On stable equivalences with endopermutation source

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Abstract

We show that a bimodule between block algebras which has a fusion stable endopermutation module as a source and which induces Morita equivalences between centralisers of nontrivial subgroups of a defect group induces a stable equivalence of Morita type; this is a converse to a theorem of Puig. The special case where the source is trivial has long been known by many authors. The earliest instance for a result deducing a stable equivalence of Morita type from local Morita equivalences with possibly nontrivial endopermutation source is due to Puig, in the context of blocks with abelian defect groups with a Frobenius inertial quotient. The present note is motivated by an application, due to Biland, to blocks of finite groups with structural properties known to hold for hypothetical minimal counterexamples to the Z_p^*-Theorem.

1 Introduction

Let p be a prime and $\mathcal{O}$ a complete discrete valuation ring having a residue field k of characteristic p; we allow the case $\mathcal{O} = k$. We will assume that k is a splitting field for all block algebras which arise in this note. Following Broué [9, §5.A], given two $\mathcal{O}$-algebras A, B, an A-B-bimodule M and a B-A-bimodule N, we say that *M and N induce a stable equivalence of Morita type between A and B* if M, N are finitely generated projective as left and right modules, and if $M \otimes_B N \cong A \oplus W$ for some projective $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$-module W and $N \otimes_A M \cong B \oplus W'$ for some projective $B \otimes_{\mathcal{O}} B^{\mathrm{op}}$-module W'. By a result of Puig in [27, 7.7.4] a stable equivalence of Morita type between block algebras of finite groups given by a bimodule with endopermutation source and its dual implies that there is a canonical identification of the defect groups of the two blocks such that both have the same local strucure and such that corresponding blocks of centralisers of nontrivial subgroups of that common defect group are Morita equivalent via bimodules with endopermutation sources. The following theorem is a converse to this result. The terminology and required background information for this statement are collected in the next two sections, together with further references.

Theorem 1.1. *Let A, B be almost source algebras of blocks of finite group algebras over $\mathcal{O}$ having a common defect group P and the same fusion system $\mathcal{F}$ on P. Let V be an $\mathcal{F}$-stable indecomposable endopermutation $\mathcal{O}P$-module with vertex P, viewed as an $\mathcal{O}\Delta P$-module through the canonical isomorphism $\Delta P \cong P$. Let M be an indecomposable direct summand of the A-B-bimodule*

$$A \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} B\ .$$

Suppose that $(M \otimes_B M^*)(\Delta P) \neq \{0\}$. Then for any nontrivial fully $\mathcal{F}$-centralised subgroup Q of P, there is a canonical $A(\Delta Q)$-$B(\Delta Q)$-bimodule M_Q satisfying $\mathrm{End}_k(M_Q) \cong (\mathrm{End}_{\mathcal{O}}(M))(\Delta Q)$. Moreover, if for all nontrivial fully $\mathcal{F}$-centralised subgroups Q of P the bimodule M_Q induces a Morita equivalence between $A(\Delta Q)$ and $B(\Delta Q)$, then M and its dual M^* induce a stable equivalence of Morita type between A and B.

For V the trivial $\mathcal{O}P$-module, variations of the above result have been noted by many authors. For principal blocks this was first pointed out by Alperin. A version for finite groups with the same local structure appears in Broué [9, 6.3], and the above theorem with V trivial is equivalent to [17, Theorem 3.1]. The first class of examples for this situation with potentially nontrivial V goes back to work of Puig [25]: it is shown in [25, 6.8] that a block with an abelian defect group P and a Frobenius inertial quotient is stably equivalent to its Brauer correspondent, using the fact that the blocks of centralisers of nontrivial subgroups of P are nilpotent, hence Morita equivalent to the defect group algebra via a Morita equivalence with endopermutation source. The above theorem is used in the proof of Biland [5, Theorem 4.1] or [7, Theorem 1]. For convenience, we reformulate this at the block algebra level.

Theorem 1.2. *Let G, H be finite groups, and let b, c be blocks of $\mathcal{O}G$, $\mathcal{O}H$, respectively, having a common defect group P. Let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ be almost source idempotents. For any subgroup Q of P denote by e_Q and f_Q the unique blocks of $kC_G(Q)$ and $kC_H(Q)$, respectively, satisfying $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$ and $\mathrm{Br}_{\Delta Q}(j)f_Q \neq 0$. Denote by $\hat{e}_Q$ and $\hat{f}_Q$ the unique blocks of $\mathcal{O}C_G(Q)$ and $\mathcal{O}C_H(Q)$ lifting e_Q and f_Q, respectively. Suppose that i and j determine the same fusion system $\mathcal{F}$ on P. Let V be an $\mathcal{F}$-stable indecomposable endopermutation $\mathcal{O}P$-module with vertex P, viewed as an $\mathcal{O}\Delta P$-module through the canonical isomorphism $\Delta P \cong P$. Let M be an indecomposable direct summand of the $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodule*

$$\mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P \times P}(V) \otimes_{\mathcal{O}P} j\mathcal{O}H \ .$$

Suppose that M has ΔP as a vertex as an $\mathcal{O}(G \times H)$-module. Then for any nontrivial subgroup Q of P, there is a canonical $kC_G(Q)e_Q$-$kC_H(Q)f_Q$-bimodule M_Q satisfying $\mathrm{End}_k(M_Q) \cong (\mathrm{End}_{\mathcal{O}}(\hat{e}_Q M \hat{f}_Q))(\Delta Q)$. Moreover, if for all nontrivial subgroups Q of P the bimodule M_Q induces a Morita equivalence between $kC_G(Q)e_Q$ and $kC_H(Q)f_Q$, then M and its dual M^* induce a stable equivalence of Morita type between $\mathcal{O}Gb$ and $\mathcal{O}Hc$.

The existence of canonical bimodules M_Q satisfying $\mathrm{End}_k(M_Q) \cong (\mathrm{End}_{\mathcal{O}}(\hat{e}_Q M \hat{f}_Q))(\Delta Q)$ in this Theorem is due to Biland [5, Theorem 3.15]. In the statement of Theorem 1.2 we let Q run over all nontrivial subgroups of P rather than only the fully $\mathcal{F}$-centralised ones; this makes no difference here since one can always achieve Q to be fully centralised through simultaneous conjugation in G and H. By contrast, in the statement of Theorem 1.1, restricting attention to fully centralised subgroups is necessary in order to ensure that $A(\Delta Q)$ and $kC_G(Q)e_Q$ are Morita equivalent. Another technical difference between the statements of the two theorems is that iMj will be an endopermutation $\mathcal{O}\Delta Q$-module, while this is not clear for $\hat{e}_Q M \hat{f}_Q$ because indecomposable $\mathcal{O}\Delta Q$-summands with vertices strictly smaller than ΔQ might not be compatible. See Biland [6, Lemma 10] for more details on this issue.

Remark 1.3. The proof of Theorem 1.1 becomes significantly shorter if one assumes that V has a fusion-stable endosplit p-permutation resolution. This concept is due to Rickard [28], who also showed the existence of such resolutions for finite abelian p-groups. As a consequence of the classification of endopermutation modules, endosplit p-permutation resolutions exist for all endopermutation modules over k and unramified $\mathcal{O}$ belonging to the subgroup of the Dade group generated by relative syzygies. For odd p this is the entire Dade group while for $p = 2$ there are some endopermutation modules which do not have endosplit permutation resolutions. See [30, Theorem 14.3] for more details. We will outline how this simplifies the proof in the Remark 5.1 below.

2 Background material on blocks and almost source algebras

Let G be a finite group. For any subgroup H of G, we denote by ΔH the 'diagonal' subgroup $\Delta H = \{(y,y) \mid y \in H\}$ of $H \times H$. Let b be a block of $\mathcal{O}G$ and P a defect group of b. That is, b is a primitive idempotent in $Z(\mathcal{O}G)$, and P is a maximal p-subgroup with the property that $\mathcal{O}P$ is isomorphic to a direct summand of $\mathcal{O}Gb$ as an $\mathcal{O}P$-$\mathcal{O}P$-bimodule. As is customary, for any p-subgroup Q of G we denote by $\mathrm{Br}_{\Delta Q} : (\mathcal{O}G)^{\Delta Q} \to kC_G(Q)$ the *Brauer homomorphism* induced by the linear map sending $x \in C_G(Q)$ to its image in $kC_G(Q)$ and $x \in G \setminus C_G(Q)$ to zero. The map $\mathrm{Br}_{\Delta Q}$ is a surjective algebra homomorphism. More generally, for Q a p-subgroup of G and M an $\mathcal{O}G$-module, we denote by $M(Q)$ the $kN_G(Q)$-module obtained from applying the Brauer construction Br_Q to M. If A is an interior P-algebra, and Q a subgroup of Q, we denote by $A(\Delta P)$ the interior $C_P(Q)$-algebra obtained from applying the Brauer construction with respect to the conjugation action of P on A; our notational conventions are as in [19, §3].

Following Green [13], for any indecomposable $\mathcal{O}Gb$-module U, if Q is a minimal subgroup of G for which there exists an $\mathcal{O}Q$-module V such that U is isomorphic to a direct summand of $\mathrm{Ind}_Q^G(V)$, then Q is a p-subgroup of G, the $\mathcal{O}Q$-module V can be chosen to be indecomposable, in which case V is isomorphic to a direct summand of $\mathrm{Res}_Q^G(U)$, and the pair (Q,V) is unique up to G-conjugacy. In that situation, Q is called a vertex of U, and V an $\mathcal{O}Q$-source of U, or simply a source of U of Q is determined by the context. Moreover, if R is a p-subgroup of G such that $\mathrm{Res}_R^G(U)$ has an indecomposable direct summand W with vertex R, then there is a vertex-source pair (Q,V) of U such that $R \subseteq Q$ and such that W is isomorphic to a direct summand of $\mathrm{Res}_R^Q(V)$. By Higman's criterion, this happens if and only of $(\mathrm{End}_{\mathcal{O}}(U))(\Delta R) \neq \{0\}$. See [22, Chapter 4] for an exposition of Green's theory of vertices and sources.

Definition 2.1 (cf. [19, Definition 4.3])**.** Let G be a finite group, let b be a block of $\mathcal{O}G$, let P be a defect group of b. An idempotent i in $(\mathcal{O}Gb)^{\Delta P}$ is called an *almost source idempotent* if $\mathrm{Br}_{\Delta P}(i) \neq 0$ and for every subgroup Q of P there is a unique block e_Q of $kC_G(Q)$ such that $\mathrm{Br}_{\Delta Q}(i) \in kC_G(Q)e_Q$. The interior P-algebra $i\mathcal{O}Gi$ is then called an *almost source algebra of the block b.*

By [23, 3.5] (see also [19, Proposition 4.1] for a proof) there is a canonical Morita equivalence between the block algebra $\mathcal{O}Gb$ and an almost source algebra $i\mathcal{O}Gi$ sending an $\mathcal{O}Gb$-module M to the $i\mathcal{O}Gi$-module iM. Regarding fusion systems, we tend to follow the conventions of [18, §2]; in particular, by a fusion system on a finite p-group we always mean a saturated fusion system (in the terminology used in [2] or [11], for instance). With the notation of the previous Definition, it follows from work of Alperin and Broué [1] that the choice of an almost source idempotent i in $(\mathcal{O}Gb)^{\Delta P}$ determines a fusion system $\mathcal{F}$ on P such that for any two subgroups Q, R of P, the set $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ is the set of all group homomorphisms $\varphi : Q \to R$ for which there is an element $x \in G$ satisfying $\varphi(u) = xux^{-1}$ for all $u \in Q$ and satisfying $xe_Qx^{-1} = e_{xQx^{-1}}$. See e. g. [18, §2], or [2, Part IV]; note that we use here our blanket assumption that k is large enough. Moreover, a subgroup Q of P is fully $\mathcal{F}$-centralised if and only if $C_P(Q)$ is a defect group of the block e_Q of $kC_G(Q)$. Given a subgroup Q of P it is always possible to find a subgroup R of P such that $Q \cong R$ in $\mathcal{F}$ and such that R is fully $\mathcal{F}$-centralised.

Proposition 2.2 (cf. [19, Proposition 4.5])**.** *Let G be a finite group, b a block of $\mathcal{O}G$, P a defect group of b, and $i \in (\mathcal{O}Gb)^{\Delta P}$ an almost source idempotent of b. with associated almost source algebra $A = i\mathcal{O}Gi$. If Q is a fully $\mathcal{F}$-centralised subgroup of P, then* $\mathrm{Br}_{\Delta Q}(i)$ *is an almost source* *idempotent of $kC_G(Q)e_Q$ with associated almost source algebra $A(\Delta Q)$; in particular, $kC_G(Q)e_Q$ and $A(\Delta Q)$ are Morita equivalent.*

By [19, 4.2], an almost source algebra A of a block with P as a defect group is isomorphic to a direct summand of $A \otimes_{\mathcal{O}P} A$ as an A-A-bimodule. Since $A \otimes_{\mathcal{O}P} A \cong (A \otimes_{\mathcal{O}} A^{\mathrm{op}}) \otimes_{\mathcal{O}P \otimes \mathcal{O}P^{\mathrm{op}}} \mathcal{O}P$ this means that as an $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$-module, A is relatively $\mathcal{O}P \otimes_{\mathcal{O}} \mathcal{O}P^{\mathrm{op}}$-projective. Since A, $\mathcal{O}P$, and hence $A \otimes_{\mathcal{O}} A^{\mathrm{op}}$, $\mathcal{O}P \otimes_{\mathcal{O}} \mathcal{O}P^{\mathrm{op}}$ are symmetric $\mathcal{O}$-algebras, it follows that A is also relatively $\mathcal{O}P \otimes_{\mathcal{O}} \mathcal{O}P^{\mathrm{op}}$-injective. Tensoring a split map $A \to A \otimes_{\mathcal{O}P} A$ by $- \otimes_A U$ implies that any A-module U is relatively $\mathcal{O}P$-projective, or equivalently, isomorphic to a direct summand of $A \otimes_{\mathcal{O}P} U$. Vertices and sources of indecomposable $\mathcal{O}Gb$-modules can be read off from almost source algebras; the following result is a slight generalisation of [15, 6.3].

Proposition 2.3. *Let G be a finite group, b a block of $\mathcal{O}G$, P a defect group of b, and i an almost source idempotent in $(\mathcal{O}Gb)^{\Delta P}$. Set $A = i\mathcal{O}Gi$. Let U be an indecomposable $\mathcal{O}Gb$-module, and let Q be a minimal subgroup of P such that the A-module iU is isomorphic to a direct summand of $A \otimes_{\mathcal{O}Q} V$ for some $\mathcal{O}Q$-module V. Then Q is a vertex of U, and U is isomorphic to a direct summand of $\mathcal{O}Gi \otimes_{\mathcal{O}Q} iU$, or equivalently, the $\mathcal{O}Q$-module V with the property that iU is isomorphic to a direct summand of $A \otimes_{\mathcal{O}Q} V$ can be chosen to be an indecomposable direct summand of $\mathrm{Res}_Q(iU)$.*

Proof. Note that iU is an indecomposable A-module. Let Q be a minimal subgroup of P such that iU is isomorphic to a direct summand of $A \otimes_{\mathcal{O}Q} V$, for some $\mathcal{O}Q$-module V. Tensoring with $\mathcal{O}Gi \otimes_A -$ implies that U is isomorphic to a direct summmand of $\mathcal{O}Gi \otimes_{\mathcal{O}Q} V$, hence of $\mathrm{Ind}_Q^G(V)$. Thus Q contains a vertex of U. By general abstract nonsense (e.g. the equivalence of the statements (i) and (ii) in [10, Theorem 6.8] applied to restriction and induction between A and $\mathcal{O}Q$), iU is then isomorphic to a direct summand of $A \otimes_{\mathcal{O}Q} iU$, thus of $A \otimes_{\mathcal{O}Q} V$ for some indecomposable direct summand V of $\mathrm{Res}_Q(iU)$. The minimality of Q implies that V has Q as a vertex. But V is isomorphic to a direct summand of $\mathrm{Res}_Q^G(U)$, and hence Q is contained in a vertex of U. The result follows. $\square$

By a result of Puig in [24], fusion systems of blocks can be read off their source algebras; this is slightly extended to almost source algebras in [19, 5.1, 5.2].

Proposition 2.4 (cf. [19, Proposition 5.1])**.** *Let G be a finite group, let b be a block of $\mathcal{O}G$ with defect group P, let $i \in (\mathcal{O}Gb)^{\Delta P}$ be an almost source idempotent and set $A = i\mathcal{O}Gi$. Denote by $\mathcal{F}$ the fusion system of A on P. Let Q be a fully $\mathcal{F}$-centralised subgroup of P and let $\varphi : Q \to P$ be a morphism in $\mathcal{F}$. Set $R = \varphi(Q)$. Denote by e_Q, e_R the unique blocks of $kC_G(Q)$, $kC_G(R)$ satisfying $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$ and $\mathrm{Br}_{\Delta R}(i)e_R \neq 0$.*

(i) For any primitive idempotent n in $(\mathcal{O}Gb)^{\Delta R}$ satisfying $\mathrm{Br}_{\Delta R}(n)e_R \neq 0$ there is a primitive idempotent m in $A^{\Delta Q}$ satisfying $\mathrm{Br}_{\Delta Q}(m) \neq 0$ such that $m\mathcal{O}G \cong {}_{\varphi}(n\mathcal{O}G)$ as $\mathcal{O}Q$-$\mathcal{O}Gb$-bimodules and such that $\mathcal{O}Gm \cong (\mathcal{O}Gn)_{\varphi}$ as $\mathcal{O}Gb$-$\mathcal{O}Q$-bimodules.

(ii) For any primitive idempotent n in $A^{\Delta R}$ satisfying $\mathrm{Br}_{\Delta R}(n) \neq 0$ there is a primitive idempotent m in $A^{\Delta Q}$ satisfying $\mathrm{Br}_{\Delta Q}(m) \neq 0$ such that $mA \cong {}_{\varphi}(nA)$ as $\mathcal{O}Q$-A-bimodules and such that $Am \cong (An)_{\varphi}$ as A-$\mathcal{O}Q$-bimodules.

Proposition 2.5 (cf. [19, Proposition 5.2])**.** *Let G be a finite group, b be a block of $\mathcal{O}G$ with defect group P, let i be an almost source idempotent in $(\mathcal{O}Gb)^{\Delta P}$ and set $A = i\mathcal{O}Gi$. Denote by $\mathcal{F}$ the fusion system of A on P. Let Q, R be subgroups of P.*

(i) Every indecomposable direct summand of A as an $\mathcal{O}Q$-$\mathcal{O}R$-bimodule is isomorphic to $\mathcal{O}Q \otimes_{\mathcal{O}S} {}_{\varphi}\mathcal{O}R$ for some subgroup S of Q and some morphism $\varphi : S \to R$ belonging to $\mathcal{F}$.

(ii) If $\varphi : Q \to R$ is an isomorphism in $\mathcal{F}$ such that R is fully $\mathcal{F}$-centralised then ${}_{\varphi}\mathcal{O}R$ is isomorphic to a direct summand of A as an $\mathcal{O}Q$-$\mathcal{O}R$-bimodule.

In particular, $\mathcal{F}$ is determined by the $\mathcal{O}P$-$\mathcal{O}P$-bimodule structure of A.

Proposition 2.6. *Let G be a finite group, b be a block of kG with defect group P, and let i be an almost source idempotent in $(kGb)^{\Delta P}$. Denote by $\mathcal{F}$ the fusion system on P determined by i. Let Q, R be subgroups of P and denote by e the unique block of $kC_G(Q)$ satisfying $\mathrm{Br}_{\Delta Q}(i)e \neq 0$. Let $\varphi : Q \to R$ be an injective group homomorphism such that ${}_{\varphi}kR$ is isomorphic to a direct summand of $ekGi$ as an kQ-kR-bimodule. Then $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$.*

Proof. Let T be a fully $\mathcal{F}$-centralised subgroup of P isomorphic to Q in the fusion system $\mathcal{F}$. That is, if f is the unique block of $kC_G(T)$ satisfying $\mathrm{Br}_{\Delta T}(i)f \neq 0$, then $C_P(T)$ is a defect group of $kC_G(T)f$, there is an element $x \in G$ such that $(T, f) = {}^{x}(Q, e)$, and the isomorphism $\psi : Q \to T$ defined by $\psi(u) = xux^{-1}$ is in $\mathcal{F}$. Since ${}_{\varphi}kR$ is a summand of e_QkGi, multiplication by x shows that the kT-kR-bimodule ${}_{\varphi\circ\psi^{-1}}kS$ is a direct summand of $xekGi = xex^{-1}kGi = fkGi$. Moreover, φ is a morphism in $\mathcal{F}$ if and only if $\varphi\circ\psi^{-1}$ is. Thus, after possibly replacing (Q, e) by (T, f) we may assume that (Q, e) is fully $\mathcal{F}$-centralised. By [19, Proposition 4.6] (ii) this implies that every local point of Q on kGb associated with e has a representative in $ikGi$. Since ${}_{\varphi}kR$ is indecomposable as a kQ-kR-bimodule with a vertex of order $|Q|$, this bimodule is isomorphic to a direct summand of $jkGi$ for some primitive local idempotent j in $(kGb)^{\Delta Q}$ appearing in a primitive decomposition of e in $(kGb)^{\Delta Q}$. But then $\mathrm{Br}_{\Delta Q}(j) \in kC_G(Q)e$, and hence, after possibly replacing j with a suitable $((kGb)^{\Delta Q})^{\times}$-conjugate, we may assume that $j \in ikGi$. It follows from Proposition 2.5 (i) that φ is a morphism in $\mathcal{F}$. ☐

Proposition 2.7. *Let G be a finite group, H a subgroup of G and A an $\mathcal{O}$-algebra. Let M be an $\mathcal{O}H$-A-bimodule and V an $\mathcal{O}H$-module. Consider $V \otimes_{\mathcal{O}} M$ as an $\mathcal{O}H$-A bimodule with H acting diagonally on the left, consider V as a module for $k\Delta H$ via the canonical isomorphism $\Delta H \cong H$ and consider $\mathrm{Ind}_{\Delta H}^{G\times H}(V)$ as an $\mathcal{O}G$-$\mathcal{O}H$-bimodule. We have a natural isomorphism of $\mathcal{O}G$-A-bimodules*

$$\mathrm{Ind}_H^G(V \otimes_{\mathcal{O}} M) \cong \mathrm{Ind}_{\Delta H}^{G\times H}(V) \otimes_{\mathcal{O}H} M$$

sending $x \otimes (v \otimes m)$ to $((x, 1) \otimes v) \otimes m$, where $v \in V$ and $m \in M$.

Proof. This is a straightforward verification. ☐

3 On fusion-stable endopermutation modules

Let P be a finite p-group. Following Dade [12] a finitely generated $\mathcal{O}$-free $\mathcal{O}P$-module V is an *endopermutation module* if $\mathrm{End}_{\mathcal{O}}(V) \cong V \otimes_{\mathcal{O}} V^*$ is a permutation $\mathcal{O}P$-module, with respect to the 'diagonal' action of P. See Thévenaz [30] for an overview on this subject and some historic background, leading up to the classification of endopermutation modules. We will use without further comment some of the basic properties, due to Dade, of endopermutation modules - see for instance [29, §28]. If V is an endopermutation $\mathcal{O}P$-module having an indecomposable direct summand with vertex P, then for any two subgroups Q, R of P such that Q is normal in R, there is an endopermutation kR/Q-module $V' = \mathrm{Defres}^P_{R/Q}(V)$ satisfying $\mathrm{End}_{\mathcal{O}}(V)(\Delta Q) \cong \mathrm{End}_k(V')$ as

R/Q-algebras, and as interior R/Q-algebras if $R \subseteq QC_P(Q)$. This construction is also known as Dade's 'slash' construction.

Definition 3.1. Let P be a finite p-group and $\mathcal{F}$ a fusion system on P. Let Q be a subgroup of P and V be an endopermutation $\mathcal{O}Q$-module. We say that V is $\mathcal{F}$*-stable* if for any subgroup R of Q and any morphism $\varphi : R \to Q$ in $\mathcal{F}$ the sets of isomorphism classes of indecomposable direct summands with vertex R of the $\mathcal{O}Q$-modules $\mathrm{Res}^Q_R(V)$ and ${}_\varphi V$ are equal (including the possibility that both sets may be empty).

With the notation of 3.1, the property of V being $\mathcal{F}$-stable does not necessarily imply that $\mathrm{Res}^Q_R(V)$ and ${}_\varphi V$ have to be *isomorphic* as $\mathcal{O}R$-modules, where $\varphi : R \to Q$ is a morphism in $\mathcal{F}$ (so this is a slight deviation from the terminology in [21, 3.3. (1)]). What the $\mathcal{F}$-stability of V means is that the indecomposable direct factors of $\mathrm{Res}^Q_R(V)$ and ${}_\varphi V$ with vertex R, if any, are isomorphic, but they may occur with different multiplicities in direct sum decompositions (in other words, in the terminology of [21, 3.3.(2)] the class of V in the Dade group is $\mathcal{F}$-stable, provided that V has an indecomposable direct summand with vertex P). By [21, 3.7], every class in $D_{\mathcal{O}}(P)$ having an $\mathcal{F}$-stable representative has a representative W satisfying the stronger stability condition $\mathrm{Res}^P_R(W) \cong {}_\varphi W$ for any morphism $\varphi : R \to P$ in $\mathcal{F}$. It follows from Alperin's fusion theorem that in order to check whether an endopermutation $\mathcal{O}P$-module V with an indecomposable direct summand of vertex P is $\mathcal{F}$-stable, it suffices to verify that $\mathrm{Res}^P_R(V)$ and ${}_\varphi V$ have isomorphic summands with vertex R for any $\mathcal{F}$-essential subgroup R of P and any p'-automorphism φ of R in $\mathrm{Aut}_{\mathcal{F}}(R)$. In particular, if P is abelian, then an indecomposable endopermutation $\mathcal{O}P$-module V with vertex P is $\mathcal{F}$-stable if and only if $V \cong {}_\varphi V$ for any $\varphi \in \mathrm{Aut}_{\mathcal{F}}(P)$. In the majority of cases where Definition 3.1 is used we will have $Q = P$. One notable exception arises in the context of bimodules, where we consider the fusion systems $\mathcal{F} \times \mathcal{F}$ on $P \times P$ with the diagonal subgroup ΔP playing the role of Q. The key argument exploiting the $\mathcal{F}$-stability of an endopermutation $\mathcal{O}P$-module V having an indecomposable direct summand with vertex P goes as follows: if Q is a subgroup of P and $\varphi : Q \to P$ a morphism in $\mathcal{F}$, then the restriction to ΔQ of $V \otimes_{\mathcal{O}} {}_\varphi V^*$ is again a permutation module, or equivalently, $V \otimes_{\mathcal{O}} V^*$ remains a permutation module for the twisted diagonal subgroup $\Delta_\varphi Q = \{(u, \varphi(u)) \mid u \in Q\}$ of $P \times P$. By a result of Broué in [8], if V is a permutation $\mathcal{O}P$-module, then $\mathrm{End}_{\mathcal{O}}(V)(\Delta P) \cong \mathrm{End}_k(V(P))$. This is not true for more general modules, but Dade's 'slash' construction from [12] for endopermutation modules yields a generalisation of this isomorphism, as follows.

Proposition 3.2. *Let A be an almost source algebra of a block of a finite group algebra over $\mathcal{O}$ with a defect group P and fusion system $\mathcal{F}$ on P. Let Q be a subgroup of P and let V be an $\mathcal{F}$-stable endopermutation $\mathcal{O}Q$-module having an indecomposable direct summand with vertex Q. Set $U = A \otimes_{\mathcal{O}Q} V$. The following hold.*

(i) As an $\mathcal{O}Q$-module, U is an endopermutation module, and U has a direct summand isomorphic to V.

(ii) Let R be a subgroup of Q. The A-module structure on U induces an $A(\Delta R)$-module structure on $U' = \mathrm{Defres}^Q_{RC_Q(R)/R}(U)$ extending the $kC_Q(R)$-module structure on U' such that we have an isomorphism $(\mathrm{End}_{\mathcal{O}}(U))(\Delta R) \cong \mathrm{End}_k(U')$ as algebras and as $A(\Delta R)$-$A(\Delta R)$-bimodules.

Proof. By [19, 5.2], every indecomposable direct summand of A as an $\mathcal{O}Q$-$\mathcal{O}Q$-bimodule is isomorphic to $\mathcal{O}Q \otimes_{\mathcal{O}R} {}_\varphi\mathcal{O}Q$ for some subgroup R of Q and some morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$, and at least one summand of A as an $\mathcal{O}Q$-$\mathcal{O}Q$-bimodule is isomorphic to $\mathcal{O}Q$. Thus every indecomposable direct summmand of U is isomorphic to $\mathrm{Ind}_R^Q({}_{\varphi}V_{\varphi(R)})$ for some subgroup R of Q and some $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R,Q)$, where $V_{\varphi(R)}$ is an indecomposable direct summand of vertex $\varphi(R)$ of $\mathrm{Res}^Q_{\varphi(R)}(V)$, and V is a summand of U as an $\mathcal{O}Q$-module. Since V is $\mathcal{F}$-stable, we have ${}_{\varphi}(V_{\varphi(R)}) \cong V_R$, which implies that the restriction to $\mathcal{O}Q$ of U is an endopermutation $\mathcal{O}Q$-module. Statement (i) follows. For statement (ii) we consider the structural algebra homomorphism $A \to \mathrm{End}_{\mathcal{O}}(U)$ given by the action of A on U. This is a homomorphism of interior Q-algebras. Applying the Brauer construction with respect to ΔR, where R is a subgroup of Q, yields a homomorphism of interior $C_Q(R)$-algebras $A(\Delta R) \to (\mathrm{End}_{\mathcal{O}}(U))(\Delta R)$. Since U is an endopermutation $\mathcal{O}Q$-module, we have $(\mathrm{End}_{\mathcal{O}}(U))(\Delta R) \cong \mathrm{End}_k(U')$ as interior $C_Q(R)$-algebras. This yields a homomorphism $A(\Delta R) \to \mathrm{End}_k(U')$, hence a canonical $A(\Delta R)$-module structure on U' with the properties as stated. □

Statement (ii) in Proposition 3.2 is particularly useful when Q is fully $\mathcal{F}$-centralised, since in that case $C_P(Q)$ is a defect group of the unique block e_Q of $kC_G(Q)$ satisfying $\mathrm{Br}_{\Delta Q}(i)e_Q \neq 0$, and the algebras $A(\Delta Q)$ and $kC_G(Q)e_Q$ are Morita equivalent. Statement (ii) of 3.2 is essentially equivalent to a result of Biland; since we will use this for proving that the Theorems 1.1 and 1.2 are equivalent we state this and sketch a proof for the convenience of the reader.

Proposition 3.3 (Biland [5, Theorem 3.15 (i)]). *Let G be a finite group, b a block of $\mathcal{O}G$, P a defect group of b and $i \in (\mathcal{O}Gb)^{\Delta P}$ an almost source idempotent. Let Q be a subgroup of P and let V be an $\mathcal{F}$-stable endopermutation $\mathcal{O}Q$-module having an indecomposable direct summand with vertex Q. Set $X = \mathcal{O}Gi \otimes_{\mathcal{O}Q} V$. Let R be a subgroup of Q, denote by e_R the unique block of $kC_G(R)$ satisfying $\mathrm{Br}_{\Delta R}(i)e_R \neq 0$, and let $\hat{e}_R$ be the block of $\mathcal{O}C_G(R)$ which lifts e_R. There is a canonical $kC_G(R)e_R$-module Y_R such that we have an isomorphism $(\mathrm{End}_{\mathcal{O}}(\hat{e}_R Y))(\Delta R) \cong \mathrm{End}_k(Y_R)$ as algebras and as $kC_G(R)e_R$-$kC_G(R)e_R$-bimodules.*

Proof. Applying Br_R to the canonical algebra homomorphism $\mathcal{O}Gb \to \mathrm{End}_k(Y)$ and cutting by e_R and $\hat{e}_R$ yields an algebra homomorphism $kC_G(R)e_r \to (\mathrm{End}_{\mathcal{O}}(\hat{e}_R Y))(\Delta R)$. In order to show that this is isomorphic to $\mathrm{End}_k(Y_R)$ for some module Y_R it suffices to observe that the indecomposable summands of $\mathrm{Res}_R(\hat{e}_R Y)$ with vertex R are all isomorphic. Note that $\hat{e}_R Y = \hat{e}_R \mathcal{O}Gi \otimes_{\mathcal{O}Q} V$. Any indecomposable direct summand of $\hat{e}_R \mathcal{O}Gi$ as an $\mathcal{O}R$-$\mathcal{O}Q$-bimodule with a vertex of order at least $|R|$ is isomorphic to ${}_{\varphi}\mathcal{O}Q$ for some group homomorphism $\varphi : R \to Q$ induced by conjugation with an element in G. In view of the fusion stability of V, it suffices to show that φ is a morphism in $\mathcal{F}$. This is an immediate consequence of 2.6, whence the result. □

As mentioned earlier, there is a technical difference between the Propositions 3.2 and 3.3: statement (i) in Proposition 3.2 may not have an an analogue at the block algebra level, since it is not clear whether $\hat{e}_R X$ is an endopermutation $\mathcal{O}R$-module, because the indecomposable direct summands with vertex strictly contained in R might not be compatible.

4 Bimodules with fusion-stable endopermutation source

Throughout this Section we fix the following notation and hypotheses. Let G, H be finite groups, b a block of $\mathcal{O}G$ and c a block of $\mathcal{O}H$. Suppose that b and c have a common defect group P. Let $i \in (\mathcal{O}Gb)^{\Delta P}$ and $j \in (\mathcal{O}Hc)^{\Delta P}$ be almost source idempotents. Set $A = i\mathcal{O}Gi$ and $B = j\mathcal{O}Hj$. Suppose that A and B determine the same fusion system $\mathcal{F}$ on P. Let V be an $\mathcal{F}$-stable indecomposable endopermutation $\mathcal{O}P$-module with vertex P. Whenever expedient, we consider V as an $\mathcal{O}\Delta P$-module through the canonical isomorphism $\Delta P \cong P$. Set

$$U = A \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} B \ ,$$

$$X = \mathcal{O}Gi \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} j\mathcal{O}H \ .$$

The $A \otimes_{\mathcal{O}} B^{\mathrm{op}}$-module U corresponds to the $\mathcal{O}(G \times H)$-module X through the canonical Morita equivalence between $A \otimes_{\mathcal{O}} B^{\mathrm{op}}$ and $\mathcal{O}Gb \otimes_{\mathcal{O}} \mathcal{O}Hc^{\mathrm{op}}$; in particular, there is a canonical bijection between the isomorphism classes of indecomposable direct summands of U and of X. This Section contains some technical statements which involve the tensor product of two bimodules. This yields a priori four module structures, and keeping track of those is essential - see Broué [10, §1] for some formal properties of *quadrimodules*. If the algebras under consideration are group algebras, we play this back to two actions via the usual 'diagonal' convention: given two finite groups G, H and two $\mathcal{O}G$-$\mathcal{O}H$-bimodules S, S', we consider $S \otimes_{\mathcal{O}} S'$ as an $\mathcal{O}G$-$\mathcal{O}H$-bimodule via the diagonal left action by G and the diagonal right action by H; explicitly, $x \cdot (s \otimes s') \cdot y = xsy \otimes xs'y$, where $x \in G$, $y \in H$, $s \in S$, and $s' \in S'$. This is equivalent to the diagonal $G \times H$-action if we interpret the $\mathcal{O}G$-$\mathcal{O}H$-bimodules as $\mathcal{O}(G \times H)$-modules in the usual way. The following result is a bimodule version of 3.2.

Proposition 4.1. *Consider U as an $\mathcal{O}\Delta P$-module, with $(u, u) \in \Delta P$ acting on U by left multiplication with u and right multiplication with u^{-1}. Then, as an $\mathcal{O}\Delta P$-module, U is an endopermutation module having V as a direct summand, and for any subgroup Q of P, the A-B-bimodule structure on U induces an $A(\Delta Q)$-$B(\Delta Q)$-bimodule structure on $U' = \mathrm{Defres}^{\Delta P}_{\Delta Q C_P(Q)/Q}(U)$ such that we have an isomorphism of $A(\Delta Q) \otimes_k B(\Delta Q)^{\mathrm{op}}$-bimodules $\mathrm{End}_{\mathcal{O}}(U)(\Delta Q) \cong \mathrm{End}_k(U')$.*

Proof. This is the special case of 3.2 with $P \times P$, $\mathcal{F} \times \mathcal{F}$, ΔP, ΔQ, $A \otimes_{\mathcal{O}} B^{\mathrm{op}}$, instead of P, $\mathcal{F}$, Q, R, A, respectively. ☐

Theorem 4.2. *Let Q be a subgroup of P, and let U' be the $A(\Delta Q)$-$B(\Delta Q)$-bimodule from 4.1 such that $(\mathrm{End}_{\mathcal{O}}(U))(\Delta Q) \cong \mathrm{End}_k(U')$. Then $\mathrm{End}_{B^{\mathrm{op}}}(U)$ is a ΔQ-subalgebra of $\mathrm{End}_{\mathcal{O}}(U)$, the algebra homomorphism*

$$\beta : \mathrm{End}_{B^{\mathrm{op}}}(U)(\Delta Q) \to \mathrm{End}_{\mathcal{O}}(U)(\Delta Q)$$

induced by the inclusion $\mathrm{End}_{B^{\mathrm{op}}}(U) \subseteq \mathrm{End}_{\mathcal{O}}(U)$ is injective, and there is a commutative diagram of algebra homomorphisms

$$\begin{tikzcd}
\mathrm{End}_{\mathcal{O}}(U)(\Delta Q) \arrow[r, "\cong"] & \mathrm{End}_k(U') \\
\mathrm{End}_{B^{\mathrm{op}}}(U)(\Delta Q) \arrow[u, "\beta"] \arrow[r, "\gamma"'] & \mathrm{End}_{B(\Delta Q)^{\mathrm{op}}}(U') \arrow[u]
\end{tikzcd}$$

where the right vertical arrow is the obvious inclusion map. In particular, the algebra homomorphism γ is injective.

Proof. For $\varphi \in \mathrm{End}_{\mathcal{O}}(U)$, $y \in Q$, and $u \in U$ we have

$$^{\Delta y}\varphi(u) = \Delta y \cdot \varphi(\Delta y^{-1} \cdot u) = y\varphi(y^{-1}uy)y^{-1} \ ,$$

where $\Delta y = (y, y)$. If $\varphi \in \mathrm{End}_{B^{\mathrm{op}}}(U)$, then in particular φ commutes with the right action by Q, and hence we have ${}^{\Delta y}\varphi(u) = y\varphi(y^{-1}u) = {}^{(y,1)}\varphi(u)$, which shows that ${}^{\Delta y}\varphi$ is again a B^{op}-homomorphism. The algebra of ΔQ-fixed points in $\mathrm{End}_{B^{\mathrm{op}}}(U)$ is equal to $\mathrm{End}_{\mathcal{O}Q\otimes_{\mathcal{O}}B^{\mathrm{op}}}(U)$. The existence of a commutative diagram as in the statement is formal: if $\varphi \in \mathrm{End}_{\mathcal{O}Q\otimes_{\mathcal{O}}B^{\mathrm{op}}}(U)$, then in particular $b\cdot\varphi = \varphi\cdot b$ for all $b \in B$, hence for all $b \in B^{\Delta Q}$, and applying $\mathrm{Br}_{\Delta Q}$ yields that the image of φ in $\mathrm{End}_{\mathcal{O}}(U)(\Delta Q)$ commutes with the elements in $B(\Delta Q)$. Since the upper horizontal map is a bimodule isomorphism, it follows that the image of φ in $\mathrm{End}_k(U')$ commutes with the elements in $B(\Delta Q)$, hence lies in the subalgebra $\mathrm{End}_{B(\Delta Q)^{\mathrm{op}}}(U')$. In order to show that β is injective, we first note that this injectivity does not make use of the left A-module structure of U but only of the left $\mathcal{O}Q$-module structure. Thus we may decompose U by decomposing A as an $\mathcal{O}Q$-$\mathcal{O}P$-bimodule. By 2.5, every summand of A as an $\mathcal{O}Q$-$\mathcal{O}P$-bimodule is of the form $\mathcal{O}Q\otimes_{\mathcal{O}R\,\varphi}\mathcal{O}P$ for some subgroup R of Q and some homomorphism $\varphi : R \to P$ belonging to the fusion system $\mathcal{F}$. Using the appropriate version of the isomorphism 2.7 of $\mathcal{O}P$-B-modules $\mathrm{Ind}_{\Delta P}^{P\times P}(V)\otimes_{\mathcal{O}P} B \cong V\otimes_{\mathcal{O}} B$ it suffices therefore to show that applying $\mathrm{Br}_{\Delta Q}$ to the inclusion map

$$\mathrm{Hom}_{B^{\mathrm{op}}}(\mathcal{O}Q\otimes_{\mathcal{O}R}\,{}_{\varphi}(V\otimes_{\mathcal{O}} B), \mathcal{O}Q\otimes_{\mathcal{O}S}\,{}_{\psi}(V\otimes_{\mathcal{O}} B)) \subseteq$$

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}Q\otimes_{\mathcal{O}R}\,{}_{\varphi}(V\otimes_{\mathcal{O}} B), \mathcal{O}Q\otimes_{\mathcal{O}S}\,{}_{\psi}(V\otimes_{\mathcal{O}} B))$$

remains injective upon applying $\mathrm{Br}_{\Delta Q}$, where R, S are subgroups of Q and where $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$, $\psi \in \mathrm{Hom}_{\mathcal{F}}(S, P)$. If one of R, S is a proper subgroup of Q, then both sides vanish upon applying $\mathrm{Br}_{\Delta Q}$. Thus it suffices to show that the map

$$\mathrm{Hom}_{B^{\mathrm{op}}}({}_{\varphi}(V\otimes_{\mathcal{O}} B), {}_{\psi}(V\otimes_{\mathcal{O}} B))(\Delta Q) \to \mathrm{Hom}_{\mathcal{O}}({}_{\varphi}(V\otimes_{\mathcal{O}} B), {}_{\psi}(V\otimes_{\mathcal{O}} B))(\Delta Q)$$

is injective, where φ, $\psi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$. The summands of ${}_{\varphi}V$, ${}_{\psi}V$ with vertices smaller than Q yield summands of $V\otimes_{\mathcal{O}} B$ which vanish on both sides upon applying $\mathrm{Br}_{\Delta Q}$. The fusion stability of V implies that indecomposable summands with vertex Q of ${}_{\varphi}V$, ${}_{\psi}V$ are all isomorphic to an indecomposable direct summand W with vertex Q of $\mathrm{Res}_Q^P(V)$. Thus it suffices to show that the map

$$\mathrm{End}_{B^{\mathrm{op}}}(W\otimes_{\mathcal{O}} B)(\Delta Q) \to \mathrm{End}_{\mathcal{O}}(W\otimes_{\mathcal{O}} B)(\Delta Q)$$

is injective, where W is an indecomposable direct summand of $\mathrm{Res}_Q^P(V)$ with vertex Q. Using the natural adjunction isomorphism

$$\mathrm{End}_{\mathcal{O}}(W\otimes_{\mathcal{O}} B) \cong \mathrm{Hom}_{\mathcal{O}}(B, W^*\otimes_{\mathcal{O}} W\otimes_{\mathcal{O}} B)$$

it suffices to show that the map

$$\mathrm{Hom}_{B}(B, W^*\otimes_{\mathcal{O}} W\otimes_{\mathcal{O}} B)(\Delta Q) \to \mathrm{Hom}_{\mathcal{O}}(B, W^*\otimes_{\mathcal{O}} W\otimes_{\mathcal{O}} B)(\Delta Q)$$

is injective. Now $W^*\otimes_{\mathcal{O}} W$ is a direct sum of a trivial $\mathcal{O}Q$-module $\mathcal{O}$ and indecomposable permutation $\mathcal{O}Q$-modules with vertices strictly smaller than Q. Thus it suffices to show that the map

$$\mathrm{End}_{B^{\mathrm{op}}}(B)(\Delta Q) \to \mathrm{End}_{\mathcal{O}}(B)(\Delta Q)$$

is injective. The canonical isomorphism $\mathrm{End}_{B^{\mathrm{op}}}(B) \cong B$ yields an isomorphism $\mathrm{End}_{B^{\mathrm{op}}}(B)(\Delta Q) \cong B(\Delta Q)$. Since B has a ΔQ-stable $\mathcal{O}$-basis, it follows that $\mathrm{End}_{\mathcal{O}}(B)(\Delta Q) \cong \mathrm{End}_k(B(\Delta Q))$. Using these isomorphisms, the last map is identified with the structural homomorphism

$$B(\Delta Q) \to \mathrm{End}_k(B(\Delta Q))\ ,$$

which is clearly injective. $\square$

Lemma 4.3. *Let W be an indecomposable direct summand of $U \otimes_B U^*$ or of $U \otimes_{\mathcal{O}P} U^*$. Then W is isomorphic to a direct summand of $A \otimes_{\mathcal{O}Q} A$ for some fully $\mathcal{F}$-centralised subgroup Q of P such that $W(\Delta Q) \neq \{0\}$. In particular, $U \otimes_B U^*$ and $U \otimes_{\mathcal{O}P} U^*$ are p-permutation bimodules.*

Proof. Since B is isomorphic to a direct summand of $B \otimes_{\mathcal{O}P} B$, it follows that $U \otimes_B U^*$ is isomorphic to a direct summand of $U \otimes_{\mathcal{O}P} U^*$. Thus it suffices to prove the statement for an indecomposable direct summnad W of $U \otimes_{\mathcal{O}P} U^*$. Using the isomorphisms from 2.7, we get isomorphisms as $\mathcal{O}P$-$\mathcal{O}P$-bimodules

$$\mathrm{Ind}_{\Delta P}^{P\times P}(V^*) \otimes_{\mathcal{O}P} B \otimes_{\mathcal{O}P} B \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V) \cong V^* \otimes_{\mathcal{O}} B \otimes_{\mathcal{O}P} B \otimes_{\mathcal{O}} V \cong (V \otimes_{\mathcal{O}} V^*) \otimes_{\mathcal{O}} (B \otimes_{\mathcal{O}P} B)\ ,$$

where the right side is to be understood as a tensor product of two $\mathcal{O}P$-$\mathcal{O}P$-bimodules with the above conventions. Every indecomposable summand of $B \otimes_{\mathcal{O}P} B$ as an $\mathcal{O}P$-$\mathcal{O}P$-bimodule is isomorphic to $\mathcal{O}P \otimes_{\mathcal{O}Q} {}_{\varphi}\mathcal{O}P \cong \mathrm{Ind}_{\Delta_\varphi Q}^{P\times P}(\mathcal{O})$ for some subgroup Q of P and some morphism $\varphi : Q \to P$ in $\mathcal{F}$, where $\Delta_\varphi Q = \{(u, \varphi(u)) \mid u \in Q\}$. Thus any indecomposable direct summand of $(V \otimes_{\mathcal{O}} V^*) \otimes_{\mathcal{O}} (B \otimes_{\mathcal{O}P} B)$ is isomorphic to a direct summand of an $\mathcal{O}P$-$\mathcal{O}P$-bimdoule of the form $\mathrm{Ind}_{\Delta_\varphi Q}^{P\times P}(V \otimes_{\mathcal{O}} V^*)$. The restriction to $\Delta_\varphi Q$ of $V \otimes_{\mathcal{O}} V^*$ is a permutation module thanks to the stability of V, and hence the indecomposable direct summands of $\mathrm{Ind}_{\Delta_\varphi Q}^{P\times P}(V \otimes_{\mathcal{O}} V^*)$ are of the form $\mathcal{O}P \otimes_{\mathcal{O}R} {}_{\varphi}\mathcal{O}P$, where R is a subgroup of Q and where we use abusively the same letter φ for the restriction of φ to any such subgroup. Thus W is isomorphic to a direct summand of $A \otimes_{\mathcal{O}R} {}_{\varphi}A$, with R and φ as before. Set $S = \varphi(R)$. The indecomposability of W implies that W is isomorphic to a direct summand of $Ar \otimes_{\mathcal{O}R} {}_{\varphi}mA$ for some primitive idempotent r in $A^{\Delta R}$ and some primitive isempotent m in $A^{\Delta S}$. By choosing R minimal, we may assume that r, m belong to local points of R and S on A, respectively. Let T be a fully $\mathcal{F}$-centralised subgroup of P and let $\psi : T \to R$ be an isomorphism in $\mathcal{F}$. Then, by 2.4, there are primitive idempotents n, s in $A^{\Delta T}$ such that $An \cong Am_\psi$ as A-$\mathcal{O}T$-bimodules and $sA \cong {}_{\psi\circ\varphi}rA$ as $\mathcal{O}T$-A-bimodules. Thus Y is isomorphic to a direct summand of $A \otimes_{\mathcal{O}T} A$. The minimality of R, hence of T, implies that ΔT is a vertex of $\mathcal{O}Gi \otimes_A W \otimes_B j\mathcal{O}H$, viewed as an $\mathcal{O}(G \times H)$-module, and Proposition 2.3 implies that a source, which has just shown to be trivial, is a summand of W restricted to ΔT, which implies that $W(\Delta T) \neq \{0\}$. $\square$

Proposition 4.4. *Let M be an indecomposable direct summand of the A-B-bimodule U. The following statements are equivalent.*

(i) A is isomorphic to a direct summand of the A-A-bimodule $M \otimes_B M^*$.

(ii) A is isomorphic to a direct summand of the A-A-bimodule $M \otimes_{\mathcal{O}P} M^*$.

(iii) $(M \otimes_B M^*)(\Delta P) \neq \{0\}$.

(iv) $(M \otimes_{\mathcal{O}P} M^*)(\Delta P) \neq \{0\}$.

Proof. Since B is isomorphic to a direct summand of the B-B-bimodule $B \otimes_{\mathcal{O}P} B$, it follows that $M \otimes_B M^*$ is isomorphic to a direct summand of $M \otimes_{\mathcal{O}P} M^*$. This yields the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv). Since $A(\Delta P) \neq \{0\}$, we trivially have the implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv). Since M is finitely generated projective as a left A-module and as a right B-module (hence also as a right $\mathcal{O}P$-module), we have $M \otimes_B M^* \cong \mathrm{End}_{B^{\mathrm{op}}}(M)$, and $M \otimes_{\mathcal{O}P} M^* \cong \mathrm{End}_{(\mathcal{O}P)^{\mathrm{op}}}(M)$. It is well-known that if A is isomorphic to a direct summand of $M \otimes_{\mathcal{O}P} M^*$, then the canonical algebra homomorphism $A \to \mathrm{End}_{(\mathcal{O}P)^{\mathrm{op}}}(M)$ is split injective as a bimodule homomorphism (see e.g. [14, Lemma 4] for a proof). This algebra homomorphism factors through the inclusion $\mathrm{End}_{B^{\mathrm{op}}}(M) \subseteq \mathrm{End}_{(\mathcal{O}P)^{\mathrm{op}}}(M)$, which implies that the canonical algebra homomorphism $A \to \mathrm{End}_{B^{\mathrm{op}}}(M)$ is also split injective as a bimodule homomorphism. This shows the implication (ii) $\Rightarrow$ (i). Suppose that (iv) holds. Set $Y = \mathcal{O}Gi \otimes_A M$. Then $Y \otimes_{\mathcal{O}P} Y^* \cong \mathcal{O}Gi \otimes_A M \otimes_{\mathcal{O}P} M^* \otimes_A i\mathcal{O}G$. It follows from 4.3 that $Y \otimes_{\mathcal{O}P} Y^*$ is a permutation $\mathcal{O}(P \times P)$-module on which $\mathcal{O}P$ acts freely on the left and on the right and that $\mathcal{O}P$ is isomorphic to a direct summand of $Y \otimes_{\mathcal{O}P} Y^* \cong \mathrm{End}_{\mathcal{O}P^{\mathrm{op}}}(Y)$. Thus $(\mathrm{End}_{\mathcal{O}P^{\mathrm{op}}}(X))(\Delta P)$ is nonzero projective as a left or right $kZ(P)$-module. It follows from [26, Proposition 3.8] that $\mathcal{O}Gb$ is isomorphic to a direct summand of $Y \otimes_{\mathcal{O}P} Y^*$ as an $\mathcal{O}Gb$-$\mathcal{O}Gb$-bimodule. Multiplying by i on the left and right implies that A is isomorphic to a direct summand of $M \otimes_{\mathcal{O}P} M^*$. Thus (iv) implies (ii), completing the proof. ☐

Proposition 4.5. *Let M be an indecomposable direct summand of the A-B-bimodule U. Then A is isomorphic to a direct summand of the A-A-bimodule $M \otimes_B M^*$ if and only if B is isomorphic to a direct summand of the B-B-bimodule $M^* \otimes_A M$.*

Proof. Suppose that A is isomorphic to a direct summand of $M \otimes_B M^*$, but that B is not isomorphic to a direct summand of $M^* \otimes_A M$. It follows from Lemma 4.3, applied to B, A, V^* instead of A, B, V, respectively, that $M^* \otimes_B M$ is a direct sum of summands of bimodules of the form $B \otimes_{\mathcal{O}Q} B$, with Q running over a family of proper subgroups of P. Thus $M \otimes_B M^* \otimes_A M \otimes_B M^*$ is a direct sum of summands of bimodules of the form $M \otimes_{\mathcal{O}Q} M^*$, with Q running over a family of proper subgroups of P. In particular, we have $(M \otimes_B M^* \otimes_A M \otimes_B M^*)(\Delta P) = \{0\}$. But $A \cong A \otimes_A A$ is a summand of $M \otimes_B M^* \otimes_A M \otimes_B M^*$, hence $(M \otimes_B M^* \otimes_A M \otimes_B M^*)(\Delta P) \neq \{0\}$. This contradiction shows that B is isomorphic to a direct summand of $M^* \otimes_A M$. Exchanging the roles of A and B yields the converse. ☐

In particular, if the equivalent statements in Proposition 4.5 hold, then the algebras A, B are separably equivalent (cf. [20, Definition 3.1]).

Proposition 4.6. *Let M be an indecomposable direct summand of the $\mathcal{O}Gb$-$\mathcal{O}Hc$-bimodule X. The following are equivalent.*

(i) $\mathcal{O}Gb$ is isomorphic to a direct summand of the $\mathcal{O}Gb$-$\mathcal{O}Gb$-bimodule $M \otimes_{\mathcal{O}Hc} M^*$.

(ii) $\mathcal{O}Hc$ is isomorphic to a direct summand of the $\mathcal{O}Hc$-$\mathcal{O}Hc$-bimodule $M^* \otimes_{\mathcal{O}Gb} M$.

(iii) M has vertex ΔP.

If these equivalent conditions hold, then V is an $\mathcal{O}\Delta P$-source of M.

Proof. Set $A = i\mathcal{O}Gi$ and $B = j\mathcal{O}Hj$. The equivalence of (i) and (ii) is a reformulation of 4.5 at the level of block algebras, via the standard Morita equivalences between block algebras and almost source algebras. The bimodule M has a vertex ΔQ contained in ΔP, for some subgroup Q of P. If this vertex is smaller than ΔP, then $(M \otimes_{\mathcal{O}Hc} M^*)(\Delta P) = \{0\}$, so also $(iMj \otimes_B jM^*i)(\Delta P) = \{0\}$. Thus 4.4 implies that $iMj \otimes_B jM^*i$ has no summand isomorphic to B, hence $M \otimes_{\mathcal{O}Hc} M^*$ has no summand isomorphic to $\mathcal{O}Hc$. This shows that (ii) implies (iii). Suppose that ΔP is a vertex of M. Then clearly V is a source of M. By 2.3, M has a vertex source pair (P', V') such that $P' \subseteq P \times P$ and such that V' is a direct summand of iMj as an $\mathcal{O}P'$-module. It follows that as an $\mathcal{O}(P \times P)$-module, iMj has an indecomposable direct summand W with vertex P' and source V'. Green's indecomposability theorem implies that $W \cong \mathrm{Ind}_{P'}^{P \times P}(V')$ is a summand of iMj as an $\mathcal{O}P$-$\mathcal{O}P$-bimodule, hence of $A \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P \times P}(V) \otimes_{\mathcal{O}P} B$. Using the bimodule structure of A and B, it follows that P' is a 'twisted' diagonal subgroup of the form $\{(\varphi(u), \psi(u)) \mid u \in P$ for some φ, $\psi \in \mathrm{Aut}_{\mathcal{F}}(P)$. Since $A \cong {}_{\varphi}A$ as $\mathcal{O}P$-A-bimodules and $B \cong B_{\psi}$ as B-$\mathcal{O}P$-bimodules, it follows that iMj has a direct summand isomorphic to to $\mathrm{Ind}_{\Delta P}^{P \times P}(V')$, and then $V' \cong V$ by the stability of V. But then $iMj \otimes_{\mathcal{O}P} jM^*i$ has a summand isomorphic to $\mathrm{Ind}_{\Delta P}^{P\times P}(V) \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(V^*) \cong \mathrm{Ind}_{\Delta P}^{P\times P}(V \otimes_{\mathcal{O}} V^*)$. Since $V \otimes_{\mathcal{O}} V^*$ has a trivial summand, it follows that $iMj \otimes_{\mathcal{O}P} jM^*i$ has a summand isomorphic to $\mathcal{O}P$, which implies that $(iMj \otimes_{\mathcal{O}P} jM^*i)(\Delta P) \neq \{0\}$. Proposition 4.4 implies that A is isomorphic to a direct summand of $iMj \otimes_B jM^*i$, and hence $\mathcal{O}Gb$ is isomorphic to a direct summand of $M \otimes_{\mathcal{O}Hc} M^*$, completing the proof. □

5 Proof of Theorem 1.1 and of Theorem 1.2

Proof of Theorem 1.1. We use the notation and hypotheses from Theorem 1.1. Since $(M \otimes_B M^*)(\Delta P) \neq \{0\}$, it follows from 4.4 that $M \otimes_B M^* \cong A \oplus X$ for some A-A-bimodule X with the property that every indecomposable direct summand of X is isomorphic to a direct summand of $A \otimes_{\mathcal{O}Q} A$ for some fully $\mathcal{F}$-centralised subgroup Q of P. In what follows we use the canonical isomorphism $M \otimes_B M^* \cong \mathrm{End}_B(M)$ and analogous versions. By 4.2, for any subgroup Q of P we have an injective algebra homomorphism

$$\mathrm{End}_{B^{\mathrm{op}}}(M)(\Delta Q) \to \mathrm{End}_{B(\Delta Q)^{\mathrm{op}}}(M_Q)$$

The left term is isomorphic to $A(\Delta Q) \oplus X(\Delta Q)$. If Q is nontrivial and fully $\mathcal{F}$-centralised, then the right term is isomorphic to $A(\Delta Q)$ by the assumptions on M_Q. This forces $X(\Delta Q) = \{0\}$ for any nontrivial fully $\mathcal{F}$-centralised subgroup Q of P. It follows from 4.3 that X is projective as an A-A-bimodule. Similarly, 4.5 implies that $M^* \otimes_A M \cong B \oplus Y$ for some B-B-bimodule Y, and the same argument with the roles of A and B exchanged shows that Y is projective. □

Proof of Theorem 1.2. By multiplying the involved bimodules with almost source idempotents, it follows using the block algebra versions 3.3 and 4.6 of 4.4 and of 3.2, respectively, that Theorem 1.2 is equivalent to Theorem 1.1. □

Remark 5.1. We sketch a proof of Theorem 1.1 under the additional assumption that the endopermutation $\mathcal{O}P$-module V has an $\mathcal{F}$-stable p-permutation resolution Y_V (cf. [28, §7]). That is, Y_V is a bounded complex of permutation $\mathcal{O}P$-modules such that the complex $Y_V \otimes_{\mathcal{O}} Y_V^*$ is split as a complex of $\mathcal{O}P$-modules with respect to the diagonal action of P, and such that $Y_V \otimes_{\mathcal{O}} Y_V^*$ has homology concentrated in degree zero and isomorphic to $V \otimes_{\mathcal{O}} V^*$. The $\mathcal{F}$-stability means that for any subgroup Q of P and any morphism $\varphi : Q \to P$ in $\mathcal{F}$ the indecomposable summands of $\mathrm{Res}_Q^P(Y_V)$ and $\mathrm{Res}_\varphi(Y_V)$ with vertex Q (as complexes) are isomorphic (this is slightly weaker than the condition stated in [16, Theorem 1.3]). The proof of [16, Theorem 1.3] yields an indecomposable direct summand Y of the complex $A \otimes_{\mathcal{O}P} \mathrm{Ind}_{\Delta P}^{P\times P}(Y_V) \otimes_{\mathcal{O}P} B$ such that $Y \otimes_B Y^*$ is split with homology concentrated in degree zero isomorphic to $M \otimes_B M^*$; similarly for $Y^* \otimes_A Y$. Note that Y is splendid in the sense of [16, 1.10] or [17, 1.1]. It follows from [28, §7.3] that if Q is a fully $\mathcal{F}$-centralised subgroup of P, then $Y(\Delta Q)$ is a bounded complex of $A(\Delta Q)$-$B(\Delta Q)$-bimodules with homology concentrated in a single degree and isomorphic to a bimodule M_Q as in the statement of the Theorem. It follows from [16, Proposition 2.4] or [19, Theorem 9.2] that for any fully $\mathcal{F}$-centralised subgroup Q of P we have $(Y \otimes_B Y^*)(\Delta Q) \cong Y(\Delta Q) \otimes_{B(\Delta Q)} Y(\Delta Q)^*$ and this complex is again split with homology concentrated in degree zero isomorphic to $M_Q \otimes_{B(\Delta Q)} M_Q^*$. Thus if M_Q induces a Morita equivalence, then $A(\Delta Q)\ M_Q \otimes_{B(\Delta Q)} M_Q^*$. Therefore, if M_Q induces a Morita equivalence for all nontrivial fully $\mathcal{F}$-centralised subgroups Q of P, then $Y(\Delta Q)$ induces in particular a derived equivalence for all such Q, and hence, by a result of Rouquier (see [19, Appendix] for a proof) the complex Y induces a stable equivalence. This implies that M induces a stable equivalence, providing thus an alternative proof of Theorem 1.1.

6 Appendix

In the proof of Proposition 4.4 we have made use of [26, Proposition 3.8]. The purpose of this section is to give a proof of a slightly more general result in this direction. We use without further comment the following standard properties of p-permutation modules: if U is an indecomposable $\mathcal{O}G$-module with vertex P and trivial source, then the $kN_G(P)$-module $U(P)$ is the Green correspondent of $k \otimes_{\mathcal{O}} U$, and we have a canonical algebra isomorphism $(\mathrm{End}_{\mathcal{O}}(U))(P) \cong \mathrm{End}_k(U(P))$. Moreover, as a $kN_G(P)/P$-module, $U(P)$ is the multiplicity module of U; in particular, $U(P)$ is projective indecomposable as a $kN_G(P)/P$-module. Any p-permutation kG-module lifts uniquely, up to isomorphism, to a p-permutation $\mathcal{O}G$-module. In particular, the isomorphism class of an indecomposable $\mathcal{O}G$-module U with vertex P and trivial source is uniquely determined by the isomorphism class of the projective indecomposable $kN_G(P)/P$-module $U(P)$. See e. g. [29, §27] for an expository account on p-permutation modules with further references. The following result is well-known (we include a proof for the convenience of the reader):

Proposition 6.1. *Let G be a finite group, P a p-subgroup, U an indecomposable $\mathcal{O}G$-module with vertex P and trivial source $\mathcal{O}$, and let M be an $\mathcal{O}G$-module such that $\mathrm{Res}^G_P(M)$ is a permutation $\mathcal{O}P$-module. Set $N = N_G(P)/P$. Let $\alpha : U \to M$ be a homomorphism of $\mathcal{O}G$-modules. The following are equivalent.*

(i) The $\mathcal{O}G$-homomorphism $\alpha : U \to M$ is split injective.

(ii) The kN-homomorphism $\alpha(P) : U(P) \to M(P)$ is injective.

Proof. The implication (i) ⇒ (ii) is trivial. Suppose that (ii) holds. Then $\alpha(P) : U(P) \to M(P)$ is split injective as a kN-homomorphism because $U(P)$ is projective, hence injective, as a kN-module. Using that $\mathrm{soc}(U(P))$ is simple it follows that M has an indecomposable direct summand M' such that the induced map $\beta(P) : U(P) \to M'(P)$ is still split injective, where β is the composition of α followed by the projection from M onto M'. The Brauer homomorphism applied to the algebra $\mathrm{End}_{\mathcal{O}}(M')$ maps $\mathrm{End}_{\mathcal{O}}(M')^G_P$ onto $(\mathrm{End}_k(M'))(P)^N_1 \cong \mathrm{End}_k(M'(P))^N_1$ (cf. [29, (27.5)]). The summand of $M'(P)$ isomorphic to $U(P)$ corresponds to a primitive idempotent in $\mathrm{End}_k(M'(P))^N_1$, hence lifts to a primitive idempotent in $\mathrm{End}_{\mathcal{O}}(M')^G_P$. Since M' is indecomposable, this idempotent is $\mathrm{Id}_{M'}$, and hence, by Higman's criterion, M' has P as a vertex. But then M' has a trivial source, and so $M'(P)$ is indecomposable as a kN-module, hence isomorphic to $U(P)$. By the Green correspondence this implies $U \cong M'$. Composing β with the inverse of this isomorphism yields an endomorphism γ of U which induces an automorphism on $U(P)$. Since $\mathrm{End}_{\mathcal{O}G}(U)$ is local, this implies that γ is an automorphism of U, and hence that $\beta : U \to M'$ is an isomorphism. It follows that α is split injective, whence the implication (ii) ⇒ (i). □

Proposition 6.2. *Let G be a finite group, P a p-subgroup, U an indecomposable $\mathcal{O}G$-module with vertex P and trivial source, and let M be an $\mathcal{O}G$-module such that $\mathrm{Res}^G_P(M)$ is a permutation $\mathcal{O}P$-module. Set $N = N_G(P)/P$. Suppose that N has a normal p-subgroup Z such that the kN-module $k \otimes_{kZ} U(P)$ is simple and such that $M(P)$ is projective as a kZ-module. Let $\alpha : U \to M$ be a homomorphism of $\mathcal{O}G$-modules. The following are equivalent.*

(i) The $\mathcal{O}G$-homomorphism $\alpha : U \to M$ is split injective.

(ii) There is a nonzero direct summand W of $\mathrm{Res}^N_Z(U(P))$ such that the kZ-homomorphism $\alpha(P)|_W : W \to \mathrm{Res}^N_Z(M(P))$ is injective.

Proof. The implications (i) $\Rightarrow$ (ii) is trivial. Suppose that (ii) holds. Since $U(P)$ is projective indecomposable as a kN-module, it has a simple socle and a simple top, and these are isomorphic. The restriction of $U(P)$ to kZ remains projective, and hence the module $k \otimes_{kZ} U(P)$ has the same dimension as the submodule $U(P)^Z$ of Z-fixed points in $U(P)$. Since Z is normal in N, it follows that $U(P)^Z$ is a kN-submodule of $U(P)$, hence that $U(P)^Z$ contains the simple socle of $U(P)$. Since $k \otimes_{kZ} U(P)$ is assumed to be simple, hence isomorphic to the top and bottom composition factor of $U(P)$, it follows that $U(P)^Z$ is equal to the socle $\mathrm{soc}(U(P))$ of $U(P)$ as a kN-module. By the assumption (ii), the kernel of the map $U(P) \to M(P)$ does not contain $U(P)^Z$. Since the socle of $U(P)$ as a kN-module is simple, this implies that $\alpha(P) : U(P) \to M(P)$ is injective, and hence that α is split injective by Proposition 6.1. This shows the implication (ii) $\Rightarrow$ (i). □

Corollary 6.3 (Puig [26, Proposition 3.8])**.** *Let G be a finite group, b a block of $\mathcal{O}G$, P a defect group of b, and A an interior G-algebra. Suppose that the conjugation action of P on A stabilises an $\mathcal{O}$-basis of A, and that $\mathrm{Br}_{\Delta P}(b) \cdot A(\Delta P) \cdot \mathrm{Br}_{\Delta P}(b)$ is projective as a left or right $kZ(P)$-module. Then the map $\alpha : \mathcal{O}Gb \to A$ induced by the structural homomorphism $G \to A^\times$ is split injective as a homomorphism of $\mathcal{O}Gb$-$\mathcal{O}Gb$-bimodules.*

Proof. After replacing A by $b{\cdot}A{\cdot}b$ we may assume that $A(\Delta P)$ is projective as a left or right $kZ(P)$-module. As an $\mathcal{O}(G \times G)$-module, $\mathcal{O}Gb$ has vertex ΔP and trivial source. By the assumptions, A is a permutation $\mathcal{O}\Delta P$-module. We have $N_{G\times G}(\Delta P) = (C_G(P) \times C_G(P)) \cdot N_{\Delta G}(\Delta P)$. Set $N = N_{G\times G}(\Delta P)/\Delta P$. Denote by Z the image of $Z(P) \times \{1\}$ in N; this is equal to the image of $\{1\} \times Z(P)$, normal in N, and canonically isomorphic to $Z(P)$. Consider the induced map $\alpha(\Delta P) : kC_G(P)\mathrm{Br}_{\Delta P}(b) \to A(\Delta P)$. If e is a block of $kC_G(P)$ occurring in $\mathrm{Br}_{\Delta P}(b)$, then $kC_G(P)/Z(P)\bar{e}$ is a matrix algebra, where $\bar{e}$ is the canonical image of e in $kC_G(P)/Z(P)$. (We use here again our assumption that k is large enough.) Thus $kC_G(P)/Z(P)\bar{e}$ is simple as a module over $k(C_G(P) \times C_G(P))$. Since the blocks e arising in this way are permuted transitively by $N_G(P)$, it follows that $k \otimes_{kZ} kC_G(P)\mathrm{Br}_{\Delta P}(b) \cong kC_G(P)/Z(P)c$ is a simple kN-module, where c is the image of $\mathrm{Br}_{\Delta P}(b)$ in $kC_G(P)/Z(P)$, or equivalently, c is the sum of the $\bar{e}$ as above. By the assumptions, $A(\Delta P)$ is projective as a kZ-module, and hence the obvious composition of algebra homomorphisms $kZ(P) \to kC_G(P)\mathrm{Br}_{\Delta P} \to A(\Delta P)$ is injective. Thus $kC_G(P)\mathrm{Br}_{\Delta P}(b)$ has a summand isomorphic to kZ, as a kZ-module, which is mapped injectively into $A(\Delta P)$ by $\alpha(\Delta P)$. The result follows from the implication (ii) $\Rightarrow$ (i) in Proposition 6.2. □

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