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# The Minimal Design Problem on Dynamic Polynomial Combinants

Nicos Karcanias \*

*\* Systems and Control Centre, School of Engineering and  
Mathematical Sciences, City University London, Northampton Square  
EC1V 0HB, UK (e-mail: N.Karcanias@city.ac.uk).*

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**Abstract:** The theory of dynamic polynomial combinants is linked to the linear part of the Dynamic Determinantal Assignment Problems, which provides the unifying description of the pole and zero dynamic assignment problems in Linear Systems. The fundamentals of the theory of dynamic polynomial combinants have been recently developed by examining issues of their representation, parameterization of dynamic polynomial combinants according to the notions of order and degree and spectral assignment. Central to this study is the link of dynamic combinants to the theory of "Generalised Resultants", which provide the matrix representation of the dynamic combinants. The paper considers the case of coprime set polynomials for which spectral assignability is always feasible and provides a complete characterisation of all assignable combinants with order above and below the Sylvester order. A complete parameterization of combinants and respective Generalised Resultants is given and this leads naturally to the characterisation of the minimal degree and order combinant for which spectrum assignability may be achieved, referred to as the "Dynamic Combinant Minimal Design" (DCMD) problem. Such solutions provide low bounds for the respective Dynamic Assignment control problems.

*Keywords:* Linear Systems, Structural Properties, Polynomial Methods, Output Feedback

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## 1. INTRODUCTION

The study of determinantal type problems (such as pole zero assignment, stabilisation) has been unified by the development of a framework referred to as Determinantal Assignment Problem (DAP) [Karcanias and Giannakopoulos (1984)], [Leventides and Karcanias (1998)]. This framework belongs to the general algebro-geometric methodology [Brockett and Byrnes (1980)] and it relies on tools from algebra and algebraic geometry [Hodge and Pedoe (1952)]. DAP is a multi-linear nature problem and thus may be naturally split into a linear and multi-linear problem (decomposability of multivectors). The final solution is thus reduced to the solvability of a set of linear equations coming from the spectrum assignability of polynomial combinants [Karcanias et al. (1983)], characterising the linear problem, together with quadratics characterising the multi-linear problem of decomposability, which in turn define some appropriate Grassmann variety [Hodge and Pedoe (1952)].

Dynamic compensation problems may also be studied within the DAP framework, but their linear sub-problem depends on dynamic polynomial combinants which have much richer properties and they have been studied recently [Karcanias and Galanis (2010)]. Amongst the open issues in the area of dynamic frequency assignment problems, is defining the least complexity compensator (this is frequently defined by the McMillan degree), for which we may have solvability of the arbitrary spectrum assignment of the corresponding DAP. This is referred to as the minimal design problem of DAP. The linear part of DAP is expressed as spectrum assignability of dynamic polynomial

combinants, and thus the study of the minimal design for this problem may enable the derivation of lower bounds for the solution of the original minimal design problem defined on DAP.

The fundamental aspects of the theory of dynamic polynomial combinants have been examined in [Karcanias and Galanis (2010)], where their representation in terms of Generalized Resultants [Barnett (1970b)] [Barnett (1983)] [Vardulakis and Stoye (1978)] and Toeplitz matrices has been established. Dynamic polynomial combinants have been parameterized in terms of order and degree and this has introduced the foundations for the investigation of the property of spectrum assignability for some value of the degree and order of the dynamic combinant which are referred to as Sylvester order and degree respectively. It has been established that under the conditions of coprimeness of the set  $\mathcal{P}$  defining the combinant, there is always an order and degree such that the corresponding combinant has its spectrum assignable. The parametrisation of all dynamic combinants according to order and degree is used to show that all combinants of degree greater than the Sylvester degree have elements (corresponding to some appropriate order) which are assignable, and there is a set of degrees less than the Sylvester degree for which we have assignable combinants for some appropriate order. The latter property motivates the study for finding the least degree and order combinant that is spectral assignable. This problem is referred to here as Minimal Design Problem for Dynamic Combinants (MDP-DC) and is examined here. Using the systematic construction of the family of Generalised Resultants with order and degree less than

the Sylvester, we define the minimal solution in a finite number of tests using only rank tests. These results on the dynamic polynomial combinants are clearly necessary for the solvability of the corresponding DAP, and thus provide lower bounds for the solutions of the corresponding dynamic frequency assignment problem.

Throughout the paper the following notation is adopted: If  $\mathcal{F}$  is a field, or ring then  $\mathcal{F}^{m \times n}$  denotes the set of  $m \times n$  matrices over  $\mathcal{F}$ . If  $H$  is a map, then  $\mathcal{R}(H), \mathcal{N}_r(H), \mathcal{N}_l(H)$  denote the range, right, left null spaces respectively.  $Q_{k,n}$  denotes the set of lexicographically ordered, strictly increasing sequences of  $k$  integers from the set  $\tilde{n} \triangleq \{1, 2, \dots, n\}$ . If  $\mathcal{V}$  is a vector space and  $\{\underline{v}_{i_1}, \dots, \underline{v}_{i_k}\}$  are vectors of  $\mathcal{V}$  then  $\underline{v}_{i_1} \wedge \dots \wedge \underline{v}_{i_k} = \underline{v}_\omega \wedge, \omega = (i_1, \dots, i_k)$  denotes their exterior product and  $\wedge^r \mathcal{V}$  the  $r$ -th exterior power of  $\mathcal{V}$ . If  $H \in \mathcal{F}^{m \times n}$  and  $r \leq \min\{m, n\}$ , then  $C_r(H)$  denotes the  $r$ -th compound matrix of  $H$  [Marcus and Minc (1964)]. We shall denote by  $\mathbb{R}[s], \mathbb{R}(s)$  the ring of polynomials, rational functions over  $\mathbb{R}$  respectively.

## 2. BASIC DEFINITIONS AND PROPERTIES OF DYNAMIC COMBINANTS

### *The Determinantal Assignment and Polynomial Combinants*

A large family of problems for Linear Systems dealing with Dynamic Compensation [Kucera (1979)], [Kailath (1980)] may be reduced to a common formulation that is represented by the determinantal assignment problem (DAP) [Karcaniyas and Giannakopoulos (1984)]. This problem deals with the study of the following equation with respect to polynomial matrix  $H(s)$ :

$$\det(H(s)M(s)) = f(s) \quad (1)$$

where  $f(s)$  is a polynomial of an appropriate degree  $d$ . If  $M(s) \in \mathbb{R}^{p \times m}[s]$ ,  $r \leq p$ , such that  $\text{rank}(M(s)) = r$  and let  $\mathcal{H}$  be a family of full rank  $r \times p$  constant matrices having a certain structure. Solve with respect to  $H \in \mathcal{H}$  the equation:

$$f_M(s, H) = \det(HM(s)) = f(s) \quad (2)$$

where  $f(s)$  is a real polynomial of an appropriate degree  $d$ . If, we denote the rows of  $H(s)$ , columns of  $M(s)$  respectively, then  $\underline{h}_i(s)^t, \underline{m}_i(s), i \in \tilde{r}$ , then:

$$C_r(H(s)) = \underline{h}_1(s)^t \wedge \dots \wedge \underline{h}_r(s)^t = \underline{h}(s)^t \wedge \in \mathbb{R}^{1 \times \sigma} \quad (3)$$

$$C_r(M(s)) = \underline{m}_1(s) \wedge \dots \wedge \underline{m}_r(s) = \underline{m}(s) \wedge \in \mathbb{R}^\sigma [s], \sigma = \binom{p}{r} \quad (4)$$

then by the Binet-Cauchy theorem [Marcus and Minc (1964)]:

$$\begin{aligned} f_M(s, H) &= C_r(H)C_r(M(s)) = \\ &= \langle h(s), m(s) \rangle = \sum_{\omega \in Q_{r,p}} h_\omega(s)m_\omega(s) \end{aligned} \quad (5)$$

where  $\langle \cdot, \cdot \rangle$  denotes inner product,  $\omega = (i_1, \dots, i_r) \in Q_{r,p}$ , and  $h_\omega(s), m_\omega(s)$  are the coordinates of  $\underline{h}(s) \wedge, \underline{m}(s) \wedge$  respectively. Note that  $h_\omega(s)$  is the  $r \times r$  minor of

$H(s)$ , which corresponds to the  $\omega$  set of columns of  $H(s)$  and thus  $h_\omega(s)$ , is a multilinear alternating function of the entries  $h_{ij}(s)$  of  $H(s)$ . The study of the zero structure of the multilinear function  $f_M(s, H)$  may thus be reduced to a linear subproblem and a standard multilinear algebra problem as it is shown below.

**(i) Linear subproblem of DAP:** Set  $\underline{m}(s) \wedge = \underline{p}(s) \in \mathbb{R}^\sigma [s]$ . Determine whether there exists a  $\underline{k}(s) \in \mathbb{R}^\sigma [s]$ ,  $\underline{k}(s) \neq \underline{0}$ , such that:

$$f_M(s, \underline{k}(s)) = \underline{k}(s)^t \underline{p}(s) = \sum k_i(s)p_i(s) = f(s) \in \mathbb{R}[s] \quad (6)$$

**(ii) Multilinear subproblem of DAP:** Assume that  $\mathcal{K}$  is the family of solution vectors  $\underline{k}(s)$  of (6). Determine whether there exists  $H(s)^t = [\underline{h}_1(s), \dots, \underline{h}_r(s)]$ , where  $H(s)^t \in \mathbb{R}^{p \times r} [s]$ , such that:

$$\underline{h}_1(s) \wedge \dots \wedge \underline{h}_r(s) = \underline{h}(s) \wedge = \underline{k}(s) \in \mathcal{K} \quad (7)$$

The polynomials  $f_M(s, \underline{k}(s))$  are generated by  $\underline{p}(s) = [p_1(s), \dots, p_i(s), \dots, p_\sigma(s)]^t \in \mathbb{R}^\sigma [s]$ , or as linear combinations of the set  $\mathcal{P} = \{p_i(s) \in \mathbb{R}[s], i \in \tilde{\sigma}\}$  and they will be referred to as *Dynamic Polynomial Combinants* (DPC). The study of the spectral properties of such polynomials has been considered in [Karcaniyas and Galanis (2010)]. We may define:

*Definition 1.* Given a set of polynomials  $\mathcal{P} = \{p_i(s) : p_i(s) \in \mathbb{R}[s]; i \in \tilde{m}\}$  and a family of polynomial sets  $\langle \mathcal{K} \rangle = \{\mathcal{K}_d, \forall d \in \mathbb{Z}^+ : \mathcal{K}_d = \{k_i(s) : k_i(s) \in \mathbb{R}[s]; i \in \tilde{m}, d = \max\{\deg(k_i(s))\}\}$ , we consider:

$$f(s, \mathcal{K}; \mathcal{P}) = \sum k_i(s)p_i(s) = \phi(s) \text{ where } k_i(s) \in \mathcal{K}_d \quad (8)$$

defined as  $d$  order dynamic-polynomial combinants of  $\mathcal{P}$  and  $p = \deg\{f(s, \mathcal{K}; \mathcal{P})\}$  is as its degree.

The representation problem of a given order and degree dynamic combinant is summarised here [Karcaniyas and Galanis (2010)] and this involves the parameterization of all sets  $\langle \mathcal{K} \rangle = \{\mathcal{K}_d, \forall d \in \mathbb{N} : \mathcal{K}_d = \{k_i(s) : k_i(s) \in \mathbb{R}[s]; i \in \tilde{m}, d = \max\{\deg(k_i(s))\}\}$  which lead to a polynomial combinant of a given degree  $p$ . We assume that the maximal degree polynomial in  $\mathcal{K}, k_1(s) \neq 0$  and such sets  $\mathcal{K}$  are call *proper*. If we define  $\mathcal{P}$  as:

$$\begin{aligned} \mathcal{P} &= \{p_i(s) \in [s], i \in \tilde{m}, n = \deg\{p_1(s)\} \geq \deg\{p_i(s)\}, \\ &\quad , i = 2, \dots, m, q = \max\{\deg\{p_i(s)\}\}, i = 2, \dots, m\} \end{aligned} \quad (9)$$

$$\begin{aligned} p_1(s) &= s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0, p_i(s) = \\ &= b_{i,q}s^q + \dots + b_{i,1}s + b_{i,0}, i = 2, \dots, m \end{aligned} \quad (10)$$

$$\underline{p}(s) = \begin{bmatrix} p_1(s) \\ p_2(s) \\ \vdots \\ p_m(s) \end{bmatrix} = [p_0, p_1, \dots, p_n] \begin{bmatrix} 1 \\ s \\ \vdots \\ s^n \end{bmatrix} = P \underline{e}_n(s) \quad (11)$$

Then the set  $\mathcal{P}$  will be referred to as an  $(m; n(q))$ -ordered set of  $\mathbb{R}[s]$ . Consider now the  $(m; d)$  set  $\mathcal{K} = \{k_i(s) \in [s], i \in \tilde{m}, \deg\{k_i(s)\} \leq d\}$  with the resulting  $d$ -order polynomial combinant of  $\mathcal{P}$ , defined as:

$$f_d(s, \mathcal{K}, \mathcal{P}) = \sum_{i=1}^m k_i(s)p_i(s) = \underline{k}^t(s)\underline{p}(s) \quad (12)$$

The matrix  $P \in \mathbb{R}^{m \times (n+1)}$  generates the representative  $\underline{p}(s) \in \mathbb{R}^m[s]$  of  $\mathcal{P}$  and it is the *basis matrix* of  $\mathcal{P}$ . Clearly:

$$-\infty \leq \partial[f_d(s, \mathcal{K}, \mathcal{P})] \leq n + q \quad (13)$$

*Generalised Resultant Representations of Dynamic Combinants*

For the general  $(m; d)$  set  $\mathcal{K}$  with a representative vector:

$$\begin{aligned} \underline{k}(s)^t &= \underline{k}_0 + s\underline{k}_1^t + \dots + s^d \underline{k}_d^t = [k_1(s), k_2(s), \dots, k_m(s)], \\ \underline{k}_i(s) &= k_{i,0} + k_{i,1}s + \dots + k_{i,d}s^d \end{aligned} \quad (14)$$

then  $f_d(s, \mathcal{K}, \mathcal{P})$  may be expressed as:

$$\begin{aligned} f_d(s, \mathcal{K}, \mathcal{P}) &= \sum_{i=1}^m [k_{i,d}, \dots, k_{i,1}, k_{i,0}] \begin{bmatrix} s^d p_i(s) \\ \vdots \\ s p_i(s) \\ p_i(s) \end{bmatrix} = \\ &= [\underline{k}_{1,d}, \dots, \underline{k}_{m,d}]^t \begin{bmatrix} \underline{p}_{1,d}(s) \\ \vdots \\ \underline{p}_{m,d}(s) \end{bmatrix} \end{aligned} \quad (15)$$

*Proposition 2.* Every dynamic combinant  $f_d(s, \mathcal{K}, \mathcal{P})$  defined by an  $(m; d)$  set  $\mathcal{K}$  is equivalent to a constant polynomial combinant defined by the  $(m(d+1); 0)$  set  $\mathcal{K}^0$  and generated by the  $(m(d+1); (n+d)(q+d))$  the  $d$ -th power of the  $(m; n(q))$  set  $\mathcal{P}$ , defined by:

$$\mathcal{P}^d = \{s^d p_1(s), \dots, s p_1(s), p_1(s); \dots, s^d p_m(s), \dots, s p_m(s), p_m(s)\} \quad (16)$$

If  $\mu = n + d$ ,  $\tilde{\underline{e}}_\mu(s)^t = [s^\mu, s^{\mu-1}, \dots, s, 1]$ , then  $\partial[\underline{p}_{1,d}(s)] = n + d$ ,  $\partial[\underline{p}_{i,d}(s)] \leq q + d$  for all  $i = 2, 3, \dots, m$ , then:

$$\begin{aligned} \underline{p}_{1,d}(s) &= \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & 0 & \dots & 0 \\ 0 & 1 & a_{n-1} & \dots & a_2 & a_1 & a_0 & \dots & 0 \\ \vdots & & & \ddots & & & & & \vdots \\ 0 & 0 & \dots & 1 & a_{n-1} & \dots & \dots & a_1 & a_0 \end{bmatrix} \tilde{\underline{e}}_\mu(s) = \\ &= S_{n,d}(p_1) \tilde{\underline{e}}_\mu(s), S_{n,d}(p_1) \in \mathbb{R}^{(d+1) \times (\mu+1)} \end{aligned} \quad (17)$$

and for  $i = 2, 3, \dots, m$

$$\begin{aligned} \underline{p}_{i,d}(s) &= \begin{bmatrix} 0 & \dots & 0 & b_{i,q} & \dots & b_{i,1} & b_{i,0} & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & 0 & b_{i,q} & \dots & b_{i,1} & b_{i,0} & 0 & \dots & 0 \\ \vdots & & & 0 & \vdots & \ddots & & & & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & b_{i,q} & \dots & b_{i,1} & b_{i,0} \end{bmatrix} \tilde{\underline{e}}_\mu(s) = \\ &= S_{n,d}(p_i) \tilde{\underline{e}}_\mu(s), S_{n,d}(p_i) \in \mathbb{R}^{(d+1) \times (\mu+1)} \end{aligned} \quad (18)$$

The set  $\mathcal{P}^d$  has then a basis matrix representation as shown below:

$$\underline{p}_d(s) = \begin{bmatrix} \underline{p}_{1,d}(s) \\ \underline{p}_{2,d}(s) \\ \vdots \\ \underline{p}_{m,d}(s) \end{bmatrix} = \begin{bmatrix} S_{n,d}(p_1) \\ S_{n,d}(p_2) \\ \vdots \\ S_{n,d}(p_m) \end{bmatrix} \tilde{\underline{e}}_\mu(s) = S_{\mathcal{P},d} \tilde{\underline{e}}_\mu(s) \quad (19)$$

where  $S_{\mathcal{P},d} \in \mathbb{R}^{m(d+1) \times (\mu+1)}$  which is the  $d$ -th *Generalised Resultant representation* of the set  $\mathcal{P}$  and  $S_{\mathcal{P},d}$  is the basis

matrix of the  $\mathcal{P}^d$  set. An alternative expression for the dynamic combinant is obtained using the basis matrix description of the set  $\mathcal{P}$  and this leads to an alternative equivalent representation, referred to [Karcianas and Galanis (2010)] as the Toeplitz representation.

### 3. FIXED DEGREE AND ORDER PARAMETERISATION OF $\mathcal{K}$ SETS AND GENERALISED RESULTANTS

The general representation of dynamic combinants considered before, based on the order may lead to combinants of varying degree. An alternative characterisation based on the fixed degree of  $f_d(s, \mathcal{K}, \mathcal{P})$  but with varying order  $\mathcal{K}$  provides an explicit parameterisation of the  $\mathcal{K}$  sets. The fixed degree parameterisation of combinants is summarised below [Karcianas and Galanis (2010)]:

*Theorem 3.* Given the  $(m; q(n))$  set  $\mathcal{P}$  and a general proper  $(m; d)$  set  $\mathcal{K}$ . Then,

- (i) For all proper  $(m; d)$  sets  $\mathcal{K}$ ,  $n \leq \partial[f_d(s, \mathcal{K}, \mathcal{P})] \leq n + d$
- (ii) If  $p \in \mathbb{N}_{>0}$ ,  $p \geq n$ , then the family  $\{\mathcal{K}_p\}$  for which  $\partial[f_d(s, \mathcal{K}, \mathcal{P})] = p$ , satisfies the conditions

$$\partial[k_1(s)] \leq p - n, \quad \partial[k_i(s)] \leq p - q, \quad i = 2, \dots, m$$

where at least one of the first two conditions holds as an equality.

- (iii) The  $p$  degree family  $\{\mathcal{K}_p\}$  contains  $n - q + 1$  subfamilies parameterised by an order  $d$ . The possible values for the order are:

$$d_1 = p - q > d_2 = p - q - 1 > \dots > d_{n-q+1} = p - n$$

$$\begin{aligned} \{\mathcal{K}_p^{d_1}\} &= \{k_i(s) : \partial[k_1(s)] \leq p - n, \partial[k_2(s)] = d_1 = \\ &= p - q, \partial[k_i(s)] \leq d_1, i = 3, \dots, m\} \end{aligned}$$

$$\begin{aligned} \{\mathcal{K}_p^{d_2}\} &= \{k_i(s) : \partial[k_1(s)] \leq p - n, \partial[k_2(s)] = d_2 = \\ &= p - q - 1, \partial[k_i(s)] \leq d_2, i = 3, \dots, m\} \end{aligned}$$

$$\begin{aligned} \{\mathcal{K}_p^{d_{n-q+1}}\} &= \{k_i(s) : \partial[k_1(s)] = \partial[k_2(s)] = \\ &= d_{n-q+1} = p - n, \partial[k_i(s)] \leq d_{n-q+1} = \\ &= p - n, i = 3, \dots, m\} \end{aligned}$$

The  $(m, n(q))$  set  $\mathcal{P}$  the degree of the proper combinants takes values  $p \geq n$  and the family of proper combinants of  $\mathcal{P}$  may thus be parameterised by degree and orders. The entire set of  $\mathcal{K}$  vectors is denoted as  $\langle \mathcal{K} \rangle$  and may be partitioned as:

$$\langle \mathcal{K} \rangle = \{\mathcal{K}_n\} \cup \{\mathcal{K}_{n+1}\} \cup \dots \cup \{\mathcal{K}_{n+q-1}\} \quad (20)$$

whereas each subset  $\{\mathcal{K}_p\}$  has the structure defined by the previous result. The above suggests that  $\{\mathcal{K}_n\}$  class acts as a generator of all other classes derived simply by adding the corresponding increase in the degree. For a class  $\{\mathcal{K}_p^d\}$  we shall denote by  $\langle \mathcal{K}_p^d \rangle$  the ordered set of degrees of the  $\{k_i(s), i \in \tilde{m}\}$  polynomials.

*Corollary 4.* Given an  $(m; q)$  set  $\mathcal{P}$  and a general  $(m; d)$  set  $\mathcal{K}$ , then:

- (i) The minimal degree family  $p = n$ ,  $\{\mathcal{K}\}_n$  is:

$$\begin{aligned} \{\mathcal{K}_n^0\} &: \langle \mathcal{K}_n^0 \rangle = (0, 0, \dots, 0); \\ \{\mathcal{K}_n^1\} &: \langle \mathcal{K}_n^1 \rangle = (0, 1, \dots, 1); \\ &\vdots \\ \{\mathcal{K}_n^{n-q}\} &: \langle \mathcal{K}_n^{n-q} \rangle = (0, n-q, \dots, n-q) \end{aligned}$$

(ii) The general degree family  $p = n + d$ ,  $\{\mathcal{K}_p\}$  is:

$$\begin{aligned} \{\mathcal{K}_p\} &= \{\mathcal{K}_p^d\} : \langle \mathcal{K}_p^d \rangle = (0, \dots, 0) + (d, d, \dots, d); \\ \{\mathcal{K}_p^{d+1}\} &: \langle \mathcal{K}_p^{d+1} \rangle = (0, 1, \dots, 1) + (d, d, \dots, d); \end{aligned}$$

$$\begin{aligned} &\vdots \\ \{\mathcal{K}_p^{d+n-q}\} &: \langle \mathcal{K}_p^{d+n-q} \rangle = \\ &= (0, n-q, \dots, n-q) + (d, d, \dots, d) \end{aligned}$$

(iii) For the general degree  $p$  family,  $p \geq n$ , the values of possible orders in decreasing order are:

$$\begin{aligned} d_1 &= p - q > d_2 = p - q - 1 > \dots \\ \dots &> d_{n-q} = p - n + 1 > d_{n-q+1} = p - n \end{aligned}$$

and they are  $d_i = p - q + 1 - i$ ,  $i = 1, 2, \dots, n - q + 1$ .

Amongst all  $(m; d)$  sets  $\mathcal{K}$ , the set defined by:

$$\begin{aligned} \{\mathcal{K}_{n+q-1}^{n-1}\} &= \{k_1(s) : \partial[k_1(s)] = q - 1, k_i(s) : \\ &: \partial[k_i(s)] = n - 1, i = 2, \dots, m\} \end{aligned} \quad (21)$$

plays a particular role in our study and it is referred to as the Sylvester set of  $\mathcal{P}$ . The general  $p$  degree family may be expressed as:

$$\begin{aligned} \{\mathcal{K}_p\} &= \{\mathcal{K}_p^{\tilde{d}_i}, \tilde{d}_i = p - n - 1 + i, i = 1, 2, \dots, n - q + 1\} = \\ &= \{\mathcal{K}_p^{p-n}; \mathcal{K}_p^{p-n+1}; \dots; \mathcal{K}_p^{p-q-1}; \mathcal{K}_p^{p-q}\} \end{aligned} \quad (22)$$

The element  $\mathcal{K}_p^{p-q}$  that corresponds to the highest order  $d_1 = p - q$  will be called the generator of the family and its degrees are:  $\langle \mathcal{K}_p^{p-q} \rangle = (p - n, p - q, \dots, p - q)$ .

Similarly, the element  $\mathcal{K}_p^{p-n}$  that corresponds to the lowest order  $d_{n-q+1} = p - n$  will be called the co-generator of the family and its degrees are:  $\langle \mathcal{K}_p^{p-n} \rangle = (p - n, p - n, \dots, p - n)$ .

The above suggests that the entire family of vector sets  $\langle \mathcal{K} \rangle$  may be expressed in "direct sum" form ( $\dot{\cup}$ ) as:

$$\begin{aligned} \langle \mathcal{K} \rangle &= \{\mathcal{K}_n\} \dot{\cup} \{\mathcal{K}_{n+1}\} \dot{\cup} \dots \dot{\cup} \{\mathcal{K}_{n+q-1}\} \dot{\cup} \dots \\ \{\mathcal{K}_p\} &= \{\mathcal{K}_p^{p-n}\} \dot{\cup} \{\mathcal{K}_p^{p-n+1}\} \dot{\cup} \dots \dot{\cup} \{\mathcal{K}_p^{p-q}\} \end{aligned} \quad (23)$$

#### 4. GENERALISED RESULTANTS BASED ON FIXED DEGREE AND ORDER PARAMETRISATIONS

The parameterisation of the sets  $\mathcal{K}$  based on degree and order induces a natural parameterisation of the corresponding Generalized Resultants. For the general  $(m; d)$  set  $\mathcal{K}$  that leads to combinants of degree  $p$  its structure is explicitly defined by:

$$\begin{aligned} \{\mathcal{K}_p^d\} &= \{k_1(s) : \partial[k_1(s)] = p - n = \tilde{d}, k_2(s) : \partial[k_2(s)] = d, \\ &, \tilde{d} \leq d \leq d^* = p - q, \dots, k_i(s) : \partial[k_i(s)] \leq d, i = 3, \dots, m\} \end{aligned} \quad (24)$$

The above set  $\{\mathcal{K}_p^d\}$ ,  $p \geq n$  and with  $d$  taking values as above, represents the general set generating dynamic combinants of a given degree  $d$  and order  $p$  and for all  $k_i(s)$ ,  $i = 3, \dots, m$ ,  $\partial[k_i(s)] \leq d$ .

*Proposition 5.* The dynamic combinant  $f_d(s, \mathcal{K}_d^d, \mathcal{P})$ , generated by the set  $\{\mathcal{K}_p^d\}$  is equivalent to a constant combinant of degree  $p$  that is generated by the polynomial set  $\mathcal{P}_p^d$ ,  $\tilde{d} = p - n$ ,  $\tilde{d} \leq d \leq p - q = d^*$ , where:

$$\begin{aligned} \mathcal{P}_p^d &= \{s^{\tilde{d}}p_1(s), \dots, sp_1(s), p_1(s); s^d p_2(s), \dots \\ &\dots, sp_2(s), p_2(s); \dots; s^d p_m(s), \dots, sp_m(s), p_m(s)\} \end{aligned} \quad (25)$$

The set  $\mathcal{P}_p^d$  is the  $(p, d)$ - power of  $\mathcal{P}$  and has degree  $p$  and its polynomial vector representative is:

$$\underline{p}_d(s) = \begin{bmatrix} \underline{p}_{1, \tilde{d}}(s) \\ \underline{p}_{2, d}(s) \\ \vdots \\ \underline{p}_{m, d}(s) \end{bmatrix} = \begin{bmatrix} S_{n, \tilde{d}}(p_1) \\ S_{q, d}(p_2) \\ \vdots \\ S_{q, d}(p_m) \end{bmatrix} \tilde{e}_p(s) = S_{p, d} \tilde{e}_p(s) \quad (26)$$

where the structure of the Toeplitz type blocks above  $S_{n, \tilde{d}}(p_1), S_{q, d}(p_i)$   $i = 2, \dots, m$  is given below:

*Proposition 6.* The Generalised Resultants corresponding to the parameterized set  $\{\mathcal{K}_p^d\}$  are defined by:

(i) Given that  $\underline{p}_{1, \tilde{d}}(s)$  has degree  $\tilde{d} + n = p - n + n = p$ , then:

$$\begin{aligned} S_{n, \tilde{d}}(p_1) &= \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 & 0 & \dots & 0 \\ 0 & 1 & a_{n-1} & \dots & a_2 & a_1 & a_0 & \dots & 0 \\ \vdots & & \ddots & & & & & & \vdots \\ 0 & 0 & \dots & 1 & a_{n-1} & \dots & \dots & a_1 & a_0 \end{bmatrix} \in \\ &\in \mathbb{R}^{(\tilde{d}+1) \times (p+1)} \end{aligned}$$

(ii) Given that  $\underline{p}_{i, d}(s)$  has degree  $d + q$  which satisfies the inequality  $p - (n - q) \leq d + q \leq p$  and thus  $d + q + 1 \leq p + 1$ , the structure of  $S_{q, d}(p_i)$  is defined for all  $i = 2, \dots, m$  and  $\forall d : p - n \leq d \leq p - q$  by:

$$\begin{aligned} S_{q, d}(p_i) &= \begin{bmatrix} 0 & \dots & 0 & b_{i, q} & \dots & b_{i, 1} & b_{i, 0} & 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & 0 & b_{i, q} & \dots & b_{i, 1} & b_{i, 0} & 0 & \dots & 0 \\ \vdots & & 0 & \vdots & \ddots & \ddots & & & & & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & b_{i, q} & \dots & b_{i, 1} & b_{i, 0} \end{bmatrix} \in \\ &\in \mathbb{R}^{(p-q-d) \times (p+1)} \end{aligned}$$

The boundary case  $d = p - q$ , has is no zero block and when  $d = p - n$ , then the zero block takes its maximal dimension  $n - q$ . The matrix  $S_{p, d}(\mathcal{P}) \in \mathbb{R}^{\sigma \times (p+1)}$ ,  $\sigma = p - n - d + m(d + 1)$  will be called the  $(p, d)$ - *Generalised Resultant* of the set  $\mathcal{P}$  where the possible values of  $d$  are:  $p - n \leq d \leq p - q$ . Clearly the  $S_{p, d}(\mathcal{P})$  matrix, denoted briefly by  $S_{p, d}$ , is the basis matrix of the  $(p, d)$  power of  $\mathcal{P}$ ,  $\mathcal{P}_p^d$ .

*Remark 7.* We can parameterise all dynamic combinants in terms of the degree  $p$  and the corresponding order  $d$  as:

- (a)  $p = n$  : then  $0 \leq d \leq n - q$
- (b)  $p = n + 1$  : then  $1 \leq d \leq n - q + 1$
- (c)  $p > n + 1$  : then  $p - n \leq d \leq p - q$

and their properties are defined by the properties of corresponding  $(p, d)$ - generalised resultants  $S_{p, d}(\mathcal{P})$ .

The properties of all dynamic combinants are described by the corresponding family of matrices:

$$S(\mathcal{P}) = \{S_{p,d}(\mathcal{P}) \mid p \geq n \text{ and } \forall d : p - n \leq d \leq p - q\} \quad (27)$$

which will be referred to as the family of *Generalised Resultants* of the set  $\mathcal{P}$ . The element of  $S(\mathcal{P})$  that corresponds to  $p=n+q-1$ ,  $d=n-1$  and thus  $\partial[k_1(s)] = p - n = q - 1$ .  $S_{n+q-1,n-1}(\mathcal{P})$  is denoted in short as  $\tilde{S}_{\mathcal{P}}$  and it is the Sylvester Resultant of the set  $\mathcal{P}$ :

$$\tilde{S}_{\mathcal{P}} = \begin{bmatrix} S_{n,q-1}(p_1) \\ S_{q,n-1}(p_2) \\ \vdots \\ S_{q,n-1}(p_m) \end{bmatrix} \in \mathbb{R}^{\tau \times (n+q)}, \quad \tau = [q + (m-1)n] \quad (28)$$

where  $S_{n,q-1}(p_1) \in \mathbb{R}^{q \times (n+q)}$ ,  $S_{q,n-1}(p_i) \in \mathbb{R}^{n \times (n+q)}$ ,  $j = 2, \dots, m$  and  $\tau = [q + (m-1)n]$ .

## 5. SPECTRUM ASSIGNMENT OF DYNAMIC COMBINANTS AND THE SYLVESTER RESULTANT

We now consider the problem of arbitrary assignment of the spectrum of dynamic combinants for some appropriate order and degree. This is part of the more problem dealing with the parameterisation of all possible degree and order combinants for which assignment may be achieved. The results in this section follow from the equivalence of dynamic combinants to constant combinants, which imply reduction of the problem to a linear matrix equation. We may summarise the results from [Karcianas and Galanis (2010)] below:

*Lemma 8.* [Barnett (1970a)], [Fatouros and Karcianas (2003)]. Let  $\mathcal{P}$  be an  $(m,n(q))$  set with Sylvester Resultant  $\tilde{S}_{\mathcal{P}}$ . The set  $\mathcal{P}$  is coprime, if and only if  $\tilde{S}_{\mathcal{P}}$  has full rank.

*Theorem 9.* Let  $\mathcal{P}$  be an  $(m,n(q))$  set. There exists a  $d$  such that  $f_d(s, \mathcal{K}, \mathcal{P})$  is completely assignable, if and only if the set  $\mathcal{P}$  is coprime.

*Corollary 10.* For the  $(m,n(q))$  coprime set  $\mathcal{P}$  the following properties hold true:

- (i) There exists a combinant  $\tilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P})$  of degree  $p = n + q - 1$  and order  $d = n - 1$  which is completely assignable.
- (ii) All combinants  $\tilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P})$  of order  $d = n - 1$  and degree  $p : n + q - 1 \leq p \leq 2n - 1$  are also completely assignable.
- (iii) (iii) All combinants  $\tilde{f}_p(s, \mathcal{K}, \mathcal{P})$  of degree  $p > p_s = n + q - 1$  have an assignable element by selection of some appropriate order  $p - n \leq d \leq p - q$ .

The special combinant of order  $d=n-1$  and degree  $p = n + q - 1$  is the Sylvester combinant of the set  $\mathcal{P}$ , it is denoted by:  $\tilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P}) = \sum_{i=1}^m k_i(s)p_i(s) \partial[k_1(s)] = q-1$ , and for  $i = 2, \dots, m$ ,  $\partial[k_i(s)] = n - 1$ , and the zero assignment problem is expressed as making  $\tilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P})$  an arbitrary polynomial  $\alpha(s)$  of degree  $n + q - 1$ , i.e.  $\alpha(s) = \underline{\alpha}^t \tilde{\underline{e}}_{n+q-1}(s)$ , which is equivalent to:

$$\begin{bmatrix} \tilde{k}_1^t & \tilde{k}_2^t & \dots & \tilde{k}_m^t \end{bmatrix} \begin{bmatrix} S_{n,q-1}(p_1) \\ S_{q,n-1}(p_2) \\ \vdots \\ S_{q,n-1}(p_m) \end{bmatrix} = \underline{\alpha}^t \text{ or } \tilde{\underline{k}}^t \tilde{S}_{\mathcal{P}} = \underline{\alpha}^t \quad (29)$$

*Remark 11.* Under coprimeness assumption the above equation has always a solution and the number of degrees of freedom is  $\rho_s = mn + 1 - 2n$ . For the case  $m = 2$  the assignment problem has a unique solution.

It is clear that two combinants of the same order  $d = n - 1$  and different degrees may be both assignable. In fact, under the coprimeness assumption, both combinants  $\tilde{f}_{n-1}(s, \mathcal{K}, \mathcal{P})$ ,  $f_{n-1}(s, \mathcal{K}, \mathcal{P})$  of degrees respectively  $n+q-1$  and  $2n - 1$  are assignable. This raises the questions of investigating the assignability of all combinants  $f_d(s, \mathcal{K}, \mathcal{P})$  with  $d < n-1$  and parameterize all combinants  $\tilde{f}_d(s, \mathcal{K}, \mathcal{P})$  of order  $d$ ,  $d \leq n - 1$  and degree  $p \leq n + q - 1$  which are assignable. The families with degree  $p > p_s$  will be called non-proper.

The family of all resultants of degree less or equal to  $p_s$  is referred to as proper subset and can be defined as:

$$S_{pr}(\mathcal{P}) = \{S_{p,d}(\mathcal{P}) : n \leq p \leq n + q - 1 = p_s, \quad d = p - q - \rho, \quad \rho = 0, 1, \dots, n - q\} \quad (30)$$

This family is partitioned by the degrees and orders as:

*Proposition 12.* The family of proper generalised resultants of the  $(m,n(q))$  set  $\mathcal{P}$  is partitioned into  $q - 1$  sets as:

$$S_{pr}(\mathcal{P}) = \{S_{p_s}\} \dot{\cup} \{S_{p_{s-1}}\} \dot{\cup} \dots \dot{\cup} \{S_n\} \quad (31)$$

where  $p_s = n + q - 1$  and each subset of a fixed degree is also partitioned by the corresponding order has  $n - q + 1$  elements.

## 6. CONSTRUCTION OF THE FAMILY OF THE PROPER SYLVESTER RESULTANTS

The construction of the generalised resultants together with the parameterisation of the  $\mathcal{K}$  sets leads to:

*Proposition 13.* The proper combinant of the  $(m,n(q))$  set  $\mathcal{P}$  that has  $p_s = n + q - 1$  degree and order  $d = n - 1 - \rho$ ,  $\rho = 1, 2, \dots, n - q$  is defined by the generalised resultant  $S_{p_s, n-1-\rho}$  defined as in (16), (17) and (18) which is also expressed as:

$$S_{p_s, n-1-\rho} = \begin{bmatrix} S_{n,q-1}(p_1) \\ 0_\rho \quad \vdots \quad S_{q,n-1-\rho}(p_2) \\ \vdots \\ 0_\rho \quad \dots \quad S_{q,n-1-\rho}(p_m) \end{bmatrix} \quad (32)$$

where  $S_{n,q-1}(p_1)$ ,  $S_{q,n-1-\rho}(p_i)$ ,  $i = 2, \dots, m$  are the standard Sylvester blocks. Furthermore, any two successive combinants of degree  $p_s$  and order  $d = n - 1 - \rho$  and  $d' = n - \rho - 2$  are related as:

$$S_{p_s, n-1-\rho} = \begin{bmatrix} S_{n,q-1}(p_1) \\ 0_\rho \quad \vdots \quad S_{q,n-1-\rho}(p_2) \\ \vdots \\ 0_\rho \quad \dots \quad S_{q,n-1-\rho}(p_m) \end{bmatrix} \cong \begin{bmatrix} x \dots x \\ \vdots \\ x \dots x \\ S_{p_s, n-\rho-2} \end{bmatrix} \quad (33)$$

where  $\cong$  denotes row equivalence (permutations).

*Corollary 14.* If  $S_{p_s, n-\rho-1}$ ,  $S_{p_s, n-\rho-2}$  are two generalised Sylvester matrices corresponding to combinants of degree  $p_s$  and orders  $d = n - \rho - 1$ ,  $d' = n - \rho - 2$  respectively, then:

$$\text{rank}(S_{p_s, n-\rho-1}) \geq \text{rank}(S_{p_s, n-\rho-2}) \quad (34)$$

Furthermore, if  $S_{p_s, n-\rho-1}$  has full rank then all higher order generalised resultants are also full rank.

The above result describes rank properties of generalised resultants that have the same degree and different orders. The investigation of links between generalised resultants of different degree is considered next. In the following we will use the notation  $S_{q, n-1-\rho}^\rho(p_i) = \begin{bmatrix} 0_\rho \\ \vdots \\ S_{q, n-1-\rho}(p_i) \end{bmatrix}$ .

With this notation for  $p_s$  and  $p_s - 1$  degrees we have:

$$S_{p_s, n-\rho-1} = \begin{bmatrix} S_{n, q-1}(p_1) \\ S_{q, n-\rho-2}^\rho(p_2) \\ \vdots \\ S_{q, n-\rho-2}^\rho(p_m) \end{bmatrix} \quad (35)$$

where  $d = n-1-\rho$ ,  $q-1 \leq d \leq n-1$ ,  $\rho = 0, 1, 2, \dots, n-q$ . For the  $p_s-1$  degree with  $q-2 \leq d' \leq n-2$ ,  $d' = n-2-\rho'$ ,  $\rho' = 0, 1, 2, \dots, n-q$  we have:

$$S_{p_s-1, n-1-\rho'} = \begin{bmatrix} S_{n, q-2}(p_1) \\ S_{q, n-2-\rho'}^{\rho'}(p_2) \\ \vdots \\ S_{q, n-2-\rho'}^{\rho'}(p_m) \end{bmatrix} \quad (36)$$

*Remark 15.* The definition of Generalised Resultants readily establishes the following relationship:

$$S_{p_s, n-1} = \begin{bmatrix} S_{n, q-1}(p_1) \\ S_{q, n-1}(p_2) \\ \vdots \\ S_{q, n-1}(p_m) \end{bmatrix} = \begin{bmatrix} 1 & x \dots x \\ 0 & S_{n, q-2}(p_1) \\ x & x \dots x \\ 0 & S_{n, q-2}(p_2) \\ \vdots & \vdots \\ x & x \dots x \\ 0 & S_{n, q-2}(p_m) \end{bmatrix} \cong \begin{bmatrix} 0 & X \\ 1 & x \dots x \\ 0 & S_{p_s-1, n-2} \end{bmatrix} \quad (37)$$

The above clearly leads to the following result:

*Proposition 16.* The maximal order generalised resultants  $S_{p_s, n-1}$  and  $S_{p_s-1, n-2}$ ,  $S_{p_s-2, n-3}$  etc, of degrees  $p_s, p_s-1, p_s-2, \dots$  etc they are related as:

$$S_{p_s, n-1} \cong \begin{bmatrix} 1 & x \dots x \\ 0 & X \\ 0 & S_{p_s-1, n-2} \end{bmatrix} \cong \begin{bmatrix} I_1 & X \\ 0 & X \\ 0 & S_{p_s-2, n-3} \end{bmatrix} \cong \dots \\ \dots \cong \begin{bmatrix} I_\mu & X \\ 0 & X \\ 0 & S_{p_s-\mu, n-\mu-1} \end{bmatrix}, \quad \mu = 0, 1, \dots, q-1 \quad (38)$$

and thus:

$$\text{rank}(S_{p_s, n-1}) \geq 1 + \text{rank}(S_{p_s-1, n-2}) \geq 2 + \\ + \text{rank}(S_{p_s-2, n-3}) \geq \dots \geq q-1 + \text{rank}(S_{n, n-q}) \quad (39)$$

The above result establishes an important rank property for the generators of each of the given degree  $p_s - \mu$  classes which has implications for the determination of the least degree solution. The analysis so far indicates a systematic process for construction of the family of generalised resultants.

## Construction of a Family of Generalised Resultants

Given the  $(m, n(q))$  set  $\mathcal{P}$ , which is assumed to be coprime, we construct the Sylvester resultant that corresponds to  $p_s = n + q - 1$  degree combinant and has order  $d = n - 1$ . If  $S_{p_s, n-1}$  is the Sylvester resultant, then the family of proper generalised resultants is defined from  $S_{p_s, n-1}$  by transformations on this matrix. Thus, if:

$$S_{p_s, n-1} = \begin{bmatrix} S_{n, q-1}(p_1) \\ S_{q, n-1}(p_2) \\ \vdots \\ S_{q, n-1}(p_m) \end{bmatrix} \quad (40)$$

is the Sylvester Resultant, then the construction of the different degree and order families is described below:

**(a) The construction of  $\{S_{p_s}\}$  family:** This family has degree  $p_s = n + q - 1$  and has  $n-q+1$  generalised resultants of respective order  $d = n - \rho - 1$ ,  $\rho = 0, 1, \dots, n-q$ , where for  $\rho = 0$  we have  $S_{p_s, n-1}$  as the generator of the family. The element  $S_{p_s, n-\rho-1}$  of  $\{S_{p_s}\}$  is constructed from  $S_{p_s, n-1}$  by keeping the first block  $S_{n, q-1}(p_1)$  and then eliminating the first  $\rho$  rows from each of the blocks  $S_{q, n-1}(p_i)$ ,  $i = 2, \dots, m$ . This leads to the construction of:

$$S_{p_s, n-1-\rho} = \begin{bmatrix} S_{n, q-1}(p_1) \\ 0_\rho \vdots S_{q, n-1-\rho}(p_2) \\ \vdots \\ 0_\rho \vdots S_{q, n-1-\rho}(p_m) \end{bmatrix} = \begin{bmatrix} S_{n, q-1}(p_1) \\ S_{q, n-1-\rho}^\rho(p_2) \\ \vdots \\ S_{q, n-1-\rho}^\rho(p_m) \end{bmatrix} \quad (41)$$

The above family is denoted by  $\langle S_{p_s} \rangle = \{S_{p_s, n-1-\rho}, \rho = 0, 1, \dots, n-q\}$  and for  $\rho = 0$  we have the generator of the family, the Sylvester Resultant  $S_{p_s, n-1}$ .

**(b) The construction of  $\{S_{p_s-1}\}$  family:** This family has degree  $p'_s = n + q - 2$  and has  $n-q+1$  generalised resultants of respective order  $d' = n - 2 - \rho$ ,  $\rho = 0, 1, \dots, n-q$ , where for  $\rho = 0$  we have  $S_{p'_s, n-2}$  as the generator of the family which is constructed from as follows:

**The generator of  $\{S_{p_s-1}\}$  family:** Eliminate the first row for each of the  $S_{n, q-1}(p_1)$ ,  $S_{q, n-1}(p_i)$ ,  $i = 2, \dots, m$  blocks that results in matrix blocks  $[0, S_{n, q-2}(p_1)]$ ,  $[0, S_{q, n-2}(p_i)]$ ,  $i = 2, 3, \dots, m$ . The generator of the  $p'_s = n + q - 2$  family has order  $d' = n - 2$  and it is defined from these blocks by eliminating the first zero columns. This leads to:

$$S_{p'_s, n-2} = \begin{bmatrix} S_{n, q-2}(p_1) \\ S_{q, n-2}(p_2) \\ \vdots \\ S_{q, n-2}(p_m) \end{bmatrix} \quad (42)$$

Having defined the generator  $S_{p'_s, n-2}$  of dimension  $\tau' \times (n + q - 1)$  where  $\tau' = \tau - m = q + (m - 1)n - m$  we can proceed with the construction of the rest of the elements of the  $p'_s$  family by following a similar process as before i.e..

**The general element of the  $\{S_{p_s-1}\}$  family:** The general element  $S_{p'_s, n-2-\rho}$ ,  $\rho = 1, 2, \dots, n-q$  is constructed from the generator  $S_{p'_s, n-2}$  by keeping the first block  $S_{n, q-2}(p_1)$  and by eliminating the first  $\rho$  rows from each of the  $S_{q, n-2}(p_i)$ ,  $i = 2, \dots, m$  blocks. This leads to

the construction of:

$$S_{p_s', n-2-\rho} = \begin{bmatrix} S_{n, q-2}(p_1) \\ 0_\rho \vdots S_{q, n-2-\rho}(p_2) \\ \vdots \\ 0_\rho \vdots S_{q, n-2-\rho}(p_m) \end{bmatrix} = \begin{bmatrix} S_{n, q-2}(p_1) \\ S_{q, n-2-\rho}^\rho(p_2) \\ \vdots \\ S_{q, n-2-\rho}^\rho(p_m) \end{bmatrix} \quad (43)$$

The above family is denoted by:  $\langle \mathcal{S}_{p_s-1} \rangle = \{S_{p_s', n-2-\rho}, \rho = 0, 1, 2, \dots, n-q, d' = n-2-\rho\}$  where for  $\rho = 0$  we have the generator of the family.

**(c) The construction of  $\{\mathcal{S}_{p_s-\mu}\}$  family:** The family with degree  $p_s^\mu = n+q-1-\mu$ ,  $\mu = 0, 1, \dots, q-1$  follows a similar construction process that involves the construction of the generator  $S_{p_s^\mu, n-\mu-1}$  and then the elements of the family by deleting the first  $\rho$  rows  $\rho = 0, 1, \dots, n-q$  form each of the  $i = 2, \dots, m$  blocks. The resulting family  $\{\mathcal{S}_{p_s-\mu}\}$  has again  $n-q+1$  elements.

The above provides a systematic procedure for defining the elements of the partitioning of the proper family of the resultants of  $\mathcal{P}$ .

## 7. THE SEARCH FOR THE MINIMAL DEGREE AND ORDER SOLUTION

The results on the rank properties of the generalised resultants provide the basis for the development of a procedure that may lead to determining the least degree and order solution of the spectral assignment problem.

**Problems:** For an  $(m, n(q))$  coprime set  $\mathcal{P}$  with Sylvester degree  $p_s = n+q-1$  and generators for the different degree families  $\{S_{p_s, n-1}; S_{p_s-1, n-2}; \dots; S_{p_s-\mu, n-1-\mu}\}$ ,  $\mu = 0, 1, \dots, n-q$  define:

- The least value of  $\mu$ , say  $\mu^*$  such that the  $S_{p_s-\mu^*, n-\mu^*-1} \in \mathbb{R}^{\tau \times (p_s-\mu^*+1)}$  has  $\tilde{\tau} \geq p_s - \mu^* + 1$ .
- Having defined the value of such a  $\mu^*$  consider the  $\{\mathcal{S}_{p_s-\mu^*}\}$  and define the least order element  $S_{p_s-\mu^*, n-\mu^*-1-\rho^*} \in \mathbb{R}^{\tau' \times (p_s-\mu^*+1)}$  for which  $\tau' \geq p_s - \mu^* + 1$ .

The resulting values for  $\mu^*, \rho^*$  define the boundaries for the searching process and are considered next. We first note that the partition:

$$\mathcal{S}_{p_r}(\mathcal{P}) = \{\mathcal{S}_{p_s}\} \dot{\cup} \{\mathcal{S}_{p_s-1}\} \dot{\cup} \dots \dot{\cup} \{\mathcal{S}_{p_n}\} \quad (44)$$

has  $\nu+1 = q$  elements since  $p_s - \nu = n$ . Each of the  $\{\mathcal{S}_{p_s-\mu}\}$ ,  $\mu = 0, 1, \dots, q-1$  families has a generator  $S_{p_s-\mu, n-1-\mu} \in \mathbb{R}^{[q+(m-1)n-m\mu] \times [n+q-1-\mu]}$ . These relations readily lead to:

*Proposition 17.* The least degree generator  $S_{p_s-\mu, n-1-\mu} \in \mathbb{R}^{\tau' \times \rho'}$  for which  $\tau \geq \rho'$ :

$$\mu^* = \min\left\{\frac{1+mn-2n}{m-1}, q-1\right\} \quad (45)$$

*Remark 18.* The suggested computation of  $\mu^*$  above indicates that none of the classes  $\{\mathcal{S}_{p_s-\mu}\}$  with  $\mu \geq \mu^*$  contain an element that is assignable and thus assignment has to be investigated only for the classes:

$$\{\mathcal{S}_{p_s}\}, \{\mathcal{S}_{p_s-1}\}, \dots, \{\mathcal{S}_{p_s-\mu^*}\} \quad (46)$$

Given that  $\{\mathcal{S}_{p_s-\mu^*}\}$  contains  $n-q+1$  elements, it is worth finding the element  $\{S_{p_s-\mu^*, n-1-\mu^*-\rho}\} \in \mathbb{R}^{\tau'' \times \rho''}$  for which  $\tau'' \geq \rho''$ . This is established next.

*Proposition 19.* The least degree and order generalised resultant  $\{S_{p_s-\mu^*, n-1-\mu^*-\rho^*}\} \in \mathbb{R}^{\tau'' \times \rho''}$  for which  $\tau'' \geq \rho''$  is defined by:

$$\mu^* = \min\left\{\frac{1+mn-2n}{m-1}, q-1\right\} \quad (47)$$

$$\rho^* = \min\left\{n-q+1, \frac{m(n-\mu^*)-2n+1+\mu^*}{m-1}\right\} \quad (48)$$

The above results provide a lower bound for the degrees and the order for a combinant to be assignable.

*Theorem 20.* For the coprime  $(m, n(q))$  set  $\mathcal{P}$  with Sylvester degree  $p_s = n+q-1$  the least degree combinant  $p^*$  that may be assignable and the least order  $d^*$  with the assignability property are:

$$p^* = n+q-1-\mu^*, \quad d^* = n-1-\mu^*-\rho^* \quad (49)$$

where  $\mu^*$  and  $\rho^*$  are defined by (47),(48) respectively.

Clearly,  $p^*, d^*$  define lower bounds and thus specify the values where the test of the rank properties makes sense. In principle we expect the minimal values of degree and order,  $\tilde{p}, \tilde{d}$  to be higher than the corresponding  $p^*, d^*$ . The above provide the basis for the development of the searching process considered next. We first state the following result:

*Proposition 21.* Let  $\{\mathcal{S}_{p_s-\mu}\}$ ,  $\mu = 0, 1, \dots, q-1$  be the family with degree  $p_s - \mu$ . If the generator  $S_{p_s-\mu, n-1-\mu}$  is rank deficient, then all elements of the family  $\{\mathcal{S}_{p_s-\mu}\}$  are rank deficient.

*Remark 22.* The search for the least degree and least order solution is restricted only to those families with full rank generators.

We may now state the main result:

*Theorem 23.* Consider the  $(m, n(q))$  coprime set  $\mathcal{P}$  with Sylvester degree  $p_s = n+q-1$ . The following properties hold true:

- The least degree assignable combinant  $\tilde{p} = p_s - \tilde{\nu}$  is defined by the maximal index for  $\tilde{\nu}$  for which

$$0 \leq \tilde{\nu} \leq \mu^* = \min\left\{\frac{1+mn-2n}{m-1}, q-1\right\} \quad (50)$$

where  $\tilde{\nu}$  is the maximal index for which the generator  $S_{p_s-\tilde{\nu}, n-1-\tilde{\nu}}$  has full rank.

- The least order assignable combinant corresponds to the least degree  $\tilde{p} = p_s - \tilde{\nu}$  and to the least order  $\tilde{d} = n-1-\tilde{\nu}-\tilde{\rho}$ , where  $\tilde{\rho}$  is the maximal index for which:

$$0 \leq \tilde{\rho} \leq \rho^* = \min\left\{n-q+1, \frac{m(n-\mu^*)-2n+1+\mu^*}{m-1}\right\} \quad (51)$$

and  $S_{p_s-\tilde{\nu}, n-1-\tilde{\nu}-\tilde{\rho}}$  has full rank.

The results so far lead to the following algorithm for computing the least degree and least order solution.

**Procedure for Determining the Least Degree and Order Solutions.** For some coprime  $(m, n(q))$  set  $\mathcal{P}$  with



Sylvester degree  $p_s = n+q-1$  determining the least degree and least order solutions involves the following steps:

**Step(1):** Compute the numbers  $\mu^*$  and  $\rho^*$  by:

$$\mu^* = \min\left\{\frac{1+mn-2n}{m-1}, q-1\right\}$$

$$\rho^* = \min\left\{n-q+1, \frac{m(n-\mu^*)-2n+1+\mu^*}{m-1}\right\}$$

which define the lower bounds for the assignable degree and order:

$$p^* = p_s - \mu^* = n+q-1-\mu^*, \quad d^* = n-1-\mu^*-\rho^* \quad (52)$$

**Step(2):** Define the generators of the proper family  $\mathcal{S}_{p_r}(\mathcal{P}) = \{\mathcal{S}_{p_s}\} \cup \dots \cup \{\mathcal{S}_{p_s-\nu}\} \cup \dots$  for  $\nu \leq \mu^*$  in reverse order i.e.  $\mathcal{S}_{p_s-\nu, n-1-\nu}, \nu = \mu^*, \mu^*-1, \dots, 0$ . Then, test successively the ranks of  $\mathcal{S}_{p_s-\mu^*+j, \nu-1-\mu^*+j}, j = 0, 1, \dots, \mu^*$  and determine the least index  $j = \alpha$  for which the matrix generator  $\mathcal{S}_{p_s-\mu^*+\alpha, n-1-\mu^*+\alpha}$  has full rank. Then the least assignable degree is:

$$\tilde{p} = p_s - \mu^* + \alpha = p_s - \tilde{\nu} \quad (53)$$

**Step(3):** Having defined the least degree assignable generator  $\mathcal{S}_{p_s-\tilde{\nu}, n-1-\tilde{\nu}}$  we consider the corresponding class  $\{\mathcal{S}_{p_s-\tilde{\nu}}\}$  and for  $\rho \leq \rho^*$  we list its elements in reverse order:  $\mathcal{S}_{p_s-\tilde{\nu}, n-1-\tilde{\nu}-\rho}, \rho = \rho^*, \rho^*-1, \dots, 0$ . Then, we test successively the ranks of  $\mathcal{S}_{p_s-\tilde{\nu}, n-1-\tilde{\nu}-\rho^*+i}, i = 0, 1, \dots, \rho^*$  and determine the least index  $i = \beta$  for which the corresponding Generalised Resultant  $\mathcal{S}_{p_s-\tilde{\nu}, n-1-\tilde{\nu}-\rho^*+\beta}$  has full rank. Then, the least assignable order for the least degree  $\tilde{p}$  is defined by:

$$\tilde{d} = n-1-\tilde{\nu}-\rho^*+\beta = n-1-\tilde{\nu}-\tilde{\rho} \quad (54)$$

The above process involves a small number of rank tests starting from smaller order generalised resultants and leads to the minimal degree  $\tilde{p}$  and least order  $\tilde{d}$  in a finite number of steps.

## 8. CONCLUSIONS

The fundamentals of the theory of dynamic polynomial combinants have been reviewed and their representation in terms of Generalized Resultants has been established. The conditions for existence of spectrum assignable combinants have been established and these are equivalent to the coprimeness of the generating set  $\mathcal{P}$ . The parameterization of combinants in terms of order and degree has been shown to be central in the study of their properties and this lays the foundations for investigating the properties of the family of Generalised Resultants. Amongst the key problems in this area is the minimal design problem dealing with finding the least order and degree for which spectrum assignability may be guaranteed. Conditions for the characterisation of the minimal order and degree combinant for which arbitrary assignment is possible have been derived and a simple algorithm that produces such solutions in few steps is given. The current framework allows the further development of the theory of dynamic combinants that may answer questions related to the zero distribution of dynamic combinants, in the cases where complete assignability (due to order and degree) is not possible. The computation of the GCD of polynomials [Karcianas (1987)] [Mitrouli and Karcianas (1993)] is an

essential part from such investigations and leads to non-assignability. The study of non-assignable combinants is also important and it is linked to the existence of an "approximate GCD" [Karcianas et al. (2006)].

## REFERENCES

- Barnett, S. (1970a). Greatest common divisor of several polynomials. *Proc. Camb. Phil. Soc.*, 70, 263–268.
- Barnett, S. (1970b). Greatest common divisor of two polynomials. *Linear Algebra Appl.*, 3, 7–9.
- Barnett, S. (1983). *Polynomials and Linear Control Systems*. Marcel Dekker, Inc.
- Brockett, R.W. and Byrnes, C.I. (1980). Multivariable nyquist criterion, root loci and pole placement: Geometric viewpoint. *IEEE Trans. Aut. Control*, AC-26, 271–283.
- Fatouros, S. and Karcianas, N. (2003). Resultant properties of gcd of many polynomials and a factorisation representation of gcd. *International Journal of Control*, 76, 1666–1683.
- Hodge, W. and Pedoe, P. (1952). *Methods of algebraic geometry*. Cambridge University Press, 2.
- Kailath, T. (1980). *Linear Systems*. Englewood Cliffs, New Jersey, Prentice Hall.
- Karcianas, N. (1987). Invariance properties, and characterization of the greatest common divisor of a set of polynomials. *International Journal of Control*, 46(5), 1751–1760.
- Karcianas, N. and Galanis, G. (2010). Dynamic polynomial combinants and generalised resultants. *Systems & Control Centre, Research Report, City University London*.
- Karcianas, N. and Giannakopoulos, C. (1984). Grassmann invariants, almost zeros and the determinantal pole-zero assignment problems of linear systems. *International Journal of Control*, 673–698.
- Karcianas, N., Giannakopoulos, C., and Hubbard, M. (1983). Almost zeros of a set of polynomials of  $\mathbb{R}[s]$ . *International Journal of Control*, 38(6), 1213–1238.
- Karcianas, N., Mitrouli, M., Fatouros, S., and Halikias, G. (2006). Approximate gcd of many polynomials, generalised resultants and strength of approximation. *Computers and Mathematics with Applications*, 51, 1817–1830.
- Kucera, V. (1979). *Discrete Linear Control: The Polynomial Equation Approach*. John Wiley & Sons, New York.
- Leventides, J. and Karcianas, N. (1998). Dynamic pole assignment using global, blow up linearisation: Low complexity solutions. *Journal of Optimisation Theory and Applications*, 96, 57–86.
- Marcus, M. and Minc, M. (1964). *A survey of matrix theory and matrix inequalities*. Allyn and Bacon, Boston.
- Mitrouli, M. and Karcianas, N. (1993). Computation of the gcd of polynomials using gaussian transformations and shifting. *Int. J. Control*, 58, 211–228.
- Vardulakis, A.I.G. and Stoyly, P.N.R. (1978). Generalised resultant theorem. *Journ. of IMA*, 22, 331–335.