A dynamic route choice model for public transport networks with boarding queues

The concepts of optimal strategy and hyperpath were born within the framework of static frequency-based public transport assignment, where it is assumed that travel times and frequencies do not change over time and no overcrowding occurs. However, the formation of queues at public transport stops can prevent passengers from boarding the first vehicle approaching and can thus lead to additional delays in their trip. Assuming that passengers know from previous experience that for certain stops/lines they will have to wait for the arrival of the 2nd, 3rd, ..., $k^{\text{-th}}$ vehicle, they may alter their route choices, thus resulting in a different assignment of flows across the network. The aim of this paper is to investigate route choice behaviour changes as a result of the formation and dispersion of queues at stops within the framework of optimal travel strategies. A new model is developed, based on modifications of existing algorithms.

1. Introduction

It has been largely acknowledged in the last decades that urban sustainable development needs to overcome the dependence on the private car (Newman and Kenworthy 1999, European Commission 2009) and requires a modal shift towards public transport, as this leads to better performance over private transport with regards to the six *sub-objectives for sustainability* (May 2001 cited Black *et al*. 2002).

In the context of promoting a transfer of passenger mobility from individual to collective means, it is important to improve the quality of service provided (i.e. total travel time reduction, increased service regularity, increased comfort on board). When this scope cannot be pursued by building more suitable infrastructure, which can be politically, financially, and environmentally constrained, Advanced Traveller Information Systems (ATIS) could play a major role if they were able to provide travellers with accurate route guidance which can consider updated and useful information about network conditions. At the present time this is not the case, as journey planners do not currently consider bus travel time which may vary with the time of the day due to recurrent road congestion. The current journey planners also do not take into account queuing time at stops and stations due to passenger congestion on the transit network. From an assignment perspective, this may lead to passengers changing their route and mode choice, time of departure and sometimes even their final destination, and while this is a major problem in large cities' public transport networks, there does not seem to be any broad agreement in the literature on how this phenomenon should be modelled.

Consequently, a step forward in the public transport modelling may be achieved by developing a new route choice model, as presented in this work, which is capable of considering travel time variability as well as the formation and dispersion of passenger queues at stops in densely connected transit networks. On the other hand, although the effects on departure time choice and mode choice are crucial for the development of a multimodal and dynamic assignment procedure, they are out of the scope of the present study.

The application of this route choice model is twofold. On one hand it can be embedded in a dynamic assignment model to capture the formation and dispersion of passenger queues at transit stops and evaluate the resulting increase in waiting times. On the other hand, when queues length is estimated by means of transit assignment models, it can benefit *dynamic* journey planners, enabling them consider congestion patterns on the transit network.

The rest of the paper is organised as follows. In next section, the background research is presented, while the methodology is explained in Section 3. In Section 4, numerical examples will be presented and conclusions will be drawn in Section 5. Details about the solution algorithm are provided in the Appendix.

2. Background research

Much of previous research on route choice in public transport networks has focussed on static conditions, where it is assumed the relevant model variables, such as travel times and line frequencies, are fixed, and passengers can always board the first approaching carrier. In this context, if no service schedule is published or services are highly frequent and/or unreliable, users have no explicit knowledge about carriers' arrival time at transit stops. So, if there are two or more competing lines, so-called *common* lines (Chriqui and Robillard 1975), they may face the question if it is more convenient to board the first approaching line or wait for another, they consider more convenient.

Some authors (Spiess 1983, Spiess 1984, Nguyen and Pallottino 1988, Spiess and Florian 1989) prove that, when there is this uncertainty, rather than the single shortest itinerary between origin and destination, route choice can be modelled as the selection of the *optimal strategy* (Spiess and Florian 1989), and graphically represented as the *shortest hyperpath* on an oriented hypergraph[1] modelling the transit network (Nguyen and Pallottino 1988, Wu *et al*. 1994, Nguyen *et al*. 1998).

The optimal strategy is selected *pre-trip* and, starting from the origin, involves the iterative sequence of: walking to a public transport stop or to the destination, selecting the *attractive* (Nguyen and Pallottino 1988) lines to board and, for each of them, the stop where to alight. Once travelling towards the destination, if two or more attractive lines are available at a stop, the best option is to board the first approaching (Spiess 1983, 1984).

It is well known (Billi *et al*. 2004, Nökel and Wekeck 2007) that this definition of optimal travel strategy only applies when:

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[1] For a detailed review of directed hypergraphs and their applications, we refer the reader to Gallo *et al.* (1993).

- passengers have no explicit knowledge about the carrier's arrival times at transit stops and the available capacities of arriving carriers;
- the vehicle arrivals of different lines at the stop are not synchronized, and, for each line, follow a Poisson distribution;
- passenger arrivals at stops are not timed to coincide with vehicle arrivals; and
- passengers try to minimize their total expected travel time to their destinations.

As such, formation and dispersion of passenger queues are not captured, and the dependency of waiting time and passengers' distribution among the different *attractive* lines from overcrowding is not considered. Recurrent passenger congestion is one of the major problems faced by large city public transport networks and the distortions brought about by neglecting this phenomenon can be significant, as explained by the following example.

Consider a small network with two nodes, origin and destination, and two lines that have the same travel time to destination upon boarding (10 minutes). As detailed in Table I, Line 1 is more frequent but is also congested and passengers cannot board the first vehicle approaching.

Table I: Line 1 and Line 2 congestion levels, average headways and travel time upon boarding. Line 1 is congested and passengers are not able to board the first vehicle.

	Average frequency [min^{-1}]	Travel time to destination upon boarding [min]	Congestion level
Line 1	1/3	10	Congested
Line 2	1/6	10	Not congested

If congestion is disregarded, passengers' distribution (i.e. the probability to board Line 1 or Line 2) and total expected waiting time solely depend on the average frequencies of the services, as in (Nguyen Pallottino 1988, Spiess and Florian 1989), and their values are shown in Table II. The results are clearly distorted because, although congested, Line 1 attracts the major percentage of passengers and its congestion would become even more severe. Thus, if embedded in an assignment procedure, this route choice would not lead to equilibrium conditions.

Table II: Line 1 and Line 2 boarding probability and total expected waiting time at the considered stop.

	Line 1	Line 2	Origin Stop
Boarding probability	0.67	0.33	-
Total Expected waiting time [min]	-	-	2

Most of the research carried out to overcome this shortcoming in the context dense public transport networks, where *common* lines may be available from the same stop, has dealt with overcrowding in case of passengers *mingling* at the stop (De Cea and Fernandez 1993, Marcotte and Nguyen 1998, Cominetti and Correa 2001, Kurauchi *et al*. 2003, Cepeda *et al*. 2006, Schmöcker *et al*. 2008, Leurent *et al*. 2011, Leurent and Benezech 2011). This behavioural assumption implies that no waiting priority is respected and, in case of oversaturation, all passengers waiting at a stop have the same probability to board the next carrier to approach (provided the carrier is attractive). Thus a possible simple solution (De Cea and Fernandez 1993) would consider the *effective frequency*, namely the line frequency perceived by waiting passengers that decreases as the probability of not boarding its first arriving carrier increases. If the stop in the previous example is considered and we assume the 'fail-to-board' (Schmöcker *et al*. 2008) probability for Line 1 is 0.5, while passengers are always able to board a carrier of Line 2, then effective frequencies, passengers' distribution and total expected waiting time are those displayed in Table III. The share of Line 1 decreases of about 30% in favour of Line 2 and, as expected, the consideration of overcrowding leads to an increase of the total expected waiting time.

Table III: Effective frequency and boarding probability of Line 1 and Line 2; total expected waiting time at the considered stop.

	Line 1	Line 2	Origin Stop
Effective frequency [min^{-1}]	1/6	1/6	-
Boarding probability	0.5	0.5	-
Total Expected waiting time [min]	-	-	3

For stations and stops with large platforms, it is acceptable to assume passengers mingle and apply results from the afore-mentioned studies. However, in urban bus networks the stop layout is usually such that, when passenger congestion occurs, users arriving at the stop would join a FIFO (*first-in first-out*) queue and board the first line of their attractive set that becomes available. Models based on the mingling queuing protocol are clearly not applicable to the latter scenario. Early attempts to overcome this shortcoming are made by Gendreau (1984) and Bouzaïene-Ayari (1988) by using a bulk queue model, but its complexity prevents practical applications of the models. In a latter study, Leurent and Benezech (2011) developed a model based on the ‘attractivity threshold’, but in this case rather than assuming the route is chosen *pre-trip*, authors consider a completely *adaptive* behaviour at transit stops.

On the other hand, we prove that in the context of commuting passengers, who know by previous experience the number of carrier passages they must let go before being able to board, expected waiting times and passenger distribution among attractive lines may be calculated by extending the formulas presented by Nguyen and Pallottino (1988) and Spiess and Florian (1989). Results obtained with our model will be compared with those detailed in Table II (uncongested model) and Table III (effective frequency model), and it will be shown that the FIFO queue model penalises the congested line, in terms of passengers' distribution, more than the uncongested and the effective frequency model, while the total expected waiting time increases.

Prior to that, the proposed route choice model will be introduced in the next section, together with the mathematical framework and the graph representing the transit network.

3. Methodology

3.1 Network model: notation and definitions

The transit network comprises a set of lines $\Im \subseteq \aleph$ and, together with the pedestrian network, it is represented by a directed *hypergraph* (Gallo *et al.* 1993) $HG = \{N, E\}$, where $N = \{i \mid i = 1, 2, \ldots, n\}$ is the node set and $E = \{e \mid e = 1, 2, \ldots, m\}$ is the hyperarc set.

- $N_i^+ = \{j \mid (i,j) \in E\}$
- $N_i^- = \{j \mid (j,i) \in E\}$
- e: generic hyperarc, with $TL(e) \in N$ defined as the tail and $HD(e) \subset N$ defined as the head of the hyperarc. It should be noted that normal arcs are a sub-set of hyperarcs for which $|HD(e)| = 1$, where $|HD(e)|$ is defined as the cardinality of the hyperarc (Nielsen 2004). For reasons of clarity and simplicity, all the hyperarcs for which $|HD(e)| = 1$ will henceforth be called 'arcs', and 'hyperarcs' only those for which $|HD(e)| > 1$. A generic hyperarc may also be defined as $e = (i, L_{ip})$, where $i \in N$ and $L_{ip} \subseteq N_i^+$
- H_p: hyperpath connecting a single origin destination pair (r,s).

The following sub-sets of N and E are defined:

- *CN* centroid nodes
- *PN*: pedestrian nodes;
- *RN*: stop nodes;
- *LN*: line nodes.
- *PE*: pedestrian arcs;
- *LE*: line arcs. Each arc $e \in LE$ is uniquely associated with a line $\ell \in \Im$;
- *DE*: dwelling arcs;
- *DuE*: dummy arcs, connecting the stop nodes to the pedestrian network
- *SE*: support hyperarcs (Bellei *et al.* 2000), representing all the lines sharing a stop. Namely, for the generic stop node i, a support hyperarc a is defined such that $TL(a) = i$ and $HD(a) = N_i^+$. Each branch $b = (i,j)$ of a support hyperarc is also called boarding arc ($b \in BE$);
- *WE*: waiting hyperarcs (Gentile *et al.* 2005a), representing only the *attractive* lines serving the stop. Considering the same stop node as before, the associated waiting hyperarc c is defined such that $TL(c) = i$ and $HD(c) = L_{ip}^*(\xi) \subseteq N_i^+$, where $L_{ip}^*(\xi)$ represents the set of attractive lines serving the stop node i for passengers travelling along the hyperpath H_p at time ξ.
- *AE*: alighting arcs.

It should be noted that in the dynamic and congested scenario the branches of the hyperarc (Figure1) do not only represent the 'average delay due to the fact that the transit service is not continuously available over time', but also the 'time spent by users queuing at the stop and waiting that the service become actually available to them' (Meschini *et al* 2007). Moreover, for modelling purposes, we assume that passengers arriving at the stop join a unique, mixed queue regardless of their particular attractive line set. In this case, overtaking is possible among passengers having different attractive sets; however, any competition among passengers sharing the same attractive set is solved by applying the FIFO rule. (Trozzi *et al* 2010).

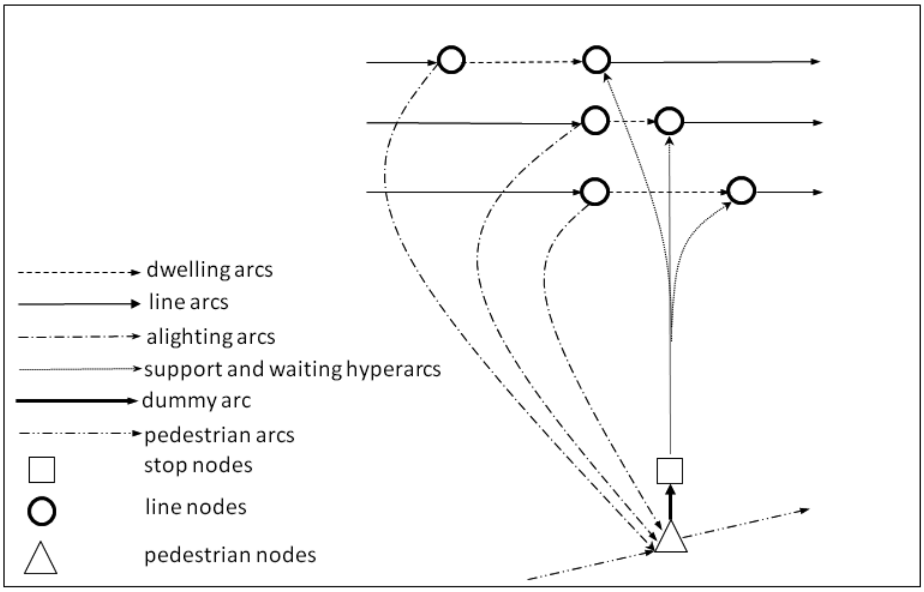

Figure 1: Representation of a stop in the hypergraph.

In order to represent time-dependent travel times, waiting times, etc., the following dynamic variables are also introduced:

- $k_{ij}(\xi)$: number of vehicle arrivals that passengers, arriving at stop node i at time ξ, have to wait before boarding the attractive line associated with the line node j;
- $t_{ij}(\xi)$: exit time from arc $h = (i,j) \in E$ for users entering it at time ξ;
- $c_{ij}(\xi)$: travel time of arc $h = (i,j) \in E$ for users entering it at time ξ. Namely, if the arc is entered at time ξ and the exit time is $t_{ij}(\xi)$, then $c_{ij}(\xi) = t_{ij}(\xi) - \xi$;
- $\varphi_{ij}(\xi)$: mean frequency at time ξ and stop i of the public transport line associated with line node j;

- $\pi_{ije}(\xi)$: diversion probability. This is the probability of using at time ξ the boarding arc $b = (i,j)$, which is a branch of hyperarc $e = (i, L_{ip})$;
- $\theta_{ie}(\xi)$: expected waiting time at the stop node i and time ξ, if the considered hyperarc is $e=(i, L_{ip})$;
- $\theta_{ije}(\xi)$: partial waiting time at the stop node i and time ξ, if the considered hyperarc is $e=(i, L_{ip})$;
- $\theta_{i|je}(\xi)$: expected waiting time at the stop node i and time ξ conditional to the event of boarding line represented by node j out of the set represented by hyperarc $e=(i, L_{ip})$;
- $t_{i|je}(\xi)$: boarding time for users arriving at stop i at time ξ, conditional to the event of boarding line represented by node j out of the set represented by hyperarc $e=(i, L_{ip})$. Namely: $t_{i|je}(\xi) = \xi + \theta_{i|je}(\xi)$;
- $g_p^{is}(\xi)$: expected actual cost of the hyperpath H_p to destination s for users leaving node i at time ξ;
- $S^{is}(\xi)$: expected travel cost of the *minimal* hyperpath to destination s for users leaving node i at time ξ.

3.2 Problem definition

In the original formulation of *shortest hyperpaths* (Nguyen and Pallottino 1988, Spiess and Florian 1989), it is acknowledged that at any intermediate stop, passengers travelling towards a specific destination would only consider the subset of *attractive lines*, which are used in order to minimize the total expected travel time to destination, and would board whichever attractive line comes first. Consequently, each elemental itinerary is a particular realisation of the optimal strategy or, from a graphical point of view, a single path of the shortest hyperpath.

Definition

A subgraph $H_p = (N_p, E_p, \pi_p(\xi))$, where $N_p \subset N$, $E_p \subset E$ and $\pi_p(\xi) = (\pi_{ije}(\xi))$ a real value vector of dimension (E_p x 1) is a dynamic hyperpath connecting origin $r \in CN_p$ and destination $s \in CN_p$, if:

- H_p is acyclic with at least one arc;
- node r has no predecessors and node s has no successors;
- for every node $i \in N_p \setminus \{r, s\}$ there is a hyperpath from r to s traversing i. Each node has at most one immediate successor hyperarc: if $i \notin RN_p$ then its successor has cardinality equal to one, otherwise the successor hyperarc has cardinality equal or greater than one.;
- the elements of the characteristic vector $\pi_p(\xi)$ satisfy the conditions:

$$\begin{aligned} &\sum_{j \in L_{ip}(\xi)} \pi_{ije}(\xi) = 1, \;\; \forall i \in RN_p \\ &\pi_{ije}(\xi) > 0 \end{aligned} \qquad (1)$$

  and the value of its components depends on the time ξ they are evaluated;
- travel times $c_{ij}(\xi)$ associated to arcs $(i,j) \in E_p$ and waitinig times $\theta_{ie}(\xi)$ associated to nodes $i \in RN_p$ depend on the entering time ξ they are evaluated; waiting times $\theta_{ie}(\xi)$ also depend on the considered hyperarc $e = (i, L_{ip}(\xi)) \in E_p$.

In the static context, it is shown that the total travel time of the generic hyperpath H_p can be computed by explicitly taking into account all the elemental paths ℓ forming it (Nguyen and Pallottino 1988, 1989). Therefore, if Q_p is the set of such paths, λ_ℓ is the probability of choosing the elemental path ℓ, and γ_ℓ is its travel time, the travel time of hyperpath H_p is:

$$g_p = \sum_{\ell \in Q_p} \lambda_\ell \cdot \gamma_\ell \tag{2}$$

On the other hand, γ_ℓ can be expressed as the sum of travel and waiting times on the path's arcs and nodes:

$$\gamma_\ell = \sum_{(i,j) \in E_p} c_{ij} \cdot \delta_{ij\ell} + \sum_{i \in NR_p} \theta_i \cdot \delta'_{i\ell} \tag{3}$$

where $\delta_{ij\ell} = 1$ if arc $h = (i,j)$ belongs to path ℓ, otherwise $\delta_{ij\ell} = 0$, and $\delta'_{i\ell} = 1$ if path ℓ traverses node i, otherwise $\delta'_{i\ell} = 0$. Thus the following expression of the hyperpath's total travel time can be obtained:

$$g_p = \sum_{\ell \in Q_p} \lambda_\ell \cdot \left[\sum_{(i,j) \in E_p} c_{ij} \cdot \delta_{ij\ell} + \sum_{i \in RN_p} \theta_i \cdot \delta'_{i\ell} \right] \tag{4}$$

In a network with overcrowding, as considered here, travel times depend on the time the arc is entered. Consequently, it can happen that the same node is traversed by different paths at different times and the travel cost associated with it has different values. Hence, the above definition of the hyperpath's total travel time does not apply to the dynamic scenario. Nevertheless, by extending the local recursive formula (the so-called *generalised Bellman equation*) given in Nguyen and Pallottino (1988, 1989) to the dynamic context, a sequential definition of the dynamic hyperpath's travel time structure is obtained. What is more, the equation enables obtaining the result whilst avoiding path enumeration, thus decreasing the computational burden.

Definition

The total travel time of the dynamic hyperpath H_p connecting r to s at time ξ is sequentially defined in reverse topological order as:

$$g_p^{is}(\xi)=\begin{cases}0, & \text{if } i=s\\ c_{ij}(\xi)+g_p^{js}\left(t_{ij}(\xi)\right), & \text{if } i\notin RN_p\\ \theta_{ie}(\xi)+g_p^{HD(e)s}\left(t_{iHD(e)}(\xi)\right), & \text{if } i\in RN_p\end{cases} \tag{5}$$

$$g_p^{is}(\xi)=\begin{cases}0, & \text{if } i=s\\ c_{ij}(\xi)+g_p^{js}\left(t_{ij}(\xi)\right), & \text{if } i\notin RN_p\\ \theta_{ie}(\xi)+\sum_{j\in L_{ip}(\tau)}\pi_{ije}(\xi)\cdot\left[g_p^{js}\left(t_{i|je}(\xi)\right)\right], & \text{if } i\in RN_p\end{cases} \tag{5b}$$

where:

- $\theta_{ie}(\xi)$ is the total expected waiting (and queuing) time at stop i associated with hyperarc $e = (i, L_{ip})$;
- $g_p^{HD(e)s}(t_{iHD(e)}(\xi))$ is the remaining cost of the hyperpath, upon boarding one of the attractive line(s) from stop node i;
- L_{ip} is the particular set of competing lines, that passengers might consider boarding at stop i when travelling along hyperpath H_p.

3.3 Stop model

3.3.1 Probability distribution functions of the waiting times

It should be noted that the above formulation of the hyperpath travel time structure and computation is independent of specific values given to the diversion probabilities $\pi_{ije}(\xi)$, and expected waiting costs $\theta_{ie}(\xi)$. These variables are specified by the stop model and depend on the particular assumptions adopted for modelling the passenger and the public transport vehicle arrivals at the stop node, and on the passenger boarding mechanism.

In our model, the basic hypotheses about carrier and passenger arrivals (Nguyen and Pallottino 1988, Spiess and Florian 1989) are not changed, but passengers waiting at a stop may be prevented from boarding an approaching carrier because of overcrowding. If the stop layout is such that passengers have to join a FIFO queue, then the user coming at stop i at time ξ has to wait for the $k_{ij}(\xi)$-th carrier of line j.

As proved by Larson and Odoni (1981, p. 54), the waiting time before the $k_{ij}(\xi)^{\text{th}}$ carrier arrival occurs is Erlang-distributed with parameters $k_{ij}(\xi)$ and $1/\varphi_{ij}(\xi)$,

$$f_{ij}(w,\xi)=\begin{cases}\dfrac{\varphi_{ij}(\xi)^{k_{bi}j(\xi)}\cdot\exp\left(\varphi_{ij}(\xi)\cdot w\right)\cdot w^{\left[k_{ij}(\xi)-1\right]}}{\left[k_{ij}(\xi)-1\right]!}, & \text{if } w\geq 0\\ 0, & \text{otherwise}\end{cases} \tag{6}$$

where w is the stochastic variable representing the waiting time.

As such, considering hyperarc e = (i, $L_{ip}(\xi)$), the diversion probability can be expressed by the following formula:

$$\pi_{ije}(\xi)=\int_0^{+\infty} f_{ij}(w,\xi)\cdot\prod_{z\in L_{ip}\setminus\{j\}}\overline{F}_{iz}(w,\xi)\,dw \tag{7}$$

where $\overline{F}_{iz}(w,\xi)$ is the survival function of the Erlang-distributed waiting time for the branch b=(i, z) of hyperarc e = (i, L_{ip}).

On the other hand, the total expected waiting time for the considered stop i and hyperarc e is equal to:

$$\theta_{ie}(\xi)=\int_0^{+\infty}\prod_{j\in L_{ip}}\overline{F}_{ij}(w,\xi)\,dw \tag{8}$$

Recalling the definition of conditional expected value (Loève 1978, Melotto 2004), it is also possible to write:

$$\theta_{ie}(\xi) = \sum_{j \in L_{ip}} \pi_{ije}(\xi) \cdot \theta_{i|je}(\xi) = \sum_{j \in L_{ip}} \theta_{ije}(\xi) \tag{9}$$

$$\theta_{i|je}(\xi) = \frac{1}{\pi_{ije}(\xi)} \int_0^{+\infty} w \cdot f_{ij}(w,\xi) \prod_{z \in L_{ip} \setminus \{j\}} \overline{F}_{iz}(w,\xi)\, dw \tag{10}$$

$$\theta_{ije}(\xi) = \int_0^{+\infty} w \cdot f_{ij}(w,\xi) \prod_{z \in L_{ip} \setminus \{j\}} \overline{F}_{iz}(w,\xi)\, dw \tag{11}$$

To further stress the implication of the proposed stop model, consider the small example network of Section 2 and the four scenarios summarised in Table IV.

Table IV: Average headways, *k* values and travel time upon boarding for the two lines in the considered scenarios.

	Line 1 Average frequency [min^{-1}]	Line 1 *k*	Line 1 Travel time upon boarding [min]	Line 2 Average frequency [min^{-1}]	Line 2 *k*	Line 2 Travel time upon boarding [min]
S 1	1/6	1	10	1/6	1	10
S 2	1/3	2	10	1/6	1	10
S 3	1/2	3	10	1/6	1	10
S 4	1	6	10	1/6	1	10

Table V Boarding probabilities, conditional expected waiting times and total expected waiting time at the stop for the considered scenarios.

		$\pi_{ije}(\xi)$	$\theta_{i\|je}(\xi)$ [min]	$\theta_{ie}(\xi)$ [min]
S 1	Line 1	0.50	3.00	3.00

	Line 2	0.50	3.00	3.00
S 2	Line 1	0.45	3.99	3.33
	Line 2	0.55	2.81	3.33
S 3	Line 1	0.42	4.50	3.47
	Line 2	0.58	2.72	3.47
S 4	Line 1	0.40	5.13	3.62
	Line 2	0.60	2.62	3.62

Scenario 2 is the same considered in Section 2. As displayed in Table V, the probability to board Line 1 is not only lower than the value calculated with the uncongested model (Table II), but also lower than the value calculated with the effective frequency model (Table III). On the other hand, the opposite holds for the total expected waiting time.

This phenomenon may be explained with the properties of the Erlang and exponential distributions. In fact, if k = 2 and φ = 1/3min for Line 1, this service would be boarded only in case both vehicles of Line 1 pass with an headway shorter than the average value of three minutes. However, this event is less probable than one vehicle of Line 2 arriving before its average inter-arrival time (six minutes). Moreover, if the mean of the Erlang distribution is constant (in our case, $k/\varphi = 6$) and $k \to \infty$, the waiting time before boarding tends to be deterministic, thus $\theta_{i|je}(\xi)$ for Line 1 increases and tends to $k_{line1}/\varphi_{line1}$.

On the other hand, the boarding probability for Line 2 tends to equation (12),

$$\pi_{line2}(\xi) = \int_0^{\frac{k_{line1}}{\varphi_{line1}}} f_{line2}(w, \xi)\, dw \qquad (12)$$

where, $k_{line1}/\varphi_{\ line1} = 1/\varphi_{\ line2}$ is also the expected value of $f_{line2}(w,\ \xi)$. Because the expected value of an exponential distribution is always lower than its median, the boarding probability of Line 2 progressively increases with $k_{\ line1}$ (for constant $k_{\ line2}$ = 1), while $\pi_{\ line1}$ decreases.

3.3.2 *Attractive Set*

In general, the above expressions of the diversion probabilities and expected waiting times can be applied to any subset of N_i^+, however, only a specific subset $L^*_{ip}(\xi) \subseteq N_i^+$ is associated with the minimum travel time to destination at time ξ. This set is defined *attractive* and includes all the lines nodes that make up the head of the waiting hyperarc associated to the stop node considered.

Recalling the definition of an *attractive set* given in (Nguyen and Pallottino 1988), we can write:

$$\begin{aligned} &\exists\, L^*_{ip}(\xi) \subseteq N_i^+ : \\ &\sum_{j \in L^*_{ip}(\xi)} \theta_{ije}(\xi) + \sum_{j \in L^*_{ip}(\xi)} \pi_{ije}(\xi) \cdot \left[g_p^{js}\left(t_{i|je}(\xi) \right) \right] = \\ &\min_{L_{ip} \subseteq N_i^+} \left\{ \sum_{j \in L_{ip}} \theta_{ije}(\xi) + \sum_{j \in L_{ip}} \pi_{ije}(\xi) \cdot \left[S_p^{js}\left(t_{i|je}(\xi) \right) \right] \right\} \end{aligned} \tag{13}$$

Consequently, in order to determine $L^*_{ip}(\xi)$, it is in general necessary to compute $g^{is}_{p}(\xi)$ for all the possible subsets of N_i^+. However, at least for the uncongested static case, it is counter-intuitive to exclude a line from $L^*_{ip}(\xi)$ if it has a shorter remaining travel time than any other attractive one. Therefore, it is possible to solve the combinatorial problem described above through a greedy approach (Spiess and Florian 1989, Nguyen and Pallottino 1988, Chriqui and Robillard 1975). Namely, the lines are processed in ascending order of their travel time upon boarding and the progressive calculation of the values of π_{ije}, θ_{ie} and $g^{is}{}_p$ is stopped as soon as the addition of the next line increases the value of $g^{is}{}_p$. At this point, the cost is minimal (S^{is}) and the set of lines corresponds to the attractive set.

The correctness of the greedy method, in the static case, depends on the shape of the waiting time pdf (exponential). By contrast, in the dynamic scenario the only exact method of finding $L^*{}_{ip}(\xi)$ requires the enumeration of all the possible combination of lines serving the considered stop. Greedy-type heuristics could be applied also in the time-dependent case to overcome the computational complexity arising from the exact solution of the problem, however this is out of the scope of the present work.

4. Numerical Example

The Decreasing Order of Time (DOT) method, presented by Chabini (1998), which has been analytically proved to be the most efficient solution method for the all-to-one search for every possible arrival time, was extended to the time-dependent shortest hyperpath problem in order to devise a solution algorithm for the example considered in this section.

Although the proposed model has a continuous time representation, a discrete-time representation for its numerical solution has been adopted. The main idea is to divide the analysis period $AP = [0, \Theta]$ into T time intervals, such that $AP = \{\xi^0, \xi^1, ..., \xi^\tau, ..., \xi^{T-1}\}$, with $\xi^0 = 0$ and $\xi^{T-1} = \Theta$, and to replicate the network along the time dimension, forming a time-expanded hypergraph HG_T containing *vertexes* in the form (i, ξ), and *edges* in the form $((i, \xi), (j, t_{ij}(\xi)))$.

If time intervals are short enough to ensure that the exit time of a generic edge $t_{ij}(\xi^\tau)$ is not earlier than the next interval $\xi^{\tau+1}$, for $\tau \leq T$-2, it is ensured that the network is cycle-free and the *vertex* chronological ordering is equivalent to the topological one.

Thus, HG_T is scanned starting from the last temporal layer to the value assumed for $\xi = \xi^0$ and, within the generic layer, no topological order is respected. When a generic vertex (i, ξ^τ) is visited, its forward star is scanned in order to set the minimal travel cost to destination and the successive edge by means of equation (13) (refer to the appendix for a detailed formulation of the algorithm).

Such algorithm has been applied to the network used by Spiess and Florian (1989) in their seminal work (Figure 2) in order to find optimal travel strategies and calculate travel times from each node to destination (node 16).

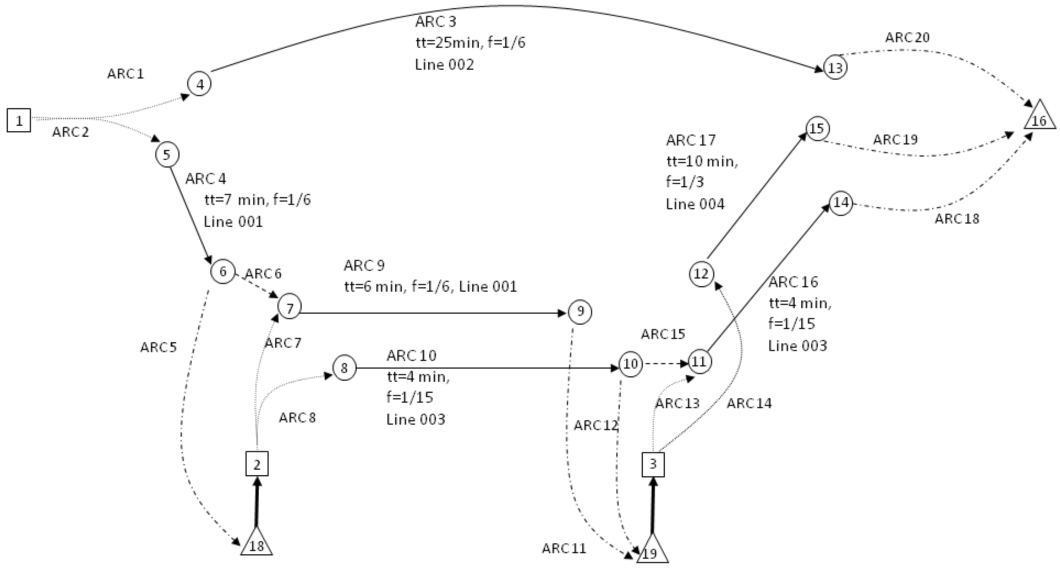

Figure 2: Example network, the graphic representation is consistent with Figure 1.

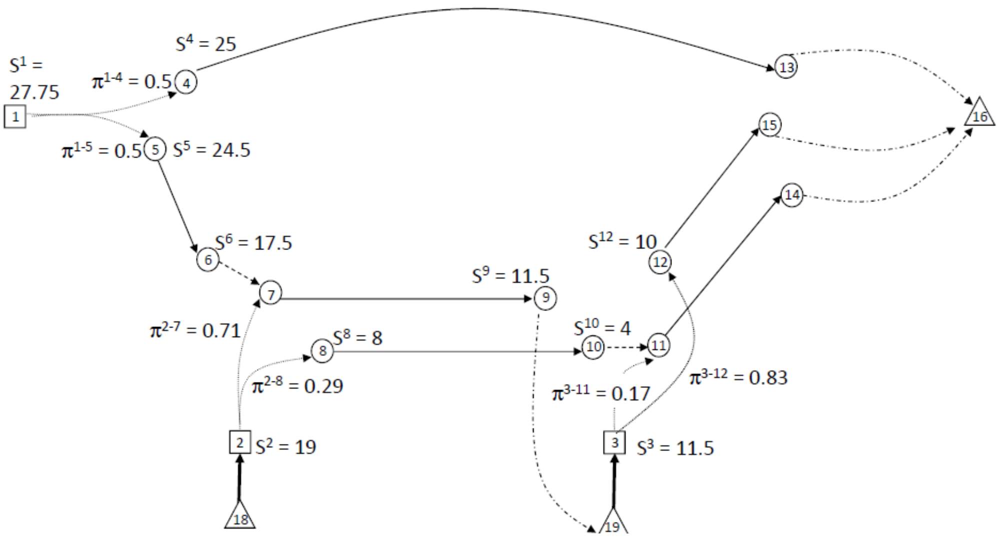

Figure 3: Optimal strategy found for the static case, expected total travel time from each intermediate node to the destination and diversion probabilities at the stop nodes.

The results obtained in the static and uncongested scenario, with reference to destination node 16, are summarised in Figure 3: for each stop the waiting hyperarc is depicted, S^i indicates the total expected travel time from node i to destination and $\pi^{i\text{-}j}$ is the probability to traverse the boarding arc (i, j).

Table VI: Time dependent travel variables for each line arc of the example network: in-vehicle travel time, average frequency and number of passages to be waited before boarding (k).

Time of the day	Travel variable	Arc 3 (4,13)	Arc 4 (5,6)	Arc 9 (7,9)	Arc 10 (8,10)	Arc 16 (11,14)	Arc 17 (12,15)
8:00-08:30	Travel time [min]	30	7	6	4	4	15
8:00-08:30	Freq. [min^{-1}]	1/3	1/6	1/6	1/15	1/15	1/3
8:00-08:30	k	2	1	1	1	2	1
8:30-09:00	Travel time [min]	35	7	6	4	4	15
8:30-09:00	Freq.[min^{-1}]	1/3	1/6	1/6	1/15	1/10	1/3
8:30-09:00	k	3	1	1	1	2	2
9:00-09:30	Travel time [min]	35	7	6	4	4	10
9:00-09:30	Freq. [min^{-1}]	1/3	1/6	1/6	1/15	1/5	1/3
9:00-09:30	k	2	1	1	1	2	1

In order to compare static conditions with results obtained when hypothesising time-dependence of passenger congestion and travel variables within the day, the morning peak [08:00 –9:30] is divided into one-minute intervals. Changes in travel conditions are assumed to occur only at 08:00, 08:30, and 09:00, as shown in Table VI, while outside the analysis period travel variables are assumed to remain constant and have the same values as in the uncongested scenario.

Results, summarised in Figures 4-5-6, show that the change in travel variables does not only affect the expected travel time to the destination and the boarding probability, but also the alternatives included in the optimal strategy. Notably, between 08:00 and 08:30, because of congestion at stop 3, passengers waiting at stop 2 do not include Line 001 in their attractive set. Instead, they prefer boarding Line 003 so as to avoid a transfer at stop 3 and the associated long waiting time. Also, between 08:30 and

09:00, because of the increased travel time upon boarding and heavy (k = 3) congestion, Line 002 is excluded from the attractive set at stop 1 and is not even re-included after 09:00, when congestion slightly decreases (k = 2). Finally, between 09:00 and 09:30, the total expected waiting time at stop 3 remarkably decreases, because Line 004 becomes more frequent and congestion on Line 003 has dissipated. Consequently, passengers waiting at stop 2 now choose to make a transfer at stop 3 and both lines (001 and 003) are included in the attractive set.

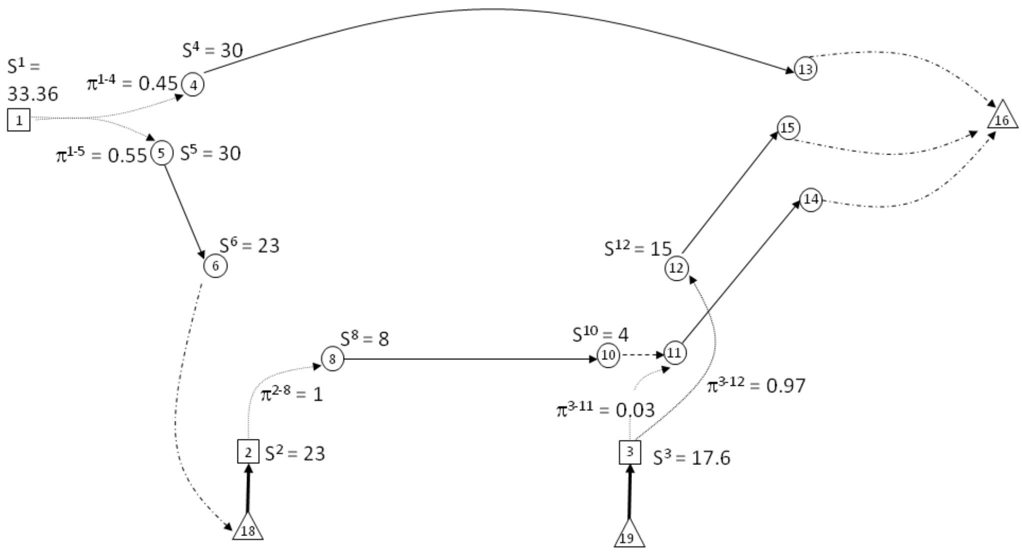

Figure 4: Optimal strategy found between 08:00 and 08:30, expected total travel time from each intermediate node to the destination and diversion probabilities at the stop nodes.

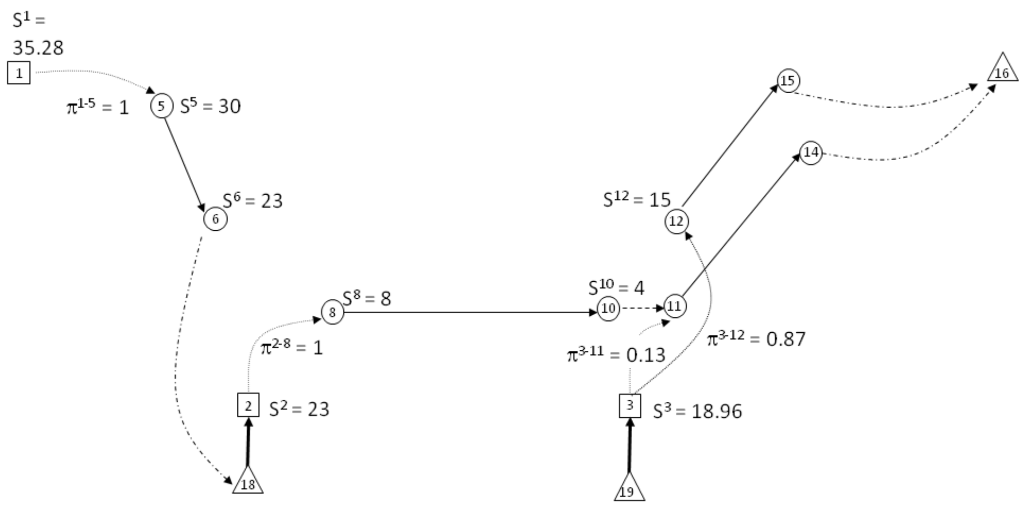

Figure 5: Optimal strategy found between 08:30 and 09:00, expected total travel time from each intermediate node to the destination and diversion probabilities at the stop nodes.

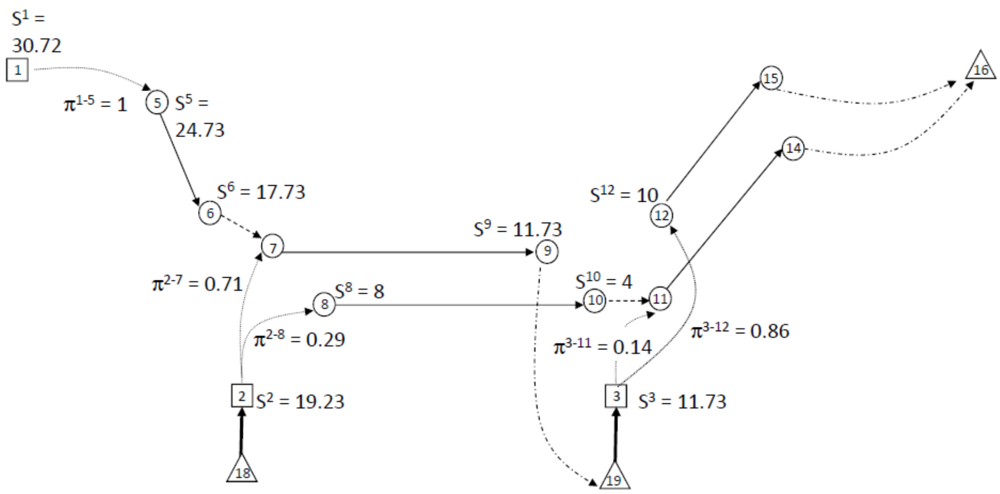

Figure 6: Optimal strategy found between 09:00 and 09:30, expected total travel time from each intermediate node to the destination and diversion probabilities at the stop nodes.

5. Conclusions

Given the importance of travel time variability in the traveller route and mode decisions, this paper presents a dynamic route choice model for public transport networks, in which travel variables, such as on-board travel times and frequencies, vary with time, and in which queues may form at public transport stops due to overcrowding, hindering passengers to board the first available line of their choice.

The model reproduces route choice of commuting passengers that know by previous experience the number of passages they must let go, at each stop, before being able to board every line of their *attractive set*. In case of constant variables, the results from the static algorithm by Spiess and Florian (1989) are reproduced. However changes in the shortest hyperpath are clearly shown when travel variables become dynamic and/or queuing at stops is considered.

In the example given, the combinatorial problem of selecting, at each stop, the set of attractive lines, is solved by means of an exact procedure. However, when applying the model in large-scale scenarios, using real network and traffic variables data, heuristics are needed to speed up the computation.

Although the expected travel time is an important factor affecting the travellers' route choice, as proved by the numerical example, many studies have found that the reliability of travel time due to non-recurrent congestion can be even more important. Indeed a "wrong" choice of route or mode can result in a significant delay, which may have severe consequences for the traveller (e.g. late arrival at the workplace). Equally, some groups of travellers may be particularly adverse to discomfort on board due to overcrowding (e.g. elderly travellers, parents travelling with your children …) and decide to change their route and/or departure time according to network conditions. As such, future developments will concentrate in the inclusion of these factors in a multi-class route choice model coupled with departure time choice model in public transport networks.

Also, within the same context, the method will be tested in large-scale scenarios, using real network and traffic variables data, and its potential of application to real-time journey planning will be investigated.

6. Appendix: the solution algorithm

The variable list of the algorithm is:

- τ: temporal layer index
- τInt: temporal layer length in minutes
- s: destination node
- i: generic node
- $FS(i)$: set of (hyper)arcs belonging to the forward star of node i

- a: generic hyperarc, where $a \in FS(i)$
- b: branch (i,j) of the generic hyperarc $a \in FS(i)$, where $i \in RN$
- $c(i,\xi^{\tau})$: waiting hyperarc from stop node i at time interval ξ^{τ}.

  $HD(c(i,\xi^{\tau})) = L_{ip}^{*}(\xi^{\tau})$
- $suc(i, \xi^{\tau})$:successor arc of the generic node i at time interval ξ^{τ}, where $i \notin RN$
- h: generic arc, where $h \in FS(i)$ and $i \notin RN$
- $g^{is}(\xi^{\tau})$: current travel cost from generic node i to destination s at time interval ξ^{τ}
- $S^{is}(\xi^{\tau})$: minimum travel cost from generic node i to destination s at time interval ξ^{τ}
- S^{is}_{stat}: minimum travel cost from generic node i to destination s at time interval $\xi^{\tau} \geq \xi^{T-1}$
- $g^{is}(j,a,\xi^{\tau})$: travel cost from i to s if passengers travel along branch $b=(i,j)$ of hyperarc a at time interval ξ^{τ}

The solution algorithm for the time-dependent all-to-one shortest hyperpath problem for every possible arrival time is detailed here.

Step 0 (Static pre-processing – Initialisation): $\forall\ i \in N \setminus \{s\}$

Calculate $S^{is}(\xi^{T-1}) = S^{is}_{\mathrm{stat}}$

$\forall\ \tau \in [0,T-2]$

Set $S^{ss}(\xi^{\tau}) = 0$, $suc(s, \xi^{\tau}) = \varnothing$

$\forall\ i \in N \setminus \{s\}$

Set $S^{is}(\xi^{\tau}) = \infty$

Step 1 (Hyperarcs' dynamic attributes): $\forall\ \tau \in [0,T-2]$

$\forall\ i \in RN$

$\forall \ a \in FS(i)$

$\forall \ b = (i,j) \subseteq a$

Calculate $\pi_{ija}(\xi^{\tau})$ with eq. (7)

Calculate $\theta_{ija}(\xi^{\tau})$ with eq. (11)

$\theta_{ia}(\xi^{\tau}) = \theta_{ia}(\xi^{\tau}) + \theta_{ija}(\xi^{\tau})$

Step 2 (Calculate hyperpath travel time): $\forall \ \tau \in [0,T\text{-}2]$

$\forall \ i \in N \setminus \{s\}$

If $i \in RN$, $\forall \ a \in FS(i)$

$\forall \ b = (i,j) \subseteq a$

If $[\![\theta_{i|ja}(\xi^{\tau}) \ / \ \tau\mathrm{Int}]\!] \geq 1$

$t_{i|ja}(\xi^{\tau}) = [\![\theta_{i|ja}(\xi^{\tau}) \ / \ \tau\mathrm{Int}]\!] + \xi^{\tau}$

Else

$t_{i|ja}(\tau) = \xi^{\tau} + 1.$

If $t_{i|ja}(\xi^{\tau}) \leq \xi^{T}$ and $g^{j}(t_{i|ja}(\xi^{\tau})) < \infty$

$g^{is}(j,a,\xi^{\tau}) = g^{j}(t_{ija}(\xi^{\tau}))$

Else

$g^{is}(j,a,\xi^{\tau}) = S^{js}{}_{\mathrm{stat}}.$

$g^{is}(\xi^{\tau}) = g^{i}(\xi^{\tau}) + g^{is}(j,a,\xi^{\tau})\cdot\pi_{ija}(\xi^{\tau})$

If $S^{is}(\xi^{\tau}) > g_{p}{}^{is}(\xi^{\tau})$ then

$S^{is}(\xi^{\tau}) = g_{p}{}^{is}(\xi^{\tau})$ And $c(i,\xi^{\tau}) = a$

Else if $i \notin RN$, $\forall \ h \in FS(i)$

If $[\![c_{ij}(\xi^{\tau}) \ / \ \tau\mathrm{Int}]\!] \geq 1$

$t_{ij}(\xi^{\tau}) = [\![c_{ij}(\xi^{\tau}) / \tau \mathrm{Int}]\!] + \xi^{\tau}$

Else

$t_{ij}(\xi^{\tau}) = \xi^{\tau} + 1$

End if

$g^{is}(\xi^{\tau}) = c_{ij}(\xi^{\tau}) + g^{js}(t_{ij}(\xi^{\tau}))$

If $S^{is}(\xi^{\tau}) > g^{is}(\xi^{\tau})$

$S^{is}(\xi^{\tau}) = g^{is}(\xi^{\tau})$ and

$suc(i, \xi^{\tau}) = h$

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