O_p\left(\sqrt{\frac{T}{n}}\right), \quad (4)$$

$$\frac{1}{T}\sum_{t=1}^{T}\left(\tilde{f}_t - H'f_t\right) u_{it} = O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{T^{3/2}}\right). \quad (5)$$

Under $\frac{n}{T^3} \to 0$, $\sqrt{n}\left(\tilde{f}_t - H'f_t\right) \xrightarrow{d} QN(0, \Upsilon_t)$, where Q is defined in Theorem 2 in Bai (2004, p. 148) and $\Upsilon_t \equiv \lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} E\left(\lambda_i\lambda_j' u_{it}u_{jt}\right)$.

Lemma 1 states that rates and uniform convergence of $\tilde{f}_t - H'f_t$ are the same as for $\hat{F}_t - H'F_t$ - see Lemma 2 in Bai (2004). This can also be compared with the results in Theorem 2 in Bai (2003), where it is shown that $\hat{f}_t - H_1'f_t = O_p\left(n^{-1/2}\right) + O_p\left(T^{-1}\right)$ - in general, the rotation matrices H and H_1 are different. Therefore, heuristically, $\tilde{f}_t$ should be a better estimator than $\hat{f}_t$ for the space spanned by f_t, especially when T is small. Lemma 1 is a complement, regarding the properties of $\tilde{f}_t$, to Lemma A.1 in Trapani (2012).

We now turn to presenting results on the estimation of the loadings λ_i. To this end, it is possible to use the estimator of λ_i from (1), say $\hat{\lambda}_i$. Bai (2004, p. 148-149) shows that $\hat{\lambda}_i$ is "superconsistent", viz. $\hat{\lambda}_i - H^{-1}\lambda_i = O_p\left(T^{-1}\right)$; also, the rate of convergence does not depend on n. Alternatively, it is possible to estimate loadings as $\tilde{\lambda}_i = \left[\sum_{t=1}^{T} \tilde{f}_t \tilde{f}_t'\right]^{-1} \left[\sum_{t=1}^{T} \tilde{f}_t x_{it}\right]$. Let $\Sigma_\varepsilon \equiv E\left(\varepsilon_t \varepsilon_t'\right) = E\left(f_t f_t'\right)$; it holds that:

Proposition 1 *As $(n, T) \to \infty$ it holds that $\tilde{\lambda}_i - H^{-1}\lambda_i = O_p\left(n^{-1}\right) + O_p\left(T^{-1/2}\right)$. Under $\frac{\sqrt{T}}{n} \to 0$, $\sqrt{T}\left(\tilde{\lambda}_i - H^{-1}\lambda_i\right) \xrightarrow{d} N\left(0, V_i\right)$ with $V_i = \left(H'\Sigma_\varepsilon H\right)^{-1}\left(H'\Phi_i H\right)\left(H\Sigma_\varepsilon H'\right)^{-1}$ and $\Phi_i = \lim_{T\to\infty} E\left(f_t f_s' u_{it} u_{is}\right)$.*

Proposition 1 states that the properties of $\tilde{\lambda}_i$ are (apart from the rotation matrix H) the same as in Theorem 2 in Bai (2003), where estimation of λ_i is based on using (2). This can be compared with $\hat{\lambda}_i$, whose convergence rate does not depend on n and it is faster in T.

Based on Lemma 1 and Proposition 1, consider the first-differenced estimator of the common components c_{it}, $\tilde{c}_{it} \equiv \hat{\lambda}_i' \tilde{f}_t = \hat{C}_{it} - \hat{C}_{it-1} = \hat{\lambda}_i'\left(\hat{F}_t - \hat{F}_{t-1}\right)$. By combining the results above, and using Lemma 3 in Bai (2004), we have $\tilde{c}_{it} - c_{it} = \hat{\lambda}_i' \tilde{f}_t - \lambda_i' f_t = \left(\hat{\lambda}_i - H^{-1}\lambda_i\right)' \tilde{f}_t + \left(\tilde{f}_t - H' f_t\right)' H^{-1}\lambda_i + \left(\hat{\lambda}_i - H^{-1}\lambda_i\right)'\left(\tilde{f}_t - H' f_t\right) = O_p\left(n^{-1/2}\right) + O_p\left(T^{-1}\right)$. Using Theorem 3 in Bai (2004) on the limiting distribution of $T\left(\hat{\lambda}_i - H^{-1}\lambda_i\right)$, the asymptotic distribution of $\tilde{c}_{it} - c_{it}$ has the same properties as in Theorem 4 in Bai (2004, p. 149).

The results in Lemma 1 and Proposition 1 can be combined in order to estimate the LRV of F_t and C_{it}. Let Σ_F be the LRV of F_t, and define similarly the LRV of C_{it} as Σ_C. A rotation of Σ_F can be estimated as

$$\hat{\Sigma}_F = \hat{\gamma}_0^F + \sum_{j=1}^{h}\left(1 - \frac{j}{h+1}\right)\left(\hat{\gamma}_j^F + \hat{\gamma}_j^{F\prime}\right),$$

where h is a bandwidth parameter and $\hat{\gamma}_j^F \equiv T^{-1}\sum_{t=j+1}^{T} \tilde{f}_t \tilde{f}_{t-j}'$. Of course, $\hat{\Sigma}_F$ does not estimate Σ_F consistently due to rotational indeterminacy; it can be expected that $\left\|\hat{\Sigma}_F - H'\Sigma_F H\right\| = o_p(1)$. Similarly, Σ_C can be estimated either as $\hat{\Sigma}_C = \hat{\lambda}_i' \hat{\Sigma}_F \hat{\lambda}_i$, or as

$\tilde{\Sigma}_C = \tilde{\lambda}_i' \hat{\Sigma}_F \tilde{\lambda}_i$. By virtue of Proposition 1, $\hat{\Sigma}_C$ should be better, and we focus our attention on it.

Theorem 1 *Assume that* $\sum_{j=0}^{\infty} j^s \left|\gamma_j^F\right| < \infty$. *It holds that*

$$\left\|\hat{\Sigma}_C - \Sigma_C\right\| = O_p\left(\frac{h}{\sqrt{T}}\right) + O_p\left(\frac{h}{n}\right) + O_p\left(\frac{1}{h}\right). \tag{6}$$

Theorem 1 contains rates of convergence for $\hat{\Sigma}_C$, which is consistent provided that $h \to \infty$ and $h/\min\left\{n, \sqrt{T}\right\} \to 0$. This also gives a selection rule for h; the choice of the bandwidth that maximizes the speed of convergence is $h^* = O\left(\min\left\{T^{1/4}, n^{1/2}\right\}\right)$.

We point out that $\hat{\Sigma}_C$ is not the only possible estimator for Σ_C. One could consider estimating a rotation of Σ_F using $\hat{f}_t$ calculated from (2). Given that H differs depending on whether (1) or (2) is used, in this case it is necessary to employ the estimated loadings from model (2), which have the same properties as $\tilde{\lambda}_i$ in Proposition 1. Based on this, and on Lemma 1, it can be expected that this estimator does not converge as fast as $\hat{\Sigma}_C$. Similarly, it is possible to estimate Σ_C using the x_{it}s directly. Theoretically, this estimator should work, since e_{it} is stationary, although this may introduce some noise in the estimation of Σ_C.

Proofs

Proof of Lemma 1. See the online material.

Proof of Proposition 1. Let $\delta_{nT} \equiv \min\{\sqrt{n}, T\}$. By definition, $\tilde{\lambda}_i - H^{-1}\lambda_i = \left(\sum_{t=1}^T \tilde{f}_t \tilde{f}_t'\right)^{-1} \times \left[\sum_{t=1}^T H' f_t u_{it} + \sum_{t=1}^T \tilde{f}_t' \left(\tilde{f}_t - H' f_t\right)\lambda_i + \sum_{t=1}^T \left(\tilde{f}_t - H' f_t\right) u_{it}\right] = \left(\sum_{t=1}^T \tilde{f}_t \tilde{f}_t'\right)^{-1} (I + II + III)$. Consider the denominator. By Lemma A.1 in Trapani (2012), $\sum_{t=1}^T \left\|\tilde{f}_t - H' f_t\right\|^2 = O_p\left(T\delta_{nT}^{-2}\right)$ and $\sum_{t=1}^T \left(\tilde{f}_t - H' f_t\right)' f_t = O_p\left(\sqrt{T}\delta_{nT}^{-1}\right) + O_p\left(\frac{\sqrt{T}}{n}\right)$. Hence, $\sum_{t=1}^T \tilde{f}_t \tilde{f}_t' = H' \sum_{t=1}^T f_t f_t' H + o_p(T) = O_p(T)$. As regards the numerator, $I = O_p\left(\sqrt{T}\right)$ by a CLT. Using the same arguments as for the denominator, $II = O_p\left(\sqrt{T}\delta_{nT}^{-1}\right) + O_p\left(\frac{\sqrt{T}}{n}\right)$. Hence, $\tilde{\lambda}_i - H^{-1}\lambda_i = O_p\left(T^{-1/2}\right) + O_p\left(n^{-1}\right)$. Finally, $III = O_p\left(n^{-1/2}\right) + O_p\left(T^{-3/2}\right)$ using (5). The limiting distribution follows from noting that, when $\frac{\sqrt{T}}{n} \to 0$, the dominating $O_p\left(T^{-1/2}\right)$ term is $\left(H' \sum_{t=1}^{T} f_t f_t' H\right)^{-1} \left(\sum_{t=1}^{T} H' f_t u_{it}\right)$.

Proof of Theorem 1. We omit H for simplicity when this does not cause ambiguity. We start by showing that $\left\|\hat{\Sigma}_F - H'\Sigma_F H\right\| = O_p\left(\frac{h}{\sqrt{T}}\right) + O_p\left(\frac{h}{n}\right) + O_p\left(\frac{1}{h}\right)$. By definition, $\Sigma_F = \gamma_0^F + \sum_{j=1}^{\infty}\left(\gamma_j^F + \gamma_j^{F\prime}\right)$, whence

$$
\begin{aligned}
\hat{\Sigma}_F - \Sigma_F &= \left(\hat{\gamma}_0^F - \gamma_0^F\right) + \sum_{j=1}^{h}\left(1 - \frac{j}{h+1}\right)\left[\left(\hat{\gamma}_j^F + \hat{\gamma}_j^{F\prime}\right) - \left(\gamma_j^F + \gamma_j^{F\prime}\right)\right] \\
&\quad - \sum_{j=1}^{h}\left(\frac{j}{h+1}\right)\left(\gamma_j^F + \gamma_j^{F\prime}\right) - \sum_{j=h+1}^{\infty}\left(\gamma_j^F + \gamma_j^{F\prime}\right) \\
&= I - II - III.
\end{aligned}
$$

Consider I. We have $\hat{\gamma}_0^F - \gamma_0^F = T^{-1}\sum_{t=j+1}^{T} \tilde{f}_t \tilde{f}_t' - \gamma_0^F = \left(T^{-1}\sum_{t=j+1}^{T} f_t f_t' - \gamma_0^F\right) - T^{-1}\sum_{t=j+1}^{T}\left(\tilde{f}_t - f_t\right) f_t' - T^{-1}\sum_{t=j+1}^{T} f_t\left(\tilde{f}_t - f_t\right)' + T^{-1}\sum_{t=j+1}^{T}\left(\tilde{f}_t - f_t\right)\left(\tilde{f}_t - f_t\right)' = I_a + I_b + I_b' + I_c$. The CLT yields $I_a = O_p\left(T^{-1/2}\right)$; as far as I_b and I_c are concerned, Lemma A.1 in Trapani (2012) entails that they are both $O_p\left(n^{-1}\right) + O_p\left(T^{-2}\right)$. The same holds for $\hat{\gamma}_j^F - \gamma_j^F$; putting all together $I = O_p\left(hT^{-1/2}\right) + O_p\left(hn^{-1}\right)$. Standard arguments yield $II = O\left(h^{-1}\right)$ and $III = o\left(h^{-s}\right)$. The Theorem follows from $\hat{\lambda}_i - H^{-1}\lambda_i = O_p\left(T^{-1}\right)$.

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