City Research Online

City St George's, University of London

Citation: Faria, C. F. D. M. & Fring, A. (2006). Isospectral Hamiltonians from Moyal products. Czechoslovak Journal of Physics, 56(9), pp. 899-908. doi: 10.1007/s10582-006-0386-x

This is the unspecified version of the paper.

This version of the publication may differ from the final published version. To cite this item please consult the publisher's version.

Permanent repository link: https://openaccess.city.ac.uk/id/eprint/697/

Link to published version: https://doi.org/10.1007/s10582-006-0386-x

Copyright and Reuse: Copyright and Moral Rights remain with the author(s) and/or copyright holders. Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge, unless otherwise indicated, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way. For full details of reuse please refer to City Research Online policy.

City Research Online: http://openaccess.city.ac.uk/ publications@citystgeorges.ac.uk

arXiv:quant-ph/0607154v1 22 Jul 2006

Isospectral Hamiltonians from Moyal products

C. Figueira de Morisson Faria and A. Fring

Centre for Mathematical Science, City University,
Northampton Square, London EC1V 0HB, UK

Received 11 July 2006;
final version 21 July 2006

Recently Scholtz and Geyer proposed a very efficient method to compute metric operators for non-Hermitian Hamiltonians from Moyal products. We develop these ideas further and suggest to use a more symmetrical definition for the Moyal products, because they lead to simpler differential equations. In addition, we demonstrate how to use this approach to determine the Hermitian counterpart for a Pseudo-Hermitian Hamiltonian. We illustrate our suggestions with the explicitly solvable example of the $-x^4$-potential and the ubiquitous harmonic oscillator in a complex cubic potential.

PACS: 03.65.-w,02.30.Mv
Key words: pseudo-Hermiticity, PT invariance, Moyal products

1 Introduction

Many non-Hermitian Hamiltonians H are known to possess real discrete spectra, e.g. [1, 2], which make them potential candidates for physical systems. Unlike as for Hermitian Hamiltonians h, the conventional inner products of the corresponding wavefunctions are usually indefinite and the central problem in this context is to construct meaningful inner products serving to formulate a consistent quantum theory. Besides from compensating wrong signs by hand [3], there are essentially three different, albeit in many cases equivalent, ways to achieve this: i) by employing bi-orthonormal eigenstates [4], ii) by constructing the so-called C-operator [5, 6] or iii) by restricting to pseudo-Hermitian Hamiltonians H and constructing their Hermitian counterparts h related to H by a similarity transformation $\eta = \eta^\dagger$ [7, 8]

$$h = \eta H \eta^{-1} = h^\dagger \qquad \Leftrightarrow \qquad H^\dagger = \eta^2 H \eta^{-2}. \tag{1}$$

See also [9] for an earlier discussion of these issues. In many respects the last possibility is the most direct and straightforward approach on which we will almost exclusively concentrate here. The natural starting point is usually a given non-Hermitian Hamiltonian H for which one needs to construct η. Subsequently one may formulate all relevant physical questions for the non-Hermitian system in terms of the conventional formulation of quantum mechanics associated to the Hermitian system. Thus a key task in this approach is to find η^2 and η. Unfortunately this is only possible in some exceptional cases in an exact manner and otherwise one has to resort to perturbation theory. Building on earlier work [6, 10], we found in [11]

the closed expressions

$$h = h_0 + \sum_{n=1}^{[\frac{\ell}{2}]} \frac{g^{2n}(-1)^n E_n}{4^n (2n)!} c_q^{(2n)}(h_0), \quad H = h_0 - \sum_{n=1}^{[\frac{\ell+1}{2}]} \frac{g^{2n-1}\kappa_{2n-1}}{(2n-1)!} c_q^{(2n-1)}(h_0), \quad (2)$$

which are related according to (1) perturbatively. Here $[x]$ denotes the integer part of a number x, the non-Hermitian Hamiltonian is assumed to be of the form

$$H = h_0 + igh_1, \quad (3)$$

$h_0 = h_0^\dagger$, $h_1 = h_1^\dagger$ and $c_q^{(n)}(x)$ denotes the n-fold commutator of the operator q with some operator x. In case one has the condition $c_q^{(\ell+1)}(h_0) = 0$ for some finite integer ℓ the expressions (2) are exact and otherwise they are just correct up to the stated order in the coupling constant g. The E_n are Euler's numbers and the κ_{2n-1} may be constructed from them [11]. One should note, however, that, in practice, the above-stated procedure may lead to rather cumbersome relations involving commutators. This fact poses a major problem for determining closed formulae for specific isospectral pairs of Hamiltonians, or for carrying out perturbative computations up to higher orders.

The main purpose of this paper is to propose a practical scheme for overcoming this difficulty. We elaborate further on a recent proposal by Scholtz and Geyer [12] to solve (1) by means of Moyal products instead of computing commutators. The central idea is to exploit isomorphic relations between commutator relations and real valued functions multiplied by Moyal products, which correspond to differential equations. We shall demonstrate that this approach is rather practical and allows to compute pairs of isospectral Hamiltonians $h = h^\dagger$ and $H \neq H^\dagger$.

2 Similarity transformations from Moyal Products

2.1 Generalities on Moyal products

Moyal products are applied in a wide field of research, such as non-commutative(nc) M-theory [15], nc-string theory [16], nc-integrable field theories [17, 18, 19, 20], etc. The key idea is to transfer the noncommutative nature of some operators to real valued functions. Technically one may set up such an isomorphism in various different ways. In the present context of studying non-Hermitian Hamiltonians such possibilities have been exploited in [12]. The authors defined the Moyal product of real valued functions depending on the variables x and p as

$$f(x,p) * g(x,p) = f(x,p) e^{i \overleftarrow{\partial}_x \overrightarrow{\partial}_p} g(x,p) = \sum\nolimits_{s=0}^{\infty} \frac{i^s}{s!} \partial_x^s f(x,p) \partial_p^s g(x,p). \quad (4)$$

The classical, more widespread and symmetrical definition is, see e.g. [14, 21, 22]

$$\begin{aligned} f(x,p) \star g(x,p) &= f(x,p) e^{\frac{i}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} g(x,p) \quad (5) \\ &= \sum_{s=0}^{\infty} \frac{(-i/2)^s}{s!} \sum_{t=0}^{s} (-1)^t \left(\begin{array}{c} s \\ t \end{array} \right) \partial_x^t \partial_p^{s-t} f(x,p) \partial_x^{s-t} \partial_p^t g(x,p), \end{aligned}$$

or exchange the roles of x and p in (4). In order to achieve a proper isomorphism between the operator expressions and those computed with Moyal products one requires different types of prescriptions to translate real valued functions into operator valued expressions. Products computed with definition (4) must be viewed as ordered products in which all $\hat{p}$-operators are moved to the left of all $\hat{x}$-operators. In expressions computed with definition (5) on the other hand one should replace each monomial $p^m x^n$ or $x^n p^m$ by the totally symmetric polynomial $S_{m,n}$ in the m operators $\hat{p}$ and n operators $\hat{x}$

$$\hat{S}_{m,n} = \frac{m!n!}{(m+n)!} \sum\nolimits_{\pi} \hat{p}^m \hat{x}^n. \tag{6}$$

Here π indicates the sum over the entire permutation group. The simplest example to illustrate this is

$$[\hat{x}^2, \hat{p}^2] = 4i\hat{p}\hat{x} - 2 \;\;\cong\;\; x^2 * p^2 - p^2 * x^2 = 4ipx - 2 \;\;\cong\;\; x^2 \star p^2 - p^2 \star x^2 = 4ipx. \tag{7}$$

We use here the standard canonical commutation relation $[\hat{x}, \hat{p}] = i$ and throughout the paper we employ atomic units $\hbar = e = m_e = c\alpha = 1$. We observe that the $*$-product yields the correct operator expression upon replacing $x \to \hat{x}$, $p \to \hat{p}$. The $\star$-product on the other hand corresponds to the correct commutation relations for $px \to S_{1,1} = (\hat{p}\hat{x} + \hat{x}\hat{p})/2$. The defining relation (5) is slightly more complicated than (4), but it is more symmetrical and leads therefore to cancellations of various terms as one can easily convince oneself. Loosely speaking, it has the advantage that it incorporates already more of the noncommutative nature of $\hat{x}$ and $\hat{p}$ from the very beginning. As we shall see below, this is the reason why it leads to simpler differential equations for the quantities we wish to determine. These statements are also supported by comparing the equations resulting from (5) and (4) for all examples calculated up to now with this method [12][1]).

2.2 Hermitian counterparts from Pseudo-Hermitian Hamiltonians

Let us now briefly discuss in general how we proceed to construct h from a given H by first solving (1) for η^2 and η. The explicit knowledge of η is vital, since once it is known one may control the entire quantum mechanical formalism, such as inner products, observables, time evolution, etc. However, in general one can not compute η exactly and has to rely on perturbative methods [6, 10, 11] in which one has to solve the commutator relations occurring in (2) order by order. This is a very cumbersome procedure and up-to-now it has only been carried out for few cases to lowest order. Here we present a simple and more efficient scheme which leads to the exact determination of η by employing Moyal products. We build on

[1]) For instance for the example $H = p^2 + igx^3$, definition (4) yields

$$2gx^3\eta^2(x,p) + 3igx^2\partial_p\eta^2(x,p) - 3gx\partial_p^2\eta^2(x,p) - ig\partial_p^3\eta^2(x,p) + 2p\partial_x\eta^2(x,p) + i\partial_x^2\eta^2(x,p) = 0,$$

whereas (5) gives the simpler form $4gx^3\eta^2(x,p) - 3gx\partial_p^2\eta^2(x,p) + 4p\partial_x\eta^2(x,p) = 0$.

suggestions of [12], but as a starting point we use instead of the definition (4) the definition (5) and write the second equation in (1) isomorphically as

$$H^\dagger \star \eta^2 = \eta^2 \star H. \tag{8}$$

With definition (5) this will then lead to a differential equation in η^2, whose order depends on the degree of x and p in H. In most cases this equation can not be solved exactly and under these circumstances we assume a perturbative expansion for η^2 in the coupling constant g, which was introduced in (3)

$$\eta^2(x,p) = \sum\nolimits_{n=0}^{\infty} g^n c_n(x,p). \tag{9}$$

Combining (8) and (9) then leads to a differential equation, which involves the functions c_n in a recursive manner and can therefore be solved order by order. Ultimately we aim at exact expressions, which yield η^2 to all orders in perturbation theory [13]. Having then solved various differential equations, we naturally expect some ambiguities in the general solutions, which mirror the possibility of different boundary conditions. We would like to stress that this is not a drawback, which is only present when using Moyal products, but it is rather a reflection of a general feature of perturbation theory. The same kind of ambiguity occurs in the perturbative approach based on commutators. In that context one may only fix the operator corresponding to the $c_n(x,p)$ up to any operator which commutes with the Hermitian part of H, that is h_0. This means that, in (2), the expressions are insensitive to any replacement $q \to q + \tilde{q}$ with $[\tilde{q}, h_0] = 0$. A further type of ambiguity, which is always present irrespective of an exact or perturbative treatment, is a multiplication of η^2 by operators which commute with H, i.e. we could re-define $\eta^2 \to \eta^2 Q$ for any Q, which satisfies $[Q, H] = 0$.

We may fix these ambiguities by invoking a further property of η. As discussed in [11], assuming a dependence on the coupling constant g of the form

$$\eta(-g) = \eta(g)^{-1}, \qquad h(g) = h(-g) \qquad \text{and} \qquad H^\dagger(g) = H(-g), \tag{10}$$

will guarantee the pseudo-Hermiticity relations (1). Therefore we require next

$$\eta^2(g) \star \eta^2(-g) = 1, \tag{11}$$

which may be solved systematically order by order when we already know the expansion (9) up to the ambiguities. Note that (11) is automatically satisfied if we take $\eta^2 = e^q$ and $q = \sum_{n=1}^{\infty} g^{2n-1} q_{2n-1}$ as assumed in many cases on the grounds of PT-invariance [6]. Having determined η^2 the transformation η is subsequently computed easily order by order from

$$\eta \star \eta = \eta^2 = \sum\nolimits_{n=0}^{\infty} g^n c_n(x,p), \tag{12}$$

when assuming a further power series expansion

$$\eta(x,p) = \sum\nolimits_{n=0}^{\infty} g^n q_n(x,p). \tag{13}$$

Finally, having obtained an explicit expression for η, we may compute the Hermitian counterpart to H from (1) as

$$h \star \eta = \eta \star H. \tag{14}$$

According to the above arguments, i.e. the second equation in (10), we should find an expression of the general form

$$h(x,p) = \sum\nolimits_{k=0}^{\infty} g^{2k} h_{2k}(x,p). \tag{15}$$

Let us now illustrate the above with two explicit examples, by starting with one for which the differential equation may be solved exactly and thereafter complicating it to a case which requires perturbation theory.

3 The non-Hermitian -x^4 potential

As a straightforward example we consider the non-Hermitian Hamiltonian

$$H(\hat{x},\hat{p}) = \hat{p}^2 - \frac{\hat{p}}{2} + \alpha\left(\hat{x}^2 - 1\right) + ig\left(\frac{\{\hat{x},\hat{p}^2\}}{2} - 2\,\alpha\hat{x}\right), \tag{16}$$

which results from $H = -d^2/dz^2 - \varepsilon z^4$ when using $z = -2i\sqrt{1+ix}$ as transformation, $\alpha = 16\varepsilon$ and the introduction of the coupling constant g to separate off the non-Hermitian part [23]. The exact similarity transformation η for this Hamiltonian was recently constructed by Jones and Mateo [23] using perturbation theory in terms of commutators in the spirit of equation (2). See also [23] for further reasoning on how the Hamiltonian (16) can be used to make sense of the $-\varepsilon z^4$-potential despite its unappealing property of being unbounded from below.

In order to illustrate the method let us see how to determine η by using Moyal products. First we notice that we wish to treat $H(\hat{x},\hat{p})$ as a real valued function and we therefore have to replace the anti-commutator with the appropriate Moyal products. When using definition (5) we have to replace $\{\hat{x},\hat{p}^2\}$ by $x\star p^2 + p^2\star x = 2xp^2$ and the differential equation (8) for the Hamiltonian (16) reads

$$\begin{aligned} 0 \;=\; & 4gp^2x\eta^2(x,p) - 8gx\alpha\eta^2(x,p) - 4x\alpha\partial_p\eta^2(x,p) \\ & -\partial_x\eta^2(x,p) + 4p\partial_x\eta^2(x,p) + 2gp\partial_p\partial_x\eta^2(x,p) - gx\partial_x^2\eta^2(x,p). \end{aligned} \tag{17}$$

As a comparison we also present the differential equation resulting from (8) when using the $*$-product instead of the $\star$-product. When converting the operator valued Hamiltonians into a function, we have to pay attention to the fact that $(f*g)^* \neq f^**g^*$. Thus in H we replace $i\{\hat{x},\hat{p}^2\}$ by $i(x*p^2+p^2*x) = 2ip^2x - 2p$, whereas in $H^\dagger$ we substitute $-i\{\hat{x},\hat{p}\}$ by $-i(x*p^2+p^2*x) = -2ip^2x + 2p$, which is of course not the same as converting first the anti-commutator with a subsequent conjugation. The resulting differential equation reads

$$\begin{aligned} 0 \;=\; & 4gp^2x\eta^2(x,p) - 8gx\alpha\eta^2(x,p) - 4x\alpha\partial_p\eta^2(x,p) \\ & +i2gp^2\partial_p\eta^2(x,p) - i2\alpha\partial_p^2\eta^2(x,p) - i4g\alpha\partial_p\eta^2(x,p) + i4gp\eta^2(x,p) \\ & -(1+2g-4p-4pxig)\partial_x\eta^2(x,p) + (2i-2gx)\partial_x^2\eta^2(x,p). \end{aligned} \tag{18}$$

Obviously, equation (18) is more complicated than (17), which illustrates our assertion that the $\star$-product is more advantageous than the $*$-product. Ultimately they should lead, however, to the same result. Indeed, each line in (17) as well as (18) vanishes separately for

$$\eta^2(x,p) = e^{\frac{g\,p^3}{3\,\alpha}-2\,g\,p}, \tag{19}$$

which, when compensating a slight difference in convention, is precisely the same expression as found in [23]. To find $\eta(x,p)$ from $\eta^2(x,p)$ by means of (12) is trivial in this case as we just have to take the square root, i.e. $\eta(x,p) = e^{\frac{g\,p^3}{6\,\alpha}-g\,p}$. Using (14) thereafter we find the Hermitian counterpart of H

$$h(x,p,g) = p^2 - \frac{p}{2} + \alpha\left(x^2-1\right) + g^2\frac{\left(p^2-2\,\alpha\right)^2}{4\,\alpha}. \tag{20}$$

When setting the artificially introduced parameter g to 1, we recover precisely the expression found in [23]

$$h(x,p,g=1) = \frac{p^4}{4\,\alpha} - \frac{p}{2} + x^2\,\alpha. \tag{21}$$

In [23] also the interesting massive case $H = -d^2/dz^2 + m^2z^2 - \varepsilon z^4$ has been discussed. The transformation from the z-variable to the x-variable will add in (16) a term $-m^2(1+4igx)$. The resulting similarity transformation is then $\eta(x,p) = e^{\frac{g\,p^3}{6\,\alpha}-g\,p(1+2m^2/\alpha)}$ and the Hermitian counterpart reads

$$h(x,p,g) = p^2 - \frac{p}{2} + \alpha\left(x^2-1\right) - 4m^2 + g^2\frac{\left(p^2-2\,\alpha-4\,m^2\right)^2}{4\,\alpha}, \tag{22}$$

which for $g \to 1$ reduces precisely to the expression reported in [23].

Let us next embark on an example for which the differential equation (17) can not be solved exactly.

4 Harmonic oscillator perturbed by a complex cubic potential

The prototype example for the study of non-Hermitian Hamiltonian systems is the harmonic oscillator perturbed with a complex cubic potential

$$H = \frac{p^2}{2} + \frac{x^2}{2} + igx^3. \tag{23}$$

It was the discovery [1] that this Hamiltonian possesses a positive real discrete spectrum, which led to the current interest in this subject. This Hamiltonian is obviously non-Hermitian, but PT-invariant and pseudo-Hermitian [5, 8, 7, 10]. The latter property means that the relations (1) hold. For the Hamiltonian (23) the

conjugation relation (8) translates into the differential equation[2])

$$0 = 4gx^3\eta^2(x,p) - 3gx\partial_p^2\eta^2(x,p) - 2x\partial_p\eta^2(x,p) + 2p\partial_x\eta^2(x,p). \qquad (24)$$

Using the expansion (12) we find the recursive equation

$$0 = 4x^3 c_{n-1}(x,p) - 3x\partial_p^2 c_{n-1}(x,p) - 2x\partial_p c_n(x,p) + 2p\partial_x c_n(x,p). \qquad (25)$$

Notice the occurrence of the aforementioned ambiguities. It is easy to see that we may add to $c_n(x,p)$ any arbitrary function $\zeta[(p^2+x^2)/2]$. This type of ambiguity just reflects the fact that in the perturbative formulation in terms of operators we may add to q any function of $h_0 = (p^2+x^2)/2$. We write $c_n(x,p) = \tilde{c}_n(x,p) + \zeta[(p^2+x^2)/2]$ and determine first the functions $\tilde{c}_n(x,p)$ which also obey (25).

Taking as the initial condition $\tilde{c}_0(x,p) = 1$, we find order by order

$$\begin{aligned}
\tilde{c}_1(x,p) &= \frac{4\,p^3}{3} + 2\,p\,x^2, \qquad \tilde{c}_2(x,p) = 6\,x^2 - \frac{2\,p^2\,x^4}{3} - \frac{8\,x^6}{9},\\
\tilde{c}_3(x,p) &= \frac{112p^5}{15} - \frac{64p^9}{81} + \frac{56p^3x^2}{3} - \frac{32p^7x^2}{9} + 14px^4 - \frac{56p^5x^4}{9} - \frac{140p^3x^6}{27}\\
&\quad - \frac{16px^8}{9},\\
\tilde{c}_4(x,p) &= 112p^2x^2 - \frac{128p^6x^2}{3} + 98x^4 - \frac{1856p^4x^4}{15} + \frac{32p^8x^4}{81} - \frac{5392p^2x^6}{45}\\
&\quad + \frac{416p^6x^6}{243} - \frac{1768x^8}{45} + \frac{230p^4x^8}{81} + \frac{176p^2x^{10}}{81} + \frac{160x^{12}}{243},\\
\tilde{c}_5(x,p) &= -224p^3 + \frac{2752p^7}{5} - \frac{19456p^{11}}{405} + \frac{2048p^{15}}{3645} - 336px^2 + \frac{9632p^5x^2}{5}\\
&\quad - \frac{107008p^9x^2}{405} + \frac{1024p^{13}x^2}{243} + 1928p^3x^4 - \frac{26752p^7x^4}{45} + \frac{3328p^{11}x^4}{243}\\
&\quad + \frac{8332px^6}{15} - \frac{272896p^5x^6}{405} + \frac{18304p^9x^6}{729} - \frac{154132p^3x^8}{405} + \frac{760p^7x^8}{27}\\
&\quad - \frac{11488px^{10}}{135} + \frac{23524p^5x^{10}}{1215} + \frac{5536p^3x^{12}}{729} + \frac{320px^{14}}{243},
\end{aligned}$$

This can be expanded effortlessly to higher orders, but we will not report these functions here. Next we demand the dependence on the coupling constant g to be of the form (11) in order to fix the ambiguities. Since we may add to $c_a(x,p)$ any arbitrary function $\zeta_0^a[h_0 = (p^2+x^2)/2]$, due to the recursive equation (25) this

[2]) For comparison definition (4) yields

$$\begin{aligned}
0 &= 4\,g\,x^3\,\eta^2(x,p) - 2\,x\,\partial_p\eta^2(x,p) + 6\,i\,g\,x^2\,\partial_p\eta^2(x,p) - i\partial_p^2\,\eta^2(x,p)\\
&\quad -6\,g\,x\,\partial_p^2\eta^2(x,p) - 2\,i\,g\,\partial_p^3\eta^2(x,p) + 2\,p\,\partial_x\eta^2(x,p) + i\,\partial_x^2\eta^2(x,p).
\end{aligned}$$

function will produce descendents at higher level in c_n for $n > a$. Thus we have the general form

$$c_n(x,p) = \tilde{c}_n(x,p) + \sum\nolimits_{k=0}^{n-a} \zeta_k^{(a)}[h_0]. \tag{26}$$

We can now use this function and achieve that (11) is satisfied order by order. Explicitly,

$$\begin{aligned} c_1(x,p) &= \frac{4p^3}{3} + 2px^2, \\ c_2(x,p) &= \tilde{c}_2(x,p) + \zeta_0^{(2)}[h_0] = \tilde{c}_2(x,p) + \frac{8}{9}h_0^3 - 4h_0. \end{aligned}$$

As we note, indeed the additional function just depends on h_0. In a similar fashion we can compute the higher order functions

$$\begin{aligned} c_3(x,p) &= 12p - \frac{248p^5}{15} + \frac{32p^9}{81} - \frac{64p^3x^2}{3} + \frac{16p^7x^2}{9} - 2px^4 + \frac{8p^5x^4}{3} + \frac{4p^3x^6}{3}, \\ c_4(x,p) &= 152p^4 - \frac{832p^8}{45} + \frac{32p^{12}}{243} + (56p^2 - \frac{2368p^6}{45} + \frac{64p^{10}}{81})x^2 \\ &\quad -(26 + \frac{128p^4}{3} - \frac{16p^8}{9})x^4 - (8p^2 - \frac{16p^6}{9})x^6 + \frac{2p^4x^8}{3}, \\ c_5(x,p) &= -1024p^3 + \frac{2144p^7}{5} - \frac{4672p^{11}}{405} + \frac{128p^{15}}{3645} + (-168p + \frac{10864p^5}{15} \\ &\quad -\frac{20416p^9}{405} + \frac{64p^{13}}{243})x^2 + (\frac{712p^3}{3} - \frac{3488p^7}{45} + \frac{64p^{11}}{81})x^4 \\ &\quad -(28p + 48p^5 - \frac{32p^9}{27})x^6 - (\frac{28p^3}{3} - \frac{8p^7}{9})x^8 + \frac{4p^5x^{10}}{15}, \\ c_6(x,p) &= 3584p^2 - \frac{98336p^6}{15} + \frac{340256p^{10}}{675} - \frac{6016p^{14}}{1215} + \frac{256p^{18}}{32805} - (1024 \\ &\quad +6160p^4 - \frac{70592p^8}{45} + \frac{35392p^{12}}{1215} - \frac{256p^{16}}{3645})x^2 - (216p^2 - \frac{70288p^6}{45} \\ &\quad +\frac{26816p^{10}}{405} - \frac{64p^{14}}{243})x^4 + (340 + \frac{1520p^4}{3} - \frac{3232p^8}{45} + \frac{128p^{12}}{243})x^6 \\ &\quad +(14p^2 - \frac{328p^6}{9} + \frac{16p^{10}}{27})x^8 - (\frac{20p^4}{3} - \frac{16p^8}{45})x^{10} + \frac{4p^6x^{12}}{45}, \end{aligned}$$

We observed that always $\zeta_0^{(2n+1)}[h_0] = 0$. Making next the ansatz (13), we compute the $q_n(x,p)$ by reading off the corresponding powers in the expression (12). We find order by order

$$\begin{aligned} q_1(x,p) &= \frac{2\,p^3}{3} + px^2, \quad q_2(x,p) = -p^2 + \frac{2p^6}{9} + \frac{x^2}{2} + \frac{2p^4x^2}{3} + \frac{p^2x^4}{2}, \\ q_3(x,p) &= 6p - \frac{79p^5}{15} + \frac{4p^9}{81} - \frac{23p^3x^2}{3} + \frac{2p^7x^2}{9} - \frac{13px^4}{4} + \frac{p^5x^4}{3} + \frac{p^3x^6}{6}, \\ q_4(x,p) &= \frac{67p^4}{2} - \frac{148p^8}{45} + \frac{2p^{12}}{243} + (\frac{37p^2}{2} - \frac{442p^6}{45} + \frac{4p^{10}}{81})x^2 \end{aligned}$$

$$
\begin{aligned}
& -(\frac{61}{8}+\frac{29p^4}{3}-\frac{p^8}{9})x^4-(\frac{7p^2}{2}-\frac{p^6}{9})x^6+\frac{p^4x^8}{24}, \\
q_5(x,p) &= \frac{-997p^3}{3}+\frac{443p^7}{5}-\frac{434p^{11}}{405}+\frac{4p^{15}}{3645}-\frac{355px^2}{2}+\frac{4681p^5x^2}{30} \\
& -\frac{1952p^9x^2}{405}+\frac{2p^{13}x^2}{243}+\frac{1199p^3x^4}{12}-\frac{361p^7x^4}{45}+\frac{2p^{11}x^4}{81} \\
& +\frac{149px^6}{8}-6p^5x^6+\frac{p^9x^6}{27}-\frac{43p^3x^8}{24}+\frac{p^7x^8}{36}+\frac{p^5x^{10}}{120}, \\
q_6(x,p) &= \frac{1677p^2}{2}-\frac{103649p^6}{90}+\frac{84373p^{10}}{1350}-\frac{286p^{14}}{1215}+\frac{4p^{18}}{32805}-(\frac{1131}{4} \\
& +\frac{16811p^4}{12}-\frac{8987p^8}{45}+\frac{1717p^{12}}{1215}-\frac{4p^{16}}{3645})x^2-(\frac{2503p^2}{8}-\frac{43021p^6}{180} \\
& +\frac{1361p^{10}}{405}-\frac{p^{14}}{243})x^4+(\frac{1861}{16}+\frac{1625p^4}{12}-\frac{359p^8}{90}+\frac{2p^{12}}{243})x^6 \\
& +(\frac{943p^2}{32}-\frac{173p^6}{72}+\frac{p^{10}}{108})x^8-(\frac{29p^4}{48}-\frac{p^8}{180})x^{10}+\frac{p^6x^{12}}{720}
\end{aligned}
$$

Finally, having obtained an explicit expression for the η, we may compute the Hermitian counterpart to H from (14). We find order by order

$$
\begin{aligned}
h_0(\hat{x},\hat{p}) &= \frac{\hat{p}^2}{2}+\frac{\hat{x}^2}{2}, \quad h_2(\hat{x},\hat{p})=-\frac{1}{2}+3\hat{S}_{2,2}+\frac{3\,\hat{x}^4}{2} \\
h_4(\hat{x},\hat{p}) &= 27\hat{p}^2+2\hat{p}^6+\frac{15\hat{x}^2}{2}-36\hat{S}_{4,2}-\frac{51\hat{S}_{2,4}}{2}-\frac{7\,\hat{x}^6}{2} \\
h_6(\hat{x},\hat{p}) &= 128-984\hat{p}^4-72\hat{p}^8-1464\hat{S}_{2,2}+768\hat{S}_{6,2}-84\,\hat{x}^4+660\hat{S}_{4,4} \\
& \quad +288\hat{S}_{2,6}+\frac{27\hat{x}^8}{2},
\end{aligned}
\tag{27}
$$

where we have already converted from real valued functions to operators. Notice that $h(\hat{x},\hat{p})$ is of the form (15) and as we expect all terms $h_{2k-1}=0$, such that indeed the second relation in (10) holds. Up to $h_4(\hat{x},\hat{p})$ our results agree precisely with [24]. Higher orders do not seem to appear in the literature.

5 Conclusions

We have shown that Moyal products can be used as a very powerful tool to compute the similarity transformation η relating a pseudo-Hermitian Hamiltonian H to its Hermitian counterpart h. As concrete examples, we have applied such products to a situation for which η can be calculated exactly, namely the non-Hermitian $-x^4$ potential, and to a case for which it can only be computed perturbatively, that is the harmonic oscillator with a complex cubic perturbation. In the latter case, we profited considerably from the fact that the Moyal products involve only real-valued functions, instead of commutators, and lead to differential equations which can be solved recursively. Relation (11) was crucial to fix the ambiguities, which result from operators which commute with the unperturbed Hamiltonian h_0.

In a more general context, this is a far less cumbersome procedure than those involving commutators, and allows one to evaluate isospectral pseudo-Hermitian–Hermitian pairs perturbatively up to a very high order. Even more, Moyal products can be employed to obtain closed formula for the similarity transformation η to all orders in perturbation theory. Such formulae have been obtained for several specific isospectral pairs, and will be discussed in a subsequent publication [13].

Despite the practical use of this approach, there are clear limitations, such as for instance when the potential is of a nonpolynomial nature. In that case the sum in the definition for the Moyal product does not terminate and one has to deal with differential equations of infinite order. Hence, there is a clear need for further alternative methods, such as recently proposed in [25].

Acknowledgments. We are grateful to Hugh Jones for useful comments.

References

[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).

[2] M. Znojil, Phys. Lett. **A259**, 220 (1999); Phys. Lett. **A264**, 108 (1999).

[3] B. Bagchi, C. Quesne and M. Znojil, Mod. Phys. Lett. **A16** 2047 (2001).

[4] S. Weigert, Phys. Rev. **A68**, 062111(4) (2003).

[5] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401(4) (2002).

[6] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. **D70**, 025001(19) (2004).

[7] A. Mostafazadeh, J. Math. Phys. **43**, 205 (2002); J. Math. Phys. **43**, 2814 (2002); J. Math. Phys. **43**, 3944 (2002); J. Phys. **A36**, 7081 (2003).

[8] S. Weigert, J. Phys. **B5**, S416 (2003).

[9] F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. **213**, 74 (1992).

[10] A. Mostafazadeh, J. Phys. **A38**, 6557 (2005).

[11] C. Figueira de Morisson Faria and A. Fring, J. Phys. **A39**, 9269 (2006).

[12] F. G. Scholtz and H. B. Geyer, Phys. Lett. **B634**, 84 (2006); quant-ph/0602187.

[13] C. Figueira de Morisson Faria and A. Fring, in preparation.

[14] J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1949).

[15] D. B. Fairlie, Mod. Phys. Lett. **A13**, 263 (1998).

[16] N. Seiberg and E. Witten, JHEP **09**, 032 (1999).

[17] A. Dimakis and F. Müller-Hoissen, Int. J. Mod. Phys. **B14**, 2455 (2000).

[18] M. T. Grisaru and S. Penati, Nucl. Phys. **B655**, 250 (2003).

[19] I. Cabrera-Carnero and M. Moriconi, Nucl. Phys. **B673**, 437 (2003).

[20] O. Lechtenfeld, L. Mazzanti, S. Penati, A. D. Popov, and L. Tamassia, Nucl. Phys. **B705**, 477 (2005).

[21] D. B. Fairlie, J. of Chaos, Solitons and Fractals **10**, 365 (1999).

[22] R. Carroll, North-Holland Mathematics Studies **186**, (Elsevier, Amsterdam) (2000).

[23] H. F. Jones and J. Mateo, Phys. Rev. **D73**, 085002(2006).

[24] H. Jones, J. Phys. **A38**, 1741 (2005).

[25] A. Mostafazadeh, quant-ph/0603023 (2006).