From real fields to complex Calogero particles

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Abstract: We provide a novel procedure to obtain complex $\mathcal{PT}$-symmetric multi-particle Calogero systems. Instead of extending or deforming real Calogero systems, we explore here the possibilities for complex systems to arise from real nonlinear field equations. We exemplify this procedure for the Boussinesq equation and demonstrate how singularities in real valued wave solutions can be interpreted as N complex particles scattering amongst each other. We analyze this phenomenon in more detail for the two and three particle case. Particular attention is paid to the implemention of $\mathcal{PT}$-symmetry for the complex multi-particle systems. New complex $\mathcal{PT}$-symmetric Calogero systems together with their classical solutions are derived.

1. Introduction

The analytic continuation of real physical systems into the complex plane is a principle which has turned out to be very fruitful, since many new features can be revealed in this manner which might otherwise be undetected. A famous and already classical example, proposed more than half a century ago, is for instance Heisenberg's programme of the analytic S-matrix [1]. Here our main concern will be complex multi-particle Calogero systems, in particular those exhibiting $\mathcal{PT}$-symmetry [2].

Quantum systems are said to be $\mathcal{PT}$-symmetric when they are invariant under simultaneous parity $\mathcal{P}$ and time reversal $\mathcal{T}$ transformations. When the Hamiltonian, not necessarily Hermitian, exhibits this symmetry, i.e. $[H, \mathcal{PT}] = 0$, and moreover when all wave-functions are also invariant under such an operation this property is referred to as unbroken $\mathcal{PT}$-symmetry. The virtue of this feature is that it is a sufficient property to guarantee the spectrum of the Hamiltonian H to be real. The underlying mechanisms responsible for this are by now well understood [3, 4, 5, 6] and may be formulated alternatively in terms of pseudo/quasi-Hermiticity; for definitions see for instance [7] and references therein.

There are two fundamentally different possibilities to view complex systems: One may either regard the complexified version just as a broader framework, as in the spirit of the analytic S-matrix, and restrict to the real case in order to describe the underlying physics or alternatively one may try to give a direct physical meaning to the complex models.

With the latter motivation in mind complex $\mathcal{PT}$-symmetric Calogero systems have been introduced and studied recently [8, 9, 10, 11, 12]. The hope for a direct physical interpretation stems from the fact that unbroken $\mathcal{PT}$-symmetry will guarantee their eigenspectra to be real and allows for a consistent quantum mechanical description, i.e. such systems constitute well defined quantum systems which have been overlooked up to now. Nonetheless, so far any such proposal lacks a direct physical meaning and the complexifications are generally introduced in a rather ad hoc manner. Here our main purpose is to demonstrate that various complex Calogero models appear rather naturally from *real* valued nonlinear field equations and thus we provide a well defined physical origin for these systems.

The solutions for the real Calogero systems were found in the reverse order when compared to the usual way progress is made, i.e. the quantum theory was solved before the classical one. Calogero solved first the quantized one-dimensional three-body problem with pairwise inverse square interaction [13] and subsequently constructed the ground state of the N-body generalization [14] described by the Hamiltonian

$$H_C = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2}\sum_{i\neq j}^{N} \frac{g}{(x_i - x_j)^2}, \tag{1.1}$$

with $g \in \mathbb{R}$ being the coupling constant. Marchioro [15, 16] investigated thereafter the classical analogue of these models obtaining a solution to which we will appeal below. The integrability of these classical counterparts was established later by Moser [17], using a Lax pair consisting of matrices L, M, with entries

$$L_{ij} = p_i \delta_{ij} + \frac{\imath\sqrt{g}}{x_i - x_j}(1-\delta_{ij}), \tag{1.2}$$

$$M_{ij} = \sum_{k\neq i}^{N} \frac{\imath\sqrt{g}}{(x_i - x_k)^2}\delta_{ij} - \frac{\imath\sqrt{g}}{(x_i - x_j)^2}(1-\delta_{ij}), \tag{1.3}$$

constructed in such a manner that the Lax equation

$$\frac{dL}{dt} + [M, L] = 0 \tag{1.4}$$

becomes equivalent to the Calogero equations of motion,

$$\ddot{x}_i = \sum_{j\neq i}^{N} \frac{2g}{(x_i - x_j)^3}. \tag{1.5}$$

We use the notation $\imath \equiv \sqrt{-1}$ throughout the manuscript and abbreviate time derivatives as usual by $dx_i/dt = \dot{x}_i$ and $d^2x_i/dt^2 = \ddot{x}_i$. Integrability follows in the standard fashion by noting that all quantities of the for $I_n = \mathrm{tr}(L^n)\ /n$ are integrals of motion and conserved in time by construction.

Calogero systems have become very important in theoretical physics, having been explored in various contexts ranging from condensed matter physics to cosmology, e.g. [18, 19, 20]. The main focus of our interest here are the complex extensions which have been studied recently in connection with $\mathcal{PT}$-symmetric models [8, 9, 10, 11, 12].

The idea of exploiting $\mathcal{PT}$-symmetry in order to obtain models with real energies can be adapted to classical systems as well and has been used to formulate various complex extensions of nonlinear wave equations, such as the Korteweg-de Vries (KdV) and Burgers equations [21, 22, 23, 24]. In the classical case the reality of the energy is ensured in an even simpler way, as in that case the $\mathcal{PT}$-symmetry of the Hamiltonian is sufficient. Remarkably these systems allow for the existence of solitons and compacton solutions [25, 26].

Here we shall explore $\mathcal{PT}$-symmetry in a context where the complex extensions or deformations do not need to be imposed artificially, but instead we investigate whether this symmetry is already naturally present in the system, albeit hidden. To achieve this goal we exploit the fact that nonlinear equations, such as Benjamin-Ono and Boussinesq, can be associated to Calogero particle systems. We explore these connections and are then naturally led to complex $\mathcal{PT}$-symmetric Calogero systems.

In the next section we shall demonstrate how a complex one-dimensional $\mathcal{PT}$-symmetric Calogero system is embedded in a real solitonic solution of the Benjamin-Ono wave equation and how constrained $\mathcal{PT}$-symmetric Calogero particles emerge from real solutions of the Boussinesq equation. Thereafter we construct the explicit solution of the three-particle configuration with the aforementioned constraint and show that the resulting motion, unlike in the unconstrained situation, cannot be restricted to the real line. We shall also establish that a subclass of this constrained Calogero motion is related to the poles in the solution of different nonlinear KdV-like differential equation. The relation of these complex particles with previously obtained $\mathcal{PT}$-symmetric complex extensions of the Calogero model [27] is discussed in section 5, where we demonstrate that they are different from those proposed here. Our conclusions are drawn in section 6.

2. Poles of nonlinear waves as interacting particles

The assumption of rational real valued functions as multi-soliton solutions of nonlinear wave equations was studied more than three decades ago by various authors, see e.g. [28]. We take some of these findings as a setting for the problem at hand. In order to illustrate the key idea we present what is probably the simplest scenario in which corpuscular objects emerge as poles of nonlinear waves, namely in the Burgers equation

$$u_t + \alpha u_{xx} + \beta (u^2)_x = 0. \tag{2.1}$$

Assuming that this equation admits rational solutions of the form

$$u(x,t) = \frac{2\alpha}{\beta} \sum_{i=1}^{N} \frac{1}{x - x_i(t)}, \tag{2.2}$$

it is straightforward to see that surprisingly the N poles interact with each other through a Coulombic inverse square force

$$\ddot{x}_i(t) = -2\alpha \sum_{j \neq i}^{N} \frac{1}{[x_i(t) - x_j(t)]^2}. \tag{2.3}$$

This pole structure survives even after making modifications in the ansatz for the wave equation, although the nature of the interaction may change. By acting on the second derivative in Burgers equation with a Hilbert transform

$$\hat{H}u(x) = \frac{1}{\pi} PV \int_{-\infty}^{+\infty} dz \frac{u(z)}{z - x}, \tag{2.4}$$

we obtain the Benjamin-Ono equation [29, 30]

$$u_t + \alpha \hat{H} u_{xx} + \beta (u^2)_x = 0. \tag{2.5}$$

As shown in [31], the ansatz proposed for the equation above which will allow for similar conclusions has a slightly different form,

$$u(x,t) = \frac{\alpha}{\beta} \sum_{k=1}^{N} \left(\frac{\imath}{x - z_k(t)} - \frac{\imath}{x - z_k^*(t)} \right) \tag{2.6}$$

being, however, still a real valued solution with the only restriction that the complex poles satisfy complex Calogero equations of motion

$$\ddot{z}_k(t) = 8\alpha^2 \sum_{k \neq j}^{N} \frac{1}{(z_k(t) - z_j(t))^3}. \tag{2.7}$$

Note that there is a difference in the power laws appearing in (2.3) and (2.7), but more importantly that equation (2.2) has real poles, whereas (2.6) has complex ones. We stress once more that the field $u(x,t)$ is real in both cases. Hence, this viewpoint provides a nontrivial mechanism which leads to particle systems defined in the complex plane.

Interesting observations of this kind can be made for other nonlinear equations as well, but not always will the ansatz work directly, that is without any further requirements as in the previous cases. In some situations additional conditions might be necessary. Examples of nonlinear integrable wave equations for which such type of constraints occur are the KdV and the Boussinesq equations,

$$u_t + \left(\alpha u_{xx} + \beta u^2\right)_x = 0 \qquad \text{and} \qquad u_{tt} + \left(\alpha u_{xx} + \beta u^2 - \gamma u\right)_{xx} = 0, \tag{2.8}$$

respectively. For both of these equations one can have "N-soliton" solutions[1] of the form

$$u(x,t) = -6 \frac{\alpha}{\beta} \sum_{k=1}^{N} \frac{1}{(x - x_k(t))^2}, \tag{2.9}$$

[1] Soliton is to be understood here in a very loose sense in analogy to the Painlevé type ideology of indistructable poles. In the strict sense not all solution possess the N-soliton solution characteristic, that is moving with a preserved shape and regaining it after scattering though each other.

as long as in each case two sets of constraints are satisfied

$$\dot{x}_k(t) = -12\alpha \sum_{j\neq k}^{N} (x_k(t) - x_j(t))^{-2} \;, \qquad 0 = \sum_{j\neq k}^{N} (x_k(t) - x_j(t))^{-3}, \tag{2.10}$$

and

$$\ddot{x}_k(t) = -24\alpha \sum_{j\neq k}^{N} (x_k(t) - x_j(t))^{-3} \;, \qquad \dot{x}_k(t)^2 = 12\alpha \sum_{j\neq k}^{N} (x_k(t) - x_j(t))^{-2} + \gamma, \tag{2.11}$$

respectively. Naturally these constraints might be incompatible or admit no solution at all, in which case (2.9) would of course not constitute a solution for the wave equations (2.8). Notice that if the $x_k(t)$ are real or come in complex conjugate pairs the solution (2.9) for the corresponding wave equations is still real.

Airault, McKean and Moser provided a general criterium, which allows us to view these equations from an entirely different perspective, namely to regard them as constrained multi-particle systems [28]:

Given a multi-particle Hamiltonian $H(x_1, ..., x_N, \dot{x}_1, ..., \dot{x}_N)$ with flow $x_i = \partial H/\partial \dot{x}_i$ and $\dot{x}_i = -\partial H/\partial x_i$ together with conserved charges I_n in involution with H, i.e. vanishing Poisson brackets $\{H, I_n\} = 0$, then the locus of grad$(I_n) = 0$ is invariant with respect to time evolution. Thus it is permitted to restrict the flow to that locus provided it is not empty.

Taking the Hamiltonian to be the Calogero Hamiltonian H_C it is well known that one may construct the corresponding conserved quantities from the Calogero Lax operator (1.2) as mentioned from $I_n = \mathrm{tr}(L^n)/n$. The first of these charges is just the total momentum, the next is the Hamltonian followed by non trivial ones

$$I_1 = \sum_{i=1}^{N} p_i, \qquad I_2 = H_C(g), \qquad I_3 = \frac{1}{3}\sum_{i=1}^{N} p_i^3 + g \sum_{i\neq j}^{N} \frac{p_i + p_j}{(x_i - x_j)^2} \;, \ldots \tag{2.12}$$

According to the above mentioned criterium we may therefore consider an I_3-flow restricted to the locus defined by $\mathrm{grad}(I_2) = 0$ or an I_2-flow subject to the constraint $\mathrm{grad}(I_3 - \gamma I_1) = 0$. Remarkably it turns out that the former viewpoint corresponds exactly to the set of equations (2.10), whereas the latter to (2.11) when we identify the coupling constant as $g = -12\alpha$. Thus the solutions of the Boussinesq equation are related to the constrained Calogero Hamiltonian flow, whereas the KdV soliton solutions arise from an I_3-flow subject to constraining equations derived from the Calogero Hamiltonian.

As our main focus is on the Calogero Hamiltonian flow and its possible complexifications we shall concentrate on possible solutions of the systems (2.11) and investigate whether these type of equations allow for nontrivial solutions or whether they are empty. It will be instructive to commence by looking first at the unconstrained system. The classical solutions of a two-particle Calogero problem are given by

$$x_{1,2}(t) = 2R(t) \pm \sqrt{\frac{g}{E} + 4E(t - t_0)^2}, \tag{2.13}$$

with E, t_0 being initial conditions and $\dot{R}(t) = 0$ the centre of mass velocity. Relaxing this condition by allowing boosts will only shifts the energy scale since the total momentum is conserved. Depending therefore on the initial conditions we may have either real or complex solutions.

The three particle model, i.e. taking $N = 3$ in (1.1), is slightly more complicated. Marchioro [15] found the general solution by expressing the dynamical variables in terms of Jacobi relative coordinates R, X, Y in polar form via the transformations $R(t) = (x_1(t)+x_2(t)+x_3(t))/3$, $X(t) = r(t)\sin\phi(t) = (x_1(t)-x_2(t))/\sqrt{2}$ and $Y(t) = r(t)\cos\phi(t) = (x_1(t)+x_2(t)-2x_3(t))/\sqrt{6}$. The variables may then be separated and the resulting equations are solved by

$$x_{1,2}(t) = R(t) + \frac{1}{\sqrt{6}}r(t)\cos\phi(t) \pm \frac{1}{\sqrt{2}}r(t)\sin\phi(t), \tag{2.14}$$

$$x_3(t) = R(t) - \frac{2}{\sqrt{6}}r(t)\cos\phi(t), \tag{2.15}$$

where

$$R(t) = R_0 + t\tilde{R}_0 \tag{2.16}$$

$$r(t) = \sqrt{\frac{B^2}{E} + 2E(t-t_0)^2}, \tag{2.17}$$

$$\phi(t) = \frac{1}{3}\cos^{-1}\left\{\varphi_0 \sin\left[\sin^{-1}(\varphi_0\cos 3\phi_0) - 3\tan^{-1}\left(\frac{\sqrt{2}E}{B}(t-t_0)\right)\right]\right\}. \tag{2.18}$$

The solutions involve 7 free parameters: The total energy E, the angular momentum type constant of motion B, the integration constants t_0, ϕ_0, R_0, $\tilde{R}_0$ and the coupling constant g, with the abbreviation $\varphi_0 = \sqrt{1 - 9g/2B^2}$. We note that, depending on the choice of these parameters, both real and complex solutions are admissible, a feature which might not hold for the Calogero system restricted to an invariant submanifold.

Let us now elaborate further on the connection between the field equations and the particle system and restrict the general solution (2.14)-(2.16) by switching on the additional constraints in (2.11) and subsequently study the effect on the soliton solutions of the nonlinear wave equation. Notice that the second constraint in (2.11) can be viewed as setting the difference between the kinetic and potential energy of each particle to a constant. Adding all of these equations we obtain $H_C = N\gamma/2$, which provides a direct interpretation of the constant γ in the Boussinesq equation as being proportional to the total energy of the Calogero model.

3. The motion of Boussinesq singularities

The two particle system, i.e. $N = 2$, is evidently the simplest I_2-Calogero flow constrained with $\mathrm{grad}(I_3 - \gamma I_1) = 0$ as specified in (2.11). The solution for this system was already provided in [28],

$$x_{1,2}(t) = \kappa \pm \sqrt{\gamma(t-\tilde{\kappa})^2 - 3\alpha/\gamma}, \tag{3.1}$$

with $\kappa, \tilde{\kappa}$ taken to be real constants. In fact this solution is not very different from the unconstrained motion shown in the previous section (2.13). The restricted one may be obtained via an identification between the coupling constant and the parameter in the Boussinesq equation as $\kappa = 2R(t)$, $E = \gamma/4$, $\tilde{\kappa} = t_0$ and $g = -3\alpha/4$. The two soliton solution for the Boussinesq equation (2.9) then acquires the form

$$u(x,t) = -12\frac{\alpha}{\beta}\gamma\frac{\gamma(x-\kappa)^2 + \gamma^2(t-\tilde{\kappa})^2 - 3\alpha}{[\gamma(x-\kappa)^2 - \gamma^2(t-\tilde{\kappa})^2 + 3\alpha]^2}, \tag{3.2}$$

which, in principle, is still real-valued when keeping the constants to be real. When inspecting (3.2) it is easy to see that the two singularities repel each other on the x-axis as time evolves, thus mimicking a repulsive scattering process. However, we may change the overall behaviour substantially when we allow the integration constants to be complex, such that the singularities become regularized. In that case we observe a typical solitonic scattering behaviour, i.e. two wave packets keeping their overall shape while evolving in time and when passing though each other regaining their shape when the scattering process is finished, albeit with complex amplitude. A special type of complexification occurs when we take the integration constants $\kappa, \tilde{\kappa}$ to be purely imaginary, in which case (3.2) becomes a solution for the $\mathcal{PT}$-symmetrically constrained Boussinesq equation, with $\mathcal{PT} : x \to -x, t \to -t, u \to u$. We depict the described behaviour in figure 1 for some special choices of the parameters.

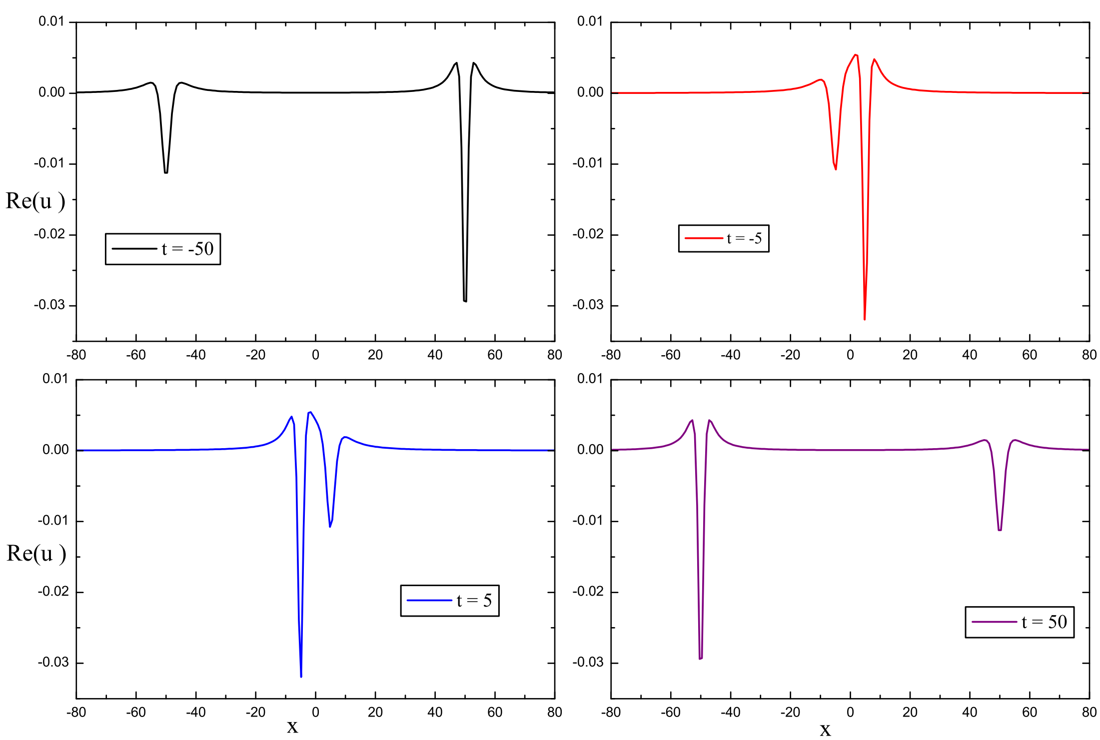


Figure 1: Time evolution of the real part of the constraint Boussinesq two soliton solution (3.2) with $\kappa = \imath 2.3$, $\tilde{\kappa} = -\imath 0.6$, $\alpha = -1/6$,$\beta = 5/8$ and γ= 1.

For larger numbers of particles the solutions have not been investigated and it is not even clear whether the locus of interest is empty or not. Let us therefore embark on solving this problem systematically. Unfortunately we can not simply imitate Marchioro's method of separating variables as the additional constraints will destroy this possibility. However, we notice that (2.11) can be represented in a different way more suited for our purposes. Differentiating the second set of equations in (2.11) and making use of the first one, we arrive at the set of expressions

$$\sum_{k\neq j}^{N} \frac{(\dot{x}_k(t)+\dot{x}_j(t))}{(x_k(t)-x_j(t))^3} = 0, \tag{3.3}$$

which are therefore consistency equations of the other two.

We now focus on the case $N = 3$. Inspired by the general solution of the unconstrained three particle solution (2.14) and (2.15), we adopt an ansatz of the general form

$$x_{1,2}(t) = A_0(t) + A_1(t) \pm A_2(t), \tag{3.4}$$

$$x_3(t) = A_0(t) + \lambda A_1(t), \tag{3.5}$$

with $A_i(t)$, $i = 0, 1, 2$ being some unknown functions and λ a free constant parameter. We note that $\lambda \neq 1$, since otherwise the three coordinates could be expressed in terms of only two linearly independent functions, $A_0(t) + A_1(t)$ and $A_2(t)$, and we would not able to express the normal mode like functions $A_i(t)$ in terms of the original coordinates $x_i(t)$. Calogero's choice, $\lambda = -2$, in equation (2.15), allows an elegant map of Cartesian coordinates into Jacobi's relative coordinates, but other possibilities might be more convenient in the present situation. Here we keep λ to be free for the time being.

Substituting this ansatz for the $x_i(t)$ into the second set of equations in (2.11) and using the compatibility equation (3.3), we are led to six coupled first order differential equations for the unknown functions $A_0(t), A_1(t), A_2(t)$

$$\frac{(\dot{A}_0(t)+\lambda\dot{A}_1(t))^2-\gamma}{2g} + \frac{1}{2A_+(t)^2} + \frac{1}{2A_-(t)^2} = 0, \tag{3.6}$$

$$\frac{(\dot{A}_0(t)+\dot{A}_1(t)\pm\dot{A}_2(t))^2-\gamma}{2g} + \frac{1}{8A_2(t)^2} + \frac{1}{2A_\mp(t)^2} = 0, \tag{3.7}$$

$$\frac{2\dot{A}_0(t)+(\lambda+1)\dot{A}_1(t)+\dot{A}_2(t)}{A_-(t)^3} - \frac{2\dot{A}_0(t)+(\lambda+1)\dot{A}_1(t)-\dot{A}_2(t)}{A_+(t)^3} = 0, \tag{3.8}$$

$$\frac{\dot{A}_0(t)+\dot{A}_1(t)}{4A_2(t)^3} + \frac{2\dot{A}_0(t)+(\lambda+1)\dot{A}_1(t)\pm\dot{A}_2(t)}{A_\mp(t)^3} = 0. \tag{3.9}$$

For convenience we made the identifications $A_\pm(t) = A_2(t) \pm (\lambda-1)A_1(t)$.

From the latter set of equations above, (3.8) and (3.9), we can now eliminate two of the first derivatives together with the use of the conservation of momentum. Depending on the choice, the remaining $\dot{A}_i(t)$ are eliminated with the help of the first three equations (3.6) and (3.7). The two equations left then become multiples of each other depending only on $A_1(t)$ and $A_2(t)$. Subsequently we can express $A_2(t)$, and consequently $\dot{A}_0(t), \dot{A}_1(t)$, in terms of $A_1(t)$ as the only unknown quantity. In this manner we arrive at

$$A_2(t) = \frac{\sqrt{-g - 4\gamma(\lambda-1)^2 A_1(t)^2}}{2\sqrt{3\gamma}}, \tag{3.10}$$

$$\dot{A}_0(t) = \sqrt{\gamma} + \frac{3g\sqrt{\gamma}(2+\lambda)}{(\lambda-1)[g+16\gamma(\lambda-1)^2 A_1(t)^2]}, \tag{3.11}$$

$$\dot{A}_1(t) = \frac{9g\sqrt{\gamma}}{(1-\lambda)[g+16\gamma(\lambda-1)^2 A_1(t)^2]}, \tag{3.12}$$

with $g = -12\alpha$. This means that once we have solved the differential equation (3.12) for $A_1(t)$ the complete solution is determined up to the integration of $\dot{A}_0(t)$ in (3.11) and a simple substitution in (3.10). In other words we have reduced the problem to solve the set of coupled nonlinear equations (2.11) to solving one first order nonlinear equation.

Let us now make a comment on the number of free parameters, that is integration constants, occurring in this solution. In the original formulation of the problem we have started with 3 second order differential equations, so that we expect to have 6 integration constants for the determination of x_1, x_2 and x_3. However, together with the additional 3 constraining equations this number is reduced to 3 free parameters. Finally we can invoke the conservation of total momentum from (2.12), which yields $3\ddot{A}_0(t) + (\lambda+2)\ddot{A}_1(t) = 0$ and we are left with only 2 free parameters. We choose them here to be the two arbitrary constants attributed to the integration of $\dot{A}_0(t)$ in (3.11) and $\dot{A}_1(t)$ in (3.12), respectively.

In turn this also means that, without loss of generality, we may freely choose the constant λ introduced in (3.5). Indeed, keeping it generic we observe that the solutions for the $A_i(t)$ do not depend on it despite its explicit presence in the equations (3.10), (3.11) and (3.12). The most convenient choice is to take $\lambda = -2$ as in that case the equations simplify considerably.

Let us now solve (3.10), (3.11) and (3.12) and substitute the result into the original expressions (3.4) and (3.5) in order to see how the particles behave. We find

$$x_{1,2}(t) = c_0 + \sqrt{\gamma}t + \frac{1}{12}\left(\frac{g}{\xi(t)} - \frac{\xi(t)}{\gamma}\right) \pm \frac{\imath}{4\sqrt{3}}\left(\frac{g}{\xi(t)} + \frac{\xi(t)}{\gamma}\right), \tag{3.13}$$

$$x_3(t) = c_0 + \sqrt{\gamma}t - \frac{1}{6}\left(\frac{g}{\xi(t)} - \frac{\xi(t)}{\gamma}\right), \tag{3.14}$$

where for convenience we introduced the abbreviation

$$\xi(t) = \left[-54\gamma^2(\sqrt{\gamma}gt + c_1) + \sqrt{g^3\gamma^3 + [54\gamma^2(\sqrt{\gamma}gt + c_1)]^2}\right]^{\frac{1}{3}}. \tag{3.15}$$

The above mentioned two freely choosable constants of integration are denoted by c_0 and c_1. As in the two particle case, we may once again compare this solution with the unconstrained one in (2.14), (2.15) when considering the Jacobi relative coordinates

$$R(t) = c_0 + t\sqrt{\gamma}, \quad r^2(t) = -\frac{g}{6\gamma} \quad \text{and} \quad \tan\phi(t) = i\frac{g\gamma + \xi^2(t)}{g\gamma - \xi^2(t)}. \tag{3.16}$$

We observe that the solution is now constrained to a circle in the XY-plane with real radius when $g\gamma \in \mathbb{R}^-$. The values for $\phi(t)$ lead to the most dramatic consequence, namely that the particles are now forced to move in the complex plane, unlike as in unconstrained Calogero system or the $N = 2$ case where all options are open.

Interestingly, despite the poles being complex, we may still have real wave solutions for the Boussinesq equation. Provided that $\xi(t), \gamma, g, c_0, c_1 \in \mathbb{R}$ the pole $x_3(t)$ is obviously real whereas $x_1(t)$ and $x_2(t)$ are complex conjugate to each other, such that the ansatz (2.9) yields a real solution

$$\begin{aligned} u(x,t) = & -\frac{6\alpha}{\beta} \frac{1}{\left(\varphi - \frac{1}{6}\left(\frac{g}{\xi(t)} - \frac{\xi(t)}{\gamma}\right)\right)^2} + \\ & + \frac{216\alpha}{\beta}\gamma^2\xi(t)^2 \left[\frac{g^2\gamma^2 - 12g\gamma^2\varphi\xi(t) - 4\gamma(18\gamma\varphi^2 - g)\xi(t)^2 + 12\gamma\varphi\xi(t)^3 + \xi(t)^4}{(g^2\gamma^2 + 6g\gamma^2\varphi\xi(t) + \gamma(36\gamma\varphi^2 + g)\xi(t)^2 - 6\gamma\varphi\xi(t)^3 + \xi(t)^4)^2}\right] \end{aligned} \tag{3.17}$$

with $\varphi \equiv c_0 + \sqrt{\gamma}t - x$.

Due to the non-meromorphic form of $\xi(t)$ it is not straightforward to determine how the solutions transforms under a $\mathcal{PT}$-transformation. Nonetheless, the symmetry of the relevant combinations appearing in (3.13) and (3.14) can be analyzed well for $c_0, c_1 \in i\mathbb{R}$ and $\gamma > 0$. In that case the time reversal acts as $\mathcal{T}: \left(\frac{g}{\xi(t)} \pm \frac{\xi(t)}{\gamma}\right) \to \pm\left(\frac{g}{\xi(t)} \pm \frac{\xi(t)}{\gamma}\right)$, which implies $\mathcal{PT}: x_i(t) \to -x_i(t)$ for $i = 1, 2, 3$. Thus, the solutions to the constrained problem are not only complex, but in addition they can also be $\mathcal{PT}$-symmetric for certain choices of the constants involved.

4. Different types of constraints in nonlinear wave equations

It is clear from the above that the class of complex ($\mathcal{PT}$-symmetric) multi-particle systems which might arise from nonlinear wave equations could be much larger. We shall demonstrate this by investigating one further simple example which was previously studied in [33] and also refer to the literature [32] for additional examples. One very easy nonlinear wave equations which, because of its simplicity, serves as a very instructive toy model is

$$u_t + u_x + u^2 = 0. \tag{4.1}$$

We may now proceed as above and seek for a suitable ansatz to solve this equation, possibly leading to some constraining equations in form a multi-particle systems. Making therefore a similar ansatz for $u(x,t)$ as in (2.6) or (2.9) we take

$$u(x,t) = \sum_{i=1}^{N} \frac{1 - \dot{z}_i(t)}{x - z_i(t)}. \tag{4.2}$$

It is then easy to verify that this solves the nonlinear equation (4.1) provided the $z_i(t)$ obey the constraints

$$\ddot{z}_i(t) = 2\sum_{j \neq i}^{N} \frac{(1 - \dot{z}_i(t))(1 - \dot{z}_j(t))}{z_i(t) - z_j(t)}. \tag{4.3}$$

We could now proceed as in the previous section and try to solve this differential equation, but in this case we may appeal to the general solution already provided in [33], where it was found that

$$u(x,t) = \frac{f(x-t)}{1+tf(x-t)}. \tag{4.4}$$

solves (4.1) for any arbitrary function $f(x)$ with initial condition $u(x,0) = f(x)$. Comparing (4.4) and (4.2) it is clear that the $z_i(t)$ can be interpreted as the poles in (4.4), which becomes singular when $x \to z_i(t) = t + f_i^{-1}(-1/t)$, with $i \in \{1, N\}$ labeling the different branches which could result when assuming that f is invertible but not necessarily injectively. Making now the concrete choice for f to be rational of the form

$$f(x) = \sum_{i=1}^{N} \frac{a_i}{\alpha_i - x}, \qquad \text{with } \alpha_i, a_i \in \mathbb{C}, \tag{4.5}$$

we can determine the poles concretely by inverting this function. First of all we obtain from the initial condition that

$$z_i(0) = \alpha_i, \qquad \dot{z}_i(0) = 1 + a_i \qquad \text{and} \qquad \ddot{z}_i(0) = \sum_{j \neq i}^{N} \frac{2a_i a_j}{\alpha_i - \alpha_j}. \tag{4.6}$$

The first two conditions simply follow from the comparison of (4.4) and (4.2), but also follow, as so does the latter, from taking the appropriate limit in (4.5). We note that the total momentum is conserved for this system $\sum_{i=1}^{N} \dot{z}_i(t) = N + \sum_{i=1}^{N} a_i$.

Let us now see how to obtain explicit expressions for the poles. Inverting (4.5) for $N = 2$ it is easy to find that for generic values of t the poles take on the form

$$z_{1,2}(t) = t + \frac{\bar{\alpha}_{12}}{2} + \frac{\bar{a}_{12}}{2}t \pm \frac{1}{2}\sqrt{\alpha_{12}^2 + 2\alpha_{12}a_{12}t + \bar{a}_{12}^2 t^2}, \tag{4.7}$$

where we introduced the notation $\alpha_{ij} = \alpha_i - \alpha_j$, $\bar{\alpha}_{ij} = \alpha_i + \alpha_j$ and analogously for $\alpha \to a$. We note that in the case $N = 2$ the constraint (4.3) can be changed into two-particle Calogero systems constraint with the identification $g = a_1 a_2 \alpha_{12}^2$.

Next we consider the case $N = 3$ for which we obtain the solution

$$z_1(t) = t - \frac{a(t)}{3} + s_+(t) + s_-(t) \tag{4.8}$$

$$z_{2,3}(t) = t - \frac{a(t)}{3} - \frac{1}{2}\left[s_+(t) + s_-(t)\right] \pm i\frac{\sqrt{3}}{2}\left[s_+(t) - s_-(t)\right], \tag{4.9}$$

where we abbreviated

$$s_\pm(t) = \left[r(t) \pm \sqrt{r^2(t) + q^3(t)}\right]^{1/3}, \tag{4.10}$$

$$r(t) = \frac{9a(t)b(t) - 27c(t) - 2a^3(t)}{54}, \quad q(t) = \frac{3b(t) - a^2(t)}{9}, \tag{4.11}$$

$$a(t) = -a_1 - \alpha_2 - \alpha_3 - t(a_1 + a_2 + a_3), \tag{4.12}$$

$$b(t) = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3 + t[a_1\bar{\alpha}_{23} + a_2\bar{\alpha}_{31} + a_3\bar{\alpha}_{21}], \tag{4.13}$$

$$c(t) = -t(a_1\alpha_2\alpha_3 + a_2\alpha_3\alpha_1 + a_3\alpha_1\alpha_2) - \alpha_1\alpha_2\alpha_3. \tag{4.14}$$

In terms of Jacobi's relative coordinates this becomes

$$R(t) = t - \frac{1}{3}a(t), \quad r^2(t) = 6s_+(t)s_-(t) \quad \text{and} \quad \tan\phi(t) = i\frac{s_-(t) - s_+(t)}{s_-(t) + s_+(t)}, \tag{4.15}$$

which makes a direct comparison with the constrained Calogero system (3.16) straightforward. As the system (4.8), (4.9) involves more free parameters than the constrained Calogero system (3.16), we expect to observe some relations between the parameters α_i, a_i to produce the right number of free parameters. Indeed, we find that for

$$a_i = -\frac{g}{2}\prod_{j\neq i}(\alpha_i - \alpha_j)^{-2} \tag{4.16}$$

and the additional constraints

$$c_0 = \frac{1}{3}\sum_{i=1}^{3}\alpha_i, \quad c_1 = \frac{2}{27}\prod_{\substack{1\leq j<k\leq 3 \\ j,k\neq l}}(\alpha_j + \alpha_k - 2\alpha_l), \quad g = 4\sum_{\substack{i=1 \\ i<j}}^{3}\alpha_i\alpha_j - \alpha_i^2, \quad \gamma = 1. \tag{4.17}$$

the two systems become identical. Thus we have obtained an identical singularity structure for two quite different nonlinear wave equations.

5. Complex Calogero models and $\mathcal{PT}$-deformations

We have demonstrated in section 3 that the solutions of the constrained Calogero models are intrinsically of a complex nature. As there have been various proposals before in the literature suggesting complex Calogero systems in form of $\mathcal{PT}$-symmetric deformations, we shall now compare them with the above outcome. We will argue that the deformations presented here are new and different to those suggested up to now.

The simplest $\mathcal{PT}$-deformation of any model is obtained just by adding a $\mathcal{PT}$-invariant term to the original Hamiltonian. For a many-body situation, this was proposed for the first time in the framework of A_n Calogero models by introducing the Hamiltonian [8]

$$H(q,p) = H_C(q,p) + \sum_{i\neq j}^{N}\frac{i\tilde{g}p_i}{(x_i - x_j)^2}. \tag{5.1}$$

In [10] it was shown that this simply corresponds to shifting the momenta in the standard Calogero Hamiltonian together with a re-definition of the coupling constant, which means the above construction is certainly quite different from the proposal (5.1).

The second type of deformation [12] consists of replacing directly the set of ℓ-dynamical variables $q = \{q_1, \ldots, q_\ell\}$ and their conjugate momenta $p = \{p_1, \ldots, p_\ell\}$ by means of a deformation map $\varepsilon : (q,p) \to (\tilde{q}, \tilde{p})$, whereby the map is constructed in such a way that the original invariance under the Weyl group $\mathcal{W}$ is replaced by an invariance under a $\mathcal{PT}$-symmetrically deformed version of the Weyl group $\mathcal{W}^{\mathcal{PT}}$. In terms of roots the map is defined by replacing each root α by a deformed counterpart $\tilde{\alpha}$ as $\varepsilon : \alpha \to \tilde{\alpha}$, whereby the precise form of the deformation ensures the invariance under $\mathcal{W}^{\mathcal{PT}}$ as specified in

[12]. Expanding the momenta as $p = \sum_i \kappa_i \alpha_i$, with $\kappa_i \in \mathbb{R}$, this means for the Calogero Hamiltonian

$$\varepsilon : H_C(q,p) \to H_{\mathcal{PT}}(\tilde{q},\tilde{p}) = \frac{1}{2}\sum_{i,j} \kappa_i \kappa_j \tilde{\alpha}_i \tilde{\alpha}_j + \frac{1}{2}\sum_{\tilde{\alpha}\in\tilde{\Delta}} \frac{g}{(\tilde{\alpha}\cdot q)^2}. \tag{5.2}$$

In order to find the concrete forms for $\tilde{q}$ and $\tilde{p}$ we need to be more specific about the algebras involved. Let us therefore examine the models based on the rank 2 algebras A_2, B_2 and G_2. Depending on the dimensionality of the representation for the simple roots, we obtain either a two or a three particle systems and may therefore compare with the solutions found in the previous sections. In all cases the deformations of the simple roots α_1 and α_2 take on the general form

$$\tilde{\alpha}_1(\varepsilon) = R(\varepsilon)\alpha_1 + iI(\varepsilon)K_{12}\lambda_2, \qquad \text{and} \qquad \tilde{\alpha}_2(\varepsilon) = R(\varepsilon)\alpha_2 - iI(\varepsilon)K_{21}\lambda_1, \tag{5.3}$$

with λ_1, λ_2 being fundamental weights obeying $2\lambda_i \cdot \alpha_j/\alpha_j^2 = \delta_{ij}$, the functions $R(\varepsilon)$, $I(\varepsilon)$ satisfy $\lim_{\varepsilon\to 0} R(\epsilon) = 1$, $\lim_{\varepsilon\to 0} I(\epsilon) = 0$ and $K_{ij} = 2\alpha_i \cdot \alpha_j/\alpha_j^2$ are the entries of the Cartan matrix. Let us now take the following two dimensional representations for the simple roots and fundamental weights

$$\begin{array}{llll} A_2: \alpha_1 = (1,-\sqrt{3}), & \alpha_2 = (1,\sqrt{3}), & \lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, & \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2, \\ B_2: \alpha_1 = (1,-1), & \alpha_2 = (0,1), & \lambda_1 = \alpha_1 + \alpha_2, & \lambda_2 = \frac{1}{2}\alpha_1 + \alpha_2, \\ G_2: \alpha_1 = (-\sqrt{\frac{3}{2}},\sqrt{\frac{1}{2}}), & \alpha_2 = (\sqrt{6},0), & \lambda_1 = 3\alpha_1 + \alpha_2, & \lambda_2 = 3\alpha_1 + 2\alpha_2. \end{array} \tag{5.4}$$

We easily verify that this reproduces the correct entries for the Cartan matrices A_2 : $K_{11} = K_{22} = 2$, $K_{12} = K_{21} = -1$, B_2 : $K_{11} = K_{22} = 2$, $K_{12}/2 = K_{21} = -1$ and G_2 : $K_{11} = K_{22} = 2$, $K_{12} = K_{21}/3 = -1$. Having constructed the deformed roots we compute next the deformed conjugate momenta and coordinates. In the representations (5.4) the kinetic energy term changes just by an overall factor as

$$\tilde{p}^2 = \left[R(\varepsilon) - \nu_{\mathbf{g}}^2 I(\varepsilon)\right] p^2 \qquad \text{with } \nu_{A_2} = 1/\sqrt{3}, \nu_{B_2} = 1, \nu_{G_2} = -\sqrt{3}. \tag{5.5}$$

The specific choice $R(\varepsilon) = \cosh\varepsilon$ and $I(\varepsilon) = \nu_{\mathbf{g}}^{-2}\sinh\varepsilon$, used in [12], keeps the kinetic energy term completely invariant, in the sense that the original and deformed momenta are identical. The dual canonical coordinates $\tilde{q}$ are computed from

$$\tilde{\alpha}\cdot q = \tilde{q}\cdot\alpha, \qquad \alpha, q \in \mathbb{R},\ \tilde{\alpha},\tilde{q} \in \mathbb{R}\oplus i\mathbb{R}. \tag{5.6}$$

We find

$$\tilde{q}_1 = R(\varepsilon)q_1 + i\nu_{\mathbf{g}} I(\varepsilon) q_2, \qquad \text{and} \qquad \tilde{q}_2 = R(\varepsilon)q_2 - i\nu_{\mathbf{g}} I(\varepsilon) q_1. \tag{5.7}$$

We will now argue that (5.7) is always different from the constrained two particle solution of the Calogero model (3.1). In order to see this we recall first of all that for the solution to be $\mathcal{PT}$-symmetric we require $\kappa, \tilde{\kappa} \in i\mathbb{R}$. Equating now the sums $x_1 + x_2 = \tilde{q}_1 + \tilde{q}_2$ we conclude that $q_1(t) = -q_2(t) = -\kappa/\nu_{\mathbf{g}} I(\varepsilon) = \text{const}$. Next we compute $(x_1 - x_2)^2$, which yields

$$\gamma(t-\tilde{\kappa})^2 - 3\alpha/\gamma = 2R^2(\varepsilon)q_1^2(t). \tag{5.8}$$

This equation is inconsistent as the right had side is real and time independent, a condition which can not be achieved for the left hand side. This proves our statement that the deformation method suggested here is genuinely different from the proposal in [12] in the two particle case.

Keeping the deformed roots to be of the form (5.3), the three dimensional repesentations for the simple roots

$$A_2 : \alpha_1 = (1, -1, 0), \quad \alpha_2 = (0, 1, -1), \qquad G_2 : \alpha_1 = (1, -1, 0), \quad \alpha_2 = (-2, 1, 1), \tag{5.9}$$

yield the same result for the kinetic energy term (5.5), but obviously have to produce different dual canonical coordinates $\tilde{q}$. In this case we obtain

$$\tilde{q}_1 = R(\varepsilon)q_1 + i\zeta_{\mathbf{g}} I(\varepsilon)(q_2 - q_3), \tag{5.10}$$

$$\tilde{q}_2 = R(\varepsilon)q_2 + i\zeta_{\mathbf{g}} I(\varepsilon)(q_3 - q_1), \tag{5.11}$$

$$\tilde{q}_3 = R(\varepsilon)q_3 + i\zeta_{\mathbf{g}} I(\varepsilon)(q_1 - q_2), \tag{5.12}$$

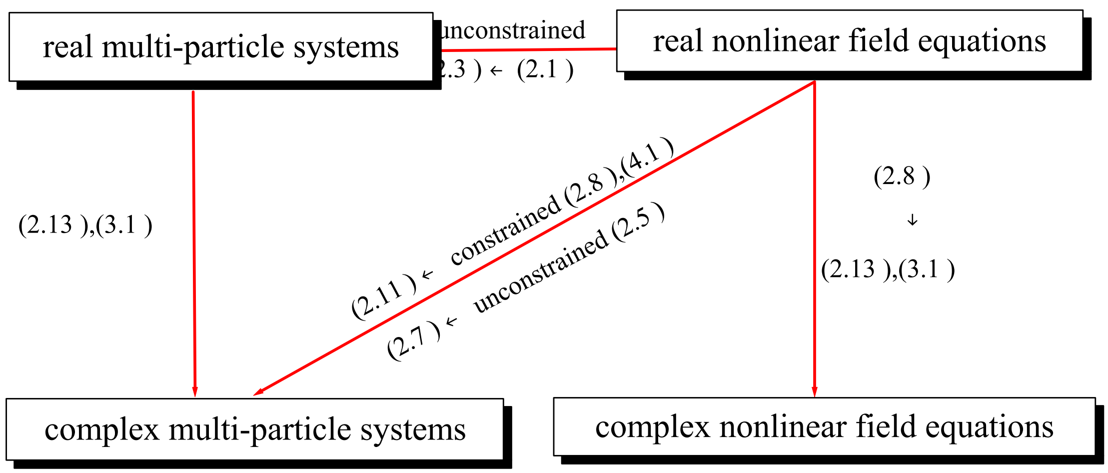
where $\zeta_{A_2} = 1/3$ and $\zeta_{G_2} = -1$. Equating these solutions with (3.13), (3.14) and solving the resulting equations for the q_i with $i = 1, 2, 3$, it is easy to argue that the q_i can not be made real, which establishes the claim that the solutions are also intrisically different for the three particle model.

6. Conclusions

Hitherto there have been two different types of procedures to complexify Calogero models. As explained in section 5 one may either add $\mathcal{PT}$-symmetric terms to the original Hamiltonian [8], which have turned out to be simple shifts in the momenta [10] or one may directly deform the root system on which the formulation of the model is based [12]. In all these approaches the deformation is introduced in a rather ad hoc fashion. In this paper we have provided a novel mechanism, which has real solutions of physically motivated nonlinear wave equations as the starting point. The constrained motion of some solitonic solutions of these models then led to complex Calogero models, some of them being $\mathcal{PT}$-symmetric.

There are some obvious open problems left. For instance it would naturally be very interesting to study systems involving larger numbers of particles, which would correspond to higher soliton solutions for the nonlinear wave equations. Clearly the study of different types of wave equations, such as the KdV etc and their $\mathcal{PT}$-symmetrically deformed versions would complete the understanding.

Our analysis is schematically summarized in figure 2.



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