



# City Research Online

## City St George's, University of London

**Citation:** Kessar, R., Koshitani, S. & Linckelmann, M. (2015). On symmetric quotients of symmetric algebras. *Journal of Algebra*, 442, pp. 423-437. doi: 10.1016/j.jalgebra.2014.05.035

This is the accepted version of the paper.

This version of the publication may differ from the final published version. To cite this item please consult the publisher's version.

**Permanent repository link:** <https://openaccess.city.ac.uk/id/eprint/6903/>

**Link to published version:** <https://doi.org/10.1016/j.jalgebra.2014.05.035>

**Copyright and Reuse:** Copyright and Moral Rights remain with the author(s) and/or copyright holders. Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge, unless otherwise indicated, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way. For full details of reuse please refer to [City Research Online policy](#).

# ON SYMMETRIC QUOTIENTS OF SYMMETRIC ALGEBRAS

RADHA KESSAR, SHIGEO KOSHITANI, AND MARKUS LINCKELMANN

ABSTRACT. We investigate symmetric quotient algebras of symmetric algebras, with an emphasis on finite group algebras over a complete discrete valuation ring  $\mathcal{O}$ . Using elementary methods, we show that if an ordinary irreducible character  $\chi$  of a finite group  $G$  gives rise to a symmetric quotient over  $\mathcal{O}$  which is not a matrix algebra, then the decomposition numbers of the row labelled by  $\chi$  are all divisible by the characteristic  $p$  of the residue field of  $\mathcal{O}$ .

## 1. INTRODUCTION

Let  $p$  be a prime and  $\mathcal{O}$  a complete discrete valuation ring having a residue field  $k$  of characteristic  $p$  and a quotient field  $K$  of characteristic zero. Unless stated otherwise, we assume that  $K$  and  $k$  are splitting fields for all finite groups under consideration. Let  $G$  be a finite group. Any subset  $M$  of the set  $\text{Irr}_K(G)$  of irreducible  $K$ -valued characters of  $G$  gives rise to an  $\mathcal{O}$ -free quotient algebra, namely the image of a structural homomorphism  $\mathcal{O}G \rightarrow \text{End}_{\mathcal{O}}(V)$ , where  $V$  is an  $\mathcal{O}$ -free  $\mathcal{O}G$ -module having character  $\sum_{\chi \in M} \chi$ . This image is isomorphic to  $\mathcal{O}G(\sum_{\chi \in M} e(\chi))$ , where  $e(\chi)$  denotes the primitive idempotent in  $Z(KG)$  corresponding to  $\chi$ . Any  $\mathcal{O}$ -free quotient algebra of  $\mathcal{O}G$  arises in this way; in particular,  $\mathcal{O}G$  has only finitely many  $\mathcal{O}$ -free algebra quotients. Any quotient of  $\mathcal{O}G$  admits a decomposition induced by the block decomposition, and hence finding symmetric quotients of  $\mathcal{O}G$  is equivalent to finding symmetric quotients of the block algebras of  $\mathcal{O}G$ . We denote by  $\text{IBr}_k(G)$  the set of irreducible Brauer characters of  $G$ , and by  $d_G : \mathbb{Z}\text{Irr}_K(G) \rightarrow \mathbb{Z}\text{IBr}_k(G)$  the decomposition map, sending a generalised character of  $G$  to its restriction to the set  $G_{p'}$  of  $p'$ -elements in  $G$ . For  $B$  a block algebra of  $\mathcal{O}G$ , we denote by  $\text{Irr}_K(B)$  and  $\text{IBr}_k(B)$  the sets of irreducible  $K$ -characters and Brauer characters, respectively, associated with  $B$ . We denote by  $d_B : \mathbb{Z}\text{Irr}_K(B) \rightarrow \mathbb{Z}\text{IBr}_k(B)$  the decomposition map obtained from restricting  $d_G$ . We denote by  $D_G = (d_{\varphi}^{\chi})$  the decomposition matrix of  $\mathcal{O}G$ , with rows indexed by  $\chi \in \text{Irr}_K(G)$  and columns indexed by  $\varphi \in \text{IBr}_k(G)$ ; that is, the  $d_{\varphi}^{\chi}$  are the nonnegative integers satisfying  $d_G(\chi) = \sum_{\varphi \in \text{IBr}_k(G)} d_{\varphi}^{\chi} \cdot \varphi$ , for any  $\chi \in \text{Irr}_K(G)$ . Equivalently,  $d_{\varphi}^{\chi} = \chi(i)$ , where  $i$  is a primitive idempotent in  $\mathcal{O}G$  such that  $\mathcal{O}Gi$  is a projective cover of a simple  $kG$ -module with Brauer character  $\varphi$ . If  $\chi, \varphi$  belong to different blocks, then  $d_{\varphi}^{\chi} = 0$ . For  $B$  a block algebra of  $\mathcal{O}G$ , we denote by  $D_B$  the submatrix of  $D_G$  labelled by  $\chi \in \text{Irr}_K(B)$  and  $\varphi \in \text{IBr}_k(B)$ . We say that  $\chi \in \text{Irr}_K(G)$  lifts the irreducible Brauer character  $\varphi \in \text{IBr}_k(G)$  if  $d_G(\chi) = \varphi$ , or equivalently, if  $d_{\varphi}^{\chi} = 1$  and  $d_{\varphi'}^{\chi} = 0$  for all  $\varphi' \in \text{IBr}_k(G)$  different from  $\varphi$ . In that case,  $\mathcal{O}Ge(\chi)$  is a matrix algebra over  $\mathcal{O}$ , hence trivially symmetric (see 4.1 below).

**Theorem 1.1.** *Let  $G$  be a finite group and  $\chi \in \text{Irr}_K(G)$ . Suppose that  $\mathcal{O}Ge(\chi)$  is symmetric. Then either  $\chi$  lifts an irreducible Brauer character, or  $d_{\varphi}^{\chi}$  is divisible by  $p$  for all  $\varphi \in \text{IBr}_k(G)$ .*

This will be proved as an application of 4.3 below. We note some obvious consequences.

**Corollary 1.2.** *Let  $G$  be a finite group and  $\chi \in \text{Irr}_K(G)$  such that  $d_\varphi^\chi$  is prime to  $p$  for some  $\varphi \in \text{IBr}_k(G)$ . Then the  $\mathcal{O}$ -algebra  $\mathcal{O}Ge(\chi)$  is symmetric if and only if  $\chi$  lifts  $\varphi$ .*

**Corollary 1.3.** *Let  $G$  be a finite group,  $B$  a block of  $\mathcal{O}G$ , and  $\chi \in \text{Irr}_K(B)$ . Suppose that  $\chi$  has height zero. Then the  $\mathcal{O}$ -algebra  $\mathcal{O}Ge(\chi)$  is symmetric if and only if  $\chi$  lifts an irreducible Brauer character.*

**Corollary 1.4.** *Let  $G$  be a finite group and  $B$  a block of  $\mathcal{O}G$ . Suppose that each row of the decomposition matrix  $D_B$  has at least one entry prime to  $p$ . Then for any  $\chi \in \text{Irr}_K(B)$ , the algebra  $\mathcal{O}Ge(\chi)$  is symmetric if and only if  $\chi$  lifts an irreducible Brauer character in  $\text{IBr}_k(B)$ .*

The hypotheses of 1.4 apply to many blocks of quasi-simple finite groups. By a result of Dade [2] they also apply to blocks with cyclic defect groups.

**Corollary 1.5.** *Let  $G$  be a finite group and  $B$  a block with cyclic defect groups. Let  $\chi \in \text{Irr}_K(B)$ . Then  $\mathcal{O}Ge(\chi)$  is symmetric if and only if  $\chi$  corresponds to a nonexceptional vertex at the end of a branch of the Brauer tree of  $B$ .*

By Erdmann's results in [3], the hypotheses of 1.4 apply to all nonnilpotent tame blocks.

**Corollary 1.6.** *Suppose that  $p = 2$ . Let  $G$  be a finite group and  $B$  a nonnilpotent block of  $\mathcal{O}G$  having a defect group  $P$  which is either generalised dihedral, quaternion, or semidihedral. Then for any  $\chi \in \text{Irr}_K(B)$ , the algebra  $\mathcal{O}Ge(\chi)$  is symmetric if and only if  $\chi$  lifts an irreducible Brauer character in  $\text{IBr}_k(B)$ .*

**Proposition 1.7.** *Let  $P$  be a finite  $p$ -group having a normal cyclic subgroup of index  $p$ . Then  $\mathcal{O}Pe(\chi)$  is symmetric for any  $\chi \in \text{Irr}_K(P)$ .*

This will be proved in §3. Since a nilpotent block is isomorphic to a matrix algebra over one of its defect group algebras, this proposition, specialised to  $p = 2$ , has the following consequence (which includes nilpotent tame blocks).

**Corollary 1.8.** *Suppose that  $p = 2$ . Let  $G$  be a finite group and  $B$  a nilpotent block of  $\mathcal{O}G$  having a defect group  $P$  which is either generalised dihedral, quaternion, semidihedral, or quasidihedral. Then for any  $\chi \in \text{Irr}_K(B)$ , the algebra  $\mathcal{O}Ge(\chi)$  is symmetric.*

The symmetric algebras arising in 1.4 and subsequent corollaries are matrix algebras (see 4.1 below). By contrast, the symmetric algebras obtained from nonlinear characters in 1.7 are not isomorphic to matrix algebras. Further examples of characters  $\chi$  with symmetric quotient  $\mathcal{O}Ge(\chi)$  which are not isomorphic to matrix algebras can be obtained from characters of central type. An irreducible character  $\chi$  of a finite group  $G$  is of central type if  $\chi(1)^2 = |G : Z(G)|$ .

**Proposition 1.9.** *Let  $G$  be a finite group and  $\chi \in \text{Irr}_K(G)$  a character of central type. Then  $\mathcal{O}Ge(\chi)$  is symmetric.*

This is shown as a special case of a slightly more general situation in 2.4. As a consequence of 1.7 and 1.9, if  $P$  is a finite  $p$ -group of order at most  $p^3$ , then  $\mathcal{O}Pe(\chi)$  is symmetric for all  $\chi \in \text{Irr}_K(P)$ . In §6 we give an example showing that this is not the case in general for irreducible characters of finite  $p$ -groups of order  $p^4$ .

If  $M$  is the set of all nontrivial characters of  $G$ , then the corresponding quotient algebra is  $\mathcal{O}G/\mathcal{O}(\sum_{x \in G} x)$ , because the augmentation ideal  $I(\mathcal{O}G)$  has a character in which all nontrivial

characters of  $G$  appear, and  $\mathcal{O}(\sum_{x \in G} x)$  is the annihilator in  $\mathcal{O}G$  of  $I(\mathcal{O}G)$ . The hypothesis of  $K$ ,  $k$  being large enough is not necessary for the following result.

**Proposition 1.10.** *Let  $G$  be a finite group. The following are equivalent.*

- (i) *The  $\mathcal{O}$ -algebra  $\mathcal{O}G/\mathcal{O}(\sum_{x \in G} x)$  is symmetric.*
- (ii) *The group  $G$  is  $p$ -nilpotent and has a cyclic Sylow- $p$ -subgroup.*

The arguments used in the proof of 1.10 can be adapted to yield the following block theoretic version; we need  $k$  to be large enough for the block  $B$  in the next result.

**Proposition 1.11.** *Let  $G$  be a finite group,  $B$  a block algebra of  $\mathcal{O}G$ , and let  $\chi \in \text{Irr}_K(B)$ . Suppose that  $\chi$  lifts an irreducible Brauer character. Set  $I = B \cap (K \otimes_{\mathcal{O}} B)e(\chi)$ , where we identify  $B$  with its image  $1 \otimes B$  in  $K \otimes_{\mathcal{O}} B$ . The following are equivalent.*

- (i) *The algebra  $B/I$  is symmetric.*
- (ii) *The block  $B$  is nilpotent with cyclic defect groups.*

## 2. NOTATION AND BASIC FACTS

If  $A$  is an  $\mathcal{O}$ -algebra which is free of finite rank as an  $\mathcal{O}$ -module, we denote by  $\text{Irr}_K(A)$  the set of characters of the simple  $K \otimes_{\mathcal{O}} A$ -modules. Taking characters of  $K \otimes_{\mathcal{O}} A$ -modules yields an isomorphism between the Grohendieck group  $R_K(A)$  of finitely generated  $K \otimes_{\mathcal{O}} A$ -modules and the free abelian group with basis  $\text{Irr}_K(A)$ , inducing a bijection between the isomorphism classes of simple  $K \otimes_{\mathcal{O}} A$ -modules and  $\text{Irr}_K(A)$ . If in addition the  $K$ -algebra  $K \otimes_{\mathcal{O}} A$  is semisimple, hence a direct product of simple  $K$ -algebras corresponding to the isomorphism classes of simple  $K \otimes_{\mathcal{O}} A$ -modules, we denote by  $e(\chi)$  the primitive idempotent of  $Z(K \otimes_{\mathcal{O}} A)$  which acts as identity on the simple  $K \otimes_{\mathcal{O}} A$ -modules with character  $\chi$  and which annihilates all other simple  $K \otimes_{\mathcal{O}} A$ -modules. In this case we have  $K \otimes_{\mathcal{O}} A \cong \prod_{\chi \in \text{Irr}_K(A)} (K \otimes_{\mathcal{O}} A)e(\chi)$ , and each factor  $(K \otimes_{\mathcal{O}} A)e(\chi)$  is a simple  $K$ -algebra. An  $\mathcal{O}$ -algebra  $A$  is *symmetric* if  $A$  is finitely generated free as an  $\mathcal{O}$ -module and if  $A$  is isomorphic to its  $\mathcal{O}$ -dual  $A^\vee = \text{Hom}_{\mathcal{O}}(A, \mathcal{O})$  as an  $A$ - $A$ -bimodule. Let  $A$  be an  $\mathcal{O}$ -algebra. A submodule  $V$  of an  $A$ -module  $U$  is called  *$\mathcal{O}$ -pure* if  $V$  is a direct summand of  $U$  as an  $\mathcal{O}$ -module. We use without further comments the following well-known facts. See e. g. [4, 17.2], and also [7, Theorem 1] for an application in the context of blocks with cyclic defect groups.

**Lemma 2.1.** *Let  $A$  be an  $\mathcal{O}$ -algebra and let  $U$  be an  $A$ -module which is finitely generated free as an  $\mathcal{O}$ -module. Let  $V$  be a submodule of  $U$ . Then  $U \cap (K \otimes_{\mathcal{O}} V)$  is the unique minimal  $\mathcal{O}$ -pure submodule of  $U$  containing  $V$ , where we identify  $U$  with its image  $1 \otimes U$  in  $K \otimes_{\mathcal{O}} U$ . Moreover, the following are equivalent.*

- (i) *The  $A$ -module  $V$  is  $\mathcal{O}$ -pure in  $U$ .*
- (ii) *The  $A$ -module  $U/V$  is  $\mathcal{O}$ -free.*
- (iii) *We have  $J(\mathcal{O})V = J(\mathcal{O})U \cap V$ .*
- (iv) *The image of  $V$  in  $k \otimes_{\mathcal{O}} U$  is isomorphic to  $k \otimes_{\mathcal{O}} V$ .*
- (v) *We have  $V = U \cap (K \otimes_{\mathcal{O}} V)$ , where we identify  $U$  to its image  $1 \otimes U$  in  $K \otimes_{\mathcal{O}} U$ .*

Thus if  $I$  is an ideal in an  $\mathcal{O}$ -algebra  $A$  which is finitely generated free as an  $\mathcal{O}$ -module, then the quotient algebra  $A/I$  is  $\mathcal{O}$ -free if and only if  $I$  is  $\mathcal{O}$ -pure in  $A$ . Any  $\mathcal{O}$ -pure ideal  $I$  of  $A$  is equal to  $A \cap M_I$  for a unique ideal  $M_I$  in  $K \otimes_{\mathcal{O}} A$ , where we have identified  $A$  to its canonical image  $1_K \otimes A$  in  $K \otimes_{\mathcal{O}} A$ . If  $K \otimes_{\mathcal{O}} A$  is in addition semisimple, hence a direct product of simple algebras, then

every ideal of  $K \otimes_{\mathcal{O}} A$  is a product of a subset of those simple algebras. In particular, in that case the set of  $\mathcal{O}$ -pure ideals in  $A$  is finite and corresponds bijectively to the set of subsets of a set of representatives of the isomorphism classes of simple  $K \otimes_{\mathcal{O}} A$ -modules.

For symmetric algebras over a field, the following result is due to Nakayama [5]; the generalisation to  $\mathcal{O}$ -algebras is straightforward (we include a proof for the convenience of the reader). Note that the left and right annihilators of an ideal  $I$  in a symmetric algebra  $A$  are always equal, denoted by  $\text{ann}(I)$ .

**Proposition 2.2** (cf. [5, Theorem 13]). *Let  $A$  be a symmetric  $\mathcal{O}$ -algebra, and let  $I$  be an  $\mathcal{O}$ -pure ideal in  $A$ . The quotient algebra  $A/I$  is a symmetric  $\mathcal{O}$ -algebra if and only if there is an element  $z \in Z(A)$  such that  $\text{ann}(I) = Az$ .*

*Proof.* Set  $\bar{A} = A/I$ , and denote by  $\pi : A \rightarrow \bar{A}$  the canonical surjection. Let  $s : A \rightarrow \mathcal{O}$  be a symmetrising form for  $A$ ; that is,  $s$  is symmetric and the map sending  $a \in A$  to the form  $a \cdot s$  defined by  $(a \cdot s)(b) = s(ab)$  for all  $b \in A$  is an isomorphism  $A \cong A^* = \text{Hom}_{\mathcal{O}}(A, \mathcal{O})$ . (Since  $s$  is symmetric, this map is automatically a homomorphism of  $A$ - $A$ -bimodules.) Suppose that  $\text{ann}(I) = Az$  for some  $z \in Z(A)$ . Then  $t = z \cdot s$  annihilates  $I$ , hence induces a form  $\bar{t} \in \bar{A}^*$  defined by  $\bar{t}(\bar{a}) = s(za)$ , where  $a \in A$  and  $\bar{a} = a + I \in \bar{A}$ . Since  $s$  is symmetric and  $z \in Z(A)$ , the forms  $t$  and  $\bar{t}$  are again symmetric. It suffices to show that the map sending  $\bar{a} \in \bar{A}$  to  $\bar{a} \cdot \bar{t} \in \bar{A}^*$  is surjective. Let  $\bar{u} \in \bar{A}^*$ . Then  $u = \bar{u} \circ \pi \in A^*$ , hence  $u = a \cdot s$  for a uniquely determined element  $a \in A$ . Since  $I \subseteq \ker(u)$ , we have  $s(aI) = \{0\}$ , hence  $a \in \text{ann}(I)$ . Thus  $a = cz$  for some  $c \in A$ . It follows that  $u = c \cdot (z \cdot s) = c \cdot t$ , and hence  $\bar{u} = \bar{c} \cdot \bar{t}$ , which shows that  $\bar{A}$  is symmetric. Suppose conversely that  $\bar{A}$  is symmetric. Let  $\bar{t} : \bar{A} \rightarrow \mathcal{O}$  be a symmetrising form, and set  $t = \bar{t} \circ \pi$ . Then  $t$  is symmetric (because  $\bar{t}$  is) and hence  $t = z \cdot s$  for some  $z \in Z(A)$ . Since  $I \subseteq \ker(t)$  we have  $s(zI) = \{0\}$ , hence  $z \in \text{ann}(I)$ . Let  $b \in \text{ann}(I)$ . Then  $u = b \cdot s$  has  $I$  in its kernel, hence induces a form  $\bar{u}$  on  $\bar{A}$  satisfying  $\bar{u}(\bar{a}) = s(ab)$ . Since  $\bar{A}$  is symmetric, there is  $\bar{c} \in \bar{A}$  such that  $\bar{u} = \bar{c} \cdot \bar{t}$ , hence such that  $u = c \cdot t = (cz) \cdot s$ . This  $(cz) \cdot s = b \cdot s$ , and hence  $b = cz \in Az$ , whence the equality  $\text{ann}(I) = Az$ .  $\square$

Let  $A$  be a symmetric  $\mathcal{O}$ -algebra. If  $z \in Z(A)$  such that  $Az$  is the annihilator of an ideal  $I$  in  $A$ , then  $Az$  is  $\mathcal{O}$ -pure. Thus finding symmetric quotients of  $A$  is equivalent to finding elements  $z \in Z(A)$  with the property that  $Az$  is  $\mathcal{O}$ -pure in  $A$ .

**Corollary 2.3.** *Let  $A$  be a symmetric  $\mathcal{O}$ -algebra and  $z \in Z(A)$ . If  $Az$  is  $\mathcal{O}$ -pure, then the annihilator  $I = \text{ann}(z) = \text{ann}(Az)$  is an  $\mathcal{O}$ -pure ideal satisfying  $\text{ann}(I) = Az$ , and the  $\mathcal{O}$ -algebra  $A/I$  is symmetric. Moreover, any symmetric  $\mathcal{O}$ -algebra quotient of  $A$  arises in this way.*

*Proof.* This follows from 2.2 and the preceding remarks.  $\square$

**Proposition 2.4.** *Let  $G$  be a finite group and  $N$  a normal subgroup of  $G$ . Suppose that  $K$  is a splitting field for  $N$ . Let  $\eta \in \text{Irr}_K(N)$ , and suppose that  $\eta$  is  $G$ -stable. If  $\mathcal{O}Ne(\eta)$  is symmetric, then  $\mathcal{O}Ge(\eta)$  is symmetric.*

*Proof.* Suppose that  $\mathcal{O}Ne(\eta)$  is symmetric. By 2.2 there exists an element  $z \in Z(\mathcal{O}N)$  such that  $\mathcal{O}Nz$  is the annihilator of the kernel of the map  $\mathcal{O}N \rightarrow \mathcal{O}Ne(\eta)$ . Thus  $z \in Z(\mathcal{O}N)e(\eta) = \mathcal{O}e(\eta)$ , where we use that  $K$  is large enough for  $N$ . In particular, there is  $\lambda \in \mathcal{O}$  such that  $z = \lambda e(\eta)$ , and hence  $z \in Z(\mathcal{O}G)$ . Since  $\mathcal{O}G$  is free as a right  $\mathcal{O}N$ -module, it follows that  $\mathcal{O}Gz$  is  $\mathcal{O}$ -pure in  $\mathcal{O}G$ . The annihilator of the kernel of the map  $\mathcal{O}G \rightarrow \mathcal{O}Ge(\eta)$  is therefore equal to  $\mathcal{O}Gz$ . The result follows from 2.2.  $\square$

*Proof of Proposition 1.9.* Since  $\chi$  is of central type, there is a unique (hence  $G$ -stable) linear character  $\zeta : Z(G) \rightarrow \mathcal{O}^\times$  such that  $e(\chi) = e(\zeta)$ . We have  $\mathcal{O}Z(G)e(\zeta) \cong \mathcal{O}$ , which is trivially symmetric, and hence  $\mathcal{O}Ge(\zeta) = \mathcal{O}Ge(\chi)$  is symmetric by 2.4.  $\square$

The argument in the proof of 2.4 to describe a central element  $z$  which generates a pure ideal admits the following generalisation.

**Proposition 2.5.** *Let  $A$  be a symmetric  $\mathcal{O}$ -algebra such that  $K \otimes_{\mathcal{O}} A$  is semisimple. Let  $\chi \in \text{Irr}_K(A)$  be the character of an absolutely simple  $K \otimes_{\mathcal{O}} A$ -module, and denote by  $e(\chi)$  the corresponding primitive idempotent in  $Z(K \otimes_{\mathcal{O}} A)$ . Let  $\lambda \in \mathcal{O}$  be an element having minimal valuation such that  $\lambda e(\chi) \in A$ , where we identify  $A$  with its image  $1 \otimes A$  in  $K \otimes_{\mathcal{O}} A$ . The following are equivalent.*

- (i) *The  $\mathcal{O}$ -algebra  $Ae(\chi)$  is symmetric.*
- (ii) *The  $A$ -module  $A\lambda e(\chi)$  is  $\mathcal{O}$ -pure in  $A$ .*
- (iii) *We have  $A\lambda e(\chi) = A \cap (K \otimes_{\mathcal{O}} A)e(\chi)$ .*

*Proof.* The equivalence of (ii) and (iii) is a general fact (cf. 2.1). Let  $I$  be the kernel of the algebra homomorphism  $A \rightarrow Ae(\chi)$  sending  $a \in A$  to  $ae(\chi)$ . Multiplication by  $e(\chi)$  in  $K \otimes_{\mathcal{O}} A$  yields the projection of  $K \otimes_{\mathcal{O}} A = \prod_{\psi \in \text{Irr}_K(A)} (K \otimes_{\mathcal{O}} A)e(\psi)$  onto the factor  $(K \otimes_{\mathcal{O}} A)e(\chi)$ ; this is a matrix algebra as  $\chi$  is the character of an absolutely simple  $K \otimes_{\mathcal{O}} A$ -module. Thus  $I$  is the  $\mathcal{O}$ -pure ideal corresponding to the complement of  $\{\chi\}$  in  $\text{Irr}_K(A)$ ; that is,  $I = A \cap \prod_{\psi} (K \otimes_{\mathcal{O}} A)e(\psi)$ , where  $\psi$  runs over the set  $\text{Irr}_K(A) - \{\chi\}$ . It follows that the annihilator of  $I$  is equal to  $A \cap (K \otimes_{\mathcal{O}} A)e(\chi)$ , and hence  $Z(A) \cap (K \otimes_{\mathcal{O}} A)e(\chi) \subseteq \mathcal{O}e(\chi)$ . Thus  $A/I$  is symmetric if and only if  $Az$  is  $\mathcal{O}$ -pure for some  $z \in A \cap \mathcal{O}e(\chi)$ , hence if and only if  $A\lambda e(\chi)$  is  $\mathcal{O}$ -pure for some  $\lambda \in \mathcal{O}$ . In that case,  $\lambda$  has necessarily the smallest possible valuation such that  $\lambda e(\chi) \in A$ . The result follows.  $\square$

**Remark 2.6.** Let  $A$  be an  $\mathcal{O}$ -algebra which is finitely generated free as an  $\mathcal{O}$ -module, and let  $I$  be an  $\mathcal{O}$ -pure ideal in  $A$ . Then the image of the canonical map  $A \rightarrow \text{End}_{\mathcal{O}}(A/I)$  sending  $a \in A$  to left multiplication by  $a + I$  in  $A/I$  has kernel  $I$ , hence image isomorphic to  $A/I$ . Thus any  $\mathcal{O}$ -free algebra quotient of  $A$  is isomorphic to the image of the structural homomorphism  $A \rightarrow \text{End}_{\mathcal{O}}(V)$  sending  $a \in A$  to left multiplication by  $a$  on  $V$ , for some  $A$ -module  $V$  which is free of finite rank as an  $\mathcal{O}$ -module. Since  $V$  is  $\mathcal{O}$ -free, this image is isomorphic to the canonical image of  $A$  in  $\text{End}_K(K \otimes_{\mathcal{O}} V)$ . Thus, if  $K \otimes_{\mathcal{O}} V$  is a semisimple  $K \otimes_{\mathcal{O}} A$ -module, then this image depends only on the isomorphism classes of simple  $K \otimes_{\mathcal{O}} A$ -modules occurring in a decomposition of  $K \otimes_{\mathcal{O}} V$ , but not on the multiplicity of the simple factors of  $K \otimes_{\mathcal{O}} V$ .

**Remark 2.7.** It follows from the formal properties of Morita equivalences that a Morita equivalence between two algebras induces a bijection between quotients of the two algebras, in such a way that quotients corresponding to each other are again Morita equivalent. In particular, symmetric quotients are preserved under Morita equivalences. We describe this briefly for the convenience of the reader. Let  $A, B$  be Morita equivalent  $\mathcal{O}$ -algebras; that is, there is an  $A$ - $B$ -bimodule  $M$  and a  $B$ - $A$ -bimodule  $N$  such that  $M, N$  are finitely generated projective as left and right modules, and such that we have isomorphisms of bimodules  $M \otimes_B N \cong A$  and  $N \otimes_A M \cong B$ . The functor  $M \otimes_B -$  induces an equivalence between the categories of  $B$ - $B$ -bimodules and of  $A$ - $B$ -bimodules, sending  $B$  to  $M$ . Thus this functor induces a bijection between ideals in  $B$  and subbimodules of  $M$ , sending an ideal  $J$  in  $B$  to the subbimodule  $MJ$ . Note that since  $M$  is finitely generated projective as a right  $B$ -module, we have  $MJ \cong M \otimes_B J$ . Similarly, we have a bijection between ideals in  $A$

and subbimodules in  $M$  sending an ideal in  $I$  in  $A$  to the subbimodule  $IM \cong I \otimes_A M$ . Combining these bijections yields a bijection between ideals in  $A$  and ideals in  $B$ , with the property that the ideal  $I$  in  $A$  corresponds to the ideal  $J$  in  $B$  if and only if  $IM = MJ$ , which in turn holds if and only if  $JN = NI$ . If  $I, J$  correspond to each other through this bijection, then  $M/IM$  and  $N/JN$  induce a Morita equivalence between  $A/I$  and  $B/J$ . Indeed,  $M/IM$  is finitely generated projective as a left  $A/I$ -module, since  $M$  is finitely generated as a left  $A$ -module. Using  $IM = MJ$ , this argument shows that  $M/IM$  is finitely generated projective as a right  $B/J$ -module. Similarly,  $N/JN$  is finitely generated projective as a left and right module. Moreover, we have  $M/IM \otimes_{B/J} N/JN \cong M/MJ \otimes_B N/JN \cong M \otimes_B N/JN \cong M \otimes_B N/M \otimes_B JN \cong M \otimes_B N/M \otimes_B NI \cong A/I$ . The same argument with reversed roles yields  $N/NI \otimes_{A/I} M/IM \cong B/J$ . Since Morita equivalences preserve the property of being symmetric, this shows that  $A/I$  is symmetric if and only if  $B/J$  is symmetric.

### 3. PROOF OF PROPOSITION 1.7

Let  $G$  be a finite  $p$ -group having a cyclic normal subgroup  $H$  of index  $p$ , and let  $\chi \in \text{Irr}_K(G)$ . In order to prove Proposition 1.7 we may assume that  $G$  is nonabelian, hence  $H$  has order at least  $p^2$ . Suppose first that  $p$  is odd. Then the automorphism of  $H$  induced by an element  $t \in G - H$  acts trivially on the subgroup  $H^p$  of index  $p$  in  $H$ , hence  $Z(G) = H^p$  has index  $p^2$  in  $G$ . Any nonlinear character of  $G$  has degree  $p$ , hence is a character of central type. It follows from 1.9 that  $\mathcal{O}Ge(\chi)$  is symmetric.

Suppose now that  $p = 2$ . The previous arguments remain valid so long as the action of  $G$  on the cyclic normal 2-subgroup  $H$  of index 2 is trivial on the subgroup  $H^2$  of index 2 in  $H$ . This includes the case of semidihedral 2-groups (where  $t$  is an involution which acts on the cyclic subgroup  $H$  of order  $2^n$  by sending a generator  $s$  of  $H$  to  $s^{1+2^{n-1}}$ ). If  $n = 2$ , then  $|G| = 8$ , hence  $\chi$  is a character of central type, and so the symmetry of  $\mathcal{O}Ge(\chi)$  follows from 1.9. Suppose that  $n \geq 3$  and that  $G$  does not act trivially on the subgroup of index 2 in  $H$ . Then  $G$  is generalised dihedral or generalised quaternion (corresponding in both cases to the action of an element  $t \in G - H$  of order either 2 or 4 on  $H$  sending  $s$  to  $s^{-1}$ ) or quasidihedral (corresponding to the action of  $t$  sending  $s$  to  $s^{-1+2^{n-1}}$ ). We will need the following elementary facts (a proof is included for the convenience of the reader).

**Lemma 3.1.** *Let  $n \geq 3$  and let  $\zeta$  be a primitive  $2^n$ -th root of unity in  $\mathcal{O}$ . Let  $a, b \in \mathbb{Z}$  such that  $b - a$  is even. The following hold.*

(i) *The numbers  $\zeta^a \pm \zeta^b$  are divisible by  $(1 - \zeta)^2$  in  $\mathcal{O}$ .*

(ii) *The numbers  $\frac{\zeta \pm \zeta^{-1}}{(1 - \zeta)^2}$  are invertible in  $\mathcal{O}$ .*

*Proof.* The integer 2 is divisible by  $(1 - \zeta)^3$  in  $\mathcal{O}$ , as  $n \geq 3$ . In particular,  $\frac{2}{(1 - \zeta)^2} \in J(\mathcal{O})$ , and hence, in order to prove (i), it suffices to show that  $\zeta^a + \zeta^b$  is divisible by  $(1 - \zeta)^2$ . Write  $b - a = 2c$  for some integer  $c$ . Then  $\zeta^a + \zeta^b = \zeta^a(1 + \zeta^{b-a}) = \zeta^a(1 + \zeta^{2c})$ . Thus we may assume  $a = 0$  and  $b = 2c$ . We have  $1 + \zeta^{2c} = 1 - \zeta^{2c} + 2\zeta^{2c} = (1 - \zeta^c)(1 + \zeta^c) + 2\zeta^{2c} = (1 - \zeta^c)(1 - \zeta^c + 2\zeta^c) + 2\zeta^{2c}$ , which is divisible by  $(1 - \zeta)^2$  because  $1 - \zeta$  divides  $1 - \zeta^c$  and  $(1 - \zeta)^2$  divides 2. This shows (i). For (ii), observe that  $\frac{1 - \zeta^2}{(1 - \zeta)^2} = \frac{1 + \zeta}{1 - \zeta} = 1 + \frac{2\zeta}{1 - \zeta}$  is invertible in  $\mathcal{O}$ . Multiplying this by  $\zeta^{-1}$  shows (ii).  $\square$

We complete the proof of 1.7. Let  $G$  be a generalised dihedral or quaternion group of order  $2^{n+1}$ . In order to show that  $\mathcal{O}Ge(\chi)$  is symmetric, we may assume that  $\chi(1) > 1$ , hence  $\chi(1) = 2$ . Let  $H$  be the cyclic subgroup of order  $2^n$  of  $G$ . Since  $\chi(1) = 2$  we have  $\chi = \text{Ind}_H^G(\eta)$  for some nontrivial linear character  $\eta \in \text{Irr}_K(H)$ . Arguing by induction, we may assume that  $\eta$  is faithful. Let  $\zeta$  be a root of unity of order  $2^n$  and let  $s$  be a generator of  $H$  such that  $\eta(s) = \zeta$ . Let  $\bar{\eta}$  be the character of  $H$  sending  $s$  to  $\zeta^{-1}$ . Then  $\text{Res}_H^G(\chi) = \eta + \bar{\eta}$ , and  $\chi$  vanishes outside  $H$ . In particular,

$$e(\chi) = e(\eta) + e(\bar{\eta}) = \frac{1}{2^n} \sum_{a=0}^{2^n-1} (\zeta^a + \zeta^{-a}) s^a .$$

By 3.1 the coefficients  $\zeta^a + \zeta^{-a}$  in this sum are all divisible by  $(1 - \zeta)^2$ . Set  $\lambda = \frac{2^n}{(1-\zeta)^2}$ , and set  $z = \lambda e(\chi)$ . By the above, we have  $z \in \mathcal{O}G$ . We need to show that  $\mathcal{O}Gz$  is  $\mathcal{O}$ -pure in  $\mathcal{O}G$ . Since  $e(\chi) \in \mathcal{O}H$ , and hence  $z \in \mathcal{O}H$ , it suffices to show that  $\mathcal{O}Hz$  is  $\mathcal{O}$ -pure in  $\mathcal{O}H$ . For any  $a$  such that  $0 \leq a \leq 2^n - 1$  we have

$$s^a z = \lambda (s^a e(\eta) + s^a e(\bar{\eta})) = \lambda (\zeta^a e(\eta) + \zeta^{-a} e(\bar{\eta})) .$$

We claim that the set  $\{z, 2^n e(\eta)\}$  is an  $\mathcal{O}$ -basis of  $\mathcal{O}Hz$ . Note first that

$$sz - \zeta^{-1}z = \zeta^{-1}(\zeta^2 - 1)\lambda e(\eta) = \mu 2^n e(\eta)$$

for some  $\mu \in \mathcal{O}$  because  $\zeta^2 - 1$  is divisible, in  $\mathcal{O}$ , by  $(1 - \zeta)^2$ . Thus both  $z$  and  $2^n e(\eta)$  belong to  $\mathcal{O}Hz$ . We need to show that any of the elements  $s^a z$  with  $a$  as before is an  $\mathcal{O}$ -linear combination of  $z$  and  $2^n e(\eta)$ . We have

$$s^a z - \zeta^{-a}z = \lambda \zeta^{-a} (\zeta^{2a} - 1) e(\eta) = \nu 2^n e(\eta)$$

for some  $\nu \in \mathcal{O}$ , whence the claim. It remains to show that  $\mathcal{O}Hz$  is  $\mathcal{O}$ -pure in  $\mathcal{O}H$ . Let  $u = \sum_{a=0}^{2^n-1} \mu_a s^a$  be an element in  $\mathcal{O}Hz$  such that all coefficients  $\mu_a \in \mathcal{O}$  are divisible by  $1 - \zeta$ . Write  $u = \beta z + \gamma 2^n e(\eta)$  with  $\beta, \gamma \in \mathcal{O}$ . We need to show that  $\beta, \gamma$  are divisible by  $1 - \zeta$ . Comparing coefficients for the two expressions of  $u$  above yields

$$\mu_a = \beta \frac{\zeta^a + \zeta^{-a}}{(1 - \zeta)^2} + \gamma \zeta^a$$

for  $0 \leq a \leq 2^n - 1$ . If  $a = 2^{n-2}$ , then  $\zeta^a + \zeta^{-a} = 0$ , hence  $\mu_a = \gamma \zeta^a$ , which implies that  $\gamma$  is divisible by  $1 - \zeta$ , as  $\mu_a$  is divisible by  $1 - \zeta$ . We consider the above equation for  $a = 1$ , which reads

$$\mu_1 = \beta \frac{\zeta + \zeta^{-1}}{(1 - \zeta)^2} + \gamma \zeta .$$

By 3.1 we have  $\frac{\zeta + \zeta^{-1}}{(1 - \zeta)^2} \in \mathcal{O}^\times$ . Since  $\mu_1$  and  $\gamma$  are divisible by  $1 - \zeta$ , this implies that  $\beta$  is divisible by  $1 - \zeta$ . Thus  $\mathcal{O}Hz$  is  $\mathcal{O}$ -pure in  $\mathcal{O}H$ , and hence  $\mathcal{O}Ge(\chi)$  is symmetric.

Let finally  $G$  be quasidihedral; that is, conjugation by the involution  $t$  sends  $s$  to  $s^{-1+2^{n-1}}$ . The calculations are similar to the previous case; we sketch the modifications. Let  $\bar{\eta}$  be the character of  $H$  sending  $s$  to  $\zeta^{-1+2^{n-1}} = -\zeta^{-1}$ . Then  $e(\chi) = e(\eta) + e(\bar{\eta}) = \sum_{a=0}^{2^n-1} (\zeta^a + (-1)^a \zeta^{-a}) s^a$ . As before, setting  $z = \frac{2^{n-1}}{(1-\zeta)^2} e(\chi)$ , it suffices to show that  $\mathcal{O}Hz$  is  $\mathcal{O}$ -pure in  $\mathcal{O}H$ . One verifies as before, that  $\{z, 2^n e(\eta)\}$  is an  $\mathcal{O}$ -basis of  $\mathcal{O}Hz$ . Let  $u = \sum_{a=0}^{2^n-1} \mu_a s^a$  be an element in  $\mathcal{O}Hz$  such that all coefficients  $\mu_a \in \mathcal{O}$  are divisible by  $1 - \zeta$ . Write  $u = \beta z + \gamma 2^n e(\eta)$  with  $\beta, \gamma \in \mathcal{O}$ . We need to

show that  $\beta, \gamma$  are divisible by  $1 - \zeta$ . Comparing coefficients yields  $\mu_a = \beta \frac{\zeta^a + (-1)^a \zeta^{-a}}{(1-\zeta)^2} + \gamma \zeta^a$ . If  $a = 2^{n-2}$ , then  $a$  is even and  $\zeta^a + \zeta^{-a} = 0$ , implying  $\mu_a = \gamma \zeta^a$ , which in turn implies that  $\gamma$  is divisible by  $1 - \zeta$ . Comparing coefficients for  $a = 1$  yields that  $\beta$  is divisible by  $1 - \zeta$ . Thus  $\mathcal{O}Hz$  is  $\mathcal{O}$ -pure in  $\mathcal{O}H$ , and hence  $\mathcal{O}Ge(\chi)$  is symmetric. This completes the proof of 1.7.

#### 4. ON SYMMETRIC SUBALGEBRAS OF MATRIX ALGEBRAS

Let  $G$  be a finite group. By the above, the  $\mathcal{O}$ -free  $\mathcal{O}$ -algebra quotients of  $\mathcal{O}G$  correspond to  $\mathcal{O}$ -pure ideals, hence to subsets of  $\text{Irr}_K(G)$ , and any  $\mathcal{O}$ -free quotient of  $\mathcal{O}G$  is the image of a structural map  $\alpha : \mathcal{O}G \rightarrow \text{End}_{\mathcal{O}}(V)$  for some finitely generated  $\mathcal{O}$ -free  $\mathcal{O}G$ -module  $V$ . By 2.6, the kernel of this map depends only on the set of irreducible characters of  $G$  arising as constituents of the character of  $V$ . If the character  $\chi$  of  $V$  is irreducible, then the image of  $\alpha$  is isomorphic to the algebra  $\mathcal{O}Ge(\chi)$ , where  $e(\chi) = \frac{\chi(1)}{|G|} \sum_{x \in G} \chi(x^{-1})x$ . Moreover, in that case,  $\text{Im}(\alpha) \cong \mathcal{O}Ge(\chi)$  has the same  $\mathcal{O}$ -rank  $\chi(1)^2$  as  $\text{End}_{\mathcal{O}}(V)$ . If  $\chi \in \text{Irr}_K(G)$  has degree one, then  $\mathcal{O}Ge(\chi) \cong \mathcal{O}$  is trivially symmetric. If  $\chi$  has defect zero, then  $\mathcal{O}Ge(\chi)$  is a matrix algebra over  $\mathcal{O}$ , hence also symmetric. Both examples are special cases of the following well-known situation:

**Proposition 4.1.** *Let  $G$  be a finite group,  $K$  a splitting field for  $G$ , and  $\chi \in \text{Irr}_K(G)$ . The algebra  $\mathcal{O}Ge(\chi)$  is isomorphic to a matrix algebra over  $\mathcal{O}$  if and only if  $\chi$  lifts an irreducible Brauer character.*

*Proof.* We include a proof for the convenience of the reader. Let  $X$  be an  $\mathcal{O}$ -free  $\mathcal{O}G$ -module with character  $\chi$ . The character  $\chi$  lifts an irreducible Brauer character if and only if  $k \otimes_{\mathcal{O}} X$  is a simple  $kG$ -module. The  $kG$ -module  $k \otimes_{\mathcal{O}} X$  is simple if and only if the structural map  $kG \rightarrow \text{End}_k(k \otimes_{\mathcal{O}} X)$  is surjective. By Nakayama's Lemma, this is the case if and only if the structural map  $\mathcal{O}G \rightarrow \text{End}_{\mathcal{O}}(X)$  is surjective. The result follows.  $\square$

If  $\mathcal{O}Ge(\chi)$  is symmetric but not a matrix algebra, then the following observation narrows down the possible symmetrising forms.

**Proposition 4.2.** *Let  $V$  be a free  $\mathcal{O}$ -module of finite rank  $n$ , and let  $A$  be a symmetric subalgebra of  $\text{End}_{\mathcal{O}}(V)$  of rank  $n^2$ . Then  $Z(A) \cong \mathcal{O}$ , and there is an integer  $r \geq 0$  such that the restriction to  $A$  of the map  $\pi^{-r} \text{tr}_V : \text{End}_{\mathcal{O}}(V) \rightarrow K$  sends  $A$  to  $\mathcal{O}$  and induces a symmetrising form on  $A$ . Moreover,  $r$  is the smallest nonnegative integer satisfying  $\pi^r \text{End}_{\mathcal{O}}(V) \subseteq A$ .*

*Proof.* Since the  $\mathcal{O}$ -rank of  $A$  is  $n^2$ , we have  $K \otimes_{\mathcal{O}} A \cong \text{End}_K(K \otimes_{\mathcal{O}} V)$ , whence  $Z(A) \cong \mathcal{O}$ . Any symmetrising form on  $\text{End}_K(K \otimes_{\mathcal{O}} V)$  is a nonzero linear multiple of the trace map  $\text{tr}_{K \otimes_{\mathcal{O}} V}$ , and hence a symmetrising form on  $A$  is of the form  $\pi^{-r} \text{tr}_V$  for some integer  $r$  such that  $\pi^{-r} \text{tr}_V(A) \subseteq \mathcal{O}$ . This forces  $r \geq 0$ . Let  $s$  be the smallest nonnegative integer satisfying  $\pi^s \text{End}_{\mathcal{O}}(V) \subseteq A$ . Let  $e$  be a primitive idempotent in  $\text{End}_{\mathcal{O}}(V)$ . Then  $\text{tr}_V(e) = 1$ . We have  $\pi^s e \in A$ , hence  $\pi^{-r} \text{tr}_V(\pi^s e) = \pi^{s-r} \in \mathcal{O}$ . This implies that  $s \geq r$ . Since  $\pi^{s-1} \text{End}_{\mathcal{O}}(V)$  is not contained in  $A$ , there is an element  $c \in A$  such that  $\pi^{-1}c \notin A$  and  $c \in \pi^s \text{End}_{\mathcal{O}}(V)$ . Thus  $\mathcal{O}c$  is a pure  $\mathcal{O}$ -submodule of  $A$ . Thus there is an  $\mathcal{O}$ -basis of  $A$  containing  $c$ . By considering the dual  $\mathcal{O}$ -basis with respect to the symmetrising form  $\pi^{-r} \text{tr}_V$ , it follows that  $\pi^{-r} \text{tr}_V(cA) = \mathcal{O}$ . Since  $c \in \pi^s \text{End}_{\mathcal{O}}(V)$ , we have  $\text{tr}_V(cA) \subseteq \pi^s \mathcal{O}$ , hence  $\pi^{-r} \text{tr}_V(cA) \subseteq \pi^{s-r} \mathcal{O}$ . This yields the inequality  $s \leq r$ , whence the equality  $s = r$ .  $\square$

**Corollary 4.3.** *Let  $V$  be a free  $\mathcal{O}$ -module of finite rank  $n$ , and let  $A$  be a proper symmetric subalgebra of  $\text{End}_{\mathcal{O}}(V)$  of rank  $n^2$ . Then for any idempotent  $i \in A$ , the integer  $\text{tr}_V(i)$  is divisible by  $p$ . In particular,  $p$  divides  $n$ .*

*Proof.* If  $A$  is a proper symmetric subalgebra of  $\text{End}_{\mathcal{O}}(V)$ , then the smallest nonnegative integer  $r$  satisfying  $\pi^r \text{End}_{\mathcal{O}}(V) \subseteq A$  is positive. It follows from 4.2 that  $A$  has a symmetrising form which is the restriction to  $A$  of  $\pi^{-r} \text{tr}_V$  for some positive integer  $r$ . In particular, if  $i$  is an idempotent in  $A$ , then  $\pi^{-r} \text{tr}_V(i)$  is an element in  $\mathcal{O}$ . Since  $\text{tr}_V(i)$  is an integer and  $r > 0$ , this implies that  $p$  divides  $\text{tr}_V(i)$ . Applied to  $i = 1_A = \text{Id}_V$  yields that  $p$  divides  $n = \text{tr}_V(\text{Id}_V)$ .  $\square$

**Corollary 4.4.** *Let  $V$  be a free  $\mathcal{O}$ -module of finite rank  $n$ . Suppose that  $(n, p) = 1$ . Then  $\text{End}_{\mathcal{O}}(V)$  has no proper symmetric subalgebra of  $\mathcal{O}$ -rank  $n^2$ .*

*Proof.* This is clear by 4.3.  $\square$

**Corollary 4.5.** *Let  $V$  be a free  $\mathcal{O}$ -module of finite rank  $p$ . Every proper symmetric subalgebra of rank  $p^2$  of  $\text{End}_{\mathcal{O}}(V)$  is local.*

*Proof.* If  $i$  is an idempotent in a proper symmetric subalgebra  $A$  of  $\text{End}_{\mathcal{O}}(V)$ , then  $p$  divides  $\text{tr}_V(i) \leq \text{tr}_V(\text{Id}_V) = p$ . Thus  $i = \text{Id}_V$ . The result follows.  $\square$

*Proof of Theorem 1.1.* Let  $V$  be an  $\mathcal{O}$ -free  $\mathcal{O}G$ -module with an irreducible character  $\chi \in \text{Irr}_K(G)$ . Suppose that  $\chi$  does not lift an irreducible Brauer character. Then the image  $E$  of the structural map  $\mathcal{O}G \rightarrow \text{End}_{\mathcal{O}}(V)$  is a proper subalgebra of  $\text{End}_{\mathcal{O}}(V)$ . Note that  $E \cong \mathcal{O}Ge(\chi)$ , by the preceding remarks. Thus, if  $\mathcal{O}Ge(\chi)$  is symmetric, then it follows from 4.3 that for any idempotent  $i \in \mathcal{O}G$ , the image  $j$  of  $i$  in  $E$  has the property that  $\text{tr}_V(j)$  is divisible by  $p$ . By the choice of  $V$ , we have  $\chi(i) = \text{tr}_V(j)$ . Thus  $\chi(i)$  is divisible by  $p$ , for any primitive idempotent  $i$ , and hence so is  $d_{\varphi}^{\chi}$  for any  $\varphi \in \text{IBr}_k(G)$ .  $\square$

## 5. PROOFS OF PROPOSITIONS 1.10 AND 1.11

If  $\chi : G \rightarrow \mathcal{O}^{\times}$  is a linear character, then the corresponding  $\mathcal{O}$ -pure ideal  $I_{\chi} = KGe(\chi) \cap \mathcal{O}G$  has  $\mathcal{O}$ -rank 1 and is equal to  $\mathcal{O}(\sum_{x \in G} \chi(x^{-1})x)$ . There is a unique  $\mathcal{O}$ -algebra automorphism of  $\mathcal{O}G$  sending  $x \in G$  to  $\chi(x)x$ . This automorphism sends  $\sum_{x \in G} \chi(x^{-1})x$  to  $\sum_{x \in G} x$ . Thus the linear characters of  $\mathcal{O}G$  are permuted transitively by the group of  $\mathcal{O}$ -algebra automorphisms of  $\mathcal{O}G$ , and therefore, in order to address the question whether  $\mathcal{O}G/I_{\chi}$  is symmetric, it suffices to consider the case where  $\chi = 1$  is the trivial character, in which case the corresponding pure ideal is  $I_1 = \mathcal{O}G(\sum_{x \in G} x) = \mathcal{O}(\sum_{x \in G} x)$ . The annihilator of  $I_1$  in  $\mathcal{O}G$  is the augmentation ideal  $I(\mathcal{O}G)$ . Combining these observations with Proposition 2.2 and some block theory yields a proof of 1.10, which we restate in a slightly more precise way.

**Proposition 5.1.** *Let  $G$  be a finite group. The following are equivalent.*

- (i) *The  $\mathcal{O}$ -algebra  $\mathcal{O}G/\mathcal{O}(\sum_{x \in G} x)$  is symmetric.*
- (ii) *There exists an element  $z \in Z(\mathcal{O}G)$  such that  $I(\mathcal{O}G) = \mathcal{O}Gz$ .*
- (iii) *The group  $G$  is  $p$ -nilpotent and has a cyclic Sylow- $p$ -subgroup.*

*Proof.* The equivalence of (i) and (ii) is clear by 2.2. Suppose that (ii) holds. Let  $b$  be the principal block idempotent of  $\mathcal{O}G$ . Then  $I(\mathcal{O}G)b = \mathcal{O}Gzb$  is a proper ideal in  $\mathcal{O}Gb$ ; in particular,  $zb$  is not invertible in  $Z(\mathcal{O}Gb)$ . Since  $Z(\mathcal{O}Gb)$  is local, it follows that  $zb$  is in the radical of  $Z(\mathcal{O}Gb)$ , and hence the ideal  $\mathcal{O}Gzb$  is contained in  $J(\mathcal{O}Gb)$ . Since  $\mathcal{O}Gb/I(\mathcal{O}G)b \cong \mathcal{O}G/I(\mathcal{O}G) \cong \mathcal{O}$ , this implies that  $\mathcal{O}Gb$  is a local algebra, and that  $kG\bar{z}\bar{b} = J(kG\bar{b})$ , where  $\bar{z}, \bar{b}$  are the canonical images of  $z, b$  in  $kG$ , respectively. It follows from results due to Broué and Puig in [1, §1] that  $b$  is a nilpotent block and that the finite group  $G$  is  $p$ -nilpotent. Since  $J(kG\bar{b}) = kG\bar{z}\bar{b}$  is a principal ideal, it follows

from a result of Nakayama [6] that  $kG\bar{b}$  is uniserial, and hence  $P$  is cyclic. Thus (ii) implies (iii). Conversely, if (iii) holds, then the principal block algebra  $\mathcal{O}Gb$  of  $\mathcal{O}G$  is isomorphic to  $\mathcal{O}P$ , where  $P$  is a Sylow- $p$ -subgroup of  $G$ , and if  $y$  is a generator of  $P$ , then  $I(\mathcal{O}P) = \mathcal{O}P(y-1)$ . Note that  $I(\mathcal{O}G)$  contains all nonprincipal block algebras of  $\mathcal{O}G$ . Set  $z = (y-1)b + \sum_{b'} b'$ , where in the sum  $b'$  runs over all nonprincipal block idempotents. By the above, this is an element in  $Z(\mathcal{O}G)$  satisfying  $I(\mathcal{O}G) = \mathcal{O}Gz$ , completing the proof.  $\square$

In a similar way, we obtain a proof of 1.11.

*Proof of Proposition 1.11.* Suppose that  $B/I$  is symmetric. By 2.2 the annihilator  $J$  of  $I$  is of the form  $J = Bz$  for some  $z \in Z(B)$ . Let  $X$  be an  $\mathcal{O}$ -free  $B$ -module with character  $\chi$ . By the assumptions,  $\bar{X} = k \otimes_{\mathcal{O}} X$  is a simple module over  $\bar{B} = k \otimes_{\mathcal{O}} B$ . By 4.1, and using that  $k$  is large enough, the structural map  $B \rightarrow \text{End}_{\mathcal{O}}(X)$  is surjective, and the kernel of this map is  $J$ . Since  $J = Bz$  is a proper ideal in  $B$  and  $Z(B)$  is local, it follows that  $z \in J(B)$ . Thus the image  $\bar{J}$  of  $J$  in  $\bar{B}$  is equal to  $J(\bar{B})$ , and  $\bar{B}$  has a single isomorphism class of simple modules. Since  $J(\bar{B}) = \bar{B}\bar{z}$ , where  $\bar{z}$  is the image of  $z$  in  $Z(\bar{B})$ , it follows as before from [6] that  $\bar{B}$  is uniserial. Thus  $B$  has cyclic defect groups. A block with cyclic defect and a single isomorphism class of simple modules is nilpotent, which shows that (i) implies (ii). Conversely, if (ii) holds, then  $B$  is Morita equivalent to  $\mathcal{O}P$ , where  $P$  is a defect group of  $B$  (and  $P$  is cyclic by the assumptions). Since  $\chi$  lifts an irreducible Brauer character it follows that under some Morita equivalence,  $\chi$  corresponds to the trivial character of  $\mathcal{O}P$ , and hence (i) holds by 1.10 applied to  $P$ .  $\square$

## 6. EXAMPLES

**Example 6.1.** Let  $P$  be a finite abelian  $p$ -group and  $E$  an abelian  $p'$ -subgroup of  $\text{Aut}(P)$ . Denote by  $G = P \rtimes E$  the corresponding semidirect product, and let  $\chi \in \text{Irr}_K(G)$ . Then  $\mathcal{O}Ge(\chi)$  is symmetric if and only if  $\chi(1) = 1$ . Indeed, since  $P$  is abelian, it follows that  $\chi(1)$  divides  $|E|$ , which is prime to  $p$ , and hence  $d_{\chi}^p$  is prime to  $p$  for some  $\varphi \in \text{IBr}_k(G)$ . The statement follows from 1.1.

The next example shows that there are finite  $p$ -groups having at least one irreducible character which does not have the symmetric quotient property.

**Example 6.2.** Let  $p$  be an odd prime.  $G = Q \wr R = H \rtimes R$ , where  $Q, R$  are cyclic of order  $p$ , and where  $H$  is a direct product of  $p$  copies of  $Q$  which are transitively permuted by  $R$ . Let  $s$  be a generator of  $Q$  and  $\zeta$  be a primitive  $p$ -th root of unity. For  $1 \leq i \leq p$  let  $\psi_i : H \rightarrow \mathcal{O}^\times$  be the linear character sending  $(s^{a_1}, s^{a_2}, \dots, s^{a_p}) \in H$  to  $\zeta^{a_i}$ ; that is, the kernel of  $\psi_i$  contains all but the  $i$ -th copy of  $Q$  in  $H$ , and the  $\psi_i$  are permuted transitively by the action of  $G$ . Set  $\chi = \text{Ind}_H^G(\psi_1)$ . Then  $\chi \in \text{Irr}_K(G)$ , and the  $\mathcal{O}$ -algebra  $\mathcal{O}Ge(\chi)$  is not symmetric.

In order to show this, observe first that  $\text{Res}_H^G(\chi) = \sum_{i=1}^p \psi_i$  because the  $\psi_i$  form a  $G$ -orbit in  $\text{Irr}_K(H)$ . We have

$$e(\chi) = \sum_{i=1}^p e(\psi_i) = \frac{1}{|H|} \sum_{h \in H} \left( \sum_{i=1}^p \psi_i(h^{-1}) \right) h.$$

The coefficients  $\sum_{i=1}^p \psi_i(h^{-1})$  are divisible by  $1 - \zeta$  because they are sums of  $p$  (arbitrary) powers of  $\zeta$ . Moreover, for  $h = (s, 1, \dots, 1) \in H$  one sees that  $1 - \zeta$  is the highest power of  $1 - \zeta$  dividing this coefficient. Thus if  $\mathcal{O}Ge(\chi)$  were symmetric, then  $\mathcal{O}Gz$  would have to pure in  $\mathcal{O}G$ , where  $z =$

$\frac{|H|}{1-\zeta}e(\chi)$ . Since  $z \in \mathcal{O}H$ , this is the case if and only if  $\mathcal{O}Hz$  is pure in  $\mathcal{O}H$ . We will show that  $\mathcal{O}Hz$  is not pure in  $\mathcal{O}H$ . If  $u = (s^{a_1}, s^{a_2}, \dots, s^{a_p}) \in H$ , then

$$uz = \frac{|H|}{1-\zeta} \sum_{i=1}^p ue(\psi_i) = \frac{|H|}{1-\zeta} \sum_{i=1}^p \zeta^{a_i} e(\psi_i).$$

Applied to the identity element and  $v = (1, 1, \dots, 1, s, 1, \dots, 1)$ , with  $s$  in the  $i$ -th component, and taking the difference yields  $z - vz = |H|e(\psi_i) \in \mathcal{O}Gz$ . For any  $u \in H$ , the above formula yields  $uz - z \in \oplus_{i=1}^p \mathcal{O}|H|e(\psi_i)$ . Thus the set

$$\{z, |H|e(\psi_i) \ (2 \leq i \leq p)\}$$

is an  $\mathcal{O}$ -basis of  $\mathcal{O}Hz$ . Since  $p$  is odd, this basis has at least three elements. Suppose that  $w = \sum_{h \in H} \mu_h h$  is an element in  $\mathcal{O}Hz$ . Write  $w = \alpha z + \sum_{i=2}^p \beta_i |H|e(\psi_i)$  with  $\alpha, \beta_i \in \mathcal{O}$ . Thus

$$\mu_h = \alpha \frac{\sum_{i=1}^p \psi_i(h^{-1})}{1-\zeta} + \sum_{i=2}^p \beta_i \psi_i(h^{-1}).$$

Note that if  $\sum_{i=2}^p \beta_i$  is divisible by  $1-\zeta$  then any sum of the form  $\sum_{i=2}^p \beta_i \psi_i(h^{-1})$  is divisible by  $1-\zeta$  because any character value  $\psi_i(h^{-1})$  is a power of  $\zeta$ . This shows that if  $1-\zeta$  divides both  $\alpha$  and the sum  $\sum_{i=2}^p \beta_i$ , then  $1-\zeta$  divides  $\mu_h$  for all  $h \in H$ . But since  $p > 2$  we may choose invertible elements  $\beta_i$  satisfying  $\sum_{i=2}^p \beta_i = 0$ . This shows that even if all  $\mu_h$  are divisible by  $1-\zeta$ , this does not imply that  $\alpha$  and all  $\beta_i$  are divisible by  $1-\zeta$ , hence  $\mathcal{O}Hz$  is not  $\mathcal{O}$ -pure in  $\mathcal{O}H$ .

## REFERENCES

- [1] M. Broué, L. Puig, *A Frobenius theorem for blocks*. Invent. Math. **56** (1980), 117–128.
- [2] E. C. Dade, *Blocks with cyclic defect groups*, Ann. Math. **84** (1966), 20–48.
- [3] K. Erdmann, *Blocks of tame representation type and related algebras*, Lecture Notes Math. **1428**, Springer Verlag, Berlin Heidelberg (1990).
- [4] W. Feit, *The Representation Theory of Finite Groups*. North Holland, Amsterdam (1982).
- [5] T. Nakayama, *On Frobeniusean Algebras. I*. Annals Math. **40** (1939), 611–633.
- [6] T. Nakayama, *Note on uni-serial and generalized uni-serial rings*. Proc. Imp. Acad. Tokyo XVI (1940) 285–289.
- [7] J. Thompson, *Vertices and sources*. J. Algebra **6** (1967), 1–6.

CITY UNIVERSITY LONDON, DEPARTMENT OF MATHEMATICAL SCIENCE, NORTHAMPTON SQUARE, LONDON, EC1V 0HB, UNITED KINGDOM

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, CHIBA UNIVERSITY, 1-33 YAYOI-CHO, INAGE-KU, CHIBA, 263-8522, JAPAN

CITY UNIVERSITY LONDON, DEPARTMENT OF MATHEMATICAL SCIENCE, NORTHAMPTON SQUARE, LONDON, EC1V 0HB, UNITED KINGDOM