Particles versus fields in $\mathcal{PT}$-symmetrically deformed integrable systems

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Abstract: We review some recent results on how $\mathcal{PT}$-symmetry, that is a simultaneous time-reversal and parity transformation, can be used to construct new integrable models. Some complex valued multi-particle systems, such as deformations of the Calogero-Moser-Sutherland models, are shown to arise naturally from real valued field equations of non-linear integrable systems. Deformations of complex non-linear integrable field equations, some of them even allowing for compacton solutions, are also investigated. The integrabilty of various systems is established by means of the Painlevé test.

1. Introduction

There are many examples of non-Hermitian integrable systems in the literature pre-dating the paper by Bender and Boettcher [1], which gave rise to the recent wider interest in non-Hermitian Hamiltonian systems. A well studied class of quantum field theories is for instance affine Toda field theories (ATFT)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{m^2}{\beta^2}\sum_{k=0}^{\ell} n_k \exp(\beta \alpha_k \cdot \phi), \tag{1.1}$$

involving ℓ scalar fields ϕ. The n_k are integers often called Kac labels and the α_k for $k = 1, \ldots, \ell$ are simple roots with α_0 being the negative of the highest root. When the coupling constant β is taken to be purely imaginary these models have interesting and richer features than their real counterparts. The classical solitons were found [2] to have real masses despite the fact that the model is a non-Hermitian Hamiltonian system. Unlike as for real coupling, the scattering of their fundamental particles allows for backscattering such that the associated Yang-Baxter equations give rise to solutions in terms of representations of quasi-triangle Hopf algebras (quantum groups). For the simplest example, the A_1-model, corresponding to complex Liouville theory, a rigorous proof for the reality of the spectrum was found by Faddeev and Tirkkonen [3] by relating it to the Hermitian XXZ-quantum spin chain using Bethe ansatz techniques.

In addition, integrable quantum spin chains of non-Hermitian type have been investigated in the past for instance by von Gehlen [4]. The Ising quantum spin chain in an imaginary field corresponds in the continuous limit to the Yang-Lee model ($A_2^{(2)}$-minimal ATFT)

$$\mathcal{H} = \frac{1}{2}\sum_{i=1}^{N} \sigma_i^x + \lambda \sigma_i^z \sigma_{i+1}^z + i\kappa\sigma_i^z, \qquad \lambda, \kappa \in \mathbb{R}, \tag{1.2}$$

and may be used to describe phase transitions. Here N denotes the length of the spin chain and the σ_i^z, σ_i^x are the usual Pauli matrices describing spin 1/2 particles and acting on the site i in the state space of the form $(\mathbb{C}^2)^{\otimes N}$.

It is by now well understood how to explain the spectral properties of such models by means of $\mathcal{PT}$-symmetry, i.e. symmetry of the Hamiltonian and the wavefunction with respect to a simultaneous parity transformation and time reversal, pseudo-Hermiticity or quasi-Hermiticity [5, 6, 7]. In addition is also established how to formulate a consistent quantum mechanical description via the definition of a new metric, although this is only worked out in detail for very few solvable models, e.g. [8]. Interesting questions regarding the uniqueness of the physical observables still need further investigations and remain unanswered. Once of the new feature is that unlike as for Hermitian systems the Hamiltonian alone is no longer sufficient enough to define the set of observables [9].

For classical systems, which are the main subject of this article, one may also use $\mathcal{PT}$-symmetry to establish the reality of the energy [10]

$$E = \int_{-a}^{a} \mathcal{H}[u(x)]dx = -\int_{a}^{-a} \mathcal{H}[u(-x)]dx = \int_{-a}^{a} \mathcal{H}^\dagger[u(x)]dx = E^\dagger. \tag{1.3}$$

Note that unlike as for the quantum case the reality can be established from the Hamiltonian alone, albeit together with some appropriate boundary conditions.

1.1 $\mathcal{PT}$-guided deformations

Let us now turn to the question of how to use the above mentioned arguments to construct new consistent models with real spectra. In principle we could use any of them, but clearly to exploit $\mathcal{PT}$-symmetry is most transparent, especially for classical models as we indicated in (1.3). Keeping in mind that the effect of a $\mathcal{PT}$-transformation is $\mathcal{PT}: x \to -x, p \to p$ and $i \to -i$, we may deform any $\mathcal{PT}$-symmetric function in the following way

$$f(x) \to f[-i(ix)^\varepsilon], \;\; f[-i(ix)^\varepsilon p^{\varepsilon-1}], \;\; f(x) + \tilde{f}[(ix)^\varepsilon] + \hat{f}[(ix)^\varepsilon p^\varepsilon], \tag{1.4}$$

while keeping its invariance. The deformation parameter $\varepsilon \in \mathbb{R}$ is choosen in such a way that the undeformed case is recovered for $\varepsilon = 1$. The same principle may be applied to derivatives of $\mathcal{PT}$-symmetric functions

$$\partial_x f(x) \to f_{x;\varepsilon} := \partial_{x,\varepsilon} f(x) = -i(if_x)^\varepsilon, \;\; \partial_{x,\varepsilon}^n := \partial_x^{n-1}\partial_{x,\varepsilon}, \tag{1.5}$$

such that

$$f_{xx;\varepsilon} := \partial^2_{x,\varepsilon} f \;=\; -i\varepsilon(if_x)^{\varepsilon}\frac{f_{xx}}{f_x}, \tag{1.6}$$

$$f_{xxx;\varepsilon} := \partial^3_{x,\varepsilon} f \;=\; -i\varepsilon(if_x)^{\varepsilon}\left[\frac{f_{xxx}}{f_x} + (\varepsilon-1)\left(\frac{f_{xx}}{f_x}\right)^2\right], \tag{1.7}$$

and even to supersymmetric derivative of a $\mathcal{PT}$-symmetric functions

$$D = \theta\partial_x + \partial_\theta \rightarrow D_\varepsilon := \theta\partial_{x,\varepsilon} + \partial_\theta, \tag{1.8}$$

with θ being the usual anti-commuting superspace variable. Remarkably it can be shown that the latter deformation can be carried out without breaking the supersymmetry of the models [11]. We shall apply these deformations in section 3

2. From real fields to complex particle systems

The above mentioned principle appears at times somewhat ad hoc and often the only motivation provided is that such models are likely to have real spectra. However, in the context of integrable systems some complex particle systems arise very naturally when taking systems for real valued fields as starting points.

2.1 No restrictions, ℓ-soliton solution of the Benjamin-Ono equation

Let us consider a field equation for a real valued field $u(x,t)$ of the form

$$u(x,t) = \frac{\lambda}{2}\sum_{k=1}^{\ell}\left(\frac{i}{x-z_k(t)} - \frac{i}{x-z_k^*(t)}\right), \qquad \lambda\in\mathbb{R}. \tag{2.1}$$

Chen, Lee and Pereira showed thirty years ago [12, 13] that this Ansatz constitutes an ℓ-soliton solution for the Benjamin-Ono equation [14]

$$u_t + uu_x + \lambda H u_{xx} = 0, \tag{2.2}$$

with $Hu(x)$ denoting the Hilbert transform $Hu(x) = \frac{P}{\pi}\int_{-\infty}^{\infty}\frac{u(x)}{z-x}dz$, provided the z_k in (2.1) obey the *complex* A_ℓ-Calogero equation of motion

$$\ddot{z}_k = \frac{\lambda^2}{2}\sum_{j\neq k}(z_j - z_k)^{-3}, \qquad z_k\in\mathbb{C}. \tag{2.3}$$

This is certainly the easiest example to demonstrate of how complex valued particle systems arise naturally from real valued fields.

2.2 Restriction to a submanifold

Obviously we do not expect the above procedure to produce complex valued particle systems when starting with any type of field equation. Dropping for instance in equation (2.2) the Hilbert transform and considering therefore Burgers equation instead will not lead to the desired result. However, we may consistently impose some additional constraints and make use of the following theorem found more than thirty years ago by Airault, McKean and Moser [15]:

Given a Hamiltonian $H(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n)$ with flow

$$x_i = \partial H/\partial \dot{x}_i \qquad \text{and} \qquad \ddot{x}_i = -\partial H/\partial x_i \qquad i = 1, \ldots, n \tag{2.4}$$

and conserved charges I_j in involution with H, i.e. vanishing Poisson brackets $\{I_j, H\} = 0$. Then the locus of $\texttt{grad}\, I = 0$ is invariant with regard to time evolution. Thus it is permitted to restrict the flow to that locus provided it is not empty.

In fact, often there are no real solutions to $\texttt{grad}\, I = 0$ and one is once again naturally led to consider complex particle systems. We consider the Boussinesq equation, that is a set of coupled KdV type equations, as an example

$$v_{tt} = a(v^2)_{xx} + bv_{xxxx} + v_{xx} \qquad a, b \in \mathbb{R}. \tag{2.5}$$

Then the real valued field

$$v(x,t) = \lambda \sum_{k=1}^{\ell} (x - z_k(t))^{-2}, \qquad \lambda \in \mathbb{R} \tag{2.6}$$

satisfies the Boussinesq equation (2.5) if and only if $b = 1/12$, $\lambda = -a/2$ and z_k obeys the constraining equations

$$\ddot{z}_k = 2\sum_{j\neq k} (z_j - z_k)^{-3} \qquad \Leftrightarrow \qquad \ddot{z}_k = -\frac{\partial H_{Cal}}{\partial z_i}, \tag{2.7}$$

$$\dot{z}_k^2 = 1 - \sum_{j\neq k} (z_j - z_k)^{-2} \qquad \Leftrightarrow \qquad \texttt{grad}(I_3 - I_1) = 0. \tag{2.8}$$

Here $I_3 = \sum_{j=1}^{\ell} [\dot{z}_j^3/3 + \sum_{k\neq j} \dot{z}_j (z_j - z_k)^2]$ and $I_1 = \sum_{j=1}^{\ell} \dot{z}_j$ are two conserved charges in the A_ℓ-Calogero model. In principle it could be that there is no solution to these equations, meaning that the imposition of the additional constraint (2.8), besides the equation of motion (2.7), will produce an empty locus. However, this is not the case and some genuine non-trivial solutions may be found. For $n = 2$ a solution was already reported in [15]

$$z_1 = \kappa + \sqrt{(t + \tilde{\kappa})^2 + 1/4}, \qquad z_2 = \kappa - \sqrt{(t + \tilde{\kappa})^2 + 1/4} \tag{2.9}$$

such that the Boussinesq solution acquires the form

$$v(x,t) = 2\lambda \frac{(x-\kappa)^2 + (t+\tilde{\kappa})^2 + 1/4}{[(x-\kappa)^2 - (t+\tilde{\kappa})^2 - 1/4]^2}. \tag{2.10}$$

Note that $v(x,t)$ is still a real solution. Without any complication we may change κ and $\tilde{\kappa}$ to be purely imaginary in which case, and only in this case, (2.10) becomes a solution for the $\mathcal{PT}$-symmetric equation (2.5) in the sense that $\mathcal{PT} : x \to -x, t \to -t$ and $v \to v$. Different types of solutions and also for other values of n will be reported elsewhere [16].

3. $\mathcal{PT}$-deformed particle systems

Having presented some examples of how to obtain complex many particle systems in a very natural way from real valued field equations, it appears less ad hoc to start directly by deforming some integrable many-body problems according to the principles described in section 1.1, having in mind that there might exist a corresponding real valued field equation.

3.1 Complex extended Calogero-Moser-Sutherland (CMS) models

The simplest way to deform a given model is just by adding a term to it along the lines indicated in equation (1.4). For a many body-system this was first proposed for the A_ℓ-Calogero model in [17]

$$\mathcal{H}_{BK} = \frac{p^2}{2} + \frac{\omega^2}{2}\sum_i q_i^2 + \frac{g^2}{2}\sum_{i\neq k}\frac{1}{(q_i - q_k)^2} + i\tilde{g}\sum_{i\neq k}\frac{1}{(q_i - q_k)}p_i, \tag{3.1}$$

with $g, \tilde{g} \in \mathbb{R}, q, p \in \mathbb{R}^{\ell+1}$. The Hamiltonian $\mathcal{H}_{BK}$ differs from the usual Calogero model by the last term. There are some immediate questions to be raised with regard to (3.1). Is it possible to have a representation independent formulation for $\mathcal{H}_{BK}$? May one use other algebras or Coxeter groups besides A_ℓ and B_ℓ? Is it possible to use non-rational potentials? Can one have more coupling constants? Are the extensions still integrable? These questions were answered in [18], where it was noticed that one may generalize the Hamiltonian $\mathcal{H}_{BK}$ to

$$\mathcal{H}_\mu = \frac{1}{2}p^2 + \frac{1}{2}\sum_{\alpha\in\Delta} g_\alpha^2 V(\alpha\cdot q) + i\mu\cdot p, \tag{3.2}$$

with Δ being any root system and the new vector $\mu = 1/2\sum_{\alpha\in\Delta}\tilde{g}_\alpha f(\alpha\cdot q)\alpha$, with $f(x) = 1/x$ and $V(x) = f^2(x)$. It is not so obvious, in fact no case independent proof is known, that one can further re-write the Hamiltonian such that it becomes the standard Hermitian Calogero Hamiltonian with shifted momenta

$$\mathcal{H}_\mu = \frac{1}{2}(p + i\mu)^2 + \frac{1}{2}\sum_{\alpha\in\Delta}\hat{g}_\alpha^2 V(\alpha\cdot q), \tag{3.3}$$

and re-defined coupling constant

$$\hat{g}_\alpha^2 = \begin{cases} g_s^2 + \alpha_s^2\tilde{g}_s^2 & \alpha\in\Delta_s \\ g_l^2 + \alpha_l^2\tilde{g}_l^2 & \alpha\in\Delta_l. \end{cases} \tag{3.4}$$

Here Δ_l and Δ_s refer to the root system of the long and short roots, respectively.

Thus we trivially have $\mathcal{H}_\mu = \eta^{-1}h_{\text{Cal}}\eta$ with $\eta = e^{-q\cdot\mu}$. Integrability follows then immediately by acting adjointly with η on the Calogero Lax pair $\dot{L}_{\text{Cal}} = [L_{\text{Cal}}, M_{\text{Cal}}]$, such that the new pair is obtained by $L_\mu(p) = L_{\text{Cal}}(p + i\mu)$ and $M_\mu = M_{\text{Cal}}$. An interesting statement is obtained by computing backwards and allowing in (3.3) any kind of Calogero-Moser-Sutherland potential, i.e. $V(x) = 1/x^2$, $V(x) = 1/sinh^2 x$ or $V(x) = 1/sin^2 x$

$$\mathcal{H}_\mu = \frac{1}{2}p^2 + \frac{1}{2}\sum_{\alpha\in\Delta}\hat{g}_\alpha^2 V(\alpha\cdot q) + i\mu\cdot p - \frac{1}{2}\mu^2. \tag{3.5}$$

By construction the Hamiltonian (3.5) corresponds to an integrable model, but it turns out [18] that the relation $\mu^2 = \alpha_s^2 \tilde{g}_s^2 \sum_{\alpha\in\Delta_s} V(\alpha\cdot q) + \alpha_l^2 \tilde{g}_l^2 \sum_{\alpha\in\Delta_l} V(\alpha\cdot q)$ is only valid for rational potentials. Thus without the μ^2-term only the deformed version of the Calogero model remains integrable and not its generalizations.

3.2 Complex deformed Calogero-Moser-Sutherland models

Having seen that merely adding terms to complex Hamiltonians leads to rather simple models, we comment on some of the other possibilities indicated in (1.4), which were explored in [19]. One of the symmetries of the CMS-models is its invariance with respect to the entire Coxeter group $\mathcal{W}$ resulting from the fact that we sum over all roots and the property that Weyl reflections preserve inner products. Interpreting now each Weyl reflection as a parity transformation across a particular hyperplane, we may try to seek models which remain invariant with regard to the action of a across these hyperplanes deformed version of the Weyl group $\mathcal{W}^{\mathcal{PT}}$ associated with some newly defined complex roots $\tilde{\alpha}$

$$\mathcal{H}_{\mathcal{PT}\mathrm{CMS}} = \frac{p^2}{2} + \frac{m^2}{16}\sum_{\tilde{\alpha}\in\tilde{\Delta}_s} (\tilde{\alpha}\cdot q)^2 + \frac{1}{2}\sum_{\tilde{\alpha}\in\tilde{\Delta}} g_{\tilde{\alpha}} V(\tilde{\alpha}\cdot q), \tag{3.6}$$

where $m, g_{\tilde{\alpha}} \in \mathbb{R}$. We outline the main features of the construction of the complex root system $\tilde{\alpha} \in \mathbb{R}^n \oplus i\mathbb{R}^n$ with the desired features. First recall that to each simple root α_i there is an associated Weyl reflections $\sigma_i(x) = x - 2\alpha_i(x\cdot\alpha_i)/(\alpha_i^2)$. The aim is then to construct a new complex root system $\tilde{\Delta}$ in one-to-one correspondence to the standard one Δ, which may be recovered in the limit $\epsilon = \varepsilon - 1 \to 0$

$$\lim_{\epsilon\to 0} \tilde{\alpha}_i(\epsilon) = \alpha_i \qquad \text{for } \tilde{\alpha}_i(\epsilon) \in \tilde{\Delta}(\epsilon), \alpha_i \in \Delta. \tag{3.7}$$

We define a $\mathcal{PT}$-Weyl reflection as $\tilde{\sigma}_{\alpha_i} := \sigma_{\alpha_i}\mathcal{T}$, where the time reversal $\mathcal{T}$ has the effect of a complex conjugation. We may then use the action on a generic complex root $\tilde{\alpha}$ to determine their form

$$\begin{aligned}
\tilde{\sigma}_{\alpha_j}(\tilde{\alpha}_j(\epsilon)) &= \sigma_{\alpha_j}\mathcal{T}\,(\mathtt{Re}\tilde{\alpha}_j(\epsilon)) + \sigma_{\alpha_j}\mathcal{T}\,(i\mathtt{Im}\tilde{\alpha}_j(\varepsilon)) &(3.8)\\
&= \sigma_{\alpha_j}(\mathtt{Re}\tilde{\alpha}_j(\epsilon)) - i\sigma_{\alpha_j}(\mathtt{Im}\tilde{\alpha}_j(\epsilon)) &(3.9)\\
&= -\mathtt{Re}\tilde{\alpha}_j(\epsilon) - i\mathtt{Im}\tilde{\alpha}_j(\epsilon) &(3.10)\\
&= -\tilde{\alpha}_j(\epsilon). &(3.11)
\end{aligned}$$

As a solution to these equations we find

$$\mathtt{Re}\tilde{\alpha}_i(\epsilon) = R(\epsilon)\alpha_i \qquad \text{and} \qquad \mathtt{Im}\tilde{\alpha}_i(\epsilon) = I(\epsilon)\sum_{j\neq i}\kappa_j\lambda_j, \tag{3.12}$$

$$\lim_{\epsilon\to 0} R(\epsilon) = 1 \qquad \text{and} \qquad \lim_{\epsilon\to 0} I(\epsilon) = 0, \tag{3.13}$$

with λ_j denoting fundamental roots and $\kappa_j \in \mathbb{R}$. Concrete examples for some specific algebras are for instance the deformed roots for A_2

$$\tilde{\sigma}_1\tilde{\alpha}_1(\epsilon) = -R(\epsilon)\alpha_1 \mp iI(\varepsilon)\lambda_2 =: -\tilde{\alpha}_1(\varepsilon), \tag{3.14}$$

$$-\tilde{\sigma}_1\tilde{\sigma}_2\tilde{\sigma}_1\tilde{\alpha}_1(\varepsilon) = R(\varepsilon)\alpha_2 \mp iI(\varepsilon)\lambda_1 =: \tilde{\alpha}_2(\varepsilon), \tag{3.15}$$

or those for G_2

$$\tilde{\alpha}_1(\epsilon) = R(\epsilon)\alpha_1 \pm iI(\epsilon)\lambda_2, \tag{3.16}$$

$$\tilde{\alpha}_2(\epsilon) = R(\epsilon)\alpha_2 \mp i3I(\epsilon)\lambda_1. \tag{3.17}$$

Having assembled the mathematical tools, we may substitute the deformed roots into the model (3.6) and study its properties. In [19] it was found that the A_2 and G_2 deformed Calogero models can still be solved by separation of variables, analogously to the undeformed case. However, some of the physical properties change, most notably the energy spectrum is different. The reason for this difference is that some restrictions cease to exist. For instance the wavefunctions are now regularized, such that no restriction arises from demanding finiteness. The original energy spectrum $E = 2\,|\omega|\,(2n + \lambda + 1)$ becomes in the deformed case [19]

$$E_{n\ell}^{\pm} = 2|\omega|\left[2n + 6(\kappa_s^{\pm} + \kappa_l^{\pm} + \ell) + 1\right] \qquad \text{for } n, \ell \in \mathbb{N}_0, \tag{3.18}$$

with $\kappa_{s/l}^{\pm} = (1 \pm \sqrt{1 + 4g_{s/l}})/4$ and s, l referring to coupling constants multiplying terms involving short and long roots, respectively.

In section 2 our starting point were real fields and we naturally ended up with complex particle systems, whereas in this section we started directly from the latter. It remains to comment on how the relation may be established in the reverse procedure. Having constructed the deformed roots we may compute the dual canonical coordinates $\tilde{q}$ from

$$\tilde{\alpha} \cdot q = \tilde{q} \cdot \alpha, \qquad \alpha, q \in \mathbb{R},\ \tilde{\alpha}, \tilde{q} \in \mathbb{R} \oplus i\mathbb{R}, \tag{3.19}$$

and subsequently simply replace in (3.1) the Hamiltonian $\mathcal{H}_{\mathcal{PT}\text{CMS}}(p, q, \tilde{\alpha})$ by $\mathcal{H}_{\mathcal{PT}\text{CMS}}(\tilde{p}, \tilde{q}, \alpha)$. The freedom in the choice of the functions $R(\epsilon), I(\epsilon)$ may then be used to satisfy the constraint $\texttt{grad}I = 0$. The dual canonical coordinates for A_2 are for instance easily computed to

$$\begin{aligned} \tilde{q}_1 &= R(\varepsilon)q_1 + iI(\varepsilon)/3(q_2 - q_3), \\ \tilde{q}_2 &= R(\varepsilon)q_2 + iI(\varepsilon)/3(q_3 - q_1), \\ \tilde{q}_3 &= R(\varepsilon)q_3 + iI(\varepsilon)/3(q_1 - q_2). \end{aligned} \tag{3.20}$$

At this point it might still not be possible to satisfy $\texttt{grad}I = 0$. As the construction outlined is by no means unique one still has the additional freedom to employ an alternative one. These issues are further elaborated on in [16].

4. Complex field equations

Naturally we may also start directly by considering complex field equations.

4.1 $\mathcal{PT}$-deformed field equations

Taking the symmetries of the Korteweg deVries (KdV) equation into account one may deform the derivatives according to (1.4) either in the second or the third term

$$u_t - 6uu_{x;\varepsilon} + u_{xxx;\mu} = 0, \qquad\qquad \varepsilon, \mu \in \mathbb{R}. \tag{4.1}$$

The first possibility, i.e. $\mu = 1$, was investigated in [20], leading to a not Galilean invariant, non-Hamiltonian system with at least two conserved charges in form of infinite sums allowing for steady wave solutions. Shortly afterwards the second option was investigated in [10], that is $\varepsilon = 1$, giving rise to Galilean invariant Hamiltonian system with at least three simple charges and steady state solutions. The question of whether these systems are integrable was thereafter addressed in [22] by carrying out the Painlevé test.

4.2 The Painlevé test

Let briefly summarize the main steps of this analysis proposed originally in [21]. The starting point is a series expansion for the field

$$u(x,t) = \sum_{k=0}^{\infty} \lambda_k(x,t)\phi(x,t)^{k+\alpha}, \tag{4.2}$$

with α being the leading order singularity in the field equation and $\lambda_k(x,t)$ and $\phi(x,t)$ some newly introduced fields. The substitution of this so-called Painlevé expansion into the partial differential equation (PDE) under investigation leads to recurrence relations of the general form

$$g(j, \phi_t, \phi_x, \phi_{xx}, \ldots)\lambda_j = f(\lambda_{j-1}, \lambda_{j-2}, \ldots, \lambda_1, \lambda_0, \phi_t, \phi_x, \phi_{xx}, \ldots), \tag{4.3}$$

with f and g being some functions depending on the individual system under consideration. Solving these equations recursively might then lead at some level, say k, to $g = 0$. For that level we may then compute the right hand side of (4.3) and find that either $f \neq 0$ or $f = 0$. In the former case the Painlevé test fails and the equation under investigation is not integrable, whereas in the latter case λ_k is found to be a free parameter, a so-called resonance. In case the number of resonances equals the order of the PDE the Painlevé test is passed, since in that scenario the expansion (4.2) has enough free parameters to accommodate all possible initial conditions. A slightly stronger statement is made when the series is also shown to converge. In that case one speaks of the Painlevé property of the PDE, which is conjectured to be equivalent to integrability.

4.3 Painlevé test for Burgers equation

Instead of deformed KdV-equation (4.1) let comment first on a simpler $\mathcal{PT}$-symmetrically deformed model, i.e. Burgers equations

$$u_t + u u_{x;\varepsilon} = i\kappa u_{xx;\mu} \qquad \text{with} \quad \kappa, \varepsilon, \mu \in \mathbb{R}, \tag{4.4}$$

for which the Painlevé test was carried out in [22]. In there it was found that leading order singularities only cancel for $\alpha = -1$ and $\varepsilon = \mu$ when taken to be integers. Keeping the ε generic and starting with the lowest order, the recurrence relations lead to the following equations

$$\begin{array}{lr} \text{order } -(2\varepsilon+1): & \lambda_0 + i2\varepsilon\kappa\phi_x = 0, \\ \text{order } -2\varepsilon: & \phi_t\delta_{\varepsilon,1} + \lambda_1\phi_x - i\kappa\varepsilon\phi_{xx} = 0, \\ \text{order } -(2\varepsilon-1): & \partial_x(\phi_t\delta_{\varepsilon,1} + \lambda_1\phi_x - i\kappa\varepsilon\phi_{xx}) = 0. \end{array} \tag{4.5}$$

This means that

$$\lambda_0 = -i2\varepsilon\kappa\phi_x, \quad \lambda_1 = (i\varepsilon\kappa\phi_{xx} - \phi_t\delta_{\varepsilon,1})/\phi_x, \quad \lambda_2 \equiv \text{arbitrary}, \tag{4.6}$$

such that we have already one of the desired free parameters. A more generic argument can be used [22] to derive a necessary condition for a resonance to exist

$$i2^{\varepsilon-1}\varepsilon^{\varepsilon}\lambda_r(r+1)(r-2)\kappa^{\varepsilon}\phi_x^{2\varepsilon} = 0. \tag{4.7}$$

The requirement is that the parameter λ_r becomes free, which for (4.7) is obviously the case when $r = -1, 2$. The values 2 was already found in (4.6) and -1 corresponds to the so-called fundamental resonance, which seems to be always present. Thus according to the strategy outlined in section 4.2, we have the desired amount of free parameters and conclude that the deformed Burgers equation (4.4) with $\varepsilon = \mu$ passes the Painlevé test.

For the case $\varepsilon = 2$ the convergence of the series was proven in [22] and concluded that the system even possess the Painlevé property and is therefore integrable. In that case the expansion becomes

$$u(x,t) = -\frac{4i\kappa}{\phi} + \lambda_2\phi + \frac{\xi'}{8\kappa}\phi^2 - \frac{i\lambda_2^2}{20\kappa}\phi^3 - \frac{i\lambda_2\xi'}{96\kappa^2}\phi^4 + \mathcal{O}(\phi^5). \tag{4.8}$$

One may even find a closed solution when making the further assumption $u(x,t) = \zeta(z) = \zeta(x - vt)$ and thus reducing the PDE to an ODE

$$\zeta(z) = e^{i\pi 5/3}(2v\kappa)^{1/3}\frac{\tilde{c}\ Ai'(\chi) + Bi'(\chi)}{\tilde{c}\ Ai(\chi) + Bi(\chi)} \tag{4.9}$$

with $\chi = e^{i\pi/6}(vz - c)(2v\kappa)^{-2/3}$ and Ai, Bi denoting Airy functions.

For the deformed KdV-equation (4.1) is was concluded that they are only integrable when $\varepsilon = \mu$, albeit the Painlevé series was found to be defective meaning that is does not contain enough resonances to match the order of the PDE.

4.4 Compactons versus Solitons

A further interesting $\mathcal{PT}$-symmetric deformation was proposed [23] for the generalized KdV equations [24], which are known to possess compacton solutions

$$\mathcal{H}_{l,m,p} = -\frac{u^l}{l(l-1)} - \frac{g}{m-1}u^p(iu_x)^m. \tag{4.10}$$

The corresponding equations of motions are

$$\begin{aligned} &u_t + u^{l-2}u_x + gi^m u^{p-2}u_x^{m-3}\left[p(p-1)u_x^4\right. \\ &\left.+2pmuu_x^2u_{xx} + m(m-2)u^2u_{xx}^2 + mu^2u_xu_{xxx}\right] = 0. \end{aligned} \tag{4.11}$$

The system (4.10) is yet another generalization of the generalized KdV system and the $\mathcal{PT}$-symmetric deformation of the KdV equation suggested in [10], i.e. $\mu = 1$ in (4.1), corresponding to $\mathcal{H}_{l,2,p}$ and $\mathcal{H}_{3,\varepsilon+1,0}$, respectively. These models were found to admit compacton solutions, which, depending on l, p or m, are either unstable, stable with independent width A and amplitude β or stable with amplitudes β depending on the width A. An interesting question to ask is whether these systems also allow for soliton solutions, which would be the case when they pass the Painlevé test. The test was carried out in [25] and the results are summarized in the following table:

$\mathcal{H}_{l,m,p}$	compactons	solitons
$l = p + m$	stable, independent A, β	no
$2 < l < p + 3m$	stable, dependent A, β	yes
$l \leq 2$ or $l \geq p + 3m$	unstable	no

Interestingly, it was found that there is no distinction between the generalized KdV equations and $\mathcal{PT}$-symmetric extensions of generalized KdV equations with regard to the Painlevé test.

5. Conclusions

We have shown that $\mathcal{PT}$-symmetry can be used as a guiding principle to define new interesting models, some of which are even integrable. Some complex particle systems, possibly restricted to some submanifold, are shown to be equivalent to some real valued fields obeying non-linear field equations. We also investigated some complex field equations, which for certain choices of the parameters involved turn out to be integrable admitting soliton as well as compacton solutions.

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