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DISCUSSION PAPER PI-1111

Age-Dependent Investing: Optimal Funding and Investment Strategies in Defined Contribution Pension Plans when Members are Rational Life Cycle Financial Planners

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September 2011

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Age-Dependent Investing: Optimal Funding and Investment Strategies in Defined Contribution Pension Plans when Members are Rational Life Cycle Financial Planners

David Blake,[+] Douglas Wright[*] and Yumeng Zhang[#]

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Abstract

A defined contribution pension plan allows consumption to be redistributed from the plan member's working life to retirement in a manner that is consistent with the member's personal preferences. The plan's optimal funding and investment strategies therefore depend on the desired profile of consumption over the lifetime of the member. We investigate these strategies under the assumption that the member is a rational life cycle financial planner and has an Epstein-Zin utility function, which allows a separation between risk aversion and the elasticity of intertemporal substitution. We also take into account the member's human capital during the accumulation phase of the plan and we allow the annuitisation decision to be endogenously determined during the decumulation phase.

We show that the optimal funding strategy involves a contribution rate that is not constant over the life of the plan but is age-dependent and reflects the trade-off between the desire for current versus future consumption, the desire for stable consumption over time, the member's attitude to risk, and changes in the level of human capital over the life cycle. We also show that the optimal investment strategy during the accumulation phase of the plan is 'stochastic lifestyling', with an initial high weight in equity-type investments and a gradual switch into bond-type investments as the retirement date approaches in a way that depends on the realised outcomes for the stochastic processes driving the state variables. The optimal investment strategy during the decumulation phase of the plan is to exchange the bonds held at retirement for life annuities and then to gradually sell the remaining equities and buy more annuities, i.e., a strategy known as 'phased annuitisation'.

Key words: defined contribution pension plan, funding strategy, investment strategy, Epstein-Zin utility, stochastic lifestyling, phased annuitisation, dynamic programming

JEL: G11, G23

[Typos corrected 15/9/11]

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1 Introduction

1.1 The role of the pension plan in allocating consumption across the life cycle

A typical individual's life cycle consists of a period of employment followed by a period of retirement. Most individuals therefore need to reallocate consumption from their working life to retirement if they wish to avoid poverty in old age. A defined contribution (DC) pension plan can achieve this reallocation in a way that is consistent with the preferences of the individual plan member.[1]

There are three key preferences to take into account. The first relates to the desire to smooth consumption across different possible states of nature within any given time period. The second relates to the desire to smooth consumption across different time periods. The third relates to the desire for current versus future consumption; saving for retirement involves the sacrifice of certain consumption today in exchange for uncertain consumption in the future. This uncertainty arises because both future labour income and the returns on the assets in which the retirement savings are invested are uncertain. The plan member therefore needs to form a view on both the trade-off between consumption in different states of nature in the same time period and the trade-off between consumption and consumption variability in different time periods. Attitudes to these trade-offs will influence the optimal funding and investment strategies of the pension plan.

In a DC pension plan, the member allocates part of his labour income earned each year to the pension plan in the form of a plan contribution and, thus, builds up a pension fund prior to retirement. Then, at retirement, the member uses the accumulated pension fund to finance consumption in retirement by purchasing a life annuity, by keeping the fund invested and drawing an income from it, or some combination of these.[2] The decisions

[1] The extent of this reallocation will be influenced by the level of pension benefits provided by the state and by the level of non-pension (e.g., housing) wealth owned by the individual.

[2] Some jurisdictions place restrictions on some of these options. Some plan members might wish to exercise a further option, one which arises from a 'bequest motive', i.e., the desire to leave a bequest on

regarding the level of the contribution rate in each year before retirement[3] (i.e., the funding strategy) is driven by the member's preference between current and future consumption. As a consequence, the optimal funding strategy might involve a contribution rate into the plan that is not, as in most extant plans, a fixed percentage of labour income, but is, instead, age-related.

The investment strategy prior to retirement (i.e., the decision about how to invest the accumulating fund across the major asset categories, such as equities and bonds) will influence the volatility of the pension fund (and, hence, the amount available for consumption in future periods), and so will depend on the member's attitude to that volatility, both across states of nature and across time. After retirement, hedging longevity risk becomes an important additional consideration, so the investment strategy will now include annuities as well as the traditional asset categories.

In this paper, we investigate the optimal funding and investment strategies in a DC pension plan assuming the member is a rational life cycle financial planner. To do this, we use a model that differs radically from existing studies in this field in three key respects.

The first key feature of the model is the assumption of Epstein-Zin (1989) recursive preferences by the plan member. This allows us to separate relative risk aversion (RRA) from the elasticity of intertemporal substitution (EIS). Risk aversion is related to the desire to stabilise consumption across different states of nature in a given time period[4] and EIS measures the desire to smooth consumption over time.[5] Thus, risk aversion and EIS are conceptually distinct and, ideally, should be parameterised separately.

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death. We do not consider this further here, since bequests are usually satisfied outside of a pension savings framework and pension wealth is typically not bequeathable.

[3] In the case where the plan member can exercise some choice.

[4] An individual with a high degree of risk aversion wishes to avoid consumption uncertainty in a particular period and, more specifically, the reduction in consumption that would be required in an unfavourable state of nature, such as a large fall in equity prices.

[5] An individual with a low EIS wishes to avoid consumption volatility over time and, in particular, a reduction in consumption relative to the previous time period. EIS is defined as:

Within the commonly used power utility framework, the EIS is given by the reciprocal of the coefficient of relative risk aversion (e.g., see Campbell and Viceira (2002)). This restriction has been criticised because it does not appear to reflect empirical observations. For example, based on the consumption capital asset pricing model of Breeden (1979), Schwartz and Torous (1999) disentangle these two concepts using the term structure of asset returns. Using US data on discount Treasury bond returns, equity market returns and aggregate consumption for 1964-97, their best estimate for the coefficient of RRA is 5.65 (with a standard error of 0.22) and their best estimate of the EIS is 0.226 (with a standard error of 0.008). Thus, a high coefficient of RRA tends to be associated with a low level of EIS, but the estimated parameter values do not have the exact reciprocal relationship assumed in the power utility framework. Similarly, Blackburn (2006) rejects the reciprocal relationship on the basis of a time series of RRA and EIS parameters estimated from observed S&P 500 option prices for a range of different expiry dates between 1996 and 2003.[6]

The second key feature of the model is the recognition that the optimal investment strategy will depend not just on the properties of the available financial assets, but also on the plan member's 'human capital', defined as the net present value of an individual's

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$$\varphi = -\frac{\mathrm{d}\left[\ln\left(c_{t_1} / c_{t_2}\right)\right]}{\mathrm{d}\left[\ln\left(U'\left(c_{t_1}\right) / U'\left(c_{t_2}\right)\right)\right]}$$

where c_{t_i} is consumption in period t_i and $U'(c_{t_i})$ is the marginal utility of c_{t_i}. The sign and size of the EIS reflects the relationship between the substitution effect and income effect of a shock to a state variable, such as an increase in the risk-free interest rate. The substitution effect is always negative, since current consumption decreases when the risk-free rate increases because future consumption becomes relatively cheap and this encourages an increase in savings. The income effect will be positive if an increase in the risk-free rate (which induces an increase in the income from savings) leads to an increase in current consumption; it will be negative otherwise. If the income effect dominates, the EIS will be negative and an increase in the risk-free rate leads to an increase in current consumption. If the substitution effect dominates (which is the usual assumption), the EIS will be positive and an increase in the risk-free rate leads to a decrease in current consumption. If the income and substitution effects are of equal and opposite sign, the EIS will be zero and current consumption will not change in response to an increase in the risk-free rate: in other words, consumption will be smooth over time in the presence of interest rate volatility.

[6] In particular, Blackburn (2006) found that, over the period 1996 to 2003, the RRA changed dramatically, whilst the EIS stayed reasonably constant.

future labour income.[7, 8] A commonly used investment strategy in DC pension plans is 'deterministic lifestyling'.[9] With this strategy, the pension fund is invested entirely in high risk assets, such as equities, when the member is young. Then, at a pre-set date (e.g. 5 to 10 years prior to retirement is quite common in practice), the assets are switched gradually (and often linearly) into lower risk assets such as bonds and cash. However, whilst intuitively appealing, there is no strong empirical evidence to date demonstrating that this is an optimal strategy.

If equity returns are assumed to be mean reverting over time, then the lifestyling strategy of holding the entire fund in equities for an extended period prior to retirement might be justified, as the volatility of equity returns can be expected to decay over time (as a result of the 'time diversification of risk'). However, there is mixed empirical evidence about whether equity returns are genuinely mean reverting: for example, Lo and Mackinley (1988), Poterba and Summers (1988) and Blake (1996) find supporting evidence in both US and UK markets, while Kim et al. (1991) and Howie and Davies (2002) find little support for the proposition in the same countries. We would therefore not wish an optimal investment strategy to rely on a debatable assumption of mean reversion holding true in practice.

A more appealing justification for a lifestyling investment strategy comes from recognising the importance of human capital in individual financial planning. Human capital can be interpreted as a bond-like asset in which future labour income is fairly stable over time and can be interpreted as the 'dividend' on the individual's implicit holding of human capital.[10] Most young pension plan members are likely to have a significant holding of (bond-like) human capital, but a negligible holding of financial assets, especially equity. Their pension fund should initially compensate for this with a

[7] We use the individual's personal discount factor to determine the present value. Our results are not sensitive to the choice of discount factor used.

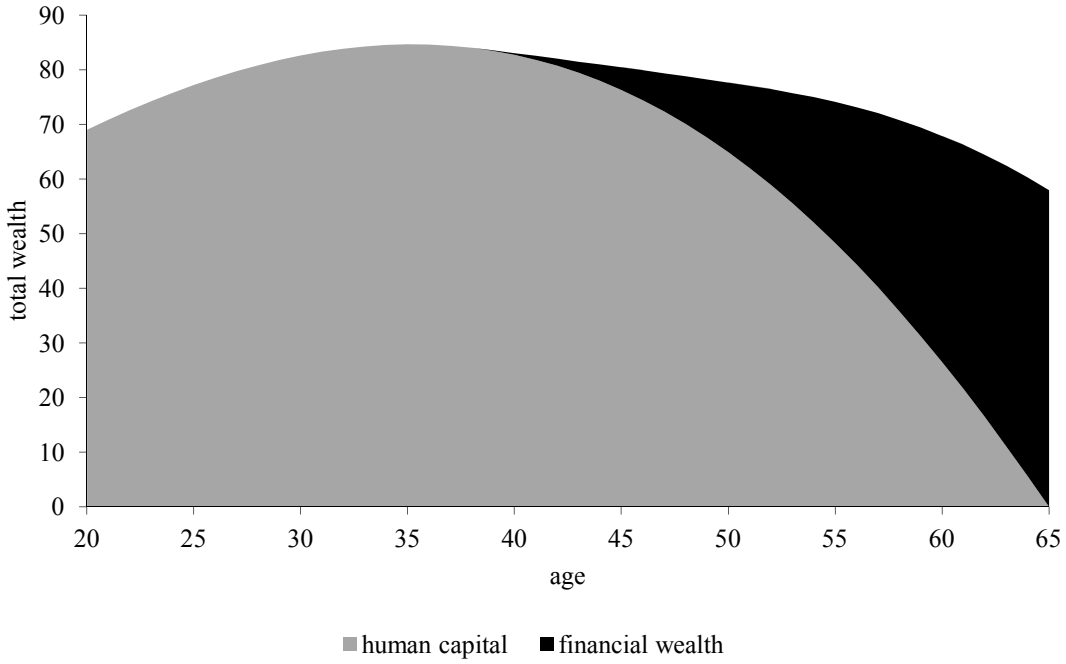
[8] The importance of human capital in a general portfolio choice setting has been emphasised by, e.g., Viceira (2001), Campbell and Viceira (2002), Cocco et al. (2005) and Gomes et al. (2008).

[9] Also known as ' lifecycling' or 'age phasing' (Samuelson (1989)).

[10] As shown by Cairns et al. (2006), the real long-term average growth rate in labour income in developed countries over the last century is very similar to the long-run real average return on government bonds, hence labour income can be thought of as an implicit substitute for risk-free bonds.

heavy weighting in equity-type assets.[11] The ratio of human to financial wealth will therefore be a crucial determinant of the optimal lifecycle portfolio composition. At younger ages, as shown in Figure 1, this ratio is large since the individual has had little time to accumulate financial wealth and expects to receive labour income for many years to come. Over time, as human capital decays and the value of financial assets in the pension fund grows, this ratio will fall and the pension fund should be rebalanced away from equities towards bonds. However to date, there has been no quantitative research exploring the human capital dimension in a DC pension framework.

Figure 1 – Decomposition of total wealth over the life cycle



The third key feature of the model is the endogeneity of the annuitisation decision. In some jurisdictions, such as the UK, there is a mandatory requirement to purchase an annuity with the pension fund up to a specified limit. The limit in the UK, for example, is

[11] By contrast, the human capital of entrepreneurs is much more equity-like in its potential volatility and so it is optimal for entrepreneurs to have a high bond weighting in their pension funds.

£20,000 per annum (as of 2011),[12] and the annuity has to be purchased at the time of retirement. However, in many jurisdictions, including the US, Japan, Australia and most continental European countries, there is no requirement to purchase an annuity at all. In this study, we determine the optimal annuitisation strategy for the member.[13]

1.2 Epstein-Zin utility

The classical dynamic asset allocation optimisation model under uncertainty was introduced by Merton (1969, 1971). With a single risky asset (equities), a constant investment opportunity set, and ignoring labour income, the optimal portfolio weight in the risky asset for an investor with a power utility function, $U(F) = F^{1-\gamma}/(1-\gamma)$, where F is the value of the fund of wealth and γ is RAA, is given by:

$$\alpha = \frac{\mu}{\gamma\sigma^2} \tag{1}$$

where μ and σ^2 are the risk premium (i.e., mean excess return over the risk-free rate of interest) and the variance of the return on the risky asset, respectively.

Equation (1) is appropriate for a single-period myopic investor, rather than a long-term investor, such as a pension plan member. Instead of focusing on the level of wealth itself, long-term investors focus on the consumption stream that can be financed from a given level of wealth. As described by Campbell and Viceira (2002, page 37), 'they consume out of wealth and derive utility from consumption rather than wealth'. Consequently, current saving and investment decisions are driven by preferences between current and future consumption.

To account for this, Epstein and Zin (1989) proposed the following discrete-time recursive utility function, which has become a standard tool in intertemporal investment models, but has not hitherto been applied to pension plans:

[12] State and occupational defined benefit pensions count towards this limit.

[13] There is a positive voluntary demand for annuities in our model. See Inkmann et al. (2011) for a recent empirical analysis of the voluntary annuity market in the UK.

$$U_t = \left((1-\beta) \times (C_t)^{1-\frac{1}{\varphi}} + \beta \times \left(E_t \left[(U_{t+1})^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right)^{\frac{1}{1-\frac{1}{\varphi}}} \quad (2)$$

where

- U_t is the utility level at time t,
- C_t is the consumption level at time t,
- γ is the coefficient of relative risk aversion (RRA),
- φ is the elasticity of intertemporal substitution (EIS),
- β is the individual's personal one-year discount factor.

The recursive preference structure in Equation (2) is helpful in two ways: firstly, it allows a multi-period decision problem to be reduced to a series of one-period problems (i.e., from time t to time $t+1$) and, secondly, as mentioned previously, it enables us to separate RRA and EIS.

Ignoring labour income, for an investor with Epstein-Zin utility, there is an analytical solution for the optimal portfolio weight in the risky asset (in the general case of a time-varying investment opportunity set) given by:[14]

$$\alpha_t = \frac{\mu_t}{\gamma \sigma_t^2} + \left(1 - \frac{1}{\gamma} \right) \times \frac{\mathrm{cov}_t \left(R_{t+1}, -\left(U_{t+1} / F_{t+1} \right) \right)}{\sigma_t^2} \quad (3)$$

This shows that the demand for the risky asset is based on the weighted average of two components. The first component is the short-term demand for the risky asset (or myopic demand, in the sense that the investor is focused on wealth in the next period). The second component is the intertemporal hedging demand, which depends on the covariance between the risky asset return, R_{t+1}, and the investor's utility per unit of

[14] For more details, see Merton (1973) and Campbell and Viceira (2002, Equation (3.15)).

wealth, $\left(U_{t+1}/F_{t+1}\right)$, over time.[15] The optimal portfolio weights, $\{\alpha_t\}$, are constant over time, provided that the investment opportunity set remains constant over time (i.e., $\mu_t = \mu$, $\sigma_t^2 = \sigma^2$ and $\text{cov}_t\left(R_{t+1}, -\left(U_{t+1}/F_{t+1}\right)\right) = -k$ in Equation (3) above).

A realistic lifecycle saving and investment model cannot, however, ignore labour income. Our aim in this study is to investigate the optimal asset allocation strategy for a DC plan member (during both the accumulation and decumulation stages of the plan) with Epstein-Zin utility who faces stochastic labour income and investment returns. We also derive the optimal profile of contribution rates over the accumulation stage of a DC plan.

The rest of the paper is structured as follows. Section 2 outlines the model with Epstein-Zin utility. In Section 3, we generate simulations of the two key state variables (equities and labour income) and derive the optimal funding and investment strategies for a DC pension plan member; we also conduct a sensitivity analysis of the key results. Finally, Section 4 contains the conclusions and discusses the issue of the issue of practical implementation.

2 The model

This section presents the model for solving the lifecycle asset allocation problem for a DC pension plan member. The model assumes two pre-retirement financial assets (a risky equity fund and a risk-free bond fund),[16] a constant investment opportunity set, a stochastic labour income process, and the availability of an additional financial asset, namely a life annuity after retirement. We consider two aspects of labour income risk: the

[15] Thus, since the coefficient of relative risk aversion, γ, will typically be greater than 1, the investor will reduce the equity weighting (relative to a myopic investor) as $\text{cov}_t\left(R_{t+1}, -\left(U_{t+1}/F_{t+1}\right)\right)$ falls in order to reduce the fall in utility when the return on the risky asset falls.

[16] In our model, the only form of savings we allow is long-term savings in a pension plan and these are not accessible prior to retirement, so we implicitly assume precautionary savings are not needed in the model. Pension savings will be allocated to either an equity fund or a bond fund. As a consequence, financial wealth and pension wealth are equivalent and we use these terms interchangeably.

systematic volatility of labour income and the correlation between labour income growth and equity returns which determines the extent to which labour income affects portfolio choice.

2.1 Model structure

2.1.1 Constraints

The DC pension plan member faces the following constraints:

- in any year prior to retirement, contributions into the pension plan must be positive or zero;
- members are not allowed to borrow from future contributions, implying that, prior to retirement, consumption must be lower than labour income; and
- borrowing from the pension fund or short selling of pension fund assets is not allowed, and pension wealth can never be negative.[17]

We will work with age rather than year as our temporal measure. The member is assumed to join the pension plan at age 20 without bringing in any transfer value from a previous plan and retire at age 65.

2.1.2 Preferences

The DC plan member is assumed to possess Epstein-Zin (1989) preferences, as described in Section 1.2 above, but adapted to allow for mortality risk at age x:

[17] These constraints recognize that savings in a pension plan are irreversible – this is what makes pension plans unique as an asset class. There can be additional saving outside the pension plan, but the immediate reversibility of this means that it can be treated as a form of (deferred) consumption and hence lumped together with 'consumption' for our purposes This allows us to focus on pension savings which are assumed to be allocated to an equity fund or a bond fund (any differences in the tax treatment of pension and non-pension savings are outside the scope of this study as these are jurisdiction specific). Given our categorization of reversible savings, we will treat financial wealth and pension wealth as equivalent and we use these terms interchangeably.

$$U_x = \left\{ (1-\beta) \times (C_x)^{1-\frac{1}{\varphi}} + \beta p_x \times \left(E_x \left[(U_{x+1})^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}} \tag{4}$$

where

- U_x is the utility level at age x,
- C_x is the consumption level at age x, and
- p_x is the (non-stochastic) one-year survival probability at age x, i.e., the probability that a member who is alive at age x survives to age $(x+1)$.

We assume that the member has a maximum potential age of 120. Thus, in the final year of age, we assume that $p_{120} = 0$ and, hence, Equation (4) reduces to:

$$U_{120} = \left\{ (1-\beta) \times (C_{120})^{1-\frac{1}{\varphi}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}} \tag{5}$$

which provides the terminal condition for the utility function.

2.1.3 Financial assets

Prior to retirement, the member has the choice of investing in a bond fund with a constant annual real return, r, and an equity fund with a return in the year of age x to $(x+1)$ given by:

$$R_x = r + \mu + \sigma Z_{1,x} \qquad \text{for } x = 20, 21, \ldots, 120 \tag{6}$$

where

- μ is the annual risk premium on the risky asset,
- σ is the annual volatility of return on the risky asset, and
- $\{Z_{1,x}\}$ is a series of independent and identically distributed (iid) standard normal random variables.

Whilst not necessarily corresponding precisely with the real world, the simplified assumption of independent and identically distributed returns on the risky asset considerably simplifies the numerical optimisation problem.

2.1.4 Labour and pension income

Prior to retirement, the member receives an annual salary at the start of each year of age x to $(x+1)$, for $x = 20, 21, \ldots, 64$, and contributes a proportion π_x of this into the pension plan.

We adopt the stochastic labour income process used in Cairns et al. (2006), where the growth rate in labour income over the year of age x to $(x+1)$ is given by:

$$I_x = r_I + \frac{S_{x+1} - S_x}{S_x} + \sigma_1 Z_{1,x} + \sigma_2 Z_{2,x} \qquad \text{for } x = 20, 21, \ldots, 64 \qquad (7)$$

where

- r_I is the long-term average annual real rate of salary growth (reflecting productivity growth in the economy as a whole),
- S_x is the career salary profile (CSP) at age x, so that the term $(S_{x+1} - S_x)/S_x$ reflects the promotional salary increase during the year of age x to $(x+1)$,
- σ_1 represents the volatility of a shock that is correlated with equity returns,
- σ_2 represents the volatility of the annual rate of salary growth, and
- $\{Z_{2,x}\}$ is a series of iid standard normal random variables (independent of $\{Z_{1,x}\}$).

The labour income received at age $(x+1)$, denoted by Y_{x+1}, is given by:

$$Y_{x+1} = Y_x \times \exp(I_x) \qquad \text{for } x = 20, 21, \ldots, 64$$

with normalisation such that $Y_{20} = 1.0$.

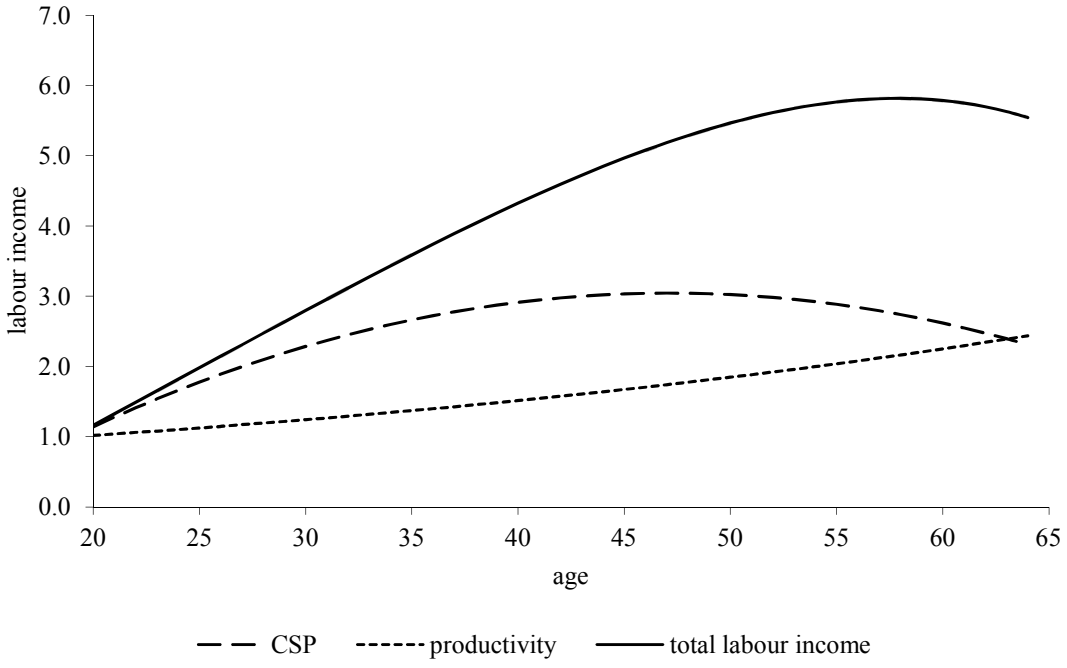
Equations (6) and (7) are subject to a common stochastic shock, $Z_{1,x}$, implying that the contemporaneous correlation between the growth rate in labour income and equity returns is given by $\sigma_1 \Big/ \sqrt{\left(\sigma_1^2 + \sigma_2^2\right)}$.

Following the work of Blake et al. (2007), we use a quadratic function to model the CSP:

$$S_x = 1 + h_1 \times \left[-1 + \frac{(x-20)}{45} \right] + h_2 \times \left[-1 + \frac{4 \times (x-20)}{45} - \left(\frac{\sqrt{3} \times (x-20)}{45} \right)^2 \right] \qquad (8)$$

Based on average male salary data (across all occupations) reported in the 2005 *Annual Survey of Hours and Earnings*, Blake et al. (2007) estimate parameter values of $h_1 = -0.1865$ and $h_2 = 0.7537$. Figure 2 shows the resulting labour income process, $\{Y_x : x = 20, 21, \ldots, 65\}$, assuming $r_I = 2\%$ and $Z_{1,x} = Z_{2,x} = 0$ for $x = 20, 21, \ldots, 64$.

Figure 2 – Labour income process



When the plan member retires at age 65, we assume that he draws at least part of his pension in the form a life annuity, thereby hedging his own mortality risk. The annual amount of pension income received depends on the accumulated wealth level at retirement, the optimal 'annuitisation ratio' (i.e., the proportion of the accumulated fund used to purchase an annuity) and the price of a life annuity. The price of a life annuity (or the 'annuity factor') at age x is calculated using the risk-free return, r, as follows:

$$\ddot{a}_x = \sum_{s=0}^{120-x} \frac{{}_s p_x}{(1+r)^s} \tag{9}$$

where ${}_s p_x$ is the probability that a life of age x survives to age $(x+s)$.

We assume the annuity factor is constant over time for each age x, so we do not explore the additional risk faced as a result of volatility in the price of a life annuity (as a result of changes over time in the underlying interest rate and the mortality assumption used). The member invests the residual wealth that is not annuitised in the risky asset and, at each future age, decides whether to consume some of this residual wealth (in addition to the annuity income received) or to use some of it to purchase additional annuity income. After retirement, the only choice of financial asset will be between life annuities and the equity fund, since the bond fund is a dominated asset (see below).

2.1.5 Pension fund dynamics

Before retirement, the growth in the member's pension wealth will depend on the investment strategy adopted, the investment returns on the equity and bond funds, and the chosen contribution rate.

The contribution rate at age x is given by:

$$\pi_x = \frac{Y_x - C_x}{Y_x} \qquad \text{for } x = 20, 21, \ldots, 64 \tag{10}$$

We require the contribution rate to be non-negative, so that $Y_x \geq C_x$ before retirement. The contribution rate is allowed to vary over time, so that consumption in any period can adjust to changes in income level and investment performance.

A proportion, α_x, of the member's pension fund is assumed to be invested in the risky asset at age x and, prior to retirement, we have the following recursive relationship for the dynamics of the pension fund:

$$F_{x+1} = \left(F_x + \pi_x Y_x\right) \times \left[1 + r + \alpha_x \times \left(\mu + \sigma Z_{1,x}\right)\right] \qquad \text{for } x = 20, 21, \ldots, 64 \qquad (11)$$

The short-selling restriction requires that $0 \le \alpha_x \le 1$.

At the start of the year of age 65 to 66, the member is assumed to retire and chooses to continue to hold a proportion, α_{65}, of the accumulated wealth in the risky asset, with the remaining proportion of $\left(1-\alpha_{65}\right)$ being used to purchase a life annuity at a current price of $\ddot{a}_{65}$. At each future age, the member can choose to use some of the residual wealth (plus the annuity income received) to purchase an additional life annuity, allowing for the possibility that the annuitisation decision is itself dynamic. Thus, for $x = 65, 66, \ldots, 120$, the pension fund dynamics equation is given by:

$$F_{x+1} = \frac{\left(1-\alpha_x\right) \times F_x}{\ddot{a}_x} \times \ddot{a}_{x+1} + \left[\alpha_x F_x + \frac{\left(1-\alpha_x\right) \times F_x}{\ddot{a}_x} - C_x\right] \times \left(1 + r + \mu + \sigma Z_{1,x}\right) \ge 0 \qquad (12)$$

where:

- $\left(1-\alpha_x\right) \times F_x / \ddot{a}_x$ is the annual income from the annuity at age x and $\left[\left(1-\alpha_x\right) \times F_x / \ddot{a}_x\right] \times \ddot{a}_{x+1}$ is the capitalised value of this income stream (i.e., the value of the annuity) at age $\left(x+1\right)$; and
- $\alpha_x F_x$ represents the non-annuitised pension wealth at age x, immediately before receiving the current annuity income of $\left(1-\alpha_x\right) \times F_x / \ddot{a}_x$ and consuming the chosen amount of C_x; this net amount is then invested in equities over the coming year, so the second term on the right hand side of Equation (12) is the value of the equity investment at age $\left(x+1\right)$.

As the plan member's age increases, the return from purchasing an annuity increases, provided that the member survives to receive the additional income. This component of the return on the annuity is known as the 'mortality premium'.[18] Eventually, the return from the annuity will exceed the return from the risky asset and then it becomes optimal to switch all remaining pension wealth into annuities.[19] As will be seen later, based on the chosen investment and mortality parameters, the life annuity becomes the dominant asset class by age 76. Similarly, as a result of the mortality premium, it is unnecessary to include the risk-free bond fund within the asset allocation decision after retirement, as this asset is immediately dominated by the return on the life annuity.

Finally, we must constrain annual consumption after retirement such that it does not exceed the annual income from the annuity plus any remaining residual wealth:

$$C_x \leq \frac{(1-\alpha_x)\times F_x}{\ddot{a}_x} + \alpha_x F_x \qquad \text{for } x = 65, 66, \ldots, 120$$

2.1.6 The optimisation problem and solution method

The model has two control variables at each age x, for $x = 20, 21, \ldots, 120$: the equity allocation, α_x, and the consumption level, C_x.

The optimisation problem is:

---

[18] This is also known variously as the 'mortality drag', 'mortality credit' or 'survivor credit'. Consider the post-retirement wealth dynamics given in Equation (12). Suppose that we set $\alpha_x = 0$ (i.e., assume the full amount of the wealth is invested in the life annuity) and assume that $C_x = F_x / \ddot{a}_x$ (i.e., the member consumes the full amount of annual annuity income), then Equation (12) can be re-written as:

$$F_{x+1} = \frac{F_x}{\ddot{a}_x} \times \ddot{a}_{x+1} = (F_x - C_x) \times (1+r) \times \left(\frac{1}{p_x}\right) = (F_x - C_x) \times \left[1 + r + \left(\frac{q_x}{1-q_x}\right) \times (1+r)\right]$$

where $q_x = 1 - p_x$ is the probability that a life of age x dies before reaching age $(x+1)$. The term $\left(q_x / (1-q_x)\right) \times (1+r)$ is the 'mortality premium' and represents the additional return above the risk-free rate arising from the redistribution of annuity wealth from annuitants who died during the year to those who survive.

[19] In the absence of a bequest motive. The value of an annuity is reduced to zero, the moment the plan member dies.

$$\max_{\alpha_x, C_x} \; U_x$$

with U_x defined as in Equation (4), subject to the following constraints:

(i) for $x = 20, 21, \ldots, 64$, we have:

a) a wealth dynamics equation satisfying:

$$F_{x+1} = \left(F_x + \pi_x Y_x\right) \times \left[1 + r + \alpha_x \times \left(\mu + \sigma Z_{1,x}\right)\right] \geq 0,$$

b) an allocation to the risky asset satisfying $0 \leq \alpha_x \leq 1$, and

c) a contribution rate satisfying $0 \leq \pi_x \leq 1$; and

(ii) for $x = 65, 66, \ldots, 120$, we have:

a) a wealth dynamics equation satisfying Equation (12),

b) an allocation to the risky asset satisfying $0 \leq \alpha_x \leq 1$, and

c) consumption satisfying $C_x \leq \dfrac{\left(1-\alpha_x\right) \times F_x}{\ddot{a}_x} + \alpha_x F_x$.

The Bellman equation at age x is:

$$V_x = \max_{\alpha_x, C_x} \left\{ \left(1-\beta\right) \times \left(C_x\right)^{1-\frac{1}{\varphi}} + \beta p_x \times \left(E_x \left[\left(V_{x+1}\right)^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}} \tag{13}$$

An analytical solution to this problem does not exist, because there is no explicit solution for the expectation term in the above expression. Instead, we must use a numerical solution method to derive the value function and the corresponding optimal control parameters. We use the terminal utility function at age 120 to compute the corresponding value function for the previous period and iterate this procedure backwards, following a standard dynamic programming strategy. [20]

[20] See the Appendix for more details. Applications of the solution method include Weil (1990), Campbell and Viceira (2001) and Gomes and Michaelides (2005).

2.2 Parameter calibration

We begin with a standard set of baseline parameter values (all expressed in real terms) presented in Table 1. The constant real risk-free interest rate, r, is set at 2% per annum, while, for the equity return process, we use a mean equity risk premium, μ, of 4% per annum[21] and a standard deviation, σ, of 20% per annum. We use the projected PMA92 table[22] as the standard male mortality table, and hence, using a real interest rate of 2% per annum, the price of a whole life annuity paying one unit per annum at the start of each year of age from age 65 is $\ddot{a}_{65} = 15.87$.

Table 1 – Baseline parameter values

Asset returns	
Real risk-free rate, r	0.02
Equity premium, μ	0.04
Volatility of annual equity return, σ	0.2
Preference parameters	
RRA, γ	5.0
EIS, φ	0.2
Discount factor, β	0.96
Labour income process	
Starting salary at age 20, Y_{20}	1.0
Average real salary growth, r_1	0.02
Volatility of shock correlated with equity returns, σ_1	0.05
Volatility of annual rate of salary growth, σ_2	0.02
Career salary profile parameter, h_1	−0.276
Career salary profile parameter, h_2	0.75835

Our baseline plan member has the following preference parameters: RRA = 5.0, EIS = 0.2, and discount factor of $\beta = 0.96$.[23] The starting salary at age 20 is normalised on

[21] In line with the recent literature, see, e.g., Fama and French (2002) and Gomes and Michaelides (2005).

[22] PMA92 is a mortality table for male pension annuitants in the UK based on experience between 1991 and 1994. We use the projected rates for the calendar year 2010, i.e., the table PMA92(C2010), published by the Continuous Mortality Investigation (CMI) Bureau in February 2004. We assume that there are no longevity improvements in the current version of the model.

[23] This parameter constellation is common in the literature (e.g., Gourinchas and Parker (2002), Vissing-Jørgensen (2002), Gomes and Michaelides (2004)). The values of RRA and EIS are also consistent with power utility for the baseline case.

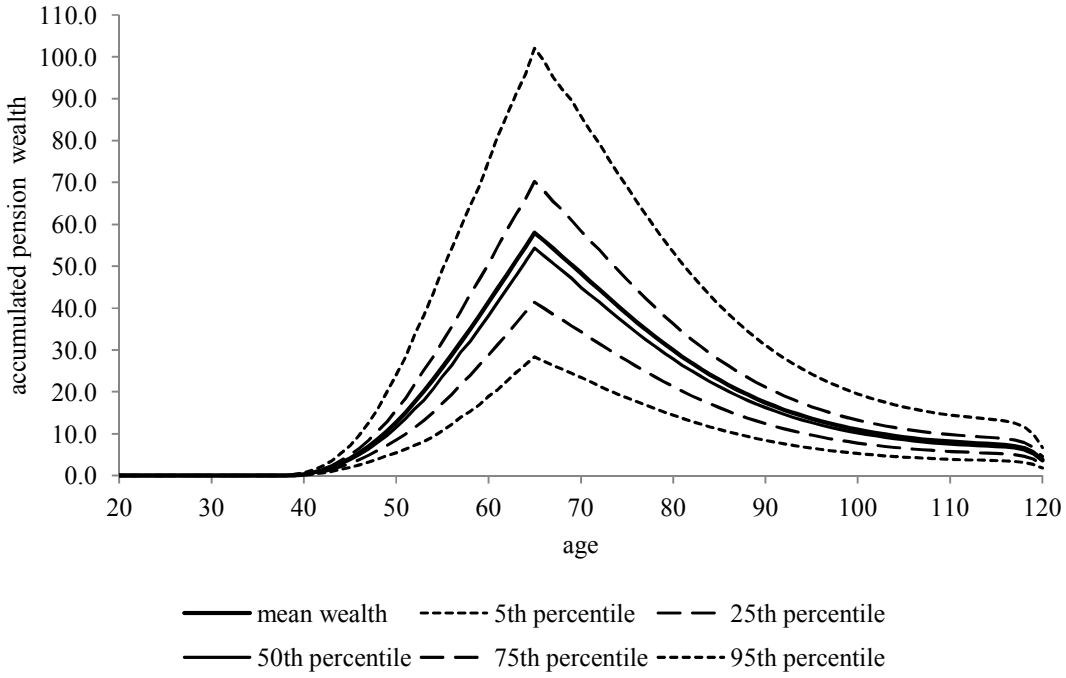
unity. All absolute wealth and income levels are measured in units of the starting salary. In line with post-war UK experience, the annualised real growth rate of national average earnings, r_I, is assumed to be 2% per annum with a standard deviation of 2% per annum.

3 Results

3.1 Baseline case

The output from the optimisation exercise is a set of optimal control variables (i.e., equity allocations, $\{\alpha_x\}$, and consumption levels, $\{C_x\}$) for each age $x = 20, 21, \ldots, 120$. We generate a series of random variables for both the equity return and labour income shocks, and then generate 10,000 independent simulations of wealth and labour income levels.

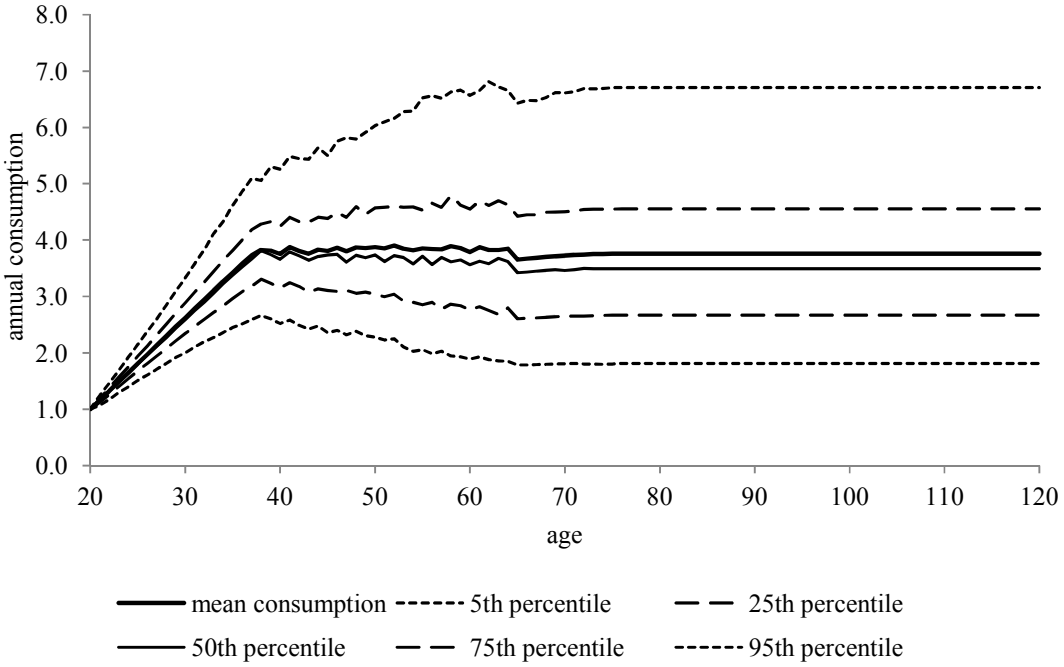
Figure 3 – Accumulated pension wealth



Based on these simulations, Figures 3 and 4 show the distribution of the accumulated pension wealth and the optimal consumption level for ages 20, 21, …, 120. In the early

years of the life cycle (i.e. up to age 35 or so), labour income is low and the desire to accumulate pension wealth to be consumed later is outweighed by the desire for current consumption and, as a consequence, the plan member makes no pension contributions at this stage. This conforms with observed practice, where younger plan members (especially those with a young family) seem unwilling (or unable) to contribute to their retirement savings on a voluntary basis.

Figure 4 – Optimal annual consumption



From Figure 1, we can see that human capital increases until about age 35. This is because of the very high rate of salary growth in the early years (relative to the discount factor, $\beta = 0.96$, applied to future labour income). Thus, whilst the member's human capital is increasing, it is optimal to consume most (if not all) of the labour income received.

However, when salary growth rates begin to slow down (after age 35) and human capital begins to fall, the retirement motive becomes more important as the member recognises

the need to build up the pension fund in order to support consumption after retirement. As a result, as can be seen from Figure 4, consumption remains largely constant from age 35 onwards (despite the continuing, but slower, growth in labour income), with the additional income saved to fund post-retirement consumption.[24]

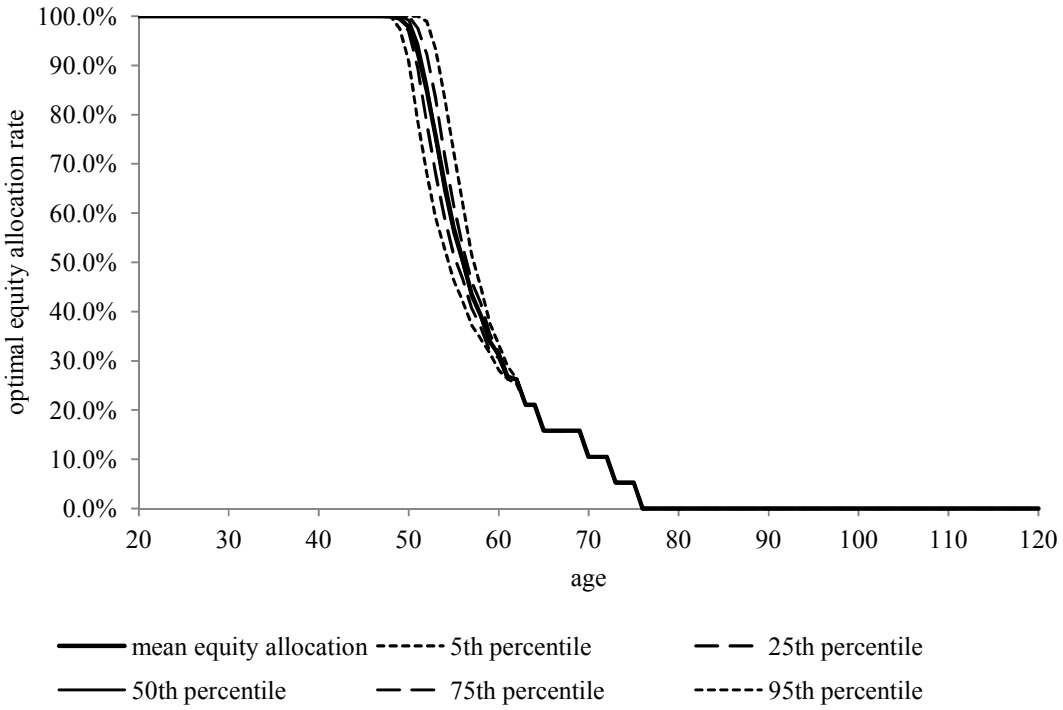
After retirement, the member receives no further labour income, but instead starts to receive pension income (from any annuities purchased on or after retirement or from drawing down an income from the fund) and, hence, to enjoy consumption in retirement financed by running down the assets in the pension fund for the remainder of his lifetime. It can be seen from Figure 4 that the pension wealth accumulated at retirement is sufficient to maintain consumption at the pre-retirement level (and, thus, the strong desire for consumption smoothing, as reflected in the low baseline EIS value of $\varphi = 0.2$, is satisfied).[25]

Figure 5 shows the distribution of the optimal equity allocation at each age, $\{\alpha_x : x = 20, 21, \ldots, 120\}$, again based on 10,000 simulations. There is a high equity weighting at younger ages with a gradual switch from equities to bonds as the retirement age approaches. Prior to around age 45, the member optimally invests all pension wealth in the risky asset to counterbalance his implicit holding of bond-like human capital. After age 45 or so, human capital starts to decline very steeply and the member responds to this by rebalancing the pension fund towards bonds. This is because bonds and human capital are substitutes for most plan members, with the degree of substitutability inversely related to the correlation between labour income growth and equity returns, $\sigma_1 \Big/ \sqrt{\left(\sigma_1^2 + \sigma_2^2\right)}$.

[24] The variability in consumption levels across different scenarios for $\{Z_{1,x}\}$ and $\{Z_{2,x}\}$ shown in Figure 4 is largely due to the variability in fund size (as shown in Figure 3), since the ratio (C_x / F_x) is fairly constant.

[25] The slight dip in consumption on retirement seen in Figure 4 is explained by the coarseness of the grid used to discretise the space spanned by the consumption control variable before and after retirement (as a result of the different constraints placed on consumption in these different stages of the lifecycle, see Section 2.1.6). Use of an ever finer grid would remove this effect, but as noted in the appendix, this would considerably increase the run time for solving the dynamic programming exercise.

Figure 5 – Optimal equity allocation



This investment strategy is known as 'stochastic lifestyling', because the optimal equity weighting over the life cycle depends on the realised outcomes for the stochastic processes driving the state variables, namely the annual equity return and labour income growth rate, and will be different for each of the 10,000 simulations generated. It is important to note that the profiles in Figure 5 are not consistent with (nor, indeed, a justification for) the more traditional 'deterministic lifestyling' strategy.

The member retires at age 65, but Figure 5 shows no immediate change in the optimal allocation to equities.[26] However, the bond holdings are exchanged for life annuities which pay the retirement income. Figure 5 shows that some of the equity fund is sold off each year and the proceeds used to purchase more life annuities, a strategy known as

[26] The lack of smoothness in the equity allocation above age 60 is again due to the discretisation procedure used by the solution method.

'phased annuitisation'.[27] This is to benefit from the mortality premium which increases with age and exceeds the equity risk premium from age 76 onwards, at which point it is optimal for the member to invest the entire residual value of the pension fund in life annuities, regardless of risk attitude.

Figure 6 shows the distribution of the optimal contribution rate, corresponding to the wealth accumulation and consumption distributions shown in Figures 3 and 4 above. Until around age 35, when labour income is low but rising rapidly, it is optimal for the plan member to consume the entire labour income (resulting in no saving towards retirement). Thus, the individual is effectively trading off a lower income in retirement in return for the ability to consume more in the early years when income is low. However, once human capital begins to decline, consumption no longer increases in line with labour income. Instead, consumption remains reasonably constant, allowing the additional labour income received each year to be saved. Thus, from age 35, the optimal contribution rate is back-loaded, increasing steadily with age to a rate of 30-35% at age 55 (and remaining at this level until retirement at age 65). Whilst there is evidence that people do begin to save much more for their retirement once their children have left home and they have paid off their mortgage, it appears to be uncommon for people in most countries to save at the rate that we have found to be optimal. On the other hand, people also accumulate non-pension assets which can be used to finance retirement consumption and it should be remembered that our model does not include any other forms of savings or wealth holding (e.g., bank accounts and investment vehicles such as mutual funds, housing etc.).[28]

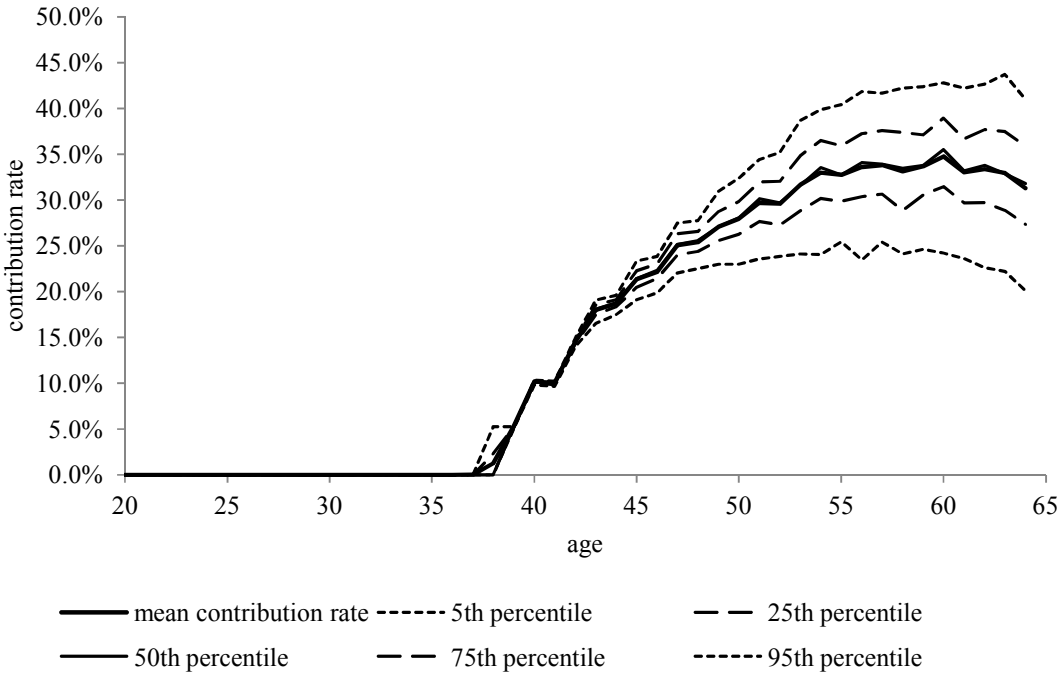
Age-related contribution rates are not common in real-world DC plans. Much more common is a fixed rate throughout the life of the plan: for example, in the UK, the

[27] Other studies which show the optimality of gradual annuitisation over time include Milevsky and Young (2007) and Horneff et al. (2008).

[28] It is worth noting that the pattern of consistently increasing real earnings assumed for the pension plan member considered here will not apply to certain occupations, such as manual labourers. To compensate for this, it will optimal for such workers to have a much more front-loaded pattern of pension contributions.

(combined employer and employee) contribution rate is typically between 8 and 10% per annum (GAD (2006, Table 8.2)).

Figure 6 – Optimal contribution rate prior to retirement



3.2 Sensitivity analysis

In this section, we conduct a sensitivity analysis on the key parameters in the model.

3.2.1 Coefficient of relative risk aversion

Figure 7 shows the mean optimal contribution rate for different levels of RRA. In all cases, contributions begin between ages 35 and 40. Members with the lowest level of risk aversion (RAA = 2) begin saving for retirement slightly later than those with the highest level of risk aversion (RAA = 10) and save around 5% less of labour income each year prior to retirement.

As a result of the lower mean contribution rate, risk-tolerant members, *ceteris paribus*, will accumulate a lower mean level of pension wealth. They therefore need (and are willing to accept) a higher average equity allocation in the pension fund in an attempt to generate the desired higher level of retirement savings. As shown in Figure 8, for such members, the mean equity allocation decreases both later and more gradually, remaining at around 50% at retirement (compared with around 20% for the baseline member and around 10% for a member with RRA = 10). However, after retirement, the mean equity allocation reduces quickly and, regardless of the level of risk aversion, all pension wealth is held in a life annuity from age 76 onwards.

Figure 7 – Mean optimal contribution rate: Effect of changing RRA

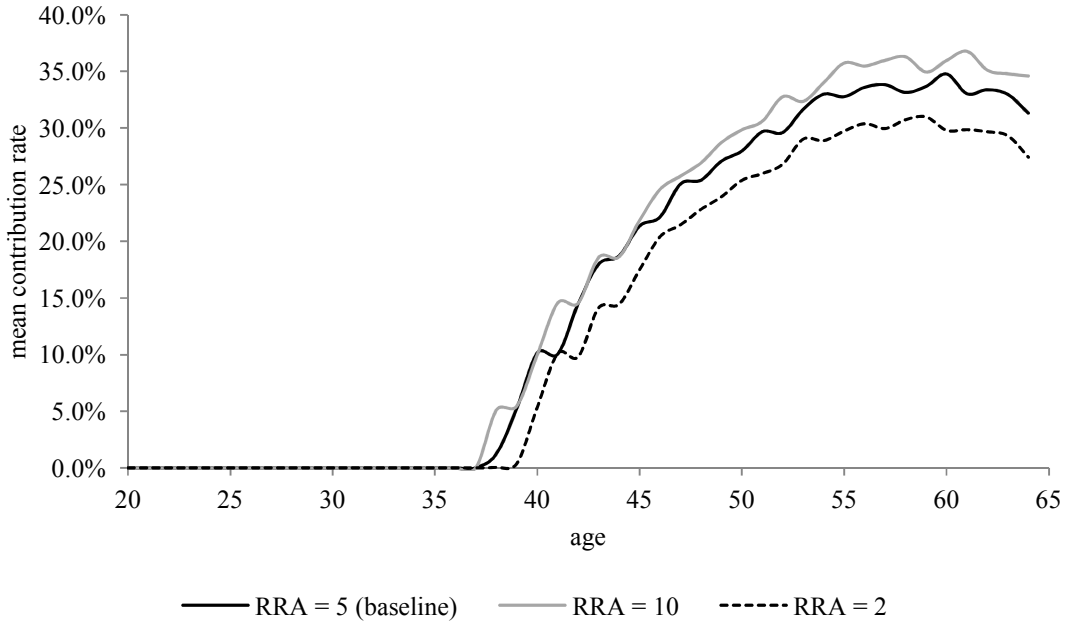
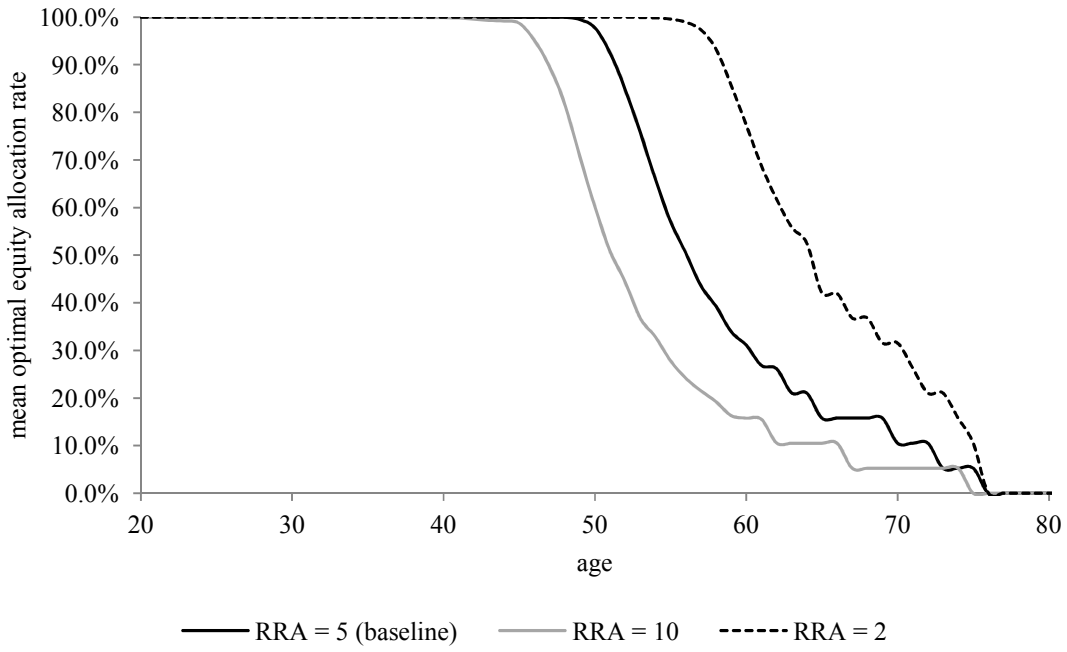


Figure 9 shows the mean consumption profile with different RRA levels.[29] The lower level of pension saving associated with lower levels of risk aversion enables higher consumption during the working life. Lower risk aversion after retirement and the associated greater equity weighting in the post-retirement pension fund will also result in

[29] The apparent drop in consumption at age 65 is again due to the discretisation procedure used by the solution method.

higher average consumption at older ages in comparison with a more risk-averse member. However, both pension wealth and the level of consumption supported by this wealth are significantly more volatile than when risk aversion is higher.[30]

Figure 8 – Mean optimal equity allocation: Effect of changing RRA

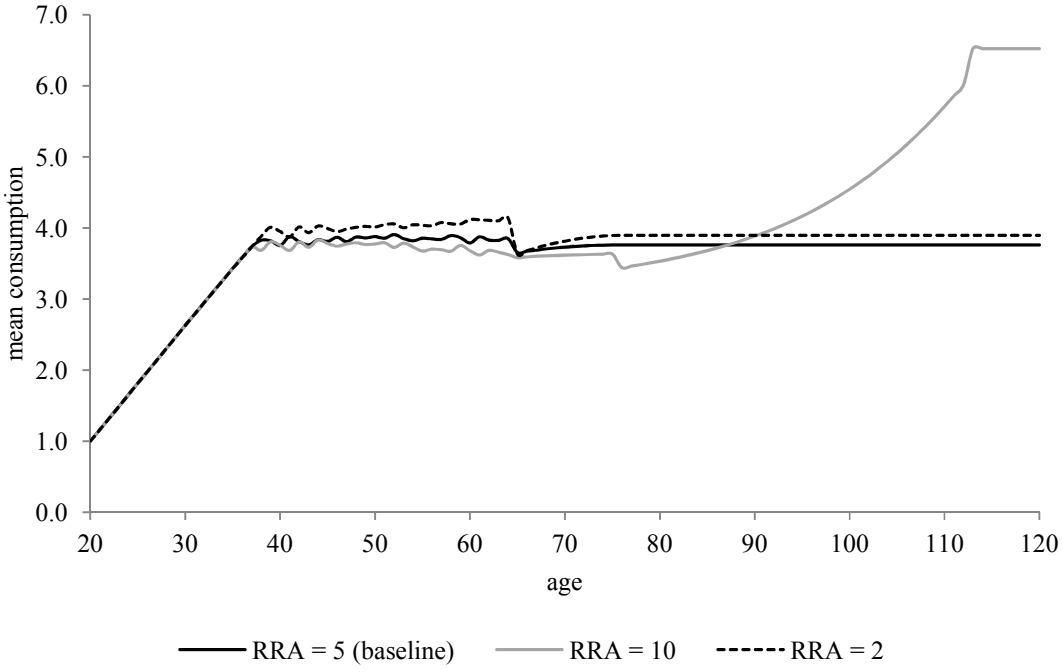


The increase in consumption at older ages for RRA = 10 can be attributed to the fact that, in this case, $\text{EIS} = 0.2 > (1/\text{RRA}) = 0.1 \Leftrightarrow \varphi > 1/\gamma \Rightarrow \left[1-(1/\varphi)\right]/(1-\gamma) < 1$. Thus, from Equation (4), the utility at age x, U_x, is increased by reducing current consumption, C_x, and increasing future consumption, since, with the above relationship between EIS and RRA, the present value of the expected utility of future consumption is increased by more than the utility of current consumption is reduced. As a consequence of this, plan members with such characteristics choose not to consume all of the annuity income

[30] For RAA = 2, the inter-quartile range of the simulated distribution of accumulated pension wealth levels at retirement is $71.56 - 36.86 = 34.70$ (compared with $70.22 - 41.36 = 28.86$ for the baseline case of RAA = 5), while for the simulated distribution of annual consumption from age 76 onwards (when the full amount of the remaining wealth is invested in the life annuity), the inter-quartile range is $4.86 - 2.42 = 2.44$ (compared with $4.55 - 2.67 = 1.88$ for the baseline case).

received. Instead, it is optimal for them to use some of this income to purchase additional annuities, thereby providing higher income (and, thus, enabling higher consumption and hence utility) in future (provided, of course, that the individual survives to receive this additional income). This is considered further below when we analyse the sensitivity of the results to changes in the EIS parameter.

Figure 9 – Mean optimal consumption: Effect of changing RRA

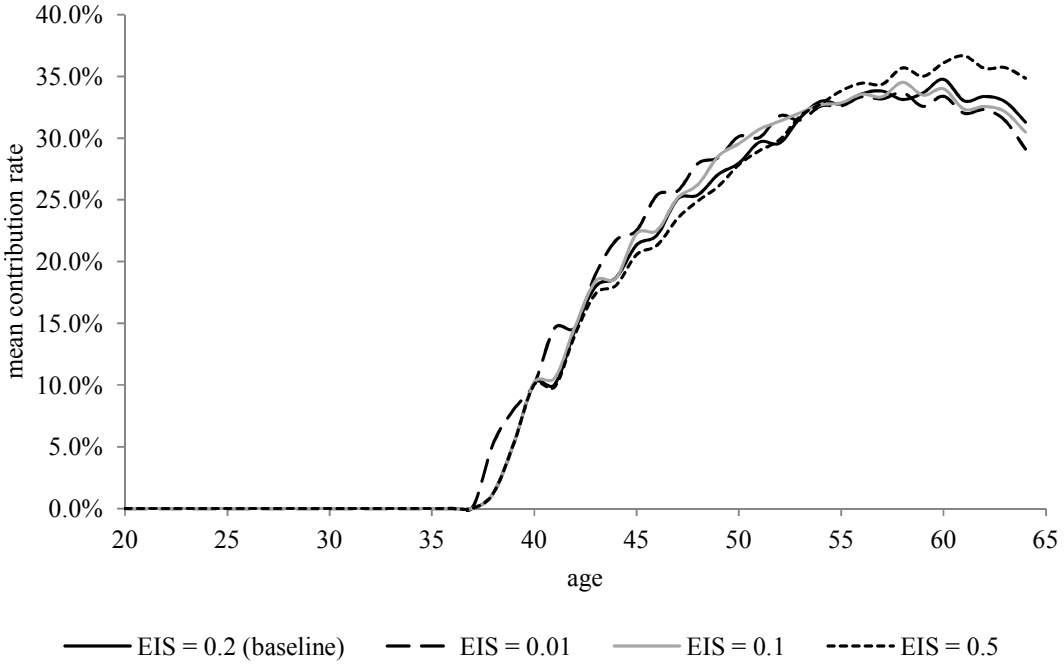


3.2.2 Elasticity of intertemporal substitution

Figure 10 shows the mean optimal contribution rate for different levels of EIS. In the middle stages of the life cycle (i.e., between age 35 and age 55), a member with a lower level of EIS will tend to save slightly more towards retirement (about 1-2% of income more per annum). This can be explained by the fact that a member with a lower EIS is less willing to accept a fall in consumption in future (particularly after retirement) and is, thus, prepared to contribute slightly more now to build up a higher fund at retirement (thereby reducing the likelihood of requiring such a decrease in consumption

subsequently). However, in the last 10 years or so before retirement, this effect is reversed. By this stage, a typical member's labour income can be expected to begin to decline slightly as retirement approaches (see Figure 2). Thus, a member with the low EIS is less willing to cut current consumption in response to this fall in income (and so contributes less to the pension plan at this time). In comparison, a member with a high EIS of 0.5 is able to maintain an annual contribution rate that is about 4-5% higher at this time, which makes up much of the deficit built up as a result of the lower contributions prior to age 55. The overall result is that the fund built up at retirement and, thus, the post-retirement consumption supported by this fund are relatively insensitive to the EIS level, as can be seen in Figure 12.

Figure 10 – Mean optimal contribution rate: Effect of changing EIS



For a given level of risk aversion, Figure 11 shows that a low EIS of 0.01 leads to a slightly lower equity weighting (of about 2-3%) at each age prior to retirement compared with a high EIS of 0.5. This follows because a member with a low EIS prefers more stable consumption and will therefore accept less equity risk.

Figure 11 – Mean optimal equity allocation: Effect of changing EIS

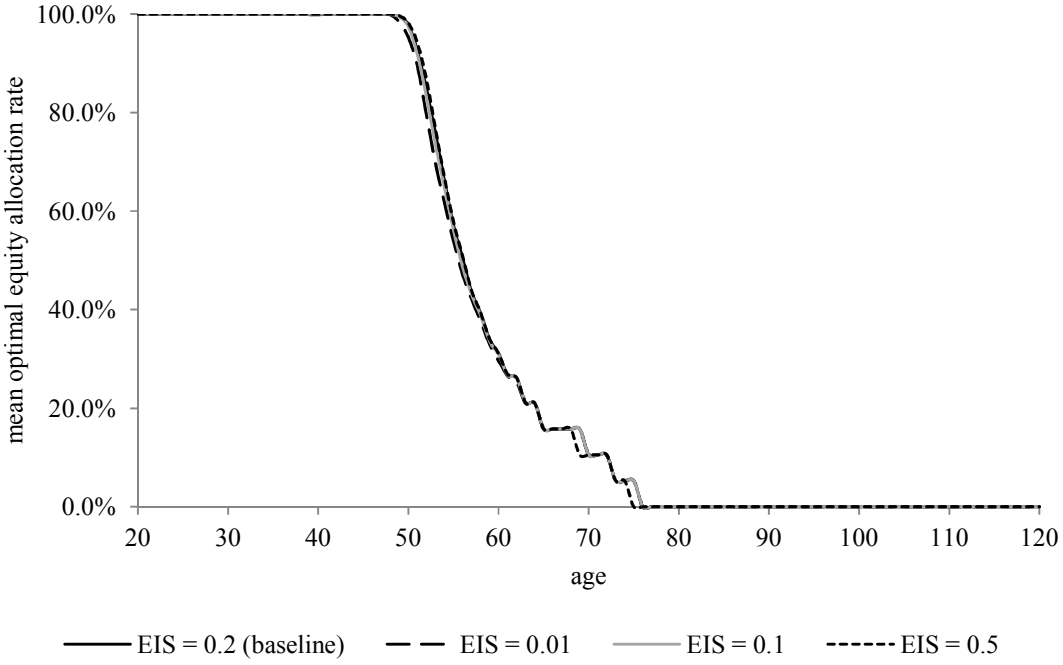


Figure 12 – Mean optimal consumption: Effect of changing EIS

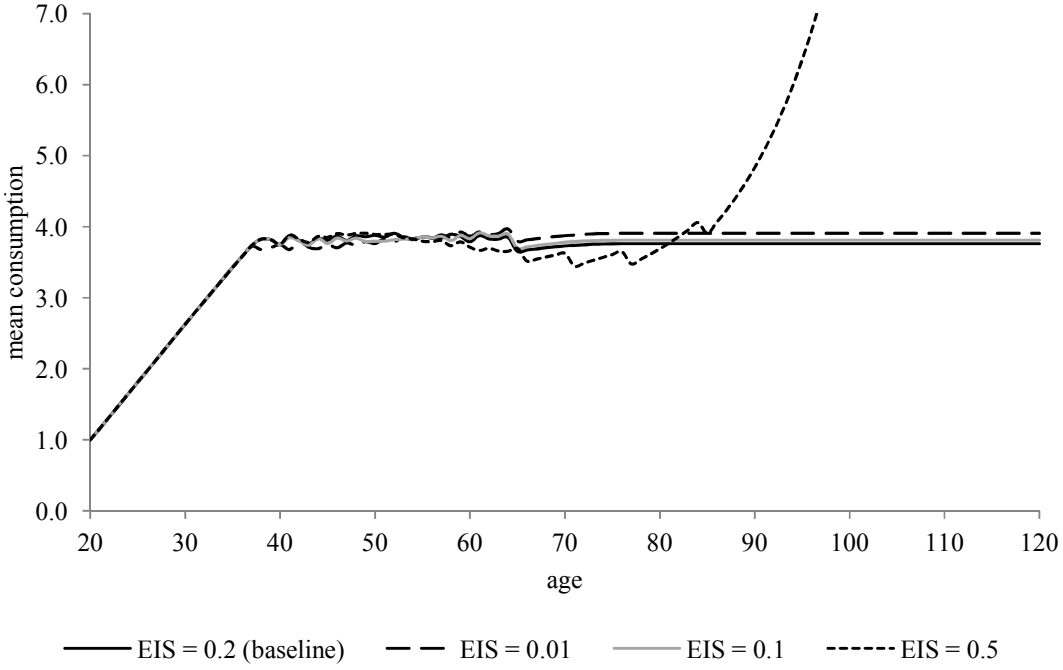


Figure 12 shows the mean consumption profile for different EIS levels. The desire of a member with a low EIS to achieve consumption stability from one time period to the next is clearly evident. In contrast, a member with a higher level of EIS is more willing to cut consumption slightly prior to retirement (when labour income begins to fall), thereby maintaining a higher contribution rate into the pension plan (as seen in Figure 10 above). Similarly, after retirement, a higher level of EIS encourages the member to consume slightly less than the full amount of the annuity income received and to use some of the resulting savings to purchase additional annuity income. If the member survives to high ages, the size of the mortality premium in the annuities purchased allows consumption to increase substantially if EIS is high relative to RRA (in particular, if $\text{EIS} > (1/\text{RRA})$). However, the probability of the member surviving to such high ages is extremely low. Figure 13 shows the mean expected consumption profile at each future age (allowing for the effects of mortality risk) for a life of age 20. In this case, it can be seen that the effect of changing the EIS level on expected consumption is minimal (unless the member survives to a very high age and benefits from the effects of the mortality premium as shown in Figure 12). Further, such behaviour appears to be uncommon in practice, suggesting that we are unlikely to observe many individuals with $\text{EIS} > (1/\text{RRA})$.

3.2.3 Personal discount factor

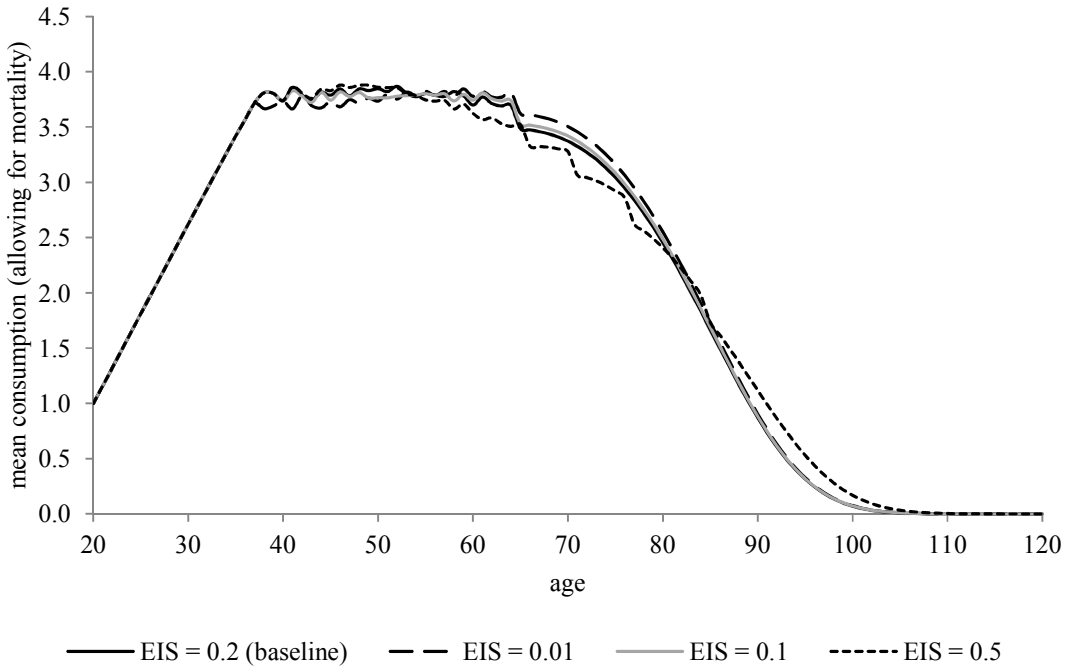
Figures 14, 15 and 16 show the outcomes from conducting a sensitivity analysis on β, the individual's personal discount factor, on the mean optimal contribution rate, consumption profile and equity weighting, respectively.

Individuals with a low personal discount factor (or high personal discount rate) value current consumption more highly than future consumption in comparison with individuals with a high personal discount factor. This will lead, *ceteris paribus*, to both a lower average contribution rate into the pension plan prior to retirement, as shown in Figure 14[31] and a downward-sloping consumption profile after around age 40, as shown

[31] Although in the years immediately prior to retirement, pension contributions are belatedly increased to

in Figure 15. There will be a correspondingly slower accumulation of financial wealth and therefore a higher ratio of human to financial wealth throughout the working life. This, in turn, leads to an optimal lifestyle strategy with a higher allocation to the risky asset throughout the working life, together with a shorter switching period, as shown in Figure 16. The figure also shows that, whatever the size of the personal discount factor, there is a common equity weighting in the fund by the time the member reaches age 65. This indicates that the optimal investment strategy after retirement will not be influenced by the size of the personal discount factor.

Figure 13 – Mean optimal expected consumption at age 20 (allowing for mortality risk): Effect of changing EIS



support consumption in retirement.

Figure 14 – Mean optimal contribution rate: The effect of changing the personal discount factor, β

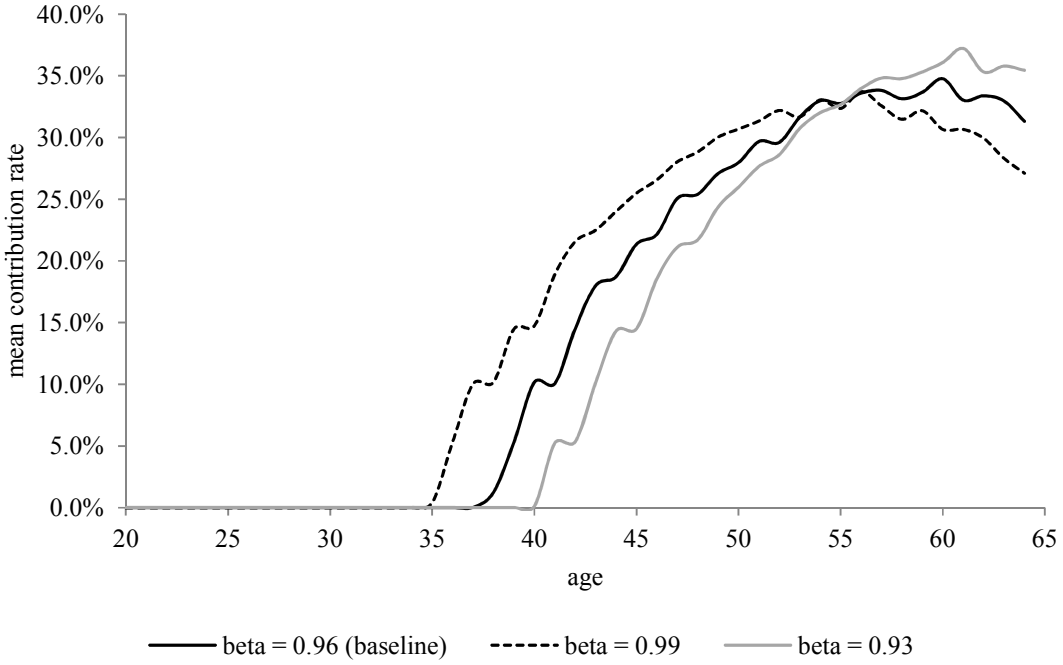


Figure 15 – Mean optimal consumption The effect of changing the personal discount factor, β

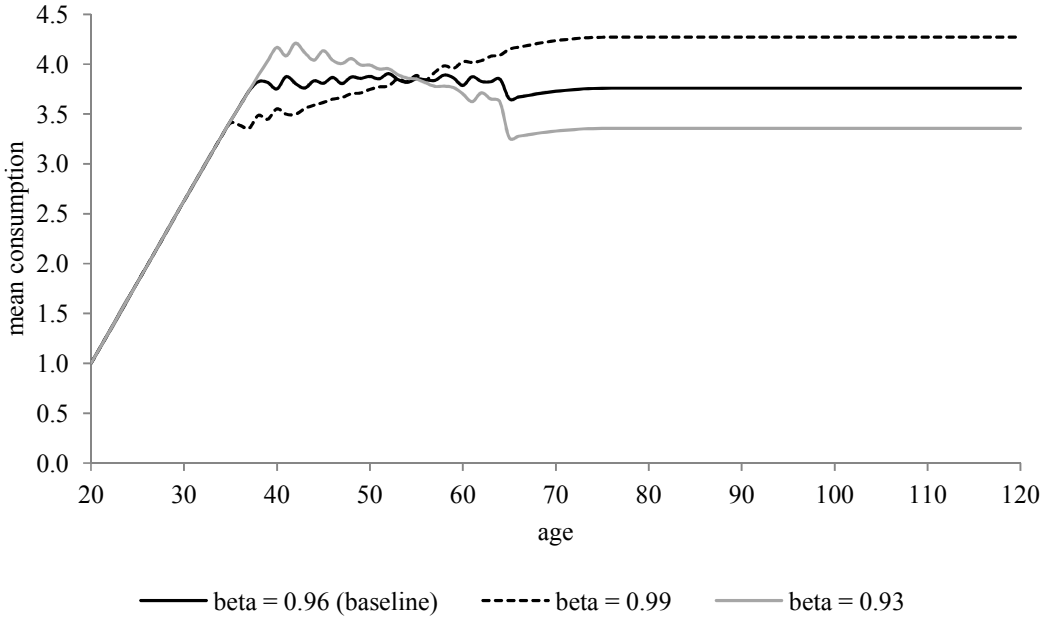
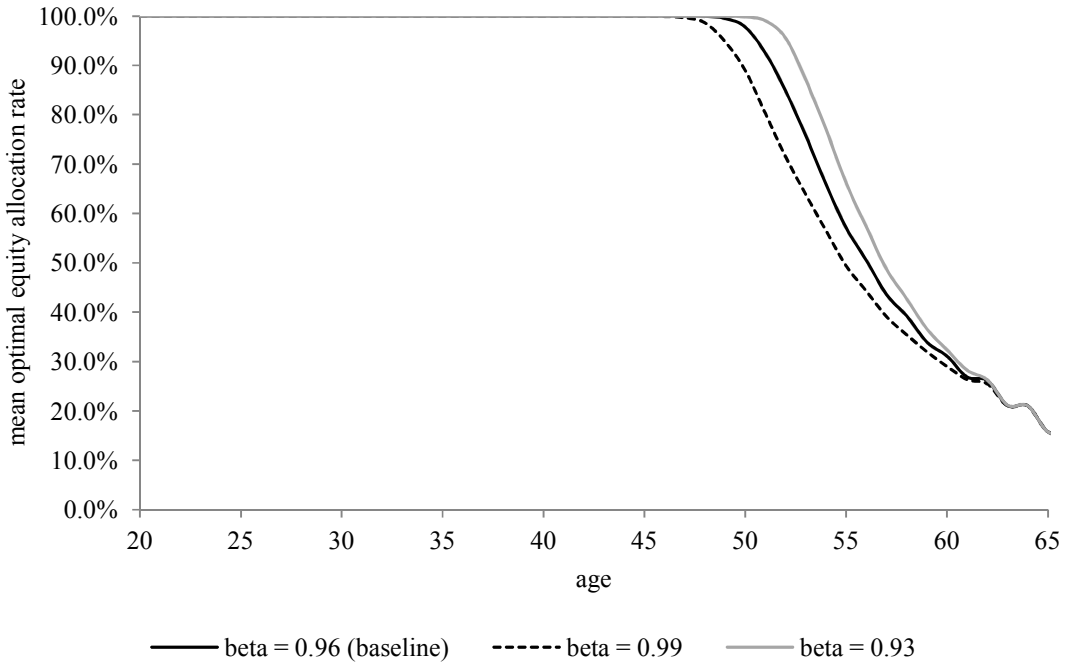


Figure 16 – Mean optimal equity allocation: The effect of changing the personal discount factor, β



4 Conclusion

In this paper, we have examined optimal funding and investment strategies in a DC pension plan using a life cycle model that has been extended in three significant ways:

- the assumption of Epstein-Zin recursive preferences by the plan member which enables a separation between relative risk aversion and the elasticity of intertemporal substitution,
- the recognition of human capital as an asset class along with financial assets, such as equities and bonds, and
- endogenising the decision about how much to annuitise in retirement.

Our key findings with respect to funding are:

- The optimal funding strategy involves an age-dependent annual contribution rate, increasing steadily from zero prior to age 35 to around 30-35% by age 55 (but then reducing slightly as labour income falls in the years immediately prior to retirement, so as to maintain pre-retirement consumption levels).

- The effect of lower risk aversion is to reduce the level of pension contributions at all ages (in the expectation of achieving higher investment returns on those contributions during the accumulation phase). However, as would be expected, the downside of this is greater uncertainty in both the pension fund at retirement and the retirement consumption supported by this fund.
- Prior to retirement, a lower level of EIS leads to a slightly higher contribution rate prior to around age 55 (with the aim of building up a higher pension fund and, thus, reducing the risk of a fall in consumption after retirement), but then to a lower contribution rate in the years immediately prior to retirement (when the member is less willing to cut consumption as labour income falls and, thus, the contribution rate into the pension plan must be reduced). The overall effect of a lower EIS is greater consumption stability over the life cycle. The effect of a higher EIS is to increase the willingness of the plan member to accept consumption volatility over time (and, in particular, a fall in the level of consumption from one time period to the next). A high level of EIS in relation to RRA (such that $EIS > (1/RRA)$) implies that, after retirement, it is optimal for the member to spend less than the pension income received and use the resulting savings each year to purchase additional annuities (thereby benefiting from the mortality premium inherent in the return on life annuities).
- A lower personal discount factor implies a preference for current (rather than future) consumption, leading to a lower contribution rate until the last 10 years or so before retirement when current consumption has to be reduced sharply each year, and thus retirement savings increased, to ensure a minimal level of pension wealth.

Our key findings with respect to investment strategy are:

- The optimal investment strategy is also age-dependent. Pre-retirement, the optimal strategy is stochastic lifestyling, rather than the more conventional deterministic lifestyling. While the optimal weighting in equities is initially very high and subsequently declines as the retirement date approaches, it does not do so in a predetermined manner as in the case of deterministic lifestyling. Instead,

the optimal equity weighting over the life cycle depends on the realisations of the stochastic processes determining equity returns and labour income. Stochastic lifestyling is justified by recognising the importance of human capital and interpreting it as a bond-like asset which depreciates over the working life. An initial high weighting in equities is intended to counterbalance human capital in the combined 'portfolio' of human capital and financial wealth. In time, the weighting in equities falls stochastically, while that in bonds rises as human capital decays over time.

- Another difference with deterministic lifestyling is that the portfolio is not completely switched into bonds by the retirement date. Depending on the member's risk aversion, there could still be significant equity holdings in the pension fund on the retirement date. For the ranges of risk aversion that we considered in this study, the optimal equity weighing at retirement varied between 20% and 50%.
- The optimal investment strategy at retirement is phased annuitisation. The first stage of this strategy is to exchange the bond fund for a life annuity, thereby securing lifelong income protection for the member as well as benefiting from the mortality premium in the return on the annuity. The optimal weight in the equity fund does not immediately change. However, each year that the member survives, the return from buying additional annuities increases and the equity weighting falls until a point is reached when the mortality premium exceeds the equity risk premium and it becomes optimal to switch the entire residual pension fund into annuities whatever the member's attitude to risk.
- The effects of lower risk aversion and a lower personal discount factor are to increase the length of time over which the pension fund is fully invested in equities and to reduce the length of the switchover period into bonds prior to retirement. Lower risk aversion leads to a higher post-retirement equity weighting, but does not affect the age at which it is optimal to switch the remaining pension fund assets into annuities (this decision depends purely on the relationship between the relative sizes of the mortality premium and the equity risk premium). The size of the personal discount factor has no effect on the optimal asset

allocation after retirement. The size of the EIS has a marginal impact on the optimal asset allocation both before and after retirement.

The results in this paper have some important implications for the optimal design of DC pension plans:

- They provide some justification for age-related contribution rates in DC pension plans. Because individuals tend to prefer relatively smooth consumption growth, a plan design involving a zero contribution rate prior to around age 35 with an increasing age-dependent contribution rate thereafter (reaching, on average, around 30 to 35% per annum in the period immediately prior to retirement) offers higher expected lifetime utility than one with fixed age-independent contribution rates. Greater contribution rate flexibility would allow for the preferences of individual members (with regard to their desire for consumption smoothing, risk attitude and relative preference for current over future consumption) to be recognised. While high, heavily back-loaded, age-related contribution rates might be optimal for 'econs' (i.e., rational life cycle financial planners), they might not be optimal for 'humans' with their behavioural difficulties in starting and maintaining long-term savings programmes (see, e.g., Thaler and Bernartzi (2004), Mitchell and Utkus (2004) and Thaler and Sunstein (2008)). A compromise solution might be a compulsory minimum contribution rate at all ages (to ensure that all plan members have some minimal pension fund to support consumption in retirement) together with age-related additional voluntary contributions (AVCs) at higher ages.
- It is important to get reliable measures of the member's risk aversion and personal discount factor. This can be achieved using appropriately designed questionnaires (see, e.g., Coller and Williams (1999), Holt and Laury (2002), Andersen et al. (2008) and Laury et al. (2011)). However, EIS seems to be less important according to our sensitivity analysis. This is helpful, since it is unlikely that we would be able to design a questionnaire that could elicit a member's EIS even if the member understood what an EIS meant! The lack of sensitivity of both the contribution rate and investment strategy to the EIS suggests that we could fix the

EIS at a level that happened to be convenient for us. A particularly convenient level would be to choose the EIS to equal the inverse of the RRA, i.e., at a level consistent with power utility. The study by Schwartz and Torous (1999) cited above showed that while there was an inverse relationship between EIS and RRA, the relationship is not exactly reciprocal. Nevertheless, it was fairly close for a typical individual, so using a reciprocal relationship in a practical application might be a reasonable approximation for most people and it would also help to speed up the numerical solution algorithm.

- It is very important to incorporate the salary process in the optimal design of a DC pension plan. For most people, their human capital will be bond-like in nature and this will have a direct impact on the optimal contribution rate and asset allocation decisions, in particular, justifying a high weight for equities in the pension plan. However, for senior plan members whose salary levels (including bonus and dividends from their own stock holdings) may have a strong link with corporate profitability, their labour income growth rate might be much higher than the return on bonds and might also be more volatile. In this case, their human capital will be more equity-like in nature and so the optimal investment strategy will be more heavily geared towards bonds.
- An investment strategy involving a switch from equities to bonds as members approach retirement is appropriate for DC pension plans, even when equity returns are not mean reverting. However, the switch away from equities is stochastic rather than predetermined, and is dependent on past investment and salary growth experience. Nevertheless, the switch should typically be made earlier than in traditional lifestyle strategies (i.e., from age 45 or so rather than age 55, which is more common in practice). Also, unlike most traditional lifestyle investment strategies, the optimal equity weight in the portfolio immediately prior to retirement is not reduced to zero (rather it depends on the risk attitude of the individual). The practical implementation of such an investment strategy would not actually be that challenging if we had reliable measures of the member's risk aversion and discount factor and could assume that the EIS was equal to the reciprocal of the RRA. Figure 5 shows that for a given RRA, discount factor and

EIS, the distribution of the optimal equity allocation is very narrow. An approximate solution for the optimal equity weighting could be that derived by Campbell and Viceira (2002, Equation (6.1)) in the case where labour income is deterministic rather than stochastic (where H_x is the value of human capital at age x):

$$\alpha_x = \frac{\mu + \sigma^2/2}{\gamma\sigma^2}\left(1 + \frac{H_x}{F_x}\right) \tag{14}$$

Since the optimal investment strategy depends on the member's RRA, discount factor and EIS, and since these differ across members, it is unlikely that a single default investment fund will be appropriate for all plan members.

- A life annuity is a critically important component of a well-designed pension plan. As a result of the mortality premium inherent in the return on a life annuity, the full amount of the pension fund should eventually be annuitised in old age (regardless of an individual's RRA, EIS or personal discount factor).[32] This is true despite the well-known aversion to annuitisation by 'humans' documented in Friedman and Warshawsky (1990) and Mitchell and Utkus (2004), and despite both their theoretical usefulness (Yaari (1965), Davidoff et al. (2005)) and money's worth (Mitchell et al. (1999), Finkelstein and Poterba (2002)) and their recognized value by annuitants once purchased (Panis (2004)).

[32] In the absence of a bequest motive.

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Appendix – Numerical solution of the dynamic programming problem

From Equation (4), the Epstein-Zin utility function at age x is as follows:

$$U_x = \left\{ (1-\beta) \times (C_x)^{1-\frac{1}{\varphi}} + \beta p_x \times \left(E_x \left[(U_{x+1})^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}$$

We assume that the member is subject to mortality risk with a maximum possible age of 120. Thus, in the final year of age, we assume that $p_{120} = 0$ and, thus, the terminal utility function is given by (c.f., Equation (5)):

$$U_{120} = \left\{ (1-\beta) \times (C_{120})^{1-\frac{1}{\varphi}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}$$

The optimisation problem is then:

$$\max_{\alpha_x, C_x} \; U_x$$

subject to the constraints given by:

- $C_x \le Y_x$ for $x = 20, 21, \ldots, 64$;
- $C_x \le \dfrac{(1-\alpha_x) \times F_x}{\ddot{a}_x} + \alpha_x F_x$ for $x = 65, 66, \ldots, 120$, and
- $0 \le \alpha_x \le 1$ for $x = 20, 21, \ldots, 120$.

Then, from Equation (13), the Bellman equation at age x is given by:

$$V_x = \max_{\alpha_x, C_x} \left\{ (1-\beta) \times (C_x)^{1-\frac{1}{\varphi}} + \beta p_x \times \left(E_x \left[(V_{x+1})^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\varphi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\varphi}}}$$

Because there is no explicit solution for the expectation term in the above expression, an analytical solution to this problem does not exist.

The most popular numerical solution method is value function iteration (see, e.g., Judd (1998, page 257-266)). This involves the discretisation of the state variables by setting up a standard equally-spaced grid and solving the optimisation for each grid point for the penultimate age. The expectation term in the Bellman equation is then approximated by using quadrature integration and then the dynamic optimisation problem can be solved by backward recursion.

To avoid choosing a local maximum, we discretise the control variables (i.e., asset allocation and consumption) into 20 equally-spaced intervals (with corresponding grid points) and optimise using a standard grid search. As an important step in solving the stochastic dynamic programming problem, we need to discretise both the state space and shocks in the stochastic processes (i.e., equity return and labour income growth). Pension wealth and labour income (prior to retirement) are discretised into 30 and 10 evenly-spaced intervals, respectively, in the computation.[33]

It is possible that the values of the state variable from the previous time period are not represented by a grid point, in which case, an interpolation method (e.g., bilinear, cubic spline, etc.) must be employed to approximate the value function. The approach requires knowledge of the distribution of each of the shocks to the system, so that appropriate quadrature integration (e.g., Gauss-Hermite quadrature) can be used.

Thus, to solve the non-linear expectation part in the Bellman equation above (i.e., $E_x\left[\left(V_{x+1}\right)^{1-\gamma}\right]$), we discretise the standard normal random variables, $Z_{1,x}$ and $Z_{2,x}$, representing the shocks to the equity return and labour income growth processes in year of age x to $\left(x+1\right)$, respectively, into 9 nodes,[34] giving (where π is the mathematical constant):

[33] Clearly, the choice concerning the number of intervals is subjective, but we felt that the choices made here represent an appropriate trade-off between accuracy and speed of computation.

[34] Nine nodes is a standard setting in the existing literature.

$$E_x\left(V_{x+1}^{1-\gamma}\right) \approx \pi^{-1}\sum_{m=1}^{9}\sum_{n=1}^{9} w_{Z_{1,x}(m)} \times w_{Z_{2,x}(n)} \times \left[V_{x+1}\left(\sqrt{2}Z_{1,x}\left(m\right), \sqrt{2}Z_{2,x}\left(n\right)\right)\right]^{1-\gamma}$$

where:

- $\left\{Z_{1,x}\left(m\right): m = 1, 2, \ldots, 9\right\}$ and $\left\{Z_{2,x}\left(n\right): n = 1, 2, \ldots, 9\right\}$ are the Gauss-Hermite quadrature nodes for the random variables $Z_{1,x}$ and $Z_{2,x}$ respectively; and
- $\left\{w_{Z_{1,x}(m)} : m = 1, 2, \ldots, 9\right\}$ and $\left\{w_{Z_{2,x}(n)} : n = 1, 2, \ldots, 9\right\}$ are the corresponding Gauss-Hermite quadrature weights.

Then, the resulting state variables dynamics are given by:

- for $x = 20, 21, \ldots, 64$, we have:

$$\begin{aligned} F_{x+1} &= \left(F_x + \pi_x Y_x\right) \times \left[1 + r + \alpha_x \times \left(\mu + \sigma \times \sqrt{2}Z_{1,x}\left(m\right)\right)\right] \quad \text{with } F_{20} = 0 \\ I_x &= r_I + \frac{S_{x+1} - S_x}{S_x} + \sigma_1 \times \sqrt{2}Z_{1,x}\left(m\right) + \sigma_2 \times \sqrt{2}Z_{2,x}\left(n\right) \end{aligned}$$

- for $x = 65, 66, \ldots, 120$, we have:

$$F_{x+1} = \frac{\left(1-\alpha_x\right) \times F_x}{\ddot{a}_x} \times \ddot{a}_{x+1} + \left[\alpha_x F_x + \frac{\left(1-\alpha_x\right) \times F_x}{\ddot{a}_x} - C_x\right] \times \left(1 + r + \mu + \sigma \times \sqrt{2}Z_{1,x}\left(m\right)\right)$$

Substituting the Gauss-Hermite approximation for the expectation term in the Bellman equation above, we derive the value function and the corresponding optimal control variables at each grid point. We then iterate the procedure back from age 119 to age 20. The computations were performed in MATLAB.[35]

[35] http://www.mathworks.com/products/matlab/.