Integrable models from $\mathcal{PT}$-symmetric deformations

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ABSTRACT: We address the question of whether integrable models allow for $\mathcal{PT}$-symmetric deformations which preserve their intgrability. For this purpose we carry out the Painlevé test for $\mathcal{PT}$-symmetric deformations of Burgers and the Korteweg-De Vries equation. We find that the former equation allows for infinitely many deformations which pass the Painlevé test. For a specific deformation we prove the convergence of the Painlevé expansion and thus establish the Painlevé property for these models, which are therefore thought to be integrable. The Korteweg-De Vries equation does not allow for deformations which pass the Painlevé test in complete generality, but we are able to construct a defective Painlevé expansion.

1. Introduction

Classical as well as quantum mechanical models, which are invariant under a simultaneous parity transformation $\mathcal{P} : x \to -x$ and time reversal $\mathcal{T} : t \to -t$, can be deformed in a controlled manner to produce new $\mathcal{PT}$-symmetric theories [1, 2, 3, 4, 5, 6, 7, 8]. The crucial feature of these models is that the $\mathcal{PT}$-symmetry can be utilized to guarantee the reality of the energy spectra, which is due to the fact that its operator realization is a specific example of an anti-linear operator [9]. In contrast to standard textbook wisdom, this means when the systems are Hamiltonian, they are non-dissipative despite being non-Hermitian. An important question to answer in this context is whether it is possible to deform models in a symmetry preserving manner. Regarding supersymmetry, it was recently shown [10] that this is indeed possible. Here we will focus on the question of whether this is also accomplishable with regard to the symmetry underlying integrability. In other words, do integrable $\mathcal{PT}$-symmetric models allow for deformations which do not destroy the integrability? A positive answer to this question will naturally lead to new integrable models. For some cases partial results already exist [11, 12, 13, 14, 15, 16, 7, 8, 10]. Here we will focus on two prototype models of integrable systems, the Burgers equation and the Korteweg-deVries (KdV) equation. We will carry out the Painlevé test for $\mathcal{PT}$-symmetric deformations of these models, establish thereafter in some cases the Painlevé property and draw conclusions about their integrability.

As there exist various notions and definitions about integrability, the Painlevé test, the Painlevé property, etc, let us briefly indicate which ones we are going to adopt in this manuscript. To start with, there is clearly no doubt that integrability is an extremely desirable property to have in a physical system, as it usually leads to exact solvability rather than to mere perturbative results. In the context of 1+1 dimensional quantum field theories the notion of integrability is usually used synonymously to the factorization of the scattering matrix, where the latter can be achieved simply by making use of one non-trivial charge [17]. Unlike as in most scenarios when one compares quantum and classical theories, the latter appear to be more complicated in this particular regard. In classical systems the definitions of integrability are much more varied and non-uniform. A common notion is so-called Liouville integrability, which assumes for a system with N degrees of freedom the existence of N analytic single valued global integrals of motion in involution. The equations of motion are then separable and exact solutions can be obtained, at least in principle. Focussing on differential equations, as we do in this paper, one calls them integrable when, given a sufficient amount of initial data, they are solvable via an associated linear problem. The problem with all these definitions is that one does not know a priori whether a system is integrable or not without having computed all integrals of motion, mapped the problem to a linear one or actually solved the equations of motion. A general method to identify integrable models before this, often very difficult, task is completed does not exist. The closest one may get to such a method is to check whether the system possesses the Painlevé property. One can then assume that the Painlevé property implies integrability in the above specified sense, albeit this connection is not rigorously proven. To make matters worse, there exist even definitions which include the notion of integrability into the definition of the Painlevé property [18].

The concept of the Painlevé property can be traced back more than a century to the original investigations of Painlevé et al. [19], who set out to construct new functions from the solutions of ordinary differential equations (ODE). The notion of a function implies immediately that the solutions one is seeking ought to be single valued, which leads to a natural definition: An ODE whose (general) solutions have no movable[1] critical[2] singularities is said to possess the (generalized) Painlevé property [18, 20, 21]. The classification of possible solutions to this problem can be organised into equivalence classes obtained from linear fractional (Möbius) transformations and has been completed only to some degree. It is proven that all linear ODE posses the Painlevé property, first order algebraic nonlinear equations lead to Weierstrass functions and second order algebraic nonlinear equations lead to the famous six Painlevé transcendental functions. The classification of algebraic ODEs with Painlevé property of order greater than two is still an open problem, albeit some partial results exist [22, 23, 24].

The situation is somewhat less structured for partial differential equations (PDE). Extrapolating the previous notions one defines: A PDE whose solutions have no mov-

[1]Movable means that the solution depends on the initial values.

[2]A critical singularity is multivalued in its neighbourhood.

able critical singularities near any noncharacteristic[3] manifold is said to possess the Painlevé property. In general this is difficult to establish, however, there exists a more applicable necessary, albeit not sufficient, condition for a PDE to possess the Painlevé property, which was developed by Weiss, Tabor and Carneval [25] and is usually referred to as the Painlevé test. This method is extremely practical and can be carried out in a very systematic fashion. Roughly speaking the main idea is that one expands the solution for a PDE (or ODE) in a power series starting with some single valued leading order terms. In case the series can be computed and involves as many free parameters as the order of the PDE then it is said that the PDE passes the Painlevé test. In order to extrapolate from the Painlevé test to the Painlevé property one should also establish the convergence of the series, which, however, has been carried out only in very rare cases.

For our purposes the relation between the Painlevé property (test) and integrability is the most interesting. Ablowitz, Ramani and Segur [26] conjectured almost thirty years ago: Any ODE which arises as a reduction of an integrable PDE, possibly accompanied by a variable transformation, possesses the Painlevé property. To this day this conjecture has not been proven rigorously, but is supported by a huge amount of evidence. On one hand one has verified this property for almost all known integrable PDEs [25, 27, 28, 6, 29] and in turn, which is more impressive, one has also used it to identify new integrable ODEs [30, 31]. The latter is what we hope to achieve in this manuscript.

In summary, we will adopt here the logic that a PDE which passes the Painlevé test and whose Painlevé expansion converges also possesses the Painlevé property. We take this as a very good indication that the system is integrable.

We briefly explain the deformation procedure in section 2 and carry out the analysis for Burgers and the KdV equation in subsection 2.1 and 2.2, respectively. We state our conclusions in section 3.

2. $\mathcal{PT}$-symmetrically deformed integrable models

Given a $\mathcal{PT}$-symmetric PDE as a starting point, we adopt the deformation principle of [7, 8, 10] to define new $\mathcal{PT}$-symmetric extensions of this model by replacing ordinary derivatives by their deformed counterparts

$$\partial_x f(x) \to -i(if_x)^\varepsilon =: f_{x;\varepsilon} \qquad \text{with } \varepsilon \in \mathbb{R}. \tag{2.1}$$

Clearly the original $\mathcal{PT}$-symmetry is preserved. In general the deformations will continue real derivatives into the complex plane, unless $\varepsilon = 2n-1$ with $n \in \mathbb{Z}$. We do not make use here of the possibility to deform also the higher derivatives via the deformation (2.1), i.e. replacing for instance $\partial_x^2 f(x)$ by $f_{x;\varepsilon} \circ f_{x;\varepsilon}$, but simply define them as successive action of ordinary derivatives on one deformation only

$$\partial_x^n f(x) \to i^{\varepsilon-1}\partial_x^{n-1}(f_x)^\varepsilon = \partial_x^{n-1} f_{x;\varepsilon} =: f_{nx;\varepsilon}. \tag{2.2}$$

This deformation preserves the order of the PDE. We can now employ this prescription to introduce new $\mathcal{PT}$-symmetric models.

[3] On a characteristic manifold we can not apply Cauchy's existence theorem and therefore we do not have a unique solution for a given initial condition.

2.1 Painlevé test for the $\mathcal{PT}$-symmetrically deformed Burgers' equation

Burgers' equation is extensively studied in fluid dynamics and integrable systems, as it constitutes the simplest PDE involving a nonlinear as well as a dispersion term

$$u_t + uu_x = \sigma u_{xx}. \tag{2.3}$$

Obviously equation (2.3) remains invariant under the transformation $t \to -t, x \to -x$, $u \to u$ and $\sigma \to -\sigma$. Taking the constant σ to be purely imaginary, i.e. $\sigma \in i\mathbb{R}$, this invariance can be interpreted as a $\mathcal{PT}$-symmetry, which was also noted recently by Yan [32]. A similar complex, albeit not $\mathcal{PT}$-symmetric, version of Burgers' equations plays an important role in the study of two-dimensional Yang-Mills theory with an SU(N) gauge group [33, 34]. The models considered in [33, 34] become $\mathcal{PT}$-symmetric after a Wick rotation, i.e $t \to it$.

Let us now consider the $\mathcal{PT}$-symmetrically deformed Burgers' equation

$$u_t + uu_{x;\varepsilon} = i\kappa u_{xx;\mu} \qquad \text{with} \quad \kappa, \varepsilon, \mu \in \mathbb{R}, \tag{2.4}$$

where for the time being we allow two different deformation parameters ε and μ.

Our first objective is to test whether this set of equations passes the Painlevé test. Following the method proposed in [25], we therefore assume that the solution of (2.4) acquires the general form of the Painlevé expansion

$$u(x,t) = \sum_{k=0}^{\infty} \lambda_k(x,t)\phi(x,t)^{k+\alpha}. \tag{2.5}$$

Here $\alpha \in \mathbb{Z}_-$ is the leading order singularity in the limit $\phi(x,t) = (\varphi(x,t) - \varphi_0) \to 0$, with $\varphi(x,t)$ being an arbitrary analytic function characterizing the singular manifold, φ_0 being an arbitrary complex constant which can be utilized to move the singularity mimicking the initial condition and the $\lambda_k(x,t)$ are analytic functions, which have to be computed recursively.

2.1.1 Leading order terms

As a starting point we need to determine all possible values for α by substituting the first term of the expansion (2.5), that is $u(x,t) \to \lambda_0(x,t)\phi(x,t)^{\alpha}$, into (2.4) and reading off the leading orders. For the three terms in (2.4) they are $u_t \sim \phi^{\alpha-1}$, $uu_{x;\varepsilon} \sim \phi^{\alpha+\alpha\varepsilon-\varepsilon}$ and $u_{xx;\mu} \sim \phi^{\alpha\mu-\mu-1}$. In order for a non-trivial solution to exist the last two terms have to match each other in powers of ϕ, which immediately yields $\alpha = (\varepsilon-\mu-1)/(\varepsilon-\mu+1) \in \mathbb{Z}_-$. Thus $\alpha = -1$ and $\varepsilon = \mu$ is the only possible solution. This means we observe from the very onset of the procedure that only the models in which all x-derivatives are deformed with the same deformation parameter have a chance to pass the Painlevé test. Therefore we can conclude already at this stage that one of the deformations of (2.3) studied in [32], i.e. $\varepsilon = 1$ and μ generic, can not pass the Painlevé test. Hence they do not possess the Painlevé property and are therefore not integrable.

2.1.2 Recurrence relations

Substituting next the Painlevé expansion (2.5) for $u(x,t)$ with $\alpha = -1$ into (2.4) with $\varepsilon = \mu$ gives rise to the recursion relations for the λ_k by identifying powers in $\phi(x,t)$. We find

$$\begin{array}{lr} \text{at order } -(2\varepsilon+1): & \lambda_0 + i2\varepsilon\kappa\phi_x = 0, \\ \text{at order } -2\varepsilon: & \phi_t\delta_{\varepsilon,1} + \lambda_1\phi_x - i\kappa\varepsilon\phi_{xx} = 0, \\ \text{at order } -(2\varepsilon-1): & \partial_x(\phi_t\delta_{\varepsilon,1} + \lambda_1\phi_x - i\kappa\varepsilon\phi_{xx}) = 0, \end{array} \tag{2.6}$$

such that

$$\lambda_0 = -i2\varepsilon\kappa\phi_x, \qquad \lambda_1 = (i\varepsilon\kappa\phi_{xx} - \phi_t\delta_{\varepsilon,1})/\phi_x \qquad \text{and} \qquad \lambda_2 \text{ is arbitrary.} \tag{2.7}$$

This means the number of free parameters, i.e. φ_0 and λ_2, at our disposal equals the order of the PDE, such that (2.4) passes the Painlevé test provided the series (2.5) makes sense and we can determine all λ_j with $j > 2$. To compute the remaining λ_j we need to isolate them on one side of the equation and those involving λ_k with $k < j$ on the other side. We expect to find some recursion relations of the form

$$g(j, \phi_t, \phi_x, \phi_{xx}, \ldots)\lambda_j = f(\lambda_{j-1}, \lambda_{j-2}, \ldots, \lambda_1, \lambda_0, \phi_t, \phi_x, \phi_{xx}, \ldots), \tag{2.8}$$

with g and f being some functions characteristic for the system under consideration. We will not present here these recursion relations for generic values of ε as they are rather cumbersome and we shall only present the first non-trivial deformation, that is the case $\varepsilon = 2$.

2.1.3 Resonances

For some particular values of j, say $j = r_1, \ldots, r_\ell$, we might encounter that the function g in (2.8) vanishes. Clearly this leads to an inconsistency and a failure of the Painlevé test unless f also vanishes. In case this scenario occurs, it implies that the recursion relation (2.8) does not fix λ_j and the compatibility conditions $g = f = 0$ lead to ℓ so-called resonances λ_{r_i} for $i = 1, \ldots \ell$. When $\ell + 1$ is equal to the order of the differential equation we can in principle produce a general solution which allows for all possible initial values. It might turn out that some missing free parameters are located before the start of the expansion (2.5), i.e. at $j < 0$, so-called negative resonances which can be treated following arguments developed in [35]. When not enough additional free parameters exist to match the order of the differential equation, the series is still of Painlevé type and is called defective.

It is straightforward to determine all possible resonances by following a standard argument. The first term in the expansion (2.5) gives rise to the leading order singularity which needs to be cancelled by some yet unknown term in the expansion. Let us carry out the calculation for Burgers equation. Using the expression for λ_0 from (2.7) and making the ansatz

$$\tilde{u}(x,t) = -2i\varepsilon\kappa\frac{\phi_x}{\phi} + \vartheta\phi^{r-1}, \tag{2.9}$$

we can compute all possible values of r for which ϑ becomes a free parameter. Substituting $\tilde{u}(x,t)$ into (2.4) and reading off the terms of the highest order, i.e. $\phi^{-2\varepsilon-1+r}$, we find the necessary condition

$$i2^{\varepsilon-1}\varepsilon^{\varepsilon}\vartheta(r+1)(r-2)\kappa^{\varepsilon}\phi_x^{2\varepsilon}=0, \tag{2.10}$$

for a resonance to exist. This yields precisely to two resonances, one at $r=2$, corresponding to the third equation in (2.6), and the so-called universal resonance at $r=-1$. This means also at higher order we can not encounter any inconsistencies or possible breakdowns of the Painlevé test for any value of the deformation parameter ε.

2.1.4 From the Painlevé test via Painlevé property to integrability

Once it is established that a PDE passes the Painlevé test one needs to be cautious about the conclusions one can draw as it is only a necessary but not sufficient condition for the Painlevé property. In case one can also guarantee the convergence of the series the PDE possess the Painlevé property, which is taken as very strong evidence for the equation to be integrable. This step has only been carried out rigorously in very rare cases, e.g. in [36, 37]. Here we establish the convergence for one particular deformation.

2.1.5 The $\varepsilon=2$ deformation

As already mentioned, the details of the recursion relation for generic values of ε are rather lengthy and we shall therefore only present the case $\varepsilon=2$ explicitly. In that case the deformed Burgers' equation (2.4) becomes

$$u_t+iuu_x^2+2\kappa u_x u_{xx}=0 \tag{2.11}$$

The substitution of the Painlevé expansion (2.5) into (2.11) and the subsequent matching of equal powers in ϕ then yields the recursion relation

$$\begin{aligned}
&i\lambda_0\phi_x^2\left\{\lambda_j\left[(2j-3)\lambda_0-2i((j-5)j+4)\kappa\phi_x\right]+2\lambda_0\left(\lambda_0+2i\kappa\phi_x\right)\delta_{0,j}\right\}= \qquad (2.12)\\
&+\sum_{n,m=1}^{j}\left\{\lambda_{j-m-n-2}\lambda_{m,x}\lambda_{n;x}+(m-1)\lambda_m\phi_x\left[(n-1)\lambda_{j-m-n}\lambda_n\phi_x+2\lambda_{j-m-n-1}\lambda_{n;x}\right]\right\}\\
&+\sum_{n=1}^{j-1}\{2\lambda_{0,x}\left[(n-1)\lambda_{j-n-1}\lambda_n\phi_x+\lambda_{j-n-2}\lambda_{n;x}\right]-2\lambda_0\phi_x\left[(n-1)\lambda_{j-n}\lambda_n\phi_x+\lambda_{j-n-1}\lambda_{n;x}\right]\\
&\quad-2i\kappa\left\{\lambda_{j-n,x}\left[\lambda_{n-3;xx}+(n-3)\left((n-2)\lambda_{n-1}\phi_x^2+2\lambda_{n-2,x}\phi_x+\lambda_{n-2}\phi_{xx}\right)\right]\right.\\
&\quad\left.\left.+\ (j-n-1)\lambda_{j-n}\phi_x\left[\lambda_{n-2;xx}+(n-2)\left((n-1)\lambda_n\phi_x^2+2\lambda_{n-1,x}\phi_x+\lambda_{n-1}\phi_{\mathrm{xx}}\right)\right]\right\}\right\}\\
&+\ 2\lambda_{0,x}\left[(j-5)j+6\right]\kappa\lambda_{j-1}\phi_x^2+\lambda_{j-2}\left[2(j-3)\kappa\phi_{xx}+i\lambda_{0,x}\right]\\
&-\ 2\lambda_0\phi_x\left\{\lambda_{j-1}\left[(j-2)\kappa\phi_{xx}+i\lambda_{0;x}\right]+\kappa\left[\lambda_{j-2;xx}+2(j-2)\phi_x\lambda_{j-1;x}\right]\right\}\\
&+\ (j-4)\lambda_{j-3}\phi_t+\lambda_{j-4;t}+2\kappa\lambda_{0,x}\left[\lambda_{j-3;xx}+2(j-3)\phi_x\lambda_{j-2;x}\right],
\end{aligned}$$

which is indeed of the general form (2.8). Having brought all λ_j with $j>k$ to the left hand side of (2.12), we may now successively determine the λ_j to any desired order. Starting with the lowest value $j=0$ the equation (2.12) reduces to

$$\lambda_0^2\phi_x^2\left(\lambda_0+i4\kappa\phi_x\right)=0, \tag{2.13}$$

which leads to $\lambda_0 = -i4\kappa\phi_x$ and thus simply reproduces the expression in (2.7) for $\varepsilon = 2$. For $j = 1$ the equation (2.12) simplifies to

$$-\lambda_0^2\lambda_1\phi_x^2 = 2\lambda_0\phi_x\left[i\kappa\lambda_0\phi_{xx} + (\lambda_0 + i4\kappa\phi_x)\lambda_{0;x}\right], \tag{2.14}$$

such that $\lambda_1 = i2\kappa\phi_{xx}/\phi_x$, which coincides with (2.7) for $\varepsilon = 2$. When $j = 2$ the equation acquires the form

$$\begin{aligned}\lambda_0\lambda_2\phi_x^2(\lambda_0 + 4\sigma\phi_x) = {}& 2\phi_x\lambda_{1;x}\lambda_0^2 - \lambda_{0;x}^2\lambda_0 + 2\lambda_1\phi_x\lambda_{0;x} - 2i\kappa\phi_{xx}\lambda_{0;x} - 4i\kappa\phi_x\lambda_{0;x}^2 \\ & -2i\kappa\phi_x(\lambda_{0,xx} - 2\phi_x\lambda_{1;x})\lambda_0.\end{aligned} \tag{2.15}$$

It is evident that the left hand side vanishes identically and upon substitution of the values for λ_0 and λ_1. We can verify that this also holds for the right hand side of (2.15), thus leading to the first resonance at level 2 and therefore to an arbitrary parameter λ_2. One may now continue in this fashion to compute the expansion to any finite order, but before we embark on this task we make a few further simplification.

As the singularity has to be a noncharacteristic analytic movable singularity manifold, we employ the implicit function theorem and make a further assumption about the specific form of $\lambda_k(x,t) = \lambda_k(t)$ and $\phi(x,t) = x - \xi(t)$, with $\xi(t)$ being an arbitrary function. Then the equation (2.12) simplifies to a much more transparent form

$$\begin{aligned}8\kappa^2\left(8\kappa\delta_{0,j} + i(j-2)(j+1)\lambda_j(t)\right) = {}& \sum_{n,m=1}^{j} i(1-m)(n-1)\lambda_m(t)\lambda_{j-m-n}(t)\lambda_n(t) \qquad (2.16)\\ + \sum_{n=1}^{j-1}\left[2\kappa(n-1)\left(n^2 - n - j(n-2) + 2\right)\lambda_{j-n}(t)\lambda_n(t)\right] & + (j-4)\lambda_{j-3}(t)\xi'(t) - \lambda'_{j-4}(t).\end{aligned}$$

Solving this equation recursively leads to the Painlevé expansion

$$u(x,t) = -\frac{4i\kappa}{\phi} + \lambda_2\phi + \frac{\xi'}{8\kappa}\phi^2 - \frac{i\lambda_2^2}{20\kappa}\phi^3 - \frac{i\lambda_2\xi'}{96\kappa^2}\phi^4 + \mathcal{O}(\phi^5). \tag{2.17}$$

Clearly we can use (2.16) to extend this expansion to any desired order. For the ordinary Burgers equations, i.e. $\varepsilon = 1$, there exist a simple choice for the free parameters, which terminates the expansion, such that one may generate Bäcklund and Cole-Hopf transformations in a very natural way. Unfortunately (2.17) does not allow an obvious choice of this form. Taking for instance $\lambda_2 = 0$ yields the expansion

$$\begin{aligned}u(x,t) = {}& -\frac{4i\kappa}{\phi} + \frac{\xi'\phi^2}{2^3\kappa} - \frac{i\xi'^2\phi^5}{7\times 2^8\kappa^3} + \frac{i\xi''\phi^6}{5\times 2^9\kappa^3} - \frac{\xi'^3\phi^8}{35\times 2^{13}\kappa^5} - \frac{23\xi'\xi''\phi^9}{385\times 2^{13}\kappa^5} - \frac{\xi^{(3)}\phi^{10}}{135\times 2^{14}\kappa^5} \\ & + \frac{19i\xi'^4\phi^{11}}{3185\times 2^{18}\kappa^7} - \frac{51i\xi'^2\xi''\phi^{12}}{385\times 2^{19}\kappa^7} - \frac{i\left(43641\xi''^2 + 16460\xi'\xi^{(3)}\right)\phi^{13}}{779625\times 2^{20}\kappa^7} + \mathcal{O}(\phi^{14}).\end{aligned} \tag{2.18}$$

Being even more specific and assuming a travelling wave solution, the general form of the movable singularity is $\xi(t) = \omega t$, which gives

$$\begin{aligned} u(x,t) &= -\frac{4i\kappa}{\phi} + \frac{\omega\phi^2}{2^3\kappa} - \frac{i\omega^2\phi^5}{7\times 2^8\kappa^3} - \frac{\omega^3\phi^8}{35\times 2^{13}\kappa^5} + \frac{19i\omega^4\phi^{11}}{3185\times 2^{18}\kappa^7} + \frac{\omega^5\phi^{14}}{3185\times 2^{21}\kappa^9} \\ &\quad - \frac{561i\omega^6\phi^{17}}{2118025\times 2^{28}\kappa^{11}} - \frac{93\omega^7\phi^{20}}{3328325\times 2^{32}\kappa^{13}} + \frac{625011i\omega^8\phi^{23}}{53003575625\times 2^{38}\kappa^{15}} \\ &\quad + \frac{32971\omega^9\phi^{26}}{53003575625\times 2^{41}\kappa^{17}} - \frac{1509727i\omega^{10}\phi^{29}}{11501775910625\times 2^{46}\kappa^{19}} + \mathcal{O}(\phi^{30}). \end{aligned} \tag{2.19}$$

Clearly we can carry on with this procedure to any desired order.

Convergence of the Painlevé expansion Having established that the deformed Burgers equations pass the Painlevé test for any value of the deformation parameter ε, let us now see whether the obtained series converges such we may conclude that these equations also posses the Painlevé property. It suffices to demonstrate this for some specific cases. Taking for this purpose $\lambda_2 = 0$, we can express the expansion (2.18) in the general form

$$u(x,t) = -\frac{4i\kappa}{\phi} + \phi \sum_{n=1}^{\infty} \alpha_n \phi^n \tag{2.20}$$

and employ Cauchy's root test, i.e. $\sum_{n=1}^{\infty} \gamma_n$ converges if and only if $\lim_{n\to\infty} |\gamma_n|^{1/n} \leq 1$, to establish the convergence of the series. We can easily find an upper bound for the real and imaginary parts of α_n

$$|\mathrm{Re}\, \alpha_{3n-\nu}| \leq \frac{|\mathrm{Re}\, p_{3n-\nu}(\xi', \xi'', \xi''', \ldots)|}{2^{3n+4-\nu}\Gamma(\frac{3n-\nu}{2})\, |\kappa|^{2n-1}} \qquad \text{for } \nu = 0, 1, 2, \tag{2.21}$$

where the $p_n(\xi', \xi'', \xi''', \ldots)$ are polynomials of finite order in t, that is $\sum_{n=0}^{\ell} \omega^n t^n$ with $\ell < \infty$ and $\omega \in \mathbb{C}$. The same expression holds when we the replace real part by the imaginary part on both sides of the inequality. We should also comment that this point of the proof is not entirely rigorous in the strict mathematical sense as we have only verified the estimate (2.21) up to order thirty. Approximating now the gamma function in (2.21) by Stirling's formula as $n \to \infty$

$$\Gamma\left(\frac{n}{2}\right) \sim \sqrt{2\pi} e^{-n/2} \left(\frac{n}{2}\right)^{\frac{n-1}{2}} \tag{2.22}$$

we obtain

$$\lim_{n\to\infty} |\mathrm{Re}\, \alpha_{3n-\nu}|^{\frac{1}{2}} \sim \frac{|\mathrm{Re}\, p_{3n-\nu}|^{1/n}}{2^{3+\frac{4-\nu}{n}} (2\pi)^{\frac{1}{2n}} e^{-\frac{1}{2}} (\frac{3n-\nu}{2})^{\frac{1}{2}-\frac{1}{2n}} |\kappa|^{2-\frac{1}{n}}} = 0. \tag{2.23}$$

The same argument holds for the imaginary part, such that the series (2.20) converges for any value of κ and choices for $\xi(t)$ leading to finite polynomials $p_n(\xi', \xi'', \xi''', \ldots)$. It is straightforward to repeat the same argument for $\lambda_2 \neq 0$.

Alternatively we can identify the leading order term in (2.11) and integrate the deformed Burgers equation twice. In this way we change the ODE into an integral equation

$$u(x,t) = 2\kappa \left\{ g(t) + \int_{x_1}^{x} d\hat{x} \left[\frac{i}{2} + \frac{1}{u^2(\hat{x},t)} \left(f(t) + \int_{x_0}^{\hat{x}} d\tilde{x} \frac{u_t(\tilde{x},t)}{u_{\tilde{x}}(\tilde{x},t)} \right) \right] \right\}^{-1}, \tag{2.24}$$

where $g(t), f(t)$ are some functions of integration. When discretising this equation, i.e. taking the left hand side to be $u_{n+1}(x,t)$ and replacing all the $u(x,t)$ on the right hand side of this equation by $u_n(x,t)$, we may iterate (2.24) with $u_0(x,t) = -4i\kappa/[x-\xi(t)]$ as initial condition and recover precisely the expansion (2.17). Exploiting the Banach fixed point theorem one may also use (2.24) as a starting point to establish the convergence of the iterative procedure and therefore the Painlevé expansion, similarly as was carried out for instance in [36, 37].

Reduction from PDE to ODE Making further assumptions on the dependence of $u(x,t)$ on x and t we can reduce the PDE to an ODE, and attempt to solve the resulting equation by integration. A common assumption is to require the solution to be of the form of a travelling wave $u(x,t) = \zeta(z) = \zeta(x - vt)$ with v being constant. When v is taken to be real, even solutions will be invariant under the original $\mathcal{PT}$-symmetry. With this ansatz the deformed Burgers' equation for $\varepsilon = 2$ (2.11) acquires the form

$$-v\zeta_z + i\zeta\zeta_z^2 + 2\kappa\zeta_z\zeta_{zz} = 0. \tag{2.25}$$

When $\xi_z \neq 0$ we can re-write this equation as

$$\frac{d}{dz}\left(c - vz + \frac{i}{2}\zeta^2 + 2\kappa\zeta_z\right) = 0, \tag{2.26}$$

which can be integrated to

$$\zeta(z) = e^{i\pi 5/3}(2v\kappa)^{1/3}\frac{\tilde{c}Ai'(\chi) + Bi'(\chi)}{\tilde{c}Ai(\chi) + Bi(\chi)} \tag{2.27}$$

with c, $\tilde{c}$ being constants, $\chi = e^{i\pi/6}(vz - c)(2v\kappa)^{-2/3}$ and $Ai(\chi)$, $Bi(\chi)$ denoting Airy functions.

2.2 Painlevé test for the $\mathcal{PT}$-symmetrically deformed KdV-equation

The KdV-equation was found to be $\mathcal{PT}$-symmetric and was the first equation for which deformations have been studied [7, 8]. Next we investigate the $\mathcal{PT}$-symmetrically deformed version of the KdV-equation with two different deformation parameters ε and μ

$$u_t - 6uu_{x;\varepsilon} + u_{xxx;\mu} = 0 \qquad \text{with} \quad \varepsilon, \mu \in \mathbb{R}. \tag{2.28}$$

The case $\mu = 1$ and ε generic was considered in [7] and the case $\varepsilon = 1$ and μ generic was studied in [8].

2.2.1 Leading order terms

As in the previous section we substitute $u(x,t) \to \lambda_0(x,t)\phi(x,t)^\alpha$ into (2.28) in order to determine the leading order term. From $u_t \sim \phi^{\alpha-1}$, $uu_{x;\varepsilon} \sim \phi^{\alpha+\alpha\varepsilon-\varepsilon}$ and $u_{xxx;\mu} \sim \phi^{\alpha\mu-\mu-2}$ we deduce $\alpha = (\varepsilon - \mu - 2)/(\varepsilon - \mu + 1) \in \mathbb{Z}_-$, such that the only solution is $\alpha = -2$ with $\varepsilon = \mu$. This means neither the case $\mu = 1$ and ε generic nor the case $\varepsilon = 1$ and μ generic can pass the Painlevé test, but the hitherto uninvestigated deformation with $\varepsilon = \mu$ has at this point still a chance to pass it.

2.2.2 Recurrence relations

Substituting the Painlevé expansion (2.5) for $u(x,t)$ with $\alpha=-2$ into (2.28) with $\varepsilon=\mu$ gives rise to the recursion relations for the λ_k by identifying powers in $\phi(x,t)$. We compute

$$
\begin{aligned}
&\text{order } -(3\varepsilon+2): && \lambda_0=\tfrac{1}{2}\varepsilon(3\varepsilon+1)\phi_x^2,\\
&\text{order } -(3\varepsilon+1): && \lambda_1=-\tfrac{1}{2}\varepsilon(3\varepsilon+1)\phi_{xx},\\
&\text{order } -3\varepsilon: && \lambda_2=\tfrac{\varepsilon(3\varepsilon+1)}{24}\left(\tfrac{4\phi_x\phi_{xxx}-3\phi_{xx}^2}{\phi_x^2}\right)+\delta_{\varepsilon,1}\tfrac{\phi_t}{6\phi_x},\\
&\text{order } -(3\varepsilon-1): && \lambda_3=\tfrac{\varepsilon(3\varepsilon+1)}{24}\left(\tfrac{4\phi_x\phi_{xx}\phi_{xxx}-3\phi_{xx}^3-\phi_x^2\phi_{4x}}{\phi_x^4}\right)+\delta_{\varepsilon,1}\tfrac{\phi_t\phi_{xx}-\phi_x\phi_{xt}}{6\phi_x^3},\\
&\text{order } -(3\varepsilon-2): && \lambda_4=\tfrac{\varepsilon(3\varepsilon+1)}{24}\left(\tfrac{6\phi_x\phi_{xx}^2\phi_{xxx}-\frac{15}{4}\phi_{xx}^4-\frac{3}{2}\phi_x^2\phi_{xx}\phi_{4x}}{\phi_x^6}+\tfrac{\phi_x\phi_{5x}-5\phi_{xxx}^2}{5\phi_x^4}\right).
\end{aligned}
\tag{2.29}
$$

We find that the relation at order $-(3\varepsilon-2)$ becomes an identity only for $\varepsilon=1$, which makes us suspect that also at higher order we will not encounter compatibility conditions and therefore will not have enough parameters equaling the order of the differential equation. To test whether new compatibility conditions arise at higher levels we can use the same general argument as in subsection 2.1.3.

2.2.3 Resonances

We try once again to match the first term in the expansion (2.5) with some term of unknown power. Using the expression for λ_0 in (2.29) and making the ansatz

$$
\tilde{u}(x,t)=\frac{1}{2}\varepsilon(3\varepsilon+1)\frac{\phi_x^2}{\phi^2}+\vartheta\phi^{r-2}, \tag{2.30}
$$

we compute all possible values of r for which ϑ becomes a free parameter. Substituting $\tilde{u}(x,t)$ into (2.28) and reading off the terms of the highest order, i.e. $\phi^{-3\varepsilon-2+r}$, we find the necessary condition

$$
\varepsilon^{\varepsilon}(-i)^{\varepsilon-1}(3\varepsilon+1)^{\varepsilon-1}(r+1)\left[6(1+3\varepsilon)-2(2+3\varepsilon)r+r^2\right]\vartheta\phi_x^{3\varepsilon}=0, \tag{2.31}
$$

for a resonance to exist. We observe the presence of the universal resonance at $r=-1$. The bracket containing the quadratic term in r can be factorized as $(r-r_-)(r-r_+)$ with $r_\pm=-(2+3\varepsilon)\pm\sqrt{9\varepsilon^2-6\varepsilon-2}$, such that $r_\pm\in\mathbb{Z}$ for $9\varepsilon^2-6\varepsilon-2=n^2$ with $n\in\mathbb{N}$. For the solution of this equation $\varepsilon_\pm=(1\pm\sqrt{n^2+3})/3$ to be an integer we need to solve a Diophantine equation $3+n^2=m^2$ with $n,m\in\mathbb{N}$, which only admits $n=1$ and $m=2$ as solution. Thus the bracket only factorises in the case $\varepsilon=1$ into $(r-6)(r-4)$. Hence, only in that case the system can fully pass the Painlevé test. Nonetheless, we may still be able to obtain a defective series if all remaining coefficients λ_j may be computed recursively. This is indeed the case as we demonstrate in detail for one particular choice of the deformation parameter.

2.2.4 $\varepsilon=2$ deformation

For $\varepsilon=\mu=2$ the deformed KdV equation (2.28) acquires the form

$$
u_t-6iuu_x^2+2iu_{xx}^2+2iu_xu_{xxx}=0 \tag{2.32}
$$

Since the expression become rather lengthy for generic values in the expansion we will present here only the case $\lambda_k(x,t) = \lambda_k(t)$ and $\phi(x,t) = x - \xi(t)$, with $\xi(t)$ being an arbitrary function. We find a recursion relation of the form (2.8)

$$-28i(1+j)(j^2-16j+42)\lambda_j(t) = -6i\sum_{n=1}^{j}\sum_{m=1}^{j-n-1}\{(m-2)(n-2)\lambda_m(t)\lambda_n(t)\lambda_{j-m-n}(t)\}$$
$$+2i\sum_{n=1}^{j-1}\left\{[(7-k)n^3+(k-4)kn^2+(18-5k)kn+6k(5+k)-28(6+n)]\lambda_{j-n}(t)\lambda_n(t)\right\}$$
$$+\lambda'_{j-6}(t)+(j-7)\lambda'_{j-5}(t). \tag{2.33}$$

The recursive solution of this equation leads to the expansion

$$u(x,t) = \frac{7}{\phi^2}+\frac{i\xi'\phi^3}{156}+\frac{(\xi')^2\phi^8}{192192}-\frac{\xi''\phi^9}{681408}+\frac{i(\xi')^3\phi^{13}}{73081008}-\frac{725i\xi'\xi''\phi^{14}}{216449705472}+\frac{i\xi'''\phi^{15}}{20262348288}$$
$$-\frac{340915(\xi')^4\phi^{18}}{23989859332927488}+\frac{1867(\xi')^2\xi''\phi^{19}}{758331543121152}+\mathcal{O}(\phi^{20}). \tag{2.34}$$

Thus we have obtained a solution of Painlevé type for the deformed KdV equation, albeit without enough free parameters, i.e. without the possibility to accommodate all possible initial values. This means we have a so-called defective series. As in the case of the deformed Burgers equation it is instructive to consider the series for travelling wave solutions, i.e. taking $\xi(t) = \omega t$, which yields

$$u(x,t) = \frac{7}{\phi^2}+\frac{i\omega\phi^3}{156}+\frac{\omega^2\phi^8}{192192}+\frac{i\omega^3\phi^{13}}{73081008}-\frac{340915\omega^4\phi^{18}}{23989859332927488}+\frac{391907i\omega^5\phi^{23}}{5676000718170643660 8}$$
$$-\frac{38892808841\omega^6\phi^{28}}{507260097462393341102260224}+\mathcal{O}(\phi^{33}). \tag{2.35}$$

Clearly we can carry on with this analysis to any desired order. The convergence of the expansion can be established in a similar fashion as we demonstrated for Burgers equation in the previous subsection or by making use of an integral equation of the type (2.24). We find a similar behaviour for other values of ε.

3. Conclusion

We have carried out the Painlevé test for $\mathcal{PT}$-symmetric deformations of the Burgers equation and the KdV equation. When deforming both terms involving space derivatives, we found that the deformations of the Burgers equation pass the test. In specific cases we have also established the convergence of the series, such that these equations have in addition the Painlevé property. Based on the conjecture by Ablowitz, Ramani and Segur we take this as very strong evidence that these equations are integrable. Regarding these models as new integrable systems leads immediately to a sequence of interesting new problems related to features of integrability, which we intend to address in a future publication [38]. It is very likely that these systems admit soliton solutions and it should be possible to compute the higher charges by means of Lax pairs, Dunkl operators or other methods. We should point out that most of our arguments will still hold when we start in (2.4) with the usual Burgers equation, which has broken $\mathcal{PT}$-symmetry, i.e. with $\sigma = i\kappa \in \mathbb{R}$. However, when embarking on the computation of charges and in particular energies we expect to find a severe difference as then the $\mathcal{PT}$-symmetry has a bearing on the reality of the eigenvalues of the charges.

For the KdV equation our findings suggest that their $\mathcal{PT}$-symmetric deformations are not integrable, albeit they allow for the construction of a defective series.

In future work one could also include deformations of the term involving the time derivative and it would clearly be very interesting to investigate other $\mathcal{PT}$-symmetrically integrable systems in the manner in order to establish their integrability.

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