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Approximate Greatest Common Divisor of Many Polynomials and Pseudo-Spectrum

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Abstract: The paper is concerned with establishing the links between the approximate GCD of a set of polynomials and the notion of the pseudo-spectrum defined on a set of polynomials. By examining the pseudo-spectrum of the structured matrix we will derive estimates of the area of the approximate roots of the initial polynomial set. We will relate the strength of the GCD to the weighted strength of the pseudo-spectra and we investigate under which conditions the roots of the approximate GCDs are a subset of the pseudo-spectra.

Keywords: Linear systems; approximate GCD; pseudo-spectrum; strength; algebraic computations.

1. INTRODUCTION

The computation of the GCD of a set of polynomials is a problem representative of the class of nongeneric computations (Karcanias & Mitrouli, 1999). In fact, the set of polynomials for which there exists a nontrivial GCD (different than one) is a subvariety of a projective space and this makes the GCD computation a hard problem. The need for defining a notion of "almost zero" for a set of polynomials has been recognised in (Karcanias et al., 1983) where it has been shown that "almost zeros" behave in similar way to exact way to exact zeros, as far as solutions of polynomial Diophantine equations. Defining an *approximate GCD* has been subsequently considered within the framework of GCD computations and amongst the methods considered have been the ERES method (Mitrouli & Karcanias, 1993), the matrix pencil method (Karcanias & Mitrouli, 1994) and the different variants of the Euclid algorithm (Noda & Sasaki, 1991).

The essence of the computation of approximate solutions is that they are based on the relaxation of exact conditions which characterise the GCD. Until recently there has been no formal framework that may allow the evaluation of quality of the approximation. A novel framework has been introduced recently (Karcanias et al., 2003) for defining in a parametric way all given order approximate GCDs and evaluate their quality of approximation, or strength by solving an optimisation problem. This approach is based on recent results (Fatouros & Karcanias, 2003) on the representation of the GCD of many polynomials in terms of the factorisation of the generalised resultant into a reduced resultant and a Toeplitz matrix representing the GCD. These results allow the parameterisation of all perturbations which are required to make a selected approximate GCD, an exact GCD of the perturbed set of polynomials. The notion of the approximate GCD is introduced by considering polynomial set perturbations which result in perturbed sets of polynomials of the nominal set and are characterised by the existence of a given degree GCD. The new notion is introduced as the solution of a distance problem between points and certain varieties in an appropriate projective space. The resulting optimisation problem is linked to the definition of a given order approximate GCD. This problem is reduced to two independent and standard form optimisation problems on certain functions defined by the original set of polynomials. The results allow the derivation of given order *optimal* approximate solutions and permit the evaluation of their respective strengths.

In this paper we establish the links between *approximate GCD*, *almost zeros* and the *pseudo-spectrum* (Lancaster & Psarrakos, 2005) of a polynomial matrix. By examining the pseudo-spectrum of the structured matrix we will derive estimates of the area of the approximate roots of the initial polynomial set. We will relate the strength of the GCD to the weighted strength of the pseudo-spectra and we will investigate under which conditions the roots of the approximate GCDs are a subset of the pseudo-spectra *F*. The final aim is to build a hybrid algorithm using the pseudo-spectra and the resultant algorithm for the location and the evaluation of the approximate GCD. Issues under investigation relate to the area of approximate common roots to the pseudo-spectra. The research aims to find a direct algorithm for the GCD using the pseudo-spectra algorithm and then extend the research to the almost zeroes of the polynomial algorithms.

2. DEFINITIONS AND PRELIMINARY RESULTS

For a set $\mathcal{P}=\{a(s),b_i(s)\in\mathbb{R}[s],\ i\in\underset{\sim}{h}\}$ of polynomials which has $h+1$ elements and with the two largest values of degrees (n,p), which is also denoted as $\mathcal{P}_{h+1,n}$. Without loss of generality we may assume $a(s)$ monic and represent the polynomials with respect to the *n* degree as

$$\begin{aligned} a(s) &= a_n s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0, \quad a_0 \neq 0 \\ b(s) &= b_{in} s^n + b_{in-1}s^{n-1} + \ldots + b_{i1} s + b_{i0}, \quad i = 1,2,\ldots,h \end{aligned} \tag{1}$$

The greatest common divisor (GCD) of $\mathcal{P}$ will be denoted by $\varphi(s)$. For any $\mathcal{P}_{h+1,n}$ set we define a vector (vr) $\underline{p}_{h+1}(s)$ and a matrix (bm) P_{h+1} as

$$\begin{aligned} \underline{p}_{h+1}(s) &= \left[a(s),\ b_1(s),\ \cdots,\ b_h(s)\right]^t \\ &= \left[\underline{p}_n, \underline{p}_{n-1},\ldots,\underline{p}_0\right] \underline{e}_n(s) = P_{h+1}\,\underline{e}_n(s) \end{aligned} \tag{2}$$

$P_{h+1}\in\mathbb{R}^{(h+1)\times(n+1)}$, $\underline{e}_n(s)=\left[s^n,s^{n-1},\ldots,s,1\right]^t$. The classical approaches for the determination of the GCD make use of the Sylvester Resultant defined by (Barnett, 1990):

Definition 1: Let

$$\mathcal{P}_{h+1,n}=\{a(s),b_i(s),i\in\underset{\sim}{h},\ n=\deg\{a(s)\},$$
$$n\geq\deg\{b_i(s)\}\ \ \forall i\in\underset{\sim}{h},\ \ p=\max\{\deg\{b_i(s)\},\ i\in\underset{\sim}{h}\}\}$$

where $a(s),b(s)$ are described as in (1). Then, we can define a $p\times(n+p)$ matrix S_0 associated with $a(s)$ and a $n\times(n+p)$ matrix S_i associated with $b_i(s)$ for each $i=1,2,\ldots,h$:

$$S_0=\begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & & \ddots & & & & & & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} & \cdots & & a_1 & a_0 \end{bmatrix}$$

$$S_i=\begin{bmatrix} b_{i,p} & b_{i,p-1} & b_{i,p-2} & \cdots & b_{i,1} & b_{i,0} & 0 & \cdots & 0 \\ 0 & b_{i,p} & b_{i,p-1} & \cdots & b_{i,2} & b_{i,1} & b_{i,0} & \cdots & 0 \\ \vdots & & \ddots & & & & & & \vdots \\ 0 & \cdots & 0 & b_{i,p} & b_{i,p-1} & \cdots & & b_{i,1} & b_{i,0} \end{bmatrix}$$

and $S_{\mathcal{P}}=\begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_h \end{bmatrix}\in\mathbb{R}^{(p+hn)\times(n+p)}$ is an *extended Sylvester matrix* for the set $\mathcal{P}$. ■

The resultant properties are summarised below (Barnett, 1990), (Fatouros & Karcanias, 2003).

Theorem 1: *For a set of polynomials $\mathcal{P}_{h+1,n}$ with resultant $S_{\mathcal{P}}$ the following properties hold true:*

(i) *Necessary and sufficient condition for a set of polynomials to be coprime is that:*

$$\rho(S_{\mathcal{P}})=n+p \tag{3}$$

(ii) *Let $\varphi(s)$ be the GCD of $\mathcal{P}$. Then*

$$\rho(S_{\mathcal{P}})=n+p-\deg\varphi(s) \tag{4}$$

If we reduce $S_{\mathcal{P}}$, by using elementary row operations, to its row echelon form, the last non vanishing row defines the coefficients of the GCD.

Certain properties of extraction of divisors from the set $\mathcal{P}$, which are equivalently expressed as factorisation of resultant matrices are summarised below (Fatouros & Karcanias, 2003) and these establish a matrix based representation of the GCD, which is equivalent to the standard algebraic factorisation of the GCD in the original set of polynomials.

Theorem 2: *Let $\mathcal{P}=\{a(s),b_1(s),\ldots,b_h(s)\}$ be a 0-order set, $\deg a(s)=n$, $\deg b_i(s)\leq p\leq n$, $i=1,\ldots,h$ be a polynomial set, $S_{\mathcal{P}}$ the respective Sylvester matrix, $\varphi(s)=\lambda_k s^k+\cdots+\lambda_1 s+\lambda_0$ be the greatest common divisor of the set and let k be its degree. Then there exists transformation matrix $\Phi_{\varphi}\in\mathbb{R}^{(n+p)\times(n+p)}$ such that:*

$$\overline{S}^{(k)}_{\mathcal{P}^*}=S_{\mathcal{P}}\Phi_{\varphi}=\left[\mathbf{0}_k\ \middle|\ \overline{S}_{\mathcal{P}^*}\right],$$

$$S_{\mathcal{P}}=\overline{S}^{(k)}_{\mathcal{P}^*}\,\hat{\Phi}_{\varphi}=\left[\mathbf{0}_k\ \middle|\ \overline{S}_{\mathcal{P}^*}\right]\hat{\Phi}_{\varphi} \tag{5}$$

where $\Phi_{\varphi}=\hat{\Phi}_{\varphi}^{-1}$, $\hat{\Phi}_{\varphi}$ being the Toeplitz form of $\varphi(s)$ and $\hat{\Phi}_{\varphi}$ and $S_i^{(k)}$, $i=1,2,\ldots,h$ the Toeplitz blocks, which correspond to the coefficients of the coprime polynomials obtained from the original set after the division by the GCD, is expressed as

$$\hat{\Phi}_{\varphi}=\begin{bmatrix} \lambda_0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \lambda_1 & \lambda_0 & \ddots & & & & & \vdots \\ \lambda_2 & \lambda_1 & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & & \vdots \\ \lambda_k & & & & \ddots & \ddots & & \vdots \\ 0 & \lambda_k & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \lambda_1 & \lambda_0 & 0 \\ 0 & \cdots & 0 & \lambda_k & & \lambda_2 & \lambda_1 & \lambda_0 \end{bmatrix}$$

and

$$\overline{S}_{\mathcal{P}^*}^{(k)} = \begin{bmatrix} \mathbf{0} & S_0^{(k)} \\ \mathbf{0} & S_1^{(k)} \\ & \vdots \\ \mathbf{0} & S_h^{(k)} \end{bmatrix} = [\mathbf{0} \;\; \tilde{S}_{\mathcal{P}}^{(k)}] \tag{6}$$

■

If $a(s) = a'(s)\varphi(s)$, $b_i(s) = b_i'(s)\varphi(s)$, $i = 1,..,h$, $\mathcal{P}^* = \{a'(s), b_1'(s), \ldots, b_h'(s)\}$, and $S_{\mathcal{P}}$, $\overline{S}_{\mathcal{P}^*}^{(k)}$ are the generalised resultants of $\mathcal{P}$, $\mathcal{P}^*$, then $\overline{S}_{\mathcal{P}^*}$ is the *(n,p)-expanded resultant of* $\mathcal{P}^*$.

We shall denote by $\mathcal{T}(n, p; h+1)$ the set of all polynomial sets $\mathcal{P}_{h+1,n}$ with the (n,p) the maximal two degrees and h+1 elements. If $\mathcal{P}_{h+1,n} \in \mathcal{T}(n, p; h+1)$ we can define an (n,p)-ordered perturbed set

$$\mathcal{P}'_{h+1,n} = \mathcal{P}_{h+1,n} - \mathcal{Q}_{h+1,n} \in \mathcal{T}(n, p; h+1)$$
$$= \{p_i'(s) = p_i(s) - q_i(s) : \deg\{q_i(s)\} \le \deg\{p_i(s)\}\}$$

Given a set $\mathcal{P}_{h+1,n} \in \mathcal{T}(n, p; h+1)$ and an $\omega(s) \in \mathbb{R}[s]$ with $\deg\{\omega(s)\} \le p$, there always exists a family of (n, p)-ordered perturbations $\mathcal{Q}_{h+1,n}$ and for every element of this family $\mathcal{P}'_{h+1,n} = \mathcal{P}_{h+1,n} - \mathcal{Q}_{h+1,n}$ has a GCD divisible by $\omega(s)$. If $\Sigma_\omega = \{\mathcal{Q}_{h+1,n}\}$ is the set of all (n, p)-order perturbations

$$\mathcal{P}'_{h+1,n} = \mathcal{P}_{h+1,n} - \mathcal{Q}_{h+1,n} \in \mathcal{T}(n, p; h+1) \tag{7}$$

with the property that $\omega(s)$ is a common factor of the elements of $\mathcal{P}'_{h+1,n}$. If $\mathcal{Q}^*_{h+1,n}$ is the minimal norm element of the set Σ_ω, then $\omega(s)$ is referred as an *r-order almost common factor* of $\mathcal{P}_{h+1,n}$, and the norm of $\mathcal{Q}^*_{h+1,n}$, denoted by $\|\mathcal{Q}^*\|$, as the *strength* of $\omega(s)$. If $\omega(s)$ is the GCD of $\mathcal{P}^*_{h+1,n} = \mathcal{P}_{h+1,n} - \mathcal{Q}^*_{h+1,n}$ then $\omega(s)$ will be called an *r-order almost GCD* of $\mathcal{P}_{h+1,n}$ with strength $\|\mathcal{Q}^*\|$. The above definition suggests that any polynomial $\omega(s)$ may be considered as an approximate GCD, as long as $\deg\{\omega(s)\} \le p$. Two important problems in this context are the evaluation of the strength of approximation of a given approximate GCD, $\omega(s)$, and the computation of the given order *Optimal Almost GCD*. These two problems have been considered in (Karcanias et al., 2006). This approach is now linked to the notion of pseudo-spectrum developed in (Tisseur & Higham, 2001), (Lancaster & Psarrakos, 2005):

Definition 2: We consider the spectra of perturbations of the regular polynomial $P(s)$ of the form

$$P_\Delta(s) = (A_n + \Delta_n)s^n + (A_{n-1} + \Delta_{n-1})s^{n-1} + \ldots + (A_1 + \Delta_1)s + (A_0 + \Delta_0) \tag{8}$$

where the matrices $\Delta_0, \Delta_1, \ldots, \Delta_n \in C^{(h+1)\times(h+1)}$ are arbitrary. For a given $\varepsilon > 0$ and a given set of nonnegative weights $w = \{w_0, w_1, \ldots, w_n\}$ with at least one nonzero element, we define the set of perturbed matrix polynomials

$$B(P, e, w) = \{P_\Delta(s) : \|\Delta_j\|_2 \le \varepsilon w_j,\ j = 0, 1, \ldots, n\}.$$

The (weighted) ε-pseudo-spectrum of $P(s)$ (Tisseur & Higham, 2001) is then defined as

$$\sigma_{\varepsilon,w}(\mathcal{P}) = \{s \in \mathbb{C} : \det P_\Delta(s) = 0,\ P_\Delta(s) \in B(P, e, w)\}. \quad ■$$

Theorem 3 (Tisseur & Higham, 2001): If we consider the scalar polynomial $w(\lambda) = w_m\lambda^m + \ldots + w_1\lambda + w_0$ and let $s_1(P(\lambda)) \ge s_2(P(\lambda)) \ge \ldots \ge s_n(P(\lambda))$ be the singular values of $P(\lambda)$ then

$$\sigma_{\varepsilon,w}(\mathcal{P}) = \{\lambda \in \mathbb{C} : s_n(P(\lambda)) \le \varepsilon w(|\lambda|)\}. \quad ■$$

Any point $\lambda_0 \in \mathbb{C}$ that is not on the bounds of the pseudo-spectrum, i.e. $s_n(P(\lambda)) \ne \varepsilon w(|\lambda|)$ lies in the centre of an open disk that does not intersect the boundaries of the pseudo-spectrum (Psarrakos, 2007), (Fatouros & Psarrakos, 2009). Based on that an algorithm that uses exclusion disks is introduced in (Fatouros & Psarrakos, 2009) for the computation of the pseudo-spectra.

3. ESTIMATING THE APPROXIMATE COMMON ROOTS OF A SET OF POLYNOMIALS

For the set of monic polynomials

$$\mathcal{P}_{h,n} = \{p_1(s), \ldots, p_h(s), \deg p_1(s) = n, \deg p_i(s) \le n, i = 2, \ldots, h\}$$

we may define the matrix

$$P(s) = \left[\begin{array}{ccc|c} & & & -(s^n - k^n) \\ & (s^n - k^n) I_h & & -(s^n - k^n) \\ & & & \vdots \\ & & & -(s^n - k^n) \\ \hline p_1(s) & p_2(s) & \ldots \; p_h(s) & 0 \end{array}\right] \tag{9}$$

The matrix $P(s)$ can also be written in the form:

$$P(s) = A_n s^n + A_{n-1}s^{n-1} + \ldots + A_1 s + A_0$$

where $A_i \in \mathbb{C}^{(h+1)\times(h+1)}$, $i = 0, 1, \ldots, n$.

The leading coefficient matrix is nonsingular and it has the following structure

$$A_n = \left[\begin{array}{ccc|c} & I_h & & \begin{matrix}-1\\-1\\\vdots\\-1\end{matrix} \\ \hline 1 & p_{2,n}(s) \quad ... \quad & p_{h,n}(s) & 0 \end{array}\right] \tag{10}$$

$$A_j = \left[\begin{array}{ccc|c} & 0_h & & \begin{matrix}0\\0\\\vdots\\0\end{matrix} \\ \hline p_{1,j}(s) & p_{2,j}(s) \quad ... \quad & p_{h,j}(s) & 0 \end{array}\right] \tag{11}$$

for $j = 1, 2, ..., n-1$ and the constant coefficient matrix will have the form

$$A_0 = \left[\begin{array}{ccc|c} & -k^n\, I_h & & \begin{matrix}k^n\\k^n\\\vdots\\k^n\end{matrix} \\ \hline p_{1,0} & p_{2,0} \quad \cdots \quad & p_{h,0} & 0 \end{array}\right] \tag{12}$$

The block

$$\left[\begin{array}{c|c} \left(s^n - k^n\right) I_h & \begin{matrix}-\left(s^n-k^n\right)\\\vdots\\\left(s^n-k^n\right)\end{matrix} \end{array}\right]$$

is constructed in this form in order to have a non-singular leading coefficient for the polynomial matrix $P(s)$. If we select the parameter k outside the investigation area, the corresponding eigenvalues lie outside that area and they do not interfere with the roots of the polynomials in the last row.

Consider a set of polynomials in $\Sigma_\omega = \left\{\mathcal{Q}_{h+1,n}\right\}$, $e_1(s) = \sum_{i=0}^{n} e_{1,i}\, s^i$, $e_k(s) = \sum_{j=0}^{\deg e_k} e_{k,j}\, s^j$, $k = 1, ..., h$. Then there exists a GCD $u(s)$ of degree $d \geq 1$ corresponding to the perturbed set

$\left\{\tilde{p}_1(s), \tilde{p}_2(s), ..., \tilde{p}_h(s) \mid \tilde{p}_i(s) = p_i(s) + e_i(s),\ i = 1, ..., h\right\}$.

The $\tilde{F}$ matrix corresponding to the perturbed set will have the form

$$\tilde{F}(s) = \left[\begin{array}{ccc|c} & s^n I_n & & \begin{matrix}-s^n\\\vdots\\-s^n\end{matrix} \\ \hline p_1(s)+e_1(s) & \cdots & p_h(s)+e_h(s) & 0 \end{array}\right] \tag{13}$$

where

$$\tilde{F}(s) = \left[\begin{array}{ccc|c} & s^n I_n & & \begin{matrix}-s^n\\\vdots\\-s^n\end{matrix} \\ \hline p_1(s) & \cdots & p_h(s) & 0 \end{array}\right] + \left[\begin{array}{ccc|c} & 0_n & & \begin{matrix}0\\\vdots\\0\end{matrix} \\ \hline e_1(s) & \cdots & e_h(s) & 0 \end{array}\right]$$

$$= \left(F_n + E_n\right) s^n + ... + \left(F_1 + E_1\right) s + \left(F_0 + E_0\right)$$

The coefficient matrices will have the form

$$E_i(s) = \left[\begin{array}{ccc|c} & 0_n & & \begin{matrix}0\\\vdots\\0\end{matrix} \\ \hline e_{i1} & \cdots & e_{ih} & 0 \end{array}\right],\ i = 0, 1, ..., n$$

and their 2-norms equal

$$\left\|E_j\right\|_2 = e_{j1}^2 + e_{j2}^2 + ... + e_{jk}^2 \tag{14}$$

Proposition 1: Let $\varepsilon = \max_j \left\{\left\|E_j\right\|_2\right\}$. The roots of the approximate GCD of the set of polynomials $\mathcal{P}_{h,n}$ lie within the pseudo-spectrum $\sigma_{\varepsilon,1}(\mathcal{P})$. ■

4. EXAMPLE

We consider an example to demonstrate the above concepts:

Example 1: Consider the following set of four polynomials with maximum degree equal to 4.

$$\mathcal{P}_{4,4} = \left\{\begin{array}{l} p_1(s) = s^4 - 6.002 s^3 - 3.9960 s^2 + 54.022 s - 45.024 \\ p_2(s) = s^4 - 8.010 s^3 + 14.040 s^2 + 8.0100 s - 15.040 \\ p_3(s) = s^4 - 8.001 s^3 + 10.999 s^2 + 32.004 s - 59.996 \\ p_4(s) = s^4 - 3.993 s^3 - 13.003 s^2 + 27.904 s + 59.876 \end{array}\right\}$$

The polynomials of the above set are coprime in standard double precision floating-point arithmetic. However, since the coefficients of the polynomials are given in 4-digits precision, we might look for an approximate ε_t-GCD for accuracy $\varepsilon_t = 10^{-4}$.

A recently developed method for estimating approximate GCDs of sets of several real univariate polynomials is the Hybrid-ERES method (Christou et al., 2010) which in the present case, for $\varepsilon_t = 10^{-4}$, it gives the approximate solution:

$$\varphi(s) = s^2 - 8.0119 s + 15.0482 \tag{15}$$

In order to evaluate the quality of this solution, we have to compute its 'strength'. The strength of a GCD is obtained by the minimization problem (Fatouros et al., 2006):

$$\min_{\mathcal{P}^*} \left\| S_{\mathcal{P}} - \overline{S}_{\mathcal{P}^*}^{(k)} \cdot \hat{\Phi}_{\varphi} \right\|_F \tag{16}$$

which is actually non-convex when the GCD is considered unknown. However, when an approximation $\varphi(s)$ is given, the above problem is equivalent to solving a multiple unconstrained linear least-squares (LLS) problem of the form

$$\min \left\| \hat{\Phi}_{\varphi}^{t} \cdot X - S_{\mathcal{P}}^{t} \right\| \tag{17}$$

Since the above LLS problem being solved is always convex, we can find (under certain conditions) a global solution which is a measure of the strength of the approximation $\varphi(s)$. For example, the strength of the approximation $\varphi(s)$ given in (15) is 0.1120.

The benefit from the computation of the pseudo-spectrum of the matrix $P(s)$, which corresponds to the set $\mathcal{P}_{4,4}$, is that we can estimate areas where the roots of an approximate GCD lie. The contour in Figure 1 shows the eigenvalues of the matrix $P(s)$ and the pseudo-spectrum areas around them. For accuracy $\varepsilon = 0.1$, there are two distinct areas (inner areas in contour plot) around the eigenvalues marked with '+'. The boundaries on the real axis of these areas (pseudo-spectrum intervals) are given in Table 1.

In Figure 2 we can see how the strength of a simple common factor $\varphi_1(s) = s - c$ changes when c runs through the pseudo-spectrum intervals J_1, J_2 with accuracy $\varepsilon_t = 10^{-4}$. We notice that there are two values of c, $c_1 = 2.9969$ and $c_2 = 5.0001$ (Table 2), where the strength is much closer to zero than other values. The factors $\varphi_1(s) = s - 2.9969$ and $\varphi_2(s) = s - 5.0001$ can be regarded as first degree approximate GCDs. However, if we combined them together, we get a second degree approximation

$$\hat{\varphi}(s) = s^2 - 7.997\,s + 14.9848 \tag{18}$$

with strength equal to 0.0375, which can be considered much better approximation than $\varphi(s)$ in (15).

Table 1. Pseudo-spectrum intervals for $\varepsilon = 0.1$

Eigenvalue - Root	Interval
2.9983	J_1 = [2.7824 , 3.4424]
5.0009	J_2 = [4.1563 , 5.6781]

Table 2. Minimum strength of a simple factor $\varphi(s) = s - c$

Interval	J_1	J_2
Root c	2.9969	5.0001
Strength	0.0010671	0.0044139

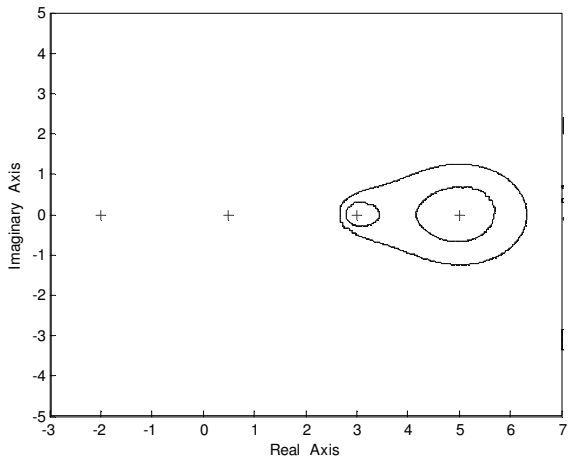

Fig. 1. The pseudo-spectra $\sigma_{\varepsilon,1}(\mathcal{P})$ for $\varepsilon = 0.2, 0.1$

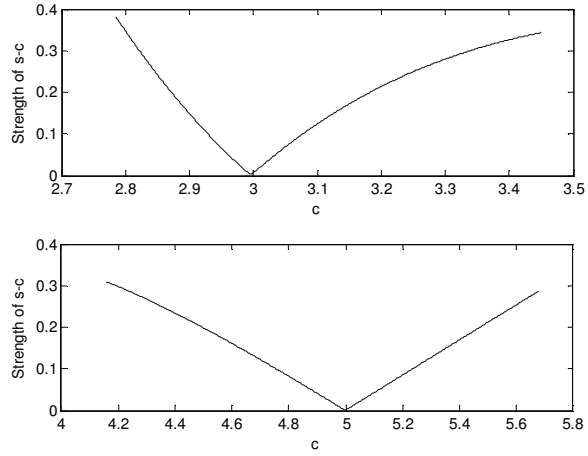

Fig. 2. Strength of a simple common factor of the form *s-c* in the pseudo-spectrum intervals.

5. CONCLUSIONS

The relationship between the approximate GCD and the notion of the pseudo-spectrum has been considered and the current study demonstrates that the pseudo spectrum provides an estimate for the region containing the roots of all approximate GCDs. This is a preliminary study aiming to extend the property of almost zeros (Karcanias et al., 1983) as trapping disks of polynomial combinants to the case of trapping regions related to the roots of the approximate GCD.

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