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Integrable models with unstable particles

O.A. Castro-Alvaredo and A. Fring

*Institut für Theoretische Physik, Freie Universität Berlin,
Arnimallee 14, D-14195 Berlin, Germany
E-mail: Olalla/Fring@physik.fu-berlin.de*

ABSTRACT: We review some recent results concerning integrable quantum field theories in 1+1 space-time dimensions which contain unstable particles in their spectrum. Recalling first the main features of analytic scattering theories associated to integrable models, we subsequently propose a new bootstrap principle which allows for the construction of particle spectra involving unstable as well as stable particles. We describe the general Lie algebraic structure which underlies theories with unstable particles and formulate a decoupling rule, which predicts the renormalization group flow in dependence of the relative ordering of the resonance parameters. We extend these ideas to theories with an infinite spectrum of unstable particles. We provide new expressions for the scattering amplitudes in the soliton-antisoliton sector of the elliptic sine-Gordon model in terms of infinite products of q -deformed gamma functions. When relaxing the usual restriction on the coupling constants, the model contains additional bound states which admit an interpretation as breathers. For that situation we compute the complete S-matrix of all sectors. We carry out various reductions of the model, one of them leading to a new type of theory, namely an elliptic version of the minimal $SO(n)$ -affine Toda field theory.

1. Introduction

The structure of integrable quantum field theories (IQFT) in 1+1 space-time dimensions has been unravelled to a very large extend. Many theories can be solved even exactly, that is to all orders in perturbation theory, in this context. However, the large majority of investigations concentrates on theories which involve exclusively stable particles, despite the fact that in nature most particles are unstable. Since of course one of the motivations to study IQFT is to reproduce realistic features, there is an apparent need to investigate also theories which have unstable particles in their spectrum. The aim of this talk is to review some recent results which deal with such theories.

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2. Analytic scattering theory of factorizable integrable models

Since not all participants of this conference work directly on integrable quantum field theories, we briefly recall some well known facts on analytic scattering theories in 1+1 space-time dimensions. Having in mind to emphasize features related to unstable particles this will also be useful to the experts. As a starting point in every scattering theory one requires a complete set of asymptotic in and out states ($t \rightarrow \pm\infty$). These states consist of operators $Z_\mu(p)$ acting on the vacuum $|0\rangle$ and creating in this way a stable particle of the type μ with momentum p . Already at this point enters the fundamental difference between stable and unstable particles. Even though experimentally unstable particles with a very long lifetime can very often not be distinguished from stable ones, mathematically they are very distinct. They can never be associated to an asymptotic state, even when they have an extremely long lifetime, as by their very nature they will have decayed in the infinite future or were never produced in the infinite past. Then the scattering matrix is defined to be the operator which relates a stable n-particle in state to a stable m-particle out state

$$Z_{\mu_m}(\theta'_m) \dots Z_{\mu_1}(\theta'_1) |0\rangle_{\text{out}} = S_{\mu_1\mu_2\dots\mu_m}^{\mu_1\mu_2\dots\mu_n}(\theta'_1, \dots, \theta_n) Z_{\mu_1}(\theta_1) \dots Z_{\mu_n}(\theta_n) |0\rangle_{\text{in}}. \quad (2.1)$$

Conveniently one parameterizes the two-momentum by the rapidity θ as $\vec{p} = m(\cosh \theta, \sinh \theta)$. Now there are some very special features happening in integrable (that means here there exists at least one non-trivial conserved charge) quantum field theories in 1+1 dimensions [1, 2, 3, 4, 5]. There is no particle production and furthermore the incoming and outgoing momenta coincide

$$\{\theta'_1, \theta'_2, \dots, \theta'_m\} = \{\theta_1, \theta_2, \dots, \theta_n\} \quad \text{with } n = m. \quad (2.2)$$

In addition, the n-particle S-matrix factorizes into a set of two-particle S-matrices

$$S_{\mu_1\mu_2\dots\mu_m}^{\mu_1\mu_2\dots\mu_n}(\theta'_1, \dots, \theta_n) = \prod_{1 \leq i < j \leq n} S_{\mu_i\mu_j}(\theta_i, \theta_j). \quad (2.3)$$

Obviously, this is a considerable simplification in comparison with the general situation (2.1), as it means that once we know the two-particle S -matrix, we control the entire scattering matrix. Because of this fact, we refer from now on to the two-particle scattering matrix as *the* S -matrix.

How does one construct this S -matrix? In general one is limited to the use of perturbation theory in the coupling constant. In particular in higher dimensions that is essentially the only method available. In contrast, two dimensions are very special as they miraculously allow to determine S exactly to all orders in perturbation theory. This is one of the major successes of this area of research and one of the reasons for the continued interest in such theories. The original ideas which lead to explicit expressions for S go back to what is called the bootstrap approach [6, 1, 2]. It consists of using various properties for the scattering matrix, which one motivates by some physical principles in order to set up an axiomatic system for S in the hope that it might be so constraining that it determines S completely. Indeed, these hopes are not in vain.

We recall the S -matrix properties:

i) Lorentz invariance

Dealing with relativistic scattering theories, we expect the scattering matrix to be Lorentz invariant, i.e. it should depend only on covariant combinations of the momenta. The Mandelstam variables are precisely such quantities, see e.g. [7]. In 1+1 dimensions only one of them is independent, usually taken to be $s_{ab} = (p_a + p_b)^2 = m_a^2 + m_b^2 + 2m_a m_b \cosh(\theta_a - \theta_b)$. Hence, Lorentz invariance is simply guaranteed when S depends either only on s_{ab} or the rapidity difference $\theta_{ab} := \theta_a - \theta_b$

$$S_{ab}(p_a, p_b) = S_{ab}(\theta_a, \theta_b) = S_{ab}(s_{ab}) = S_{ab}(\theta_{ab}). \quad (2.4)$$

Since s_{ab} admits the interpretation as the total energy in the centre of mass system, θ_{ab} has to be real for a physical process, such that $s_{ab} \geq (m_a + m_b)^2$.

ii) Hermitian analyticity

As a central assumption of analytic S -matrix theory [7] one assumes that the S -matrix can be continued to the complex plane and depends on $s_{ab}, \theta_{ab} \in \mathbb{C}$. Physical scattering amplitudes are then assumed to be real boundary values of analytic functions, which can be obtained from a generalization of Feynman's $i\varepsilon$ prescription of perturbation theory

$$S_{ab}^{\text{physical}} = \lim_{\varepsilon \rightarrow 0} S_{ab}(s + i\varepsilon) = S_{ab}(\theta) \quad s \in \mathbb{R}, \varepsilon, \theta \in \mathbb{R}^+. \quad (2.5)$$

The choice of the signs is important and relates to causality. Since a two-particle wavefunction, having here plane waves in mind modulated by some enveloping function, will depend on the sum of the momenta, i.e. on $\sqrt{s_{ab}}$, one has lost the single valuedness of the scattering matrix by an analytic continuation. This is remedied by branch cuts along the real axis at $s_{ab} \geq (m_a + m_b)^2$ and $s_{ab} \leq (m_a - m_b)^2$, the latter being motivated by crossing (see iv). Hermitian analyticity is now a postulate which states how to continue over these cuts [8, 9]

$$\lim_{\varepsilon \rightarrow 0} S_{ab}(s + i\varepsilon) = \lim_{\varepsilon \rightarrow 0} S_{ab}(s - i\varepsilon) \quad \Leftrightarrow \quad S_{ab}(\theta) = [S_{ba}(-\theta^*)]^* . \quad (2.6)$$

once more for $s \in \mathbb{R}, \varepsilon, \theta \in \mathbb{R}^+$. The equivalence is due to the fact that the analytic continuation $s + i\varepsilon \leftrightarrow s - i\varepsilon$ corresponds to $\theta \leftrightarrow -\theta$. Often one merely uses real analyticity $S_{ab}(\theta) = [S_{ab}(-\theta^*)]^*$ instead of (2.6), which only coincides when $S_{ab} = S_{ba}$, that is in parity invariant theories. This difference is very important with regard to the theories consider below, which involve unstable particles as they unavoidably break parity invariance. Further support for (2.6) comes from perturbation theory [8], general considerations in analytic S -matrix theory [10, 7] and explicitly constructed examples.

iii) Unitarity

Assuming that the states in (2.1) are complete and orthogonal, the operator which maps them to each other has to be unitary

$$SS^\dagger = S^\dagger S = 1 . \quad (2.7)$$

The combination of (2.6) and (2.7) leads to the simpler relation $S_{ab}(\theta)S_{ba}(-\theta) = 1$, which may also be derived from applying twice the Zamolodchikov algebra $Z_a(\theta_1)Z_b(\theta_2) = S_{ab}(\theta_{12})Z_b(\theta_2)Z_a(\theta_1)$.

iv) Crossing symmetry

Crossing symmetry can be motivated by the Lehmann-Symanzik-Zimmermann (LSZ) formalism [11] and consists of the replacement of an incoming particle a by its anti-particle \bar{a} with reversed momentum. A discussion of the anti-particle theorem can be found in [10]. The prescription amounts to the continuation of the Mandelstam variable s_{ab} to the variable $t_{ab} = (p_a - p_b)^2$

$$\lim_{\varepsilon \rightarrow 0} S_{ab}(s + i\varepsilon) = \lim_{\varepsilon \rightarrow 0} S_{b\bar{a}}(t - i\varepsilon) \quad \Leftrightarrow \quad S_{b\bar{a}}(\theta) = S_{ab}(i\pi - \theta) . \quad (2.8)$$

It is easy to check that the analytic continuation $s + i\varepsilon \leftrightarrow t - i\varepsilon$ corresponds to $\theta \leftrightarrow i\pi - \theta$.

v) Yang-Baxter equation

In (2.3) we already indicated that the conserved charge(s) of an integrable theory can be used to disentangle an n -particle scattering process into a consecutive scattering of two particles only. An additional consequence of this argument is that the order in which this takes place does not matter, such that two different orderings are taken to be equivalent. As in general the S -matrices do not commute, this leads to a new constraint. In other words this amounts to say that the operators Z in (2.1) obey an associative algebra. As a result of this one obtains the Yang-Baxter equation [12, 13]

$$S(\theta_{12}) \otimes S(\theta_{13}) \otimes S(\theta_{23}) = S(\theta_{23}) \otimes S(\theta_{13}) \otimes S(\theta_{12}) . \quad (2.9)$$

For diagonal theories, i.e. when backscattering is absent, we simply have $S_{ab}^{cd}(\theta) \rightarrow S_{ab}(\theta)$ such that (2.9) is trivially satisfied.

vi) Fusing bootstrap equation

By the same reasoning as in v) integrability, i.e. factorizability, yields a further constraining equation, when two particles are allowed (what that means is discussed in vii)) to form bound state. For instance, the particles a, b fuse to a third particle \bar{c} , i.e. $a + b \rightarrow \bar{c}$. One makes now a further assumption, sometimes referred to as nuclear democracy, namely that also the particle of type \bar{c} exists asymptotically. Then, by integrability, it is equivalence if a third particle, say l , scatters with the bound state \bar{c} or consecutively with the two particles a, b . For S this reads

$$S_{l\bar{c}}(\theta) = S_{la}(\theta + i\bar{\eta}_{ac}^b) S_{lb}(\theta - i\bar{\eta}_{bc}^a) . \quad (2.10)$$

The $\bar{\eta}_{ac}^b \in \mathbb{R}^+$ are the fusing angles specific to the individual theory considered. It is clear that the assumption of nuclear democracy does not hold if \bar{c} is an unstable particle, such that (2.10) can not be valid in the form stated for that case. We will now indicate the origin for the possibility to form bound states, which is the

vii) Pole structure

In general, the S -matrix can have a quite intricate singularity structure consisting of poles of finite order distributed all over the complex s, θ -plane. A strong further constraint is to assume that all singularities which emerge in S admit a consistent explanation. As a slightly weaker assumption one could suppose that all explainable poles form a coherent system, in the sense that the bootstrap (2.10) closes etc., and allow some redundant poles.

Single order poles are most important as they determine the particle spectrum of the theory. In the s -plane they might be on the real axis between the two branch cuts at $s = m_{\bar{c}}^2$, interpreted as an on-shell bound state particle, or in the second Riemann sheet at $s = (m_{\bar{c}} - i\Gamma_{\bar{c}}/2)^2$ corresponding to an unstable particle with finite lifetime $\tau = 1/\Gamma_{\bar{c}}$. The discussion is more conveniently carried out in the θ -plane, since $S(\theta)$ is a meromorphic function unlike $S(s)$. Near the singularity S has to be of the form

$$S_{ab}(\theta) \sim \frac{iR_{ab}^c}{(\theta - i\eta_{ab}^c + \sigma_{ab}^c)} . \quad (2.11)$$

Depending on the location and signs of the residues we have the following interpretations

$$\begin{aligned} \text{s-channel bound state: } & R_{ab}^c \in \mathbb{R}^+, \eta_{ab}^c \in \mathbb{R}^+, \sigma_{ab}^c = 0 \\ \text{t-channel bound state: } & R_{ab}^c \in \mathbb{R}^-, \eta_{ab}^c \in \mathbb{R}^+, \sigma_{ab}^c = 0 \\ \text{unstable particle: } & R_{ab}^c \in \mathbb{R}, \eta_{ab}^c \in \mathbb{R}^-, \sigma_{ab}^c \in \mathbb{R}^- \end{aligned}$$

The relation between the poles in the s and θ planes are the Breit-Wigner (BW) equations [14]

$$m_{\bar{c}}^2 - \frac{\Gamma_{\bar{c}}^2}{4} = m_a^2 + m_b^2 + 2m_a m_b \cosh \sigma_{ab}^{\bar{c}} \cos \eta_{ab}^{\bar{c}} \quad (2.12)$$

$$m_{\bar{c}} \Gamma_{\bar{c}} = 2m_a m_b \sinh \sigma_{ab}^{\bar{c}} \sin \eta_{ab}^{\bar{c}} , \quad (2.13)$$

which allow to express the mass $m_{\bar{c}}$ and decay width $\Gamma_{\bar{c}}$ of the unstable particle as functions of $m_a, m_b, \eta_{ab}^{\bar{c}}, \sigma_{ab}^{\bar{c}}$. For the stable particle formation we have the following relation between the fusing angles in (2.10) and the poles in (2.11): $\bar{\eta}_{ac}^b = \pi - \eta_{ab}^c$. Note further that for the unstable particle formation in (2.11) we made the definite choice that the unstable particle \bar{c} is formed in the process

$$a + b \rightarrow \bar{c} \quad (2.14)$$

rather than in the not equivalent one $b + a$. It is clear that parity has to be broken, as with the choice $\eta_{ab}^c, \sigma_{ab}^c \in \mathbb{R}^-$ the amplitude $S_{b\bar{a}}(\theta)$ will have a pole at $i\pi - i\eta_{ab}^c + \sigma_{ab}^c$, leaving in (2.13) the choice that either $m_{\bar{c}} < 0$ or $\Gamma_{\bar{c}} < 0$, which is of course both non-physical.

Below we will be particularly interested in the situation for large resonance parameters $\sigma_{ab}^{\bar{c}}$, when the mass of the unstable particles can be approximated as

$$m_{\bar{c}} \sim \sqrt{m_a m_b} e^{-\sigma_{ab}^{\bar{c}}/2} . \quad (2.15)$$

In terms of perturbation theory in a coupling constant β , the pole (2.11) would be of second order, i.e. $R_{ab}^c(\beta^2)$, corresponding to a tree diagram. Similarly, higher order poles admit interpretations in form of more complicated singular Feynman diagrams. In some simple theories, such as for example sine-Gordon, the highest order of the poles is two. In that context it was suggested [15] that such type of poles are of order β^4 and may be viewed as box diagrams. For quite some time higher order poles were ignored and also here we will not enter into a deeper discussion of them, which can be found for instance in [16, 17]. From what is said it is clear that such poles will not alter the particle spectrum. Nonetheless, one should be able to draw the relevant Feynman diagrams, which means one

needs certain three-point couplings to be non-vanishing. Consequently this is a constraint on the existence of certain three point couplings R_{ab}^c .

Remarkably, *the constraints i)-vii) allow to determine the S-matrix exactly, that is to all orders in perturbation theory.* However, one should say that the solution constructed this way is not unique, as there exists always the possibility to multiply with so-called CDD-factors [18]. To fix them requires additional arguments beyond the scheme outlined above, such as ultraviolet limits, certain inputs from Lagrangians, etc.

2.1 A proposal for a construction principle of unstable particle spectra

We have seen in the previous section, that there exists a powerful construction principle for the spectrum of stable particles, consisting of solving the equations (axioms) i)-vii). For unstable particles we do not have yet such a construction tool, as by now they emerge rather passively as poles in the unphysical sheet as by-products in the scattering process of two stable particles. Furthermore, a description of the scattering process of an unstable particle with another stable or unstable particle is entirely missing in this context. Obviously, scattering processes involving unstable particles do occur in nature, such that the quest for a proper prescription is of physical relevance. In addition, one aims of course always at a description which has predictive power.

From what has been said, it is clear that such a description can not be a scattering theory in the usual sense, since for that one requires the particles involved to exist asymptotically. Any unstable particle will vanish in this limit rendering such formulation meaningless at first sight. Nonetheless, some particles have extremely long lifetimes, and seem to exist quasi infinitely long from an experimentalists point of view. It appears therefore natural to seek a principle closely related to the conventional bootstrap for stable particles. Inspired by this we proposed [19] the following construction principle:

Let us assume that in the time interval $0 < t < \tau_{\bar{c}}$ we can formally associate to the unstable particle some creation operator $\tilde{Z}_{\bar{c}}^\dagger(\theta)$, with $\lim_{t \rightarrow \infty} \tilde{Z}_{\bar{c}}^\dagger(\theta) = 1$ if $\tau_{\bar{c}} < \infty$. It is clear that these operators do not exist asymptotically, but for the stated time interval they can mimic an asymptotic state. Let us now further suppose that these operators satisfy a Zamolodchikov algebra

$$Z_a(\theta_1)\tilde{Z}_b(\theta_2) = \tilde{S}_{ab}(\theta_{12})\tilde{Z}_b(\theta_2)Z_a(\theta_1) \quad (2.16)$$

$$\tilde{Z}_a(\theta_1)\tilde{Z}_b(\theta_2) = \tilde{S}_{ab}(\theta_{12})\tilde{Z}_b(\theta_2)\tilde{Z}_a(\theta_1) \quad (2.17)$$

which can be used to generate an S-matrix type of amplitude \tilde{S}_{ab} , describing the scattering of one unstable particle with a stable one (2.16) or the scattering of two unstable particles (2.17). We may proceed as before and ask which type of properties might be satisfied for \tilde{S} .

i) Lorentz invariance

As already indicated in (2.16), (2.17) it is natural to expect Lorentz invariance also for this amplitude such that \tilde{S} depends only the rapidity difference θ_{ab}

$$\tilde{S}_{ab}(p_a, p_b) = \tilde{S}_{ab}(\theta_a, \theta_b) = \tilde{S}_{ab}(s_{ab}) = \tilde{S}_{ab}(\theta_{ab}). \quad (2.18)$$

ii,iii) Hermitian analyticity, unitarity

We will not make any assumption on hermitian analyticity here and in fact we do not expect unitarity to hold, since the states formed with the \tilde{Z} are not complete. However, applying (2.16) or (2.17) twice yields

$$\tilde{S}_{ab}(\theta)\tilde{S}_{ba}(-\theta) = 1, \quad (2.19)$$

which also holds for S , derivable from combining (2.6) and (2.7) in that case. In fact, also for the construction of S it is really only the corresponding equation to (2.19) which is employed, rather than individually (2.6) and (2.7).

iv) Crossing symmetry

The validity of crossing can also be argued as before, but now we have to continue as

$$\lim_{\varepsilon \rightarrow 0} \tilde{S}_{ab}(s - i\varepsilon) = \lim_{\varepsilon \rightarrow 0} \tilde{S}_{b\bar{a}}(t + i\varepsilon) \Leftrightarrow \tilde{S}_{b\bar{a}}(-\theta) = \tilde{S}_{ab}(i\pi + \theta), \quad (2.20)$$

which in the θ -plane amounts to the same equation as the one for S .

v) Yang-Baxter equation

Supposing the algebra related to (2.16), (2.17) is associative we have by the same reasoning as for stable particles the Yang-Baxter equation

$$\tilde{S}(\theta_{12}) \otimes \tilde{S}(\theta_{13}) \otimes \tilde{S}(\theta_{23}) = \tilde{S}(\theta_{23}) \otimes \tilde{S}(\theta_{13}) \otimes \tilde{S}(\theta_{12}). \quad (2.21)$$

vi) Fusing bootstrap equation

We commence with the fusing of two stable particles to create an unstable particle as in the process (2.14). To this process we can associate bootstrap equations almost in the usual way. We scatter for this with an additional stable or unstable particle, say of type l , and obtain the \tilde{S} bootstrap equations

$$\tilde{S}_{la}(\theta - \bar{\gamma}_{ca}^b) \tilde{S}_{lb}(\theta + \bar{\gamma}_{bc}^a) = \tilde{S}_{l\bar{c}}(\theta), \quad (2.22)$$

where $\bar{\gamma} = \pm i\pi - \gamma$, $\gamma = i\eta - \sigma$ and also $\bar{\gamma} \rightarrow -\bar{\gamma}$ is not a symmetry. The angles should be measured anti-clockwise, which explains the signs. We also note that we do not assume parity invariance, such that in general $\bar{\gamma}_{ba}^c \neq \bar{\gamma}_{ab}^c$. With the help of (2.19), (2.20) one derives the bootstrap equations for the opposite parity and the ones for the crossed processes $a + c \rightarrow \bar{b}$ and $b + c \rightarrow \bar{a}$ and from (2.22)

$$\tilde{S}_{\bar{a}l}(\theta) = \tilde{S}_{al}(\theta + \bar{\gamma}_{ca}^b) \tilde{S}_{bl}(\theta - \bar{\gamma}_{bc}^a), \quad (2.23)$$

$$\tilde{S}_{l\bar{b}}(\theta) = \tilde{S}_{lc}(\theta - \bar{\gamma}_{bc}^a) \tilde{S}_{la}(\theta \pm i\pi - \bar{\gamma}_{ca}^b - \bar{\gamma}_{bc}^a), \quad (2.24)$$

$$\tilde{S}_{l\bar{c}}(\theta) = \tilde{S}_{lc}(\theta + \bar{\gamma}_{ca}^b) \tilde{S}_{lb}(\theta \pm i\pi + \bar{\gamma}_{ca}^b + \bar{\gamma}_{bc}^a). \quad (2.25)$$

From the crossing relation for the “scattering matrix” and (2.24) or (2.25) one obtains some relations between the various fusing angles

$$\bar{\gamma}_{ab}^c + \bar{\gamma}_{ca}^b + \bar{\gamma}_{bc}^a = \pm i\pi. \quad (2.26)$$

At first sight this looks very much like the usual bootstrap prescription, but there are some differences. As is clear from the scattering process of two stable particles producing

an unstable one, the angle $\bar{\gamma}_{ab}^c$ is not purely complex any longer as it is for the situation when exclusively stable particles scatter. As a consequence, this property then extends to the other angles $\bar{\gamma}_{ca}^b$ and $\bar{\gamma}_{bc}^a$ in (2.22), which also possess some non-vanishing real parts. Note that (2.26) implies that the real parts of the three angles involved add up to zero. At this point we do not have an entirely compelling reason for demanding that, but this formulation will turn out to work well.

Of course the above equations are only a proposal, which needs to be put on more solid ground. Nonetheless, at this point our proposal gains support from self-consistency and its predictive power, which may be double checked: a) The bootstrap closes consistently for many non-trivial examples, which we calculated. As for stable particles this is never guaranteed and by no means self-evident. b) The bootstrap yields the amount of unstable particles together with their mass. This prediction can be used to explain a mass degeneracy of some unstable particles which can not be seen in a thermodynamic Bethe ansatz (TBA) analysis for the concrete example of the homogeneous sine-Gordon (HSG) models, see below. c) The bootstrap is in agreement with a general Lie algebraic decoupling rule, which we also present below, describing the behaviour when certain resonance parameters tend to infinity. d) The bootstrap yields the three-point couplings of all possible interactions, that is, involving stable as well as unstable particles.

2.2 An example: The \mathfrak{g}_k -HSG model

The \mathfrak{g}_k -homogeneous sine-Gordon models (HSG) [20, 21], with \mathfrak{g} being a simple Lie algebra of rank ℓ and level k , will be our standard example in what follows. In fact they have been the first models with a well defined Lagrangian containing unstable particles which have been the subject of a systematic analysis [22, 23, 24, 25, 26, 27, 28, 19, 29]. They can be viewed as perturbed conformal field theories (CFTs)¹

$$\mathcal{H}_{G_k\text{-HSG}} = \mathcal{H}_{G_k/U(1)^\ell\text{-CFT}} - \lambda \int d^2x \phi(x, t). \quad (2.27)$$

The underlying ultraviolet CFT is a Wess-Zumino-Novikov-Witten- $G_k/U(1)^\ell$ -coset theory [31, 32]. The corresponding Virasoro central charge c is computed with standard arguments of [32] and the perturbing operator ϕ is identified with a primary field of conformal dimensions $\Delta, \bar{\Delta}$. One finds

$$c = \ell \frac{k h - h^\vee}{k + h^\vee} \quad \text{and} \quad \Delta = \bar{\Delta} = \frac{h^\vee}{k + h^\vee}. \quad (2.28)$$

Here $(h^\vee) h$ is the (dual) Coxeter number of \mathfrak{g} . For simplicity we will drop in the following the explicit mentioning of the subalgebra $U(1)^\ell$ which were indicated in (2.27).

The scattering matrix for \mathfrak{g}_k -HSG-models with \mathfrak{g} simply laced algebras was constructed in [22]. For $k = 2$ it can be brought into the simple form

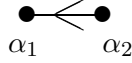
$$S_{ij}(\theta, \sigma_{ij}) = (-1)^{\delta_{ij}} \varepsilon(\sigma_{ij}) (\sigma_{ij}, 2)^{I_{ij}}, \quad 1 \leq i, j \leq \ell \quad (2.29)$$

¹For the particular case of the $SU(3)_2$ -HSG model it was shown [30] that it can be described alternatively as a perturbation of a tensor product of two minimal CFTs.

where I denotes the incidence matrix of \mathbf{g} and $\varepsilon(x)$ is the step-function, i.e. $\varepsilon(x) = 1$ for $x \geq 0$, $\varepsilon(x) = -1$ for $x < 0$. It is convenient to use the abbreviation

$$(\sigma, x) := \tanh(\theta + \sigma - i\pi x/4)/2. \quad (2.30)$$

Let us now consider the concrete case $SU(3)_2$. We can start with the known part of the scattering matrix (2.29) for the stable particles, and leave the remaining entries which involve unstable particles unknown. From this we construct consistent solutions to the bootstrap equations (2.22), (2.24) and (2.25). We can fix the imaginary parts of the fusing angles by the requirement that for vanishing resonance parameters we want to reproduce the masses predicted by the Breit-Wigner formula. Choosing the masses of the stable particles to be $m_1 = m_2 = m$ the one for the unstable results to $m_{(12)} = \sqrt{2}m$. This argument does not constrain the real parts of the fusing angles, such that they are not completely fixed and still contain a certain ambiguity. The different choices of these parameters give rise to slightly different theories. First we consider the case $\sigma_{21} > 0$.



For this choice of the resonance parameter, we then find the following bootstrap equations

$$\tilde{S}_{l(12)}(\theta) = \tilde{S}_{l1}(\theta + (1 - \nu)\sigma_{12} + i\pi/4)\tilde{S}_{l2}(\theta - \nu\sigma_{12} - i\pi/4) \quad (2.31)$$

from which we construct

$$\tilde{S}_{SU(3)}(\theta, \sigma_{12}) = \begin{pmatrix} -1 & -(\sigma_{12}, 2) & -((1 - \nu)\sigma_{12}, 3) \\ (\sigma_{21}, 2) & -1 & -(\nu\sigma_{21}, 1) \\ -((\nu - 1)\sigma_{12}, 1) & -(\nu\sigma_{12}, 3) & -1 \end{pmatrix}. \quad (2.32)$$

Here we label the rows and columns in the order $\{1, 2, (12)\}$. According to the principles outlined above, the \tilde{S} -matrix (2.32) allows for the processes

$$1 + 2 \rightarrow (12), \quad 2 + (12) \rightarrow 1, \quad (12) + 1 \rightarrow 2. \quad (2.33)$$

The related fusing angles are read off from (2.32) as

$$\gamma_{12}^{(12)} = -\frac{i\pi}{2} + \sigma_{21}, \quad \gamma_{(12)1}^2 = -\frac{3i\pi}{4} + (1 - \nu)\sigma_{12}, \quad \gamma_{2(12)}^1 = -\frac{3i\pi}{4} + \nu\sigma_{12} \quad (2.34)$$

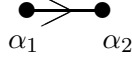
and are interrelated through equation (2.26), which still holds even though the γ 's have non-vanishing real parts. We can employ these fusing angles and compute the masses and decay widths by means of the Breit-Wigner formulae (2.12) and (2.13). Taking again for simplicity $m_1 = m_2 = m$ and in addition $\nu = 1/2$, we obtain for the first process in (2.33)

$$m_{(12)} = \sqrt{2}m \cosh \sigma_{21}/2 \quad \text{and} \quad \Gamma_{(12)} = 2\sqrt{2}m \sinh \sigma_{21}/2. \quad (2.35)$$

Employing now also in the process $2 + (12) \rightarrow 1$ the Breit-Wigner formula, we reproduce in the limit $\sigma_{12} \rightarrow 0$ the values $m_1 = m$ and $\Gamma_1 = 0$. Likewise, in the last process in (2.33) we obtain $m_2 = m$ and $\Gamma_2 = 0$.

The asymptotic limit $t \rightarrow \infty$ becomes meaningful when we operate on an energy scale at which the unstable particle has not even been created yet, i.e. $\Gamma_{(12)} \rightarrow \infty \equiv \sigma_{21} \rightarrow \infty$. In that case the theory decouples into two $SU(2)_2$ -models, i.e. free Fermions, with $S_{11} = S_{22} = -1$. This is a simple version of the decoupling rule (3.3).

Next we consider a different theory with $\sigma_{12} > 0$.



Taking also in this case for simplicity $\nu = 1/2$, we find the following bootstrap satisfied

$$\tilde{S}_{l(12)}(\theta) = \tilde{S}_{l2}(\theta - \sigma_{12}/2 + i\pi/4) \tilde{S}_{l1}(\theta + \sigma_{12}/2 - i\pi/4), \quad (2.36)$$

which yields the S-matrix

$$\tilde{S}_{SU(3)}(\theta, \sigma_{21}) = \begin{pmatrix} -1 & (\sigma_{12}, 2) & -(\sigma_{12}/2, 1) \\ -(\sigma_{21}, 2) & -1 & -(\sigma_{21}/2, 3) \\ -(\sigma_{21}/2, 3) & -(\sigma_{12}/2, 1) & -1 \end{pmatrix}. \quad (2.37)$$

The S-matrix (2.37) allows for the processes

$$2 + 1 \rightarrow (12), \quad 1 + (12) \rightarrow 2, \quad (12) + 2 \rightarrow 1, \quad (2.38)$$

instead of (2.33). Now the fusing angles are read off as

$$\gamma_{21}^{(12)} = -\frac{i\pi}{2} + \sigma_{12}, \quad \gamma_{1(12)}^2 = -\frac{3i\pi}{4} - \frac{\sigma_{12}}{2}, \quad \gamma_{2(12)}^1 = -\frac{3i\pi}{4} - \frac{\sigma_{12}}{2} \quad (2.39)$$

and also satisfy (2.26). The masses and decay width are obtained again from (2.12) and (2.13) with $\sigma_{12} \rightarrow \sigma_{21}$. As a whole, we can think of this theory simply as being obtained from the \mathbb{Z}_2 -Dynkin diagram automorphism which exchanges the roles of the particles 1 and 2. However, since parity invariance is now broken this is not a symmetry any more and the two theories are different. In the asymptotic limit $\sigma_{12} \rightarrow \infty$, we obtain once again a simple version of the decoupling rule (3.3) and the theory decouples into two $SU(2)_2$ -models.

The next example, $SU(4)_2$ -HSG, is more intriguing as it leads to the prediction a new unstable particle. Proceeding in the way as before we construct the corresponding amplitudes \tilde{S} , for details see [19]. We found there the processes

$$\begin{aligned} 1 + 2 &\rightarrow (12), & (12) + 1 &\rightarrow 2, & 2 + (12) &\rightarrow 1, \\ 3 + 2 &\rightarrow (23), & (23) + 3 &\rightarrow 2, & 2 + (23) &\rightarrow 3, \end{aligned} \quad (2.40)$$

which simply correspond to two copies of $SU(3)_2$ -HSG. It is interesting to note that the amplitudes $\tilde{S}_{(12)3}$ and $\tilde{S}_{(23)1}$ contain poles at

$$\gamma_{(12)3}^{(123)} = \frac{\sigma_{21} - 2\sigma_{23}}{2} - \frac{3i\pi}{4} \quad \text{and} \quad \gamma_{(23)1}^{(123)} = \frac{\sigma_{23} - 2\sigma_{21}}{2} - \frac{3i\pi}{4}, \quad (2.41)$$

which yield the possible processes

$$\begin{aligned} (12) + 3 &\rightarrow (123), & (123) + (12) &\rightarrow 3, & 3 + (123) &\rightarrow (12), \\ (23) + 1 &\rightarrow (123), & (123) + (23) &\rightarrow 1, & 1 + (123) &\rightarrow (23). \end{aligned} \quad (2.42)$$

An interesting prediction results from the consideration of the first two processes in (2.42). Making in the first process the particle (12) and in the second the particle (23) stable, by $\sigma_2 \rightarrow \sigma_1$ and by $\sigma_2 \rightarrow \sigma_3$, respectively, both predict the mass of the particle (123) as

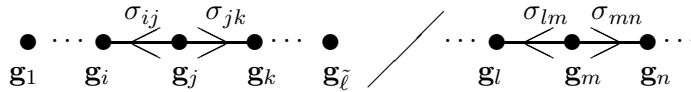
$$m_{(123)} \sim m e^{|\sigma_{13}|/2}. \quad (2.43)$$

This value is precisely the one we expect from the approximation in the Breit-Wigner formula (2.15). Note that in one case we obtain σ_{13} and in the other σ_{31} as a resonance parameter. The difference results from the fact that according to the processes (2.42), the particle (123) is either formed as $(1+2)+3$ or $3+(2+1)$. Thus the different parity shows up in this process, but this has no effect on the values for the mass.

In [19] we presented more examples and remarkably we found consistency in each case. We take the closure of the bootstrap equations as a non-trivial confirmation for our proposal.

3. Lie algebraic structure for theories with unstable particles

There exist some concrete Lagrangian formulations for integrable theories with unstable particles, such as the aforementioned HSG-models (2.27). Inspired by the structure of these models, we present here a slightly more general Lie algebraic picture. We keep the discussion here abstract and supply below concrete examples. For our formulation we need an arbitrary simply laced Lie algebra $\tilde{\mathfrak{g}}$ (possibly with a subalgebra $\tilde{\mathfrak{h}}$) with rank $\tilde{\ell}$ together with its associated Dynkin diagram (see for instance [33]). To each node we attach a simply laced Lie algebra \mathfrak{g}_i with rank ℓ_i and to each link between the nodes i and j a resonance parameter $\sigma_{ij} = \sigma_i - \sigma_j$, as depicted in the following $\tilde{\mathfrak{g}}/\tilde{\mathfrak{h}}$ -coset Dynkin diagram



Besides the usual rules for Dynkin diagrams, we adopt here the convention that we add an arrow to the link, which manifests the parity breaking and allows to identify the signs of the resonance parameters. An arrow pointing from the node i to j simply indicates that $\sigma_{ij} > 0$. Since we are dealing exclusively with simply laced Lie algebras, this should not lead to confusion. To each simple root of the algebras \mathfrak{g}_i , we associate now a stable particle and to each positive non-simple root of $\tilde{\mathfrak{g}}$ an unstable particle, such that

$$\# \text{ of stable particles} = \sum_{i=1}^{\tilde{\ell}} \ell_i, \quad \# \text{ of unstable particles} = \frac{\tilde{\ell}(\tilde{h}-2)}{2}. \quad (3.1)$$

From the discussion above, we expect that the σ 's will be associated to unstable particles, but we note that the

$$\# \text{ of resonance parameters} = \frac{\tilde{\ell}(\tilde{\ell}-1)}{2} \quad (3.2)$$

only agrees with the amount of unstable particles for $\tilde{h} = \tilde{\ell} + 1$, e.g. for $\tilde{\mathfrak{g}} = SU(\tilde{\ell} + 1)$. Since the resonance parameters govern the mass of the unstable particles, this discrepancy is interpreted as an unavoidable mass degeneracy.

Concrete examples for this formulations are the $\tilde{\mathbf{g}}_k$ -homogeneous sine-Gordon models [20, 21], for which one can choose $\tilde{\mathbf{g}}$ to be simply laced and $\mathbf{g}_1 = \dots = \mathbf{g}_{\tilde{\ell}} = SU(k)$. This is generalized [34] when taking instead $\tilde{\mathbf{g}}$ to be non-simply laced and $\mathbf{g}_i = SU(2k/\alpha_i^2)$, with α_i being the simple roots of $\tilde{\mathbf{g}}$. The choice $\mathbf{g}_1 = \dots = \mathbf{g}_{\tilde{\ell}} = \mathbf{g}$ with \mathbf{g} being any arbitrary simply laced Lie algebra gives the $\mathbf{g}|\tilde{\mathbf{g}}$ -theories [35]. An example for a theory associated to a coset is the roaming sinh-Gordon model [36], which can be thought of as $\tilde{\mathbf{g}}/\tilde{\mathbf{h}} \equiv \lim_{k \rightarrow \infty} SU(k+1)/SU(k)$ with $\mathbf{g}_1 = \dots = \mathbf{g}_{\tilde{\ell}} = SU(2)$. It is clear that the examples presented here do not exhaust yet all possible combinations and the structure mentioned above allows for more combinations of algebras, which are not yet explored. One is also not limited to Dynkin diagrams and may consider more general graphs which have multiple links, i.e. resonance parameters, between various nodes. Examples for such theories were proposed and studied in [37].

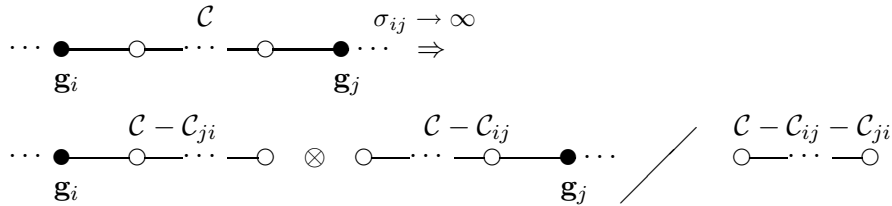
3.1 Decoupling Rule

Of special interest is to investigate the behaviour of previously defined systems when certain resonance parameters σ become very large or tend to infinity. The physical motivation for that is to describe a renormalization group (RG) flow, which we shall discuss in more detail below. Here we present first the mathematical set up.

Decoupling rule: Call the overall Dynkin diagram \mathcal{C} and denote the associated Lie group and Lie algebra by $\tilde{G}_{\mathcal{C}}$ and $\tilde{\mathbf{g}}_{\mathcal{C}}$, respectively. Let σ_{ij} be some resonance parameter related to the link between the nodes i and j . To each node i attach a simply laced Lie algebra \mathbf{g}_i . Produce a reduced diagram \mathcal{C}_{ji} containing the node j by cutting the link adjacent to it in the direction i . Likewise produce a reduced diagram \mathcal{C}_{ij} containing the node i by cutting the link adjacent to it in the direction j . Then the $\tilde{G}_{\mathcal{C}}$ -theory decouples according to the rule

$$\lim_{\sigma_{ij} \rightarrow \infty} \tilde{G}_{\mathcal{C}} = \tilde{G}_{(\mathcal{C}-\mathcal{C}_{ij})} \otimes \tilde{G}_{(\mathcal{C}-\mathcal{C}_{ji})} / \tilde{G}_{(\mathcal{C}-\mathcal{C}_{ij}-\mathcal{C}_{ji})}. \quad (3.3)$$

We depict this rule also graphically in terms of Dynkin diagrams:



According to the GKO-coset construction [32], this means that the Virasoro central charge flows as

$$c_{\tilde{\mathbf{g}}_{\mathcal{C}}} \rightarrow c_{\tilde{\mathbf{g}}_{\mathcal{C}-\mathcal{C}_{ij}}} + c_{\tilde{\mathbf{g}}_{\mathcal{C}-\mathcal{C}_{ji}}} - c_{\tilde{\mathbf{g}}_{\mathcal{C}-\mathcal{C}_{ij}-\mathcal{C}_{ji}}}. \quad (3.4)$$

The rule may be applied consecutively to each disconnected subgraph produced according to the decoupling rule (3.3). Note that this rule describes a decoupling and not a fusing, as it only predicts the flow in one direction and the limit is not reversible. From a physical point of view this is natural as also the RG flow is also irreversible. The rule (3.3) generalizes a rule proposed in [26], which was based on the assumption that unstable particles are associated exclusively to positive roots of height two.

More familiar in the mathematical literature is a decoupling rule found by Dynkin [38] for the construction of semi-simple² subalgebras $\tilde{\mathfrak{h}}$ from a given algebra $\tilde{\mathfrak{g}}$. For the more general diagrams which can be related to the $\tilde{\mathfrak{g}}_k$ -HSG models the generalized rule can be found in [39]. These rules are all based on removing some of the nodes rather than links. For our physical situation at hand this corresponds to sending the masses of all stable particles which are associated to the algebra of a particular node to infinity. As in the decoupling rule (3.3) the number of stable particles remains preserved, it is evident that the two rules are inequivalent. Letting for instance the mass scale in \mathfrak{g}_j go to infinity, the generalized (in the sense that \mathfrak{g}_j can be different from A_ℓ) rule of Kuniba is simply depicted as

$$\begin{array}{ccc} \cdots \bullet \xrightarrow{\mathcal{C}} \bullet \xrightarrow{\mathcal{C}} \bullet \cdots & m_j \rightarrow \infty \Rightarrow & \cdots \bullet \xrightarrow{\mathcal{C} - \mathcal{C}_{ji}} \bullet \quad \otimes \quad \bullet \xrightarrow{\mathcal{C} - \mathcal{C}_{jk}} \bullet \cdots \\ & & \mathfrak{g}_i \quad \quad \quad \mathfrak{g}_k \end{array}$$

Clearly this can not be produced with (3.3).

3.2 A simple example: The $SU(4)_2$ -HSG model

We illustrate the working of the rule (3.3) with a simple example. We take $\tilde{\mathfrak{g}}$ to be $SU(4)$, attach to each node simply an $SU(2)$ algebra and to the links the resonance parameters $\sigma_{12}, \sigma_{13}, \sigma_{23}$. This corresponds to the $SU(4)_2$ -HSG model. For the ordering $\sigma_{13} > \sigma_{12} > \sigma_{23}$ the rule (3.3) predicts the following flow

$$\begin{array}{ll} \begin{array}{c} \bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet \\ \rightarrow \sigma_{13} \end{array} & \begin{array}{c} \bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet \\ \rightarrow \sigma_{12} \end{array} \\ \begin{array}{c} \bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet \\ \rightarrow \sigma_{23} \end{array} & \begin{array}{c} \bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet \\ \rightarrow \sigma_{13} \end{array} \end{array}$$

The central charges are obtained from (2.28) using (3.4). Choosing instead the ordering $\sigma_{23} > \sigma_{13} > \sigma_{12}$, we compute

$$\begin{array}{ll} \begin{array}{c} \bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet \\ \rightarrow \sigma_{23} \end{array} & \begin{array}{c} \bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet \\ \rightarrow \sigma_{13} \end{array} \\ \begin{array}{c} \bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet \\ \rightarrow \sigma_{12} \end{array} & \begin{array}{c} \bullet \xrightarrow{\alpha_1} \bullet \xrightarrow{\alpha_2} \bullet \xrightarrow{\alpha_3} \bullet \\ \rightarrow \sigma_{13} \end{array} \end{array}$$

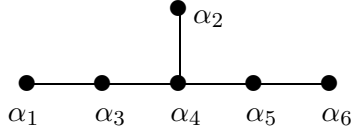
It is important to note the non-commutative nature of the limiting procedures. For more complicated algebras it is essential to keep track of the labels on the nodes, since only in this way one can decide whether they cancel against the subgroup diagrams or not.

²The subalgebras constructed in this way are not necessarily maximal and regular. A guarantee for obtaining those, except in six special cases, is only given when one manipulates adequately the extended Dynkin diagram.

3.3 A non-trivial example: The $(E_6)_2$ -HSG model

As by now we do not have a rigorous proof of the decoupling rule (3.3), we take the support for its validity from the working of various examples. We will check below the analytic predictions of the rule against some alternative method. As the previous example was a simple pedagogical one, we will consider next a non-trivial one leading to an intricate prediction for the RG-flow. The confirmative double check below can hardly be accidental and we take that as very strong support for the validity of (3.3).

We consider now the $(E_6)_2$ -HSG model. In this case we have $\tilde{\ell} = 6, \tilde{h} = 12$ such that we have 6 stable particles, 30 unstable particles and 15 resonance parameters. From the $5!$ possible orderings for the resonance parameters we present here only two concrete ones, which will predict different types of flows and mass degeneracies. Note that this degeneracy is not the unavoidable one resulting from the difference between the number of resonance parameters and non-simple positive roots that is $30 - 15$, as discussed for (3.1) and (3.2). The degeneracies discussed here are a consequence of the particular choices of the resonance parameters. The conventions for the labeling of our particles are indicated in the following Dynkin diagram:



We choose first the ordering and values for resonance parameters as

$$\sigma_{13} = 100 > \sigma_{34} = 80 > \sigma_{45} = 60 > \sigma_{56} = 40 > \sigma_{24} = 20. \quad (3.5)$$

According to the decoupling rule (3.3), we predict therefore the flow:

	E_6	$\frac{36}{7} \sim 5.14$
$\rightarrow \sigma_{16} = 280$	$SO(10)^{\otimes 2}/SO(8)$	5
$\rightarrow \sigma_{15} = 240$	$SO(10) \otimes SU(5)/SU(4)$	$\frac{34}{7} \sim 4.86$
$\rightarrow \sigma_{14}, \sigma_{36} = 180$	$SO(8) \otimes SU(5) \otimes SU(3)/SU(4) \otimes SU(2)$	$\frac{319}{70} \sim 4.56$
$\rightarrow \sigma_{12} = 160$	is already decoupled	
$\rightarrow \sigma_{35} = 140$	$SU(5) \otimes SU(4) \otimes SU(3)/SU(3) \otimes SU(2)$	$\frac{61}{14} \sim 4.36$
$\rightarrow \sigma_{26} = 120$	$SU(4)^{\otimes 3} \otimes SU(3)/SU(3)^{\otimes 2} \otimes SU(2)$	4.3
$\rightarrow \sigma_{13}, \sigma_{46} = 100$	$SU(4)^{\otimes 2} \otimes SU(3) \otimes SU(2)/SU(3) \otimes SU(2)$	4
$\rightarrow \sigma_{25}, \sigma_{34} = 80$	$SU(3)^{\otimes 3} \otimes SU(2)^{\otimes 2}/SU(2)^{\otimes 2}$	3.6
$\rightarrow \sigma_{32}, \sigma_{45} = 60$	$SU(3)^{\otimes 2} \otimes SU(2)^{\otimes 2}$	3.4
$\rightarrow \sigma_{56} = 40$	$SU(3) \otimes SU(2)^{\otimes 4}$	3.2
$\rightarrow \sigma_{24} = 20$	$SU(2)^{\otimes 6}$	3

Note that eight particles are pairwise degenerate and we therefore expect to find $15 - 8/2 = 11$ plateaux in the flow. The first step which corresponds to one of these degeneracies occurs for instance at $\sigma_{14} = \sigma_{36}$ and we have to apply the decoupling rule twice at this point before we get a new fixed point theory.

Next we arrange the couplings as

$$\sigma_{45} = 100 > \sigma_{34} = 80 > \sigma_{13} = 60 > \sigma_{56} = 40 > \sigma_{24} = 20. \quad (3.6)$$

and compute from (3.3) the flow

	E_6	$\frac{36}{7} \sim 5.14$
$\rightarrow \sigma_{16} = 280$	$SO(10)^{\otimes 2}/SO(8)$	5
$\rightarrow \sigma_{15} = 240$	$SO(10) \otimes SU(5)/SU(4)$	$\frac{34}{7} \sim 4.86$
$\rightarrow \sigma_{36} = 220$	$SU(5)^{\otimes 2} \otimes SO(8)/SU(4)^{\otimes 2}$	$\frac{33}{7} \sim 4.71$
$\rightarrow \sigma_{35} = 180$	$SU(5)^{\otimes 2}/SU(3)$	$\frac{158}{35} \sim 4.51$
$\rightarrow \sigma_{26} = 160$	$SU(4)^{\otimes 2} \otimes SU(5)/SU(3)^{\otimes 2}$	$\frac{156}{35} \sim 4.46$
$\rightarrow \sigma_{14} = \sigma_{46} = 140$	$SU(4)^{\otimes 2} \otimes SU(3)^{\otimes 2}/SU(2)^{\otimes 2} \otimes SU(3)$	4.2
$\rightarrow \sigma_{12} = \sigma_{25} = 120$	$SU(4) \otimes SU(3)^{\otimes 3}/SU(2)^{\otimes 3}$	4.1
$\rightarrow \sigma_{45} = 100$	$SU(4) \otimes SU(3)^{\otimes 2}/SU(2)$	3.9
$\rightarrow \sigma_{34} = 80$	$SU(3)^{\otimes 3}$	3.6
$\rightarrow \sigma_{13} = \sigma_{32} = 60$	$SU(3)^{\otimes 2} \otimes SU(2)^{\otimes 2}$	3.4
$\rightarrow \sigma_{56} = 40$	$SU(3) \otimes SU(2)^{\otimes 4}$	3.2
$\rightarrow \sigma_{24} = 20$	$SU(2)^{\otimes 6}$	3

In this case we have only six particles pairwise degenerate and we expect to find $15 - 6/2 = 12$ plateaux. In the next section we find that the predictions made here are confirmed even for this involved case.

4. How to detect unstable particles?

In section 2 we described several arguments which predict the spectrum of unstable particles and now we will present some methods which allow to test these predictions. In particular with regard to the bootstrap proposal this will be important, as it is not yet rigorously supported. Computing renormalization group (RG) flows will allow to detect the unstable particles. Roughly speaking, the central idea of an RG analysis is to probe different energy scales of a theory. We can flow from an energy regime so large that the unstable particle can energetically not exist to one in which it is formed. As a consequence, the particle content of the theory will be altered, which is visible in form of a typical staircase pattern of the RG scaling function.

There are various ways to compute such scaling functions, such as the evaluation of the c-theorem [40] or an analysis by means of the thermodynamic Bethe ansatz (TBA) [41]. In the first case we have to evaluate the expression

$$c(r_0) = \frac{3}{2} \int_{r_0}^{\infty} dr r^3 \langle \Theta(r) \Theta(0) \rangle. \quad (4.1)$$

The main difficulty in this approach is to evaluate the two-point correlation function $\langle \Theta(r) \Theta(0) \rangle$ for the trace of the energy-momentum tensor Θ depending on the radial distance r . Most effectively, one can do this by expanding it in terms of form factors, for a

general recent introduction see e.g. [42] and references therein. It is well known that for many, even quite non-trivial, theories such form factor expansions converge extremely fast, see [27] for the computation of (4.1) for the $SU(3)_2$ -HSG model.

Here we will concentrate more on the TBA, which is simpler to handle in most cases. As a prerequisite, one assumes to know all scattering matrices $S_{ij}(\theta)$ for the **stable** particles of the type i, j with masses m_i, m_j . Besides this dynamical interaction one also makes an assumption on the statistical interaction between the particles, which are chosen here to be of fermionic type. The TBA consists now of compactifying the space of this 1+1 dimensional relativistic model into a circle of finite circumference R , such that all energies become discrete and functions of R . The function similar to (4.1), which scales now these energies takes on the form

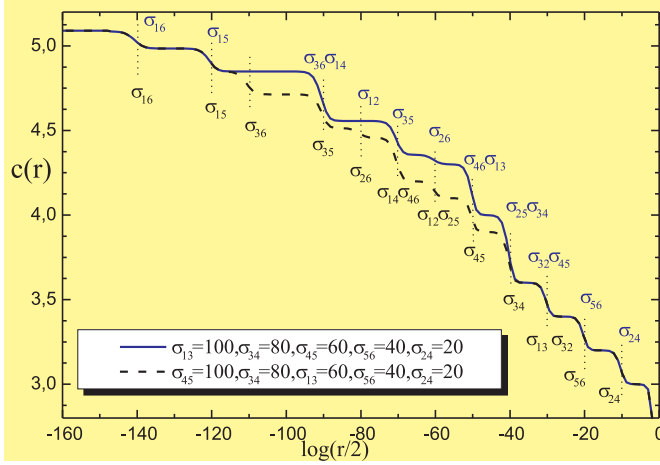
$$c_{\text{eff}}(r) = \frac{3r}{\pi^2} \sum_i m_i \int_{-\infty}^{\infty} d\theta \cosh \theta \ln(1 + e^{-\varepsilon_i(\theta, r)}). \quad (4.2)$$

One identifies the circumference R with the inverse temperature T and introduces the scaling parameter $r = m/T$, with m being an overall mass scale. The $\varepsilon_i(\theta, r)$ are the so-called the pseudo-energies which can be obtained as solutions of the thermodynamic Bethe ansatz equations

$$rm_i \cosh \theta = \varepsilon_i(\theta, r) + \sum_j [\varphi_{ij} * \ln(1 + e^{-\varepsilon_j})](\theta, r). \quad (4.3)$$

Here the $*$ denotes the convolution of two functions $(f * g)(\theta) := 1/(2\pi) \int d\theta' f(\theta - \theta') g(\theta')$ and the S (for the stable particles only!) enter via their logarithmic derivatives $\varphi_{ij}(\theta) = -id \ln S_{ij}(\theta)/d\theta$. The main difficulty in this approach is to solve (4.3), which are coupled non-linear integral equations and therefore not solvable in a systematic analytical way.

Now it is clear, that the two functions (4.1) and (4.2) can not be the same, but nevertheless they contain the same qualitative information. The functions will flow through various fixed points, at which the theory become effectively conformal field theories and the normalizations are chosen in such a way that the values of both functions coincide with the corresponding Virasoro central charges. When the theory is not unitary, (4.2) has to be corrected by an additive term to achieve this. Computing then a flow from the infrared to the ultraviolet, one passes now various CFT plateaux, where the changes are associated to the formation of unstable particles with mass (2.15). The challenge is of course to predict the positions, that is, the height and the on-set of the plateaux, as a function of the scaling parameter. The on-set is related to the energy scale of the unstable particles and thus simply determined by the formula (2.15). To predict the height is less trivial and the proposal made in [19] is that the decoupling rule (3.3) achieves this. It is important to note here that $\sigma \rightarrow \infty$ in (3.3), means in the RG context $\sigma \gg$ all other resonance parameters. In the following picture we present the numerical computation for the $(E_6)_2$ -HSG model, which precisely confirms our analytical predictions made by the decoupling rule in section 3.3



Having confirmed the predictions of our decoupling rule with a TBA-analysis, let us now discuss how the results of this analysis are compatible with our bootstrap proposal with a simple example: We consider the processes (2.40), (2.42). In order to be able to interpret the BW-formula for the production of the particle (123) from (12) + 3 or (23) + 1 one has to “make” (12) and (23) stable, which is achieved when σ_{12} or σ_{23} is zero. One has to do that as otherwise the BW can not be applied, it only makes sense for stable particles. The first not obvious result here is that the resulting mass for (123) turns out to be the same from both cases (2.42) (and in all other examples!!!). Looking at the outcome of the TBA calculation (see [19] for the numerics on this case) one finds precisely the value (2.43) reproduced by the TBA at the onset $\ln(r/2) \sim -\sigma_{13}/2 = -(\sigma_{12} + \sigma_{23})/2$. Now apparently in the TBA analysis σ_{12} or σ_{23} are not zero, which seems to contradict the previous assumptions in the bootstrap analysis. To understand this, one should keep in mind the meaning of the steps in the TBA. The formation of the particle (123) takes place when its mass becomes greater than the energy scale of the RG-flow, i.e. when $m \exp(\sigma_{13}/2) > 2m/r$. Let us chose for instance $\sigma_{12} = 30$, $\sigma_{23} = 60$, then $\exp(\sigma_{13}/2) \sim \exp(45) \sim 3.49 \times 10^{19}$. To resolve the apparent contradiction, it is now important to note that the other unstable particles are formed several orders of magnitude below at $\exp(30) \sim 1.06 \times 10^{13}$ and $\exp(15) \sim 3.72 \times 10^6$. This means in comparison to the formation energy scale of particle (123) the parameters σ_{12} , σ_{23} can be regarded as approximately zero, which is in agreement with the assumption in the bootstrap analysis!

This is just the same picture as put forward in the decoupling rule: In the formulation we say $\sigma_{13} \rightarrow \infty$, but inside the TBA analysis this is a milder statement and just means $\sigma_{13} \gg \sigma_{12}, \sigma_{23}$. Further quite non-obvious confirmation comes from the results when choosing the parameters differently, i.e. in the example discussed here $\sigma_{12} \rightarrow -\sigma_{12}$. The two pictures completely coincide. With regard to previous studies, it is very important to note that the occurrence of the step at $\exp(\sigma_{13}/2) \sim 2/r$ had no explanation at all before. Only the onsets at $\exp(\sigma_{23}/2) \sim 2/r$ and $\exp(\sigma_{12}/2) \sim 2/r$ could be explained as they correspond to the formation of unstable particles from two stable ones. The additional step (for other algebras there are far more) was a mystery pointed out first in [29]. In [19] we provided for the first time an explanation for this feature: We predict its height and on-set, thus explaining also why it is absent when the resonance parameters are chosen differently.

For all other examples studied (not even all have been presented in this proceeding, see [19] for more) this picture is completely consistent.

5. Theories with an infinite amount of unstable particles

We address now the question of how to enlarge a given finite particle spectrum of a theory to an infinite one. In general the bootstrap (2.10), which is the central construction principle for the S -matrix, is assumed to close after a finite number of steps, which means it involves a finite number of particles. However, from a physical as well as from a mathematical point of view, it appears to be natural to extend the construction in such a way that it would involve an infinite number of particles. The physical motivation for this are string theories, which admit an infinite particle spectrum. Mathematically the infinite bootstrap would be an analogy to infinite dimensional groups, in the sense that two entries of the S -matrix are combined into a third, which is again a member of the same infinite set. It appears to us that it is impossible to construct an infinite bootstrap system involving asymptotic states, although we do not know a rigorous proof of such a no-go theorem. Instead, we will demonstrate that it is possible to introduce an infinite number of unstable particles into the spectrum.

5.1 q -deformed gamma functions and Jacobian elliptic functions

In general, the S -matrix amplitudes consist of (in)finite products of hyperbolic or/and gamma functions. Here we will argue, that to enlarge the spectrum to an infinite number one should replace these functions by q -deformed quantities or elliptic functions. Let us first recall some mathematical facts in this section. We start with some properties of q -deformed quantities, which have turned out to be very useful objects as they allow for instance to carry out elegantly (semi)-classical limits when the deformation parameter is associated to Planck's constant. Here we define a deformation parameter q and its Jacobian imaginary transformed version, i.e. $\tau \rightarrow -1/\tau$, as

$$q = \exp(i\pi\tau), \quad \hat{q} = \exp(-i\pi/\tau), \quad \tau = iK_{1-\ell}/K_\ell. \quad (5.1)$$

We introduced here the quarter periods K_ℓ of the Jacobian elliptic functions depending on the parameter $\ell \in [0, 1]$, defined in the usual way through the complete elliptic integrals $K_\ell = \int_0^{\pi/2} (1 - \ell \sin^2 \theta)^{-1/2} d\theta$. Then

$$\lim_{\ell \rightarrow 0, \hat{q} \rightarrow 1} K_\ell = \lim_{\ell \rightarrow 1, q \rightarrow 1} K_{1-\ell} = \pi/2, \quad \lim_{\ell \rightarrow 0, \hat{q} \rightarrow 1} K_{1-\ell} = \lim_{\ell \rightarrow 1, q \rightarrow 1} K_\ell \rightarrow \infty. \quad (5.2)$$

It will turn out below that quantities in \hat{q} will be most relevant for our purposes and therefore we state several identities directly in \hat{q} , rather than q , even when they hold for generic values. The most basic q -deformed objects one defines are q -deformed integers (numbers), for which we take the convention

$$[n]_{\hat{q}} := \frac{\hat{q}^n - \hat{q}^{-n}}{\hat{q} - \hat{q}^{-1}}. \quad (5.3)$$

They have the obvious properties

$$\lim_{\ell \rightarrow 0} [n]_{\hat{q}} = n, \quad (5.4)$$

$$\lim_{\ell \rightarrow 0} \frac{[n + m\tau]_{\hat{q}}}{[n' + m'\tau]_{\hat{q}}} = \begin{cases} 1 & \text{for } m, m' \neq 0 \\ n/n' & \text{for } m = m' = 0 \end{cases}. \quad (5.5)$$

Next we define a q -deformed version of Euler's gamma function

$$\Gamma_{\hat{q}}(x+1) := \prod_{n=1}^{\infty} \frac{[1+n]_{\hat{q}}^x [n]_{\hat{q}}}{[x+n]_{\hat{q}} [n]_{\hat{q}}^x}. \quad (5.6)$$

The crucial property of the function $\Gamma_{\hat{q}}$, which coins also its name, is

$$\lim_{\ell \rightarrow 0} \Gamma_{\hat{q}}(x+1) = \lim_{\hat{q} \rightarrow 1} \Gamma_{\hat{q}}(x+1) = \prod_{n=1}^{\infty} \frac{n}{n+x} \left(\frac{1+n}{n} \right)^x = \Gamma(x+1). \quad (5.7)$$

We can relate deformations in q and \hat{q} through

$$\frac{\hat{q}^{(x+\tau/2-1/2)^2}}{\hat{q}^{(y+\tau/2-1/2)^2}} \frac{\Gamma_{\hat{q}}(y)\Gamma_{\hat{q}}(1-y)}{\Gamma_{\hat{q}}(x)\Gamma_{\hat{q}}(1-x)} = \frac{\Gamma_q(-y/\tau)\Gamma_q(1+y/\tau)}{\Gamma_q(-x/\tau)\Gamma_q(1+x/\tau)}. \quad (5.8)$$

Frequently we have to shift the argument by integer values

$$\Gamma_{\hat{q}}(x+1) = \hat{q}^{x-1} [x]_{\hat{q}} \Gamma_{\hat{q}}(x). \quad (5.9)$$

Relation (5.9) can be obtained directly from (5.6). As a consequence of this we also have

$$\Gamma_{\hat{q}}(x+m) = \Gamma_{\hat{q}}(x) \prod_{l=0}^{m-1} \hat{q}^{x+l-1} [x+l]_{\hat{q}} \quad m \in \mathbb{Z} \quad (5.10)$$

$$\Gamma_{\hat{q}}(x) = \Gamma_{\hat{q}}(x-m) \prod_{l=0}^{m-1} \hat{q}^{x-l-2} [x-l-1]_{\hat{q}} \quad m \in \mathbb{Z}. \quad (5.11)$$

Whereas (5.9)-(5.10) hold for generic q , the following identities are only valid for \hat{q}

$$\Gamma_{\hat{q}}(1/2 - \tau/2) \Gamma_{\hat{q}}(1/2 + \tau/2) = \ell^{1/4} \Gamma_{\hat{q}}(1/2)^2 \quad (5.12)$$

$$\frac{\Gamma_{\hat{q}}(x+2\tau)}{\Gamma_{\hat{q}}(y+2\tau)} = \frac{\Gamma_{\hat{q}}(x)}{\Gamma_{\hat{q}}(y)} \quad (5.13)$$

$$\prod_{i=1}^p \frac{\Gamma_{\hat{q}}(x_i) \Gamma_{\hat{q}}(x_i \pm \tau/2)}{\Gamma_{\hat{q}}(y_i) \Gamma_{\hat{q}}(y_i \pm \tau/2)} = \prod_{i=1}^p \frac{\Gamma_{\hat{q}^2}(x_i)}{\Gamma_{\hat{q}^2}(y_i)} \quad \text{if } \sum_{i=1}^p x_i = \sum_{i=1}^p y_i \quad (5.14)$$

$$\lim_{\hat{q} \rightarrow 1} \prod_{i=1}^p \frac{\Gamma_{\hat{q}}(x_i \pm \tau/2)}{\Gamma_{\hat{q}}(y_i \pm \tau/2)} = 1 \quad \text{if } \sum_{i=1}^p x_i = \sum_{i=1}^p y_i \quad (5.15)$$

$$\lim_{\hat{q} \rightarrow 1} \frac{1}{\ell^{1/4}} \Gamma_{\hat{q}}\left(\frac{x}{2K_{\ell}} \mp \frac{\tau}{2}\right) \Gamma_{\hat{q}}\left(1 - \frac{x}{2K_{\ell}} \pm \frac{\tau}{2}\right) = \pi \quad \text{for } x \neq 0$$

Most of these properties can be checked directly by means of the defining relation (5.6). The singularity structure will be important for the physical applications. It follows from (5.6) that the $\Gamma_{\hat{q}}$ -function has no zeros, but poles

$$\lim_{\theta \rightarrow \theta_{\Gamma,p}^m = m\tau - n} \Gamma_{\hat{q}}(\theta + 1) \rightarrow \infty \quad \text{for } m \in \mathbb{Z}, n \in \mathbb{N}. \quad (5.16)$$

Next we define

$$\{x\}_{\theta}^{\sigma} := \frac{\tanh(\theta - i\pi x + \sigma)/2}{\tanh(\theta + i\pi x + \sigma)/2}, \quad \{x\}_{\theta,\ell}^{\sigma} := \prod_{n=-\infty}^{\infty} \{x\}_{\theta-n \log q}^{\sigma} = \frac{\text{sc } \theta_- \text{dn } \theta_+}{\text{sc } \theta_+ \text{dn } \theta_-} \quad (5.17)$$

with $x \in \mathbb{Q}$, $\sigma \in \mathbb{R}$ and $\theta_{\pm} = (\theta \pm i\pi x + \sigma)iK_{\ell}/\pi$. We employed here the Jacobian elliptic functions for which we use the common notation $\text{pq}(z)$ with $p, q \in \{s, c, d, n\}$ (see e.g. [43] for standard properties). We derive important relations between the q -deformed gamma functions and the Jacobian elliptic sn-function

$$\text{sn}(x) = \frac{1}{\ell^{\frac{1}{4}}} \frac{\Gamma_{\hat{q}}(\frac{x}{2K_{\ell}} \mp \frac{\tau}{2}) \Gamma_{\hat{q}}(1 - \frac{x}{2K_{\ell}} \pm \frac{\tau}{2})}{\Gamma_{\hat{q}}(\frac{x}{2K_{\ell}}) \Gamma_{\hat{q}}(1 - \frac{x}{2K_{\ell}})}, \quad (5.18)$$

$$= \frac{q^{\frac{1}{4} - \frac{ix}{2K_{1-\ell}}} \Gamma_q(\frac{1}{2} + \frac{ix}{2K_{1-\ell}}) \Gamma_q(\frac{1}{2} - \frac{ix}{2K_{1-\ell}})}{i\ell^{\frac{1}{4}} \Gamma_q(1 - \frac{ix}{2K_{1-\ell}}) \Gamma_q(\frac{ix}{2K_{1-\ell}})}. \quad (5.19)$$

These relations can be used to obtain some of the above mentioned expressions. For instance, recalling that $\text{sn}(K_{\ell}) = 1$, we obtain (5.12). With (5.6) we recover from this the well known identity $\text{sn}(iK_{1-\ell}/2) = i/\ell^{1/4}$. The trigonometric limits

$$\lim_{\ell \rightarrow 0} \text{sn}(x) = \lim_{\hat{q} \rightarrow 1} \text{sn}(x) = \frac{\pi}{\Gamma(\frac{x}{\pi}) \Gamma(1 - \frac{x}{\pi})} = \sin(x) \quad (5.20)$$

$$\lim_{\ell \rightarrow 1} \text{sn}(x) = \lim_{q \rightarrow 1} \text{sn}(x) = \frac{1}{i} \frac{\Gamma(\frac{1}{2} + \frac{ix}{\pi}) \Gamma(\frac{1}{2} - \frac{ix}{\pi})}{\Gamma(1 - \frac{ix}{\pi}) \Gamma(\frac{ix}{\pi})} = \tanh(x). \quad (5.21)$$

can be read off directly recalling (5.2), (5.7) and presuming that (5.16) holds. We recall the zeros and poles of the Jacobian elliptic $\text{sn}(\theta)$ -function, which in our conventions are located at

$$\text{zeros:} \quad \theta_{\text{sn},0}^{lm} = 2lK_{\ell} + i2mK_{1-\ell} \quad l, m \in \mathbb{Z} \quad (5.22)$$

$$\text{poles:} \quad \theta_{\text{sn},p}^{lm} = 2lK_{\ell} + i(2m+1)K_{1-\ell} \quad l, m \in \mathbb{Z}. \quad (5.23)$$

We have now assembled the main properties of the q -deformed functions which we shall use below.

5.2 Generalizing diagonal S-matrices

Here we follow [37] and propose a quite simple principle which introduces an infinite number of unstable particles into the spectrum. We note first, that in general many scattering matrices factorize in the following form

$$S_{ab}(\theta) = S_{ab}^{\min}(\theta) S_{ab}^{\text{CDD}}(\theta). \quad (5.24)$$

Here $S_{ab}^{\min}(\theta)$ denotes the so-called minimal S-matrix which satisfies the consistency relations i)–vii) of section 2. The CDD-factor [18], only satisfies i)–vi) and has its poles in the sheet $-\pi \leq \text{Im } \theta \leq 0$, which is the “physical one” for resonance states. We note now that the minimal part is of the general form

$$S_{ab}(\theta) = \prod_{x \in \mathcal{A}} \{x\}_{\theta}^{\sigma}, \quad (5.25)$$

with \mathcal{A} being a finite set specific to each theory. Then we may define a new S -matrix

$$\hat{S}_{ab}(\theta) = \prod_{x \in \mathcal{A}} \{x\}_{\theta}^{\sigma} \{x\}_{\theta, \ell}^{\sigma} \quad (5.26)$$

and note that the additional factor in (5.26) is just of CDD-type. Therefore (5.26) constitutes a solution to the consistency relations i)–vii) of section 2, and thus a strong candidate for a scattering matrix of a proper quantum field theory. Note that whereas (5.25) was a finite product of hyperbolic functions, the new proposal (5.26) contains, according to the identity (5.17) in addition elliptic functions, which lead to the desired spectrum of infinitely many unstable particles according to the principles outlined in section 2.

5.3 Non-diagonal S-matrices

We discuss now the elliptic sine-Gordon model, which may be related to the continuum limit of the eight-vertex model. The (anti)-soliton sector was studied many years ago in [44]. In [45] we demonstrated that it is possible to associate a consistent breather sector to this model. Let us recall the argument by recalling the Zamolodchikov algebra for the soliton sector

$$Z(\theta_1)Z(\theta_2) = a(\theta_{12})Z(\theta_2)Z(\theta_1) + d(\theta_{12})\bar{Z}(\theta_2)\bar{Z}(\theta_1), \quad (5.27)$$

$$Z(\theta_1)\bar{Z}(\theta_2) = b(\theta_{12})\bar{Z}(\theta_2)Z(\theta_1) + c(\theta_{12})Z(\theta_2)\bar{Z}(\theta_1). \quad (5.28)$$

In comparison with the more extensively studied sine-Gordon model the difference is the occurrence of the amplitude d in (5.27), i.e. the possibility that two solitons change into two anti-solitons and vice versa. Invoking the consistency equations i)–v) one finds [44, 45]

$$a(\theta) = \Phi(\theta) \prod_{k=0}^{\infty} \left(\frac{\Gamma_{\hat{q}^2}[-\hat{\theta} - \frac{1+2k}{2}\lambda] \Gamma_{\hat{q}^2}[1 - \hat{\theta} - \frac{1+2k}{2}\lambda]}{\Gamma_{\hat{q}^2}[\hat{\theta} - \frac{1+2k}{2}\lambda] \Gamma_{\hat{q}^2}[1 + \hat{\theta} - \frac{1+2k}{2}\lambda]} \right. \\ \left. \times \frac{\Gamma_{\hat{q}^2}[\hat{\theta} - (k+1)\lambda] \Gamma_{\hat{q}^2}[1 + \hat{\theta} - k\lambda]}{\Gamma_{\hat{q}^2}[-\hat{\theta} - (k+1)\lambda] \Gamma_{\hat{q}^2}[1 - \hat{\theta} - k\lambda]} \right) \quad (5.29)$$

$$b(\theta) = -\frac{\text{sn}(i\theta/\nu)}{\text{sn}(i\theta/\nu + \pi/\nu)} a(\theta), \quad (5.30)$$

$$c(\theta) = \frac{\text{sn}(\pi/\nu)}{\text{sn}(i\theta/\nu + \pi/\nu)} a(\theta), \quad (5.31)$$

$$d(\theta) = -\sqrt{\ell} \text{sn}(i\theta/\nu) \text{sn}(\pi/\nu) a(\theta), \quad (5.32)$$

$$\Phi(\theta) = \frac{\Gamma_{\hat{q}}[1 + \frac{\tau}{2}] \Gamma_{\hat{q}}[-\frac{\tau}{2}] \Gamma_{\hat{q}}[1 - \hat{\theta} + \frac{\lambda}{2} + \frac{\tau}{2}] \Gamma_{\hat{q}}[\hat{\theta} - \frac{\lambda}{2} - \frac{\tau}{2}]}{\Gamma_{\hat{q}}[1 + \hat{\theta} + \frac{\tau}{2}] \Gamma_{\hat{q}}[-\hat{\theta} - \frac{\tau}{2}] \Gamma_{\hat{q}}[1 + \frac{\lambda}{2} + \frac{\tau}{2}] \Gamma_{\hat{q}}[-\frac{\lambda}{2} - \frac{\tau}{2}]} . \quad (5.33)$$

Here we used $\lambda = -\pi/K_\ell\nu$, $\hat{\theta} = i\theta/2K_\ell\nu$ with $\nu \in \mathbb{R}$ being the coupling constant of the model. With regard to property vii), it is clear that it is important to analyse the singularity structure of the amplitudes (5.29)-(5.32) to judge whether there exists a breather sector. For this we appeal to the relations (5.16), (5.22) and (5.23) and find the following pole structure inside the physical sheet

$$\begin{aligned} \theta_{a_{1,p}}^{nm} &= 2m\nu K_{1-\ell} + i2n\nu K_\ell, & \theta_{a_{2,p}}^{nm} &= (2m+1)\nu K_{1-\ell} + i(\pi - 2n\nu K_\ell), \\ \theta_{b_{1,p}}^{lm} &= 2m\nu K_{1-\ell} + i(\pi - 2l\nu K_\ell), & \theta_{b_{2,p}}^{lm} &= (2m+1)\nu K_{1-\ell} + i2l\nu K_\ell, \\ \theta_{c_{1,p}}^{lm} &= 2m\nu K_{1-\ell} + i2l\nu K_\ell, & \theta_{c_{2,p}}^{lm} &= 2m\nu K_{1-\ell} + i(\pi - 2l\nu K_\ell), \\ \theta_{d_{1,p}}^{lm} &= (2m+1)\nu K_{1-\ell} + i2l\nu K_\ell, & \theta_{d_{2,p}}^{lm} &= (2m+1)\nu K_{1-\ell} + i(\pi - 2n\nu K_\ell). \end{aligned}$$

We took $l, m \in \mathbb{Z}, n \in \mathbb{N}$ and associated always two sets of poles $\theta_{a_{1,p}}^{nm}$ and $\theta_{a_{2,p}}^{nm}$ to $a(\theta)$, $\theta_{b_{1,p}}^{nm}$ and $\theta_{b_{2,p}}^{nm}$ to $b(\theta)$ etc. One readily sees from this that if one restricts the parameter $\nu \geq \pi/2K_\ell$ all poles move out of the physical sheet into the non-physical one, where they can be interpreted in principle as unstable particles. This was already stated in [44], where the choice $\nu \geq \pi/2K_\ell$ was made in order to avoid the occurrence of non-physical states. This is clear from our discussion of property vii) in section 2, as we would have poles in the physical sheet beyond the imaginary axis, which when interpreted with the Breit-Wigner formula leave the choice that either $m_{\bar{c}} < 0$ or $\Gamma_{\bar{c}} < 0$, i.e. we either violate causality or we have Tachyons. The restriction on the parameters makes the model somewhat unattractive as this limitation eliminates the analogue of the entire breather sector which is present in the sine-Gordon model, such that also in the trigonometric limit one only obtains the soliton-antisoliton sector of that model, instead of a theory with a richer particle content. For this reason we relax here the restriction on ν and note that the poles

$$\theta_{b_{1,p}}^{n0} = \theta_{c_{2,p}}^{n0} \quad \text{for } 0 < n < n_{\max} = [\pi/2\nu K_\ell], n \in \mathbb{N} \quad (5.34)$$

are located on the imaginary axis inside the physical sheet and are therefore candidates for the analogue of the n^{th} -breather bound states in the sine-Gordon model. We indicate here the integer part of x by $[x]$. In other words, there are at most $n_{\max} - 1$ breathers for fixed ν and ℓ . The price one pays for the occurrence of these new particles in the elliptic sine-Gordon model is that one unavoidably also introduces additional Tachyons into the model as the poles always emerge in “strings”. It remains to be established whether the poles (5.34) may really be associated to a breather type behaviour.

Let us now see if the poles on the imaginary axis inside the physical sheet can be associated consistently with breathers. We proceed similarly as for the sine-Gordon model [46], even though in the latter approach the following ansatz is inspired by the classical theory and here we do not have a classical counterpart. We define the auxiliary state

$$Z_n(\theta_1, \theta_2) := \frac{1}{\sqrt{2}} [Z(\theta_1)\bar{Z}(\theta_2) + (-1)^n \bar{Z}(\theta_1)Z(\theta_2)] . \quad (5.35)$$

This state has properties of the classical sine-Gordon breather being chargeless and having parity $(-1)^n$. Choosing thereafter the rapidities such that the state (5.35) is on-shell, we can speak of a breather bound state

$$\lim_{(p_1+p_2)^2 \rightarrow m_{b_n}^2} Z_n(\theta_1, \theta_2) \equiv \lim_{\theta_{12} \rightarrow \theta + \theta_{12}^{b_n}} Z_n(\theta_1, \theta_2) = Z_n(\theta) . \quad (5.36)$$

Here $\theta_{12}^{b_n}$ is the fusing angle related to the poles in the soliton-antisoliton scattering amplitudes. We compute now with the help of (5.27) and (5.28) the exchange relation

$$Z_n(\theta_1)Z(\theta_2) = S_{b_n s}(\theta_{12})Z(\theta_2)Z_n(\theta_1) , \quad (5.37)$$

where

$$S_{b_n s}(\theta) = \frac{\text{sn}(\frac{i\theta}{\nu} - \frac{\pi}{2\nu} + nK_\ell)}{\text{sn}(\frac{i\theta}{\nu} + \frac{\pi}{2\nu} + nK_\ell)} [\ell \text{sn}^2 \frac{\pi}{\nu} \text{sn}^2 \left(\frac{i\theta}{\nu} + \frac{\pi}{2\nu} + nK_\ell \right) - 1] \bar{a} \quad (5.38)$$

and

$$\begin{aligned} \bar{a} = & \frac{\Gamma_{\hat{q}^2}[1 + \hat{\theta} + \frac{\lambda}{4} - \frac{n}{2}]\Gamma_{\hat{q}^2}[-\hat{\theta} - \frac{\lambda}{4} - \frac{n}{2}]\Gamma_{\hat{q}^2}[-\hat{\theta} + \frac{\lambda}{4} + \frac{n}{2}]\Gamma_{\hat{q}^2}[\hat{\theta} + \frac{\lambda}{4} - \frac{n}{2}]}{\Gamma_{\hat{q}^2}[1 - \hat{\theta} + \frac{\lambda}{4} - \frac{n}{2}]\Gamma_{\hat{q}^2}[\hat{\theta} - \frac{\lambda}{4} - \frac{n}{2}]\Gamma_{\hat{q}^2}[\hat{\theta} + \frac{\lambda}{4} + \frac{n}{2}]\Gamma_{\hat{q}^2}[-\hat{\theta} + \frac{\lambda}{4} - \frac{n}{2}]} \\ & \times \Phi_{13}\Phi_{23} \prod_{l=1}^{n-1} \frac{[\hat{\theta} - \frac{n}{2} + \frac{\lambda}{4} - k\lambda + l]_{\hat{q}^2}^2 [-\hat{\theta} + \frac{n}{2} - \frac{\lambda}{4} - k\lambda - l]_{\hat{q}^2}^2}{[-\hat{\theta} - \frac{n}{2} + \frac{\lambda}{4} - k\lambda + l]_{\hat{q}^2}^2 [\hat{\theta} + \frac{n}{2} - \frac{\lambda}{4} - k\lambda - l]_{\hat{q}^2}^2} \\ & \times \prod_{k=0}^{\infty} \frac{[\hat{\theta} - \frac{n}{2} + \frac{\lambda}{4} - k\lambda]_{\hat{q}^2} [-\hat{\theta} + \frac{n}{2} - \frac{\lambda}{4} - k\lambda]_{\hat{q}^2}}{[-\hat{\theta} - \frac{n}{2} + \frac{\lambda}{4} - k\lambda]_{\hat{q}^2} [\hat{\theta} + \frac{n}{2} - \frac{\lambda}{4} - k\lambda]_{\hat{q}^2}} \\ & \times \prod_{k=0}^{\infty} \frac{[\hat{\theta} + \frac{n}{2} + \frac{\lambda}{4} - k\lambda]_{\hat{q}^2} [-\hat{\theta} - \frac{n}{2} - \frac{\lambda}{4} - k\lambda]_{\hat{q}^2}}{[-\hat{\theta} + \frac{n}{2} + \frac{\lambda}{4} - k\lambda]_{\hat{q}^2} [\hat{\theta} - \frac{n}{2} - \frac{\lambda}{4} - k\lambda]_{\hat{q}^2}} \end{aligned}$$

Where $\Phi_{ij} = \Phi(\theta_{ij})$ with θ_{ij} being the difference of the on-shell rapidities. What is remarkable here and can not be anticipated a priori, is that all off-diagonal terms vanish, thus as (5.37) expresses in the soliton breather scattering there is no backscattering. Similarly, but more lengthy, we compute the scattering amplitude between the n^{th} -breather and m^{th} -breather

$$Z_n(\theta_1)Z_m(\theta_2) = S_{b_n b_m}(\theta_{12})Z_m(\theta_2)Z_n(\theta_1) \quad (5.39)$$

where

$$S_{b_n b_m}(\theta) = \left[1 - \ell \text{sn}^2 \frac{\pi}{\nu} \text{sn}^2 \left(\frac{i\theta}{\nu} + (n+m)K_\ell + \frac{\pi}{\nu} \right) \right] \quad (5.40)$$

$$\times \left[1 - \ell \text{sn}^2 \frac{\pi}{\nu} \text{sn}^2 \left(\frac{i\theta}{\nu} + (n+m)K_\ell \right) \right] \frac{\text{sn}(i\theta/\nu - \pi/\nu + (n+m)K_\ell)}{\text{sn}(i\theta/\nu + \pi/\nu + (n+m)K_\ell)} \bar{a}$$

and

$$\begin{aligned}
 \tilde{a} = & \Phi_{13}\Phi_{14}\Phi_{23}\Phi_{24} \frac{\Gamma_{\hat{q}^2}(1 + \frac{m}{2} + \frac{n}{2} + \hat{\theta} + \frac{\lambda}{2}) \Gamma_{\hat{q}^2}(\frac{-m}{2} - \frac{n}{2} - \hat{\theta} - \frac{\lambda}{2})}{\Gamma_{\hat{q}^2}(1 + \frac{m}{2} + \frac{n}{2} - \hat{\theta} + \frac{\lambda}{2}) \Gamma_{\hat{q}^2}(\frac{-m}{2} - \frac{n}{2} + \hat{\theta} - \frac{\lambda}{2})} \\
 & \times \prod_{k=1}^{\infty} \prod_{l=1}^{n-1} \frac{[\frac{m}{2} + \frac{n}{2} - l - \hat{\theta} - k\lambda + \lambda]_{\hat{q}^2} [\frac{-m}{2} - \frac{n}{2} + l + \hat{\theta} - k\lambda]_{\hat{q}^2}}{[\frac{m}{2} + \frac{n}{2} - l + \hat{\theta} - k\lambda + \lambda]_{\hat{q}^2} [\frac{-m}{2} - \frac{n}{2} + l - \hat{\theta} - k\lambda]_{\hat{q}^2}} \\
 & \times \prod_{k=0}^{\infty} \prod_{l=1}^{n-1} \frac{[\frac{m}{2} + \frac{n}{2} - l + \hat{\theta} - \frac{\lambda}{2} - k\lambda]_{\hat{q}^2} [-\frac{m}{2} - \frac{n}{2} + l - \hat{\theta} - \frac{\lambda}{2} - k\lambda]_{\hat{q}^2}}{[\frac{m}{2} + \frac{n}{2} - l - \hat{\theta} - \frac{\lambda}{2} - k\lambda]_{\hat{q}^2} [-\frac{m}{2} - \frac{n}{2} + l + \hat{\theta} - \frac{\lambda}{2} - k\lambda]_{\hat{q}^2}} \\
 & \times \prod_{k=1}^{\infty} \prod_{l=0}^{m-1} \frac{[\frac{m}{2} + \frac{n}{2} - l - \hat{\theta} - k\lambda + \lambda]_{\hat{q}^2} [-\frac{m}{2} - \frac{n}{2} + l + \hat{\theta} - k\lambda]_{\hat{q}^2}}{[\frac{m}{2} + \frac{n}{2} - l + \hat{\theta} - k\lambda + \lambda]_{\hat{q}^2} [-\frac{m}{2} - \frac{n}{2} + l - \hat{\theta} - k\lambda]_{\hat{q}^2}} \\
 & \times \prod_{k=0}^{\infty} \prod_{l=0}^{m-1} \frac{[\frac{m}{2} + \frac{n}{2} - l + \hat{\theta} - \frac{\lambda}{2} - k\lambda]_{\hat{q}^2} [-\frac{m}{2} - \frac{n}{2} + l - \hat{\theta} - \frac{\lambda}{2} - k\lambda]_{\hat{q}^2}}{[\frac{m}{2} + \frac{n}{2} - l - \hat{\theta} - \frac{\lambda}{2} - k\lambda]_{\hat{q}^2} [-\frac{m}{2} - \frac{n}{2} + l + \hat{\theta} - \frac{\lambda}{2} - k\lambda]_{\hat{q}^2}}.
 \end{aligned} \tag{5.41}$$

The latter expression (5.41) is tailored to make contact to the expressions in the literature corresponding to the trigonometric limit. Also for this amplitude the backscattering is zero.

The matrix $S_{b_n b_m}(\theta)$ also exhibits several types of poles. a) simple and double poles inside the physical sheet beyond the imaginary axis, b) double poles located on the imaginary axis, c) simple poles in the non-physical sheet and d) one simple pole on the imaginary axis inside the physical sheet at $\theta = \theta_b = i\nu(n+m)K_\ell$ which is related to the fusing process of two breathers $b_n + b_m \rightarrow b_{n+m}$. To be really sure that this pole admits such an interpretation, we have to establish according to (2.11) that the imaginary part of the residue is strictly positive, i.e.

$$-i \lim_{\theta \rightarrow \theta_b} (\theta - \theta_b) S_{b_n b_m}(\theta) > 0. \tag{5.42}$$

The explicit computation shows that this is indeed the case, see [45].

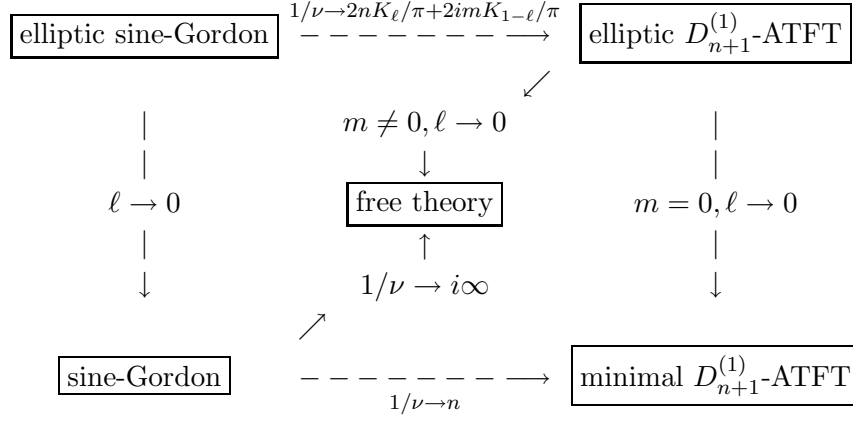
Furthermore, It is very interesting to check if also (2.10) is satisfied for the fusing process $b_n + b_m \rightarrow b_{n+m}$. For consistency, all amplitudes have to satisfy the bootstrap equations

$$S_{lb_{n+m}}(\theta) = S_{lb_n}(\theta + i\nu m K_\ell) S_{lb_m}(\theta - i\nu n K_\ell), \tag{5.43}$$

for $l \in \{b_k, s, \bar{s}\}; k, m+n < n_{\max}$. Indeed, we verify with some algebra that (5.43) holds for the above amplitudes (5.38) and (5.40).

Finally, we carry out various limits. Our formulation in terms of q-deformed quantities and elliptic functions is useful to make this task fairly easy. We state our results here only

schematically and refer the reader for details to [45]. We find



Thus we can view the elliptic sine-Gordon model as a master theory for several other models. In the limit $\ell \rightarrow 0$ we recover now all sectors, including the breathers, of the sine-Gordon model. The diagonal limit $1/\nu \rightarrow 2nK_\ell/\pi + 2imK_{1-\ell}/\pi$ is interesting as it yields a new type of theory, which we refer to as elliptic $SO(2n+2) \equiv D_{n+1}^{(1)}$ -affine Toda field theory (ATFT). To coin this name for these theories seems natural as in the trigonometric limit we obtain from it the ordinary minimal $D_{n+1}^{(1)}$ -ATFT.

6. Conclusions

We reviewed the general analytical scattering theory related to integrable quantum field theories in 1+1 space-time dimensions. We made a proposal for a construction principle of an S-matrix like object which describes the scattering between two unstable particles or an unstable particle and a stable one. We tested this proposal with various examples and found a remarkable agreement with the outcome of the thermodynamic Bethe ansatz in what concerns the particle content and the RG flow of the theories. We described the general Lie algebraic structure of theories with unstable particles and propose a decoupling rule which predicts the RG flow when some of the parameters in the theory become very large. Alternatively, we tested these analytical prediction with the TBA. Finally, we discussed how one can construct theories with and without backscattering which contain an infinite number of unstable particles.

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