"Generalised Resultants, Dynamic Polynomial Combinants and the Minimal Design Problem"

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Abstract

The theory of dynamic polynomial combinants is linked to the linear part of the Dynamic Determinantal Assignment Problems (DAP), which provides the unifying description of the dynamic, as well as static pole and zero dynamic assignment problems in Linear Systems. The assignability of spectrum of polynomial combinants provides necessary conditions for solution of the original DAP. This paper deals demonstrates the origin of dynamic polynomial combinants from Linear Systems examines issues of their representation, the parameterization of dynamic polynomial combinants according to the notions of order and degree and examines the spectral assignment of them. Central to this study is the link of dynamic combinants to the theory of *"Generalised Resultants",* which provide the matrix representation of the dynamic combinants*.* The paper considers the case of coprime set polynomials for which spectral assignability is always feasible and provides a complete characterisation of all assignable combinants with order above and below the Sylvester order. A complete parameterization of combinants and respective Generalised Resultants is given and this leads naturally to the characterisation of the minimal degree and order combinant for which spectrum assignability may be achieved, which is referred to as the "*Dynamic Combinant Minimal Design"* (DCMD) problem. An algorithmic approach based on rank tests of Sylvester matrices is given which produces the minimal order and degree solution in a finite number of steps. Such solutions provide low bounds for the respective Dynamic Assignment control problems.

Key Words: Linear Systems, Spectrum Assignment, Generalised Resultants, Polynomial Combinants, Minimal Design.

1. Introduction

The study of problems of linear feedback synthesis which are of the determinantal type (Karcanias and Giannakopoulos, 1984) (such as pole zero assignment, stabilisation) a specific school of thought based on their determinantal formulation has been developed which unifies a very large class of dynamic, as well constant compensation. This is referred to as algebro-geometric because it relies on tools from algebra and algebraic geometry and their common feature is that they are of multi-linear nature. The main difficulty of the determinantal problems in the case of frequency assignment lies in that the problem is equivalent to finding real solutions to sets of nonlinear and linear equations; in the case of stabilisation, this is equivalent to determining solutions of nonlinear equations and nonlinear inequalities (characterising the stability domain). The first of the two problems naturally belongs to the intersection theory of complex algebraic varieties, whereas, the latter belongs to the intersection theory of semi-algebraic sets. Determining real intersections is not an easy problem (Leventides and Karcanias, 1992); furthermore, it is also important to be able to compute solutions whenever such solutions exist and define “approximate solutions” when exact solutions do not exist. The use of algebraic Geometry in the study of spectrum assignment problems was originally introduced in Hermann, and Martin (1975), Brockett, and Byrnes, (1981), where an affine space approach has been used. The main emphasis in that approach has been the use of intersection theory for the development of necessary conditions and the deployment of special techniques for establishing generic sufficient conditions. Issues of dealing with non-generic cases as well as computation of solutions were not addressed.

The Determinantal Assignment Problem Approach (DAP) (Karcanias and Giannakopoulos, 1984) has been formulated as a unifying approach for all problems of frequency assignment (dynamic and constant pole zero) and its basis lies on the fact that determinantal problems are of a multi-linear nature and thus may be naturally split into a linear and multi-linear problem (decomposability of multivectors). In this framework, the final solution is thus reduced to the solvability of a set of linear equations (characterising the linear problem) together with quadratics (characterising the multi-linear problem of decomposability). The approach heavily relies on exterior algebra and this has implications on the computability of solutions (reconstruction of solutions whenever they exist) and introduces new sets of invariants (of a projective character) which, in turn, characterise the solvability of the problem. This approach has been further developed in (Leventides and Karcanias, (1995), (1998)) by the development of a “blow-up” methodology for linearization of multi-linear maps that permit the development of computations, as well as techniques for establishing the development of real solutions (Leventides and Karcanias, (1992)). The distinct advantages of the DAP approach, which is a projective space approach, are: it provides the means for computing the solutions; it can handle both generic and exact solvability investigations, and it introduces new criteria for the characterisation of solvability of different problems. Furthermore, it provides a set-up for exterior algebra computations by using the methodology of “Global Linearization” (Leventides and Karcanias, (1995), (1998)). Most of the work in the DAP framework has been on problems dealing with non-dynamic compensation, where the linear part of the problem is expressed as a constant polynomial combinants, and the study of its properties is well developed (Karcanias etc, 1983). DAP is a multi-linear nature problem and thus may be naturally split into a linear and multi-linear problem (decomposability of multivectors). The final solution is reduced to the solvability of a set of linear equations coming from the spectrum assignability of polynomial combinants (Karcanias, etc, 1983), characterising the linear problem, together with quadratics characterising the multi-linear problem of decomposability, which in turn define some appropriate Grassmann variety (Hodge & Pedoe, 1952). The study of spectrum assignment of dynamic polynomial combinants, is the linear part of the dynamic DAP and defines the properties of the linear variety involved in the overall frequency assignment study. Of course, real intersection theory of varieties is once more the central issue, but the linear varieties become more complex in the dynamic case. The current study focuses on the properties of the linear part of dynamic DAP.

This paper deals with the development of the fundamentals of the linear part of the theory of Dynamic DAP, which is linked to dynamic combinants and addresses the open problem which is referred to as *minimal design* problem (Karcanias, 2010). We first review the origins of the dynamic combinants in Control Theory problems, we introduce the basic problems related to spectrum assignment, examine the parameterization of combinants according to their order and degree, consider their representation in terms of *generalised resultants* (Barnett,1970, Vardulakis and Stoyle,1978, Fatouros and Karcanias, 2003) and finally establish the conditions for spectral assignability; the latter are equivalent to the solvability of a Diofantine Equations over

$\mathbb{R}[s]$ (Kucera, 1979). We show that all combinants of degree greater than the Sylvester degree have elements (corresponding to some appropriate order) are assignable, and there is a set of degrees less than the Sylvester degree for which we have assignable combinants for some appropriate order. This motivates the problem of searching for a least complexity solution, where complexity is defined by the degree and order of the combinant. A complete parameterization of combinants and respective generalised resultants is first given and this motivates the study for finding the least degree and order combinant that is spectrally assignable. This problem is referred to here as *Minimal Design Problem for Dynamic Combinants* (MDP-DC) (Karcanias, 2010, Karcanias and Galanis, 2010) and involves the characterisation of the minimal degree and order combinant for which spectrum assignability may be achieved.

We deploy the systematic construction of the family of Generalised Resultants with order and degree less than the Sylvester degree. The results here are based on the study of rank properties of generalised resultants of degree and order less than the Sylvester values. The partitioning of the overall family of Generalised Resultants, according to degree and order, and the fact that the rank properties of subfamilies depend on their generators is instrumental in defining the solution to the minimal design problem. The minimal degree is defined by a simple test and the minimal order is then determined by a finite number of tests. The paper develops an algorithmic approach to the MDP-DC which leads to the solution in a finite number of tests using only rank tests. The results here also determine the family of non assignable combinants the properties of which are are linked to the property that their spectrum may be partially constrained. The results on the dynamic polynomial combinants are clearly necessary for the solvability of the corresponding DAP, and thus provide lower bounds for the solutions of the corresponding dynamic frequency assignment problems. The work here provides the means for studying the properties of the linear varieties of the Dynamic DAP and sets up the appropriate framework required for the study of dynamic DAP with complexity constraints using the general algebra-geometric framework of DAP (Karcanias and Giannakopoulos, 1984, Leventides and Karcanias, 1995, 1998).

Throughout the paper the following notation is adopted: If $\mathcal{F}$ is a field, or ring then $\mathcal{F}^{m\times n}$ denotes the set of $m\times n$ matrices over $\mathcal{F}$ If H is a map, then $\mathcal{R}(H)$, $\mathcal{N}_r(H)$, $\mathcal{N}_l(H)$ denote the range, right, left null spaces respectively. $Q_{k,n}$ denotes the set of lexicographically ordered, strictly increasing sequences of k integers from the set $\tilde{n} \triangleq \{1,2,...,n\}$. If $\mathcal{V}$ is a vector space and $\{\underline{v}_{i_1},...,\underline{v}_{i_k}\}$ are vectors of $\mathcal{V}$ then $\underline{v}_{i1}\wedge ...\wedge \underline{v}_{ik} = \underline{v}_{\omega}\wedge$, $\omega=(i_1,...,i_k)$ denotes their exterior product and $\wedge^r \mathcal{V}$ the $r-$th exterior power of $\mathcal{V}$. If $H\in\mathcal{F}^{m\times n}$ and $r\le \min\{m,n\}$, then $C_r(H)$ denotes the $r-$th compound matrix of H (Marcus & Minc, 1964). We shall denote by $\mathbb{R}[s], \mathbb{R}(s), \mathbb{R}_{pr}(s)$ the ring of polynomials, rational functions and proper rational functions over $\mathbb{R}$ respectively.

2. Linear Systems and Dynamic Polynomial Combinants

Consider the linear system (Kailath, (1980)) described by $S(A,B,C,D)$:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u},\ \ \underline{y} = C\underline{x} + D\underline{u},\ \ A\in\mathbb{R}^{n\times n}, B\in\mathbb{R}^{n\times p}, C\in\mathbb{R}^{m\times n}, D\in\mathbb{R}^{m\times p} \qquad (2.1)$$

where (A,B) is controllable, (A,C) is observable, or by the transfer function matrix $G(s) = C(sI-A)^{-1}B+D$, where $\mathrm{rank}_{\mathbb{R}(s)}\{G(s)\} = \min\{m,p\}$. In terms of left, right coprime matrix fraction descriptions (LCMFD, RCMFD) (Kucera, 1979) $G(s)$ may be represented as

$$G(s) = D_l(s)^{-1} N_l(s) = N_r(s) D_r(s)^{-1} \qquad (2.2)$$

where $N_\ell(s),\ N_r(s) \in \mathbb{R}^{m\times p}[s]$, $D_\ell(s) \in \mathbb{R}^{m\times m}[s]$ and $D_r(s) \in \mathbb{R}^{p\times p}[s]$. The system will be called *square* if $m = p$ and *nonsquare* if $m \neq p$. within the state space framework we may define a number of constant, frequency assignment problems such as the Pole assignment by state feedback, Design of an n-state observer, Pole assignment by constant output feedback and Zero assignment by squaring down, which are all reduced to a Constant Determinantal assignment problem (Karcanias and Giannakopoulos, 1984). A number of dynamic assignment problems may be defined on a linear system as shown below:

Dynamic Compensation Problems

Consider the standard feedback configuration (Kucera, 1979) below

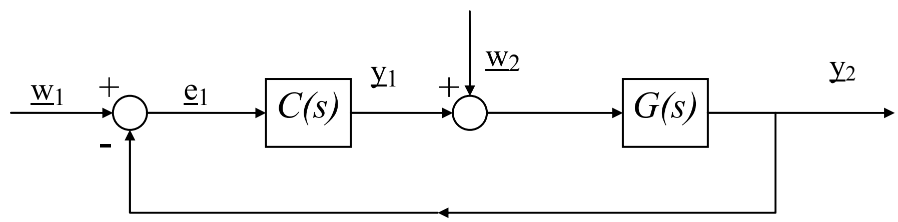

Figure (2.1): Feedback Configuration

If $G(s) \in \mathbb{R}_{pr}^{m\times p}(s)$, $C(s) \in \mathbb{R}^{p\times m}(s)$, and assume coprime MFD's as in (2. 2) and

$$C(s) = A_\ell(s)^{-1} B_\ell(s) = B_r(s)\, A_r(s)^{-1} \qquad (2.\ 3)$$

The closed loop characteristic polynomial may be expressed as [10]:

$$\text{f(s)=det}\left\{[D_\ell(s), N_\ell(s)] \begin{bmatrix} A_r(s) \\ B_r(s) \end{bmatrix}\right\} = \det\left\{[A_\ell(s), B_\ell(s)] \begin{bmatrix} D_r(s) \\ N_r(s) \end{bmatrix}\right\} \qquad (2.4)$$

i) if $p \leq m$, then $C(s)$ may be interpreted as *feedback compensator* and we will use the expression of the closed loop polynomial described by (2.4b)

ii) if $p \geq m$, the $C(s)$ may be interpreted as *pre-compensator* and we will use the expression of the closed loop polynomial described by (2.4a).

The above general dynamic formulation covers a number of important families of $C(s)$ compensators as : **(a)** Constant**,** **(b)** PI, **(c)** PD, **(d)** PID, **(e)** Bounded degree. In fact,

(a) Constant Controllers : If $p \leq m$, $A_\ell = I_p$, $B_\ell = K \in \mathbb{R}^{p\times m}$, then (2.4) expresses the constant output feedback case, whereas if $p \geq m$, $A_r = I_m$, $B_r = K \in \mathbb{R}^{p\times m}$ expresses the constant pre-compensation formulation of the problem.

(b) Proportional plus Integral Controllers: Such controllers are defined by

$$C(s) = K_0 + 1/\text{s}\ \text{K}_1 = [sI_p]^{-1}\ [\text{s}K_0 + K_1] \qquad (2.5)$$

where $K_0,\ K_1 \in \mathbb{R}^{p\times m}[s]$ and the left MFD for $C(s)$ is coprime, iff rank(K_1) = p. From the above the determinantal problem for the output feedback PI design is expressed as:

$$f(s)=\det\left\{[sI_p, sK_0+K_1]\begin{bmatrix}D_r(s)\\N_r(s)\end{bmatrix}\right\}=\det\{[I_p, K_0, K_1]\begin{bmatrix}sD_r(s)\\sN_r(s)\\N_r(s)\end{bmatrix}\} \quad (2.6)$$

(c) Proportional plus Derivative Controllers: Such controllers are expressed as

$$C(s) = sK_0 + K_1 = [I_p]^{-1}\,[sK_0 + K_1] \quad (2.7)$$

where $K_0,\ K_1 \in \mathbb{R}^{p\times m}$ and the left MFD for $C(s)$ is coprime for finite s and also for $s=\infty$ if rank(K_0) = p. From the above the determinantal output PD feedback is expressed as:

$$f(s)=\det\{[I_p, sK_0+K_1]\begin{bmatrix}D_r(s)\\N_r(s)\end{bmatrix}\}=\det\{[I_p, K_1, K_0]\begin{bmatrix}D_r(s)\\N_r(s)\\sN_r(s)\end{bmatrix}\} \quad (2.8)$$

(d) PID Controllers: These controllers are expressed as

$$C(s) = K_0 + 1/\mathrm{s}\, K_1 + sK_2 = [sI_p]^{-1}\,[s^2K_2 + sK_0 + K_1] \quad (2.9)$$

where $K_0,\ K_1 \in \mathbb{R}^{p\times m}$ and the left MFD is coprime with the only exception possibly at $s=0$, $s=\infty$ (coprimeness at $s=0$ is guaranteed by rank(K_1)=p and at s=∞ by rank(K_2)=p). From the above, the determinantal output PID feedback is expressed as :

$$f(s)=\det\left\{[sI_p, s^2K_2+sK_0+K_1]\begin{bmatrix}D_r(s)\\N_r(s)\end{bmatrix}\right\}=\det\{[I_p, K_0, K_1, K_2]\begin{bmatrix}sD_r(s)\\sN_r(s)\\N_r(s)\\s^2N_r(s)\end{bmatrix}\} \quad (2.10)$$

(e) Observability Index Bounded Dynamics (OBD) Controllers: These are defined by the property that their McMillan degree is equal to pk, where k is the observability index (Kailath, (1980)) of the controller. Such controllers are expressed by the composite MFD representation as

$$[A_l(s), B_l(s)] = T_k s^k + \dots + T_0 \quad (2.11)$$

$T_k,\ T_{k-1}\ \dots, T_0 \in \mathbb{R}^{p\times(p\times m)}$ and $T_k = [I_p,\ X]$. Note that the above representation is not always coprime, and coprimeness has to be guaranteed first for McMillan degree to be pk; otherwise, the McMillan degree is less than pk. The dynamic determinantal OBD output feedback problem is then expressed as

$$f(s)=\det\left\{[T_k s^k+\dots+T_0]\begin{bmatrix}D_r(s)\\N_r(s)\end{bmatrix}\right\}=\det\{(T_k s^k+\dots+T_0)\,M(s)\}=\det\{[T_k, T_{k-1},\dots,T_0]\begin{bmatrix}s^kM(s)\\s^{k-1}M(s)\\.\\.\\M(s)\end{bmatrix}\} \quad (2.12)$$

Remark (2.1): The above formulation of the determinantal dynamic assignment problems is based on the assumption that $p\leq m$ and thus output feedback configuration is used. If $p\geq m$, we can similarly formulate the corresponding problems as determinantal dynamic pre-compensation problems and use right coprime MFDs for $C(s)$.

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Abstract Determinantal Assignment Problem

All the problems introduced above, belong to the same problem family i.e. the determinantal assignment problem (DAP) [1]. This problem is to solve the following equation with respect t o polynomial matrix $H(s)$:

$$\det(H(s)\,M(s)) = f(s) \qquad (2.13)$$

where $f(s)$ is a polynomial of an appropriate degree d. The difficulty for the solution of DAP is mainly due to the multi-linear nature of the problem, as this is described by its determinantal character. We should note, however, that in all cases mentioned previously, all dynamics can be shifted from $H(s)$ to $M(s)$, which, in turn, transforms the problem to a constant DAP. This problem may be described as follows:

Let $M(s) \in \mathbb{R}^{p\times m}[s]$, $r \leq p$, such that rank($M(s)$) = r and let $\mathcal{H}$ be a family of full rank $r \times p$ constant matrices having a certain structure. Solve with respect to $H \in \mathcal{H}$ the equation:

$$f_M(s,H) = \det(H\,M(s)) = f(s) \qquad (2.14)$$

where f(s) is a real polynomial of an appropriate degree d.

Remark 2.2: The degree of the polynomial $f(s)$ depends firstly upon the degree of $M(s)$ and secondly, upon the structure of H. Generically, the degree of $f(s)$ is equal to the degree of $M(s)$.

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The determinantal assignment problem has two main aspects. The first has to do with the solvability conditions for the problem and the second, whenever this problem is solvable, to provide methods for constructing these solutions. If $\underline{h}_i(s)^t$, $\underline{m}_i(s)$, $i \in \tilde{r}$, we denote the rows of $H(s)$, columns of $M(s)$ respectively, then

$$C_r(H(s)) = \underline{h}_1(s)^t \wedge \ \ldots\ \wedge \underline{h}_r(s)^t = \underline{h}(s)^t \wedge \in \mathbb{R}^{1\times\sigma} \qquad (2.15a)$$

$$C_r(M(s)) = \underline{m}_1(s) \wedge \ \ldots\ \wedge \underline{m}_r(s) = \underline{m}(s) \wedge \in \mathbb{R}^{\sigma}[s],\ \sigma = \binom{p}{r} \qquad (2.15b)$$

and by Binet-Cauchy theorem (Marcus,and Minc, 1964).we have that (Karcanias and Giannakopoulos, 1984):

$$f_M(s,H) = C_r(H)C_r(M(s)) = \langle \underline{h}(s)\wedge, \underline{m}(s)\wedge \rangle = \sum_{\omega\in Q_{r,p}} h_\omega(s) m_\omega(s) \qquad (2.15c)$$

where <•, •> denotes inner product, $\omega = (i_1, \ldots, i_r) \in Q_{r,p}$, and $h_\omega(s)$, $m_\omega(s)$ are the coordinates of $\underline{h}(s)\wedge$, $\underline{m}(s)\wedge$ respectively. Note that $h_\omega(s)$ is the $r\times r$ minor of $H(s)$, which corresponds to the ω set of columns of $H(s)$ and thus $h_\omega(s)$, is a multilinear alternating function (Marcus, (1973)) of the entries $h_{ij}(s)$ of $H(s)$. The multilinear, skew symmetric nature of DAP suggests that the natural framework for its study is that of exterior algebra. The essence of exterior algebra is that it reduces the study of multilinear skew-symmetric functions to the simpler study of linear functions. The study of the zero structure of the multilinear function $f_M(s,H)$ may thus be reduced to a linear subproblem and a standard multilinear algebra problem as it is shown below.

(i) Linear subproblem of DAP: Set $\underline{m}(s)\wedge = \underline{p}(s) \in \mathbb{R}^{\sigma}[s]$. Determine whether there exists a $\underline{k}(s) \in \mathbb{R}^{\sigma}[s]$, $\underline{k}(s) \neq \underline{0}$, such that

$$f_M(s, \underline{k}(s)) = \underline{k}(s)^t\, \underline{p}(s) = \sum k_i(s) p_i(s) = f(s) \in \mathbb{R}[s] \qquad (2.16)$$

(ii) Multilinear subproblem of DAP : Assume that $\mathcal{K}$ is the family of solution vectors $\underline{k}(s)$ of (2.16). Determine whether there exists $H(s)^t = [\underline{h}_1(s), \ldots, \underline{h}_r(s)]$, where $H(s)^t \in \mathbb{R}^{p\times r}[s]$, such that

$$\underline{h}_1(s)\wedge \ldots \wedge \underline{h}_r(s) = \underline{h}(s)\wedge = \underline{k}(s) \in \mathcal{K} \qquad (2.17)$$

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The polynomials $f_M(s, \underline{k}(s))$ are generated by $\underline{p}(s) = [\, p_1(s),.., p_i(s),.., p_\sigma(s)]^t \in \mathbb{R}^\sigma[s]$, or as linear combinations of the set $\mathcal{P} = \{\, p_i(s) \in \mathbb{R}[s],\ \ i \in \tilde{\sigma}\}$ and they will be referred to as *dynamic polynomial combinants.* The study of the spectral properties of such polynomials is the objective of this paper.

3. Basic Definitions and Representation of Dynamic Combinants

Given a set of polynomials $\mathcal{P} = \{\, p_i(s): p_i(s) \in \mathbb{R}[s]\,;\, i \in \tilde{m}\,\}$ and a family of polynomial sets $<\mathcal{K}> = \{\, \mathcal{K}_{\mathbf{d}}\,, \forall\, d \in \mathbb{Z}^+: \mathcal{K}_{\mathbf{d}} = \{\, k_i(s): k_i(s) \in \mathbb{R}[s]\,;\, i \in \tilde{m}\,, d = \max\{\deg(k_i(s))\,\}$, we consider

$$f\,(s, \mathcal{K}\,; \mathcal{P}\,) = \sum k_i(s)\, p_i(s), \text{ where } k_i(s) \in \mathcal{K}_{\text{d}} \qquad (3.1)$$

which are referred to as *d order dynamic-polynomial combinants* of $\mathcal{P}$ and are polynomials with some degree p. Dynamic compensation of linear systems always involves polynomial combinants generated by the corresponding system descriptions. Concepts such as those of multivariable zeros and decoupling zeros are related to the greatest common divisor (Karcanias, 1987, Fatouros and Karcanias, 203, Karcanias etc, 2006) of certain sets $\mathcal{P}$**,** associated with the system they and define fixed zeros of the associated combinants. The pole, zero assignment and stabilizability properties of linear systems are based on properties of corresponding combinants and thus on the structure of sets $\mathcal{P}$ **,** which generate these combinants. The examination of those properties of a set $\mathcal{P}$ which affect the assignability, stabilizability and "nearly fixed" zero phenomena of the corresponding combinants $f(s,\mathcal{K}; \mathcal{P})$ is the main drive for the research here. This paper develops the fundamentals of the theory of polynomial combinants. The representation problem of given order and degree dynamic polynomial combinants is considered here, which involves a parameterization of all sets $<\mathcal{K}_{\mathbf{d}}> = \{\, \mathcal{K}_{\mathbf{d}},\ \forall\, d \in \mathbb{N}: \mathcal{K}_{\mathbf{d}} = \{k_i(s): k_i(s) \in \mathbb{R}[s]\,;\ i \in \tilde{m}\,)$, $d = \max\{\deg(k_i(s))\,\}$ which lead to a polynomial combinant of a given degree p.

Given the sets $\mathcal{P}$ with m elements and maximal degree n and the set $\mathcal{K}$ of m elements and maximal degree d of $\mathbb{R}[s]$, the generated combinant is denoted by

$$f_d(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m} k_i(s)\, p_i(s) = \phi(s) \qquad (3.2)$$

This is a polynomial generated by the set $\mathcal{P}$ and characterised by the order d of $\mathcal{K}$ and the resulted degree $\partial\, [f_d(s,\mathcal{K},\mathcal{P})]$ of the combinant. We always assume that the maximal degree polynomial in $\mathcal{K}$, $k_1(s) \neq 0$ and such sets $\mathcal{K}$ are referred to as *proper*. If we explicitly define $\mathcal{P}$ as

$$\mathcal{P} = \{p_i(s) \in [s], i \in \widetilde{m}, n = deg\{p_1(s)\} \geq deg\{p_i(s)\}, i = 2,\ldots,m,\ q = max\{deg\{p_{i(s)}\}, i = 2,\ldots,m\} \qquad (3.3a)$$

$$p_1(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0,\ \ p_i(s) = b_{i,q}s^q + \ldots + b_{i,1}s + b_{1,0},\ i = 2,\ldots,m \qquad (3.3b)$$

$$\underline{p}(s)=\begin{bmatrix} p_1(s) \\ p_2(s) \\ \vdots \\ p_m(s) \end{bmatrix}=[\underline{p}_0,\underline{p}_1,\dots,\underline{p}_n]\begin{bmatrix} 1 \\ s \\ \vdots \\ s^n \end{bmatrix}=P\underline{e}_n(s) \qquad (3.3c)$$

Then the set $\mathcal{P}$ will be referred to as an *(m;n(q))-ordered set* of $\mathbb{R}$[s] . Consider now a set of m polynomials of maximal degree d, $\mathcal{K}=\{k_i(s)\in[s], i\in\tilde{m},\ \ deg\{k_i(s)\}\le d\}$, referred to in short as an $(m;d)$ set of $\mathbb{R}$[s]. The resulting polynomial combinant is

$$f_d(s,\mathcal{K},\mathcal{P})=\sum_{i=1}^{m}k_i(s)p_i(s)=\underline{k}^t(s)\underline{p}(s) \qquad (3.4)$$

where

$$\underline{k}^t(s)=[k_1(s),k_2(s),\dots,k_m(s)]^t=\underline{k}_0^t+s\underline{k}_1^{\ t}+\dots+s^d\underline{k}_d^t \qquad (3.5)$$

is defined as a *d-order polynomial combinant* of $\mathcal{P}$, or in short as d- $\mathbb{R}$[s]-*combinant* of $\mathcal{P}$. The matrix $P\in\mathbb{R}^{m\times(n+1)}$ generates the representative $\underline{p}(s)\in\mathbb{R}^m[s]$ of $\mathcal{P}$ and it is referred to as the *basis matrix* of $\mathcal{P}$. Clearly $f_d(s,\mathcal{K},\mathcal{P})\in\mathbb{R}[s]$ and some interesting problems related to its spectrum stem from the fact that the set $\mathcal{K}$ may take arbitrary form in terms of its degree and selection of free parameters. The combinant $f_d(s,\mathcal{K},\mathcal{P})$ as a polynomial of $\mathbb{R}$[s] has degree $\partial[f_d(s,\mathcal{K},\mathcal{P})]$ that clearly satisfies the inequality

$$-\infty\le\partial[f_d(s,\mathcal{K},\mathcal{P})]\le n+q \qquad (3.6)$$

In the following we consider two different representations of $f_d(s,K,\mathcal{P})$ and the parametrisation of all combinants of different order and degree and show how these lead to standard linear algebra problem formulations. The order and degree parameterisations introduce some interesting links with the theory of generalised resultants.

Fixed Order Representations of Dynamic Combinants: Generalised Resultant Representations

For the general ($m;d$) set $\mathcal{K}$ with a representative vector

$$\underline{k}(s)^t=\underline{k}_0^t+s\underline{k}_1^t+\dots+s^d\underline{k}_d^t=[k_1(s),k_2(s),\dots,k_m(s)] \qquad (3.7a)$$

where $\underline{k}_i(s)=k_{i,0}+k_{i,1}s+\dots+k_{i,d}s^d$, then $f_d(s,\mathcal{K},\mathcal{P})$ may be expressed as

$$f_d(s,\mathcal{K},\mathcal{P})=\sum_{i=1}^{m}[k_{i,d},\dots,k_{i,1},k_{1,0}]\begin{bmatrix} s^d p_i(s) \\ \vdots \\ sp_i(s) \\ p_i(s) \end{bmatrix}=[\underline{k}_{1,d}^t,\dots,\underline{k}_{m,d}^t]\begin{bmatrix} \underline{p}_i(s)^d \\ \vdots \\ \underline{p}_m(s)^d \end{bmatrix} \qquad (3.7b)$$

The above leads to the following representation of dynamic combinants:

Proposition(3.1): Every dynamic combinant $f_d(s,\mathcal{K},\mathcal{P})$) defined by an ($m;d$) set $\mathcal{K}$ is equivalent to a constant polynomial combinant defined by the ($m(d+1);0$) set $\mathcal{K}^0$ and generated by the ($m(d+1);(n+d)(q+d)$) the d-th power of the ($m;n(q)$) set $\mathcal{P}$, defined by

$$\mathcal{P}^d = \{s^d p_1(s),...,sp_1(s), p_1(s);...;s^d p_m(s),...,sp_m(s), p_m(s)\} \tag{3.8}$$

■

The above leads to the following representation of dynamic combinants as equivalent constant combinants. If $\mu = n+d$ $\tilde{\underline{e}}_\mu(s)^t = [s^\mu, s^{\mu-1},...,s,1]$, then $\partial\,[p_{1,d}(s)] = n+d$, $\partial\,[p_{i,d}(s)] \le q+d$ for all *i=2,3,…,m* and

$$p_{1,d}(s) = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & & \ddots & & & & & & \vdots \\ 0 & 0 & \cdots & 1 & a_{n-1} & \cdots & \cdots & a_1 & a_0 \end{bmatrix} \tilde{\underline{e}}_\mu(s) = S_{n,d}(p_1)\tilde{\underline{e}}_\mu(s),\ S_{n,d}(p_1) \in \mathbb{R}^{(d+1)\times(\mu+1)} \tag{3.9a}$$

and for $i = 2,3,...,m$

$$p_{i,d}(s) = \begin{bmatrix} 0 & ... & 0 & b_{i,q} & ... & b_{i,1} & b_{i,0} & 0 & ... & ... & 0 \\ 0 & ... & 0 & 0 & b_{i,q} & ... & b_{i,1} & b_{i,0} & 0 & ... & 0 \\ \vdots & & 0 & \vdots & \ddots & \ddots & & & & & \vdots \\ 0 & ... & 0 & 0 & ... & 0 & b_{i,q} & ... & ... & b_{i,1} & b_{i,0} \end{bmatrix} \tilde{\underline{e}}_\mu(s) = S_{n,d}(p_i)\tilde{\underline{e}}_\mu(s),\ S_{n,d}(p_i) \in \mathbb{R}^{(d+1)\times(\mu+1)} \tag{3.9b}$$

The set $\mathcal{P}^d$ has then a basis matrix representation as shown below

$$\underline{p}_d(s) = \begin{bmatrix} p_{1,d}(s) \\ p_{2,d}(s) \\ \vdots \\ p_{m,d}(s) \end{bmatrix} = \begin{bmatrix} S_{n,d}(p_1) \\ S_{n,d}(p_2) \\ \vdots \\ S_{n,d}(p_m) \end{bmatrix} \tilde{\underline{e}}_\mu(s) = S_{\mathcal{P},d}\,\tilde{\underline{e}}_\mu(s) \tag{3.9c}$$

where $S_{\mathcal{P},d} \in \mathbb{R}^{m(d+1)\times(\mu+1)}$ which is the *d-th Generalised Resultant representation* of the set $\mathcal{P}$ (Karcanias and Galanis, 2010) and $S_{\mathcal{P},d}$ is the basis matrix of the $\mathcal{P}^d$ set.

Fixed Order Representations of Dynamic Combinants: Toeplitz Representation

An alternative expression for the dynamic combinant is obtained using the basis matrix description of the set $\mathcal{P}$. Thus, let us assume that

$$\underline{p}(s) = \begin{bmatrix} p_1(s) \\ \vdots \\ p_m(s) \end{bmatrix} = P\tilde{\underline{e}}_n(s), \quad P = [\underline{p}_n,...,\underline{p}_1,\underline{p}_0] \in \mathbb{R}^{m\times(n+1)} \tag{3.10}$$

where P is the basis matrix of $\mathcal{P}$. Then,

$$f_d(s,\mathcal{K},\mathcal{P}) = (\underline{k}_0^t + s\underline{k}_1^t + ... + s^d\,\underline{k}_d^t)P\,\tilde{\underline{e}}_n(s) = \underline{k}_0^t P\tilde{\underline{e}}_n(s) + s\underline{k}_1^t P\,\tilde{\underline{e}}_n(s) + ... + s^d\,\underline{k}_d^t P\,\tilde{\underline{e}}_n(s) =$$

$$= \underline{k}_0^t[\underline{0},...,\underline{0},\underline{p}]\tilde{\underline{e}}_\mu(s) + s\underline{k}_1^t[\underline{0},...,\underline{0},\underline{p}\,,\underline{0}]\tilde{\underline{e}}_\mu(s) + ... + s^d\,\underline{k}_d^t[\,\underline{0},...,\underline{0},\,\underline{p}]\tilde{\underline{e}}_\mu(s) =$$

$$=[\underline{k}_o^t,\underline{k}_1^t,\ldots,\underline{k}_d^t]\begin{bmatrix}\underline{p}_0, & \underline{p}_1, & \underline{p}_2, & \ldots, & \underline{p}_n, & \underline{0}, & \cdots & \cdots & \underline{0}\\ \underline{0} & \underline{p}_0, & \underline{p}_1, & \underline{p}_2, & \ldots, & \underline{p}_n, & \underline{0}, & \cdots & \underline{0}\\ \vdots & & \ddots & \ddots & & & \ddots & \underline{0} & \\ \underline{0}, & \cdots & \cdots & \underline{0} & \underline{p}_0, & \underline{p}_1, & \underline{p}_2, & \ldots, & \underline{p}_n,\end{bmatrix}\underline{\tilde{e}}_\mu(s) \qquad (3.11)$$

or equivalently

$$f_d(s,\mathcal{K},\mathcal{P})=\underline{k}_{d+1,m}^t Q_{\mathcal{P},d}\,\underline{\tilde{e}}_\mu(s),\quad Q_{\mathcal{P},d}\in\mathbb{R}^{m\times(d+1)\times(\mu+1)} \qquad (3.12)$$

The matrix $Q_{\mathcal{P},d}\in\mathbb{R}^{m(d+1)\times(\mu+1)}$ generating the dynamic combinant as a constant combinant is referred to as the d-th *Toeplitz Toeplitz Representation* of the set $\mathcal{P}$. From the construction of the matrices $S_{\mathcal{P},d}$, $Q_{\mathcal{P},d}$ we have:

Remark (3.1): The matrices $Q_{\mathcal{P},d}$ and $S_{\mathcal{P},d}$ associated with $\mathcal{P}$ have the same dimensions and are permutation equivalent, i.e. $\exists$ permutation matrices P_L, P_R such that

$$Q_{\mathcal{P},d}=P_L S_{\mathcal{P},d} P_R \qquad (3.13)$$

■

The above implies that establishing the rank properties of $S_{\mathcal{P},d}$ implies the same properties for $Q_{\mathcal{P},d}$ and vice versa. Thus either of the two representations may be used. In the following we shall concentrate on the Generalised Resultant representation and the general properties may be referred back to the Toeplitz Representations as well.

4. Fixed degree and order Parameterisation of $\mathcal{K}$ sets and Corresponding Resultants

The general representation of dynamic combinants considered before, based on the order may lead to combinants of varying degree. An alternative characterisation based on the fixed degree of $f_d(s,\mathcal{K},\mathcal{P})$ but with varying order $\mathcal{K}$ provides an alternative parameterisation of the $\mathcal{K}$ sets. We assume proper sets $\mathcal{K}$, (i.e. maximal degree element $k_1(s)\neq 0$) and we shall consider the generalized Resultant Representations. The fixed degree parameterisation of combinants is summarised by the following result:

Theorem (4.1): Given the *(m;q(n))* set $\mathcal{P}$ and a general proper *(m;d)* set $\mathcal{K}$. Then the following properties hold true:

(i) For all proper (*m;d*) sets $\mathcal{K}$ $n\le\partial[f_d(s,\mathcal{K},\mathcal{P})]\le n+d$

(ii) If $p\in\mathbb{N}_{>0}, p\ge n$, then the family $\{\mathcal{K}_p\}$ for which $\partial\,[f_d(s,\mathcal{K},\mathcal{P})]=p$, satisfies the conditions

$$\partial\,[k_1(s)]\le p-n,\ \ \partial\,[k_i(s)]\le p-q,\ i=2,\ldots,m \qquad (4.1a)$$

where at least one of the first two conditions holds as an equality.

(iii) The fixed degree p family $\{\mathcal{K}_p\}$ contains *n-q+1* subfamilies parameterised by a fixed order d. The possible values for the order are:

$$d_1=p-q>d_2=p-q-1>\ldots>d_{n-q+1}=p-n \qquad (4.1b)$$

and the corresponding subfamilies are

$$\{\mathcal{K}_p^{d_1}\} = \{k_i(s) : \partial\,[k_1(s)] \le p-n, \partial\,[k_2(s)] = d_1 = p-q, \partial\,[k_i(s)] \le d_1, i=3,...,m\}$$

$$\{\mathcal{K}_p^{d_2}\} = \{k_i(s) : \partial\,[k_1(s)] \le p-n, \partial\,[k_2(s)] = d_2 = p-q-1, \partial\,[k_i(s)] \le d_2, i=3,...,m\} \qquad (4.1c)$$

$$\vdots$$

$$\{\mathcal{K}_p^{d_{n-q+1}}\} = \{k_i(s) : \partial\,[k_1(s)] = \partial\,[k_2(s)] = d_{n-q+1} = p-n, \partial\,[k_i(s)] \le d_{n-q+1} = p-n,\ i=3,...,m\}$$

Proof: Parts **(i)** and **(ii)** are rather straight forward and follow from the definition of the combinant. The parameterisation implied by part (iii) follows by the construction of the combinant as indicated by the following table

$$\begin{array}{llll} p_1(s): & \partial\,[p_1(s)] = n, & k_1(s) & \partial\,[k_1(s)] \le p-n \\ p_2(s): & \partial\,[p_2(s)] = q, & k_2(s) & \partial\,[k_2(s)] \le p-q \\ \vdots & \vdots & \vdots & \vdots \\ p_m(s): & \partial\,[p_m(s)] \le q, & k_m(s) & \partial\,[k_m(s)] \le p-q \end{array} \qquad (4.2a)$$

where amongst the first two relationships at least one is an equality. The above table follows from the need to guarantee degree p to the $f_d(s,\mathcal{K},\mathcal{P})$ combinant. The condition from the above implies:

- If $\partial\,[k_2(s)] = p-q > p-n$ then we have the maximal degree $d_1 = p-q$ subfamily of $\{\mathcal{K}_p^{d_1}\}$ with degrees

$$\partial\,[k_1(s)] \le p-n,\ \ \partial\,[k_2(s)] = d_1 = p-q,\ \ \partial\,[k_i(s)] \le d_1,\ i=3,...,m$$

- If $\partial\,[k_2(s)] = p-q-1 > p-n$ then we have the next value of degree $d_2 = p-q-1$ and the $\{\mathcal{K}_p^{d_2}\}$ subfamily with degrees

$$\partial\,[k_1(s)] = p-n,\ \ \partial\,[k_2(s)] = d_1 = p-q-1,\ \ \partial\,[k_i(s)] \le d_2,\ i=3,...,m$$

the process finishes when

$$\partial\,[k_1(s)] = p-n = \partial\,[k_2(s)] = d_{n-q+1},\ \ \partial\,[k_i(s)] \le p-n,\ i=3,..,m$$

Clearly this is the last family in $\{\mathcal{K}_p\}$ for which the degree has minimal value $d_{n-q+1} = p-n$.

■

Remark (4.1): For the (*m,n(q)*) set $\mathcal{P}$ the degree of the proper combinants (corresponding to proper sets $\mathcal{K}$) takes values $p \ge n$.

■

The entire family of proper combinants of $\mathcal{P}$ may thus be parameterised by degree and orders and the entire set may be characterised by the sets of $\mathcal{K}$ vectors which will be denoted as $<\mathcal{K}>$. Clearly,

$$<\mathcal{K}> = \{\mathcal{K}_n\} \cup \{\mathcal{K}_{n+1}\} \cup ... \cup \{\mathcal{K}_{n+q-1}\} \qquad (4.3)$$

whereas each subset $\{\mathcal{K}_p\}$ has the structure defined by the previous result.

Corollary (4.1): Given an (*m;q*) set $\mathcal{P}$ and a general (*m;d*) set $\mathcal{K}$, then:

(i) The minimal degree family $p=n$, $\{\mathcal{K}\}_n$ is expressed as

$$\{\mathcal{K}_n\} = \begin{array}{l} \{\{\mathcal{K}_n^0\} :< \mathcal{K}_n^0 > \;= (0,0,...,0)\ ; \\ \{\mathcal{K}_n^1\} :< \mathcal{K}_n^1 > \;= (0,1,...,1)\ ; \\ \vdots \\ \{\mathcal{K}_n^{n-q}\} :< \mathcal{K}_n^{n-q} >= (0, n-q, ..., n-q)\} \end{array} \tag{4.4a}$$

(ii) The general degree family p=n+d, $\{\mathcal{K}_p\}$ is then expressed as

$$\{\mathcal{K}_p\} = \begin{array}{l} \{\{\mathcal{K}_p^d\} :< \mathcal{K}_p^d >= (0,...,0) + (d, d, ..., d); \\ \{\mathcal{K}_p^{d+1}\} :< \mathcal{K}_p^{d+1} >= (0,1,...,1) + (d, d, ..., d); \\ \vdots \\ \{\mathcal{K}_p^{d+n-q}\} :< \mathcal{K}_p^{d+n-q} >= (0, n-q, ..., n-q) + (d, d, ..., d)\} \end{array} \tag{4.4b}$$

(iii) For the general degree p family, $p \geq n$, the values of possible orders in decreasing order are:

$$d_1 = p - q > d_2 = p - q - 1 > .... > d_{n-q} = p - n + 1 > d_{n-q+1} = p - n \tag{4.4c}$$

and they are given as $d_i = p - q + 1 - i,\ \ i = 1, 2, ..., n - q + 1$.

■

The proof of the above result follows readily by induction and it is omitted. Amongst all ($m;d$) sets $\mathcal{K}$, the set defined by

$$\{\mathcal{K}_{n+q-1}^{n-1}\} = \{k_1(s) : \partial\ [k_1(s)] = q - 1,\ k_i(s) : \partial\ [k_i(s)] = n - 1,\ i = 2, ..., m\} \tag{4.5}$$

plays a particular role in our study and it is referred to as the *Sylvester set* of $\mathcal{P}$. The general p degree family may be expressed as

$$\begin{aligned} \{\mathcal{K}_p\} &= \{\mathcal{K}_p^{\tilde{d}_i},\ \tilde{d}_i = p - n + i,\ \ i = 1, 2, ..., n - q + 1\} = \\ &= \{\mathcal{K}_p^{p-n}; \mathcal{K}_p^{p-n+1}; ....; \mathcal{K}_p^{p-q-1}; \mathcal{K}_p^{p-q}\} \end{aligned} \tag{4.6}$$

The element $\mathcal{K}_p^{p-q}$ that corresponds to the highest order $d_1 = p - q$ will be called the *generator* of the family and its degrees are

$$< \mathcal{K}_p^{p-q} > \;= (p - n, p - q, ..., p - q) \tag{4.7a}$$

Similarly, the element $\mathcal{K}_p^{p-n}$ that corresponds to the lowest order $d_{n-q+1} = p - n$ will be called the *co-generator* of the family and its degrees are

$$< \mathcal{K}_p^{p-n} > \;= (p - n, p - n, ..., p - n) \tag{4.7b}$$

The above suggests that the entire family of vector sets $< \mathcal{K} >$ may be expressed in "direct sum" form ($\dot{\cup}$) as

$$\begin{aligned} &< \mathcal{K} > \;= \{\mathcal{K}_n\} \dot{\cup} \{\mathcal{K}_{n+1}\} \dot{\cup}\ .....\ \dot{\cup} \{\mathcal{K}_{n+q-1}\} \dot{\cup}\ ..... \\ &\{\mathcal{K}_p\} = \{\mathcal{K}_p^{p-n}\} \dot{\cup} \{\mathcal{K}_p^{p-n+1}\} \dot{\cup} ..... \dot{\cup} \{\mathcal{K}_p^{p-q}\} \end{aligned} \tag{4.8}$$

For all $p \geq n$. This parametrisation of the sets $\mathcal{K}$ leads to a corresponding parameterisation of the generalised resultants and this is considered next.

The parameterisation of the sets $\mathcal{K}$ based on degree and order (Karcanias and Galanis, 2010) induces a natural parameterisation of the corresponding Generalized Resultants. This is now considered here and this provides the basis for the study of the properties of the family of Generalised Resultants. We consider the general ($m;d$) set $\mathcal{K}$ that leads to combinants of degree p. This set is explicitly defined by:

$$\{\mathcal{K}_p^d\} = \{k_1(s): \partial\,[k_1(s)] = p-n = \tilde{d},\ k_2(s): \partial\,[k_2(s)] = d,\ \tilde{d} \le d \le d^* = p-q, ...,$$
$$k_i(s): \partial\,[k_i(s)] \le d,\ i = 3, ..., m\} \tag{4.9}$$

The above set $\{\mathcal{K}_p^d\},\ p \ge n$ and with d taking values as above, represents the general set generating dynamic combinants a given degree d and order p. Note that in the above expression we consider all $k_i(s),\ i = 3, ..., m$ as polynomials with reference degree d, $(\partial\,[k_i(s)] \le d)$ and thus we can express them as

$$k_1(s) = k_{1,\tilde{d}}s^{\tilde{d}} + ... + k_{1,1}s + k_{1,0} = [k_{1,\tilde{d}}, ..., k_{1,1}, k_{1,0}]\underline{\tilde{e}}_{\tilde{d}}(s) = \underline{\mathrm{k}}_{1\tilde{d}}^t\, \underline{\tilde{e}}_{\tilde{d}}(s)$$

$$k_i(s) = k_{i,d}s^d + ... + k_{i,1}s + k_{i,0} = [k_{i,d}, ..., k_{i,1}, k_{i,0}]\underline{\tilde{e}}_d(s) = \underline{\mathrm{k}}_{i,\tilde{d}}^t\, \underline{\tilde{e}}_{\tilde{d}}(s),\ i=2,...,m,\ \underline{\tilde{e}}_{\mu}(s)^t = [s^{\mu}, s^{\mu-1}, ..., s, 1] \tag{4.10}$$

(5.2)

Using this explicit representation for $\{\mathcal{K}_p^d\}$ the corresponding combinant becomes

$$f_d(s, \mathcal{K}, \mathcal{P}) = \sum_{i=1}^{m} k_i(s)p_i(s) = \underline{k}_{1,\tilde{d}}^t \underbrace{\begin{bmatrix} s^{\tilde{d}}p_1(s) \\ \vdots \\ sp_1(s) \\ p_1(s) \end{bmatrix}}_{\uparrow\ \underline{p}_{1,\tilde{d}}(s)} + \sum_{i=2}^{m} \underline{k}_{i,\tilde{d}}^t \underbrace{\begin{bmatrix} s^{d}p_i(s) \\ \vdots \\ sp_i(s) \\ p_i(s) \end{bmatrix}}_{\uparrow\ \underline{p}_{i,d}(s)} \tag{4.11}$$

And this readily leads to the following result.

Proposition (4.1): The dynamic combinant $f_d(s, \mathcal{K}_d^d, \mathcal{P})$, generated by the set $\{\mathcal{K}_p^d\}$ is equivalent to a constant combinant of degree p that is generated by the polynomial set $\mathcal{P}_p^d$, $\tilde{d} = p-n,\ \ \tilde{d} \le d \le p-q = d^*$, where

$$\mathcal{P}_p^d = \{s^{\tilde{d}}p_1(s), ..., sp_1(s), p_1(s); s^d p_2(s), ..., sp_2(s), p_2(s); ...; s^d p_m(s), ..., sp_m(s), p_m(s)\} \tag{4.12}$$

■

The set $\mathcal{P}_p^d$ is the (p,d)- power of $\mathcal{P}$ and has degree p and its polynomial vector representative is

$$\underline{p}_d(s) = \begin{bmatrix} \underline{p}_{1,\tilde{d}}(s) \\ \underline{p}_{2,d}(s) \\ \vdots \\ \underline{p}_{m,d}(s) \end{bmatrix} = \begin{bmatrix} S_{n,\tilde{d}}(p_1) \\ S_{q,d}(p_2) \\ \vdots \\ S_{q,d}(p_m) \end{bmatrix} \underline{\tilde{e}}_p(s) = S_{p,d}\,\underline{\tilde{e}}_p(s) \tag{4.13}$$

where the structure of the Toeplitz type blocks above $S_{n,\tilde{d}}(p_1),\ S_{q,d}(p_i)\ \ \ i = 2, ..., m$ defining the corresponding Generalised Resultants is given below:

Proposition (4.2): The Generalised Resultants corresponding to the parameterized set $\{\mathcal{K}_p^d\}$ are defined by:

(i) Given that $\underline{p}_{1,\tilde{d}}(s)$ has degree $\tilde{d} + n = p - n + n = p$, then

$$S_{n,\tilde{d}}(p_1)=\begin{bmatrix}1 & a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 & 0 & \cdots & 0\\ 0 & 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0\\ \vdots & & \ddots & & & & & & \vdots\\ 0 & 0 & \cdots & 1 & a_{n-1} & \cdots & \cdots & a_1 & a_0\end{bmatrix}\in\mathbb{R}^{(\tilde{d}+1)\times(p+1)} \tag{4.14a}$$

(ii) Given that $\underline{p}_{i,d}(s)$ has degree $d+q$ which satisfies the inequality $p-(n-q)\le d+q\le p$ and thus $d+q+1\le p+1$, the structure of $S_{q,d}(p_i)$ is defined for all $i=2,\dots,m$ and $\forall d: p-n\le d\le p-q$ by

$$S_{q,d}(p_i)=\begin{bmatrix}0 & \dots & 0 & b_{i,q} & \dots & b_{i,1} & b_{i,0} & 0 & \dots & \dots & 0\\ 0 & \dots & 0 & 0 & b_{i,q} & \dots & b_{i,1} & b_{i,0} & 0 & \dots & 0\\ \vdots & & 0 & \vdots & \ddots & \ddots & & & & & \vdots\\ 0 & \dots & 0 & 0 & \dots & 0 & b_{i,q} & \dots & \dots & b_{i,1} & b_{i,0}\end{bmatrix}\in\mathbb{R}^{(d+1)\times(p+1)} \tag{4.14b}$$

$$\underbrace{p-q-d}$$

■

Clearly in the boundary case $d=p-q$, there is no zero block and when $d=p-n$, then the zero block takes its maximal dimension $n-q$. The matrix $S_{p,d}(\mathcal{P})\in\mathbb{R}^{\sigma\times(p+1)}$, $\sigma=p-n-d+m(d+1)$ will be called the (p,d)- *Generalised Resultant* of the set $\mathcal{P}$ where the possible values of d are: $p-n\le d\le p-q$. Clearly the $S_{p,d}(\mathcal{P})$ matrix, denoted briefly by $S_{p,d}$, is the basis matrix of the (p,d) power of $\mathcal{P}$, $\mathcal{P}_p^d$. Clearly the properties of the (p,d) generalised resultant.

Remark (4.2): For the given $(m;\, n(q))$ set $\mathcal{P}$ we can parametrise all dynamic combinants in terms of the degree p and the corresponding order d as:

(a) p=n: then $0\le d\le n-q$

(b) p=n+1 : then $1\le d\le n-q+1$

(c) p>n+1: then $p-n\le d\le p-q$

and their properties are defined by the properties of corresponding (p,d)- generalised resultants $S_{p,d}(\mathcal{P})$

■

In the following we will investigate the properties of all dynamic combinants by considering the corresponding family

$$S(\mathcal{P})=\{S_{p,d}\,\forall\; p\ge n \text{ and } \forall\; d: p-n\le d\le p-q\} \tag{4.15}$$

which will be referred to as the *family of Generalised Resultants* of the set $\mathcal{P}$. Amongst the elements of $S(\mathcal{P})$ we distinguish a special element that corresponds to $p=n+q-1,\ d=n-1$ and thus $\partial\,[k_1(s)]=p-n=q-1$. This Generalised Resultant $S_{n+q-1,n-1}(\mathcal{P})$ is denoted in short as $\tilde{S}_{\mathcal{P}}$ and it is referred to as the *Sylvester Resultant* of the set $\mathcal{P}$. This matrix has the following form

$$\tilde{S}_{\mathcal{P}} = \begin{bmatrix} S_{n,\, q-1}(p_1) \\ S_{q,\, n-1}(p_2) \\ \vdots \\ S_{q,\, n-1}(p_m) \end{bmatrix} \in \mathbb{R}^{\tau\times(n+q)}, \ \ \tau = [q+(m-1)n] \qquad (4.16)$$

where $S_{n,q-1}(p_1) \in \mathbb{R}^{q\times(n+q)}$, $S_{q,n-1}(p_i) \in \mathbb{R}^{n\times(n+q)}, j = 2,...,m$ and $\tau = [q+(m-1)n]$. The characteristic of this matrix is that none of the blocks $S_{n,q-1}(p_1),\ S_{q,n-1}(p_i)$ have zero columns and that the rank of $\tilde{S}_{\mathcal{P}}$ is clearly related to algebraic properties of $\mathcal{P}$, as it will be seen subsequently.

5. Spectrum Assignment of Dynamic Combinants and the Sylvester Resultant

We now consider the problem of arbitrary assignment of the spectrum of dynamic combinants for some appropriate order and degree. This is part of the general problem dealing with the parameterisation of all possible degree and order combinants for which assignment may be achieved. We have described the link of dynamic combinants to Generalised Resultants, the structure of the family $S(\mathcal{P})$ of all generalised resultants, and we now consider the problem of arbitrary assignment of the spectrum of dynamic combinants for some appropriate order and degree. This is part of the more problem dealing with the parameterisation of all possible degree and order combinants for which assignment may be achieved. The results in this section follow from the equivalence of dynamic combinants to constant combinants, which imply reduction of the problem to a linear matrix equation involving the corresponding Generalised Resultant and the properties of the corresponding Generalised Resultants. Given that problems of spectrum assignment of dynamic combinants are always reduced to equivalent problems of constant combinants, we start our study by reviewing the basic results from the theory of constant combinants.

Spectral Properties and Assignability of Constant Polynomial Combinants

Consider the $(m;n(q))$ set $\mathcal{P}$ as described previously, with a polynomial vector representative

$$\underline{p}(s) = \begin{bmatrix} p_1(s) \\ p_2(s) \\ \vdots \\ p_m(s) \end{bmatrix} = [\underline{p}_n, \underline{p}_{n-1}, ..., \underline{p}_1, \underline{p}_0] \begin{bmatrix} s^n \\ s^{n-1} \\ \vdots \\ s \\ 1 \end{bmatrix} = \widetilde{P}\underline{\tilde{e}}_n(s) \qquad (5.1)$$

Where $\widetilde{P} \in \mathbb{R}^{m\times(n+1)}$ is the basis matrix of $\mathcal{P}$ with respect to the vector $\underline{\tilde{e}}_n(s)$. The constant polynomial combinant $f_0(s,\mathcal{K},\mathcal{P})$ is defined by

$$f_0(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m} k_i p_i(s) = [k_1, k_2, ..., k_m]\widetilde{P}\underline{\tilde{e}}_n(s) \qquad (5.2)$$

where $\mathcal{K} = \{k_i \in, i \in \tilde{m}\}$ is an arbitrary set. Clearly this is a polynomial of maximal degree n and if $k_1 \neq 0$ then it has degree n. We may thus write

$$f_0(s,\mathcal{K},\mathcal{P}) = \underline{k}^t\widetilde{P}\underline{\tilde{e}}_n(s) = \phi(s) = [\phi_n, ..., \phi_1, \phi_0]\ \underline{\tilde{e}}_n(s) \qquad (5.3)$$

The above suggests that study of properties of $f_0(s,\mathcal{K},\mathcal{P})$ is equivalent to a study of properties of degree n polynomials with real coefficients defined by a vector $\underline{\phi} \in \mathbb{R}^{n+1}$ which are defined by:

$$[k_1, k_2, ..., k_m][\underline{p}_n, \underline{p}_{n-1}, ..., \underline{p}_1, \underline{p}_0] = [\phi_n, ..., \phi_1, \phi_0] \Leftrightarrow \underline{k}^t \tilde{P} = \underline{\phi}^t, \quad \tilde{P} \in \mathbb{R}^{m\times(n+1)} \tag{5.4}$$

Lemma (5.1): For the set $\mathcal{P}$ with a basis matrix $\widetilde{P} \in \mathbb{R}^{m\times(n+1)}$ the constant combinant $f_0(s,\mathcal{K},\mathcal{P})$ is arbitrarily assignable if and only if rank$\{\tilde{P}\} = n+1$.

■

Clearly, if $f_0(s,\mathcal{K},\mathcal{P})$ is assignable a necessary condition is that m>n. The study of constant combinants has been given in (Karcanias etc 1983), where also some classification of the sets $\mathcal{P}$ has been given according to their spectra assignability properties.

Definition (5.1): If for a set $\mathcal{P}$ there exists $\underline{k}$ such that $f_0(s,\mathcal{K},\mathcal{P}) = \phi_0 \in \mathbb{R}, \ \neq 0$, then the n-th degree combinant has all its roots at $s = \infty$ and $\mathcal{P}$ may be referred to as an $\infty - assignable$ set. In the case where there is no $\underline{k}$ such that $f_0(s,\mathcal{K},\mathcal{P}) = \phi_0 \in \mathbb{R}, \ \neq 0$, then $f_0(s,\mathcal{K},\mathcal{P})$ has effective degree at least one for all vectors $\underline{k} \in \mathbb{R}^n$ and the set $\mathcal{P}$ will be called *strongly non-assignable*. For strongly non assignable sets, for all $\underline{k}$ at least one of the roots of $f_0(s,\mathcal{K},\mathcal{P})$ is finite.

Proposition (5.1): Consider the set $\mathcal{P}$ with a basis matrix $\widetilde{P} = [\underline{p}_n, \underline{p}_{n-1}, ..., \underline{p}_1, \underline{p}_0] \in \mathbb{R}^{m\times(n+1)}$. The following properties hold true:

(i) The set $\mathcal{P}$ is ∞ *-assignable,* if and only if $\mathcal{N}_\ell\{[\underline{p}_n, \underline{p}_{n-1}, ..., \underline{p}_1]\} \neq \{0\}$

(ii) The set *strongly non-assignable,* if and only if $\mathcal{N}_\ell\{[\underline{p}_n, \underline{p}_{n-1}, ..., \underline{p}_1]\} = \{0\}$.Furthermore, $f_0(s,\mathcal{K},\mathcal{P})$ has at least ν finite roots for all $\mathcal{K}$ if and only if $\mathcal{N}_\ell\{[\underline{p}_n, \underline{p}_{n-1}, ..., \underline{p}_\nu]\} = \{0\}$ and

$$\mathcal{N}_\ell\{[\underline{p}_n, \underline{p}_{n-1}, ..., \underline{p}_\nu]\} \cap \mathcal{N}_\ell\{[\underline{p}_n, \underline{p}_{n-1}, ..., \underline{p}_\nu, \underline{p}_{\nu-1}]\} = \{0\} \tag{5.5}$$

■

If we denote by $\tilde{P}^{\nu)} = [\underline{p}_n, \underline{p}_{n-1}, ..., \underline{p}_\nu]$ the submatrix of $\tilde{P}$, then if $\mathcal{N}_\ell\{\tilde{P}^{\nu)}\} = \{0\}$ and $\mathcal{N}_\ell\{\tilde{P}^{\nu-1)}\} \neq \{0\}$ then ν will be called the *index* of $\mathcal{P}$ and denotes the least number of finite zeros of $f_0(s,\mathcal{K},\mathcal{P})$ for all $\mathcal{K}$. The existence of finite roots for all $\underline{k}$ when $\nu \geq 1$ raises the question of whether there exists a region Ω of $\mathbb{C}$ that contains the ν finite roots. Such a problem has been investigated in Shan, and Karcanias, (1994). We consider next the spectrum assignment case for the dynamic case.

Spectral Assignability of Dynamic Combinants

We start our investigation of assignability by using Lemma (5.1) that establishes assignability for the case of constant combinants. This result together with the reduction of dynamic combinants to equivalent constant formulation leads to the following result:

Proposition (5.2): Given the $(m;n(q))$ set $\mathcal{P}$, then the combinant $f_d(s,\mathcal{K},\mathcal{P})$ generated by the $(m;d)$ set $\mathcal{K}$ is assignable if and only if the $m(d+1)\times(d+n+1)$ Toeplitz representation $Q_{\mathcal{P},d}$ defined by (3.11) satisfies the condition rank$\{Q_{\mathcal{P},d}\} = n+d+1$

■

The link of coprimeness of $\mathcal{P}$ to the assignability is considered next.

Proposition (5.3): If the set $\mathcal{P}$ is not coprime and $\phi(s)$ is its GCD, then for all d and all $\mathcal{K}$ sets the combinant $f_d(s,\mathcal{K},\mathcal{P})$ is not completely assignable and rank$\{Q_{\mathcal{P},d}\} < n+d+1$.

Proof:
If $\mathcal{P}$ is not coprime and $\phi(s)$ is its gcd, then if $\mathcal{P} = \{p_i(s),\ i \in \widetilde{m}\}$ we may write $p_i(s) = \phi(s)\widetilde{p}_i(s),\ \ i \in \widetilde{m}$ and thus $f_d(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m} k_i(s)p_i(s) = \phi(s)\{\sum_{i=1}^{m} k_i(s)\widetilde{p}_i(s)\}$. Clearly $f_d(s,\mathcal{K},\mathcal{P})$ has all zeros of $\phi(s)$ as fixed zeros and thus for all $\mathcal{K}$ we do not have assignability. For such sets $\mathcal{P}$ ($\phi(s)$ nontrivial gcd), we have that

$$\text{rank}\{Q_{\mathcal{P},d}\} \le \min\ (m(d+1), n+d+1) \qquad (5.6)$$

and thus rank$\{Q_{\mathcal{P},d}\} \le n+d+1$. If equality holds true, then by Lemma (5.1) we have assignability of $f_d(s,\mathcal{K},\mathcal{P})$ which contradicts the non-assignability assumption made above.

■

Corollary (5.1): Necessary condition for complete assignability of $f_d(s,\mathcal{K},\mathcal{P})$ for some d is that $\mathcal{P}$ is coprime.

■

We consider next sufficient conditions for the assignability of combinants for some appropriate order d. This study involves a study of properties of generalised resultants. For the special case of resultants with $p = n+q-1,\ d = n-1$ the Sylvester resultant $\widetilde{S}_{\mathcal{P}} = S_{n+q-1,n-1}(\mathcal{P})$ has the following well known property.

Lemma (5.2) (Barnet,1970, Fatouros and Karcanias, 2003)**:** Let $\mathcal{P}$ be an *(m, n(q))* set with Sylvester Resultant $\widetilde{S}_{\mathcal{P}}$ .The set $\mathcal{P}$ is coprime, if and only if $\widetilde{S}_{\mathcal{P}}$ has full rank .

■

We may now state the main result on the assignability of dynamic combinants:

Theorem (5.1): Let $\mathcal{P}$ be an *(m,n(q))* set. There exists a d such that $f_d(s,\mathcal{K},\mathcal{P})$ is completely assignable, if and only if the set $\mathcal{P}$ is coprime.

Proof:
The necessity has already been established by Corollary (5.1). To prove sufficiency, we consider *d=n-1*. We consider a special combinant of degree *p=n+q-1* and order *n-1* such as

$$\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m} k_i(s)p_i(s)\,,\ \ \partial\,[k_1(s)] = q-1,\ \ \partial\,[k_i(s)] = n-1,\ \ i = 2,3,\ldots,m \qquad (5.7)$$

If we now denote $k_1(s) = \underline{\widetilde{k}}_1^t\underline{\widetilde{e}}_{q-1}(s),\ \ k_i(s) = \underline{\widetilde{k}}_i^t\underline{\widetilde{e}}_{n-1}(s),\ \ i = 2,3,\ldots,m$, then

$$\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P}) = [\underline{\widetilde{k}}_1^t, \underline{\widetilde{k}}_2^t, \ldots, \underline{\widetilde{k}}_m^t]\begin{bmatrix} S_{n,q-1}(p_1) \\ S_{q,n-1}(p_2) \\ \vdots \\ S_{q,n-1}(p_m) \end{bmatrix}\underline{\widetilde{e}}_{n+q-1}(s) = \underline{\widetilde{k}}^t\widetilde{S}_{n+q-1,n-1}(\mathcal{P})\underline{\widetilde{e}}_p(s) = \underline{\widetilde{k}}^t\widetilde{S}_{\mathcal{P}}\ \underline{\widetilde{e}}_p(s) \qquad (5.8)$$

However, $\widetilde{S}_{\mathcal{P}}$ is the Sylvester resultant and by the previous Lemma (5.2) it has full rank, since the set $\mathcal{P}$ is coprime. Therefore, rank$\{\widetilde{S}_{\mathcal{P}}\} = n+q$ and given that $\widetilde{S}_{\mathcal{P}}$ and $Q_{\mathcal{P},d}$ are equivalent under column - row permutations, then by Remark (3.1) assignability is established.

■

The special combinant of order $d=n-1$ and degree $p=n+q-1$ will be referred to as the *Sylvester combinant* of the set $\mathcal{P}$, it is denoted by $\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m} k_i(s)p_i(s)$ $\partial\,[k_1(s)] = q-1,$ and for $i = 2,...,m$, $\partial\,[k_i(s)] = n-1,$ and the zero assignment problem is expressed as making $\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P})$ an arbitrary polynomial $\alpha(s)$ of degree *n+q-1*, i.e. $\alpha(s) = \underline{\alpha}^t \underline{\tilde{e}}_{n+q-1}(s)$. This is then equivalent to solving the equation

$$[\underline{\tilde{k}}_1^t;\ \underline{\tilde{k}}_2^t;...\ ;\underline{\tilde{k}}_m^t]\begin{bmatrix} S_{n,q-1}(p_1) \\ S_{q,n-1}(p_2) \\ \vdots \\ S_{q,n-1}(p_m) \end{bmatrix} = \underline{\alpha}^t \quad \text{or} \quad \underline{\tilde{k}}^t \widetilde{S}_P = \underline{\alpha}^t \tag{5.9}$$

Remark (5.1): Under coprimeness assumption the above equation has always a solution and the number of degrees of freedom is $\rho_s = mn+1-2n$. For the case m=2 the assignment problem has a unique solution.

■

Corollary (5.2): For the *(m,n(q))* coprime set $\mathcal{P}$ the following properties hold true:

(i) There exists a combinant $\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P})$ of degree $p=n+q-1$ and order $d=n-1$ which is completely assignable

(ii) All combinants $\tilde{f}_{n-1}(s,\mathcal{K},\mathcal{P})$ of order $d=n-1$ and degree $p: n+q-1 \le p \le 2n-1$ are also completely assignable.

(iii) All combinants $\tilde{f}_p(s,\mathcal{K},\mathcal{P})$ of degree $p > p_s = n+q-1$ have an assignable element by selection of some appropriate order $p-n \le d \le p-q$.

■

Proof:

Part **(i)** follows from the proof of Theorem (5.1) and by the construction of the Sylvester resultant, which in turn leads to the definition of the combinant $\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P})$ with $\partial\,[k_1(s)] = q-1$ and $\partial\,[k_i(s)] = n-1,\quad i = 2,...,m.$ Consider now the general combinant of order $d=n-1$ which has maximal degree $p=2n-1$. We can then express $k_1(s)$ as

$$\begin{aligned} k_1(s) &= k_{n-1,\,1}s^{n-1} + ... + k_{q,\,1}s^q + k_{q-1,\,1}s^{q-1} + ... + k_{1,\,1}s + k_{0,\,1} = \\ &= [k_{n-1,\,1} + ... + k_{q,\,1}; k_{q-1,\,1}, k_{1,\,1}, k_{0,\,1}]\ \underline{\tilde{e}}_{n-1}(s) = [\underline{\hat{k}}_1^t;\ \underline{\tilde{k}}_1^t]\underline{\tilde{e}}_{n-1}(s) \end{aligned} \tag{5.10}$$

Then $\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m} k_i(s)p_i(s),\quad \partial\,[k_i(s)] = n-1$ and can be expressed as

$$\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P}) = [\underline{\hat{k}}_1^t;\ \underline{\tilde{k}}_1^t,\ \underline{\tilde{k}}_2^t,...,\underline{\tilde{k}}_m^t]S_{2n-1,n-1}(\mathcal{P})\underline{\tilde{e}}_{2n-1}(s) \tag{5.11}$$

where the generalised resultant $S_{2n-1,n-1}(\mathcal{P}) = \widehat{S}_{\mathcal{P}}$ may be partitioned according to the partitioning of $[\underline{\hat{k}}_1^t;\ \underline{\tilde{k}}^t]$ and it is expressed as :

$$\hat{S}_{\mathcal{P}} = \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & \\ 0 & 1 & a_{n-1} & \cdots & \\ \vdots & \vdots & \vdots & & X \\ 0 & \dots & \cdots & 1 & \\ & 0 & & & \widetilde{S}_{\mathcal{P}} \end{bmatrix} \tag{5.12}$$

The upper block diagonal structure of $\hat{S}_{\mathcal{P}}$ and the full rank property of the Sylvester Resultant $\widetilde{S}_{\mathcal{P}}$ implies that $\hat{S}_{\mathcal{P}}$ has full rank since $\text{rank}\{\hat{S}_{\mathcal{P}}\} = n - q + \text{rank}\{\widetilde{S}_{\mathcal{P}}\} = 2n-1$ The proof for any degree $p = n+q-1 \le p < 2n-1$ follows along similar lines, as well as part (iii).

■

The matrix $\hat{S}_{\mathcal{P}}$ defined by (5.12) is an extension of the Sylvester resultant and may be referred to as *n- order extended Sylvester Resultant*. The special combinant of order d=n-1 and degree p=n+q-1 will be referred to as the *Sylvester combinant* of the set $\mathcal{P}$.

Remark(5.2): For the Sylvester combinant $\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P}) = \sum_{i=1}^{m} k_i(s)p_i(s)$ $\partial\,[k_1(s)] = q-1,$ and for $i = 2,\ldots,m$, $\partial\,[k_i(s)] = n-1,$ the zero assignment problem is making $\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P})$ an arbitrary polynomial $\alpha(s)$ of degree *n*+*q*-*1*, i.e. $\alpha(s) = \underline{\alpha}^t \underline{\tilde{e}}_{n+q-1}(s)$ This is equivalent to solving the equation

$$[\underline{\tilde{k}}_1^t;\ \underline{\tilde{k}}_2^t;\ldots;\underline{\tilde{k}}_m^t] \begin{bmatrix} S_{n,q-1}(p_1) \\ S_{q,n-1}(p_2) \\ \vdots \\ S_{q,n-1}(p_m) \end{bmatrix} = \underline{\alpha}^t \quad \text{or} \quad \underline{\tilde{k}}^t \widetilde{S}_P = \underline{\alpha}^t \tag{5.13}$$

Under coprimeness assumption the above equation has always a solution and the number of degrees of freedom is $\rho_s = mn + 1 - 2n$. For the case m=2 the assignment problem has a unique solution.

■

From Corollary (5.2) it is clear that two combinants of the same order d=n-1 and different degrees may be both assignable. In fact, under the coprimeness assumption, both combinants $\widetilde{f}_{n-1}(s,\mathcal{K},\mathcal{P}),\ f_{n-1}(s,\mathcal{K},\mathcal{P})$ of degrees respectively n+q-1 and $2n$-1 are assignable. This raises the following questions on the assignability of all combinants $f_d(s,\mathcal{K},\mathcal{P})$ with d<n-1 and the parameterisation of all combinants $\hat{f}_d(s,\mathcal{K},\mathcal{P})$ of order $d, d \le n-1$ and degree $p \le n+q-1$ which are assignable. The families with degree $p > p_s$ will be called *non-proper.*

The family of all resultants of degree less or equal to p_s is referred to as *proper* subset of the generalised resultants and can be defined as

$$\mathcal{S}_{pr}(\mathcal{P}) = \{S_{p,d}(\mathcal{P}) : n \le p \le n+q-1 = p_s,\ d = p-q-\rho,\ \ \rho = 0,1,\ldots,n-q\} \tag{5.14}$$

This family is clearly partitioned by the degrees and the orders and we may summarise this as follows:

Proposition (5.4): The family of proper generalised resultants of the (*m*,*n*(*q*)) set $\mathcal{P}$ is partitioned into q-1 sets as:

$$\mathcal{S}_{pr}(\mathcal{P}) = \{\mathcal{S}_{p_s}\} \cup \{\mathcal{S}_{p_{s-1}}\} \cup ... \cup \{\mathcal{S}_n\} \tag{5.15}$$

where $p_s = n + q - 1$ and each subset of a fixed degree is also partitioned by the corresponding order has n-q+1 elements.

■

The above readily follows from the previous analysis. The construction of the families of $\mathcal{S}_{pr}(\mathcal{P})$ of different degree and order from the Sylvester Resultant and the investigation of their property is the subject considered next.

6. Construction of the Family of the Proper Sylvester Resultants

The construction of the generalised resultants together with the parameterisation of the $\mathcal{K}$ sets leads to the following results:

Proposition (6.1): The proper combinant of the (*m,n(q)*) set $\mathcal{P}$ that has $p_s = n + q - 1$ degree and order $d = n - 1 - \rho,\ \rho = 1, 2, ..., n - q$ is defined by the generalised resultant $S_{p_s, n-1-\rho}$, (constructed as in (3.9)) and it is expressed as

$$S_{p_s, n-1-\rho} = \begin{bmatrix} S_{n,q-1}(p_1) \\ 0_\rho \vdots S_{q,n-1-\rho}(p_2) \\ \vdots \\ 0_\rho \vdots S_{q,n-1-\rho}(p_m) \end{bmatrix} \tag{6.1}$$

where $S_{n,q-1}(p_1),\ S_{n,q-1-\rho}(p_i),\ \ i = 2, ..., m$ are the standard Sylvester blocks. Furthermore, any two successive combinants of degree p_s and order $d = n - 1 - \rho$ and $d' = n - \rho - 2$ are related as

$$S_{p_s, n-1-\rho} = \begin{bmatrix} S_{n,q-1}(p_1) \\ 0_\rho \vdots S_{q,n-1-\rho}(p_2) \\ \vdots \\ 0_\rho \vdots S_{q,n-1-\rho}(p_m) \end{bmatrix} \cong \begin{bmatrix} x \ldots x \\ \vdots \ \vdots \\ x \ldots x \\ S_{p_s, n-\rho-2} \end{bmatrix} \tag{6.2}$$

where $\cong$ denotes row equivalence (permutations) on matrices.

■

The above readily leads to the following property:

Corollary (6.1): If $S_{p_s, n-\rho-1}, S_{p_s, n-\rho-2}$ are two generalised Sylvester matrices corresponding to combinants of degree p_s and orders $d = n - \rho - 1,\ d' = n - \rho - 2$ respectively, then

$$rank(S_{p_s, n-\rho-1}) \geq rank(S_{p_s, n-\rho-2}) \tag{6.3}$$

Furthermore, if $S_{p_s, n-\rho-1}$ has full rank then all higher order generalised resultants are also full rank.

■

The above result describes rank properties of generalised resultants that have the same degree and different orders. The investigation of links between generalised resultants of different degree is considered next. In the following we will use the notation $S^{\rho}_{q,n-1-\rho}(p_i) = \left[0_\rho \vdots S_{q,n-1-\rho}(p_i)\right]$. With this notation for the p_s and the $p_s - 1$ degrees we have

$$S_{p_s,n-\rho-1} = \begin{bmatrix} S_{n,q-1}(p_1) \\ S^{\rho}_{q,n-\rho-2}(p_2) \\ \vdots \\ S^{\rho}_{q,n-\rho-2}(p_m) \end{bmatrix} \tag{6.4}$$

where $d = n-1-\rho,\ q-1 \le d \le n-1,\ \rho = 0,1,2,...,n-q$. For the $p_s - 1$ degree with $q-2 \le d' \le n-2$, $d' = n-2-\rho',\ \ \rho' = 0,1,2,...,n-q$ we have

$$S_{p_s-1,n-1-\rho'} = \begin{bmatrix} S_{n,q-2}(p_1) \\ S^{\rho'}_{q,n-2-\rho'}(p_2) \\ \vdots \\ S^{\rho'}_{q,n-2-\rho'}(p_m) \end{bmatrix} \tag{6.4}$$

Remark (6.1): The definition of Generalised Resultants readily establishes the following relationship:

$$S_{p_s,n-1} = \begin{bmatrix} S_{n,q-1}(p_1) \\ S_{q,n-1}(p_2) \\ \vdots \\ S_{q,n-1}(p_m) \end{bmatrix} = \begin{bmatrix} 1 & x \ldots x \\ \underline{0} & S_{n,q-2}(p_1) \\ x & x \ldots x \\ \underline{0} & S_{n,q-2}(p_2) \\ \vdots & \vdots \\ x & x \ldots x \\ \underline{0} & S_{n,q-2}(p_m) \end{bmatrix} \cong \begin{bmatrix} \underline{0} & X \\ 1 & x \ldots x \\ \underline{0} & S_{p_s-1,n-2} \end{bmatrix} \tag{6.5}$$

■

The above clearly leads to the following result:

Proposition (6.2): For the maximal order generalised resultants $S_{p_s,n-1}$ and $S_{p_s-1,n-2}$ of degrees p_s, p_{s-1}, p_{s-2} we have the relationships:

$$S_{p_s,n-1} \cong \begin{bmatrix} 1 & x \ldots x \\ \underline{0} & X \\ \underline{0} & S_{p_s-1,n-2} \end{bmatrix} \cong \begin{bmatrix} I_1 & X \\ 0 & X \\ 0 & S_{p_s-2,n-3} \end{bmatrix} \cong \begin{bmatrix} I_\mu & X \\ 0 & X \\ 0 & S_{p_s-\mu,n-\mu-1} \end{bmatrix}, \mu = 0,1,...,q-1 \tag{6.6}$$

and thus

$$rank(S_{p_s,n-1}) \ge 1 + rank(S_{p_s-1,n-2}) \ \ge 2 + rank(S_{p_s-2,n-3}) \ge ... \ge q-1 + rank(S_{n,n-q}) \tag{6.7}$$

■

The above result establishes an important rank property for the generators of each of the given degree $p_s - \mu$ classes which has important implications for searching process and the determination of the least degree solution. The analysis so far indicates a systematic process for construction of the family of generalised resultants and this is summarised below:

Construction of a Family of Generalised Resultants

Given the $(m,n(q))$ set $\mathcal{P}$, which is assumed to be coprime, we construct the Sylvester resultant that corresponds to $p_s = n+q-1$ degree combinant and has order $d=n-1$. If $S_{p_s,n-1}$ is the Sylvester resultant, then the family of proper generalised resultants is defined from $S_{p_s,n-1}$ by transformations on this matrix. Thus, if we denote by

$$S_{p_s,n-1} = \begin{bmatrix} S_{n,q-1}(p_1) \\ S_{q,n-1}(p_2) \\ \vdots \\ S_{q,n-1}(p_m) \end{bmatrix} \tag{6.8}$$

the Sylvester Resultant, then the construction of the different degree and order families is described below:

(a) The construction of $\{\mathbf{S}_{p_s}\}$ sub-family

This family has degree $p_s = n+q-1$ and has $n\text{-}q\text{+}1$ generalised resultants of respective order $d = n-\rho-1,\ \ \rho = 0,1,\dots,n-q$, where for $\rho = 0$ we have $S_{p_s,n-1}$ as the generator of the family. The element $S_{p_s,n-\rho-1}$ of $\{\mathbf{S}_{p_s}\}$ is constructed from $S_{p_s,n-1}$ by keeping the first block $S_{n,\,q-1}(p_1)$ and then eliminating the first ρ rows from each of the blocks $S_{q,\,n-1}(p_i),\ \ i = 2,\dots,m$. This leads to the construction of

$$S_{p_s,n-1-\rho} = \begin{bmatrix} S_{n,q-1}(p_1) \\ 0_\rho \vdots S_{q,n-1-\rho}(p_2) \\ \vdots \\ 0_\rho \vdots S_{q,n-1-\rho}(p_m) \end{bmatrix} = \begin{bmatrix} S_{n,q-1}(p_1) \\ S^{\rho}_{q,n-1-\rho}(p_2) \\ \vdots \\ S^{\rho}_{q,n-1-\rho}(p_m) \end{bmatrix} \tag{6.9}$$

The above family is denoted by $<\mathbf{S}_{p_s}>=\{S_{p_s,n-1-\rho},\ \rho = 0,1,\dots,n-q\}$ and for $\rho = 0$ we have the generator of the family, the Sylvester Resultant $S_{p_s,n-1}$.

(b) The construction of $\{\mathbf{S}_{p_s-1}\}$ sub-family

This family has degree $p_s' = n+q-2$ and has $n\text{-}q\text{+}1$ generalised resultants of respective order $d' = n-2-\rho,\ \ \rho = 0,1,\dots,n-q$, where for $\rho = 0$ we have $S_{p_s',n-2}$ as the generator of the family which is constructed from as described below:

The generator of $\{\mathbf{S}_{p_s-1}\}$ family: Eliminate the first row for each of the $S_{n,q-1}(p_1),\ \ S_{q,n-1}(p_i),\ \ i = 2,\dots,m$ blocks that results in matrix blocks $[\underline{0}, S_{n,q-2}(p_1)],\ [\underline{0}, S_{q,n-2}(p_i)],\ \ i = 2,3,\dots,m.$ The generator of the $p_s' = n+q-2$ family has order $d' = n-2$ and it is defined from these blocks by eliminating the first zero columns. This leads to

$$S_{p'_s,n-2} = \begin{bmatrix} S_{n,q-2}(p_1) \\ S_{q,n-2}(p_2) \\ \vdots \\ S_{q,n-2}(p_m) \end{bmatrix} \tag{6.10}$$

Having defined the generator $S_{p'_s,n-2}$ of dimension $\tau' \times (n+q-1)$ where $\tau' = \tau - m = q + (m-1)n - m$ we can proceed with the construction of the rest of the elements of the p'_s family by following a similar process as before i.e.

The general element of the $\{\mathbf{S}_{p_s-1}\}$ sub-family: The general element $S_{p'_s,n-2-\rho}$, $\rho = 1,2,...,n-q$ is constructed from the generator $S_{p'_s,n-2}$ by keeping the first block $S_{n,q-2}(p_1)$ and by eliminating the first ρ rows form each of the $S_{q,n-2}(p_i),\ i = 2,...,m$ blocks. This leads to the construction of

$$S_{p_s',n-2-\rho} = \begin{bmatrix} S_{n,q-2}(p_1) \\ 0_\rho \vdots S_{q,n-2-\rho}(p_2) \\ \vdots \\ 0_\rho \vdots S_{q,n-2-\rho}(p_m) \end{bmatrix} = \begin{bmatrix} S_{n,q-2}(p_1) \\ S^{\rho}_{q,n-2-\rho}(p_2) \\ \vdots \\ S^{\rho}_{q,n-2-\rho}(p_m) \end{bmatrix} \tag{6.11}$$

The above family is denoted by $<\mathbf{S}_{p_s-1}>=\{S_{p'_s,n-2-\rho},\ \rho = 0,1,2,...,n-q,\ \ d' = n-2-\rho\}$ where for $\rho = 0$ we have the generator of the family.

(c) The construction of $\{\mathbf{S}_{p_s-\mu}\}$ sub-family

The family with degree $p^{\mu}{}_s = n+q-1-\mu,\ \mu = 0,1,...,q-1$ follows a similar construction process that involves the construction of the generator $S_{p_s{}^{\mu},n-\mu-1}$ and then the elements of the family by deleting the first ρ rows $\rho = 0,1,...,n-q$ form each of the $i = 2,...,m$ blocks. The resulting family $\{\mathbf{S}_{p_s-\mu}\}$ has again $n-q+1$ elements.

■

The above provides a systematic procedure for defining the partitioning and the elements of the proper family of the resultants of $\mathcal{P}$. We demonstrate the above construction with a simple example:

Example (6.1): *Consider the polynomials* $p_1(s) = s^4 + a_{31}s^3 + a_{21}s^2 + a_{11}s + a_{01}$, $p_2(s) = a_{22}s^2 + a_{12}s + a_{02}$

The Sylvester Resultant and the S_5 family is defined by:

$$\underline{d=3}:\ S_{5,3}=\begin{bmatrix} 1 & a_{31} & a_{21} & a_{11} & a_{01} & 0 \\ 0 & 1 & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . & . \\ a_{22} & a_{12} & a_{02} & 0 & 0 & 0 \\ 0 & a_{22} & a_{12} & a_{02} & 0 & 0 \\ 0 & 0 & a_{22} & a_{12} & a_{02} & 0 \\ 0 & 0 & 0 & a_{22} & a_{12} & a_{02} \end{bmatrix} \tag{6.12}$$

and the corresponding resultants of the same degree, but less order are:

$$\underline{d=2}:\ S_{5,2}=\begin{bmatrix} 1 & a_{31} & a_{21} & a_{11} & a_{01} & 0 \\ 0 & 1 & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . & . \\ 0 & a_{22} & a_{12} & a_{02} & 0 & 0 \\ 0 & 0 & a_{22} & a_{12} & a_{02} & 0 \\ 0 & 0 & 0 & a_{22} & a_{12} & a_{02} \end{bmatrix},\ \underline{d=1}:\ S_{5,1}=\begin{bmatrix} 1 & a_{31} & a_{21} & a_{11} & a_{01} & 0 \\ 0 & 1 & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . & . \\ 0 & 0 & a_{22} & a_{12} & a_{02} & 0 \\ 0 & 0 & 0 & a_{22} & a_{12} & a_{02} \end{bmatrix} \tag{6.13}$$

Clearly the family S_5 has n+q-1 elements i.e. 3 elements and thus $\{S_5\}=\{S_{5,3};S_{5,2};S_{5,1}\}$.

For the case of degree $p_s'=n+q-2=6-2=4$ which is the least degree, the corresponding order resultants define the elements of $\{S_4\}$ which are again 3 and they are defined by

$$\underline{d=2}:\ S_{4,2}=\begin{bmatrix} 1 & a_{31} & a_{21} & a_{11} & a_{01} \\ a_{22} & a_{12} & a_{02} & 0 & 0 \\ . & . & . & . & . \\ 0 & a_{22} & a_{12} & a_{02} & 0 \\ 0 & 0 & a_{22} & a_{12} & a_{02} \end{bmatrix},\ \underline{d=1}:\ S_{4,1}=\begin{bmatrix} 1 & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . \\ 0 & a_{22} & a_{12} & a_{02} & 0 \\ 0 & 0 & a_{22} & a_{12} & a_{02} \end{bmatrix}$$

and (6.14)

$$\underline{d=0}:\ S_{4,0}=\begin{bmatrix} 1 & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . \\ 0 & 0 & a_{22} & a_{12} & a_{02} \end{bmatrix}$$

■

Example (6.2): *Consider the polynomials*

$p_1(s)=s^5+a_{41}s^4+a_{31}s^3+a_{21}s^2+a_{11}s+a_{01}$, $p_2(s)=a_{32}s^3+a_{22}s^2+a_{12}s+a_{02}$,
$p_3(s)=a_{33}s^3+a_{23}s^2+a_{13}s+a_{03}$.

Here, $n=5, q=3$ and $p_s=7$. The Sylvester resultant is $S_{7,4}$ and it is the generator of the family with degree 7 defined by $\{S_7\}=\{S_{7,4};S_{7,3};S_{7,2}\}$. Similarly, the $p_s'=6$ family is generated by $S_{6,3}$ and $\{S_6\}$ is defined by $\{S_6\}=\{S_{6,3};S_{6,2};S_{6,1}\}$. The least degree family is $\{S_5\}$ and it has as a generator the generalised resultant $S_{5,2}$ Then $\{S_5\}=\{S_{5,2};S_{5,1};S_{5,0}\}$ and the family of the generalised proper resultants $\mathcal{S}_{p_r}(\mathcal{P})$ is defined by $S_{p_r}(\mathcal{P})=\{S_7\}\cup\{S_6\}\cup\{S_5\}$ where $\cup$ denotes the union of the non-intersecting sets expressing the partitioning property. The set of proper resultants is now defined below. For $p_s=7$ and $d=4$ we have the Sylvester Resultant $S_{7,4}$ from which we may construct $S_{7,3}$ and $S_{7,2}$ by the elimination of top rows of the blocks apart from the first block as shown below

Similarly for the case of order d=2 and degree $p_s=7$ we have

$$S_{7,4}=\begin{bmatrix} 1 & a_{41} & a_{31} & a_{21} & a_{11} & a_{01} & 0 & 0 \\ 0 & 1 & a_{41} & a_{31} & a_{21} & a_{11} & a_{01} & 0 \\ 0 & 0 & 1 & a_{41} & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . & . & . & . \\ 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} & 0 & 0 \\ 0 & 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} & 0 \\ 0 & 0 & 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} \\ 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} & 0 \\ 0 & 0 & 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} \end{bmatrix}=\begin{bmatrix} 1 & x & \vdots & x & \cdots & x \\ 0 & 1 & \vdots & x & \cdots & x \\ . & . & . & . & . & . \\ & 0 & \vdots & & S_{5,2} & \end{bmatrix}$$

which clearly indicates thatm $rank(S_{7,2})=2+rank(S_{5,2})$. For the set with degree $p_s'=6$ we have as a generator $S_{6,3}$ and thus $S_{6,2}$ and $S_{6,1}$ are given by

$$S_{6,3}=\begin{bmatrix} 1 & a_{41} & a_{31} & a_{21} & a_{11} & a_{01} & 0 \\ 0 & 1 & a_{41} & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . & . & . \\ a_{32} & a_{22} & a_{12} & a_{02} & 0 & 0 & 0 \\ 0 & a_{32} & a_{22} & a_{12} & a_{02} & 0 & 0 \\ 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} & 0 \\ 0 & 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} \\ . & . & . & . & . & . & . \\ a_{33} & a_{23} & a_{13} & a_{03} & 0 & 0 & 0 \\ 0 & a_{33} & a_{23} & a_{13} & a_{03} & 0 & 0 \\ 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} & 0 \\ 0 & 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} \end{bmatrix}, S_{6,2}=\begin{bmatrix} 1 & a_{41} & a_{31} & a_{21} & a_{11} & a_{01} & 0 \\ 0 & 1 & a_{41} & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . & . & . \\ 0 & a_{32} & a_{22} & a_{12} & a_{02} & 0 & 0 \\ 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} & 0 \\ 0 & 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} \\ . & . & . & . & . & . & . \\ 0 & a_{33} & a_{23} & a_{13} & a_{03} & 0 & 0 \\ 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} & 0 \\ 0 & 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} \end{bmatrix}=\begin{bmatrix} 1 & x\ldots x \\ \underline{0} & S_{5,2} \end{bmatrix}$$

$$S_{6,1}=\begin{bmatrix} 1 & a_{41} & a_{31} & a_{21} & a_{11} & a_{01} & 0 \\ 0 & 1 & & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . & . & . \\ 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} & 0 \\ 0 & 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} \\ . & . & . & . & . & . & . \\ 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} & 0 \\ 0 & 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} \end{bmatrix}$$

The least degree family $p_s''=5$ has a generator for the generalised resultant $\mathcal{S}_{5,2}$ defined by

$$S_{5,2}=\begin{bmatrix} 1 & a_{41} & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . & . \\ a_{32} & a_{22} & a_{12} & a_{02} & 0 & 0 \\ 0 & a_{32} & a_{22} & a_{12} & a_{02} & 0 \\ 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} \\ . & . & . & . & . & . \\ a_{33} & a_{23} & a_{13} & a_{03} & 0 & 0 \\ 0 & a_{33} & a_{23} & a_{13} & a_{03} & 0 \\ 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} \end{bmatrix}, S_{5,1}=\begin{bmatrix} 1 & a_{41} & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . & . \\ 0 & a_{32} & a_{22} & a_{12} & a_{02} & 0 \\ 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} \\ . & . & . & . & . & . \\ 0 & a_{33} & a_{23} & a_{13} & a_{03} & 0 \\ 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} \end{bmatrix}, S_{5,0}=\begin{bmatrix} 1 & a_{41} & a_{31} & a_{21} & a_{11} & a_{01} \\ . & . & . & . & . & . \\ 0 & 0 & a_{32} & a_{22} & a_{12} & a_{02} \\ . & . & . & . & . & . \\ 0 & 0 & a_{33} & a_{23} & a_{13} & a_{03} \end{bmatrix}$$

■

7. The Search for the Minimal Degree and Order Solution

The results on the rank properties of the generalised resultants provide the basis for the development of a procedure that may lead to determining the least degree and order solution of the spectral assignment problem. The problems to be addressed are:

Problems: For an (*m, n(q)*) coprime set $\mathcal{P}$ with Sylvester degree $p_s = n+q-1$ and generators for the different degree families $\{S_{p_s,n-1}; S_{p_s-1,n-2}; \ldots S_{p_s-\mu,n-1-\mu}\}$, $\mu = 0,1,\ldots,n-q$ define:

- The least value of μ, say μ^* such that the $S_{p_s-\mu^*,n-\mu^*-1} \in \mathbb{R}^{\tilde{\tau}\times(p_s-\mu^*+1)}$ has $\tilde{\tau} \geq p_s-\mu^*+1$
- Having defined the value of such an μ^* consider the $\{\mathbf{S}_{p_s-\mu^*}\}$ and define the least order element $S_{p_s-\mu^*,n-\mu^*-1-\rho^*} \in \mathbb{R}^{\tau'\times(p_s-\mu^*+1)}$ for which $\tau' \geq p_s-\mu^*+1$

■

The resulting values for μ^*, ρ^* define the boundaries for the searching process and are considered next. We first note that the partition

$$\mathcal{S}_{p_r}(\mathcal{P}) = \{\mathcal{S}_{p_s}\} \cup \{\mathcal{S}_{p_{s-1}}\} \cup \ldots \cup \{\mathcal{S}_{p_n}\} \tag{7.1}$$

has $\nu+1=q$ elements since $p_s-\nu=n$. Each of the $\{\mathcal{S}_{p_s-\mu}\},\ \mu=0,1,...,q-1$ families has a generator $S_{p_s-\mu,n-1-\mu}\in\mathbb{R}^{[q+(m-1)n-m\mu]\times[n+q-1-\mu]}$. These relations readily lead to:

Proposition (7.1): The least degree generator $S_{p_s-\mu,n-1-\mu}\in\mathbb{R}^{\tau'\times\rho'}$ for which $\tau\geq\rho'$

$$\mu^*=min\{\ \frac{1+mn-2n}{m-1},q-1\ \} \tag{7.2}$$

Proof: By definition, since $\mu=0,1,2,\ldots,q-1$ we have that $\mu\leq q-1$. Furthermore $q+(m-1)n-m\mu\geq n+q-1-\mu$ or $\mu-mm\geq 2n-1-mn$ and thus $\mu\leq(1+mn-2n)/(m-1)$.

■

Remark (7.1): The suggested computation of μ^* above indicates that none of the classes $\{\mathcal{S}_{p_s-\mu}\}$ with $\mu\geq\mu^*$ contain an element that is assignable and thus assignment has to be investigated only for the classes

$$\{\mathcal{S}_{p_s}\},\{\mathcal{S}_{p_s-1}\},...,\{\mathcal{S}_{p_s-\mu^*}\} \tag{7.3}$$

■

Given that $\{\mathcal{S}_{p_s-\mu^*}\}$ contains n-q+1 elements, it is worth finding the element $\{S_{p_s-\mu^*,n-1-\mu^*-\rho}\}\in\mathbb{R}^{\tau''\times\rho''}$ for which $\tau''\geq\rho''$. This is established next

Proposition (7.2): The least degree and order generalised resultant $\{S_{p_s-\mu^*,n-1-\mu^*-\rho^*}\}\in\mathbb{R}^{\tau''\times\rho''}$ for which $\tau''\geq\rho''$ is defined by:

$$\mu^*=min\{\ \frac{1+mn-2n}{m-1},\ q-1\ \}\ ,\qquad \rho^*=min\{\ n-q+1,\ \frac{m(n-\mu^*)-2n+1+\mu^*}{m-1}\ \} \tag{7.4}$$

Proof: The dimensions of $\{\mathcal{S}_{p_s-\mu^*,n-1-\mu^*-\rho}\}$ are clearly $[q(m-1)n-m\mu^*-\rho^*(m-1)]\times[n+q-1-\mu^*]$ and the above for the value of μ^* previously computed leads to $\rho^*\leq n-q+1$ and thus $mn-n-m\mu^*-\rho^*(m-1)\geq n-1-\mu^*$ and thus $\rho^*\leq[m(n-\mu^*)-2n+1+\mu^*]/(m-1)$

■

The above results provide a lower bound for the degrees and the order for a combinant to be assignable and this is expressed by the result:

Theorem (7.1): For the coprime (*m,n(q)*) set $\mathcal{P}$ with Sylvester degree $p_s=n+q-1$ the least degree combinant p^* that may be assignable and the least order d^* with the assignability property are

$$p^*=n+q-1-\mu^*,\quad d^*=n-1-\mu^*-\rho^* \tag{7.5}$$

where μ^* and ρ^* are defined by (7.4) respectively.

■

The proof of the above follows from the two previous propositions and the dimensionality arguments. Clearly, p^*, d^* define lower bounds and thus specify the values where the test of the rank properties makes sense. In principle we expect the minimal values of degree and order, $\tilde{p},\tilde{d}$ to be higher than the corresponding

p^*, d^*. The above provide the basis for the development of the searching process that is considered next. We first state the following result:

Proposition (7.3): Let $\{S_{p_s-\mu}\}, \mu = 0,1,...,q-1$ be the family with degree $p_s - \mu$. If the generator $S_{p_s-\mu,n-1-\mu}$ is rank deficient, then all elements of the family $\{S_{p_s-\mu}\}$ are rank deficient.

Proof: We first note that Proposition (6.1) and Corollary (6.1) although stated for the $\{S_{p_s}\}$ family are also true for any other family $\{S_{p_s-\mu}\}$, $\mu = 0,1,\ldots,q-1$. Furthermore we note that since all elements of $\{S_{p_s-\mu}\}$ have the same number of columns for every fixed μ, it follows that by Corollary (6.1), if the generator is rank deficient, then all elements of the family (having fewer rows) will be also rank deficient.

■

Remark (7.2): The search for the least degree and least order solution is restricted only to the families with full rank generators.

■

We may now state the main result:

Theorem (7.2): Consider the (*m, n(q)*) coprime set $\mathcal{P}$ with Sylvester degree $p_s = n+q-1$. The following properties hold true:

- The least degree assignable combinant $\tilde{p} = p_s - \tilde{\nu}$ is defined by the maximal index for $\tilde{\nu}$ for which

$$0 \le \tilde{\nu} \le \mu^* = min\{ \frac{1+mn-2n}{m-1}, q-1 \} \tag{7.6}$$

where $\tilde{\nu}$ is the maximal index for which the generator $S_{p_s-\tilde{\nu},n-1-\tilde{\nu}}$ has full rank.

- The least order assignable combinant corresponds to the least degree $\tilde{p} = p_s - \tilde{\nu}$ and to the least order $\tilde{d} = n-1-\tilde{\nu}-\tilde{\rho}$, where $\tilde{\rho}$ is the maximal index for which

$$0 \le \tilde{\rho} \le \rho^* = min\{ n-q+1, \frac{m(n-\mu^*)-2n+1+\mu^*}{m-1} \} \tag{7.7}$$

and $S_{p_s-\tilde{\nu},n-1-\tilde{\nu}-\tilde{\rho}}$ has full rank.

Proof:

Condition (7.6) follows by the theorem which defines the dimensions for which we may have assignability based on dimensions. Having specified μ^* we define successively the family of generators

$$S_{p_s-\nu,n-1-\nu}, \nu = \mu^*, \mu^*-1\ldots,0 \tag{7.8}$$

and test successively their rank. The first index for which $S_{p_s-\nu,n-1-\nu}$ has full rank, say $\tilde{\nu}$, defines the least degree family $\{S_{p_s-\tilde{\nu}}\}$. In fact, by Proposition (6.2) all degrees $p > p_s - \tilde{\nu}$ are assignable and the generator $S_{p_s-\tilde{\nu}-1,\, n-2-\tilde{\nu}}$ are rank deficient. Using the Proposition (7.3), it is clear that the $\{S_{p_s-\tilde{\nu}-1}\}$ family has no full rank element and similarly no other family of less degree has this property. Having defined the generator with least degree and full rank we consider the family that corresponds to different orders, i.e.

$$S_{p_s-\tilde{\nu},\, n-1-\tilde{\nu}-\rho}, \rho=\rho^*, \rho^*-1,...,0 \tag{7.9}$$

and test successively their rank. The first index for which $S_{p_s-\tilde{\nu},\, n-1-\tilde{\nu}-\rho}$ has full rank, say $\tilde{\rho}$ defines the least order element of $\{S_{p_s-\tilde{\nu}}\}$. By Corollary (6.1) all higher order are also assignable.

■

The results so far lead to the following algorithm for computing the least degree and least order solution

Procedure for Determining the Least Degree and Order Solutions

For some coprime (*m,n(q)*) set $\mathcal{P}$ with Sylvester degree $p_s = n+q-1$ determining the least degree and least order solutions involves the following steps:

Step(1): Compute the numbers μ^* and ρ^* by

$$\mu^* = min\{\ \frac{1+mn-2n}{m-1},\ q-1\ \}\ ,\ \rho^* = min\{\ n-q+1,\ \frac{m(n-\mu^*)-2n+1+\mu^*}{m-1}\ \} \tag{7.10}$$

which define the lower bounds for the assignable degree and order

$$p^* = p_s - \mu^* = n+q-1-\mu^*\ ,\ d^* = n-1-\mu^*-\rho^* \tag{7.11}$$

Step(2): Define the generators of the proper family $S_{p_r}(\mathcal{P}) = \{S_{p_s}\} \cup \ldots \cup \{S_{p_s-\nu}\} \cup \ldots$ for $\nu \le \mu^*$ in reverse order i.e. $S_{p_s-\nu,n-1-\nu},\ \nu = \mu^*, \mu^*-1, \ldots, 0$. Then, test successively the ranks of $S_{p_s-\mu^*+j,\nu-1-\mu^*+j},\ j = 0,1,\ldots,\mu^*$ and determine the least index $j=\alpha$ for which the matrix generator $S_{p_s-\mu^*+\alpha,n-1-\mu^*+\alpha}$ has full rank. Then the least assignable degree is

$$\tilde{p} = p_s - \mu^* + \alpha = p_s - \tilde{\nu} \tag{7.12}$$

Step(3): Having defined the least degree assignable generator $S_{p_s-\tilde{\nu},n-1-\tilde{\nu}}$ we consider the corresponding class $\{S_{p_s-\tilde{\nu}}\}$ and for $\rho \le \rho^*$ we list its elements in reverse order: $S_{p_s-\tilde{\nu},n-1-\tilde{\nu}-\rho}$, $\rho = \rho^*, \rho^*-1, \ldots, 0$. Then, we test successively the ranks of $S_{p_s-\tilde{\nu},\, n-1-\tilde{\nu}-\rho^*+i},\ i = 0,1,\ldots,\rho^*$ and determine the least index $i=\beta$ for which the corresponding generalised resultant $S_{p_s-\tilde{\nu},\, n-1-\tilde{\nu}-\rho^*+\beta}$ has full rank. Then, the least assignable order for the least degree $\tilde{p}$ is defined by

$$\tilde{d} = n-1-\tilde{\nu}-\rho^*+\beta = n-1-\tilde{\nu}-\tilde{\rho} \tag{7.13}$$

■

The above process involves a small number of rank tests starting from smaller order generalised resultants and leads to the minimal degree $\tilde{p}$ and least order $\tilde{d}$ in a finite number of steps.

Remark (7.3): The construction of all elements of the subfamilies in $S_{p_r}(\mathcal{P})$ is based on column, row elimination operations starting from the Sylvester Resultant and such conditions are readily implementable ■

8. Conclusions

The fundamentals of the theory of dynamic polynomial combinants have been reviewed and their representation in terms of Generalized Resultants has been established. The conditions for existence of spectrum assignable combinants have been established and these are equivalent to the coprimeness of the generating set $\mathcal{P}$. The parameterization of combinants in terms of order and degree has been shown to be central in the study of their properties and this lays the foundations for investigating the properties of the family of Generalised Resultants. Amongst the key problems in this area is the minimal design problem dealing with finding the least order and degree for which spectrum assignability may be guaranteed. Conditions for the characterisation of the minimal order and degree combinant for which arbitrary assignment is possible have been derived and a simple algorithm that produces such solutions in few steps is given. The current framework allows the further development of the theory of dynamic combinants that may answer questions related to the zero distribution of dynamic combinants, in the cases where complete assignability (due to order and degree) is not possible. The study of non-assignable combinants is important since this is linked to the presence of "almost fixed zeros" (Karcanias etc 1983). Current research is now focused to such cases and the notion of dynamic strong non- assignability is now examined which is linked to the property that part of the spectrum is bounded within a finite region $\Omega(P,d,\upsilon)$ of the complex plane for all $\mathcal{K}$ sets of order d. The presence of such regions is related to boundness of part of the spectrum of polynomial combinants for all $\mathcal{K}$ sets. Such studies are also linked to the properties of the "approximate GCD" of many polynomials (Karcanias, etc, 2006); the results here provide the means for examining the link between "approximate gcd", defined by the rank properties of the Sylvester Resultant and the constraints imposed on the spectrum of non-assignable dynamic combinants.

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