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# Comment on “Non-Hermitian Quantum Mechanics with Minimal Length Uncertainty”<sup>\*</sup>

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**Abstract.** We demonstrate that the recent paper by Jana and Roy entitled “Non-Hermitian quantum mechanics with minimal length uncertainty” [*SIGMA* 5 (2009), 083, 7 pages, arXiv:0908.1755] contains various misconceptions. We compare with an analysis on the same topic carried out previously in our manuscript [arXiv:0907.5354]. In particular, we show that the metric operators computed for the deformed non-Hermitian Swanson models differs in both cases and is inconsistent in the former.

**Key words:** non-Hermitian Hamiltonians; deformed canonical commutation relations; minimal length

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It is known for some time that the deformations of the standard canonical commutation relations between the position operator  $P$  and the momentum operator  $X$  will inevitably lead to a minimal length, that is a bound beyond which the localization of space-time events are no longer possible. In a recent manuscript [1] we investigated various limits of the  $q$ -deformation relations

$$[X, P] = i\hbar q^{f(N)}(\alpha\delta + \beta\gamma) + \frac{i\hbar(q^2 - 1)}{\alpha\delta + \beta\gamma}(\delta\gamma X^2 + \alpha\beta P^2 + i\alpha\delta XP - i\beta\gamma PX),$$

in conjunction with the constraint  $4\alpha\gamma = (q^2 + 1)$ , with  $\alpha, \beta, \gamma, \delta \in \mathbb{R}$  and  $f$  being an arbitrary function of the number operator  $N$ . One may consider various types of Hamiltonian systems, either Hermitian or non-Hermitian, and replace the original standard canonical variables  $(x_0, p_0)$ , obeying  $[x_0, p_0] = i\hbar$ , by  $(X, P)$ . It is crucial to note that even when the undeformed Hamiltonian is Hermitian  $H(x_0, p_0) = H^\dagger(x_0, p_0)$  the deformed Hamiltonian is inevitably non-Hermitian  $H(X, P) \neq H^\dagger(X, P)$  as a consequence of the fact that  $X$  and/or  $P$  are no longer Hermitian. Of course one may also deform Hamiltonians, which are already non-Hermitian when undeformed  $H(x_0, p_0) \neq H^\dagger(x_0, p_0)$ . In both cases a proper quantum mechanical description requires the re-definition of the metric to compensate for the introduction of non-Hermitian variables and in the latter an additional change due to the fact that the Hamiltonian was non-Hermitian in the first place.

In a certain limit, as specified in [1],  $X$  and  $P$  allow for a well-known representation of the form  $X = (1 + \tau p_0^2)x_0$  and  $P = p_0$ , which in momentum space, i.e.  $x_0 = i\hbar\partial_{p_0}$ , corresponds to the one used by Jana and Roy [2], up to an irrelevant additional term  $i\hbar\tilde{\gamma}P$ . (Whenever constants with the same name but different meanings occur in [2] and [1] we dress the former with a tilde.)

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The additional term can simply be gauged away and has no physical significance. Jana and Roy have studied the non-Hermitian displaced harmonic oscillator and the Swanson model. As we have previously also investigated the latter in [1], we shall comment on the differences. The conventions in [2] are

$$H_{\text{JR}}(a, a^\dagger) = \omega a^\dagger a + \lambda a^2 + \tilde{\delta}(a^\dagger)^2 + \frac{\omega}{2}$$

with  $\lambda \neq \tilde{\delta} \in \mathbb{R}$  and  $a = (P - i\omega X)/\sqrt{2m\hbar\omega}$ ,  $a^\dagger = (P + i\omega X)/\sqrt{2m\hbar\omega}$ , whereas in [1] we used

$$H_{\text{BF}}(X, P) = \frac{P^2}{2m} + \frac{m\omega^2}{2}X^2 + i\mu\{X, P\}$$

with  $\mu \in \mathbb{R}$  as a starting point. Setting  $\hbar = m = 1$  it is easy to see that the models coincide when  $\lambda = -\tilde{\delta}$  and  $\mu = \tilde{\delta} - \lambda$ . The Hamiltonians exhibit a “twofold” non-Hermiticity, one resulting from the fact that when  $\lambda \neq \tilde{\delta}$  even the undeformed Hamiltonian is non-Hermitian and the other resulting from the replacement of the Hermitian variables  $(x_0, p_0)$  by  $(X, P)$ . The factor of the metric operator to compensate for the non-Hermiticity of  $X$  coincides in both cases, but the factor which is required due to the non-Hermitian nature of the undeformed case differs in both cases

$$\rho_{\text{BF}} = e^{2\mu P^2} \quad \text{and} \quad \rho_{\text{JR}} = (1 + \tau P^2)^{\frac{\mu}{\omega^2 \tau}}.$$

We have made the above identifications such that  $H_{\text{JR}}(a, a^\dagger) = H_{\text{BF}}(X, P)$  and replaced the deformation parameter  $\beta$  used in [2] by  $\tau$  employed in [1]. It is well known that when given only a non-Hermitian Hamiltonian, the metric operator can not be uniquely determined. However, as argued in [1] with the specification of the observable  $X$ , which coincides in [2] and [1], the outcome is unique and we can therefore directly compare  $\rho_{\text{BF}}$  and  $\rho_{\text{JR}}$ . The limit  $\tau \rightarrow 0$  reduces the deformed Hamiltonian  $H_{\text{JR}} = H_{\text{BF}}$  to the standard Swanson Hamiltonian, such that  $\rho_{\text{JR}}$  and  $\rho_{\text{BF}}$  should acquire the form of a previously constructed metric operator. This is indeed the case for  $\rho_{\text{BF}}$ , but not for  $\rho_{\text{JR}}$ . In fact it is unclear how to carry out this limit for  $\rho_{\text{JR}}$  and we therefore conclude that the metric  $\rho_{\text{JR}}$  is incorrect.

## References

- [1] Bagchi B., Fring A., Minimal length in quantum mechanics and non-Hermitian Hamiltonian systems, arXiv:0907.5354.
- [2] Jana T.K., Roy P., Non-Hermitian quantum mechanics with minimal length uncertainty, *SIGMA* **5** (2009), 083, 7 pages, arXiv:0908.1755.