DEVELOPMENT OF HIERARCHICAL OPTIMAL CONTROL ALGORITHMS FOR INTERCONNECTED NONLINEAR DYNAMICAL SYSTEMS

By

Mohd Ismail bin Abd Aziz

A THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

CONTROL ENGINEERING RESEARCH CENTRE

DEPARTMENT OF ELECTRICAL, ELECTRONIC

AND INFORMATION ENGINEERING

CITY UNIVERSITY, LONDON

JUNE, 1999

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGMENTS

I wish to express my sincere gratitude to my research supervisor, Prof. Peter David Roberts, for his guidance, advice, understanding and kindness given throughout this work. His professionalism and high standards have provided me with a source of inspiration.

I wish to thank Dr. Victor Becerra for his friendship, assistance and continuous encouragement throughout my stay at City University.

I am grateful to my employer, Universiti Teknologi Malaysia for their financial support.

Special thanks to Mrs. J. Rivellini from City University, for her kindness and caring attitude.

Last, but not least, I want to thank my wife, Saidah, and my children, Syamir and Syireen for their constant support, patience , love and understanding.

DECLARATION

The author grants power of discretion to the University Librarian to allow this thesis to be copied in whole or in part without further reference to him. This permission covers only single copies made for study purposes only, subject to normal conditions of acknowledgement.

ABSTRACT

The main concern of this thesis is to develop and advance the knowledge of new hierarchical algorithms for optimal control of interconnected nonlinear systems. To achieve this, four basic hierarchical structures are developed by taking into account the manner in which real process measurements taken from interaction inputs are incorporated and utilized in the model-based optimal control problem. The structures are iterative in nature, and are derived using the dynamic integrated system optimization and parameter estimation (DISOPE) technique to take into account model-reality differences that may have been deliberately introduced to facilitate the solution of the complex nonlinear problem or due to uncertainty in the model used for computation.

Three of the four basic hierarchical structures are used as a basis for developing hierarchical optimal control algorithms using a linear quadratic model formulation. Two approaches are used in the coordination problem of the algorithms, price coordination approach and the direct coordination approach. The algorithms are then implemented using two techniques, the single loop and the double loop techniques. All the algorithms are implemented in software and a simulation study is carried out using two examples to investigate their effectiveness and convergence properties..

The optimality of the solution provided by the structures and the algorithms described in this research work are established. In addition, convergence analysis is carried out to provide sufficient convergence conditions of the double loop algorithms. Suggestions for future research as a continuation of the work presented in this thesis are also made.

LIST OF SYMBOLS

Symbol	Description
N_s	Number of subsystems
n_i	Order of the ith subsystem
m_i	Number of control inputs in the ith subsystem
r_i	Number of output variables in the ith subsystem
x_i	State vector for ith subsystem
x	State vector for global system
c_i	Control vector for ith subsystem
c	Control vector for global system
u_i	Interaction vector for ith subsystem
u	Interaction vector for global system
y_i	Output vector for ith subsystem
y	Output vector for global system
k	time index
$[0, N]$	time interval
f_i^*	Real state mapping for ith subsystems.
f_i	Model-based performance measure function for ith subsystems
F_i^*	Real output function for ith subsystems
F_i	Model-based output function for ith subsystems
K^*	Real output mapping for global system
q_i^*	Real performance measure function for ith subsystem
q_i	Model-based performance measure function for ith subsystem
q	Model-based performance measure function for global system
Q_i^*	Real performance index for ith subsystem
Q^*	Performance index for global system
x_o	Initial state vector for global system

x_{i0}	Initial state vector for ith subsystem
$\mathbf{Q}_{ROP}$	ROP performance index for global system
H_{ij}	Interconnection matrix for ith subsystem
H	Interconnection matrix for global system
p	Costate vector for global system
$\mathbf{H}^*$	Real Hamiltonian for global system
$\mathbf{H}$	Model-based Hamiltonian for global system
l	Price vector (Lagrange multiplier) for global system
α	State related discrete time parameter for global system
γ	Performance discrete time parameter for global system
θ	Output related discrete time parameter for global system
z	State separation vector for global system
v	Control separation vector for global system
w	Interaction separation vector for global system
$\hat{p}$	Costate separation vector for global system
μ	Lagrange multiplier vector for global system
η	Lagrange multiplier for global system
ξ	Lagrange multiplier vector for global system
λ $(\bar{\lambda})$	Lagrange multiplier vector for global system
β $(\bar{\beta})$	Lagrange multiplier vector for global system
ζ $(\bar{\zeta})$	Lagrange multiplier vector for global system
s	Iteration number
ε_l	Stepsize for price update
ε_z	Stepsize for state update
ε_v	Stepsize for control update
ε_w	Stepsize for interaction update
ε_p	Stepsize for costate update
r_{1i}	Control convexification factor

r_{2i}	State convexification factor
r_{3i}	Interaction convexification factor
Q_i $(\overline{Q}_i)$	State weighting matrix for ith subsystem (augmented)
R_i $(\overline{R}_i)$	Control weighting matrix for ith subsystem (augmented)
S_i $(\overline{S}_i)$	Interaction weighting matrix for ith subsystem (augmented)
A_i	Model-based system matrix for ith subsystem
B_i	Model-based control distribution matrix for ith subsystem
D_i	Model-based interaction distribution matrix for ith subsystem
$\mathscr{J}_i$	Model-based system output matrix for ith subsystem
$\mathscr{K}_i$	Model-based control output matrix for ith subsystem
$\mathscr{L}_i$	Model-based interaction output matrix for ith subsystem
V_i	Discrete time Ricatti matrix for ith subsystem
E_i	Discrete time matrix for ith subsystem
v_i	Lagrange multiplier for ith subsystem
h_i	Discrete time Ricatti vector
W_i	Discrete time matrix for ith subsystem
ϕ_i	Terminal weighting function for ith subsystem
Φ_i	Terminal state weighting matrix for ith subsystem
Ψ_i	State dependent terminal constraint matrix for ith subsystem
I_{q_i}	Identity matrix of order q_i
π_i	Discrete time vector for ith subsystem
G_i	State feedback gain matrix for ith subsystem
g_i	Driving input vector. for ith subsystem
M_i	Interaction feedback gain matrix for ith subsystem
m_i	Interaction driving input vector for ith subsystem
$\Re^n$	n-dimensional real space

CHAPTER 1

INTRODUCTION

1.1 CONTROL SYSTEMS

A control system is a system capable of monitoring and regulating the operations of a physical process or a plant. A control system is also defined as an interconnection of components forming a system configuration that will produce a desired goal. The system may be an economic system, a technological process, an ecological system or a chemical process. The goals are usually related to economic objectives, costs, environmental regulations or operational constraints.

A typical control configuration consists of the controlled system and the control unit. The controlled system has manipulated inputs or control variables which may cause a change in a set of measured variables which characterized the system, called the states. The controlled system is also subject to disturbances which reflect the effects of the surrounding environment. The task of the control unit is to determine the values of control variables that achieve a certain specification on the behaviour of the controlled system. The control unit based its decisions on observed variables or measurements that are related to the controlled system or the disturbance or both. In the case of a dynamic system, the decision must also be based in addition to the above, on the state of the system.

There are two major classes of control system known as open-loop and closed loop control systems. Open loop control systems are control systems in which the process output has no effect on the input quantities. In contrast, a closed-loop system utilizes a measure of actual output and compares it with the desired output in order to produce the control signal. A closed loop system is in general more capable of coping with any unexpected disturbance and uncertainties in process parameters than open-loop systems.

Consider a system with a given mathematical representation, then the problem is to find a control strategy or law so that the overall feedback system is stable and meets some desired specifications. The specification may be stated in terms of relative stability, performance measure, sensitivity, response time, interaction and so on. If the specification is a performance criterion expressed in a suitable mathematical form, then

we can formulate the problem of determining a control law which minimizes or maximizes the performance criterion. In this case we have an optimal control problem (Athans and Falb,1966; Kirk, 1970; Bryson and Ho; 1975; and Lewis and Syrmos, 1995).

Most complex processes consist of interconnected subsystems and the number of variables involved is large. Designing a centralized controlled system for such a complex process may be very difficult or almost impossible. The difficulties may be caused by limited storage capability in a single computer or due to a time consuming solution process. This problem can be overcome by using decomposition techniques and hierarchical structures.

1.2 OPTIMIZATION OF INTERCONNECTED DYNAMICAL SYSTEMS

The need for dynamic optimization arises because the system state may differ from the desired precomputed steady state conditions . The fluctuations in the state may occur due to variations in inputs, disturbances and plant parameter variation. It then may be necessary to vary the control signals on-line in order to bring the state vector back to the desired precomputed level. Here optimization implies the extremization of a suitable functional of state and control trajectories subject to constraints of system dynamics as well as constraints on states and controls.

The centralized optimization of dynamical systems with large dimensional state vectors is possible, in principle, using established methods such as Dynamic Programming (Bellman, 1957) and Pontryagin's Maximum Principle (Pontryagin et al, 1962) but the increase in the number of variables introduce problems of dimensionality. The amount of computation necessary for dynamic optimization increases rapidly with the increase in the number of state variables. For example, for the linear quadratic problem , the computation requirements increase cubically with the increase in system order (Singh,1980). Therefore application of decomposition techniques and hierarchical structures provide more efficient strategies for solving large dimensional problems. These procedures decompose the centralized problem into subsystems of an interconnected system, each having a lower order than the overall global problem in an integrated way.

Another problem concerns the original structure of systems which by nature are decomposed into subsystems or are hierarchical in structure. They often model real-life systems and their hierarchical structures depict systems dealing with society, business, management, water resources, transportation etc. These systems are separated geographically and their treatment requires consideration of not only economic cost but also issues of communication and exchange of information between the subsystems (Jamshidi, 1983; Tzafestas and Kapsiotis, 1995). Optimization of such problems requires application of hierarchical techniques to take advantage of natural decomposed structures.

Numerous hierarchical algorithms have been proposed in the last two decades for solving optimal control problem of interconnected systems. Hierarchical control schemes were first introduced in the work of Mesarovic et al (1970) which provides a general development of hierarchical system theory. The essential idea in most hierarchical techniques of the literature is to solve, independently, decomposed subsystems at the first level for a set coordination variables which are provided by a second level hierarchy. At the second level the coordinating variables are iteratively improved using information from the first level.

The hierarchical approach mentioned above has been very successful when handling a linear system with quadratic cost. In absence of state and control constraints, the centralized linear quadratic(LQ) optimal control problem has a relatively straight forward solution based on the solution of matrix differential equations or difference equations (Anderson and Moore,1989; Lewis and Syrmos, 1995). In the hierarchical approach, the solution of each subsystem in the first level in each iteration may be obtained using a similar technique if subjected to the same conditions (Singh, 1980; Singh and Titli, 1978; Mahmoud et al, 1985; Becerra, 1994).

Two major hierarchical approaches have been applied to hierarchical optimal control of linear quadratic systems. They are goal coordination and interaction prediction approaches. In the goal coordination approach (also known as the interaction balance method), the Lagrange multipliers associated with the constraints are calculated in the second level of the hierarchical structure using a gradient based formula. In the interaction prediction approach both the Lagrange multipliers and the interaction vectors are calculated in the second level based on formula determined from optimality

conditions. The values of the multipliers and interaction vectors for a current iteration are predicted using values of variables supplied from the first level calculated during the previous iteration.

Hierarchical control techniques can also be used to solve nonlinear optimization problems. The solution of nonlinear optimization problems via single level techniques leads ultimately to the solution of a nonlinear two point boundary value problem (TPBVP). The TPBVP normally requires successive approximation techniques for their solution (Sage and White, 1977). The key idea in hierarchical optimization algorithms for nonlinear system based on the prediction principle is to write the performance criterion under minimization in separable form and to write the nonlinear dynamic equations in the form of a linear part, which is blockwise separable, and another term which contains the nonlinearities and interaction terms. The role of the higher hierarchical level here is to fix the non-separable part in the performance criterion and the nonlinear part the dynamic equation. This results in a set of low order dynamic optimization problems to be solved at level one of the hierarchy. The higher level successively approximates the specification of coordination variables to their optimal values. Hierarchical optimal control algorithms based on such an approach are described in the works of Hassan and Singh(1977), Mahmoud et al (1985) and more recently Tzafestas and Kapsiotis(1995).

Another approach for solving hierarchical optimization problems of nonlinear systems is based on the principle of handling model-reality differences in dynamic optimization. The technique is an extension of the steady state optimizing technique called integrated system optimization and parameter estimation (ISOPE) introduced by Roberts (1979) .

1.3 INTEGRATED SYSTEM OPTIMIZATION AND PARAMETER ESTIMATION (ISOPE) ALGORITHMS

The basis for optimizing control is the economic objective which is quantified by a performance criterion, a steady state mathematical model of the plant and knowledge of relevant process constraints. The result from steady state optimization is a set of optimal controller set-points at which the process should be regulated until a change in economic parameters requires a new operating point. In order to design an algorithm which is capable of producing a true optimum regardless of model-reality differences, it is necessary to find a way to cater for the interaction between parameter estimation

and model-based optimization problems. ISOPE achieved this by introducing a modifier into the model-based optimization problem so that interaction between system optimization and parameter estimation is compensated at the end of the algorithm iterations. This enables the iterative technique to achieve correct optimal operating point of the real process in spite of model-reality differences. Since its conception in 1979, a considerable number of ISOPE algorithms, centralized and hierarchical have been developed. (Roberts, Wan and Lin, 1992; Roberts,1995). Of particular interest in this thesis is the ISOPE techniques extended to hierarchical process optimization with different feedback structures for the purpose of handling large scale problems (Brdys and Roberts, 1986). Various form of hierarchical ISOPE algorithms have been developed and its convergence conditions rigorously investigated (Chen et al,1986; Roberts, 1988; Brdys et al, 1989; Lin et al, 1991; Amini et al, 1992; Augustin and Roberts, 1993).

ISOPE have been extended to dynamic problems by Roberts(1992). It has been termed DISOPE (dynamic ISOPE) and the philosophy behind the techniques remains very similar. A new range of applications of DISOPE techniques has been developed (Becerra, 1994). DISOPE has been extended to hierarchical optimal control of interconnected systems with a specific interaction structure by Becerra (1992) and Becerra and Roberts (1995) .

The development of hierarchical DISOPE algorithms for optimal control of interconnected nonlinear systems with a more general interaction structure using an approach analogous to that of Brdys and Roberts (1986), and the study of their convergence properties are the central areas of research of the work described in this thesis.

1.5 SCOPE AND AIMS OF THE THESIS

As mentioned previously hierarchical ISOPE is a well established techniques for optimizing control of an interconnected steady state process. Hierarchical DISOPE is still recent. Prior to this work only one hierarchical DISOPE algorithm for an interconnected system with a specific interaction structure had been developed (Becerra, 1994; Becerra and Roberts, 1995).

The central aim of this thesis is to develop and advance the knowledge of new hierarchical DISOPE algorithms based on a more general interaction structure. Additionally an initial study of convergence properties of a specific implementation of the algorithms is carried out during the research.

The means through which the central objective is achieved are:

* To develop hierarchical structures for optimal control of interconnected dynamical systems using an approach similar to that of Brdys and Roberts (1986) and to ascertain the optimality of the solutions produced by the structures.
* To develop and implement the hierarchical algorithms for structures described above using two different approaches , the price coordination and direct coordination.
* To compare the performance of the algorithms by carrying out example simulation studies.
* To establish the optimality of the algorithms based on a linear quadratic approach and provide an initial investigation of the convergence properties of the algorithms where possible.

The scope and original contributions of this thesis are briefly summarized below.

Contribution to hierarchical optimal control structure and algorithm development

Four basic hierarchical optimal control structures are developed by taking into account the manner we incorporate and utilize real process measurements from interaction inputs in the model-based optimal control problem (Mohd_Ismail and Roberts, 1995). A total of twelve hierarchical DISOPE algorithms are developed from three of the four basic hierarchical structures. The structures are: structure with model based interaction input, structure with real interaction input in parameter estimation and structure with real input in interaction and parameter estimation. The algorithms can be separated in two categories: those developed using the price coordination approach and those using the direct coordination approach. They are then implemented using two techniques , single loop and double loop techniques. The algorithms are implemented in software

and tested with simulation examples. These works are considered as a continuation of effort made to realize the potential of DISOPE algorithms for applications in various optimal control problems.

Contribution to theoretical analysis

A study of optimality of the hierarchical algorithms described above is provided in this thesis. An initial study of convergence properties of double loop algorithms is performed in this thesis. Sufficient conditions for convergence of the double loop algorithms are derived. The convergence analysis is important in the development of further theoretical studies of the hierarchical algorithms.

Contribution to software implementation and algorithm testing

All the algorithms proposed have been implemented in software using C++ programming language based on the DMatrix class structure developed by Becerra (1995). Simulation studies were carried out on all algorithms and a comparative evaluation of the algorithms is carried out. These simulation have allowed us to obtain a greater understanding of the hierarchical DISOPE techniques and to test experimentally the algorithms developed.

1.6 OUTLINE OF THE THESIS

An outline of the thesis is given below.

Chapter 2: reviews the development and algorithmic details of the discrete DISOPE algorithm in its centralized version as first presented in Becerra (1994) and more recently by Becerra and Roberts (1996). The hierarchical decomposition and coordination procedures utilized later in the thesis are described and discussed. These concepts provide a background for developing hierarchical optimal control structures and algorithms for interconnected dynamical systems. Other different approaches for solving the optimal control of an interconnected dynamical system is also briefly described.

Chapter 3: describes the development of four basic hierarchical structures for optimal control of interconnected discrete dynamical system. The structures have different stages of incorporation of real interaction input . The structures are a dynamic analogy of structures derived by Brdys and Roberts (1986). The optimality of the hierarchical structures is then established by showing that the resulting converged solution of each model-based structures is equivalent to that of the corresponding global real optimal control problem (ROP).

Chapter 4: presents a systematic formulation of hierarchical DISOPE algorithms for optimal control of a discrete dynamical system with model-based interaction input. Four algorithms are described. Two are based on a price coordination approach, while the other two are based on the direct coordination approach. The algorithms are implemented using single and double loop techniques. The double loop technique is developed to take advantage of the global nature of the interaction structure. It is also simpler to implement in software. A simulation using two examples are carried out and a comparative evaluation is provided.

Chapter 5: establishes the optimality of the algorithms presented in chapter 4. In this chapter we also provide the sufficient convergence conditions of the double loop version of the algorithm described above. The conditions shows that convergence properties may be influenced by the length of optimization horizon and size of the price stepsize in the price coordination approach.

Chapter 6: presents the development of hierarchical algorithms for interconnected systems with real input in interaction in parameter estimation, and interconnected systems, with real input in interaction and parameter estimation. Four algorithms are developed for each system structure using techniques first described in chapter 4. The optimality conditions of all eight algorithms are also determined. Sufficient convergence conditions for the double loop implementation also provided. The algorithms are implemented in software and tested using the same simulation examples used earlier in chapter 4. A discussion of the convergence behaviour based on the

simulation studies is given. A comparison of all twelve algorithms is provided at the end of the chapter.

Chapter 7: draws the conclusions from the results obtained in this thesis and presents a series of suggestions for further work in these areas.

1.7 SUMMARY

The development of hierarchical algorithms based on a more general interaction structure is the central subject of the research work described in this thesis. The main objective is to advance and improve the existing knowledge of hierarchical dynamic optimization of interconnected systems using the hierarchical DISOPE technique, thus making the DISOPE approach more attractive for its implementation in various control problems.

In this introductory chapter, a short discussion of control systems is described. A brief review of main hierarchical optimization techniques for interconnected dynamical systems is also presented. Moreover, a brief review of the ISOPE technique, which is the predecessor of the DISOPE technique has also been given. This is followed by a discussion of aims, scope and original contributions of the work described in this thesis. Finally, the contents and the structure of the thesis have been described.

CHAPTER 2

HIERARCHICAL OPTIMAL CONTROL SYSTEMS, METHODOLOGIES AND APPROACHES

2.1 INTRODUCTION

The mathematical models of many physical and engineering systems are frequently of high dimensionality and involve interacting dynamic components. The information processing demand and requirements for experimenting with these models for control purposes are usually excessive. It is, therefore, natural to seek techniques that reduce the computational effort. Hierarchical approaches and methodologies of interconnected systems provide such techniques through manipulation of system structure in an appropriate way.

One of the techniques is based on the DISOPE algorithm which was first introduced by Roberts (1992). The centralized DISOPE was extended to solve hierarchical systems with linear state interactions (Becerra, 1994) using an interaction prediction approach (Hassan and Singh, 1976). Mohd_Ismail and Roberts (1995) proposed optimal hierarchical structures for continuous systems with a more general interaction structure using the interaction balance principle (Mesarovic et al, 1969; Findeisen et al, 1980) for coordination. In the following section we will described the centralized DISOPE algorithm in its discrete time version as presented by Becerra (1994), and Becerra and Roberts(1996). The algorithm is considered most relevant to the work described in this thesis as it provides the basis for algorithm development for hierarchical optimal control of discrete time interconnected systems. In the subsequent sections, we will then be looking at ways of dealing with optimization of interconnected dynamical systems through existing hierarchical control techniques with emphasis on its application to interconnected systems.

2.2 CENTRALIZED OPTIMAL CONTROL OF DISCRETE TIME SYSTEMS.

In this section we describe a DISOPE algorithm which have been developed to solve discrete time dynamic optimal control with model-reality differences. In this scheme it is assumed that the control input is switched between different values at discrete time

steps. The control signal is normally held constant between samples by a zero order hold.

2.2.1 DISOPE Algorithm

Problem formulation and solution approach.

Consider the following fixed time real optimal control problem (ROP)

$$\min_{c(k)} \mathbf{Q}^* = \phi(\bar{x}(N)) + \sum_{k=1}^{N-1} q^*(x(k),c(k),k)$$

subject to

$$x(k+1) = f^*(x(k),c(k),k)\,;\ \mathrm{k} \in [0, \mathrm{N\text{-}1}]$$

$$x(0) = x_0$$

$$x_t(N) = 0; \quad t \in [1,q]$$

$$\bar{x}(N) = [x_{q+1}(N),.....,x_n(N)]^T \tag{2.1}$$

where $c(k) \in \Re^m$ and $x(k) \in \Re^n$ are the discrete control and state vectors respectively, $\phi: \Re^{n-q} \to \Re$ is a given terminal measure, $q^*: \Re^n \times \Re^m \times \Re \to \Re$ is the real performance index function and $f^*: \Re^n \times \Re^m \times \Re \to \Re^n$ represents reality.

The necessary optimality conditions of the ROP are as follows (Bryson and Ho, 1969)

$$\nabla_{c(k)}\mathbf{H}^*(.) = 0$$

$$\nabla_{x(k)}\mathbf{H}^*(.) - p(k) = 0 \tag{2.2}$$

$$p_i(N) = \frac{\partial \Phi(.)}{\partial x_t(N)}; \quad t \in [q+1,n]$$

where

$$\mathbf{H}^*(.) = q^*(.) + p(k+1)^T f^*(.); \quad ,k \in [0, N-1]$$

Instead of directly solving the ROP, the following possibly simplified model-based optimal control problem (MOP) is considered

$$\min_{c(k)} \mathbf{Q} = \phi(\bar{x}(N)) + \sum_{k=1}^{N-1} q(x(k),c(k),\gamma(k))$$

subject to

$$x(k+1) = f(x(k),c(k),\alpha(k))\,;\ \mathrm{k} \in [0, \mathrm{N\text{-}1}]$$

$$x(0) = x_0$$

$$x_t(N) = 0; \quad t \in [1,q]$$

$$\bar{x}(N) = [x_{q+1}(N),.....,x_n(N)]^T \tag{2.3}$$

where $\alpha(k) \in \Re^s$ and $\gamma(k) \in \Re$ are discrete parameters, $q: \Re^n \times \Re^m \times \Re \rightarrow \Re$ is the model performance measure function and $f: \Re^n \times \Re^m \times \Re^s \rightarrow \Re^n$ represents the model.

The key to integrating system optimization and parameter estimation is to define an Expanded Optimal Control Problem (EOP) which, in spite of being model based, is made equivalent to ROP by adding the appropriate equality constraints on state equations and discrete weighting function values. In addition, state and control variables are separated between parameter estimation and optimization steps by introducing separation variables $z(k)$ and $v(k)$, for state and control respectively. The EOP which is equivalent to ROP is defined as

$$\min_{c(k)} \mathbf{Q} = \phi(\bar{x}(N)) + \sum_{k=1}^{N-1} [q(x(k),c(k),\gamma(k)) + \frac{1}{2} r_1 \|c(k) - v(k)\|^2 + \frac{1}{2} r_2 \|x(k) - z(k)\|^2]$$

subject to

$$x(k+1) = f(x(k),c(k),\alpha(k)) \; ; \; \mathrm{k} \in [0, \mathrm{N\text{-}1}]$$

$$f^*(x(k),c(k),k)) = f(x(k),c(k),\alpha(k))$$

$$q^*(x(k),c(k),k)) = q(x(k),c(k),\gamma(k))$$

$$x(0) = x_0$$

$$x_t(N) = 0; \quad t \in [1,q]$$

$$\bar{x}(N) = [x_{q+1}(N),.....,x_n(N)]^T \tag{2.4}$$

The convex terms proportional to r_1 and r_2 are introduced to augment the performance index in order to improve convergence.

Define augmented Hamiltonian $\mathbf{H}(.)$

$$\mathbf{H}(.) = q(x(k),c(k),\gamma(k)) + p(k+1)^T f(x(k),c(k),\alpha(k)) - \beta(k)^T c(k)$$
$$-\lambda(k)^T x(k) + \frac{1}{2} r_1 \|c(k) - v(k)\|^2 + \frac{1}{2} r_2 \|x(k) - z(k)\|^2 \qquad .(2.5)$$

where $p(k) \in \Re^n$ (the costate), $\beta(k) \in \Re^m$ and $\lambda(k) \in \Re^n$ are Lagrange Multipliers.

Adjoining constraints in (2.4) and using (2.5) produces

$$\overline{\mathbf{Q}} = \phi(\overline{x}(N)) + \sum_{k=0}^{N-1} \{\mathrm{H}(.) - p(k+1)^T x(k+1) + \beta(k)^T v(k) + \lambda(k)^T z(k)$$
$$+\mu(k)^T [f^*(z(k),v(k),k) - f(z(k),v(k),\alpha(k))] \qquad (2.6)$$
$$+\eta(k)^T [q^*(z(k),v(k),k) - q(z(k),v(k),\gamma(k))]\}$$

where $\mu(k) \in \Re^n$ and $\eta(k) \in \Re$ are Lagrange multipliers.

It is now desired to examine the increment in $\overline{\mathbf{Q}}$ due to increments in all variables. It is assumed that the final time N is fixed. Applying Lagrange multiplier theory, at a constrained minimum this increment Δq should be zero (Lewis and Syrmos, 1995). Therefore

$$\Delta\overline{\mathbf{Q}} = \phi(\overline{x}(N)) + \sum_{j=1}^{N-1} \{[\frac{\partial \mathbf{H}(.)}{\partial c(k)}]\Delta c(k) + [\frac{\partial \mathbf{H}(.)}{\partial x(k)} - p(k)^T]\ \Delta x(k)$$

$$+[\lambda(k)^T + \mu(k)^T \left[\frac{\partial f^*(.)}{\partial z(k)} - \frac{\partial f(.)}{\partial z(k)}\right] + \eta(k)\left[\frac{\partial q^*(.)}{\partial z(k)} - \frac{\partial q(.)}{\partial z(k)}\right]]\Delta z(k)$$

$$+[\beta(k)^T + \mu(k)^T \left[\frac{\partial f^*(.)}{\partial v(k)} - \frac{\partial f(.)}{\partial v(k)}\right] + \eta(k)\ \left[\frac{\partial q^*(.)}{\partial v(k)} - \frac{\partial q(.)}{\partial v(k)}\right]]\Delta v(k)$$

$$+[\frac{\partial \mathbf{H}(.)}{\partial \alpha(k)} - \mu(k)^T \frac{\partial^T f(.)}{\partial \alpha(k)}]\Delta\alpha(k) + [\frac{\partial \mathbf{H}(.)}{\partial \gamma(k)} - \eta(k)\ \frac{\partial\ q(.)}{\partial \gamma(k)}]\Delta\gamma(k)$$

$$+[\frac{\partial \mathbf{H}(.)}{\partial p(k+1)} - x(k+1)^T]\ \Delta p(k+1) + [\frac{\partial \mathbf{H}(.)}{\partial \lambda(k)} + z(k)^T]\Delta\lambda(k)$$

$$+[\frac{\partial \mathbf{H}(.)}{\partial \beta(k)} + v(k)^T]\Delta\beta(k) + [f^*(.) - f(.)]^T \Delta\mu(k) + [q^*(.) - q(.)]\Delta\eta(k)\} \qquad (2.7)$$

where

$$\frac{\partial \mathbf{H}(.)}{\partial c(k)} = \frac{\partial q(.)}{\partial c(k)} + p(k+1)^T \frac{\partial f(.)}{\partial c(k)} - \beta(k)^T + r_1 (c(k) - v(k))^T$$

$$\frac{\partial \mathbf{H}(.)}{\partial x(k)} = \frac{\partial q(.)}{\partial x(k)} + p(k+1)^T \frac{\partial f(.)}{\partial x(k)} - \lambda(k)^T + r_2 (x(k) - z(k))^T$$

$$\frac{\partial\, \mathbf{H}(.)}{\partial \alpha(k)} = p(k+1)^T \frac{\partial f(.)}{\partial \alpha(k)}; \qquad \frac{\partial\, \mathbf{H}(.)}{\partial \gamma(k)} = \frac{\partial q(.)}{\partial \gamma(k)};$$

$$\frac{\partial\, \mathbf{H}(.)}{\partial p(k+1)} = f^T(.); \qquad \frac{\partial\, \mathbf{H}(.)}{\partial \lambda(k)} = -x(k)^T;$$

$$\frac{\partial\, \mathbf{H}(.)}{\partial \beta(k)} = -c(k)^T; \tag{2.8}$$

Setting to zero the coefficients of independent increments in (2.7), we can conclude by inspection that $\eta(k) = 1$ and $\mu(k) = p(k+1)$, $k \in [0, N-1]$, and the following optimality conditions are obtained:

Stationarity

$$\frac{\partial^T \mathbf{H}(.)}{\partial c(k)} = 0 \tag{2.9}$$

Costate equation

$$\frac{\partial^T \mathbf{H}(.)}{\partial x(k)} - p(k) = 0 \tag{2.10}$$

State equation

$$x(k+1) = f(x(k), c(k), u(k), \alpha(k)) \;;\; k \in [0,..N-1] \tag{2.11}$$

Boundary conditions

$$\begin{aligned} p_t(N) &= v_t \text{ (i.e free)}; \; t \in [1,q] \\ p_j(N) &= \frac{\partial \phi(.)}{\partial x_j(N)}; \; j \in [q+1,n] \end{aligned} \tag{2.12}$$

Multiplier equations

$$\beta(k) = -\left[\frac{\partial f^*(.)}{\partial v(k)} - \frac{\partial f(.)}{\partial v(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial v(k)} - \frac{\partial q(.)}{\partial v(k)}\right]^T$$

$$\lambda(k) = -\left[\frac{\partial f^*(.)}{\partial z(k)} - \frac{\partial f(.)}{\partial z(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial z(k)} - \frac{\partial q(.)}{\partial z(k)}\right]^T \tag{2.13}$$

plus the following equality constraints

$$f^*(z(k), v(k), w(k), k) = f\ (z(k), v(k), w(k), \alpha(k)) \tag{2.14}$$

$$q^*(z(k), v(k), w(k), k) = q\ (z(k), v(k), w(k), \gamma(k)) \tag{2.15}$$

$$v(k)=c(k);\quad k\in[0,N-1]$$
$$z(k)=x(k);\quad k\in[0,N] \tag{2.16}$$
$$\hat{p}(k)=p(k);\quad k\in[0,N]$$

where $\hat{p}(k)$ is introduced as a costate separation variable. When grouping the terms in $\Delta v(k)$ and $\Delta z(k)$ to derive the equations for $\beta(k)$ and $\lambda(k)$, the terms $r_1(c(k)-v(k))$ and $r_2(x(k)-z(k))$ were neglected because at optimum $c(k)=v(k)$ and $x(k)=z(k)$,and thus the addition of these terms had no effect on the optimality of the EOP.

It is assumed that the structure of f and q is such that given $v(k)$ and $z(k), k\in[0,N-1]$, the values of $\alpha(k)$ and $\gamma(k), k\in[0,N-1]$ can be uniquely determined from (2.14) and (2.15). Observe that optimality conditions (2.9)-(2.11) are model-based, and that $\beta(k)$ and $\lambda(k), k\in[0,N-1]$ carry information on model-reality differences in curvature, in contrast to $\alpha(k)$ and $\gamma(k), k\in[0,N-1]$ which carry the information on model-reality in value.

Recall that , our task is to solve ROP described by (2.1). We have defined EOP which is equivalent to ROP and derived its necessary optimality conditions. Thus if we satisfy the optimality of EOP, we are also satisfying the optimality of ROP. It is intended to solve ROP by using model-based computations. Given specified values of parameters $\alpha(k)$ and $\gamma(k)$, multipliers $\beta(k)$ and $\lambda(k)$, and vectors $v(k)$ and $z(k)$, equations (2.6) and optimality conditions (2.9)-(2.11) are satisfied by solving the following modified model-based optimal control problem (MMOP) .

MMOP

$$\min_{c(k)} \mathbf{Q} = \phi(\bar{x}(N))+\sum_{k=1}^{N}\{q(x(k),c(k),u(k),\gamma(k))+$$
$$-\lambda(k)^T x(k)-\beta(k)^T c(k)\}$$

subject to

$$x(k+1)=f(x(k),c(k),u(k),\alpha(k))$$
$$x(0)=x_0$$
$$x_t(N)=0;\quad t\in[1,q] \tag{2.17}$$
$$\bar{x}(N)=[x_{q+1}(N),.....,x_n(N)]^T$$

Therefore, using prescribed values of vectors $\hat{p}(k), v(k)$ and $z(k)$ we can calculate the functions $\alpha(k)$, $\gamma(k)$, $\beta(k)$ and $\lambda(k)$ from (2.14), (2.15) and (2.13), and if the solution $c(k), x(k)$ and $p(k)$ of MMOP obtained from these function satisfy (2.16), then that solution is also the solution of ROP.

The above reasoning leads to the following centralized discrete DISOPE algorithm which, assuming convergence, achieves the solution of ROP through repeated solutions of MMOP (Roberts, 1992, Becerra, 1994, and Becerra and Roberts, 1996).

2.1.2 Centralized Discrete time DISOPE algorithm

Data: $f(.), q(.), x_o, \phi(.), N, r_1, r_2$, and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Step 0: Compute a nominal solution $c(k)^0, x(k)^0, p(k)^0$. Set $s=0$, $v(k)^0 = c(k)^0$, $z(k)^0 = x(k)^0, \hat{p}(k)^0 = p(k)^0; k \in [0,N]$

Step 1: Compute the parameters $\alpha(k)$ and $\gamma(k)$, $k \in [0,N-1]$, to satisfy (2.14) and (2.15). This is called parameter estimation step.

Step 2: Calculate the multipliers $\beta(k)^s$ and $\lambda(k)^s$; $k \in [0,N-1]$ from equation (2.13).

Step 3: With prescribed $v(k)^s$, $z(k)^s$, $\alpha(k)^s$, $\gamma(k)^s$, $\beta(k)^s$ and $\lambda(k)^s$; $k \in [0,N-1]$ solve the model-based modified optimal control problem to obtain $x(k)^{s+1}, p(k)^{s+1}; k \in [0,N]$ and $c(k)^{s+1}; k \in [0,N-1]$. This is called the system optimization step.

Step 4: Test the convergence and update the estimates (Equation 2.16) for the optimal solution of real optimization problem. If $v(k)^{s+1} = v(k)^s$; $k \in [0,N-1]$ within a defined tolerance, stop, else set $s = s+1$ and repeat from Step 1.

A version of the centralized discrete time DISOPE algorithm which uses a linear model and quadratic performance index was developed and implemented in the C++ programming language by Becerra (1994) and Becerra and Roberts (1996). The choice of a linear quadratic model enables the solution of the MMOP to be computed using the backward sweep method (Bryson and Ho, 1975; Lewis, 1986). It was observed from the simulation results that the convexification scalar r_1 and the stepsize for updating the control signal has the greatest influence on the convergence of the centralized DISOPE algorithm with linear quadratic model-based problem. For details of derivation and discussion of simulation results see the references quoted above.

The technique described above in deriving the centralize DISOPE algorithm will again be utilized in deriving algorithms for hierarchical optimal control for interconnected systems. In the next section we describe the hierarchical control concept and methodologies.

2.3 DECOMPOSITION OF AN INTERCONNECTED CONTROL SYSTEM

A large scale system may be described as a complex system composed of a number of constituents or smaller subsystems serving particular functions and sharing resources, which is governed by interconnected goals and constraints. In an interconnected system such as illustrated by figure 2.1 below, it is often impractical to design a single control system for the entire process.

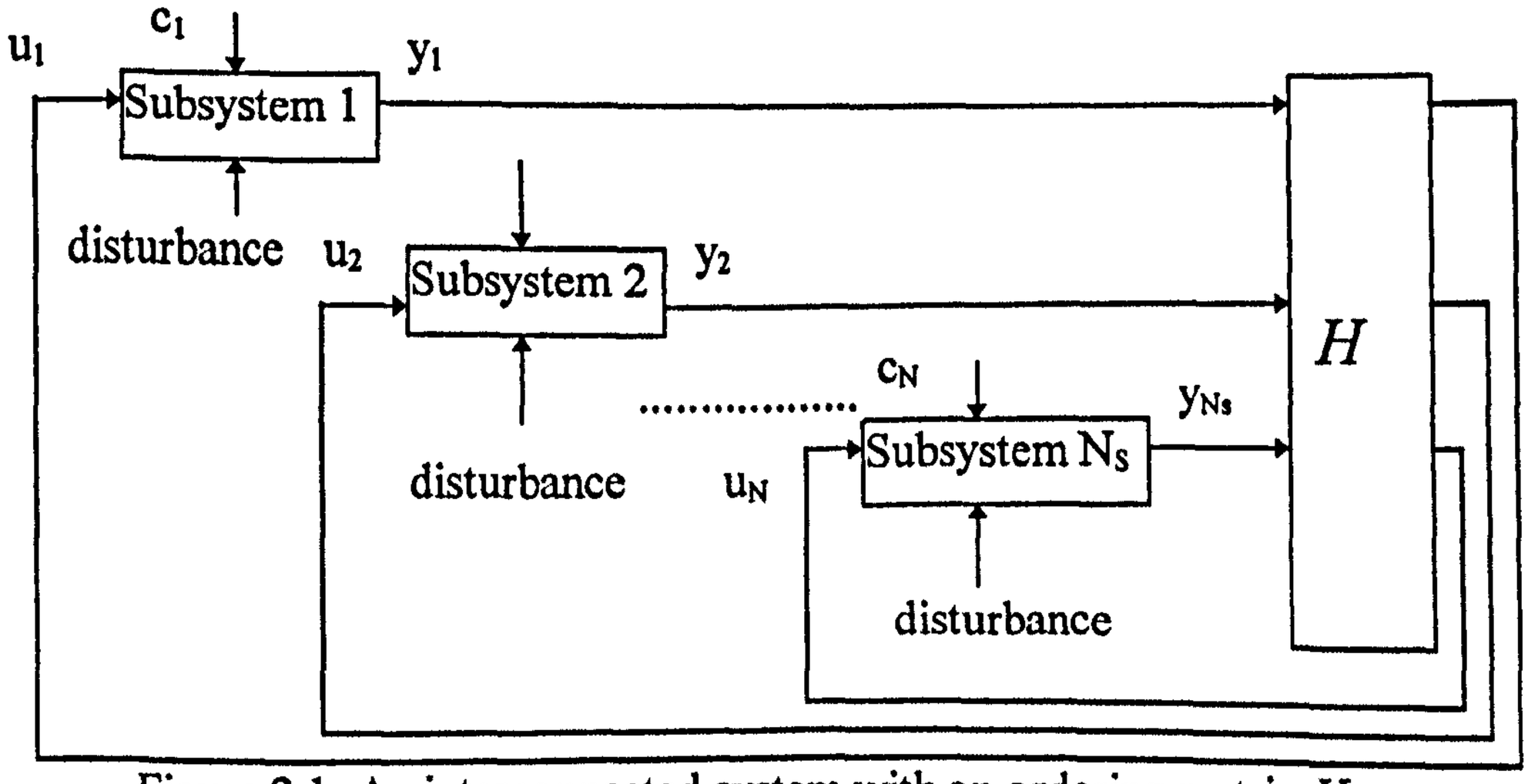

Figure 2.1. An interconnected system with an ordering matrix *H*.

Consequently, it is necessary to decompose the problem into a number of smaller problems where the individual subsystems can be solved more easily. It is important to solve the individual problems in such a way that the overall objective is achieved. The decomposition can be achieved using hierarchical structures.

The two main approaches are the multilayer concept (Lefkowitz, 1966) and the multilevel concept (Mesarovic et al, 1970). In the first concept the control task is split into layers, each of which acts at different time interval. In the second one the control of the system is divided into local goals, and local control units are introduced where their action is regulated by a coordinator in a higher level. Mesarovic(1970) proposed three different approaches to decomposition of an interconnected system which were derived using the two concepts of decomposition. They are multilevel hierarchical approach, multistrata hierarchical approach and the multilayer hierarchical approach.

2.3.1 Multilevel Hierarchical Approach

The essential idea in the multilevel technique is to solve, independently, decomposed subsystems problems at a lower level using a set variables called 'coordinating variables' prescribed by a higher level. At this higher level, the coordinating variables are iteratively improved, using information from the lower level. A two level hierarchical scheme is illustrated in figure 2.1.

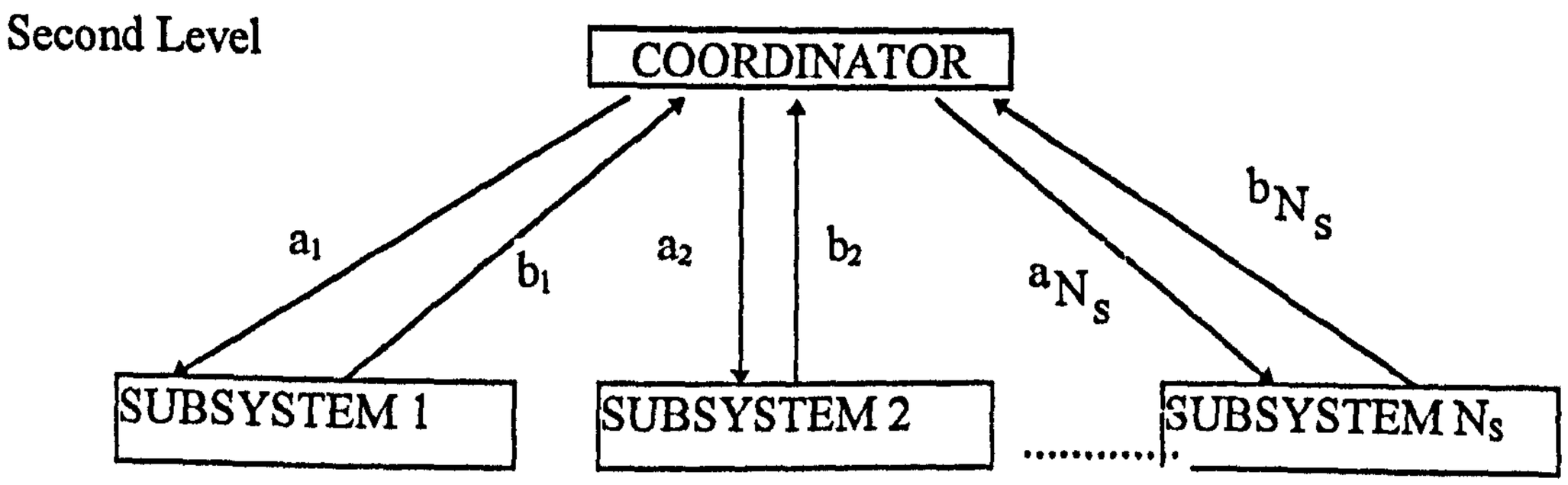

Fig. 2.2. Structure of a two-level hierarchical system.

At the first level, N_S subsystems of the original interconnected system are shown. At the second level a coordinator receives the local solutions of the N_S subsystems,

$b_i, i = [1, N_s]$, and then provides a new set of coordinating variables, $a_i, i = [1, N_s]$. The aim of the coordinator is to arrange the activities of the subsystems to provide a feasible solution to the overall problem.

Figure 2.3 show a multilevel, multi-objective system. The control is distributed into levels and arranged in a hierarchy with a pyramid structure. The controllers on the first level control each of the interconnected subsystems, whereas the controllers on the second level are then assigned the task of coordinating groups of first level controllers. Similarly, third level control may in turn control second level units which results in a pyramid hierarchy. The higher levels in the hierarchy must act accordingly so that the global solution is obtained.

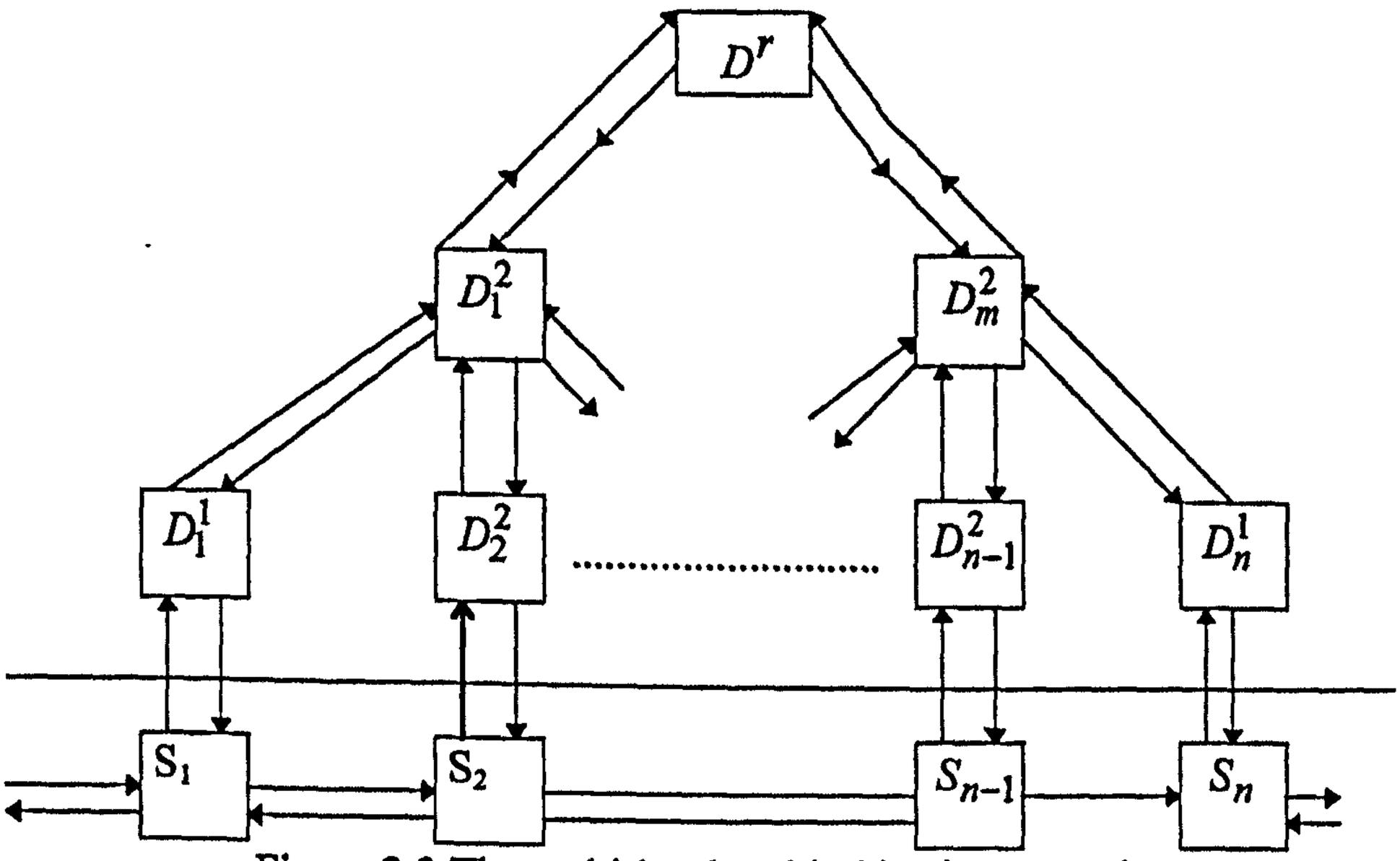

Figure 2.3 The multi-level multi-objective control structure

2.3.2 Multistrata Hierarchical Approach

In this approach ,the decomposition of the control system is based on the level of influence. The problem is separated into a number of smaller better defined subproblems and each of the subproblems is solved separately. Figure 2.4 illustrates the decomposition based on stratum, where all the strata are acting in parallel and in general the higher the level the less often control action takes place.

The main characteristics of multistrata structure are:

i) Each strata are assigned different tasks.

ii) The higher strata has a higher priority over the lower.

iii) Individual strata considers a different time horizon with the higher strata having the longer horizon.

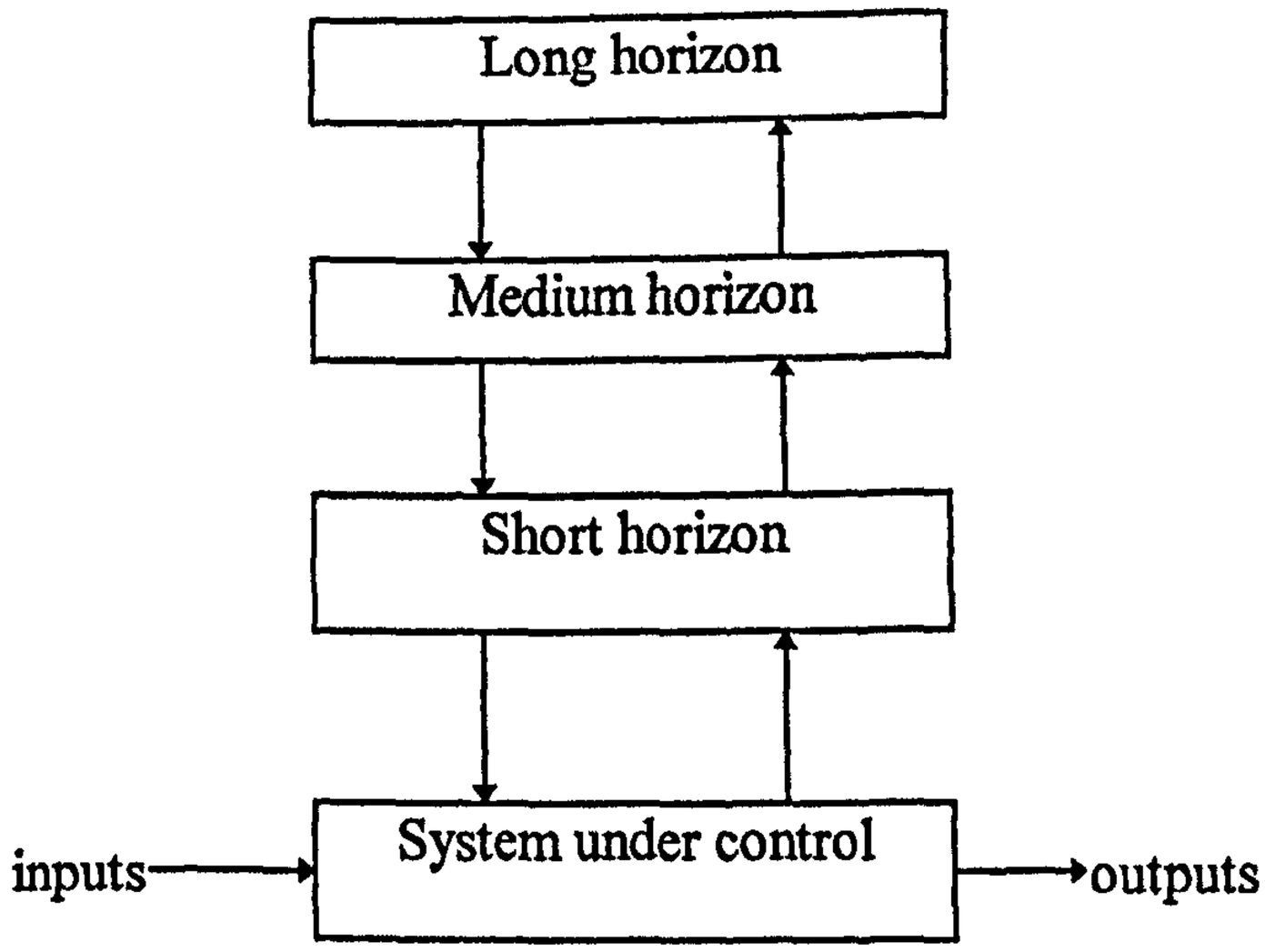

Figure 2.4 The decomposition of control system on the basis of strata.

2.3.4 The multilayer hierarchical approach

In this approach the decomposition is based on the complexity of the control tasks. The control task are distributed in a vertical division (Singh and Titli, 1978; Leigh, 1992) as shown in figure 2.5 below.

The lowest level is the of regulation layer. Its task is to maintain the chosen variables at the desired values despite fast-disturbances acting upon the process. The second level is that of optimization. The task of the control here is to determine the best values of chosen variables to ensure that some overall measure of the system performance is maximized. The parameter adaptation which is in the third level is concerned with the adaptation or updating of parameter values used in the mathematical models employed in the optimization layer. The aim of such adaptation is to maintain the validity of the model despite changes in the system.

The fourth level is that of model adaptation. This layer is responsible for selecting the structure and strategies for the lower layers such that an overall objective is achieved

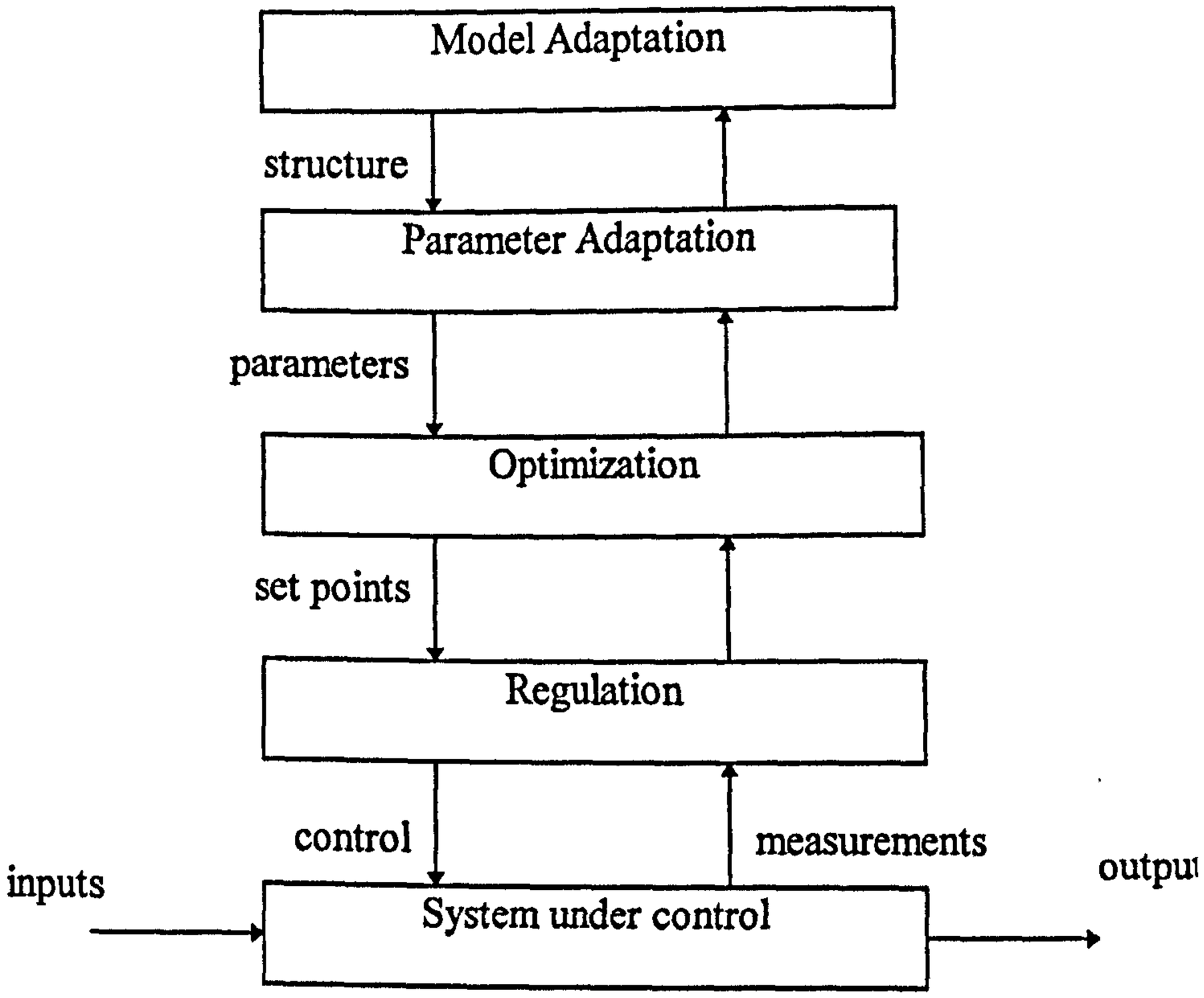

Figure 2.5 A multilayer control strategy

2.4 COORDINATION OF HIERARCHICAL STRUCTURES

It was mentioned in the previous section that an interconnected system can be hierarchically controlled by decomposing it into a number of subsystems and then coordinating the resulting subproblems to transform a given integrated system into a multilevel one. There are many ways to carry out the transformation. Most of the schemes, however, are based on two distinct approaches (Jamshidi, 1983): the model-coordination method and goal coordination method (Mesarovic et al, 1969). For simplicity we described the two coordination approaches for a two subsystem static optimization problem.

2.4.1 Model Coordination Method.

Consider the following static optimization problem (Schoeffler, 1971; Jamshidi, 1983):

$$\min \; q(x,c,u)$$

$$\text{subject to } f(x,c,u)=0 \tag{2.18}$$

where x is a vector of state variables, c is a vector of control variables and u is the interaction. Suppose the problem can be decomposed into two subsystems, i.e.

$$q(x,c,u)=\sum_{i=1} q_i(x_i,c_i,u_i) \tag{2.19}$$

$$\text{subject to } \quad f_i(x_i,c_i,u_1,u_2)=0; \;\; i\in 1,2$$

The subsystems are still interconnected through vectors u_i ; $i\in 1,2$ .The objective of the model coordination method is to convert the integrated problem (2.18) into a two-level problem by fixing interaction variables u_i; $i\in 1,2$ at some value for example w_i ; $i\in 1,2$, that is

$$\text{Constraint} \quad u_i = w_i \;\; ; i \in 1,2 \tag{2.18}$$

Therefore problem (2.16) can be decomposed into the following two level problems:

First level problem(Subsystem i)

$$\text{Find } \; P_i(w) = \min_{x_i,c_i} q_i(x_i,c_i,u_i)$$

$$\text{subject to } \quad f_i(x_i,c_i,w_1,w_2) \tag{2.19}$$

Second level problem(Coordinator)

$$\min_{w} P(w) = P_1(w) + P_2(w) \tag{2.20}$$

The minimization is to be done, respectively, over the following feasible sets:

$$S_{1,i} = \{(x_i,c_i): f_i(x_i,c_i,w) = 0\},\ i = 1,2 \tag{2.21}$$

$$S_{2,i} = \{w_i: P_i(w_i) \text{ exists}\},\ i = 1,2 \tag{2.22}$$

The hierarchical scheme for solving problem (2.16) using model coordination approach is illustrated in figure 2.7. In this procedure the coordinating variables w_i ; $i\in 1,2$ fix the interaction by adding a constraint to a mathematical model. The task of determining the coordinating variables is assigned to the coordinator in the second level. The first level problems are constructed by fixing certain interacting variables in the original optimization problem.

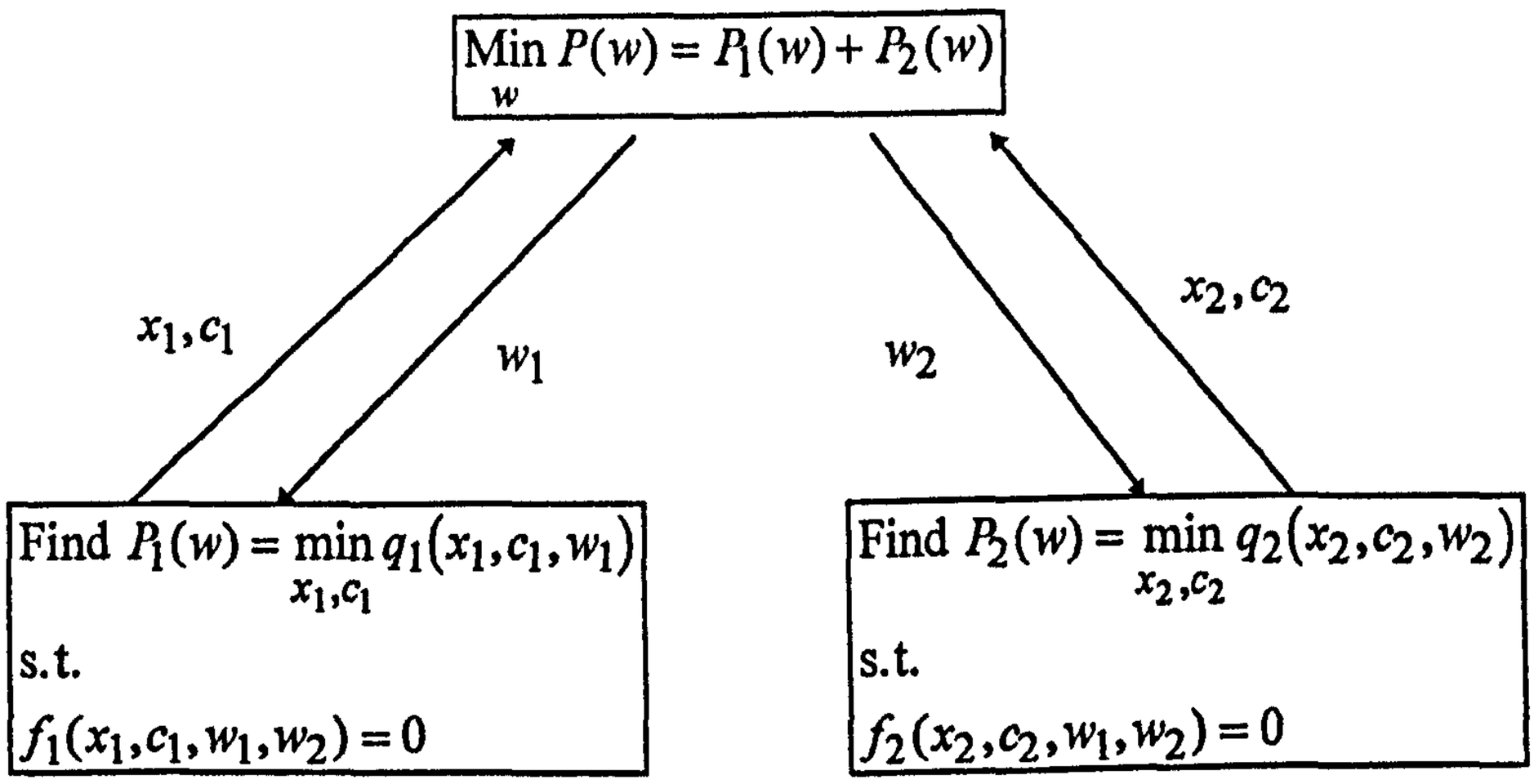

Figure 2.6 Hierarchical solution using model coordination

2.4.2 Goal Coordination Method

In this model based method the interaction between subsystems is removed by cutting all links between the subsystems. Consider again the static problem described by equation (2.16). Let y_i be the outgoing variable from the ith subsystem, while u_i denotes its incoming variable. Removal of all links between the subsystems implies that $y_i \neq u_i$. Under this condition, u_i acts as an arbitrary manipulated variable and should be chosen by the optimizing subsystems. As a result the global optimization problem (2.16) is completely decomposed into two subsystems and their performance indices are separated. In order to make sure the individual subproblems yield a solution to the original problem, it is necessary that interaction-balance principle be satisfied, i.e. $y_i = u_i$. (Mesarovic et al.,1969; Schoeffler, 1971; Jamshidi, 1983).

The procedure is to decompose problem (2.16) into decoupled subproblems which constitute the first level problem. The second level problem is to force the first level subproblems to a solution for which the interaction-balance holds. Mathematically, an additional penalty term is introduced to penalize the performance of the system if the interconnections do not balance. Hence the modified performance index (2.17) can be expressed as

$$q(x,c,u,y,l)=\sum_{i=1}^{2}\{q_i(x_i,c_i,u_i)\}+l^T(y-u) \tag{2.23}$$

where l is a Lagrange multiplier known as price vector which causes any interaction unbalance $(y-u)$ to affect the performance index.

By introducing the u variables the system's equation are given by

$$\begin{aligned} f_1(x_1,c_1,y_1,u_2)&=0\\ f_2(x_2,c_2,y_2,u_1)&=0 \end{aligned} \tag{2.24}$$

The modified performance index (2.23) is minimize over the following feasible set:

$$S_0=\{(x,c,y,u): f_1(.)=f_2(.)=0\} \tag{2.25}$$

Expanding the penalty term in (2.23),i.e.

$$l^T(y-u)=l_1^T(y_1-u_1)+l_2^T(y_2-u_2) \tag{2.26}$$

expression (2.23) can be decomposed to form individual modified performance indices. Therefore from equations (2.23), (2.24) and (2.25), the first level problem can be formulated as illustrated by figure 2.7.

The second level problem is to manipulate the coordinating variable l in order to derive the two subsystems interaction to zero, i.e.

$$\min_{l}\ e=\min_{l}(y-u) \tag{2.27}$$

The interaction balance is held by manipulating the performance indices of first level problems through variable l. Using Lagrange Duality Theorem, an algorithm for the coordinator can be determined and the following overall dual function can be obtained

$$D(l)=\sum_{i=1}^{N_s} D_i(l)=\min\ q(x,c,y,u,l) \tag{2.28}$$

It can be shown that the coordinating variable l can be interpreted as a vector of Lagrange multipliers and the second level problem can then be solved using well-known iterative methods such as Newton's or conjugate gradient methods (see Geoffrion (1971), Pearson (1971) , Singh (1981)).

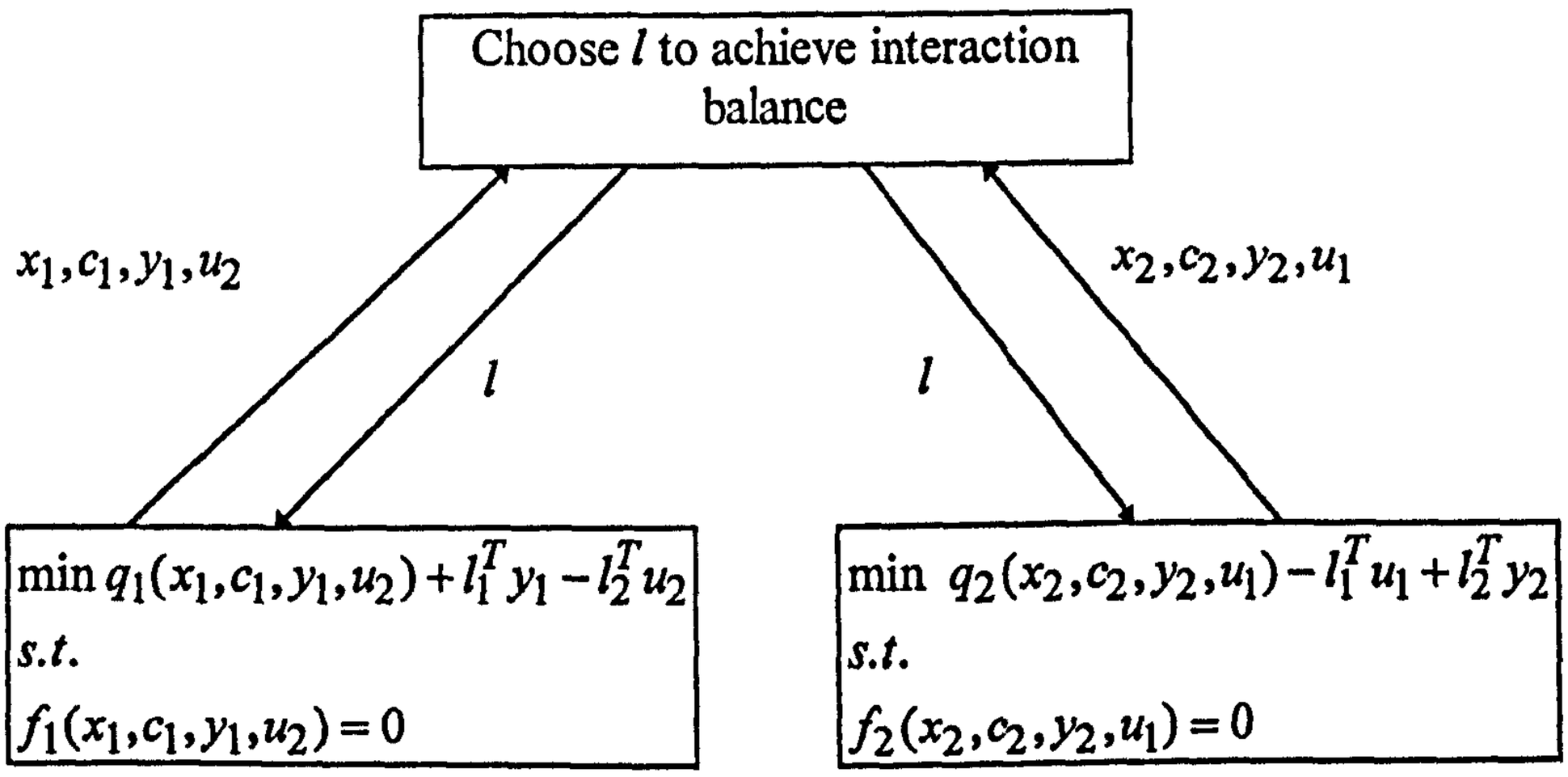

Figure 2.7 Hierarchical solution using goal coordination method.

2.5 APPLICATION OF DECOMPOSITION COORDINATION TO DYNAMICAL SYSTEMS

In this section we described briefly a technique developed, from the approaches discussed previously, for solving optimal control problem of dynamical systems. The two main existing techniques are goal coordination (Mesarovic et al, 1970) and interaction prediction techniques (Hassan and Singh, 1976). The major difference between them is that the coordination vector of the latter constitutes both the interaction vector as well as the Lagrange multipliers associated with the interconnection constraints. Application of goal coordination techniques to interconnected dynamical systems is described in Singh (1976). In the resulting procedure, the value of the coordinating vector is computed using a gradient-type method in the second level. One main disadvantage of the goal coordination approach stem from the slow numerical convergence at the second level and its adverse effects on first level calculations (Jamshidi, 1983). The interaction prediction approach does not have this convergence problem at the second level as the coordinator function is computed directly from optimality conditions of the associated Lagrangian. Note that this approach will be utilized in the development of hierarchical DISOPE algorithms as described in Chapters 4 and 6 of this thesis. In the following subsection we describe the application of the interaction prediction technique to a discrete interconnected system.

2.5.1 Interaction Prediction Technique

It is desired to solve the following linear optimal control problem with a quadratic performance index.

$$\min_{c_i(k),x_i(k)}\{\quad \mathbf{Q}^* = \sum_{i=1}^{N_s}\sum_{k=1}^{N}\frac{1}{2}\{x_i^T(k)Q_i x_i(k)+c_i^T(k)R_i c_i(k)\}\}$$

subject to

$$x_i(k+1) = A_i x_i(k) + B_i c_i(k) + D_i u_i(k) \tag{2.23}$$

$$x_i(0) = x_{io}$$

$$u_i(k) = \sum_{j=1}^{N_s}\{C_{ij}c_j(k)+W_{ij}x_j(k)\};\ i\in[1,N_s], k\in[1,N]$$

where $x_i(k)\in \Re^{n_i}$, $c_i(k)\in\Re^{m_i}$ and $u_i(k)\in\Re^{r_i}$ are the state, control and interaction vectors respectively. The interaction vector $u_i(k)$ represents the interactions between the ith subsystem and the remaining (N-1) subsystems. The integer r_i represents the number of incoming interactions to the ith subsystem. $Q_i \geq 0$, $R_i > 0$ are weighting matrices of appropriate dimensions. A_i, B_i and D_i are coefficients matrices of appropriate dimensions. C_{ij} and W_{ij} are appropriate interconnection matrices between control $c_i(k)$, state $x_i(k)$ and the state $x_i(k+1)$ respectively.

The Lagrangian of the problem defined above is given by

$$L = \sum_{i=1}^{N_s} L_i = \sum_{i=1}^{N_s}\{\sum_{k=1}^{N}\frac{1}{2}\{x_i^T(k)Q_i x_i(k)+c_i^T(k)R_i c_i(k)\}$$

$$+l_i^T(k)u_i(k) - \sum_{j=1}^{N_s} l_j^T(k)\{C_{ji}c_i(k)+W_{ji}x_i(k)\} \tag{2.24}$$

$$p_i^T(k+1)x_i(k+1) - A_i x_i(k) + B_i c_i(k) + D_i u_i(k)$$

It is assumed that Lagrangian L is additively separable for $u_i(k)$ and $l_i(k)$ trajectories. This implies that for any given $u_i(k)$ and $l_i(k)$, there are N_S independent minimization problems. A necessary condition for this is

$$\frac{\partial L}{\partial u_i} = \frac{\partial L}{\partial l_i} = 0 \tag{2.24}$$

which result in

$$l_i(k) + D_i^T p_i(k+1) = 0 \tag{2.25}$$

$$u_i(k) - \sum_{j=1}^{N_s} \{C_{ij} c_j(k) + W_{ij} x_j(k)\} = 0 \tag{2.26}$$

Therefore the coordination at the second level in this interaction prediction scheme is given by

$$\begin{bmatrix} l_i(k) \\ u_i(k) \end{bmatrix}^{s+1} = \begin{bmatrix} -D_i^T p_i(k+1) \\ \sum_{j=1}^{N_s} \{C_{ij} c_j(k) + W_{ij} x_j(k)\} \end{bmatrix}^s \tag{2.27}$$

for $k = [1, N_s]$ and s is the iteration number.

For a known set of augmented interaction vectors $[l^T(k) \mid u^T(k)]$, the ith subsystem Hamiltonian is

$$\begin{aligned} \mathbf{H}_i(.) = & \frac{1}{2}\{x_i^T(k) Q_i x_i(k) + c_i^T(k) R_i c_i(k)\} \\ & + l_i^T(k) u_i(k) - \sum_{j=1}^{N_s} l_j^T(k) \{C_{ji} c_i(k) + W_{ji} x_i(k)\} \\ & + p_i^T(k+1) x_i(k+1) - A_i x_i(k) + B_i c_i(k) + D_i u_i(k) \end{aligned} \tag{2.28}$$

The necessary condition (Lewis and Syrmos, 1995) for optimality is given by

$\nabla \mathbf{H}_{x_i(k)} - p_i(k) = 0$ i.e

$$p_i(k) = Q_i x_i(k) + A_i^T p_i(k+1) - \sum_{j=1}^{N_s} (l_j^T(k) W_{ji})^T ; p_i(N_s) = 0 \tag{2.29}$$

$\nabla \mathbf{H}_{c_i(k)} = 0$ i.e

$$c(k) = -R_i^{-1} B_i^T p_i(k+1) + R_i^T \sum_{j=1}^{N_s} (l_j^T(k) C_{ji})^T \tag{2.30}$$

$\nabla \mathbf{H}_{p_i(k+1)} = 0$ i.e.

$$x_i(k+1) = A_i x_i(k) + B_i c_i(k) + D_i u_i(k),\ x_i(0) = x_{io}; \tag{2.31}$$

Equations (2.29) and (2.31) with (2.30) constitute a two point boundary value problem the solution of which will give the optimum estimates of state $x_i(k)$ and control $c_i(k)$ for the ith subsystem for given values of $u_i(k)$ and $l_i(k)$;i∈[1,Ns]. The solution of the TPBVP can be obtained using backward sweep method (Bryson and Ho, 1975; Lewis and Syrmos, 1995).

From the above analysis the interaction prediction technique for optimal control of interconnected system can be summarized as follows:

Algorithm 2.5.1 Interaction prediction technique for interconnected system.

Step 1: At level 2, set s=1, assume initial values for $u_i(k)=u_i^a(k)$ and $l_i(k)=l_i^a(k)$. Send them to level 1, $i \in [1,N_s]$ and $k \in [1,N]$.

Step 2: Solve Ns TPBVP defined by (2.29)-(2.31) to obtain the optimum estimates of state $x_i(k)$ and control $c_i(k)$; $i \in [1,N_s]$ and $k \in [1,N]$

Step 3: Test for convergence of (2.25) and (2.26) .If within defined tolerance, then stop. Otherwise update $u_i(k)$ and $l_i(k)$;I∈[1,Ns] using (2.27), set s= s+1, and then go to step 2.

It is noted that the second level convergence of the interaction prediction method is very fast(Singh and Hassan,1976, Jamshidi, !983) when compared to the goal coordination algorithm. However one its disadvantage which it shared with the goal coordination approach is that it is sensitive to modeling errors.(Sandell et al, 1978).

2.6 SUMMARY

In this chapter we have described a centralized DISOPE algorithm for a discrete dynamical system. We have also described approaches for dealing with interconnected dynamical systems. Emphasis is made on the decomposition and coordination techniques of hierarchical systems as the techniques will be utilized in developments of algorithms later in this thesis. A brief description of the interaction prediction technique which utilizes some of the coordination-decomposition principles is also described.

In the next chapter we will developed optimal hierarchical structures for interconnected dynamical systems using the DISOPE approach and some of hierarchical control techniques described earlier .

CHAPTER 3

HIERARCHICAL STRUCTURES FOR OPTIMAL CONTROL OF INTERCONNECTED DISCRETE DYNAMICAL SYSTEMS

3.1. INTRODUCTION

In this chapter, we derive four basic hierarchical structures for solving the optimal control problem of interconnected discrete dynamical systems. Recently DISOPE was extended to hierarchical control of large scale systems by incorporating the interaction measurements in the model reality difference parameter of the subsystem's plant dynamics (Beccera, 1994). Here, a more general strategy is suggested. This is performed by taking into account the manner we incorporate and utilize real process measurements from interaction inputs in the model based optimal control problem (MOP). This approach had first been suggested by Brdys and Roberts (1986) for solving hierarchical steady state optimizing control. Through appropriate integration of model-based optimization and model parameter estimation, assuming convergence of the associated iterative scheme, it is then shown that the solution produced by each new structure converged to the solution of real optimal control problem (ROP).

With increasing sophistication and decreasing cost of microprocessors, more control schemes are being implemented digitally. In these procedures, the control input is switched between different values at discrete time steps. The control signal is normally held constant between samples by a zero-order hold. Such controls are usually designed using a discretized version of the continuous plant. There are also processes which are discrete in nature and can only be controlled by using discrete time controllers (Leigh,1992; Franklin et al,1990).

3.1.1 System Description

We assume that the interconnected system (see fig.2.1) is an arrangement of N_s subsystems. Each of which is described by a state equation

$$x_i(k+1) = f_i^*(x_i\ (k), c_i(k), u_i(k), k)\ ; \mathrm{k} \in [0, \mathrm{N\text{-}1}] \qquad (3.1)$$

with initial condition

$$x_i(0) = x_{io}$$

and an input/output equation

$$y_i(k) = F_i^*(x_i\ (k), c_i(k), u_i(k), k)\ ; \mathrm{k} \in [0, \mathrm{N}] \qquad (3.2)$$

where $f_i^*: \Re^{n_i} \times \Re^{m_i} \times \Re^{r_i} \times \Re \rightarrow \Re^{n_i}$ represents the ith subsystem's real dynamics and $F_i^*: \Re^{n_i} \times \Re^{m_i} \times \Re^{r_i} \times \Re \rightarrow \Re^{r_i}$ represents the ith subsystem's real input and output equation. $x_i\ (k) \in \Re^{n_i}$, $c_i(k) \in \Re^{m_i}$ and $u_i(k) \in \Re^{r_i}$ are the state, control and interaction vectors respectively. $y_i(k) \in \Re^{r_i}$ is the ith output vector.

The structure of the interconnection between the subsystems is given by

$$u_i(k) = H_i y = \sum_{j=1}^{N_s} H_{ij} y_j\ ; \mathrm{k} \in [0, \mathrm{N}] \qquad (3.3)$$

where $y = [y_1, \ldots y_{Ns}]$ and H_i and H_{ij} are the interconnection matrices composed of zeros and ones. The ith performance index is given by

$$\min\ \mathbf{Q}_i^* = \sum_{k=0}^{N-1} q_i^*(x_i(k), c_i(k), u_i(k), k);\ \ \mathrm{k} \in [0, \mathrm{N}] \qquad (3.4)$$

where $q_i^*: \Re^{n_i} \times \Re^{m_i} \times \Re^{r_i} \times \Re \rightarrow \Re$ is the discrete real performance function. If we define global vectors

$$x(k) = [x_1(k), \ldots, x_{N_s}(k)] \in \Re^{n_1} \times \ldots\ldots \times \Re^{n_{N_s}} \in \Re^{n}$$

$$c(k) = [c_1(k), \ldots, c_{N_s}(k)] \in \Re^{m_1} \ldots\ldots \times \Re^{m_{N_s}} \in \Re^{m}$$

$$u(k) = [u_1(k), \ldots, u_{N_s}(k)] \in \Re^{r_1} \times \ldots\ldots \times \Re^{r_{N_s}} \in \Re^{r}$$

$$y(k)=[y_1(k),...,y_{N_s}(k)]\in\Re^{r_1}\times.....\times\Re^{r_{N_s}}\in\Re^{r} \tag{3.4a}$$

then we can describe the global relationship of the whole system as follows:

$$x(k+1)=f^{*}(x(k),c(k),u(k),k)\,; \qquad k\in[0,\text{N-1}]$$

$$x(0)=x_0 \tag{3.5}$$

$$y(k)=F^{*}(x(k),c(k),u(k),k) \tag{3.6}$$

$$u(k)=Hy(k) \tag{3.7}$$

where $f^{*}=[f_1^{*},....,f_{N_s}^{*}]$, $F^{*}=[F_1^{*},.....,F_{N_s}^{*}]$ and $H=(H_{ij})$; $i,j\in[1,\ N_s]$

It is assumed that equations (3.6) and (3.7) can be uniquely solved with respect to the pair $\{x(k),c(k)\}$ so that the output function can be expressed as

$$y(k)=K^{*}(x(k),c(k)) \tag{3.8}$$

where $K^{*}:\Re^{n}\times\Re^{r}\rightarrow\Re^{r}$ and $K^{*}=[K_1^{*},....,K_{N_s}^{*}]$.

It is also assumed that the global performance index is additively separable and is given by

$$\mathbf{Q}^{*}=\sum_{i=1}^{N_s}\mathbf{Q}_i^{*}=\sum_{i=1}^{N_s}q^{*}(x(k),c(k),u(k),k) \tag{3.9}$$

where $q^{*}(x(k),c(k),u(k),k)=\sum_{i=1}^{N_s}q_i{}^{*}(x_i(k),c_i(k),u_i(k),k)$.

It is also assumed that the mappings f^{*},K^{*} and the performance index $\mathbf{Q}^{*}$ is Frechet differentiable. Hence the task of determining the optimal control of the global problem can be defined as the following real optimal control problem (ROP):

$$\min\ \mathbf{Q}^{*}=\sum_{k=1}^{N-1}q^{*}(x(k),c(k),u(k),k) \tag{3.10}$$

subject to

$$x(k+1)=f^{*}(x(k),c(k),u(k),k)\,;\ k\in[0,\text{N-1}]$$

$$x(0)=x_0$$

$$y(k)=F^{*}(x(k),c(k),u(k),k)$$

$$u(k)=Hy(k)$$

3.1.2 Necessary optimality conditions of global ROP

In this section we derive the necessary optimality conditions of the ROP. (Bryson and Ho,1975 and Lewis,1986).

Eliminating $y(k)$ we can write the global ROP as:

ROP

$$\min \mathrm{Q}^* = \sum_{k=1}^{N-1} q^*(x(k),c(k),u(k),k)$$

subject to

$$x(k+1) = f^*(x(k),c(k),u(k),k)\,;\, \mathrm{k} \in [0, \mathrm{N\text{-}1}]$$

$$x(0) = x_0$$

$$u(k) = HK^*(x(k),c(k))$$

adjoining the constraints by using Lagrange multipliers we obtain the following augmented performance index

$$Q^*_{ROP} = \sum_{i=1}^{N} [q^*(x(k),c(k),u(k),k) + p(k+1)(f^*(x(k),c(k),u(k),k) - x(k+1))$$

$$+ l(k)^T (u(k) - HK^*(x(k),c(k))] \qquad (3.11)$$

where $p(k) \in \Re^n$ and $l(k) \in \Re^r$ are multiplier vector functions. $p(k)$ is usually termed as costate and $l(k)$ is defined as the price. If we define a Hamiltonian as

$$\mathrm{H}^*(.) = q^*(x(k),c(k),u(k),k) + p(k+1)^T (f^*(x(k),c(k),u(k),k)$$

$$+ l(k)^T (u(k) - HK^*(x(k),c(k)) \qquad (3.12)$$

then we can rewrite (3.11) as:

$$\mathrm{Q}_{ROP} = \sum_{i=1}^{N} [\mathrm{H}^*(.) - p(k+1)^T (x(k+1))] \qquad (3.13)$$

By using calculus of variations and relatively straight forward algebraic manipulation, the following necessary optimality conditions of the global ROP are obtained:

$\nabla_{c(k)} \mathrm{H}^* = 0$ that is

$$\frac{\partial^T q^*(.)}{\partial c(k)}+\frac{\partial^T f^*(.)}{\partial c(k)}p(k+1)-\frac{\partial^T K^*}{\partial c(k)}H^T l(k)=0 \qquad (3.14)$$

$\nabla_{x(k)}\mathbf{H}^* - p(k) = 0$ that is

$$\frac{\partial^T q^*(.)}{\partial x(k)}+\frac{\partial^T f^*(.)}{\partial x(k)}p(k+1)-\frac{\partial^T K^*}{\partial x(k)}H^T l(k)-p(k)=0 \qquad (3.15)$$

$\nabla_{u(k)}\mathbf{H}^* = 0$ that is

$$\frac{\partial^T q^*(.)}{\partial u(k)}+\frac{\partial^T f^*(.)}{\partial u(k)}p(k+1)+l(k)=0 \qquad (3.16)$$

$$x(k+1) = f^*(x(k),c(k),u(k),k)\,;\, k\in [0, N-1] \qquad (3.18)$$

$$u(k) = HF^*(x(k),c(k),u(k),k) \qquad (3.19)$$

$$x(0) = x_0 \qquad (3.20)$$

$$p(N) = 0. \qquad (3.21)$$

It is often necessary to represent the real process with some mathematical model. This model, which is an approximation of the real process contains model reality differences. The differences maybe are introduced deliberately in order to facilitate mathematical aspects of the design or arise due to uncertainties in the knowledge of the process mathematical structures and parameters. Instead of directly solving ROP, in the next section we consider a model based optimal control problem of the interconnected system.

3.2 MODEL BASED OPTIMAL CONTROL OF INTERCONNECTED SYSTEMS

We now define the model of the ith subsystem's dynamic and input output equation as

$$x_i(k+1) = f_i(x_i(k),c_i(k),u_i(k),\alpha(k))\,;\, k\in [0, N-1] \qquad (3.22)$$

with initial condition

$$x_i(0) = x_{io}$$

and an input/output equation

$$y_i(k) = F_i\ (x_i\ (k), c_i(k), u_i(k), \theta_i(k))\,;\, k\in [0, N] \qquad (3.23)$$

where $f_i^*:\Re^{n_i}\times\Re^{m_i}\times\Re^{r_i}\times\Re\rightarrow\Re^{n_i}$ and $F_i^*:\Re^{n_i}\times\Re^{m_i}\times\Re^{r_i}\times\Re\rightarrow\Re^{r_i}$ are approximations of f_i^* and F_i^*. The ith performance index is defined as:

$$\min \mathbf{Q}_i = \sum_{k=0}^{N} q_i\,(x_i\,(k), c_i(k), u_i(k), \gamma(k)); \quad k\in[0, N] \tag{3.24}$$

where $q_i:\Re^{n_i}\times\ \Re^{m_i}\times\Re^{r_i}\times\Re\rightarrow\Re$. It is assumed that the model mappings f_i , F_i and the performance index $\mathbf{Q}_i$ are Frechet differentiable. $\alpha_i(k)\in\Re^{n_i}$, $\theta_i(k)\in\Re^{n_i}$ and $\gamma_i(k)\in\Re$ are discrete parameters which take into account the model reality differences.

In addition (3.4a), we will also need the following definition:

$$\alpha(k)=[\alpha_1(k),...,\alpha_{N_s}(k)]\in\Re^{n_1}\times.....\times\Re^{n_{N_s}}\in\Re^{n}$$

$$\theta(k)=[\theta_1(k),...,\theta_{N_s}(k)]\in\Re^{r_1}\times.....\times\Re^{r_{N_s}}\in\Re^{r} \tag{3.24a}$$

$$\gamma(k)=[\gamma_1(k),...,\gamma_{N_s}(k)]\in\Re^{1}\times.....\times\Re^{N_s}\in\Re$$

we can now define the global model based optimization problem (MOP) which is equivalent to ROP as:

MOP

$$\min\ \mathbf{Q}\ = \sum_{k=1}^{N-1} q(x(k), c(k), u(k), \gamma(k))$$

subject to

$$x(k+1)=f(x(k),c(k),u(k),\alpha(k))\ ;\ k\in[0, N\text{-}1]$$

$$x(0)=x_0$$

$$y(k)=F(x(k),c(k),u(k),\theta(k))$$

$$u(k)=Hy(k)$$

$$f^*(x(k),c(k),u(k),k)=\ f(x(k),c(k),u(k),\alpha(k))$$

$$K^*(x(k),c(k))=\ F(x(k),c(k),u(k),\theta(k))$$

$$q^*(x(k),c(k),u(k),k)=\ q(x(k),c(k),u(k),\gamma(k))$$

The solution of the global MOP provides the control $c(k)$ the function of the current parameter estimates $\alpha(k)=\alpha(c(k))$, $\theta(k)=\theta(c(k))$ and $\gamma(k)=\gamma(c(k))$. In turn such

estimates are obtained by matching model and real states, output and performance index at the current computed control $c(k)=c(\alpha(k),\theta(k),\gamma(k))$. Hence the optimization steps and the parameter estimation steps interact and, in general, because the model is only a simplification of reality, several iterations may be required before convergence is achieved. However simply iterating between optimization and parameter estimation in general does not lead to the correct optimal solution of ROP (Durbeck, 1965; Foord,1974), and it is necessary to properly integrate the two steps taking into account their mutual interaction.

3.3 THE HIERARCHICAL DISOPE APPROACH

The structures for finding an optimal control shown in this section are iterative in nature. The current controls are generated by solving a modified optimal control problem based on the availability of information from the real system and the model. Depending on the manner we use this information we can derive four basic structures which are equivalent to the MOP. These structures are the dynamic analogy of structures obtained for hierarchical steady-state integrated system optimization and parameter estimation, ISOPE (Brdys and Roberts, 1986).

Eliminating y(k) in MOP we obtain an equivalent form MOP1 that is:

MOP1

$$\min \mathbf{Q} = \sum_{i=1}^{N-1} q(x(k),c(k),u(k),\gamma(k))$$

subject to

$$x(k+1)=f(x(k),c(k),u(k),\alpha(k)) \; ; \; k\in[0, N-1]$$

$$x(0)=x_0$$

$$u(k)=HF\ (x(k),c(k),u(k),\theta(k))$$

$$f^*(x(k),c(k),u(k),k)=f(x(k),c(k),u(k),\alpha(k))$$

$$K^*(x(k),c(k))=F(x(k),c(k),u(k),\theta(k))$$

$$q^*(x(k),c(k),u(k),k)=q\ (x(k),c(k),u(k),\gamma(k))$$

Noting that $K^*(x(k),c(k)) = F(x(k),c(k),u(k),\theta(k))$ an equivalent expression for interaction input can be written as:

$$u(k) = HK^*(x(k),c(k)) \tag{3.25}$$

Utilizing this in the parameter estimation steps to replace u(k), we obtain the second equivalent form (MOP2), which is:

MOP2

$$\min Q = \sum_{i=1}^{N-1} q(x(k),c(k),u(k),\gamma(k))$$

subject to

$$x(k+1) = f(x(k),c(k),u(k),\alpha(k)) ; k \in [0, N-1]$$

$$x(0) = x_0$$

$$u(k) = HF(x(k),c(k),u(k),\theta(k))$$

$$f^*(x(k),c(k),HK^*(x(k),c(k)),k) = f(x(k),c(k),HK^*(x(k),c(k)),\alpha(k))$$

$$K^*(x(k),c(k)) = F(x(k),c(k),HK^*(x(k),c(k)),\theta(k))$$

$$q^*(x(k),c(k),HK^*(x(k),c(k)),k) = q(x(k),c(k),HK^*(x(k),c(k)),\gamma(k))$$

The third equivalent form can be obtained by using the information from the real output in the interaction term instead of the output model

$$u(k) = HF(x(k),c(k),u(k),\theta(k)) \tag{3.26}$$

As a consequence of this it is no longer necessary to include the parameter estimation for $\theta(k)$. The MOP3 can be described as:

MOP3

$$\min Q = \sum_{i=1}^{N-1} q(x(k),c(k),u(k),\gamma(k))$$

subject to

$$x(k+1) = f(x(k),c(k),u(k),\alpha(k)) ; k \in [0, N-1]$$

$$x(0) = x_0$$

$$u(k) = HK^*(x(k),c(k))$$

$$f^*(x(k),c(k),HK^*(x(k),c(k)),k) = f(x(k),c(k),HK^*(x(k),c(k)),\alpha(k))$$

$$q^*(x(k),c(k),HK^*(x(k),c(k)),k) = q(x(k),c(k),HK^*(x(k),c(k)),\gamma(k))$$

Finally , we obtain the fourth equivalent form by eliminating all interaction variables $u(k)$.

MOP4

$$\min \mathbf{Q} = \sum_{i=1}^{N-1} q(x(k),c(k),HK^*(x(k),c(k)),\gamma(k))$$

subject to

$$x(k+1) = f(x(k),c(k),HK^*(x(k),c(k)),\alpha(k)) \; ; \; k \in [0, N-1]$$

$$x(0) = x_0$$

$$f^*(x(k),c(k),HK^*(x(k),c(k)),k) = f(x(k),c(k),HK^*(x(k),c(k)),\alpha(k))$$

$$q^*(x(k),c(k),HK^*(x(k),c(k)),k) = q(x(k),c(k),HK^*(x(k),c(k)),\gamma(k))$$

Using a similar procedure to that used for developing steady-state ISOPE algorithms (Roberts, 1988) , the four equivalent MOPs can be further expanded and solved to obtain their respective hierarchical iterative structures.

3.4 STRUCTURES WITH MODEL BASED INTERACTION INPUT

It is assumed that real outputs $y(k) = K^*(x(k),c(k))$ are available together with real model dynamics and the performance measure. Expanding MOP1 by introducing three new variables $z(k), v(k)$ and $w(k)$, produces an equivalent expanded optimal control problem (EOP):

EOP1

$$\min \mathbf{Q} = \sum_{k=1}^{N-1} q(x(k),c(k),u(k),\gamma(k))$$

subject to

$$x(k+1) = f(x(k),c(k),u(k),\alpha(k)) \; ; \; k \in [0,..N-1]$$

$$x(0) = x_0$$

$$u(k) = HF\ (x(k),c(k),u(k),\theta(k))$$

$$f^*(z(k),v(k),w(k),k) = f(z(k),v(k),w(k),\alpha(k))$$

$$K^*(z(k),v(k)) = F(z(k),v(k),w(k),\theta(k))$$

$$q^*(z(k),v(k),w(k),k) = q(z(k),v(k),w(k),\gamma(k))$$

$$x(k) = z(k)$$

$$c(k) = v(k)$$

$$u(k) = w(k)$$

Adjoining the constraints produces the following augmented performance index:

$$Q_{EOP1} = \sum_{i=1}^{N-1} \{q(x(k),c(k),u(k),\gamma(k)) \; p(k+1)^T[f \; (x(k),c(k),u(k),\alpha(k)) - x(k+1)]$$

$$+ l(k)^T[u(k) - HF \; (x(k),c(k),u(k),\theta(k))]$$

$$+ \mu(k)^T[f^*(z(k),v(k),w(k),k) - f(z(k),v(k),w(k),\alpha(k))]$$

$$+ \xi(k)^T[K^*(z(k),v(k)) - F(z(k),v(k),w(k),\theta(k))]$$

$$+ \eta(k)^T[q^*(z(k),v(k),w(k),k) - q(z(k),v(k),w(k),\gamma(k))]$$

$$\lambda(k)^T[z(k) - x(k)] + \beta(k)^T[v(k) - c(k)] + \zeta(k)^T[w(k) - u(k)]\} \qquad (3.27)$$

where $p(k) \in \Re^n$ is the costate vector, $l(k) \in \Re^r$ is the price vector, $\mu(k) \in \Re^n$, $\eta(k) \in \Re$ and $\xi(k) \in \Re^r$, $\lambda(k) \in \Re^n$, $\beta(k) \in \Re^m$ and $\zeta(k) \in \Re^r$ are Lagrange multiplier functions.

Define the Hamiltonian function H(.)

$$\mathrm{H}(.) = q(x(k),c(k),u(k),\gamma(k)) + p(k+1)^T f(x(k),c(k),u(k),\alpha(k))$$

$$+ l(k)^T[u(k) - HF(x(k),c(k),u(k),\theta(k))]$$

$$- \lambda(k)^T x(k) - \beta(k)^T c(k) - \zeta(k)^T u(k) \qquad (3.28)$$

Equation (3.27) can be re-written as

$$Q_{EOP1} = \sum_{k=1}^{N-1} \{\mathrm{H}(.) - p(k+1)^T x(k+1) + \lambda(k)^T z(k) + \beta(k)^T v(k) + \zeta(k)^T w(k)$$

$$+ \mu(k)^T[f^*(z(k),v(k),w(k),k) - f(z(k),v(k),w(k),\alpha(k))]$$

$$+ \xi(k)^T[K^*(z(k),v(k)) - F(z(k),v(k),w(k),\theta(k))]$$

$$+\ \eta(k)^T[q^*(z(k),v(k),w(k),k) - q(z(k),v(k),w(k),\gamma(k))]\} \qquad (3.29)$$

Now it is desired to examine the increment in Q_{EOP1} due to increments in all the variables. According to the Lagrange multiplier theory, at a constrained minimum this increment should be zero (Lewis and Syrmos,1995). Therefore

$$\Delta Q_{EOP1} = \sum_{j=1}^{N-1} \{[\frac{\partial\, \mathbf{H}(.)}{\partial c(k)}]\Delta c(k)+[\frac{\partial\, \mathbf{H}(.)}{\partial x(k)} - p(k)^T]\ \Delta x(k)+[\frac{\partial\, \mathbf{H}(.)}{\partial u(k)}]\Delta u(k)$$

$$+[\lambda(k)^T+\mu(k)^T\left[\frac{\partial f^*(.)}{\partial z(k)}-\frac{\partial f(.)}{\partial z(k)}\right]+\eta(k)\left[\frac{\partial q^*(.)}{\partial z(k)}-\frac{\partial q(.)}{\partial z(k)}\right]+\xi(k)^T\left[\frac{\partial K^*(.)}{\partial z(k)}-\frac{\partial F(.)}{\partial z(k)}\right]]\Delta z(k)$$

$$+[\beta(k)^T+\mu(k)^T\left[\frac{\partial f^*(.)}{\partial v(k)}-\frac{\partial f(.)}{\partial v(k)}\right]+\eta(k)\left[\frac{\partial q^*(.)}{\partial v(k)}-\frac{\partial q(.)}{\partial v(k)}\right]+\xi(k)^T\left[\frac{\partial K^*(.)}{\partial v(k)}-\frac{\partial F(.)}{\partial v(k)}\right]]\Delta v(k)$$

$$+[\zeta(k)^T+\mu(k)^T\left[\frac{\partial f^*(.)}{\partial w(k)}-\frac{\partial f(.)}{\partial w(k)}\right]+\eta(k)\left[\frac{\partial q^*(.)}{\partial w(k)}-\frac{\partial q(.)}{\partial w(k)}\right]+\xi(k)^T\left[\frac{\partial K^*(.)}{\partial w(k)}-\frac{\partial F(.)}{\partial w(k)}\right]]\Delta w(k)$$

$$+[\frac{\partial\, \mathbf{H}(.)}{\partial \alpha(k)}-\mu(k)^T\frac{\partial^T f(.)}{\partial \alpha(k)}]\Delta\alpha(k)+[\frac{\partial\, \mathbf{H}(.)}{\partial \gamma(k)}-\eta(k)\ \frac{\partial\, q(.)}{\partial \gamma(k)}]\Delta\gamma(k)$$

$$+[\frac{\partial\, \mathbf{H}(.)}{\partial \theta(k)}-\xi(k)^T\frac{\partial F(.)}{\partial \theta(k)}]\ \Delta\theta(k)+[\frac{\partial\, \mathbf{H}(.)}{\partial\, p(k+1)}-x(k+1)^T]\ \Delta p(k+1)$$

$$+[\frac{\partial\, \mathbf{H}(.)}{\partial \lambda(k)}+z(k)^T]\Delta\lambda(k)+[\frac{\partial\, \mathbf{H}(.)}{\partial \beta(k)}+v(k)^T]\Delta\beta(k)+[\frac{\partial\, \mathbf{H}(.)}{\partial l(k)}]\Delta l(k)$$

$$+[\frac{\partial\, \mathbf{H}(.)}{\partial \zeta(k)}+w(k)^T]\Delta\zeta(k)+[f^*(.)-f(.)]^T\Delta\mu(k)+[K^*(.)-F(.)]^T\Delta\xi(k)$$

$$+[q^*(.)-q(.)]\Delta\eta(k) \qquad (3.30)$$

where

$$\frac{\partial\, \mathbf{H}(.)}{\partial c(k)}=\frac{\partial q(.)}{\partial c(k)}+p(k+1)^T\frac{\partial f(.)}{\partial c(k)}-l(k)^T H\frac{\partial F(.)}{\partial c(k)}-\beta(k)^T$$

$$\frac{\partial\, \mathbf{H}(.)}{\partial x(k)}=\frac{\partial q(.)}{\partial x(k)}+p(k+1)^T\frac{\partial f(.)}{\partial x(k)}-l(k)^T H\frac{\partial F(.)}{\partial x(k)}-\lambda(k)^T$$

$$\frac{\partial\, \mathbf{H}(.)}{\partial u(k)}=\frac{\partial q(.)}{\partial u(k)}+p(k+1)^T\frac{\partial f(.)}{\partial u(k)}-l(k)^T\frac{\partial[u(k)-HF(.)]}{\partial u(k)}-\zeta(k)^T$$

$$\frac{\partial \mathbf{H}(.)}{\partial \alpha(k)} = p(k+1)^T \frac{\partial f(.)}{\partial \alpha(k)}; \qquad \frac{\partial \mathbf{H}(.)}{\partial \gamma(k)} = \frac{\partial q(.)}{\partial \gamma(k)};$$

$$\frac{\partial \mathbf{H}(.)}{\partial p(k+1)} = f^T(.); \qquad \frac{\partial \mathbf{H}(.)}{\partial \theta(k)} = -l(k)^T H \frac{\partial F(.)}{\partial \alpha(k)};$$

$$\frac{\partial \mathbf{H}(.)}{\partial d(k)} = u(k) - HF(.); \qquad \frac{\partial \mathbf{H}(.)}{\partial \lambda(k)} = -x(k)^T;$$

$$\frac{\partial \mathbf{H}(.)}{\partial \beta(k)} = -c(k)^T; \qquad \frac{\partial \mathbf{H}(.)}{\partial \zeta(k)} = -u(k)^T$$

According to Lagrange multiplier theory, when the increment $\Delta \mathbf{Q}_{EOP1} = 0$, the constrained minimum can be achieved. Setting to zero the coefficient of independent increment, we obtain the following optimality conditions of EOP1:

$$\frac{\partial q(.)}{\partial c(k)} + p(k+1)^T \frac{\partial f(.)}{\partial c(k)} - l(k)^T H \frac{\partial F(.)}{\partial c(k)} - \beta(k)^T = 0 \tag{3.31}$$

$$\frac{\partial q(.)}{\partial x(k)} + p(k+1)^T \frac{\partial f(.)}{\partial x(k)} - l(k)^T H \frac{\partial F(.)}{\partial x(k)} - \lambda(k)^T - p(k) = 0 \tag{3.32}$$

$$\frac{\partial q(.)}{\partial u(k)} + p(k+1)^T \frac{\partial f(.)}{\partial u(k)} - l(k)^T \frac{\partial [u(k) - HF(.)]}{\partial u(k)} - \zeta(k)^T = 0 \tag{3.33}$$

$$\lambda(k)^T + \mu(k)^T \left[\frac{\partial f^*(.)}{\partial z(k)} - \frac{\partial f(.)}{\partial z(k)} \right] + \eta(k) \left[\frac{\partial q^*(.)}{\partial z(k)} - \frac{\partial q(.)}{\partial z(k)} \right] + \xi(k)^T \left[\frac{\partial K^*(.)}{\partial z(k)} - \frac{\partial F(.)}{\partial z(k)} \right] = 0 \tag{3.34}$$

$$\beta(k)^T + \mu(k)^T \left[\frac{\partial f^*(.)}{\partial v(k)} - \frac{\partial f(.)}{\partial v(k)} \right] + \eta(k) \left[\frac{\partial q^*(.)}{\partial v(k)} - \frac{\partial q(.)}{\partial v(k)} \right] + \xi(k)^T \left[\frac{\partial K^*(.)}{\partial v(k)} - \frac{\partial F(.)}{\partial v(k)} \right] = 0 \tag{3.35}$$

$$\zeta(k)^T + \mu(k)^T \left[\frac{\partial f^*(.)}{\partial w(k)} - \frac{\partial f(.)}{\partial w(k)} \right] + \eta(k) \left[\frac{\partial q^*(.)}{\partial w(k)} - \frac{\partial q(.)}{\partial w(k)} \right] - \xi(k)^T \left[\frac{\partial F(.)}{\partial w(k)} \right] = 0 \tag{3.36}$$

$$p(k+1)^T \frac{\partial f(x(k),c(k),u(k),\alpha(k))}{\partial \alpha(k)} - \mu(k)^T \frac{\partial f(z(k),v(k),w(k),\alpha(k))}{\partial \alpha(k)} = 0 \tag{3.37}$$

$$\frac{\partial q(x(k),c(k),u(k),\gamma(k))}{\partial \gamma(k)} - \eta(k)^T \frac{\partial q(z(k),v(k),w(k),\gamma(k))}{\partial \gamma(k)} = 0 \tag{3.38}$$

$$-l(k)^T H\frac{\partial F(x(k),c(k),u(k),\theta(k)s)}{\partial\alpha(k)} - \xi(k)^T \frac{\partial F(z(k),v(k),w(k),\theta(k))}{\partial\theta(k)} = 0 \qquad (3.39)$$

$$x(k+1) = f(x(k),c(k),u(k),\alpha(k)) \;;\; k\in[0,..N-1] \qquad (3.40)$$

$$x(0) = x_0 \qquad (3.41)$$

$$u(k) = HF(x(k),c(k),u(k),\theta(k)) \qquad (3.42)$$

$$f^*(z(k),v(k),w(k),k) = f(z(k),v(k),w(k),\alpha(k)) \qquad (3.43)$$

$$K^*(z(k),v(k)) = F(z(k),v(k),w(k),\theta(k)) \qquad (3.44)$$

$$q^*(z(k),v(k),w(k),k) = q(z(k),v(k),w(k),\gamma(k)) \qquad (3.45)$$

$$x(k) = z(k) \qquad (3.46)$$

$$c(k) = v(k) \qquad (3.47)$$

$$u(k) = w(k) \qquad (3.48)$$

Introducing $\hat{p}(k)$ as a costate separation variable, that is

$$\hat{p}(k) = p(k) \qquad (3.49)$$

and applying optimality conditions (3.46) - (3.48) into equation (3.37), (3.38) and (3.39) gives

$$\mu(k) = \hat{p}(k) \qquad (3.50)$$

$$\eta(k) = 1 \qquad (3.51)$$

$$\xi(k) = -\mathrm{H}^T l(k) \qquad (3.52)$$

Substituting these equations into (3.34),(3.35) and (3.36) will provides us with the multipliers equations:

$$\lambda(k) = -\left[\frac{\partial f^*(.)}{\partial z(k)} - \frac{\partial f(.)}{\partial z(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial z(k)} - \frac{\partial q(.)}{\partial z(k)}\right]^T + \left[\frac{\partial K^*(.)}{\partial z(k)} - \frac{\partial F(.)}{\partial z(k)}\right]^T H^T l(k) \qquad (3.53)$$

$$\beta(k) = -\left[\frac{\partial f^*(.)}{\partial v(k)} - \frac{\partial f(.)}{\partial v(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial v(k)} - \frac{\partial q(.)}{\partial v(k)}\right]^T + \left[\frac{\partial K^*(.)}{\partial v(k)} - \frac{\partial F(.)}{\partial v(k)}\right]^T H^T l(k) \qquad (3.54)$$

$$\zeta(k) = -\left[\frac{\partial f^*(.)}{\partial w(k)} - \frac{\partial f(.)}{\partial w(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial w(k)} - \frac{\partial q(.)}{\partial w(k)}\right]^T - \left[\frac{\partial F(.)}{\partial w(k)}\right]^T H^T l(k)$$

(3.55)

The parameter estimation problem is defined by optimality conditions (3.43),(3.44) and (3.45). It is assumed that the structure of $f(.), q(.)$ and $F(.)$ is such that given $z(k)$, $v(k)$ and $w(k)$, k∈ [0,..N] the values of the model reality differences parameters $\alpha(k)$, $\gamma(k)$ and $\theta(k)$, k∈ [0,..N] can be uniquely determined in this step.

$$\alpha(k) = f^*(z(k),v(k),w(k)) - f(z(k),v(k),w(k),\alpha(k)) \tag{3.56}$$

$$\theta(k) = K^*(z(k),v(k)) - F(z(k),v(k),w(k),\theta(k)) \tag{3.57}$$

$$\gamma(k) = q^*(z(k),v(k),w(k)) - q(z(k),v(k),w(k),\gamma(k)) \tag{3.58}$$

Using equation (3.42), following a procedure suggested by (Findeisen et al, 1980) a price updating formula is obtained as follows:

$$l(k+1)^{s+1} = l(k)^s + \varepsilon_l(u(k) - HF(x(k),c(k),u(k),\theta(k)) \tag{3.59}$$

where s is the iteration number.

From equations (3.46)-(3.49) we can define the following relaxation formulae for updating the variables $z(k), v(k), w(k)$ and $\hat{p}(k)$ at each iteration.

$$z(k)^{s+1} = z(k)^s + \varepsilon_z(x(k)^s - z(k)^s)$$

$$v(k)^{s+1} = v(k)^s + \varepsilon_v(c(k)^s - v(k)^s)$$

$$w(k)^{s+1} = w(k)^s + \varepsilon_w(u(k)^s - w(k)^s)$$

$$\hat{p}(k)^{s+1} = \hat{p}(k)^s + \varepsilon_p(p(k)^s - \hat{p}(k)^s) \tag{3.60}$$

For a given $\alpha(k), \gamma(k), \theta(k), \beta(k), \lambda(k), \zeta(k)$ and $l(k)$ optimality conditions (3.31), (3.32) and (3.33) are satisfied by solving the following modified model based optimal control problem :

MMOP1

$$\min \mathbf{Q} = \sum_{k=1}^{N} \{q(x(k),c(k),u(k),\gamma(k)) + l(k)^T (u(k) - HF(x(k),c(k),u(k),\theta(k))$$

$$-\lambda(k)^T x(k) - \beta(k)^T c(k) - \zeta(k)^T u(k)\}$$

subject to

$$x(k+1) = f\ (x(k),c(k),u(k),\alpha(k))$$

$$x(0) = x_0$$

which is separable and can be decomposed into N local optimization problems:

MMOP1 $_i$

$$\min \mathbf{Q}_i = \sum_{k=1}^{N-1} \{q_i(x_i(k),c_i(k),u_i(k),\gamma_i(k))$$

$$+l_i(k)^T (u_i(k) - \sum_{j=1}^{Ns} l_i(k) H_{ij} F_i(x(k),c(k),u(k),\theta(k))$$

$$-\lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k) - \zeta_i(k)^T u_i(k)\}$$

subject to

$$x_i(k+1) = f_i\ (x_i(k),c_i(k),u_i(k),\alpha_i(k))$$

$$x_i(0) = x_{i0}$$

The above analysis leads to the single iterative hierarchical structure described in Section 3.4.1 for solving EOP1.

3.4.1 Hierarchical Structures and Information Exchange of EOP1

The hierarchical structure and information exchange in solving EOP1 are given in figure 3.1. The structure consists of two levels, the coordinator and N local optimization units. each local optimization unit is made up of the ith local optimal control problem, the parameter estimation step and the variable updating step. The coordinator consists of the price updating formula given by equation (3.59) and the modifier equations given by (3.53)-(3.55). Notice that each local optimization problem can be solved independently. Thus the structure is suitable for the application of parallel processing methods.

COORDINATOR

$$l(k+1)^{s+1} = l(k)^{s} + \varepsilon_l (w(k) - HF(z(k), v(k), w(k), \theta(k))$$

$$\lambda(k) = -\left[\frac{\partial f^*(.)}{\partial z(k)} - \frac{\partial f(.)}{\partial z(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial z(k)} - \frac{\partial q(.)}{\partial z(k)}\right]^T + \left[\frac{\partial K^*(.)}{\partial z(k)} - \frac{\partial F(.)}{\partial z(k)}\right]^T H^T l(k)$$

$$\beta(k) = -\left[\frac{\partial f^*(.)}{\partial v(k)} - \frac{\partial f(.)}{\partial v(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial v(k)} - \frac{\partial q(.)}{\partial v(k)}\right]^T + \left[\frac{\partial K^*(.)}{\partial v(k)} - \frac{\partial F(.)}{\partial v(k)}\right]^T H^T l(k)$$

$$\zeta(k) = -\left[\frac{\partial f^*(.)}{\partial w(k)} - \frac{\partial f(.)}{\partial w(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial w(k)} - \frac{\partial q(.)}{\partial w(k)}\right]^T - \left[\frac{\partial F(.)}{\partial w(k)}\right]^T H^T l(k)$$

l, λ, β, ζ

w, v, z, p, derivatives

LOCAL OPTIMIZATION PROBLEM

$$\min Q_i = \sum_{k=1}^{N} \{q_i(x_i(k), c_i(k), u_i(k), \gamma_i(k))$$

$$+ l_i(k)^T (u_i(k)$$

$$- \sum_{j=1}^{Ns} l_i(k) H_{ij} F_i(x(k), c(k), u(k), \theta(k))$$

$$- \lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)$$

$$- \zeta_i(k)^T u_i(k)\}$$

subject to

$$x_i(k+1) = f_i\ (x_i(k), c_i(k), u_i(k), \alpha_i(k))$$
$$x_i(0) = x_{i0}$$

PARAMETER ESTIMATION

$$\alpha_i(k) \leftarrow f_i^*(z_i(k), v_i(k), w_i(k))$$
$$= f_i(z_i(k), v_i(k), w_i(k), \alpha_i(k))$$
$$\theta_i(k) \leftarrow K^*{}_i(z_i(k), v_i(k))$$
$$= F_i(z_i(k), v_i(k), w_i(k), \theta_i(k))$$
$$\gamma_i(k) \leftarrow q_i^*(z_i(k), v_i(k), w_i(k))$$
$$= q_i(z_i(k), v_i(k), w_i(k), \gamma_i(k))$$

$\alpha_i, \gamma_i, \theta_i$

VARIABLES UPDATES

z_i, v_i, w_i

$$z_i(k)^{s+1} = z_i(k)^s + \varepsilon_z (x_i(k)^s - z_i(k)^s)$$
$$v_i(k)^{s+1} = v_i(k)^s + \varepsilon_v (c_i(k)^s - v_i(k)^s)$$
$$w_i(k)^{s+1} = w_i(k)^s + \varepsilon_w (u_i(k)^s - w_i(k)^s)$$
$$\hat{p}_i(k)^{s+1} = \hat{p}_i(k)^s + \varepsilon_p (p_i(k)^s - \hat{p}_i(k)^s)$$

Figure 3.1. A two level structure with model based interaction input

3.5 STRUCTURES WITH REAL INTERACTION INPUT IN PARAMETER ESTIMATION.

In this technique we consider the structure given by MOP2 in which the real interaction input, $u(k) = HK^*(x(k),c(k))$; $k \in [1,..N-1]$, is utilized in the parameter estimation problem. We introduce additional constraints $z(k)=x(k)$ and $v(k)=c(k)$ in MOP2 to form the following equivalent Expanded Optimal Control Problem (EOP2).

EOP2

$$\min \mathbf{Q} = \sum_{k=1}^{N-1} q(x(k),c(k),u(k),\gamma(k))$$

subject to

$$x(k+1)=f(x(k),c(k),u(k),\alpha(k)) \text{ ; k}\in[0,..\text{N-1}]$$

$$x(0) = x_0$$

$$u(k)=HF(x(k),c(k),u(k),\theta(k))$$

$$f^*(z(k),v(k),HK^*(z(k),v(k)),k)= f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k))$$

$$K^*(z(k),v(k))= F(z(k),v(k),HK^*(z(k),v(k)),\theta(k))$$

$$q^*(z(k),v(k),HK^*(z(k),v(k)),k)=q(z(k),v(k),HK^*(z(k),v(k)),\gamma(k))$$

$$x(k)=z(k)$$

$$c(k)=v(k)\ c(k) = v(k)$$

Adjoining the constraints using Lagrange multipliers as in the previous derivation, we obtain:

$$\mathbf{Q}_{EOP2} = \sum_{i=1}^{N-1} \{q(x(k),c(k),u(k),\gamma(k)) + p(k+1)^T[f\ (x(k),c(k),u(k),\alpha(k))-x(k+1)]$$

$$+ l(k)^T[u(k)-HF(x(k),c(k),u(k),\theta(k))]$$

$$+ \mu(k)^T[f^*(z(k),v(k),HK^*(z(k),v(k)),k) - f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k))]$$

$$+ \xi(k)^T[K^*(z(k),v(k)) - F(z(k),v(k),HK^*(z(k),v(k)),\theta(k))]$$

$$+ \eta(k)^T[q^*(z(k),v(k),HK^*(z(k),v(k)),k) - q(z(k),v(k),HK^*(z(k),v(k)),\gamma(k))]$$

$$\lambda(k)^T[z(k)-x(k)]+\beta(k)^T[v(k)-c(k)]\} \qquad (3.61)$$

where $p(k) \in \Re^n$ $, l(k) \in \Re^r, \mu(k) \in \Re^n, \eta(k) \in \Re, \xi(k) \in \Re^r, \lambda(k) \in \Re^n$ and $\beta(k) \in \Re^m$, are Lagrange multiplier functions.

Define:

$$\mathbf{H}(.) = q(x(k),c(k),u(k),\gamma(k)) + p(k+1)^T f(x(k),c(k),u(k),\alpha(k))$$
$$+ l(k)^T[u(k) - HF\ (x(k),c(k),u(k),\theta(k))]\ - \lambda(k)^T x(k) - \beta(k)^T c(k) \qquad (3.62)$$

Following similar procedure outline in Section 3.4, we then use (3.62) and apply Lagrange multiplier theory to obtain the minimum of (3.61). As a consequence of this the necessary optimality conditions of EOP2 are obtained:

$$\frac{\partial q(.)}{\partial c(k)} + p(k+1)^T \frac{\partial f(.)}{\partial c(k)} - l(k)^T H \frac{\partial F(.)}{\partial c(k)} - \beta(k)^T = 0 \qquad (3.63)$$

$$\frac{\partial q(.)}{\partial x(k)} + p(k+1)^T \frac{\partial f(.)}{\partial x(k)} - l(k)^T H \frac{\partial F(.)}{\partial x(k)} - \lambda(k)^T \text{-} p(k) = 0 \qquad (3.64)$$

$$\frac{\partial q(.)}{\partial u(k)} + p(k+1)^T \frac{\partial f(.)}{\partial u(k)} - l(k)^T \frac{\partial [u(k) - HF(.)]}{\partial u(k)} = 0 \qquad (3.65)$$

$$\lambda(k)^T + \mu(k)^T \left[\frac{\partial f^*(.)}{\partial z(k)} - \frac{\partial f(.)}{\partial z(k)}\right] + \eta(k) \left[\frac{\partial q^*(.)}{\partial z(k)} - \frac{\partial q(.)}{\partial z(k)}\right] + \xi(k)^T \left[\frac{\partial K^*(.)}{\partial z(k)} - \frac{\partial F(.)}{\partial z(k)}\right] = 0$$

(3.66)

$$\beta(k)^T + \mu(k)^T \left[\frac{\partial f^*(.)}{\partial v(k)} - \frac{\partial f(.)}{\partial v(k)}\right] + \eta(k) \left[\frac{\partial q^*(.)}{\partial v(k)} - \frac{\partial q(.)}{\partial v(k)}\right] + \xi(k)^T \left[\frac{\partial K^*(.)}{\partial v(k)} - \frac{\partial F(.)}{\partial v(k)}\right] = 0$$

(3.67)

$$p(k+1)^T \frac{\partial f(x(k),c(k),u(k),\alpha(k))}{\partial \alpha(k)} - \mu(k)^T \frac{\partial f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k))}{\partial \alpha(k)} = 0$$

(3.68)

$$\frac{\partial q(x(k),c(k),u(k),\gamma(k))}{\partial \gamma(k)} - \eta(k)^T \frac{\partial f(z(k),v(k),w(k),\gamma(k))}{\partial \gamma(k)} = 0 \qquad (3.69)$$

$$-l(k)^T H\frac{\partial F(x(k),c(k),u(k),\theta(k)s)}{\partial\alpha(k)} - \xi(k)^T \frac{\partial F(z(k),v(k),HK^*(z(k),v(k)),\theta(k))}{\partial 0(k)} =0$$

(3.70)

$$x(k+1)= f(x(k),c(k),u(k),\alpha(k)) \ ; \ k\in[0,..N-1] \tag{3.71}$$

$$x(0) = x_0 \tag{3.72}$$

$$u(k)= HF(x(k),c(k),u(k),\theta(k)) \tag{3.73}$$

$$f^*(z(k),v(k),HK^*(z(k),v(k)),k)= f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k)) \tag{3.74}$$

$$K^*(z(k),v(k))= F(z(k),v(k),HK^*(z(k),v(k)),\theta(k)) \tag{3.75}$$

$$q^*(z(k),v(k),HK^*(z(k),v(k)),k)= q(z(k),v(k),HK^*(z(k),v(k)),\gamma(k)) \tag{3.76}$$

$$x(k)=z(k) \tag{3.77}$$

$$c(k)=v(k) \tag{3.78}$$

Introducing $\hat{p}(k)$ as costate separation variable, that is

$$\hat{p}(k) = p(k) \tag{3.79}$$

and applying it , equations (3.77) and (3.78) in optimality conditions (3.68)-(3.70) gives:

$$\mu(k) = \hat{p}(k)$$

$$\eta(k)= 1 \tag{3.80}$$

$$\xi(k) = -H^T l(k)$$

In this procedure we only need to calculate two modifiers which are determined by equations (3.66) and (3.67). That is:

$$\lambda(k)=-\left[\frac{\partial f^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial z(k)} - \frac{\partial f(z(k),c(k),HK^*(z(k),v(k)),\alpha(k))}{\partial z(k)}\right]^T \hat{p}(k+1)$$

$$-\left[\frac{\partial q^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial z(k)} - \frac{\partial q(z(k),c(k),HK^*(z(k),v(k)),\gamma(k))}{\partial z(k)}\right]^T$$

$$-\left[\frac{\partial K^*(z(k),c(k),k)}{\partial z(k)} - \frac{\partial F(z(k),c(k),HK^*(z(k),v(k)),\theta(k))}{\partial z(k)}\right]^T H^T l(k)$$

(3.81)

$$\beta(k)=-\left[\frac{\partial f^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial v(k)}-\frac{\partial f(z(k),c(k),HK^*(z(k),v(k)),\alpha(k))}{\partial v(k)}\right]^T \hat{p}(k+1)$$

$$-\left[\frac{\partial q^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial v(k)}-\frac{\partial q(z(k),c(k),HK^*(z(k),v(k)),\gamma(k))}{\partial v(k)}\right]^T$$

$$-\left[\frac{\partial K^*(z(k),c(k),k)}{\partial v(k)}-\frac{\partial F(z(k),c(k),HK^*(z(k),v(k)),\theta(k))}{\partial v(k)}\right]^T H^T l(k) \qquad (3.82)$$

Optimality conditions (3.74)-(3.76) define the parameter estimation step, that is:

$$\alpha(k) = f^*(z(k),v(k),HK^*(z(k),v(k)),k) - f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k))$$

$$\theta(k) = K^*(z(k),v(k)) - F(z(k),v(k),HK^*(z(k),v(k)),\theta(k)) \qquad (3.83)$$

$$\gamma(k) = q^*(z(k),v(k),HK^*(z(k),v(k)),k) - q(z(k),v(k),HK^*(z(k),v(k)),\gamma(k))$$

From equation (3.73) and using the formula previously defined in Section 3.4, the price *l(k)* is updated by:

$$l(k+1)^{s+1} = l(k)^s + \varepsilon_l(u(k) - HF(x(k),c(k),u(k),\theta(k)) \qquad (3.84)$$

At each iteration the separation variables $z(k), v(k)$ and $\hat{p}(k)$ are updated using relaxation formula define by (3.60).

Optimality conditions (3.63),(3.64) and (3.65) can be satisfied by solving the following model based optimal control problem for given $\alpha(k),\gamma(k),\theta(k),\lambda(k)$ and $\beta(k)$.

MMOP2

$$\min \mathrm{Q} = \sum_{k=1}^{N}\{q(x(k),c(k),u(k),\gamma(k)) + l(k)^T(u(k) - HF(x(k),c(k),u(k),\theta(k))$$

$$-\lambda(k)^T x(k) - \beta(k)^T c(k)\}$$

subject to

$$x(k+1) = f(x(k),c(k),u(k),\alpha(k))$$

$$x(0) = x_0$$

which is separable and can be decomposed into N local optimization problems:

MMOP2 $_i$

$$\min \mathbf{Q}_i = \sum_{k=1}^{N-1} \{q_i(x_i(k), c_i(k), u_i(k), \gamma_i(k))$$

$$+ l_i(k)^T (u_i(k) - \sum_{j=1}^{Ns} l_i(k) H_{ij} F_i(x(k), c(k), u(k), \theta(k))$$

$$- \lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)\}$$

subject to

$$x_i(k+1) = f_i \ (x_i(k), c_i(k), u_i(k), \alpha_i(k))$$

$$x_i(0) = x_{i0}$$

From the above analysis we can derive a basic hierarchical structure for solving EOP2.

3.5.1 Hierarchical Structure and Information Exchange of EOP2

The previous analysis leads to a two level structure shown in fig. 3.2. The coordinator in the upper level consists of the price updating mechanism (3.84) and the two modifier equations (3.81) and (3.82). The first level consists of N local optimization problems. Each of which contains a local optimal control unit, a parameter estimation step and a variable updating step. As in Section. 3.4.1, the local optimization unit is totally independent, thus making the whole procedure suitable for parallel processing techniques.

A comparison of these structure with that of Section 3.4 leads to the following observations. Firstly, the number of iterated variables is reduced because there is no variables $w(k)$. Secondly, computation in the coordinator is reduced because there no parameter $\zeta(k)$. Finally, it is no longer necessary for each local optimization problem to provide the coordinator with the value of parameter $\theta(k)$. These advantages are achieved at the cost of incorporating the real system interaction measurements in the parameter estimation steps.

COORDINATOR

$$l(k+1)^{s+1} = l(k)^{s} + \varepsilon_l (u(k) - HF(z(k), v(k), u(k), \theta(k))$$

$$\lambda(k) = -\left[\frac{\partial f^*(.)}{\partial z(k)} - \frac{\partial f(.)}{\partial z(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial z(k)} - \frac{\partial q(.)}{\partial z(k)}\right]^T + \left[\frac{\partial K^*(.)}{\partial z(k)} - \frac{\partial F(.)}{\partial z(k)}\right]^T H^T l(k)$$

$$\beta(k) = -\left[\frac{\partial f^*(.)}{\partial v(k)} - \frac{\partial f(.)}{\partial v(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial v(k)} - \frac{\partial q(.)}{\partial v(k)}\right]^T + \left[\frac{\partial K^*(.)}{\partial v(k)} - \frac{\partial F(.)}{\partial v(k)}\right]^T H^T l(k)$$

l, β, λ

u, z, v, p, θ derivatives

LOCAL OPTIMIZATION PROBLEM

$$\min Q_i = \sum_{k=1}^{N} \{q_i(x_i(k), c_i(k), u_i(k), \gamma_i(k))$$

$$+ l_i(k)^T u_i(k)$$

$$- \sum_{j=1}^{Ns} l_j^T(k) H_{ji} F_i(x_i(k), c_i(k), u_i(k), \theta_i(k))$$

$$- \lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)$$

subject to

$$x_i(k+1) = f_i\ (x_i(k), c_i(k), u_i(k), \alpha_i(k))$$

$$x_i(0) = x_{i0}$$

PARAMETER ESTIMATION

$$\alpha_i(k) \leftarrow f_i^*(z_i(k), v_i(k), H_i K^*(z_i(k), v_i(k)), k)$$

$$= f_i(z_i(k), v_i(k), H_i K^*(z_i(k), v_i(k)), \alpha_i(k))$$

$$\theta_i(k) \leftarrow K_i^*(z_i(k), v_i(k))$$

$$= F_i(z_i(k), v_i(k), H_i K^*(z_i(k), v_i(k)), \theta_i(k))$$

$$\gamma_i(k) \leftarrow q_i^*(z_i(k), v_i(k), H_i K^*(z_i(k), v_i(k)), k)$$

$$= q_i(z_i(k), v_i(k), H_i K^*(z_i(k), v_i(k)), \gamma_i(k))$$

$\alpha_i, \theta_i, \gamma_i$

VARIABLES UPDATES

$$z_i(k)^{s+1} = z_i(k)^s + \varepsilon_z (x_i(k)^s - z_i(k)^s)$$

$$v_i(k)^{s+1} = v_i(k)^s + \varepsilon_v (c_i(k)^s - v_i(k)^s)$$

$$\hat{p}_i(k)^{s+1} = \hat{p}_i(k)^s + \varepsilon_p (p_i(k)^s - \hat{p}_i(k)^s)$$

z_i, v_i

Figure 3.2 A two level structure with real interaction input in parameter estimation

3.6 STRUCTURE WITH REAL INPUT IN INTERACTION AND PARAMETER ESTIMATION

In this section we consider the structure based on MOP3. The structure utilizes real output $y(k) = K^*(x(k),c(k))$ as the interaction input instead of the model output function $y(k) = F(x(k),c(k),u(k),\theta(k))$. The parameter estimation remains as in Section 3.5. We introduce separation variables $z(k) = x(k)$ and $v(k) = c(k)$ to form the following Expanded Optimal Control Problem:

EOP3

$$\min \mathbf{Q} = \sum_{k=1}^{N-1} q(x(k),c(k),u(k),\gamma(k))$$

subject to

$$x(k+1) = f\ (x(k),c(k),u(k),\alpha(k)); \text{k}\in[0,..\text{N-1}]$$

$$x(0) = x_0$$

$$u(k) = HK^*(x(k),c(k))$$

$$f^*(z(k),v(k),HK^*(z(k),v(k)),k) = f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k))$$

$$q^*(z(k),v(k),HK^*(z(k),v(k)),k) = q(z(k),v(k),HK^*(z(k),v(k)),\gamma(k))$$

$$x(k) = z(k)$$

$$c(k) = v(k)$$

Notice that in the above formulation parameter $\theta\ (k)$ does not have to be computed as in formulation EOP1 and EOP2.

Adjoining the constraints using Lagrange multipliers as in the previous derivation, we obtain:

$$\mathbf{Q}_{EOP3} = \sum_{i=1}^{N-1} \{q(x(k),c(k),u(k),\gamma(k))\ p(k+1)^T[f(x(k),c(k),u(k),\alpha(k)) - x(k+1)]$$

$$+ l(k)^T[u(k) - HK^*(x(k),c(k))]$$

$$+ \mu(k)^T[f^*(z(k),v(k),HK^*(z(k),v(k)),k) - f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k))]$$

$$+ \eta(k)^T[q^*(z(k),v(k),HK^*(z(k),v(k)),k) - q\ (z(k),v(k),HK^*(z(k),v(k)),\gamma(k))]$$

$$\lambda(k)^T[z(k) - x(k)] + \beta(k)^T[v(k) - c(k)]\} \qquad (3.85)$$

where $p(k) \in \Re^n$, $l(k) \in \Re^r$, $\mu(k) \in \Re^n$, $\eta(k) \in \Re$, $\lambda(k) \in \Re^n$ and $\beta(k) \in \Re^m$, are Lagrange multiplier functions.

Define:

$$\mathrm{H}(.) = q(x(k),c(k),u(k),\gamma(k)) + p(k+1)^T f\ (x(k),c(k),u(k),\alpha(k)) + l(k)^T [u(k) - HK^*(x(k),c(k))]\ -\lambda(k)^T x(k) - \beta(k)^T c(k) \qquad (3.86)$$

Repeating the procedure outlined in Section 3.4, we then use (3.86) and apply Lagrange multiplier theory to obtain the minimum of (3.85). These will result in the following necessary optimality conditions for EOP3:

$$\frac{\partial q(.)}{\partial c(k)} + p(k+1)^T \frac{\partial f(.)}{\partial c(k)} - l(k)^T H \frac{\partial K^*(.)}{\partial c(k)} - \beta(k)^T = 0 \qquad (3.87)$$

$$\frac{\partial q(.)}{\partial x(k)} + p(k+1)^T \frac{\partial f(.)}{\partial x(k)} - l(k)^T H \frac{\partial K^*(.)}{\partial x(k)} - \lambda(k)^T - p(k) = 0 \qquad (3.88)$$

$$\frac{\partial q(.)}{\partial u(k)} + p(k+1)^T \frac{\partial f(.)}{\partial u(k)} - l(k)^T = 0 \qquad (3.89)$$

$$\lambda(k)^T + \mu(k)^T \left[\frac{\partial f^*(.)}{\partial z(k)} - \frac{\partial f(.)}{\partial z(k)}\right] + \eta(k) \left[\frac{\partial q^*(.)}{\partial z(k)} - \frac{\partial q(.)}{\partial z(k)}\right] = 0 \qquad (3.90)$$

$$\beta(k)^T + \mu(k)^T \left[\frac{\partial f^*(.)}{\partial v(k)} - \frac{\partial f(.)}{\partial v(k)}\right] + \eta(k) \left[\frac{\partial q^*(.)}{\partial v(k)} - \frac{\partial q(.)}{\partial v(k)}\right] = 0 \qquad (3.91)$$

$$p(k+1)^T \frac{\partial f(x(k),c(k),u(k),\alpha(k))}{\partial \alpha(k)} - \mu(k)^T \frac{\partial f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k))}{\partial \alpha(k)} = 0 \qquad (3.92)$$

$$\frac{\partial q(x(k),c(k),u(k),\gamma(k))}{\partial \gamma(k)} - \eta(k)^T \frac{\partial f(z(k),v(k),w(k),\gamma(k))}{\partial \gamma(k)} = 0 \qquad (3.93)$$

$$x(k+1) = f(x(k),c(k),u(k),\alpha(k)) \ ;\ k \in [0,..N-1] \qquad (3.94)$$

$$x(0) = x_0 \qquad (3.95)$$

$$u(k) = HK^*(x(k),c(k)) \qquad (3.96)$$

$$f^*(z(k),v(k),HK^*(z(k),v(k)),k) = f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k)) \qquad (3.97)$$

$$q^*(z(k),v(k),HK^*(z(k),v(k)),k) = q(z(k),v(k),HK^*(z(k),v(k)),\gamma(k)) \quad (3.98)$$

$$x(k)=z(k) \quad (3.99)$$

$$c(k)=v(k) \quad (3.100)$$

Introducing $\hat{p}(k)$ as costate separation variable, that is

$$\hat{p}(k) = p(k) \quad (3.101)$$

Using this and equations (3.99) and (3.100) in optimality conditions (3.92) and (3.93) gives:

$$\mu(k) = \hat{p}(k)$$

$$\eta(k)= 1 \quad (3.102)$$

The modifiers are given by equations (3.90) and (3.91) as:

$$\lambda(k) = -\left[\frac{\partial f^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial z(k)} - \frac{\partial f(z(k),c(k),HK^*(z(k),v(k)),\alpha(k))}{\partial z(k)}\right]^T \hat{p}(k+1)$$

$$-\left[\frac{\partial q^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial z(k)} - \frac{\partial q(z(k),c(k),HK^*(z(k),v(k)),\gamma(k))}{\partial z(k)}\right]^T \quad (3.103)$$

$$\beta(k) = -\left[\frac{\partial f^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial v(k)} - \frac{\partial f(z(k),c(k),HK^*(z(k),v(k)),\alpha(k))}{\partial v(k)}\right]^T \hat{p}(k+1)$$

$$-\left[\frac{\partial q^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial v(k)} - \frac{\partial q(z(k),c(k),HK^*(z(k),v(k)),\gamma(k))}{\partial v(k)}\right]^T \quad (3.104)$$

Notice that incorporation of real inputs in the interaction terms results in simpler calculation for the modifiers when compared to that of EOP1 and EOP2.

The parameter estimation step is defined by optimality conditions (3.97) and (3.98). The price vector is computed using the formula:

$$l(k+1)^{s+1} = l(k)^s + \varepsilon_l(u(k) - HK^*(x(k),c(k)) \quad (3.105)$$

where ε_l is a suitably chosen scalar, chosen so as to aid convergence. At each iteration the separation variables *z(k)* and *v(k)* are updated using relaxation formula defined by (3.60).

Optimality conditions (3.87)-(3.89) are satisfied by solving the following Modified Model-based Optimal Control Problem (MMOP) under specified $\alpha(k)$ and $\gamma(k)$, specified multipliers $\lambda(k)$ and $\beta(k)$, and specified *v(k)* and *z(k)*.

MMOP3

$$\min \mathbf{Q} = \sum_{k=1}^{N} \{q(x(k),c(k),u(k),\gamma(k)) + l(k)^T [u(k) - HK^*(x(k),c(k))] - \lambda(k)^T x(k) - \beta(k)^T c(k)\}$$

subject to

$$x(k+1) = f\ (x(k),c(k),u(k),\alpha(k))$$

$$x(0) = x_0$$

which is separable and can be decomposed into N_s local optimization problems:

MMOP3 $_i$

$$\min \mathbf{Q}_i = \sum_{k=1}^{N-1} \{q_i(x_i(k),c_i(k),u_i(k),\gamma_i(k)) + l_i(k)^T (u_i(k) - \sum_{j=1}^{Ns} l_i(k) H_{ij} K_i^*(x_i(k),c_i(k)) - \lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)\}$$

subject to

$$x_i(k+1) = f_i\ (x_i(k),c_i(k),u_i(k),\alpha_i(k))$$

$$x_i(0) = x_{i0}; \qquad i \in [1,..N_s],\quad k \in [1,..N]$$

The above analysis will lead to a two level hierarchical structure for solving EOP3.

3.6.1 Hierarchical Structure and Information Exchange of EOP3

The above procedure results in a single iterative two level hierarchical structure for solving EOP3. As in section 3.5, the coordinator which is in level two, is made up of price updating mechanism and the two multiplier equations. The first level is made up of N_S local optimization problems, each of which contains a local optimal control unit,

parameter estimation step and a variable update unit. Notice that, the Ns local optimization units remain independent thus making the procedure suitable for parallel processing algorithms.

The structure obtained is different from that of Section 3.5 in the sense that the amount of computation is further reduced. By incorporating real measurements in the interaction terms ,we do not have to calculate model-reality parameter $\theta(k)$ and also Lagrange multiplier vector $\xi(k)$. As a result the number of parameters involved in information exchange between the two level is significantly reduced.

COORDINATOR

$$l(k+1)^{s+1} = l(k)^s + \varepsilon_l[u(k) - HK^*(z(k),v(k))]$$

$$\lambda(k) = -\left[\frac{\partial f^*(.)}{\partial z(k)} - \frac{\partial f(.)}{\partial z(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial z(k)} - \frac{\partial q(.)}{\partial z(k)}\right]^T$$

$$\beta(k) = -\left[\frac{\partial f^*(.)}{\partial v(k)} - \frac{\partial f(.)}{\partial v(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial v(k)} - \frac{\partial q(.)}{\partial v(k)}\right]^T$$

l,β,λ

u,z,v,p *derivatives*

LOCAL OPTIMIZATION PROBLEM

$$\min Q_i = \sum_{k=1}^{N} \{q_i(x_i(k),c_i(k),u_i(k),\gamma_i(k))$$

$$+l_i(k)^T u_i(k)$$

$$-\sum_{j=1}^{N_s} l_j^T(k) H_{ji} K_i^*(x_i(k),c_i(k))$$

$$-\lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)$$

subject to

$$x_i(k+1) = f_i\ (x_i(k),c_i(k),u_i(k),\alpha_i(k))$$
$$x_i(0) = x_{i0}$$

PARAMETER ESTIMATION

$$\alpha_i(k) \leftarrow f_i^*(z_i(k),v_i(k),H_iK^*(z_i(k),v_i(k)),k)$$
$$= f_i(z_i(k),v_i(k),H_iK^*(z_i(k),v_i(k)),\alpha_i(k))$$

$$\gamma_i(k) \leftarrow q_i(z_i(k),v_i(k),H_iK^*(z_i(k),v_i(k)),k)$$
$$= q_i(z_i(k),v_i(k),H_iK^*(z_i(k),v_i(k)),\gamma_i(k))$$

α_i, γ_i

VARIABLES UPDATES

z_i, v_i

$$z_i(k)^{s+1} = z_i(k)^s + \varepsilon_z(x_i(k)^s - z_i(k)^s)$$
$$v_i(k)^{s+1} = v_i(k)^s + \varepsilon_v(c_i(k)^s - v_i(k)^s)$$
$$\hat{p}_i(k)^{s+1} = \hat{p}_i(k)^s + \varepsilon_p(p_i(k)^s - \hat{p}_i(k)^s)$$

Figure 3.3 A two level structure with real input in interaction and parameter estimation

3.7 STRUCTURE WITH TOTAL REAL INTERACTION MEASUREMENTS

We now consider MOP4 in which the interaction vector $u(k)$ is totally eliminated by substituting with the real input $HK^*(x(k),c(k))$. The real interaction measurement is now incorporated in the performance measure, the plant dynamics and the parameter estimation. There is no free interaction variable in these procedure.

MOP4 is expanded by including two separation variables $z(k)=x(k)$ and $v(k)=c(k)$ to an equivalent form, giving the Expanded Optimal Control Problem :

EOP4

$$\min \mathbf{Q} = \sum_{k=1}^{N-1} q(x(k),c(k),HK^*(x(k),c(k)),\gamma(k))$$

subject to

$$x(k+1)=f(x(k),c(k),HK^*(x(k),c(k)),\alpha(k)) \; ; \; k\in[0,..N-1]$$

$$x(0)=x_0$$

$$f^*(z(k),v(k),HK^*(z(k),v(k)),k)= f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k))$$

$$q^*(z(k),v(k),HK^*(z(k),v(k)),k)=q(z(k),v(k),HK^*(z(k),v(k)),\gamma(k))$$

$$x(k)=z(k)$$

$$c(k)=v(k)$$

Adjoining the constraints using Lagrange multipliers as before, we obtain:

$$\mathbf{Q}_{EOP4} = \sum_{i=1}^{N-1} \{q(x(k),c(k),HK^*(x(k),c(k)),\gamma(k))$$

$$p(k+1)^T[f(x(k),c(k),HK^*(x(k),c(k)),\alpha(k))-x(k+1)]$$

$$+ \mu(k)^T[f^*(z(k),v(k),HK^*(z(k),v(k)),k) - f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k))]$$

$$+ \eta(k)^T[q^*(z(k),v(k),HK^*(z(k),v(k)),k) - q(z(k),v(k),HK^*(z(k),v(k)),\gamma(k))]$$

$$+\lambda(k)^T[z(k)-x(k)]+\beta(k)^T[v(k)-c(k)]\} \qquad (3.106)$$

where $p(k)\in\Re^n$, $\mu(k)\in\Re^n$, $\eta(k)\in\Re$, $\lambda(k)\in\Re^n$ and $\beta(k)\in\Re^m$, are Lagrange multiplier functions.

Define:

$$\mathbf{H}(.) = q(x(k),c(k),HK^{*}(x(k),c(k)),\gamma(k)) + p(k+1)^{T} f(x(k),c(k),HK^{*}(x(k),c(k)),\alpha(k)) - \lambda(k)^{T} x(k) - \beta(k)^{T} c(k) \quad (3.107)$$

Repeating the process shown in Section 3.4, we then use (3.107) and apply Lagrange multiplier theory to obtain the minimum of (3.108). These will result in the following necessary optimality conditions for EOP4:

$$\frac{\partial q(x(k),c(k),HK^{*}(x(k),c(k),\gamma(k))}{\partial c(k)} + p(k+1)^{T} \frac{\partial f(x(k),c(k),HK^{*}(x(k),c(k),\alpha(k))}{\partial c(k)} - \beta(k)^{T} = 0 \quad (3.108)$$

$$\frac{\partial q(x(k),c(k),HK^{*}(x(k),c(k),\gamma(k))}{\partial x(k)} + p(k+1)^{T} \frac{\partial f(x(k),c(k),HK^{*}(x(k),c(k),\gamma(k))}{\partial x(k)} - \lambda(k)^{T} - p(k) = 0 \quad (3.109)$$

$$\lambda(k)^{T} + \mu(k)^{T}\left[\frac{\partial f^{*}(z(k),v(k),HK^{*}(z(k),v(k),k)}{\partial z(k)} - \frac{\partial f(z(k),v(k),HK^{*}(z(k),v(k),\alpha(k))}{\partial z(k)}\right] + \eta(k)\left[\frac{\partial q^{*}(z(k),v(k),HK^{*}(z(k),v(k),k)}{\partial z(k)} - \frac{\partial q(z(k),v(k),HK^{*}(z(k),v(k),\gamma(k))}{\partial z(k)}\right] = 0 \quad (3.110)$$

$$\beta(k)^{T} + \mu(k)^{T}\left[\frac{\partial f^{*}(z(k),v(k),HK^{*}(z(k),v(k),k)}{\partial v(k)} - \frac{\partial f(z(k),v(k),HK^{*}(z(k),v(k),\alpha(k))}{\partial v(k)}\right] + \eta(k)\left[\frac{\partial q^{*}(z(k),v(k),HK^{*}(z(k),v(k),k)}{\partial v(k)} - \frac{\partial q(z(k),v(k),HK^{*}(z(k),v(k),\gamma(k))}{\partial v(k)}\right] = 0 \quad (3.111)$$

$$p(k+1)^{T} \frac{\partial f(x(k),c(k),u(k),\alpha(k))}{\partial \alpha(k)} - \mu(k)^{T} \frac{\partial f(z(k),v(k),HK^{*}(z(k),v(k)),\alpha(k))}{\partial \alpha(k)} = 0 \quad (3.112)$$

$$\frac{\partial q(x(k),c(k),u(k),\gamma(k))}{\partial \gamma(k)} - \eta(k)^{T} \frac{\partial f(z(k),v(k),w(k),\gamma(k))}{\partial \gamma(k)} = 0 \quad (3.113)$$

$$x(k+1) = f(x(k),c(k),HK^*(x(k),c(k),\alpha(k)) \;;\; k\in[0,..N\text{-}1] \tag{3.114}$$

$$x(0) = x_0 \tag{3.115}$$

$$f^*(z(k),v(k),HK^*(z(k),v(k)),k) = f(z(k),v(k),HK^*(z(k),v(k)),\alpha(k)) \tag{3.116}$$

$$q^*(z(k),v(k),HK^*(z(k),v(k)),k) = q(z(k),v(k),HK^*(z(k),v(k)),\gamma(k)) \tag{3.117}$$

$$x(k)=z(k) \tag{3.118}$$

$$c(k)=v(k) \tag{3.119}$$

Introducing $\hat{p}(k)$ as costate separation variable, that is

$$\hat{p}(k) = p(k) \tag{3.120}$$

Using this and equations (3.118) and (3.119) in optimality conditions (3.112) and (3.113) gives:

$$\mu(k) = \hat{p}(k)$$

$$\eta(k) = 1 \tag{3.121}$$

The modifiers are given by equations (3.110) and (3.111) as:

$$\lambda(k) = -\left[\frac{\partial f^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial z(k)} - \frac{\partial f(z(k),c(k),HK^*(z(k),v(k)),\alpha(k))}{\partial z(k)}\right]^T \hat{p}(k+1)$$

$$-\left[\frac{\partial q^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial z(k)} - \frac{\partial q(z(k),c(k),HK^*(z(k),v(k)),\gamma(k))}{\partial z(k)}\right]^T$$

(3.122)

$$\beta(k) = -\left[\frac{\partial f^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial v(k)} - \frac{\partial f(z(k),c(k),HK^*(z(k),v(k)),\alpha(k))}{\partial v(k)}\right]^T \hat{p}(k+1)$$

$$-\left[\frac{\partial q^*(z(k),c(k),HK^*(z(k),v(k)),k)}{\partial v(k)} - \frac{\partial q(z(k),c(k),HK^*(z(k),v(k)),\gamma(k))}{\partial v(k)}\right]^T$$

(3.123)

The parameter estimation step are given by optimality conditions (3.116) and (3.117). As in Section 3.6, the separation variables are updated using relaxation formula (3.60). By solving the following Modified Model based Optimal Control Problem (MMOP)

under the specified values of $\alpha(k)$ and $\gamma(k)$, specified modifier $\lambda(k)$ and $\beta(k)$ and specified $z(k)$ and $v(k)$:

MMOP4

$$\min \mathbf{Q} = \sum_{k=1}^{N-1} \{q(x(k),c(k),HK^{*}(x(k),c(k),\gamma(k)) - \lambda(k)^{T} x(k) - \beta(k)^{T} c(k)\}$$

subject to

$$x(k+1) = f(x(k),c(k),HK^{*}(x(k),c(k)),\alpha(k))$$

$$x(0) = x_0$$

which is separable and can be decomposed into N_S local optimization problems:

MMOP4 $_i$

$$\min \mathbf{Q}_i = \sum_{k=1}^{N-1} \{q_i(x_i(k),c_i(k),H_iK^{*}(x_i(k),c_i(k),\gamma_i(k)) - \lambda_i(k)^{T} x_i(k) - \beta_i(k)^{T} c_i(k)\}$$

subject to

$$x_i(k+1) = f_i(x_i(k),c_i(k),H_iK^{*}(x_i(k),c_i(k),\alpha_i(k))$$

$$x_i(0) = x_{i0}; \qquad i\in[1,..N_S],\ k\in[1,..N]$$

The previous analysis will lead to the a two level hierarchical structure for solving EOP4.

3.7.1 Hierarchical Structure and Information Exchange of EOP4

The structure proposed to solve EOP4 contains a coordinator level and N_S local optimization units in the lower level as shown in Figure 3.4. There is no price updating mechanism apart from the need to calculate the modifiers $\lambda(k)$ and $\beta(k)$ in the coordinator. The structure is completely decentralized because the real interaction measurements are utilized not only to the local estimation unit but also to the local optimal control unit.

COORDINATOR

$$\lambda(k) = -\left[\frac{\partial f^*(.)}{\partial z(k)} - \frac{\partial f(.)}{\partial z(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial z(k)} - \frac{\partial q(.)}{\partial z(k)}\right]^T$$

$$\beta(k) = -\left[\frac{\partial f^*(.)}{\partial v(k)} - \frac{\partial f(.)}{\partial v(k)}\right]^T \hat{p}(k+1) - \left[\frac{\partial q^*(.)}{\partial v(k)} - \frac{\partial q(.)}{\partial v(k)}\right]^T$$

β,λ

$z,v,p,derivatives$

LOCAL OPTIMIZATION PROBLEM

$$\min Q_i =$$

$$\sum_{k=1}^{N} \{q_i(x_i(k), c_i(k), H_i K^*(.), \gamma_i(k))$$

$$-\lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)$$

subject to

$$x_i(k+1) = f_i(x_i(k), c_i(k), H_i K^*(.), \alpha_i(k))$$

$$x_i(0) = x_{i0}$$

PARAMETER ESTIMATION

$$\alpha_i(k) = f_i^*(z_i(k), v_i(k), H_i K^*(.), v_i(k)), k)$$

$$- f_i(z_i(k), v_i(k), H_i K^*(.), \alpha_i(k))$$

$$\gamma_i(k) = q_i^*(z_i(k), v_i(k), H_i K^*(.), k)$$

$$- q_i(z_i(k), v_i(k), H_i K^*(.), \gamma_i(k))$$

$\alpha_i(k), \gamma_i(k)$

VARIABLES UPDATES $\longrightarrow$ $z_i(k), v_i(k)$

$$z_i(k)^{s+1} = z_i(k)^s + \varepsilon_z(x_i(k)^s - z_i(k)^s)$$

$$v_i(k)^{s+1} = v_i(k)^s + \varepsilon_v(c_i(k)^s - v_i(k)^s)$$

$$\hat{p}_i(k)^{s+1} = \hat{p}_i(k)^s + \varepsilon_p(p_i(k)^s - \hat{p}_i(k)^s)$$

Figure 3.4 A two level structure with total real interaction measurements

3.8 OPTIMALITY OF BASIC HIERARCHICAL STRUCTURES

In this section we will show the four structures derived to solve four equivalent problems EOP1, EOP2, EOP3 and EOP4 are optimal in the sense that their optimality conditions are equivalent to those of real optimal control problem(ROP).

The following assumptions define the existence and uniqueness of optimal solution of ROP, and the existence of appropriate derivatives. For a detail discussion of this assumption, see for example, Rubio (1986).

Assumption 3.1

The optimal solution $c^{op}(k), x^{op}(k), p^{op}(k)$ and $u^{op}(k)$; k∈[1,..N], exists and is unique under the given boundary conditions.

Assumption 3.2

The derivatives

$$\frac{\partial f^*(.)}{\partial x(k)}, \frac{\partial f^*(.)}{\partial u(k)}, \frac{\partial f^*(.)}{\partial c(k)}, \frac{\partial K^*(.)}{\partial x(k)}, \frac{\partial K^*(.)}{\partial c(k)}, \frac{\partial q^*(.)}{\partial x(k)}, \frac{\partial q^*(.)}{\partial c(k)} \text{ and } \frac{\partial q^*(.)}{\partial c(k)}$$

exist and are continuous for k∈[1,..N].

The following theorem establishes the optimality of hierarchical structures EOP1-EOP4.

Theorem 3.1-Optimality of Structures

Under Assumptions 3.1 and 3.2, and assuming convergence, the structures EOP1-EOP4 are optimal in the sense that their optimality conditions satisfies the optimality conditions of the global real optimal control problem (ROP) defined by (3.14)-(3.16)

Proof:

From optimality conditions of EOP1, substitute the modifier equations (3.53), (3.54) and (3.55) to eliminate $\lambda(k), \beta(k)$ and $\zeta(k)$ in equations (3.31),(3.32) and (3.33) to obtain the following:

$$\frac{\partial q(.)}{\partial c(k)} + p(k+1)^T \frac{\partial f(.)}{\partial c(k)} - l(k)^T H \frac{\partial F(.)}{\partial c(k)}$$

$$-p(k+1)^T\left[\frac{\partial f^*(.)}{\partial v(k)}-\frac{\partial f(.)}{\partial v(k)}\right]-\left[\frac{\partial q^*(.)}{\partial v(k)}-\frac{\partial q(.)}{\partial v(k)}\right]-l(k)^T H\left[\frac{\partial K^*(.)}{\partial v(k)}-\frac{\partial F(.)}{\partial v(k)}\right]=0 \qquad (3.124)$$

$$\frac{\partial q(.)}{\partial x(k)}+p(k+1)^T\frac{\partial f(.)}{\partial x(k)}-l(k)^T H\frac{\partial F(.)}{\partial x(k)}$$

$$-p(k+1)^T\left[\frac{\partial f^*(.)}{\partial z(k)}-\frac{\partial f(.)}{\partial z(k)}\right]-\left[\frac{\partial q^*(.)}{\partial z(k)}-\frac{\partial q(.)}{\partial z(k)}\right]-l(k)^T H\left[\frac{\partial K^*(.)}{\partial z(k)}-\frac{\partial F(.)}{\partial z(k)}\right]=0 \qquad (3.125)$$

$$\frac{\partial q(.)}{\partial u(k)}+p(k+1)^T\frac{\partial f(.)}{\partial u(k)}-l(k)^T H\frac{\partial F(.)}{\partial u(k)}$$

$$-p(k+1)^T\left[\frac{\partial f^*(.)}{\partial w(k)}-\frac{\partial f(.)}{\partial w(k)}\right]-\left[\frac{\partial q^*(.)}{\partial w(k)}-\frac{\partial q(.)}{\partial w(k)}\right]-l(k)^T H\frac{\partial F(.)}{\partial w(k)} \qquad =0 \qquad (3.126)$$

At convergence $x(k)=z(k), c(k)=v(k)$ and $u(k)=w(k)$. Substituting these in (3.116)-(3.118) produces (3.31),(3.32) and (3.33) which are a subset of optimality conditions for ROP. This show that the structure developed to solve EOP1 is optimal in sense that it will satisfy the optimality conditions of ROP.

To prove that the structure to solve EOP2 is optimal, we substitute equations (3.82) and (3.83) in (3.65) and (3.66) to eliminate $\lambda(k)$ and $\beta(k)$. Noting that at convergence $x(k)=z(k)$ and $c(k)=v(k)$ and from optimality condition (3.75) $K^*(.)=F(.)$,the equations (3.66), (3.67) and (3.68) reduce to (3.31), (3.32) and (3.33).

Similarly to show that the structure to solve EOP3 is optimal, we eliminate $\lambda(k)$ and $\beta(k)$ in equations (3.87) and (3.88) using modifier equations (3.103) and (3.104). At convergence $x(k)=z(k)$ and $c(k)=v(k)$. Substituting this in (3.87) and (3.88) leads to optimality conditions (3.31) and (3.32). From (3.97) and (3.98) we have $f^*(.)=f(.)$ and $q^*(.)=q(.)$, applying this in (3.89), produces (3.33).

To show that the hierarchical structure is optimal, we eliminate $\lambda(k)$ and $\beta(k)$ in (3.108) and (3.109) using modifier equations (3.122) and (3.123). Substituting optimality conditions (3.116) to (3.119) produces

$$\frac{\partial q^*(x(k),c(k),HK^*(x(k),c(k),\gamma(k))}{\partial c(k)} + p(k+1)^T \frac{\partial f^*(x(k),c(k),HK^*(x(k),c(k),\alpha(k))}{\partial c(k)} = 0 \tag{3.127}$$

$$\frac{\partial q^*(x(k),c(k),HK^*(x(k),c(k),\gamma(k))}{\partial x(k)} + p(k+1)^T \frac{\partial f^*(x(k),c(k),HK^*(x(k),c(k),\alpha(k))}{\partial x(k)} - p(k) = 0 \tag{3.128}$$

Optimality conditions (3.127) and (3.128) are a subset of the centralized version of ROP.

Q.E.D

3.9 SUMMARY

In this chapter new hierarchical structures for optimal control of interconnected discrete dynamical are presented and discussed. The structures presented are a dynamic analogy to the structures derived by Brdys and Roberts (1986) for solving optimizing control of steady-state systems. The hierarchical structures presented are of an iterative type and utilize real interaction input in the model when available. These new structures represent optimal methods in the sense that the converged solution will agree with the solution of the original real optimal control problem (ROP). This is achieved by employing an appropriate integration of model-based optimal control problem and parameter estimation to produce an optimizing scheme. The first structure is a fully model-based structure, where interaction inputs are based on output function models of the subsystems. The second structure is derived by incorporating the real output model in the interaction measurements. The third is obtained by further utilizing the real interaction measurements in the parameter estimation step. The final structure, which is fully decentralized, is produced by utilizing the real interaction measurements in the whole model of the systems thus eliminating the interaction variables. All the structures are derived by extending the DISOPE techniques to interconnected systems, to take into account model reality differences that may have been deliberately

introduce to facilitate the solutions of complex nonlinear optimal control problems or due to uncertainty in the model used for computation.

In chapter 4, four versions of an algorithm implementation for hierarchical structures with model based interaction input are presented. We will also present simulation results that illustrate the convergence behaviour of the four approaches. In chapter 5, we will provide the convergence analysis for two of the algorithms.

CHAPTER 4

ALGORITHMS FOR STRUCTURE WITH MODEL BASED INTERACTION MEASUREMENTS.

4.1 INTRODUCTION

In this chapter, four hierarchical algorithms for optimal control of interconnected systems with model-reality differences are developed using two different approaches. The algorithms however are similar in the sense that they are based on the structure with model based interaction measurements which is presented in Chapter 3, Section 3.4 . In the first approach we implement directly the structure shown in figure 3.4.1. This approach is called a price coordination approach to denote the role of the interconnection price by as defined by equation (3.59) in the coordination of the algorithm. The first algorithm presented is a single iterative version with a linear-quadratic model based problem which is developed and implemented in software as algorithm 4.2.1 . In the second algorithm a double iterative technique is employed. This technique utilizes the global nature of the price coordination formula (3.59) by calculating the interconnection price in a global outer loop. In the second approach, we calculate the interconnection price directly from the optimality conditions. We denote this by calling this approach a direct coordination approach. This procedure restricts the choice of the output function model and calculates the interaction vector $u(k)$ at the coordination level (level 1) instead of at a lower level as in the price coordination approach. Two versions of this approach are implemented with a linear quadratic model based problem. The first version is a single iterative algorithm and the second version employs a double iterative technique. In the double iterative technique the interaction vector and the interaction price are iterated in the algorithm at a global outer loop of the hierarchical structure. All algorithms are implemented in software and tested with two simulation examples.

4.2 PRICE COORDINATION APPROACH

PROBLEM FORMULATION AND SOLUTION APPROACH

Consider the optimal control problem defined by $MMOP1_i$ in Chapter 3, Section 3.4.

Augmenting the performance index with variable augmentation to aid convergence. Then the following equivalent augmented problem is produced:

$MMOP1_i$':

$$\min_{c(k),u(k)} \mathrm{Q}_i = \phi(\bar{x}(N)) + \sum_{k=1}^{N-1} \{q_i(x_i(k),c_i(k),u_i(k),\gamma_i(k))$$

$$+l_i(k)^T u_i(k) - \sum_{j=1}^{N_S} l_j^T(k) H_{ji} F_i(x_i(k),c_i(k),u_i(k),\theta_i(k))$$

$$-\lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k) - \zeta_i(k)^T u_i(k)$$

$$+\frac{1}{2} r_{1i} \|c_i(k) - v_i(k)\|^2 + \frac{1}{2} r_{2i} \|x_i(k) - z_i(k)\|^2 + \frac{1}{2} r_{3i} \|u_i(k) - w_i(k)\|^2 \}$$

subject to

$$x_i(k+1) = f_i(x_i(k),c_i(k),u_i(k),\alpha_i(k)); \ k \in [1,N-1]$$

$$x_i(0) = x_{i0}$$

$$x_{i,t}(N) = 0; t \in [1,q]$$

$$\bar{x}_i(N) = [x_{i,q+1}(N),.....,x_{i,n}(N)]^T$$

where r_{1i}, r_{2i} and r_{3i} are given scalar convexification factors. Notice that at the end of the iterations, $c(k)=v(k)$, $x(k)=z(k)$ and $u(k)=w(k)$ so that at this stage the augmentation terms and their derivatives are zero, so having no effect in the real optimality of the solution.

The model based optimal control problem and consequently the modified model based control problem $MMOP1_i$, can be chosen as a linear quadratic approximation of the real optimal control problem for which there are standard solution procedures (Lewis and Syrmos,1995). This provides a computational advantage. The parameters $\alpha(k)$, $\gamma(k)$ and $\theta(k)$,k∈[0,.N-1] can be chosen as shift parameters and thus we have:

$q_i(x_i(k),c_i(k),u_i(k),\gamma_i(k)) =$

$$\frac{1}{2} x_i(k)^T Q_i x_i(k) + \frac{1}{2} c_i(k)^T R_i c_i(k) + \frac{1}{2} u_i(k)^T S_i u_i(k) + \gamma_i(k)$$

$$f_i(x_i(k),c_i(k),u_i(k),\alpha_i(k))=$$

$$A_i x_i(k)+B_i c_i(k)+D_i u_i(k)+\alpha_i(k)$$

$$F_i(x_i(k),c_i(k),u_i(k),\theta_i(k))=$$

$$\mathcal{J}_i x_i(k)+\mathcal{K}_i c_i(k)+\mathcal{L}_i u_i(k)+\theta_i(k)$$

$$\phi_i(\bar{x}_i(N))=\frac{1}{2}x_i(N)^T\Phi_i x_i(N);\qquad k\in[0,N] \tag{4.1}$$

where $\Phi_i \geq 0, Q_i \geq 0, R_i > 0$ and $S_i \geq 0$ are weighting matrices of appropriate dimensions, A_i, B_i and D_i are matrices which represent a linear model of $f_i^*(.)$, $\mathcal{J}_i, \mathcal{K}_i$ and $\mathcal{L}_i$ are matrices which represent a linear model of the output function $F_i(.)$.

The corresponding Hamiltonian function is:

$$\mathbb{H}(.)=\frac{1}{2}x_i(k)^T Q_i x_i(k)+\frac{1}{2}c_i(k)^T R_i c_i(k)+\frac{1}{2}u_i(k)^T S_i u_i(k)+\gamma_i(k)$$

$$+p_i(k+1)^T[A_i x_i(k)+B_i c_i(k)+D_i u_i(k)+\alpha_i(k)]$$

$$+l_i(k)^T u_i(k)-\sum_{j=1}^{N_s} l_j^T(k)H_{ji}[\mathcal{J}_i x_i(k)+\mathcal{K}_i c_i(k)+\mathcal{L}_i u_i(k)+\theta_i(k)]$$

$$-\lambda_i(k)^T x_i(k)-\beta_i(k)^T c_i(k)-\zeta_i(k)^T u_i(k)$$

$$+\frac{1}{2}r_{1i}\|c_i(k)-v_i(k)\|^2+\frac{1}{2}r_{2i}\|x_i(k)-z_i(k)\|^2+\frac{1}{2}r_{3i}\|u_i(k)-w_i(k)\|^2,\ k\in[0,N-1] \tag{4.2}$$

Applying model based optimality conditions (3.31),(3.32),(3.33) and (3.41) for the ith subsystem, the following equations are obtained :

$$R_i c_i(k)+B_i^T p(k+1)-\mathcal{K}_i^T\sum_{j=1}^{N_s}H_{ji}l_j(k)-\beta_i(k)+r_{1i}c_i(k)-r_{1i}v_i(k)=0$$

$$Q_i x_i(k)+A_i^T p(k+1)-\mathcal{J}_i^T\sum_{j=1}^{N_s}H_{ji}l_j(k)-\lambda_i(k)-p_i(k)+r_{2i}x_i(k)-r_{2i}z_i(k)=0$$

$$S_i u_i(k)+D_i^T p(k+1)-\mathcal{L}_i^T\sum_{j=1}^{N_s}H_{ji}l_j(k)-\zeta_i(k)+l_i(k)+r_{3i}u_i(k)-r_{3i}w_i(k)=0$$

$$\bar{p}_i(N) = \Phi_i \bar{x}_i(N) \quad \text{where } \bar{p}_i(N) = [p_{i,q+1}(N) \ldots . p_{i,n}(N)]^T$$

$$p_{i,t}(N) = \upsilon_{i,t};\ t \in [1,q]; \quad k \in [0,N] \tag{4.3}$$

To facilitate the computation, we define the following augmentations;

Let

$$\bar{Q}_i = Q_i + r_{2i} I\ ;\ \bar{\lambda}_i(k) = \lambda_i(k) + \mathcal{J}_i^T \sum_{j=1}^{N_s} H_{ji}^T l_j(k) + r_{2i} z_i(k)\ ; \text{ then}$$

$$p_i(k) = \bar{Q}_i x_i(k) + A_i^T p_i(k+1) - \bar{\lambda}_i(k); \quad \bar{p}_i(N) = \Phi_i \bar{x}_i(N)$$

where $\bar{p}_i(N) = [p_{i,q+1}(N) \ldots . p_{i,n}(N)]^T$

$$p_{i,t}(N) = \upsilon_{i,t};\ t \in [1,q]; \quad k \in [0,N] \tag{4.4}$$

and let

$$\bar{R}_i = \bar{R}_i + r_{1i} I\ ;\ \bar{\beta}_i(k) = \beta_i(k) + \mathcal{K}_i^T \sum_{j=1}^{N_s} H_{ji}^T l_j(k) + r_{1i} v_i(k)\ ;$$

$$\Rightarrow c_i(k) = -\bar{R}_i^{-1}(B_i^T p_i(k+1) - \bar{\beta}_i(k)); \qquad k \in [0, N-1] \tag{4.5}$$

Also let

$$\bar{S}_i = S_i + r_{3i} I_r\ ;\ \bar{\zeta}_i(k) = \zeta_i(k) + \mathcal{L}_i^T \sum_{j=1}^{N_s} H_{ji}^T l_j(k) + r_{3i} w_i(k)\ ;$$

$$\Rightarrow u_i(k) = -\bar{S}_i^{-1}(D_i^T p_i(k+1) - \bar{\zeta}_i(k) + l_i(k)); \qquad k \in [0, N-1] \tag{4.6}$$

Substituting (4.5) and (4.6) in the model dynamic equation, results in the following:

$$x_i(k+1) = A_i x_i(k) - B_i \bar{R}_i^{-1}(B_i^T p_i(k+1) - \bar{\beta}_i(k))$$

$$- D_i \bar{S}_i^{-1}(D_i^T p_i(k+1) - \bar{\zeta}_i(k) + l_i(k)) + \alpha_i(k)$$

$$x_i(0) = x_{io}; \qquad x_{i,t}(N) = 0; \quad t \in [1, q_i]; \qquad k \in [0, N-1] \tag{4.7}$$

Equations (4.4) and (4.7) constitute a two-point boundary value problem. It can be solved by using Ricatti equation techniques (Bryson and Ho, 1975; Lewis and Syrmos, 1995). The key is to assume the relationship between costate and state as

$$p_i(k) = V_i(k) x_i(k) + E_i(k) v_i + h_i(k); \qquad k \in [0, N]$$

where $V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}$; $E_i(N) = \begin{bmatrix} I_{q_i} \\ 0 \end{bmatrix}$; $h_i(N) = 0$

$$v_i = [p_{i,1}(N) \ldots . p_{i,q}(N)]^T \tag{4.8}$$

where $V_i(k)$ is an $n_i \times n_i$ matrix, $E_i(k)$ is $n_i \times q_i$ matrix , $h_i(k) \in \Re^{n_i}$ and $v_i \in \Re^{q_i}$. Substituting this in (4.7), produces

$$\begin{aligned} x_i(k+1) = [I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} [A_i x_i(k) + B_i R_i^{-1} \overline{\beta}_i(k) \\ + D_i \overline{S}_i^{-1} (\overline{\zeta}_i(k) - l_i(k)) + \alpha_i(k) - (B_i \overline{R}_i^{-1} B_i^T + D_i \overline{S}_i^{-1} D_i^T) E_i(k+1) v_i \\ - (B_i \overline{R}_i^{-1} B_i^T + D_i \overline{S}_i^{-1} D_i^T) h_i(k+1)] \end{aligned} \tag{4.9}$$

Substituting (4.8) in (4.4) gives

$$\begin{aligned} & V_i(k) x_i(k) + E_i(k) v_i + h_i(k) \\ & = \overline{Q}_i x_i(k) + A_i^T V_i(k+1) x_i(k+1) + A_i^T E_i(k+1) v_i + A_i^T h_i(k+1) - \lambda_i(k) \end{aligned} \tag{4.10}$$

Substituting (4.9) into (4.10) and grouping terms

$$\begin{aligned} & [-V_i(k) + \overline{Q}_i + A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} A_i] x_i(k) \\ & + [A_i^T E_i(k+1) - E_i(k+1) - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet \\ & \qquad (B_i \overline{R}_i^{-1} B_i^T + D_i \overline{S}_i^{-1} D_i^T) E_i(k+1)] v_i \\ & + [A_i - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet \\ & \qquad (B_i \overline{R}_i^{-1} B_i^T + D_i \overline{S}_i^{-1} D_i^T)] h_i(k+1) - h_i(k) + \overline{\lambda}_i(k) \\ & + A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} (B_i \overline{R}_i^{-1} \beta_i(k) \\ & + D_i \overline{S}_i^{-1} (\overline{\zeta}_i(k) - l_i(k) + \alpha_i(k)] = 0; \qquad k \in [0, N-1] \end{aligned} \tag{4.11}$$

Equating coefficients to zero in (4.11) results in the following set of difference equations, which can be solved backwards from terminal conditions shown.

$$\begin{aligned} & V_i(k) = \overline{Q}_i + A_i V_i(k+1)[I_n + B_i \overline{R}_i B_i^T V_i(k+1) + D_i \overline{S}_i D_i^T V_i(k+1)]^{-1} A_i \\ & k \in [0, N-1], \qquad V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}; \end{aligned} \tag{4.12}$$

$$\begin{aligned} E_i(k) = [A_i^T - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet \\ (B_i \overline{R}_i^{-1} B_i + D_i \overline{S}_i^{-1} D_i)] E_i(k+1) \end{aligned}$$

$$k \in [0, N-1]; \qquad E_i(N) = \begin{bmatrix} I_{q_i} \\ \mathbf{0} \end{bmatrix} \tag{4.13}$$

$$h_i(k) = [A_i^T - A_i^T V_i(k+1)[I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1) + D_i \bar{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$

$$(B_i \bar{R}_i^{-1} B_i + D_i \bar{S}_i^{-1} D_i)] h_i(k+1) - \bar{\lambda}_i(k)$$

$$+ A_i^T V_i(k+1)[I_n + B_i \bar{R}_i^T B_i^T V_i(k+1) + D_i \bar{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$

$$(B_i \bar{R}_i^{-1} \beta_i(k) + D_i \bar{S}_i^{-1} (\bar{\zeta}_i(k) - l_i(k) + \alpha_i(k);$$

$$\text{where} \qquad h_i(N) = 0; \qquad k \in [0, N-1] \tag{4.14}$$

Let us assume the (assumed fixed) terminal constraints function as

$$\Psi_i(k) = E_i(k)^T x_i(k) + W_i(k) \nu_i + \pi_i(k); \qquad k \in [0, N]$$

where $W_i(k)$ is a $q_i \times q_i$ matrix and $\pi_i(k) \in \Re^{q_i}$ and

$$\Psi_i(k) = [x_{i,1}(N) \ldots\ldots x_{i,q}(N)]^T = [0 \ldots 0]^T$$

then $$\Psi_i(k+1) = E_i(k+1)^T x_i(k+1) + W_i(k+1) \nu_i + \pi_i(k+1); \tag{4.15}$$

Using (4.9) and equating $\Psi_i(k)$ and $\Psi_i(k+1)$ in (4.15) results in a repeat of (4.13) plus the following difference equations, which may be solved backwards.

$$W_i(k) = W_i(k+1) - E_i(k+1)^T [I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1) + D_i \bar{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$

$$(B_i \bar{R}_i^{-1} B_i^T + D_i \bar{S}_i^{-1} D_i^T)] E_i(k+1)$$

$$W(N) = 0; \qquad k \in [0, N-1] \tag{4.16}$$

$$\pi_i(k) = \pi_i(k+1) + E_i(k+1)^T [I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1) + D_i \bar{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$

$$[-(B_i \bar{R}_i^{-1} B_i + D_i \bar{S}_i^{-1} D_i)] h_i(k+1) + (B_i \bar{R}_i^T \beta_i(k) + D_i \bar{S}_i^{-1} (\bar{\zeta}_i(k) - l_i(k) + \alpha_i(k)]$$

$$\text{where } \pi_i(N) = 0; \qquad k \in [0, N-1] \tag{4.17}$$

We need to find an expression for multiplier ν_i. then from the terminal constraint function (4.15), we obtain the following

$$\nu_i = W_i(k)[E_i(k)^T x_i(k) - \pi_i(k)] \tag{4.18}$$

for k∈[0,N]. Noticing that $W_i(N)$ is ill conditioned

$$\tilde{p}_i(N) = \nu_i = -W_i(0)[E_i(0)^T x_{io} + \pi_i(0)] \tag{4.19}$$

Hence $\tilde{p}_i(N)$ can be obtained once $E_i(k)$,$W_i(k)$ and $\pi_i(k)$, $k \in [0, N]$. Thus from (4.8) we obtained the following

$$p_i(k) = V_i(k)x_i(k) + E_i(k)\tilde{p}_i(N) + h_i(k); \qquad k \in [0, N] \tag{4.20}$$

Substituting (4.20) in (4.5) gives

$$c_i(k) = -\bar{R}_i^{-1} B_i^T [V_i(k+1)x_i(k+1) + E_i(k+1)\tilde{p}_i(N) + h_i(k+1) + \bar{R}_i^{-1}\bar{\beta}_i(k));$$
$$k \in [0, N-1] \tag{4.21}$$

which can be expressed as

$$c_i(k) = -G_i(k)x_i(k) + g_i(k) \tag{4.22}$$

where $G_i(k) = \bar{R}_i^{-1} B_i^T V_i(k+1)[I_n + B_i \bar{R}_i B_i^T V_i(k+1) + D_i \bar{S}_i D_i^T V_i(k+1)]^{-1} A_i$

and

$$g_i(k) = -\bar{R}_i^{-1} B_i^T V_i(k+1)[I_n + B_i \bar{R}_i B_i^T V_i(k+1) + D_i \bar{S}_i D_i^T V_i(k+1)]^{-1} \bullet$$
$$[(B_i \bar{R}_i^T \beta_i(k) + D_i \bar{S}_i^{-1}(\bar{\zeta}_i(k) - l_i(k)) + \alpha_i(k)$$
$$-(B_i \bar{R}_i^{-1} B_i^T + D_i \bar{S}_i^{-1} D_i^T)](E_i(k+1)\nu + h_i(k+1))$$
$$+E_i(k+1)\tilde{p}_i(N) + h_i(k+1)] + \bar{R}_i^T \beta_i(k) \qquad k \in [0, N-1] \tag{4.23}$$

The interaction term is given by

$$u_i(k) = -\bar{S}_i^{-1} D_i^T [V_i(k+1)x_i(k+1) + E_i(k+1)\tilde{p}_i(N) + h_i(k+1)]$$
$$+\bar{S}_i^{-1}(\bar{\zeta}_i(k) - l_i(k)) \qquad k \in [0, N-1] \tag{4.24}$$

If we define

$$M_i(k) = \bar{S}_i^{-1} D_i^T V_i(k+1)[I_n + B_i \bar{R}_i B_i^T V_i(k+1) + D_i \bar{S}_i D_i^T V_i(k+1)]^{-1} A_i$$

and

$$m_i(k) = -\bar{S}_i^{-1} D_i^T V_i(k+1)[I_n + B_i \bar{R}_i B_i^T V_i(k+1) + D_i \bar{S}_i D_i^T V_i(k+1)]^{-1} \bullet$$
$$[(B_i \bar{R}_i^T \beta_i(k) + D_i \bar{S}_i^{-1}(\bar{\zeta}_i(k) - l_i(k)) + \alpha_i(k)$$
$$-(B_i \bar{R}_i^{-1} B_i^T + D_i \bar{S}_i^{-1} D_i^T)](E_i(k+1)\nu + h_i(k+1))$$
$$+E_i(k+1)\tilde{p}_i(N) + h_i(k+1)] + \bar{S}_i^{-1}(\bar{\zeta}_i(k) - l_i(k)) \qquad k \in [0, N-1] \tag{4.25}$$

then (4.24) can written as

$$u_i(k) = -M_i(k)x_i(k) + m_i(k) \tag{4.26}$$

Thus from (4.22) and (4.26), the state equation can be expressed as

$$x_i(k+1) = (A_i - B_i G_i(k) - D_i M_i(k))x_i(k) + B_i g_i(k) + D_i m_i(k) + \alpha_i(k);$$

$$x_i(0) = x_{io}, \qquad x_i(N) = 0;\ t \in [1,q]; \quad k \in [0, N-1] \tag{4.27}$$

The linear model allows the multipliers (3.54),(3.55) and (3.56) to written as

$$\lambda_i(k) = -\left[\frac{\partial f_i^*(.)}{\partial z_i(k)} - A_i\right]^T \hat{p}_i(k+1) - \left[\frac{\partial q_i^T(.)}{\partial z_i(k)} - Q_i z_i(k)\right]^T + \left[\frac{\partial K_i^*(.)}{\partial z_i(k)} - \mathcal{J}_i\right]^T H_i^T l_i(k)$$

$$\beta_i(k) = -\left[\frac{\partial f_i^*(.)}{\partial v_i(k)} - B_i\right]^T \hat{p}_i(k+1) - \left[\frac{\partial q_i^T(.)}{\partial v_i(k)} - R_i v_i(k)\right]^T + \left[\frac{\partial K_i^*(.)}{\partial v_i(k)} - \mathcal{K}_i\right]^T H_i^T l_i(k)$$

$$\zeta_i(k) = -\left[\frac{\partial f_i^*(.)}{\partial w_i(k)} - D_i\right]^T \hat{p}_i(k+1) - \left[\frac{\partial q_i^T(.)}{\partial w_i(k)} - S_i w_i(k)\right]^T - \mathcal{L}_i{}^T H_i^T l_i(k);$$

$$k \in [0,N-1] \tag{4.28}$$

The calculation of parameter $\alpha_i(k)$ and $\theta_i(k)$ becomes

$$\alpha_i(k) = f_i^*(z_i(k), v_i(k), w_i(k), k) - A_i z_i(k) - B_i v_i(k) - D_i w_i(k)$$

$$\theta_i(k) = K_i^*(z_i(k), v_i(k)) - \mathcal{J}_i z_i(k) - \mathcal{K}_i v_i(k) - \mathcal{L}_i w_i(k);$$

$$k \in [0,N-1] \tag{4.29}$$

Note that it is not necessary to calculate $\gamma_i(k)$.

The price multiplier vector l(k) is computed using formula defined by (3.60), that is

$$l_i^{s+1}(k+1) = l_i^s(k) + \varepsilon_l [u_i(k) - \sum_{j=1}^{Ns} H_{ij}(\mathcal{J}_j x_j(k) + \mathcal{K}_j c_j(k) + \mathcal{L}_j u_j(k) + \theta_j(k))] \tag{4.30}$$

where ε_l is a positive scalar suitably chosen to preserve convergence and s denotes the iteration number .

The above analysis gives rise to the following two versions of hierarchical DISOPE algorithms for interconnected systems with model based interaction input. We will denote the single loop version by PC1 and the double loop version by PC2.

4.2 .1 Single Loop Technique PC1

A single loop technique is obtained by iterating all the coordinating variables in equations (4.29) and (4.30) in the upper level simultaneously with local optimization units defined by $MMOP1_i$ in the lower level. This scheme is shown in Chapter 3 as:

figure 3.4.1, where the appropriate linear quadratic models are applied to the model-based problem as defined in Section 4.2.

The lower level consists of N_S local optimization units, each of which contains the $MMOP1_i$ unit, the parameter estimation unit and the variable update unit. Each local optimization unit is independent of other local optimization units, thus making it suitable for application of parallel processing.

The algorithm can be summarized as follows:

Algorithm 4.2.1: Single Loop Price Coordination Hierarchical DISOPE for systems with model based interaction measurements.

Data: $A_i, B_i, D_i, Q_i, R_i, S_i, x_{io}, \Phi_i, N, N_s, r_{1i}, r_{2i}, r_{3i}, \epsilon_c, \epsilon_x, \epsilon_u, \epsilon_p, \epsilon_l$ and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Step 0: At level 2, choose the initial value of coordinating variables $\lambda_i^0(k), \beta_i^0(k)$, ζ_i^0(k) and $l_i^0(k)$. At level 1, let $\alpha_i^0(k) = \theta_i^0(k) = 0$, compute or choose a nominal solution for the ith local optimal control problem. Set iteration number $s = 0$ and $v_i^0(k) = c_i^0(k)$ and $w_i^0(k) = u_i^0(k)$, k∈[0.N-1]. $z_i^0(k) = x_i^0(k)$ and $\bar{p}_i^0(k) = p_i^0(k)$, k∈[0,N]. Send them to level 2.

Step 1. At level two, calculate coordinating variables $\lambda_i^s(k), \beta_i^s(k)$, ζ_i^s(k) and $l_i^s(k)$, k∈[1,N] using equations (4.28) and (4.30). Augment them according to (4.4), (4.5) and (4.6). Send them to the local optimization problem in level 1.

Step 2: At level 1, calculate $\alpha_i^s(k)$ and $\theta_i^s(k)$ from (4.29) and send them to the ith local optimal control problem and to level 1. This is called the parameter estimation step.

Step 3: At level 1, solve the ith modified model based optimal control problem.

3.0 If s=0, from data compute $V_i(k)$, $E_i(k)$ and $W_i(k)$, k ∈[0,N] and $G_i(k)$, $M_i(k)$, k ∈[0,N-1] using equations (4.12),(4.13),(4.16), (4.23) and (4.25). The results are stored for use in the subsequent iteration.

3.1 Solve (4.14) and (4.17) backwards to obtain

$h_i^s(k)$ and $\pi_i^s(k), k \in [0, N]$.

3.2 Use (4.19) to obtain $\tilde{p}_i(N)$. Calculate $g_i(k)$ and $m_i(k)$, $k \in [0, N-1]$ from equation (4.23) and (4.25).

3.3 Solve equation (4.27) to obtain new state $x_i^{s+1}(k)$, $k \in [0, N]$.

3.4 Use (4.8) to calculate new costate $p_i^{s+1}(k)$, $k \in [0, N]$

3.5 Use (4.21) to calculate new control $c_i^{s+1}(k)$, $k \in [0, N-1]$

3.6 Use (4.24) to calculate new interaction $u_i^{s+1}(k)$, $k \in [0, N-1]$

Step 4. At level 1, update the variables

$$z_i^{s+1}(k) = z_i^s(k) + \varepsilon_z(x_i^s(k) - z_i^s(k)), \ k \in [0, N]$$

$$v_i^{s+1}(k) = v_i^s(k) + \varepsilon_v(c_i^s(k) - v_i^s(k)) , \ k \in [0, N-1]$$

$$w_i^{s+1}(k) = w_i^s(k) + \varepsilon_w(u_i^s(k) - w_i^s(k)) , \ k \in [0, N-1]$$

$$\hat{p}_i^{s+1}(k) = \hat{p}_i^s(k) + \varepsilon_p(p_i^s(k) - \hat{p}_i^s(k)) , \ k \in [0, N]$$

Send them to level 2.

Step 5 At level 2, convergence of coordinating variables is checked. If $v^{s+1}(k) = v^s(k)$, $z^{s+1}(k) = z^s(k)$ and $w^{s+1}(k) = w^s(k)$ within a defined tolerance, stop, otherwise set s = s + 1 and go to step 1. The whole procedure is repeated.

4.2.2 Double Loop Technique PC2

The double loop technique is first introduced in hierarchical ISOPE algorithms by Shao and Roberts (1983) in which a nested double iterative loop structure is employed such that the subsystem coordination is separated from the controller set-point optimization. The technique was developed for the purpose of reducing as much as possible the number of set-point changes at the expense of an increase in the total number of model based iterations for some circumstances .(See for example, Brdys and Roberts, 1985).

In the hierarchical DISOPE the technique is developed to exploit the global structure of the interconnection terms defined by (3.24). The proposed technique involves an iterative procedure of solving modifier equations (4.28) for a given value of price *l(k)*.

The outer loop task is to evaluate the global price *l(k)* such that equation (3.42) is satisfied and this is equivalent to solving the following equation :

$$u(l(k)) = HF(x(l(k)), c(l(k)), u(l(k)), \theta(x(l(k)), (c(l(k)), (u(l(k)))\ \ k \in [0, N-1] \tag{4.31}$$

where $x(l(k))$, $c(l(k))$ and $u(l(k))$ are solutions of the inner loop problem under a prescribed *l(k)*. Note that in the linear quadratic model based problem F(.) is defined by the appropriate equation in (4.1). The strategy for updating the price is given by a global version of (4.30) that is:

$$l^{s+1}(k+1) = l^{s}(k) + \varepsilon_l(u(k) - H(\mathcal{J}x(k) + \mathcal{K}c(k) + \mathcal{L}u(k) + \theta(k)) \tag{4.32}$$

The inner loop consists of two levels similar to the structure given in Section 4.2.1. On the upper level we have the modifier equations as defined by (4.28). The lower level is made up of N_S local optimization units. Each local optimization unit consists of a $MMOP_i$ unit, parameter estimation unit and the variable update unit. Notice that, the local optimization unit remains independent thus making this technique suitable for of application parallel processing methods. The software implementation of this technique is much more easier because we do not have to separate the interacting component for each subsystem at the coordinator level. This separation of global component (the price mechanism) and local components also facilitates the convergence analysis as shown in Chapter 5.

Sufficient conditions for convergence of this iterative scheme are given in Chapter 5.

The procedure for implementing the algorithm is summarized as follows:

Algorithms 4.2.2 : Double Loop Price Coordination Hierarchical DISOPE for systems with model based interaction input

---

Data: $A_i, B_i, D_i, Q_i, R_i, S_i, x_{io}, \Phi_i, N, N_s, r_{1i}, r_{2i}, r_{3i}, \epsilon_c, \epsilon_x, \epsilon_u, \epsilon_p$ and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Inner Loop

Step 0. At level 2, choose the initial values of coordinating variables $\lambda_i^0(k), \beta_i^0(k)$ and ζ_i^0(k) . At level 1, let $\alpha_i^0(k) = \theta_i^0(k) = 0$, compute or choose a nominal solution for the ith local optimal control problem. Set iteration number $s = 0$ and $v_i^0(k) = c_i^0(k)$ and $w_i^0(k) = u_i^0(k)$, $k \in [0, N]$

$z_i^0(k) = x_i^0(k)$ and $\bar{p}_i^0(k) = p_i^0(k)$, $k \in [0, N]$. Send them to level 2.

Step 1: At level 2, calculate modifier vectors $\lambda_i^s(k), \beta_i^s(k)$ and $\zeta_i^s(\mathrm{k})$, $k \in [1, N]$ using equations (4.28) under prescribed *l(k)* from the outer loop. Augment them according to (4.4),(4.5) and (4.6). Send them to the local optimization problem in level 1.

Step 2: At level 1, calculate $\alpha_i^s(k)$ and $\theta_i^s(k)$ from (4.29) and send them to the ith local optimal control problem and to level 1. This is called the parameter estimation step.

Step 3: At level 1, solve the ith modified model based optimal control problem.

3.0 If s=0, from data compute $V_i(k)$, $E_i(k)$ and $W_i(k)$, $k \in [0, N]$ and $G_i(k)$, $M_i(k)$, $k \in [0, N\text{-}1]$ using equations (4.12),(4.13),(4.16), (4.23) and (4.25). The results are stored for use in the subsequent iteration.

3.1 Solve (4.14) and (4.17) backwards to obtain $h_i^s(k)$ and $\pi_i^s(k), k \in [0, N]$.

3.2 Use (4.19) to obtain $\tilde{p}_i(N)$. Calculate $g_i(k)$ and $m_i(k)$, $k \in [0, N-1]$ from equation (4.23) and (4.25).

3.3 Solve equation (4.27) to obtain new state $x_i^{s+1}(k)$, $k \in [0, N]$.

3.4 Use (4.8) to calculate new costate $p_i^{s+1}(k)$, $k \in [0, N]$

3.5 Use (4.21) to calculate new control $c_i^{s+1}(k)$, $k \in [0, N-1]$

3.6 Use (4.24) to calculate new interaction $u_i^{s+1}(k)$, $k \in [0, N-1]$

Step 4. At level 1, update the variables

$$z_i^{s+1}(k) = z_i^s(k) + \varepsilon_z(x_i^s(k) - z_i^s(k)), \ k \in [0, N]$$

$$v_i^{s+1}(k) = v_i^s(k) + \varepsilon_v(c_i^s(k) - v_i^s(k)) , \ k \in [0, N-1]$$

$$w_i^{s+1}(k) = w_i^s(k) + \varepsilon_w(u_i^s(k) - w_i^s(k)) , \ k \in [0, N-1]$$

$$\hat{p}_i^{s+1}(k) = \hat{p}_i^s(k) + \varepsilon_p(p_i^s(k) - \hat{p}_i^s(k)) , \ k \in [0, N]$$

Send them to level 2.

Step 5 At level 2, convergence of coordinating variables is checked. If $v^{s+1}(k) = v^s(k)$, $z^{s+1}(k) = z^s(k)$ and $w^{s+1}(k) = w^s(k)$ within a defined tolerance, stop, otherwise set s = s + 1 and go to step 1. The whole procedure is then repeated.

Outer Loop:

Step 0: Set p = 0, choose a nominal value for $l^0(k)$. Send it to the inner loop.

Step 1: Using prescribed global values of $x(k)$, $c(k)$, $u(k)$ and $\theta(k)$ from the inner loop calculate the price using equation (4.31). Send it to the inner loop. Set p = p +1.

Step 2: Test for convergence using equation (4.31) . If within a defined tolerance, stop, otherwise repeat step 1.

Simulation results for the two algorithms will be shown at the end of the chapter.

4.3 DIRECT COORDINATION APPROACH PROBLEM FORMULATION AND SOLUTION APPROACH

In this section we develop hierarchical algorithms for solving EOP1 (see Chapter 3, Section 3.4) using methods based directly on the optimality conditions . We choose the model to be of a linear quadratic form. In this technique the price multiplier function is calculated directly from the optimality conditions. The interaction term is calculated directly from the interaction equation as defined in optimality condition (3.42). Because of this , **MMOP1$_i$'** in Section 4.2 is now modified as a minimization problem with respect to the control vector only . In this sense this approach is similar to the interaction-prediction approach (Mahmoud et al,1985). However, in the present problem a different and more general interaction structure is adopted . Augmenting the performance index with variable convexification terms to aid convergence, **MMOP1$_i$** in chapter 3, Section 3.4 is now modified to the following equivalent augmented problem:

MMOP1$_i$'':

$$\min_{c(k)} Q_i = \phi(\bar{x}(N)) + \sum_{k=1}^{N-1} \{q_i(x_i(k), c_i(k), u_i(k), \gamma_i(k))$$

$$+ l_i(k)^T u_i(k) - \sum_{j=1}^{N_s} l_j^T(k) H_{ji} F_i(x_i(k), c_i(k), u_i(k), \theta_i(k))$$

$$- \lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k) - \zeta_i(k)^T u_i(k)$$

$$+\frac{1}{2}r_{1i}\|c_i(k)-v_i(k)\|^2+\frac{1}{2}r_{2i}\|x_i(k)-z_i(k)\|^2+\frac{1}{2}r_{3i}\|u_i(k)-w_i(k)\|^2\}$$

subject to

$$x_i(k+1)=f_i\ (x_i(k),c_i(k),u_i(k),\alpha_i(k));\ \text{k}\in[1,\text{N-1}]$$

$$x_i(0)=x_{i0}$$

$$x_{i,t}(N)=0;t\in[1,q]$$

$$\bar{x}_i(N)=[x_{i,q+1}(N),.....,x_{i,n}(N)]^T$$

where r_{1i},r_{2i} and r_{3i} are given scalar convexification factors.

To solve **MMOP1$_i$''** we choose the following linear quadratic approximation of ROP for which there are standard procedures for its solution (Lewis and Syrmos, 1995). The parameters $\alpha_i(k)$, $\gamma_i(k)$ and $\theta_i(k)$, k∈[0,N-1] can be chosen as shift parameters.

Define

$q_i(x_i(k),c_i(k),u_i(k),\gamma_i(k))=$

$$\frac{1}{2}x_i(k)^T Q_i x_i(k)+\frac{1}{2}c_i(k)^T R_i c_i(k)+\frac{1}{2}u_i(k)^T S_i u_i(k)+\gamma_i(k)$$

$f_i(x_i(k),c_i(k),u_i(k),\alpha_i(k))=$

$$A_i x_i(k)+B_i c_i(k)+D_i u_i(k)+\alpha_i(k)$$

$F_i(x_i(k),c_i(k),u_i(k),\theta_i(k))=$

$$\mathcal{J}_i x_i(k)+\mathcal{K}_i c_i(k)+\theta_i(k)$$

$$\phi_i(\bar{x}_i(N))=\frac{1}{2}x_i(N)^T\Phi_i x_i(N);\qquad \text{k}\in[0,\text{N}] \tag{4.33}$$

Note that the model for $F_i(.)$ does not contain any interaction vector $u_i(k)$ in order to satisfy the sequence of computation used in this approach. Proceeding as in Section 4.2, we define the Hamiltonian as:

$$\text{H}(.)=\frac{1}{2}x_i(k)^T Q_i x_i(k)+\frac{1}{2}c_i(k)^T R_i c_i(k)+\frac{1}{2}u_i(k)^T S_i u_i(k)+\gamma(k)$$

$$+p_i(k+1)^T[A_i x_i(k)+B_i c_i(k)+D_i u_i(k)+\alpha_i(k)]$$

$$+l_i(k)^T u_i(k)-\sum_{j=1}^{N_S} l_j(k)H_{ji}[\mathcal{J}_i x_i(k)+\mathcal{K}_i c_i(k)+\theta_i(k)]$$

$$-\lambda_i(k)^T x_i(k) \dot{-} \beta_i(k)^T c_i(k)-\zeta_i(k)^T u_i(k)$$

$$+\frac{1}{2}r_{1i}\|c_i(k)-v_i(k)\|^2+\frac{1}{2}r_{2i}\|x_i(k)-z_i(k)\|^2+\frac{1}{2}r_{3i}\|u_i(k)-w_i(k)\|^2, k\in[0,N-1] \quad (4.33a)$$

Applying optimality conditions (3.31)-(3.33) and (3.41) gives the following equations,

$$R_i c_i(k)+B_i^T p_i(k+1)-\mathcal{K}_i^T\sum_{j=1}^{N_S} H_{ji}l_j(k)-\beta_i(k)-r_{1i}c_i(k)-r_{1i}v_i(k)=0$$

$$Q_i x_i(k)+A_i^T p_i(k+1)-\mathcal{J}_i^T\sum_{j=1}^{N_S} H_{ji}l_j(k)-\lambda_i(k)-p_i(k)+r_{2i}x_i(k)-r_{2i}z_i(k)=0$$

$$S_i u_i(k)+D_i^T p_i(k+1)-\zeta_i(k)+l_i(k)+r_{3i}u_i(k)-r_{3i}w_i(k)=0$$

$$\bar{p}_i(N)=\Phi_i\bar{x}_i(N) \quad \text{where } \bar{p}_i(N)=[p_{i,q+1}(N)....p_{i,n}(N)]^T$$

$$p_{i,t}(N)=\upsilon_{i,t};\ t\in[1,q];\quad k\in[0,N] \quad (4.34)$$

To simplify the derivation and computation we make the following augmentation.

Let

$$\bar{Q}_i=Q_i+r_{2i}I\ ;\ \bar{\lambda}_i(k)=\lambda_i(k)+\mathcal{J}_i^T\sum_{j=1}^{N_S} H_{ji}^T l_j(k)+r_{2i}z_i(k)\ ;\ \text{then}$$

$$p_i(k)=\bar{Q}_i x_i(k)+A_i^T p_i(k+1)-\lambda_i(k);\qquad \bar{p}_i(N)=\Phi_i\bar{x}_i(N)$$

where $\bar{p}_i(N)=[p_{i,q+1}(N)....p_{i,n}(N)]^T$

$$p_{i,t}(N)=\upsilon_{i,t};\ t\in[1,q];\quad k\in[0,N] \quad (4.35)$$

Let

$$\bar{R}_i=\bar{R}_i+r_{1i}I\ ;\ \bar{\beta}_i(k)=\beta_i(k)+\mathcal{K}_i^T\sum_{j=1}^{N_S} H_{ji}^T l_j(k)+r_{1i}v_i(k)\ ;$$

$$\Rightarrow c_i(k)=-\bar{R}_i^{-1}(B_i^T p_i(k+1)-\bar{\beta}_i(k));\qquad k\in[0,N-1] \quad (4.36)$$

Let $\quad \bar{S}_i=S_i+r_{3i}I_r\ ;\ \bar{\zeta}_i(k)=\zeta_i(k)+r_{3i}w_i(k)\ ;$

$$\Rightarrow l_i(k) = \bar{\zeta}_i(k) - \bar{S}_i u_i(k) - D_i^T p_i(k+1); \qquad k \in [0, N-1] \tag{4.37}$$

where the interaction vector $u_i(k)$ is calculated from optimality condition (3.42), that is;

$$u_i(k) = \sum_{j=1}^{N_S} H_{ij} F_j(x_i(k), c_i(k), u_i(k), \theta_i(k))$$

$$= \sum_{j=1}^{N_S} H_{ij}(\mathcal{J}_j x_j(k) + \mathcal{K}_j c_j(k) + \theta_j(k)); \quad k \in [0, N-1] \tag{4.38}$$

Note that in (4.37) we calculate the Lagrange multiplier associated with the interaction terms *l(k)* directly from the optimality conditions.

Substituting (4.36) in the model dynamic equation, produces the following;

$$x_i(k+1) = A_i x_i(k) - B_i \bar{R}_i^{-1}(B_i^T p_i(k+1) - \beta_i(k)) + D_i u_i(k) + \alpha_i(k)$$

$$x_i(0) = x_{io}; \qquad x_{i,t}(N) = 0; \quad t \in [1, q_i]; \qquad k \in [0, N-1] \tag{4.39}$$

Equations (4.35) and (4.39) define a TPBVP, which can be solved by using Riccati equation methods (Bryson and Ho, 1975; Lewis and Syrmos, 1995).

The key to solution is to assume the relationship between costate and state as

$$p_i(k) = V_i(k) x_i(k) + E_i(k) \nu_i + h_i(k); \qquad k \in [0, N]$$

where $V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}$; $E_i(N) = \begin{bmatrix} I_{q_i} \\ 0 \end{bmatrix}$; $h_i(N) = 0$

$$\nu_i = [p_{i,1}(N) \ldots . p_{i,q}(N)]^T \tag{4.40}$$

where $V_i(k)$ is an $n_i \times n_i$ matrix, $E_i(k)$ is $n_i \times q_i$ matrix , $h_i(k) \in \Re^{n_i}$ and $\nu_i \in \Re^{q_i}$.

Substituting (4.40) in (4.39) produces;

$$x_i(k+1) = [I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1)]^{-1} [A_i x_i(k) + B_i R_i^{-1} \bar{\beta}_i(k)$$

$$-(B_i \bar{R}_i^{-1} B_i^T + D_i \bar{S}_i^{-1} D_i^T)(E_i(k+1)\nu_i + h_i(k+1)) + D_i u_i(k) + \alpha_i(k) \tag{4.41}$$

Applying (4.40) in (4.34) gives;

$V_i(k) x_i(k) + E_i(k) \nu_i + h_i(k)$

$$= \overline{Q}_i x_i(k) + A_i^T V_i(k+1) x_i(k+1) + A_i^T E_i(k+1)\nu + A_i^T h_i(k+1) - \lambda_i(k) \qquad (4.42)$$

Substituting (4.41) into (4.42) and grouping terms

$$[-V_i(k) + \overline{Q}_i + A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1)]^{-1} A_i] x_i(k)$$

$$+[A_i^T E_i(k+1) - E_i(k+1) - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1)]^{-1} \bullet$$

$$(B_i \overline{R}_i^{-1} B_i^T) E_i(k+1)] \nu_i$$

$$+[A_i - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1)]^{-1} \bullet (B_i \overline{R}_i^{-1} B_i^T)] h_i(k+1)$$

$$-h_i(k) + \overline{\lambda}_i(k) + A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1)]^{-1} (B_i \overline{R}_i^{-1} \beta_i(k)$$

$$+D_i u_i(k) + \alpha_i(k)] = 0; \qquad k \in [0, N-1] \qquad (4.43)$$

Equating coefficients of $x_i(k)$ and ν_i to zero in (4.43) results in the following set of difference equations, which can be solved backward from the terminal condition shown.

$$V_i(k) = \overline{Q}_i + A_i V_i(k+1)[I_n + B_i \overline{R}_i B_i^T V_i(k+1)]^{-1} A_i$$

$$k \in [0, N-1], \qquad V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}; \qquad (4.44)$$

$$E_i(k) = [A_i^T - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1)]^{-1} (B_i \overline{R}_i^{-1} B_i)] E_i(k+1)$$

$$k \in [0, N-1]; \qquad E_i(N) = \begin{bmatrix} I_{q_i} \\ 0 \end{bmatrix} \qquad (4.45)$$

$$h_i(k) = [A_i^T - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i B_i^T V_i(k+1)]^{-1} B_i \overline{R}_i^{-1} B_i] h_i(k+1) - \overline{\lambda}_i(k)$$

$$+A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^T B_i^T V_i(k+1)]^{-1} \bullet$$

$$(B_i \overline{R}_i^{-1} \beta_i(k) + D_i u_i(k) + \alpha_i(k));$$

where $\quad h_i(N) = 0; \qquad k \in [0, N-1] \qquad (4.46)$

By using the matrix inversion lemma (see, for example Lewis and Syrmos, 1995) we obtain the following equivalence:

$$[I_{n_i} + B_i \overline{R}_i^{-1} B_i^T V_i(k+1)] = I_{n_i} - B_i [\overline{R}_i + B_i^T V_i(k+1) B_i]^{-1} B_i^T V_i(k+1) \qquad (4.47)$$

Define

$$G_i(k) = [\overline{R}_i + B_i^T V_i(k+1) B_i]^{-1} B_i^T V_i(k+1) A_i \qquad (4.48)$$

The Discrete Riccati Equation (4.44) can now be expressed as

$$V_i(k) = \overline{Q}_i + A_i V_i(k+1)[A_i - B_i G_i(k)]$$

$$k \in [0, N-1], \qquad V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}; \tag{4.49}$$

Equation (4.45) becomes

$$E_i(k) = [A_i - B_i G_i(k)]^T E_i(k+1)$$

$$k \in [0, N-1]; \qquad E_i(N) = \begin{bmatrix} 0 \\ I_{q_i} \end{bmatrix} \tag{4.50}$$

and equation (4.46) reduces to

$$h_i(k) = (A_i - B_i G_i(k))^T h_i(k+1) + (A_i - B_i G_i(k+1))^T V_i(k+1)(D_i u_i(k) + \alpha_i(k))$$

$$-\bar{\lambda}_i(k) + G_i(k)^T \bar{\beta}_i(k); \qquad h_i(N) = 0; \qquad k \in [0, N-1] \tag{4.51}$$

Let assume the terminal constraint function (assumed fixed) as

$$\Psi_i(k) = E_i(k)^T x_i(k) + W_i(k) v_i + \pi_i(k); \qquad k \in [0, N]$$

where $W_i(k)$ is a $q_i \times q_i$ matrix and $\pi_i(k) \in \Re^{q_i}$ and

$$\Psi_i(k) = [x_{i,1}(N) \ldots\ldots x_{i,q}(N)]^T = [0 \ldots 0]^T$$

then $\Psi_i(k+1) = E_i(k+1)^T x_i(k+1) + W_i(k+1) v_i + \pi_i(k+1);$ (4.52)

Using (4.39) and equating $\Psi_i(k)$ and $\Psi_i(k+1)$ in (4.52) results in a repeat of (4.43) plus the following difference equations, which may be solved backwards.

$$W_i(k) = W_i(k+1) - E_i(k+1)^T [I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1)]^{-1} B_i \bar{R}_i^{-1} B_i^T] E_i(k+1)$$

$$W(N) = 0; \qquad k \in [0, N-1] \tag{4.53}$$

$$\pi_i(k) = \pi_i(k+1) + E_i(k+1)^T [I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1)]^{-1} \bullet$$

$$[(B_i \bar{R}_i^T \beta_i(k) - B_i \bar{R}_i^{-1} B_i) h_i(k+1) + D_i u_i(k) + \alpha_i(k)]$$

where $\pi_i(N) = 0; \qquad k \in [0, N-1]$ (4.54)

Noticing that $W_i(N)$ is ill conditioned

$$\tilde{p}_i(N) = v_i = -W_i(0)[E_i(0)^T x_{io} + \pi_i(0)] \tag{4.55}$$

Hence $\tilde{p}_i(N)$ can be obtained once $E_i(k)$,$W_i(k)$ and $\pi_i(k)$, $k \in [0, N]$. Thus from (4.40) we obtain the following

$$p_i(k) = V_i(k) x_i(k) + E_i(k) \tilde{p}_i(N) + h_i(k); \qquad k \in [0, N] \tag{4.56}$$

Substituting (4.56) in (4.36) gives

$$c_i(k) = -\overline{R}_i^{-1} B_i^T [V_i(k+1)x_i(k+1) + E_i(k+1)\tilde{p}_i(N) + h_i(k+1) + \overline{R}_i^{-1}\overline{\beta}_i(k));$$

$$k \in [0, N-1] \tag{4.57}$$

which can be expressed as

$$c_i(k) = -G_i(k)x_i(k) + g_i(k) \tag{4.58}$$

where

$$g_i(k) = [\overline{R}_i + B_i^T V_i(k+1)B_i]^{-1}[-B_i^T V_i(k+1)(D_i u_i(k+1) + \alpha_i(k))$$

$$-B_i^T E_i(k+1)\tilde{p}_i(N) - B_i^T h_i(k+1) + \beta_i(k) \tag{4.59}$$

Finally, the model dynamic equation (4.41) can written as

$$x_i(k+1) = (A_i - B_i G_i(k))x_i(k) + B_i g_i(k) + D_i u_i(k) + \alpha_i(k);$$

$$x_i(0) = x_{io}, \qquad x_i(N) = 0;\ t \in [1,q]; \quad k \in [0, N-1] \tag{4.60}$$

The modifier multiplier functions for $\lambda_i(k)$ and $\beta_i(k)$ remains as in equation (4.28).

Modifier function $\zeta_i(k)$ is now defined by

$$\zeta_i(k) = -\left[\frac{\partial f_i^*(.)}{\partial w_i(k)} - D_i\right]^T \hat{p}_i(k+1) - \left[\frac{\partial q_i^T(.)}{\partial z_i(k)} - S_i w_i(k)\right]^T \tag{4.61}$$

The calculation of parameter $\alpha_i(k)$ is as defined by equation (4.29). Parameter $\theta_i(k)$ is given by

$$\theta_i(k) = F_i^*(z_i(k), v_i(k), w_i(k), k) - \mathcal{J}_i z_i(k) - \mathcal{K}_i v_i(k));$$

$$k \in [0, N-1] \tag{4.62}$$

Notice that, it is not necessary to calculate $\gamma_i(k)$.

The above analysis gives rise to the following two algorithms for hierarchical optimal control of interconnected systems with model based interaction input. We will denote the single loop version by DC1 and the double loop version by DC2.

4.3.1 Single Loop Technique DC1.

In this section a two level single iterative hierarchical algorithm is presented. The coordinator which is in the upper level is made up of equation (4.37) which computes the price $l(k)$ and is followed by the modifier equations given from equation (4.28).. The interaction variable is calculated in the coordinator and is defined by equation (4.38). The lower level is made up of N_S units of local optimization problems. Each of

them consists of the local optimal control problem, the parameter estimation step and the update unit. A single iterative technique is obtained by iterating the variables in the coordinator simultaneously with local optimization units in the lower level.

Algorithm 4.3.1: Single Loop Direct Coordination Hierarchical DISOPE for system with model based interaction measurements.

Data: A_i, B_i, D_i, Q_i, R_i, S_i, x_{io}, Φ_i, N, N_s, r_{1i}, r_{2i}, r_{3i}, ϵ_c, ϵ_x, ϵ_u, ϵ_p and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Step 0: At level 2, choose the initial value of coordinating variables $\lambda_i^0(k), \beta_i^0(k)$, ζ_i^0(k) and $l_i^0(k)$. Set $u_i^0 = 0$. At level 1, let $\alpha_i^0(k) = \theta_i^0(k) = 0$, compute or choose a nominal solution for the ith local optimal control problem. Set iteration number $s = 0$ and $v_i^0(k) = c_i^0(k)$, k∈[0.N-1]. $z_i^0(k) = x_i^0(k)$ and $\overline{p}_i^0(k) = p_i^0(k)$, k∈[0,N]. Send them to level 2.

Step 1: At level 2, calculate interconnection price $l_i^s(k)$ from equation (4.37). Then compute modifiers $\lambda_i^s(k)$, β_i^s(k) and $\zeta_i^s(k)$ from equation (4.28). Calculate the interaction vector u_i^s from equation (4.38). Send them to level 1.

Step 2: At level 1, calculate $\alpha_i^s(k)$ and $\theta_i^s(k)$ from equations (4.29) and (4.54). These are send to level 2 and the ith local optimal control unit.

Step 3: At level 1, solve the ith modified model based optimal control problem.

3.0 If s=0, from data compute $G_i(k)$, k∈[0,N-1] and $V_i(k)$, $E_i(k)$ and $W_i(k)$, k∈[0,N] using equations (4.48), (4.49), (4.50) and (4.53) . The results are stored for use in subsequent iterations.

3.1 Solve (4.51) and (4.54) backwards to obtain $h_i^s(k)$ and $\pi_i^s(k), k \in [0, N]$

3.2 Use (4.55) to obtain $\tilde{p}_i(N)$. Calculate $g_i(k)$, k∈[0.N-1] from equation (4.59).

3.3 Solve equation (4.60) to obtain new state $x_i^{s+1}(k)$, $k \in [0, N]$.

3.4 Use (4.40) to calculate new costate $p_i^{s+1}(k)$, $k \in [0, N]$

3.5 Use (4.51) to calculate new control $c_i^{s+1}(k), \quad k \in [0, N-1]$

Step 4: At level 1, update the variables

$$z_i^{s+1}(k) = z_i^s(k) + \varepsilon_z(x_i^s(k) - z_i^s(k)), \ k \in [0, N]$$

$$v_i^{s+1}(k) = v_i^s(k) + \varepsilon_v(c_i^s(k) - v_i^s(k)) \ , \ k \in [0, N-1]$$

$$w_i^{s+1}(k) = w_i^s(k) + \varepsilon_w(u_i^s(k) - H_i F(x_i(k), c_i(k), \theta_i)) \ , \ k \in [0, N-1]$$

$$\hat{p}_i^{s+1}(k) = \hat{p}_i^s(k) + \varepsilon_p(p_i^s(k) - \hat{p}_i^s(k)) \ , \ k \in [0, N]$$

Send them to level 2.

Step 5: At level 2, convergence of coordinating variables are checked. If $v^{s+1}(k) = v^s(k)$, $z^{s+1}(k) = z^s(k)$ and $w^{s+1}(k) = w^s(k)$ within a defined tolerance, stop, otherwise set s = s + 1 and go to step 1. The whole procedure is then repeated.

4.3.2 Double Loop Technique DC2

In this technique, we utilize the global nature of the price and the interaction term by calculating them in an outer loop. The outer loop iteration consists of computation of the interaction term given by a global version of (4.38), that is

$$w(k) = H(\mathcal{J}z(k) + \mathcal{K}v(k) + \theta(k)) \qquad k \in [0, N-1] \tag{4.62}$$

where the global control $v(k)$,global state $z(k)$ and model reality $\theta(k)$ are prescribed by the inner loop . The interconnection price $l(k)$ is computed from the global version of equation (4.37) , that is

$$l(k) = \bar{\zeta}(k) - \bar{S}u(k) - D^T p(k+1); \qquad k \in [0, N-1] \tag{4.63}$$

where $\bar{\zeta}(k)$ and $p(k)$ are given by the inner loop. The interaction term $u(k)$ is updated using the relaxation formula

$$u^{p+1}(k) = u^p(k) + \varepsilon_u(w^p(k) - u^p(k)) \tag{4.64}$$

The inner loop is made up of a two level iterative structure consisting of modifier equations in level 2 and Ns local optimization problems in level 1. Modifiers equations are given by equation (4.28). The structure of local optimization units is similar to the

previous algorithm. It is made up of $MMOP_i$ units, the parameter estimation unit and the variable update unit. The local optimization unit remains independent thus making this approach suitable for utilization of parallel processing technique.

The double loop procedure discussed above can be summarized as follows.

Algorithms 4.3.2 Double Loop Direct Coordination Hierarchical DISOPE for system with model based interaction measurements.

Data: $A_i, B_i, D_i, Q_i, R_i, S_i, x_{io}, \Phi_i, N, N_s, r_{1i}, r_{2i}, r_{3i}, \epsilon_c, \epsilon_x, \epsilon_u, \epsilon_p$ and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Inner Loop

Step 0. At level 2, choose the initial value of coordinating variables $\lambda_i^0(k), \beta_i^0(k)$ and ζ_i^0(k) . At level 1, let $\alpha_i^0(k) = \theta_i^0(k) = 0$, compute or choose a nominal solution for the ith local optimal control problem. Set iteration number $s = 0$ and $v_i^0(k) = c_i^0(k)$ and $w_i^0(k) = u_i^0(k)$, k∈[0.N-1], $z_i^0(k) = x_i^0(k)$ and $\bar{p}_i^0(k) = p_i^0(k)$, k∈[0,N]. Send them to level 2.

Step 1: At level two, calculate modifier vectors $\lambda_i^s(k), \beta_i^s(k)$ and ζ_i^s(k) , k∈[1,N] using equations (4.28) under prescribed *l(k)* from the outer loop. Augment them according to (4.35), (4.36) and (4.37). Send them to the local optimization problem in level 1.

Step 2: At level 1, calculate $\alpha_i^s(k)$ and $\theta_i^s(k)$ from (4.29) and send them to the ith local optimal control problem and to level 1. This is called the parameter estimation step.

Step 3: At level 1, solve the ith modified model based optimal control problem.

3.0 If s=0, from data compute $G_i(k)$, k ∈[0,N - 1] and $V_i(k)$, $E_i(k)$ and $W_i(k)$,k ∈[0,N] using equations (4.48), (4.49), (4.50) and (4.53) . The results are stored for use in subsequent iterations.

3.1 Solve (4.51) and (4.54) backwards to obtain $h_i^s(k)$ and $\pi_i^s(k), k \in [0, N]$

3.2 Use (4.55) to obtain $\tilde{p}_i(N)$. Calculate $g_i(k)$,k∈[0.N-1] from equation (4.59).

3.3 Solve equation (4.60) to obtain new state $x_i^{s+1}(k), \quad k \in [0, N]$.

3.4 Use (4.40) to calculate new costate $p_i^{s+1}(k), \quad k \in [0, N]$

3.5 Use (4.51) to calculate new control $c_i^{s+1}(k), \quad k \in [0, N-1]$

Step 4. At level 1, update the variables

$$z_i^{s+1}(k) = z_i^s(k) + \varepsilon_z(x_i^s(k) - z_i^s(k)), \quad k \in [0, N]$$

$$v_i^{s+1}(k) = v_i^s(k) + \varepsilon_v(c_i^s(k) - v_i^s(k)), \quad k \in [0, N-1]$$

$$\hat{p}_i^{s+1}(k) = \hat{p}_i^s(k) + \varepsilon_p(p_i^s(k) - \hat{p}_i^s(k)), \quad k \in [0, N]$$

Send them to level 2.

Step 5 At level 2, convergence of coordinating variables is checked. If $v^{s+1}(k) = v^s(k)$ and $z^{s+1}(k) = z^s(k)$ within a defined tolerance, stop, otherwise set s = s + 1 and go to step 1. The whole procedure is then repeated.

Outer Loop:

Step 0: Set p = 0, choose a nominal value for $l^0(k)$ and for $u^0(k)$. Send it to the inner loop. Set $w^0(k) = u^0(k)$.

Step 1: Calculate $w^p(k)$ from interaction equation (4.62) using prescribed global values of $z(k)$, $v(k)$ and $\theta(k)$ from the inner loop. Calculate the price $l(k)$ using equation (4.63). Send it to the inner loop.

Step 2: Update the variable

$$w^{p+1}(k) = w^p(k) + \varepsilon_w(u^p(k) - HF(x(k), c(k), \theta)), \quad k \in [0, N-1]$$

Step 3: Test for convergence using equation (4.62) . If $w^{p+1}(k) = HF^p(.)$ within a defined tolerance, stop, otherwise set p = p +1 and go to step 1. The whole procedure is then repeated.

Simulation results for the two algorithms will be shown at the end of the chapter. Note that the inner and outer loop are run simultaneously.

4.4 SIMULATION EXAMPLES

Algorithms 4.2.1 (PC1), 4.2.2 (PC2), 4.3.1 (DC1) and 4.3.2 (DC2) were implemented in C++ programming language using object oriented and modular programming techniques. In the implementation we make use of the DMatrix class type and operators developed by Becerra (1995), which optimizes operations involving the matrix structures. The application DMatrix class types helped simplify the task of programming the algorithms.

The convergence of the appropriate vectors in each algorithm is verified by comparing the following 2-norm with a given tolerance

$$\left\| a^{s+1} - a^{s} \right\|_2 = \left(\sum_{k=0}^{N-1} \sum_{j=1}^{Ns} \left\| a_j^{s+1}(k) - a_j^{s}(k) \right\|^2 \right)^{1/2} \tag{4.62}$$

where a^s denotes the relevant iterated vectors and s denotes the iteration number.

In the simulation examples we will be discussing the efficiency of the algorithms using two examples. The number of iterations used to compare the performance of the algorithms is defined as the number of times the global performance index is evaluated. Sensitivity of the speed of convergence to tuning parameters such as ε_c, ε_x, ε_u, ε_l, r_1 ,r_2 and r_3 will also be discussed.

To ascertain if the solution achieved at the end of the iterations is the correct optimal solution of the ROP, we test that the solution satisfies the optimality conditions of ROP as defined by equations (3.14), (3.15) and (3.16). Throughout the simulation it is assumed that the values of derivatives with respect to $v(k), z(k)$ and $w(k)$ of f^* and F^* are available. A solution to this problem was proposed by Roberts and Becerra (1996) by using an extension Broyden's formula for approximating the Jacobian matrix in Quasi-Newtons method ,to define a recursion on the Jacobian trajectories given two successive control and states trajectories.

Example 4.4.1

This example consists of optimal control of an interconnected system containing three non linear subsystems. The tolerance specified for convergence is set at ε_T=0.01. The ROP is as follows:

ROP:

$$\min_{u(k),c(k)} \frac{1}{2} \sum_{i=1}^{3} \sum_{k=0}^{51} [x_i(k)^T Q_i^* x_i(k) + c_i(k)^T R_i^* c_i(k) + u_i(k)^T S_i^* u_i(k)].$$

where

$$Q_1^* = diag(0.5, 0.5) \; ; \quad R_1^* = 0.1 \; ; \; S_1^* = diag(0.5, 0.5)$$

$$Q_2^* = diag(0.5, 0.5) \; ; \quad R_2^* = 0.1 \; ; \; S_2^* = diag(0.5, 0.5)$$

$$Q_3^* = diag(0.5, 0.5, 0.5) \; ; \quad R_3^* = 0.1 \; ; \; S_3^* = diag(0.5, 0.5)$$

subject to:

Subsystem 1:

$$x_{11}(k+1) = 0.125x_{11}(k) + 0.005x_{12}(k) + 0.0025u_{11}(k) + 0.0125u_{12}(k) + 0.0025c_{11}(k) + 0.125x_{11}(k)x_{12}(k)$$

$$x_{12}(k+1) = 0.05x_{12}(k) - 0.0125u_{11}(k) + 0.005u_{12}(k) - 0.0125c_{11}(k) + 0.025u_{11}(k)u_{12}(k)$$

$$y_{11}(k) = x_{11}(k) + 0.05c_{11}(k) + 0.025x_{11}(k)c_{11}(k)$$

$$y_{21}(k) = x_{21}(k)$$

$$x_{11}(0) = 1.0 \quad x_{12}(0) = 0.8$$

Subsystem 2:

$$x_{21}(k+1) = -0.0625x_{21}(k) + 0.0125u_{21}(k) + 0.0125u_{22}(k) + 0.005c_{21}(k) + 0.05x_{21}(k)^3$$

$$x_{22}(k+1) = -0.0125x_{21}(k) + 0.975x_{22}(k) + 0.005u_{21}(k) + 0.005c_{21}(k) + 0.025u_{21}(k)x_{22}(k)$$

$$y_{21}(k) = x_{21}(k) + 0.05c_{21}(k)$$

$$y_{22}(k) = x_{22}(k)$$

$$x_{21}(0) = 0.5 \; , x_{22}(0) = 0.6$$

Subsystem 3:

$$x_{31}(k+1) = -0.00425u_{32}(k) + 0.975x_{31}(k) + 0.025x_{33}(k) + 0.025x_{31}(k)x_{33}(k)$$
$$x_{32}(k+1) = 0.0025u_{31}(k) + 0.025u_{33}(k) + 0.9875x_{32}(k)$$
$$x_{33}(k+1) = 0.01u_{31}(k) - 0.025u_{32}(k) - 0.025x_{31}(k) + 0.975x_{33}(k) + 0.0025c_{31}(k) + 0.005x_{31}(k)x_{32}(k)$$
$$y_{31}(k) = x_{31}(k) + 0.0125c_{31}(k) + \sin(x_{31}(k))$$
$$y_{32}(k) = x_{32}(k)$$
$$y_{33}(k) = x_{33}(k)$$
$$x_{31}(0) = 1.5 \quad x_{32}(k) = 1.0 \quad x_{33}(k) = 1.2$$

MOP:

$$\min_{u(k),c(k)} \frac{1}{2} \sum_{i=1}^{3} \sum_{k=0}^{51} [x_i(k)^T Q_i x_i(k) + c_i(k)^T R_i c_i(k) + u_i(k)^T S_i u_i(k) + \gamma_i(k)]$$

where

$Q_1 = diag(0.4, 0.4)$; $R_1 = 0.08$; $S_1 = diag(0.4, 0.4)$

$Q_2 = diag(0.4, 0.4)$; $R_2 = 0.08$; $S_2 = diag(0.4, 0.4)$

$Q_3 = diag(0.4, 0.4, 0.4)$; $R_3 = 0.08$; $S_3 = diag(0.4, 0.4)$

subject to:

Subsystem 1:

$$x_1(k+1) = \begin{bmatrix} 0 & -0.005 \\ 0 & 0.05 \end{bmatrix} x_1(k) + \begin{bmatrix} 0 \\ 0.0125 \end{bmatrix} c_1(k) + \begin{bmatrix} 0.025 & 0 \\ -0.0125 & 0.005 \end{bmatrix} u_1(k) + \alpha_1(k)$$

$$y_1(k) = \theta_1(k)$$
$$x_{11}(0) = 1.0 \quad x_{12}(0) = 0.8$$

Subsystem 2:

$$x_2(k+1)=\begin{bmatrix}-0.625 & 0\\ -0.0125 & 0.975\end{bmatrix}x_2(k)+\begin{bmatrix}0.005\\ 0.005\end{bmatrix}c_2(k)+\begin{bmatrix}0.0125 & 0.0125\\ 0 & 0.005\end{bmatrix}u_2(k)+\alpha_2(k)$$

$$y_2(k)=\theta_2(k)$$

$$x_{21}(0)=0.5\ ,x_{22}(0)=0.6$$

Subsystem 3:

$$x_3(k+1)=\begin{bmatrix}0.975 & 0 & 0.025\\ 0 & 0.9875 & 0\\ -0.025 & 0 & 0.975\end{bmatrix}x_3(k)+\begin{bmatrix}0\\ 0\\ 0.0025\end{bmatrix}c_3(k)$$

$$+\begin{bmatrix}0 & -0.00425 & 0\\ 0.0025 & 0.0 & 0.025\\ 0.01 & -0.025 & 0\end{bmatrix}u_3(k)+\alpha_3(k)$$

$$y_3(k)=\theta_3(k)$$

$$x_{31}(0)=1.5,\ x_{32}(0)=1.0,\ x_{33}(0)=1.2$$

The interconnection matrix H is given as

$$H=\begin{bmatrix}0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\end{bmatrix}$$

To test the effectiveness of the hierarchical algorithms in dealing with model reality differences the above choice of MOP is made. The MOP plant dynamics is a linear approximation of the ROP. The MOP performance measure is deliberately chosen to be 80% of the ROP performance index. Notice that the MOP output function $y(k)$ is assumed to be zero.

Table 4.4.1 shows the results of simulation of the algorithms for this example. The entries marked with a * indicates the best performance for each algorithm. The single loop price coordination algorithm PC1 converges after 23 iterations without addition of convexification terms. However the speed of convergence of the double loop price

coordination algorithm, PC2 improves with addition of convexification terms involving scalars r_2 and r_3. This shows that the convexification of the states and the interaction vectors are able to influence the rate the convergence for this algorithm favorably. The direct coordination algorithms converge at a much faster speed when compared to that of the price coordination version . Single loop versions of direct coordination, DC1 is slightly faster in terms of convergence when compared to the double loop version, DC2. This can be accounted by the extra computation required in the outer loop in algorithm DC2. The centralized problem is also solved using centralized DISOPE for comparison. If each subsystem in the interconnected problem is solved in parallel instead sequentially a better CPU time is expected.

Figures 4.4.1.1-4.4.1.3 show the plant dynamics of the subsystems. Notice that the dynamics of subsystem 1 is slow when compared to that of subsystem 3. This indicates that the hierarchical DISOPE algorithms presented are robust enough to solve subsystems of involving slow and fast dynamic response. Figures 4.4.1.4-4.4.1.6 show the final control signals of each subsystem. Figures 4.4.1.7-4.4.1.9 show the interaction vectors of each subsystems. Figure 4.4.1.10 compares the speed of convergence of the best performance of the direct coordination algorithm DC1 and the price coordination algorithms, PC1 and PC2. It is observed from Figure 4.4.1.10 that the convergence of the global performance index of price coordination algorithms appears to be increasing towards the final global performance index value . This is because in the price coordination approach, the interconnection constraint defined by optimality condition (3.42) is not satisfied in the iterations until the price updating mechanism defined by (4.30) has sufficiently converged. This is in contrast with the direct coordination algorithms, where the interaction variable and the price are computed in the second level using values of variables prescribed in the previous iteration which results in the satisfaction of the interconnection constraint in each iteration. The figure also shows that the direct coordination approach has better convergence at initial and tail end stages. Algorithm PC2 is observed to be slowest in terms of initial and tail end convergence. This observation is illustrated by Figures 4.4.1.11, 4.4.1.12 and 4.4.1.13 which display the convergence behaviour of the interaction norm and the control norm of the respective algorithms.

Algorithm	ε_l	r_1	r_2	r_3	Number of iterations	CPU (s)	Final Performance Index
Price Cor. single loop							
	0.3	0	0	0	31	129	121.4840
	0.5	0	0	0	27	114	121.4893
	0.3	1.0	0	0	47	193	121.4911
	0.3	0	0	1.0	65	285	121.5071
	0.3	0	1.0	0	29	126	121.5031
*	0.4	0	0	0	23	99	121.4907
Price Cor. Double Loop							
	0.2	0	0	0.5	22	105	121.4971
	0.15	0	0	0	23	108	121.4983
	0.15	0	1.0	0.5	22	106	121.5016
	0.4	1	0	0	28	115	121.4891
*	0.3	0	1.0	0.5	20	93	121.4898
Direct Cr. single loop							
*	n/a	0	0	0	11	48	121.5077
	n/a	1.0	1.0	0	45	184	121.5067
	n/a	0.5	0	0	28	110	121.4853
	n/a	0	0	0.5	11	50	121.5017
	n/a	0	0	1.0	11	48	121.5076
Direct Cr. Double Loop							
*	n/a	0	0	0	12	51	121.5028
	n/a	1.0	0	0	45	210	121.5048
	n/a	0.5	0	0.5	29	115	121.5091
	n/a	0	0	0.7	12	51	121.5064
Centralize DISOPE (Centralized Problem)	n/a	0	0	n/a	6	32	121.5032

Table 4.4.1 Algorithms performance for Example 4.4.1.

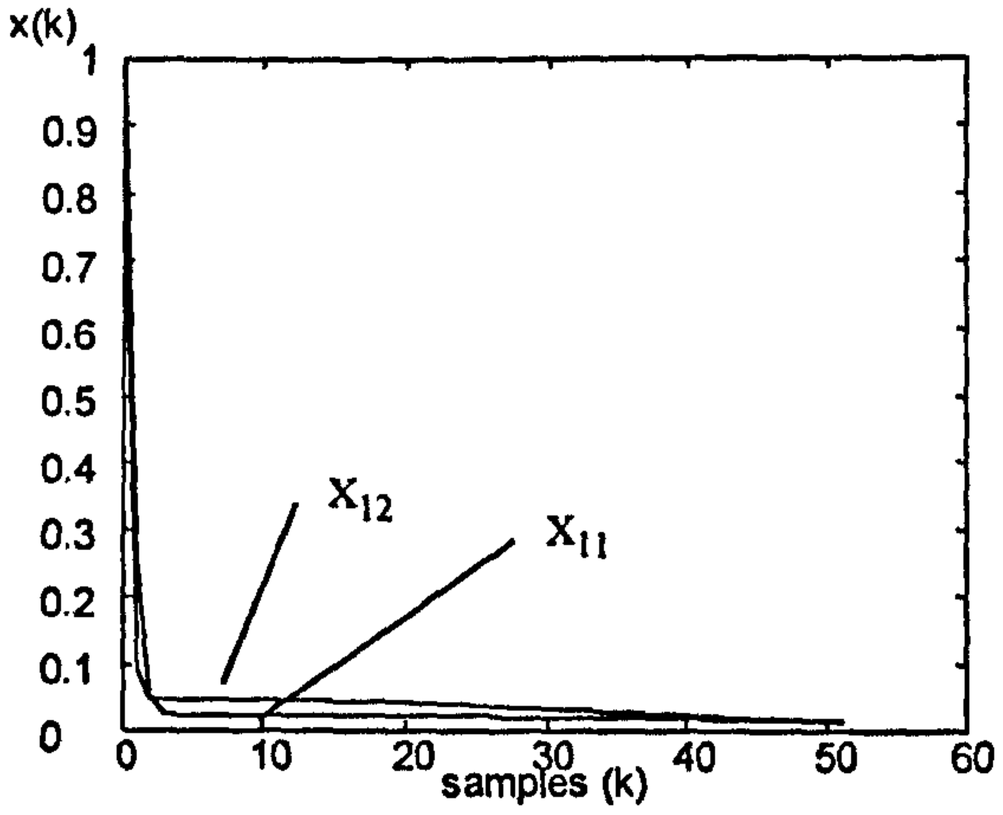

Figure 4.4.1.1 Subsystem 1, state vectors

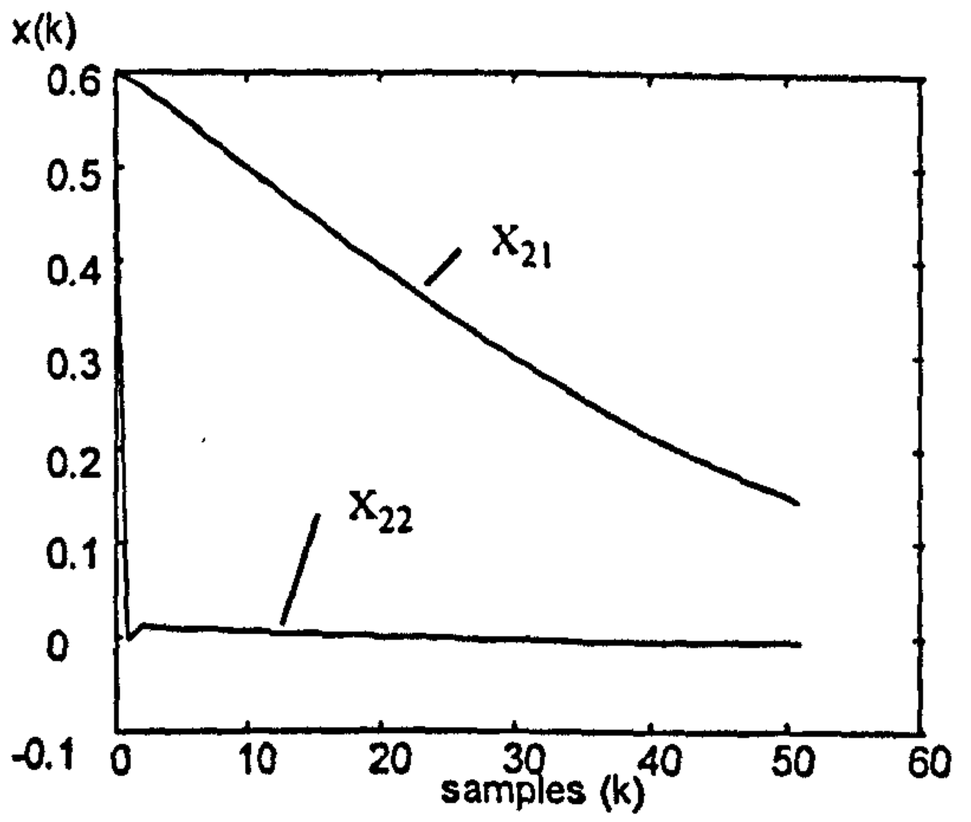

Figure 4.4.1.2 Subsystem 2, state vectors

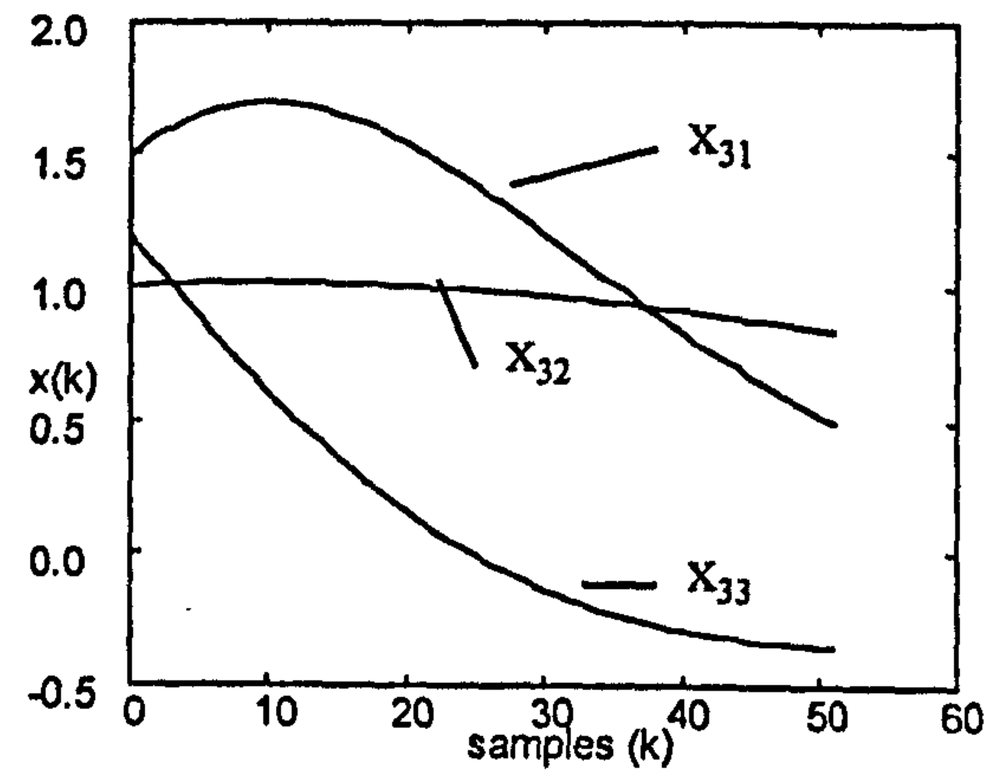

Figure 4.4.1.3 Subsystem 3, state vectors

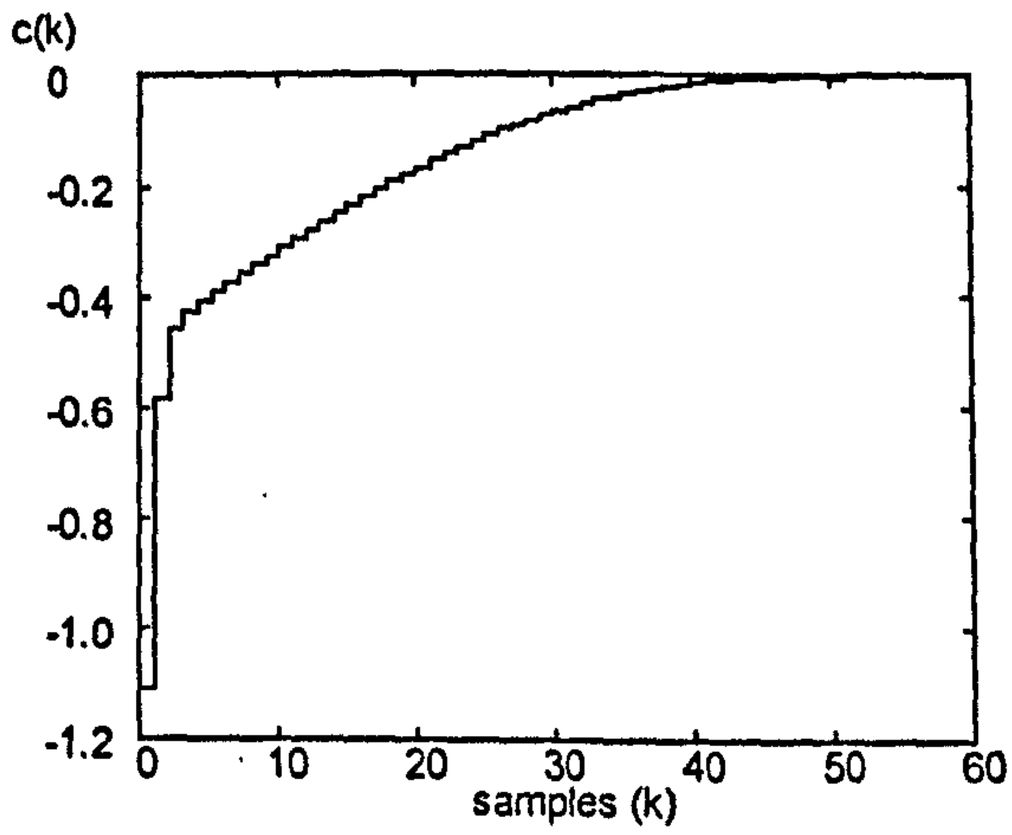

Figure 4.4.1.4 Subsystem 1, Final control signal

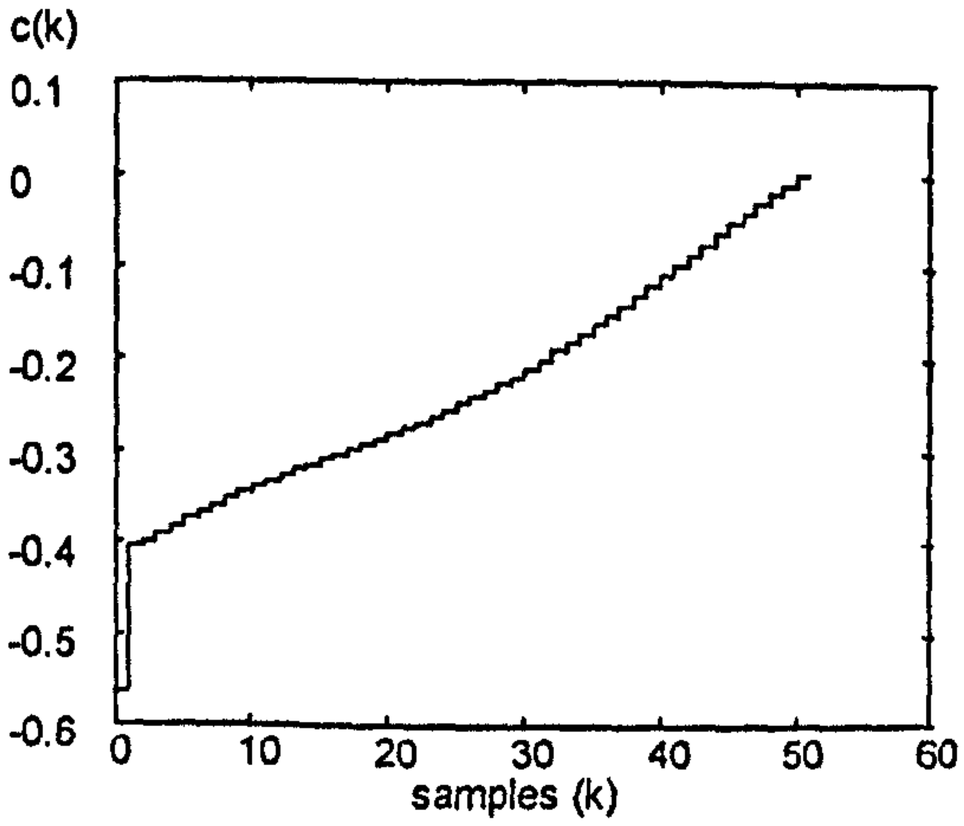

Figure 4.4.1.5 Subsystem 2, Final control signal

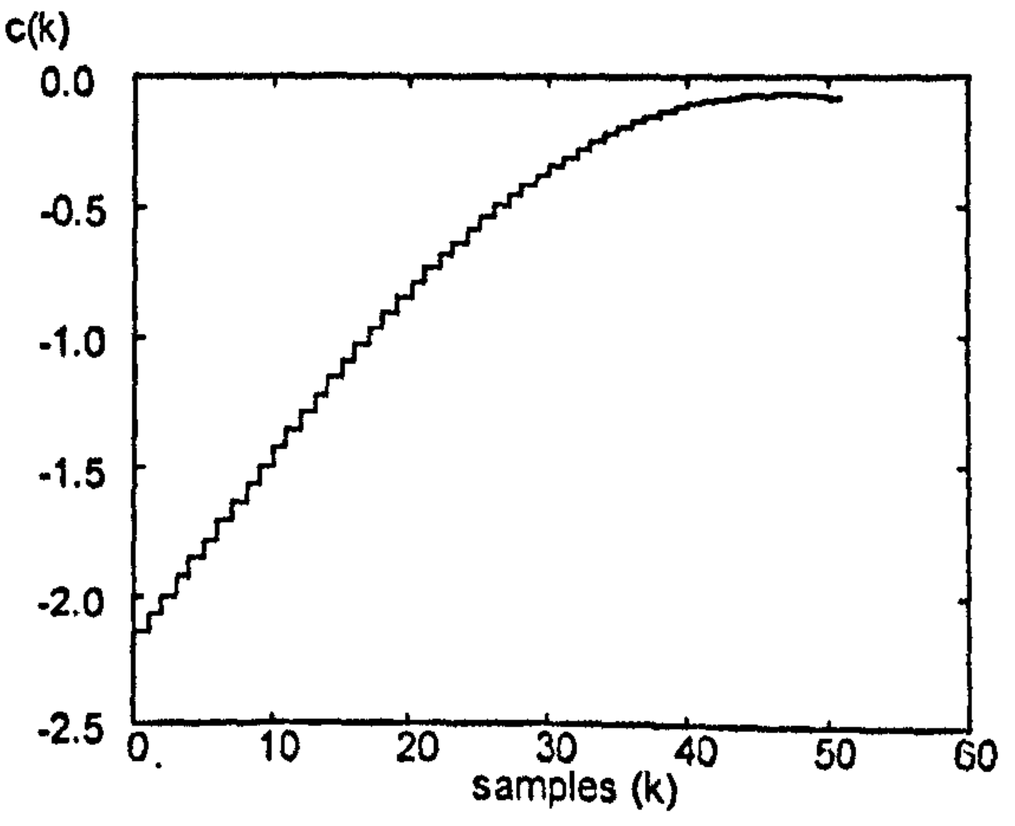

Figure 4.4.1.6 Subsystem 3, Final control signal

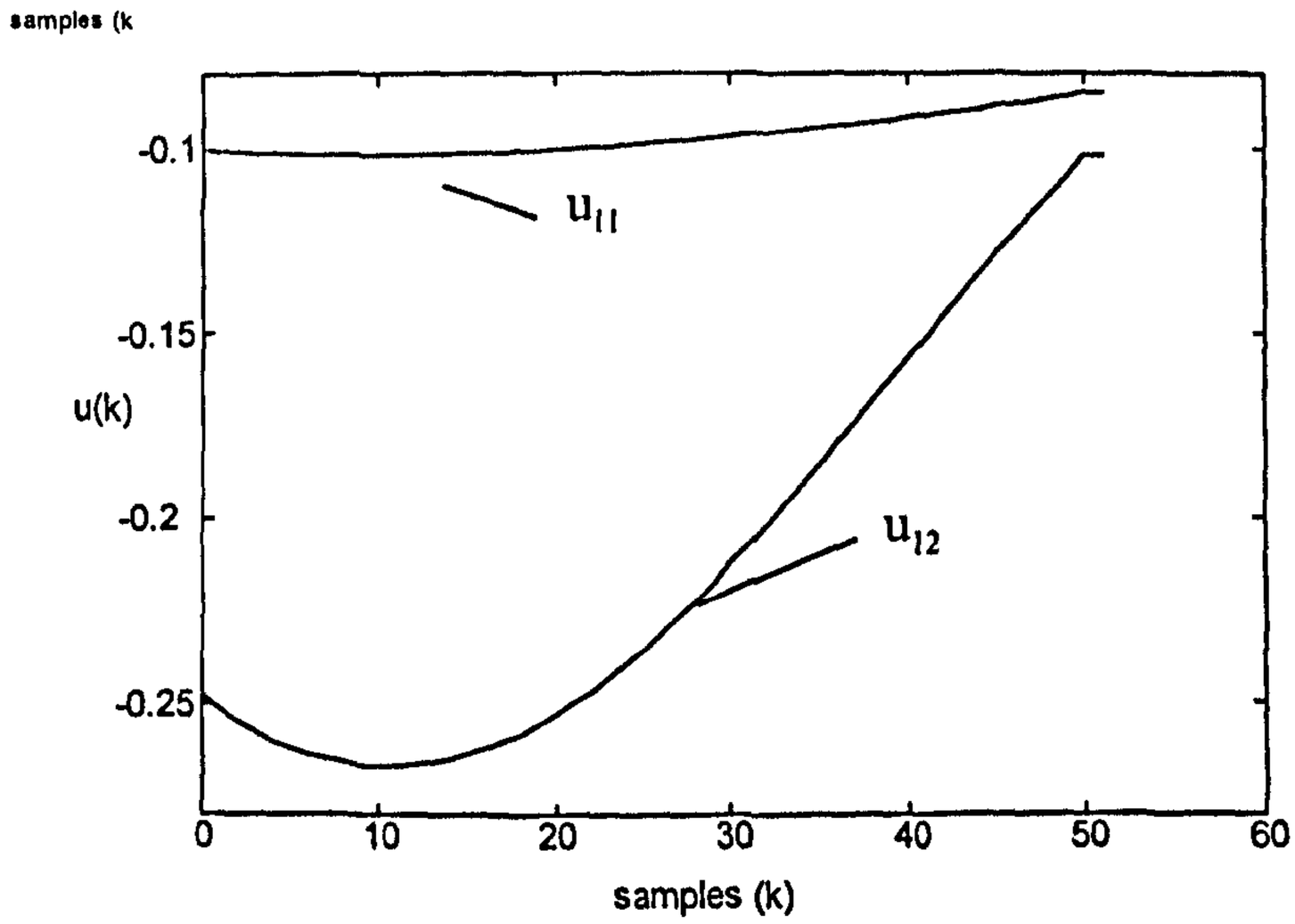

Fig.4.5.1.7 Subsystem 1 interaction vector

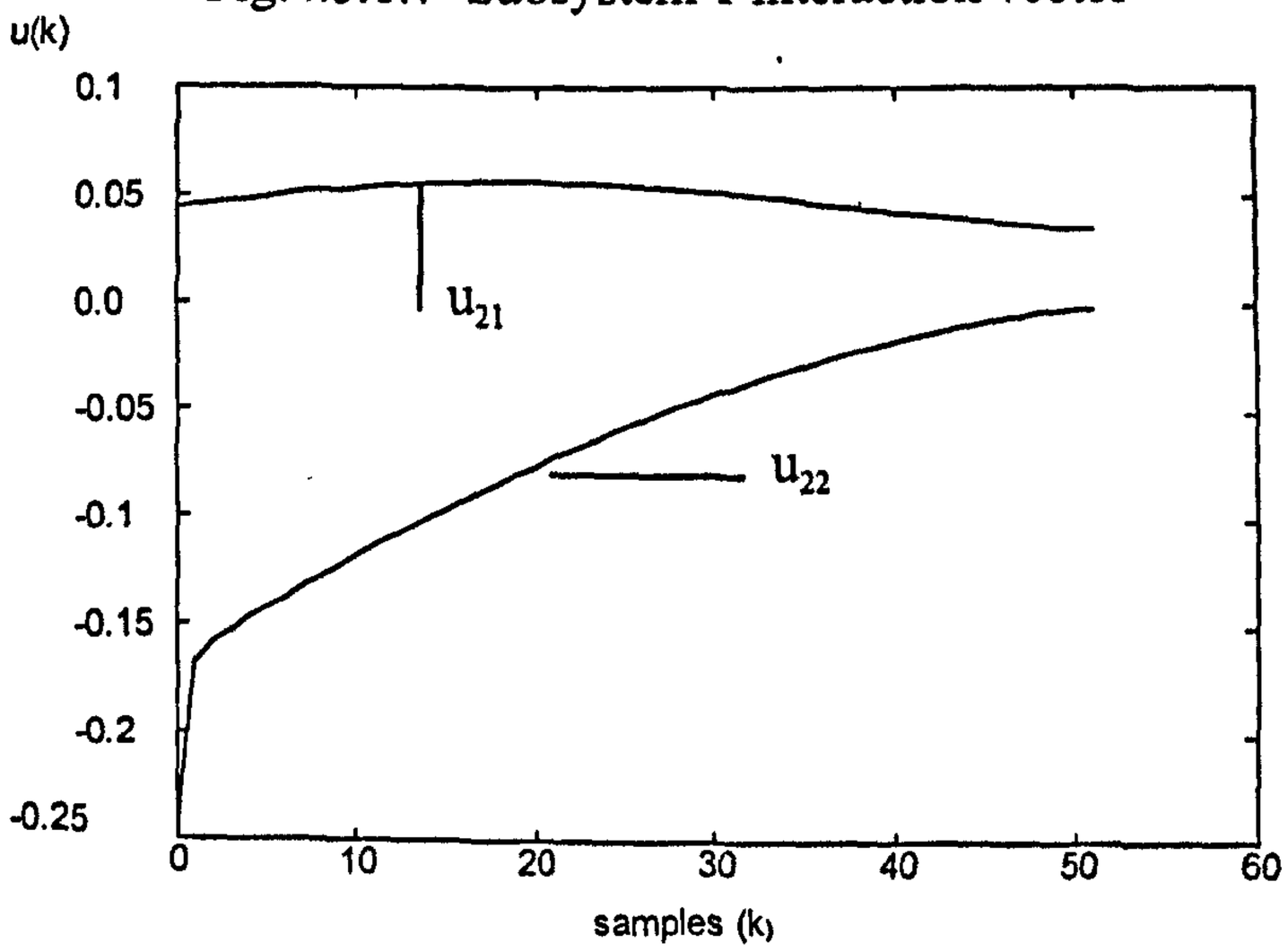

Fig.4.5.1.8 Subsystem 2 interaction vector

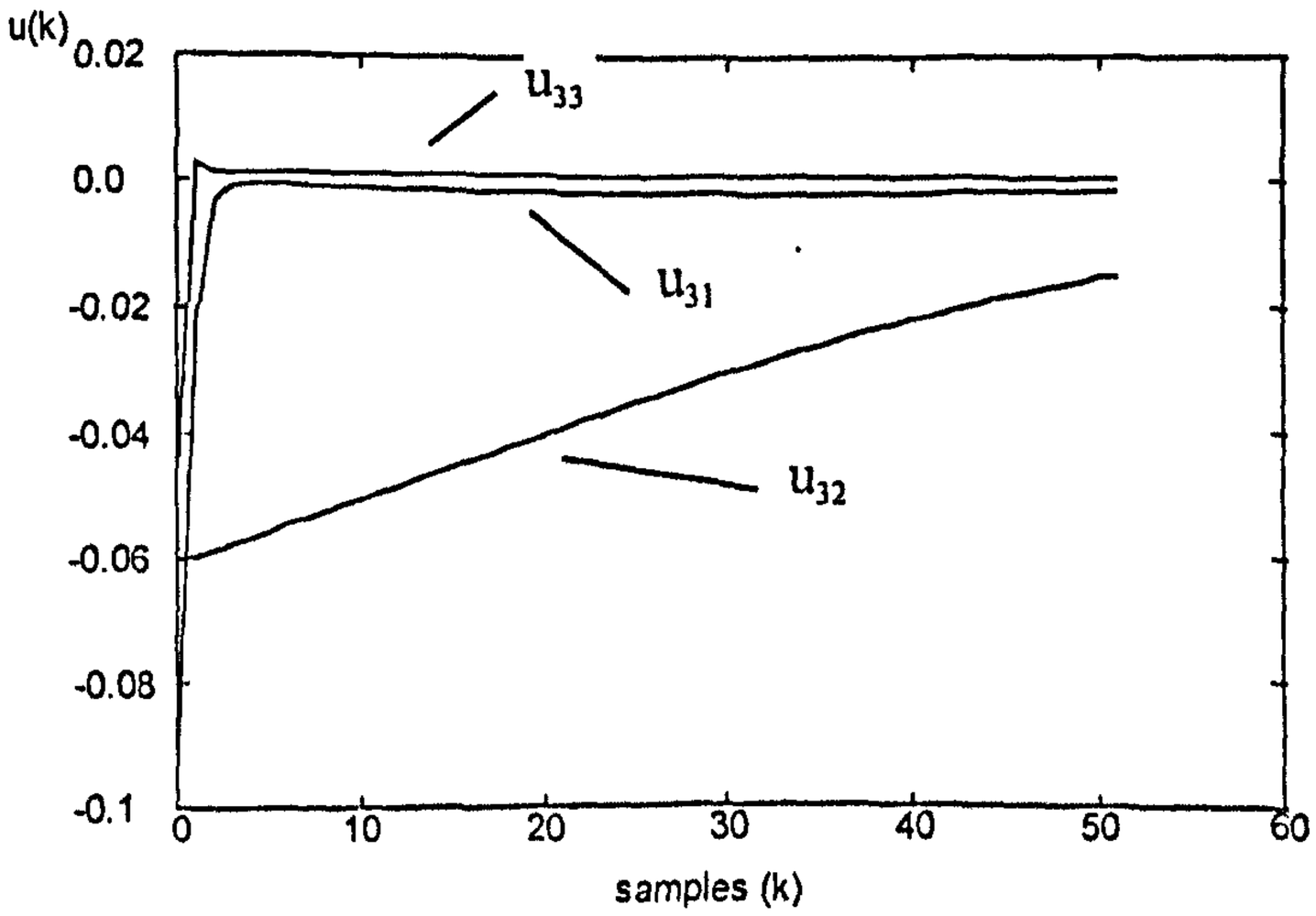

Fig.4.5.2.9 Subsystem 3 interaction vector

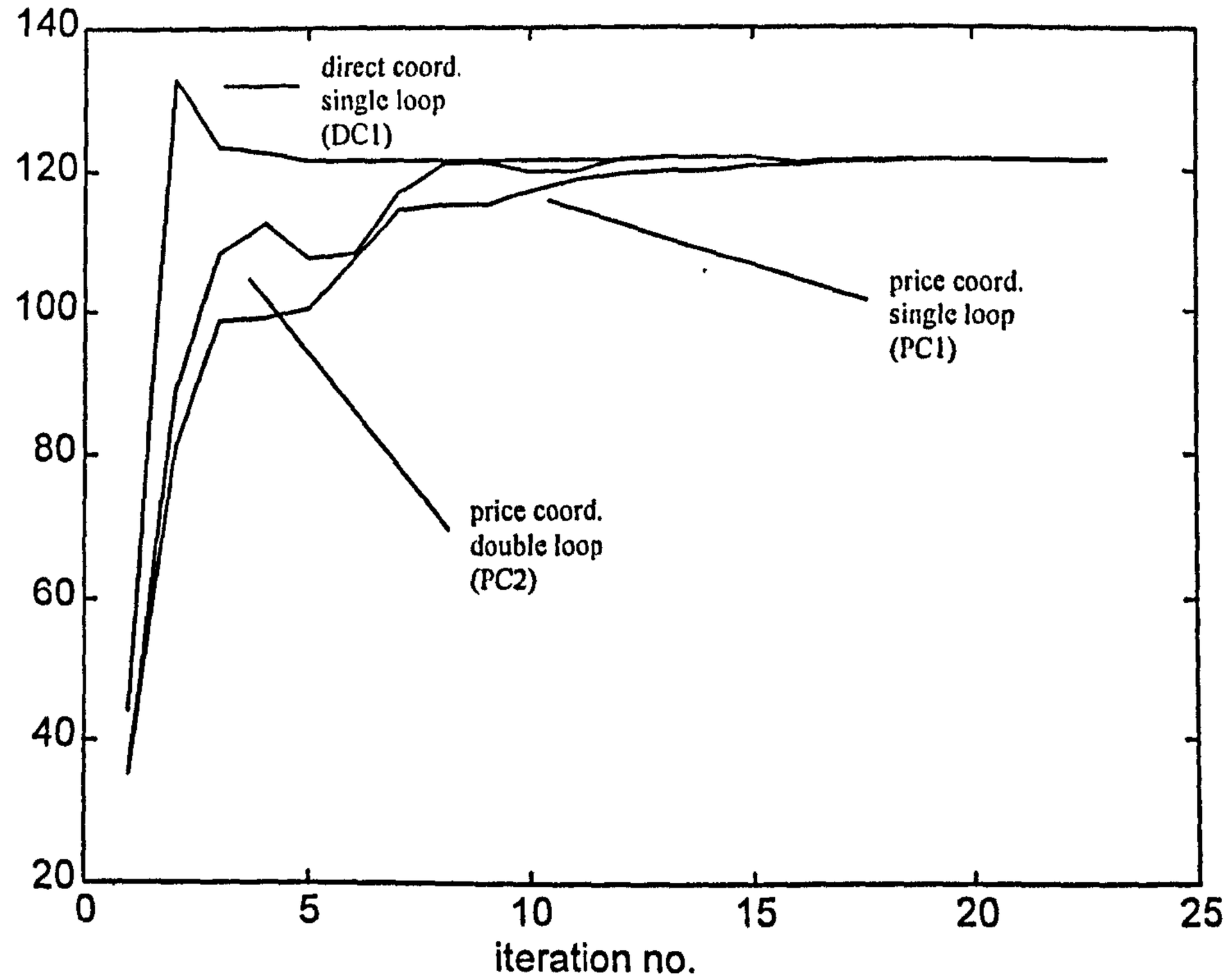

Fig.4.4.1.10 Convergence of performance index for algorithms DC1, PC1 and PC2

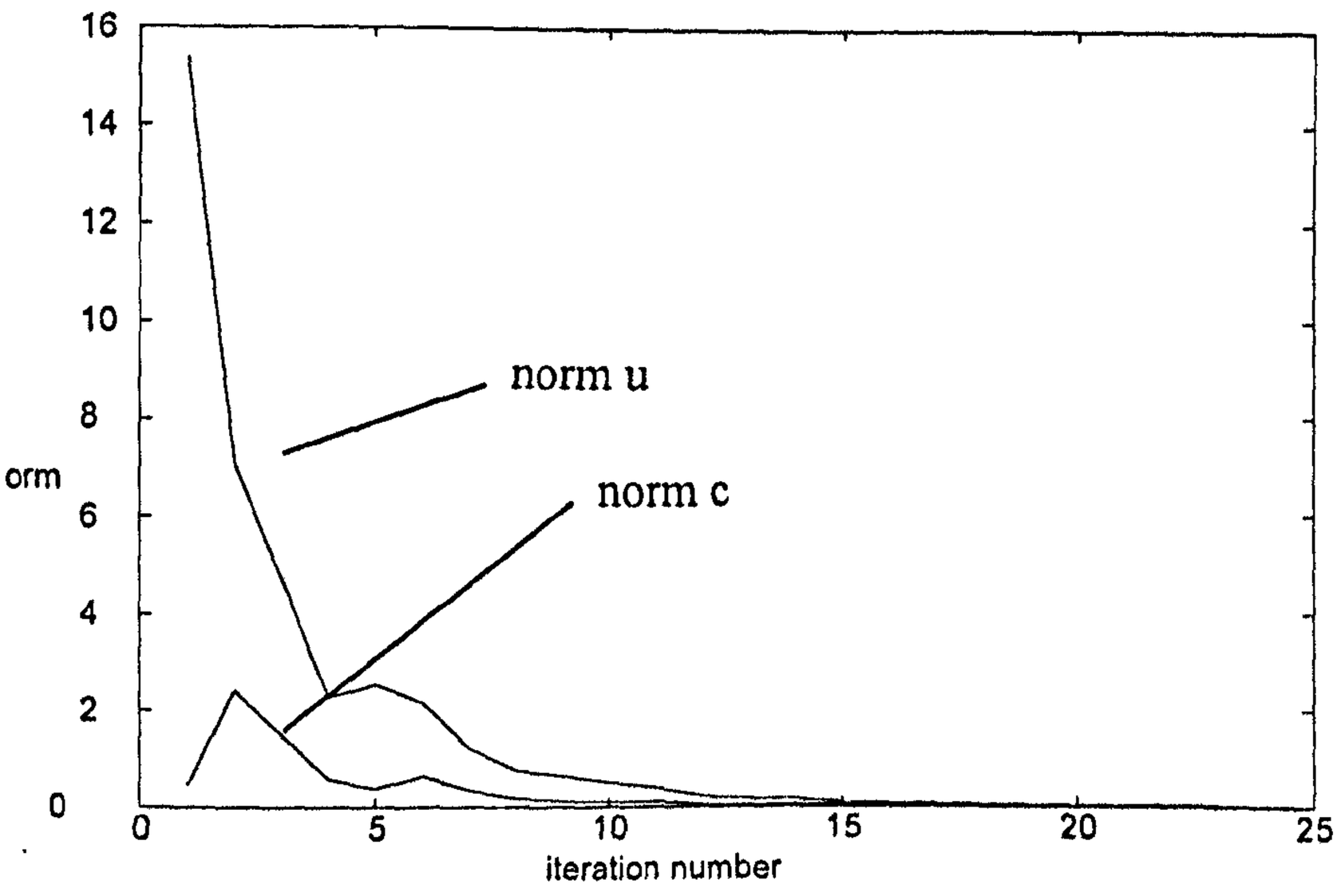

Figure 4.4.1.11 Convergence of interaction norm and control norm , price coordination double loop(PC2) algorithm

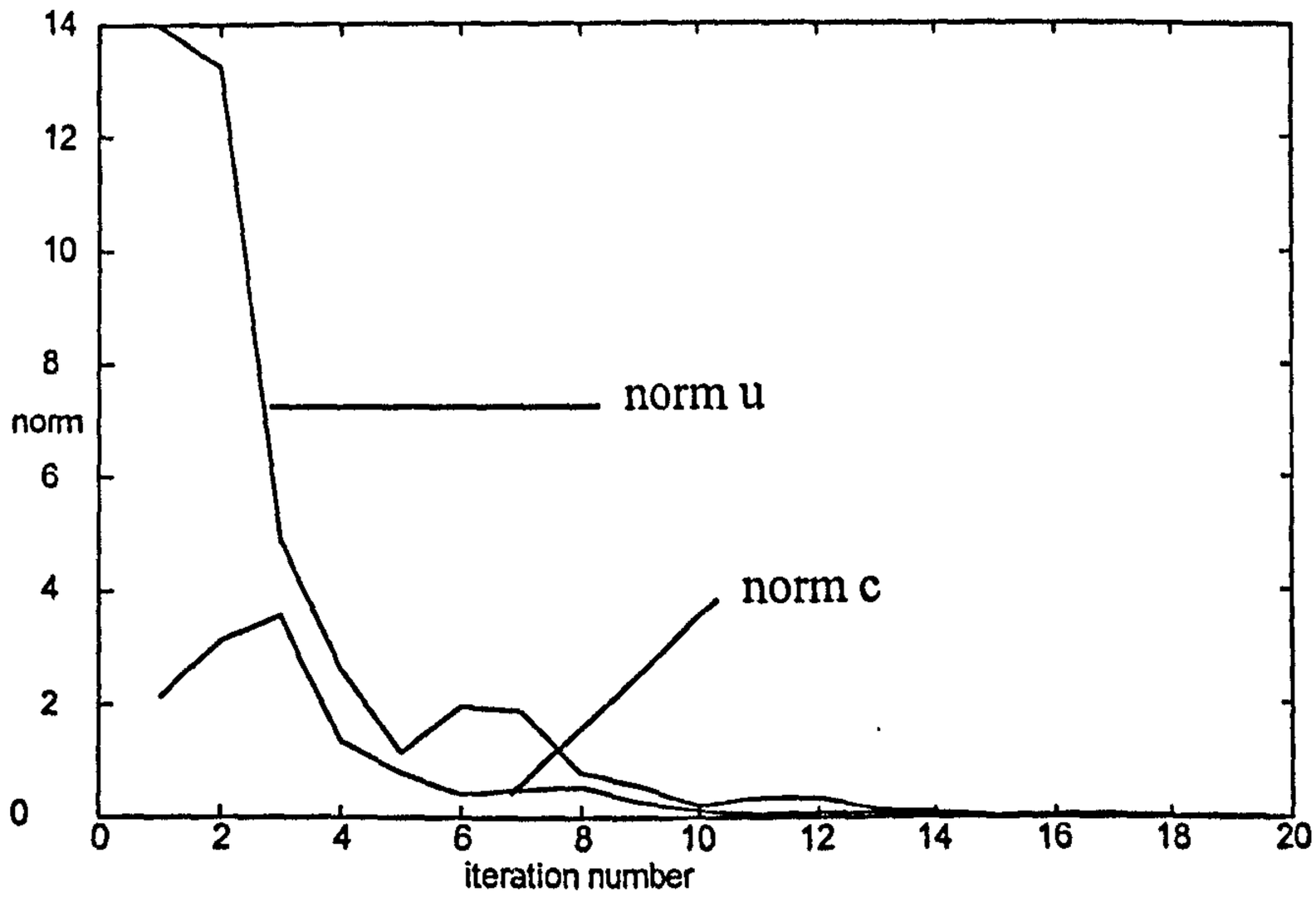

Figure 4.4.1.12 Convergence of interaction norm and control norm , price coordination single loop(PC1) algorithm

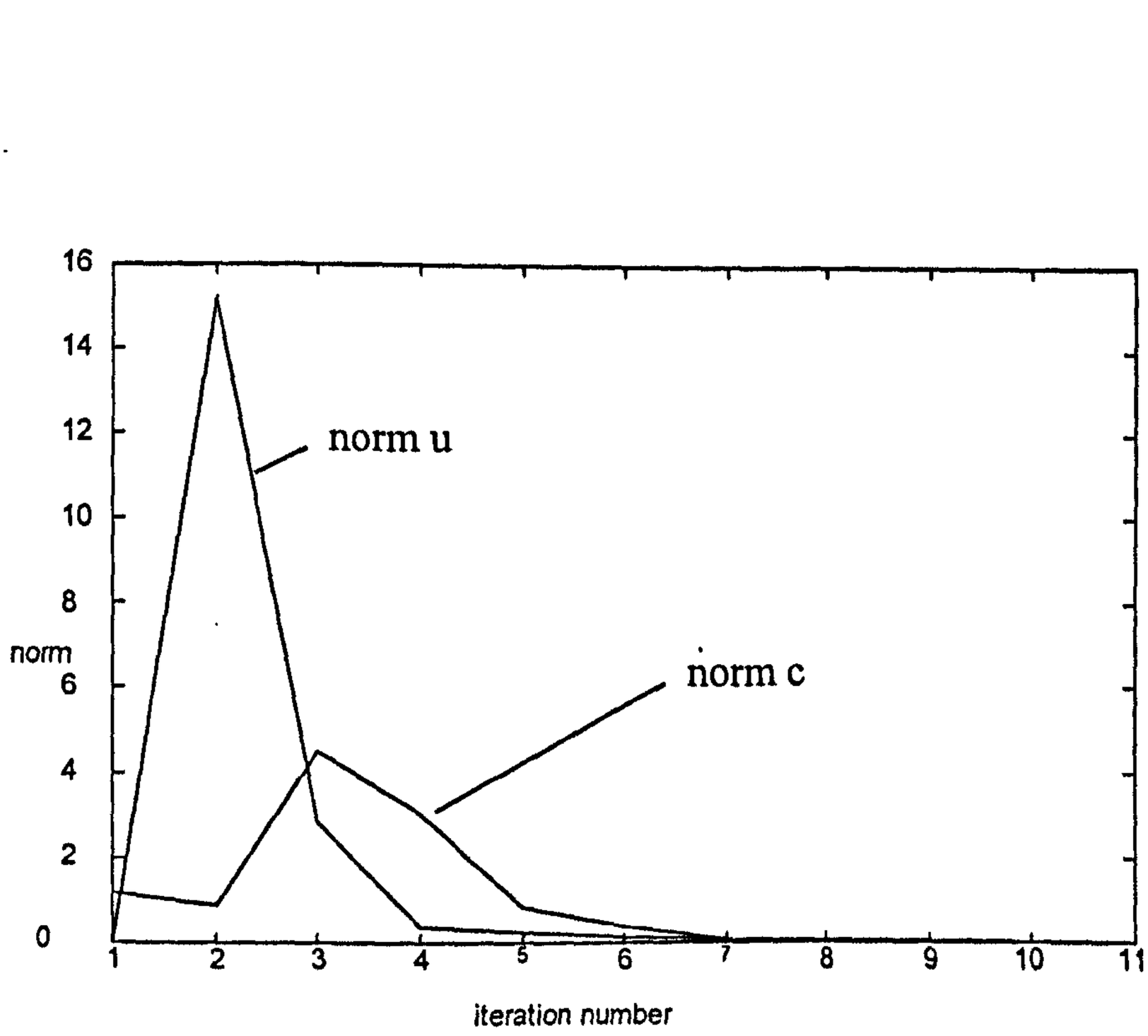

Figure 4.4.1.13 Convergence of interaction norm and control norm , direct coordination double loop(DC2) algorithm

Example 4.4.2

This example is a modification of an example from Findeisen et al (1980), consisting of an interconnected system made up three non-linear subsystems. Note that the coefficients of the control and interaction vectors in the subsystems of the ROP are nonlinear. The example is tested on the four different algorithms presented in this chapter. For comparative purposes a centralized version of the example is also solved on a centralized Discrete DISOPE algorithm. The tolerance specified for convergence is set at ε_T=0.01. $T = 0.05$

ROP:

$$\min_{u(k),c(k)} \frac{1}{2} \sum_{i=1}^{3} \sum_{k=0}^{51} [x_i(k)^T Q_i^* x_i(k) + c_i(k)^T R_i^* c_i(k) + u_i(k)^T S_i^* u_i(k)].$$

where

$Q_1^* = 0.5$; $R_1^* = 0.1$; $S_1^* = 0.5$

$Q_2^* = 0.5$; $R_2^* = 0.1$; $S_2^* = 0.5$

$Q_3^* = 0.5$; $R_3^* = 0.1$; $S_3^* = 0.5$

subject to

subsystem 1:

$$x_{1,1}(k+1) = T(1+0.5\sin(\frac{2\pi k}{5})c_{1,1}(k)) + T(0.5+0.2\sin(\frac{2\pi k}{5}))u_{1,1}(k) + x_{1,1}(k)$$

$$y_{1,1}(k) = x_{1,1}(k)$$

$$x_{1,1}(0) = 1.0$$

subsystem 2:

$$x_{2,1}(k+1) = T(1+0.5\sin(\frac{2\pi k}{5})c_{2,1}(k)) + T(0.5+0.5\sin(\frac{2\pi k}{5}))u_{2,1}(k) + 0.5x_{2,1}(k)$$

$$y_{2,1}(k) = x_{2,1}(k) + c_{2,1}(k)$$

$$x_{2,1}(0) = 0.8$$

subsystem 3:

$$x_{3,1}(k+1) = T(1+0.5\sin(\frac{2\pi k}{5})c_{3,1}(k)) + T(0.5+0.5\sin(\frac{2\pi k}{5}))u_{3,1}(k) + x_{3,1}(k)$$

$$y_{3,1}(k) = x_{3,1}(k)$$

$$x_{3,1}(0) = 0.3$$

The interconnection matrix H is given as

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The model-based problem is an Linear Quadratic approximation of ROP. The output function $y_i(k)$ is deliberately chosen to be zero to test the convergence of the algorithms.

MOP:

$$\min_{u(k),c(k)} \frac{1}{2} \sum_{i=1}^{3} \sum_{k=0}^{51} [x_i(k)^T Q_i x_i(k) + c_i(k)^T R_i c_i(k) + u_i(k)^T S_i u_i(k) + \gamma(k)]$$

where

$$Q_1 = 0.5 \; ; \quad R_1 = 0.1 \; ; \; S_1 = 0.5$$

$$Q_2 = 0.5 \; ; \quad R_2 = 0.1 \; ; \; S_2 = 0.5$$

$$Q_3 = 0.5 \; ; \quad R_3 = 0.1 \; ; \; S_3 = 0.5$$

subject to

subsystem 1:

$$x_{1,1}(k+1) = 0.05c_{1,1}(k) + 0.025u_{1,1}(k) + x_{1,1}(k)$$

$$y_{1,1}(k) = 0$$

$$x_{1,1}(0) = 1.0$$

subsystem 2:

$$x_{2,1}(k+1) = 0.05c_{2,1}(k) + 0.025u_{2,1}(k) + 0.5x_{2,1}(k)$$

$$y_{2,1}(k) = 0$$

$$x_{2,1}(0) = 0.8$$

subsystem 3:

$$x_{3,1}(k+1) = 0.05c_{3,1}(k) + 0.025u_{3,1}(k) + x_{3,1}(k)$$

$$y_{3,1}(k) = 0$$

$$x_{3,1}(0) = 0.3$$

Algorithm	ε_1	r_1	r_2	r_3	Number of iterations	CPU (s)	Final Performance Index
Price Cor. single loop	0.3	0.9	1.0	0	101	260	15.8701
	0.2	0.9	1.0	0	100	259	15.8692
	0.1	0.9	0	0	99	258	15.8679
*	0.05	0.9	0	0	94	250	15.8710
	0.01	0.9	0	0	107	275	15.8689
	0.05	0.9	0	1.0	121	307	15.8659
Price Cor. Double Loop	0.7	0.9	0	0	95	276	15.8729
*	0.3	0.9	0	0	94	265	15.8717
	0.5	0.9	1.0	0	95	269	15.8722
	0.3	1.2	0	1.0	100	288	15.8690
	0.3	2.0	0	0	135	381	15.8733
Direct Cr. single loop	N/A	1.0	1.0	0.0	102	273	15.8747
		1.0	0.0	0.0	102	274	15.8750
*		0.9	0.0	1.0	97	240	15.8721
		1.2	0.0	3.0	113	298	15.8801
		0.9	0.0	0.5	110	294	15.8831
Direct Cr. Double Loop	N/A	0.9	1.0	0.0	96	235	15.8788
*		0.9	0.0	0.0	96	232	15.8731
		0.9	0.0	0.1	114	276	15.8697
		0.9	0.0	0.5	113	274	15.8677
Centralize DISOPE (Centralized Problem)	N/A	0.0	0.0	N/A	25	40	15.8891

Table 4.4.2 Algorithms performance for Example 4.4.2

The choice for step size for variables updates are found by trial and error. The best combination for this example are $\varepsilon_v = 1.0.\ \varepsilon_z = 1.0.\ \varepsilon_u = 0.9.\ \varepsilon_p = 1.0.$

Entries marked with a * denotes the best choice of parameter settings. From table 4.4.2 it can be observed that r_1 and ε_l influence the rate of convergence of the price coordination algorithms, PC1 and PC2. In the single loop direct coordination algorithm DC1, r_1 and r_3 have the effect of improving the rate of convergence. However in the double loop versions of each algorithm only r_1 seemed to influence the rate of convergence. There is no significant difference in terms of speed of convergence between the double loop and single loop technique for the two approaches. It can be seen however that the double loop technique increases the computation time slightly for the same number of iterations. The speed of convergence is very slow when compared to the centralized approach using the centralized version of DISOPE. This can be explained by the difference in the number of variables to be iterated in the hierarchical case as compared with the centralized case. Figure 4.4.2.1-4.4.2.3 show the final state vector of each subsystem. The final control signals and the interaction vector can be observed in figures 4.4.2.4-4.4.2.9. It should be noted that changing the stepsize for update for values other than the optimum choice shown results in a slower speed of convergence.

Figure 4.4.2.10 shows the convergence of global performance index of single loop versions of the algorithms and figure 4.4.2.11 shows the convergence of double loop versions of the algorithms. It can be observed that there is no significant difference in convergence behaviour between the two approaches in solving example 4.4.2. The change in direction in the convergence pattern of the performance index can be explained by the presence of model-reality differences in the model based problem. Note that in this example we have deliberately approximated a sinusoidal time-varying function in ROP with a linear time invariant function in MOP. This convergence pattern is also exhibited by the centralized DISOPE algorithm in solving the equivalent centralized problem. This is illustrated by Figure 4.4.2.13. Figure 4.4.2.12 shows the convergence of the control variation norm of the single loop algorithms with the best choice of tuning parameter settings.

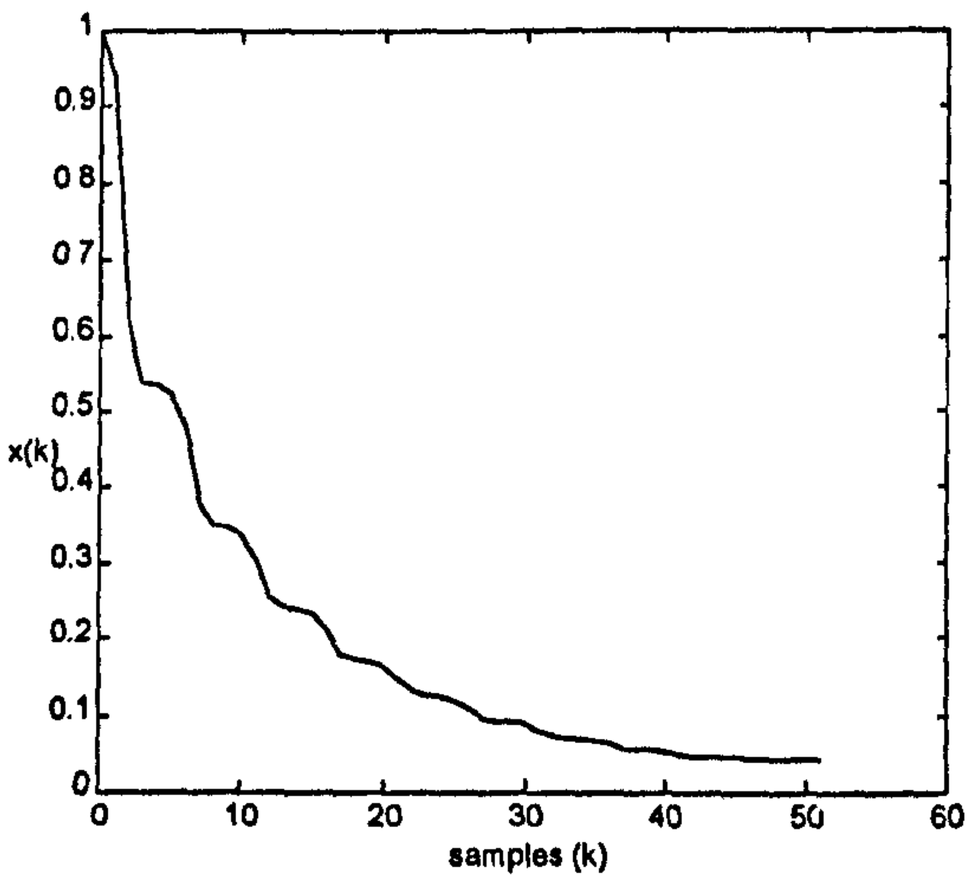

Figure 4.4.2.1: Subsystem 1, state vector

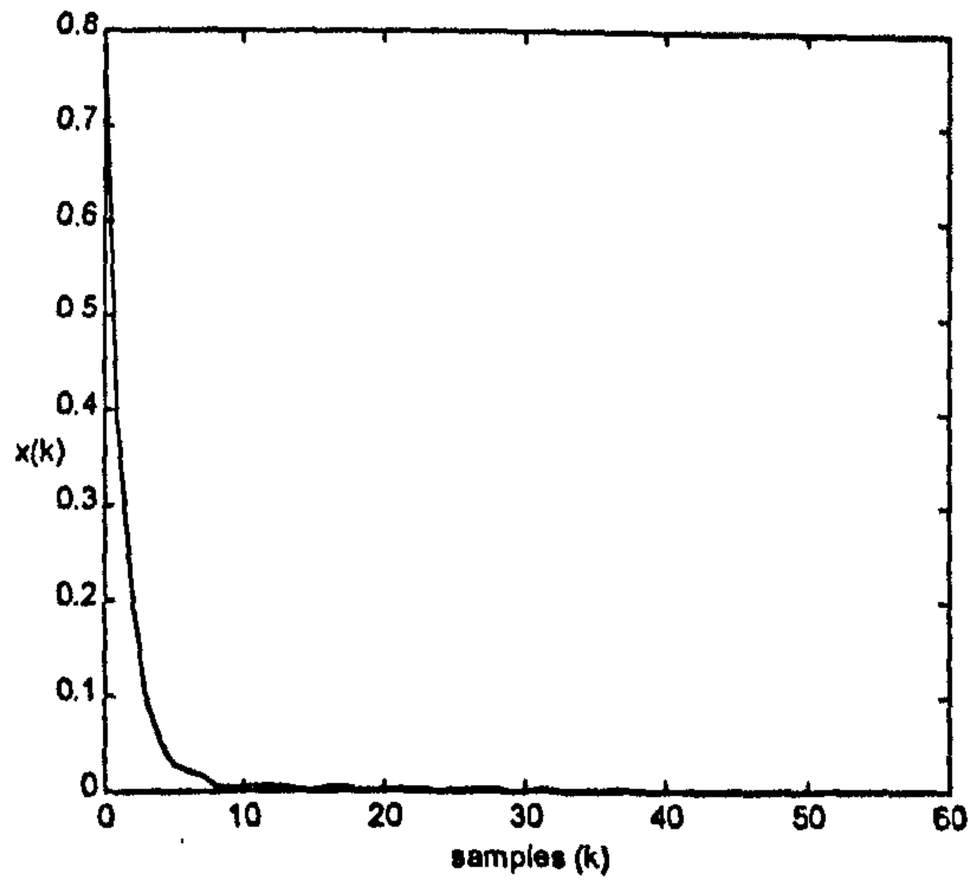

Figure 4.4.2.2: Subsystem 2, state vector

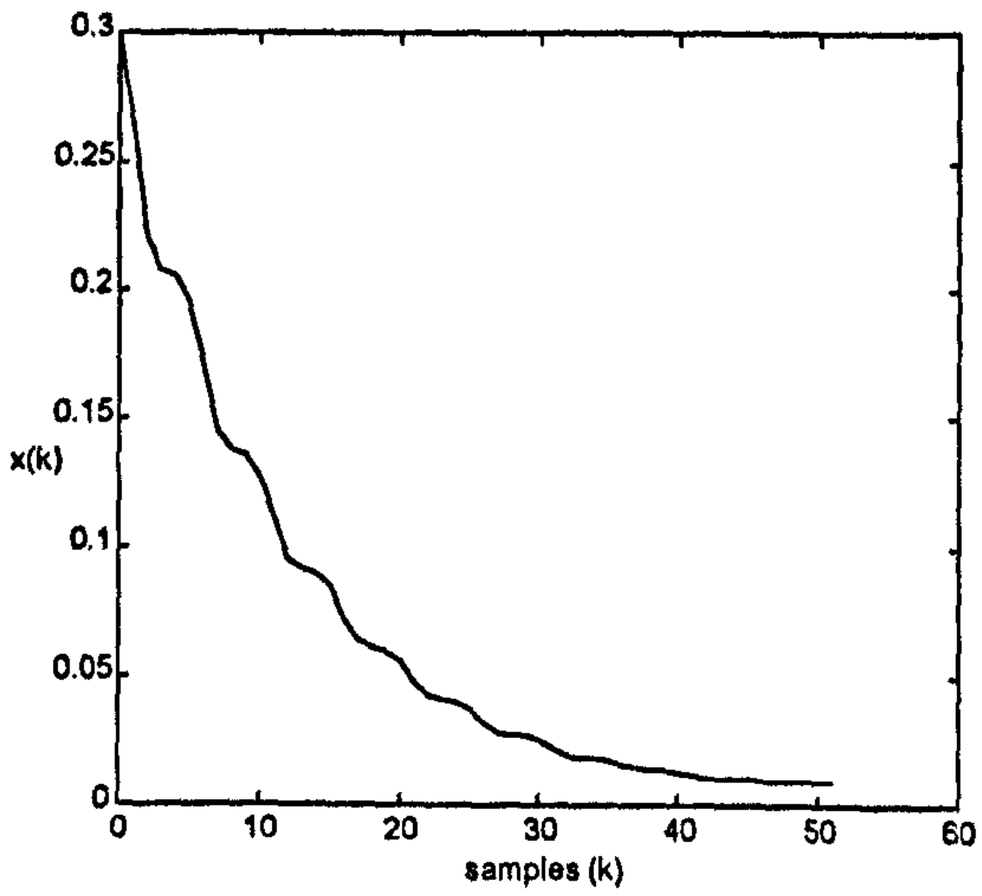

Figure 4.4.2.2 :. Subsystem 3, state vector

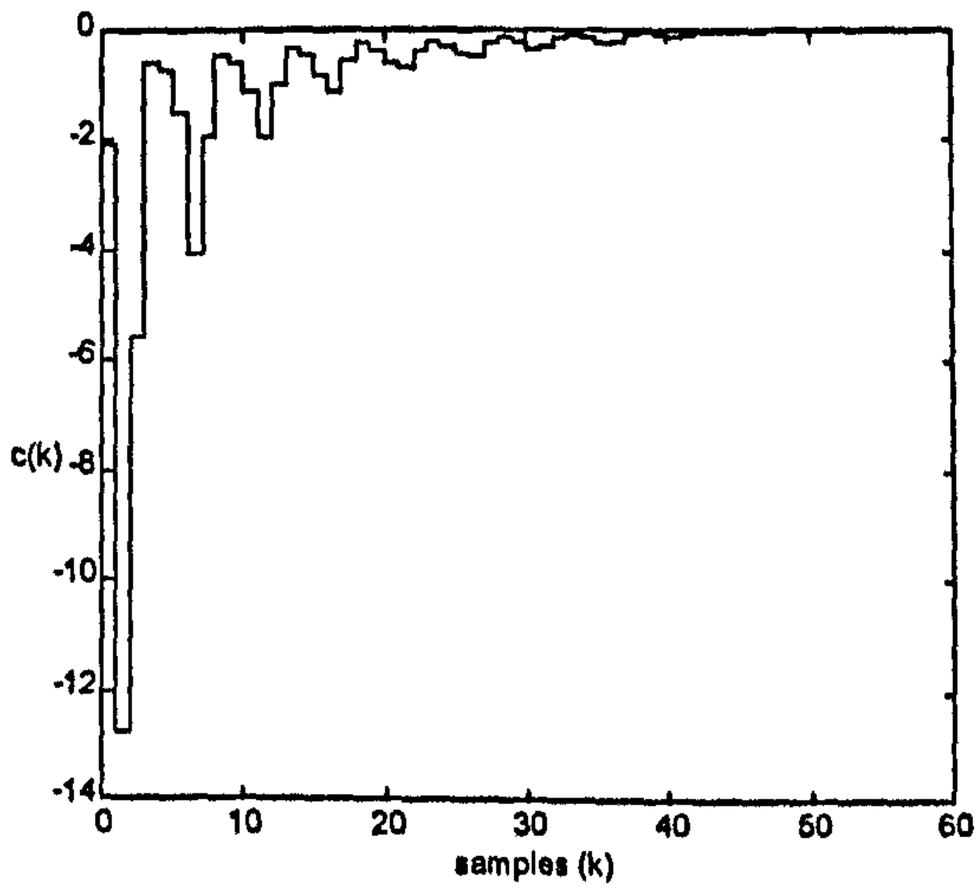

Figure 4.4.2.4: Subsystem 1, final control signal

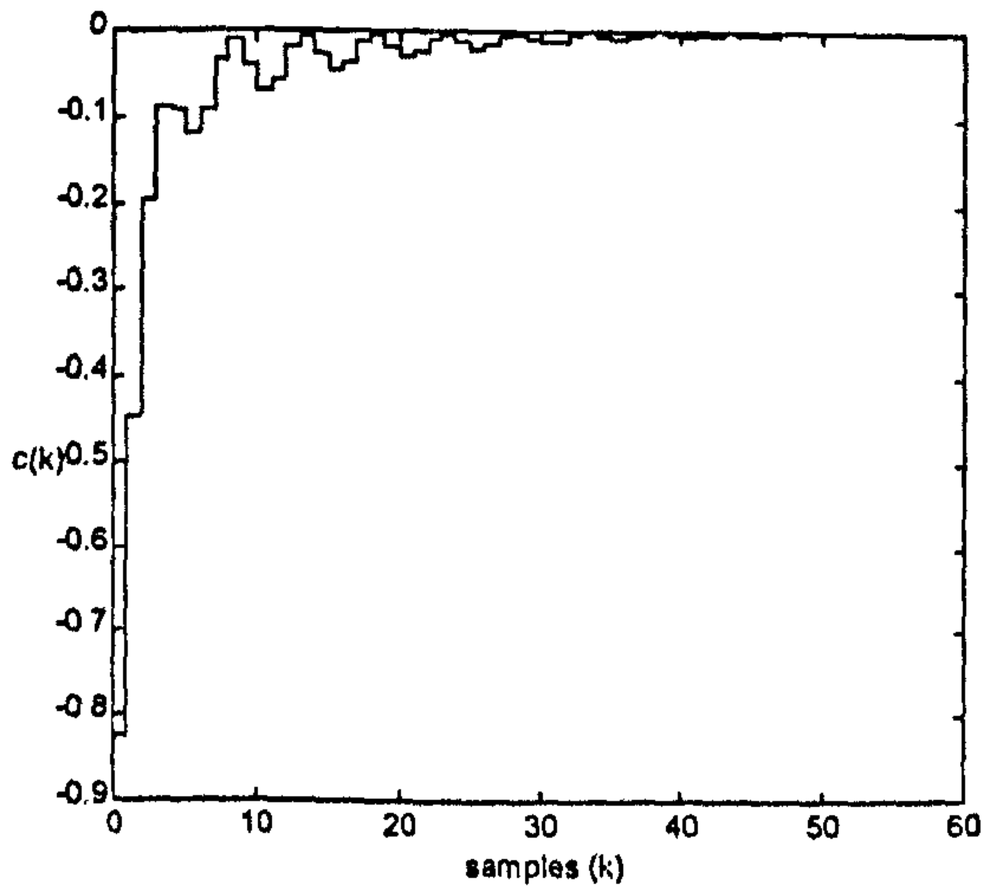

Figure 4.4.2.5: Subsystem 2 ,final control signal

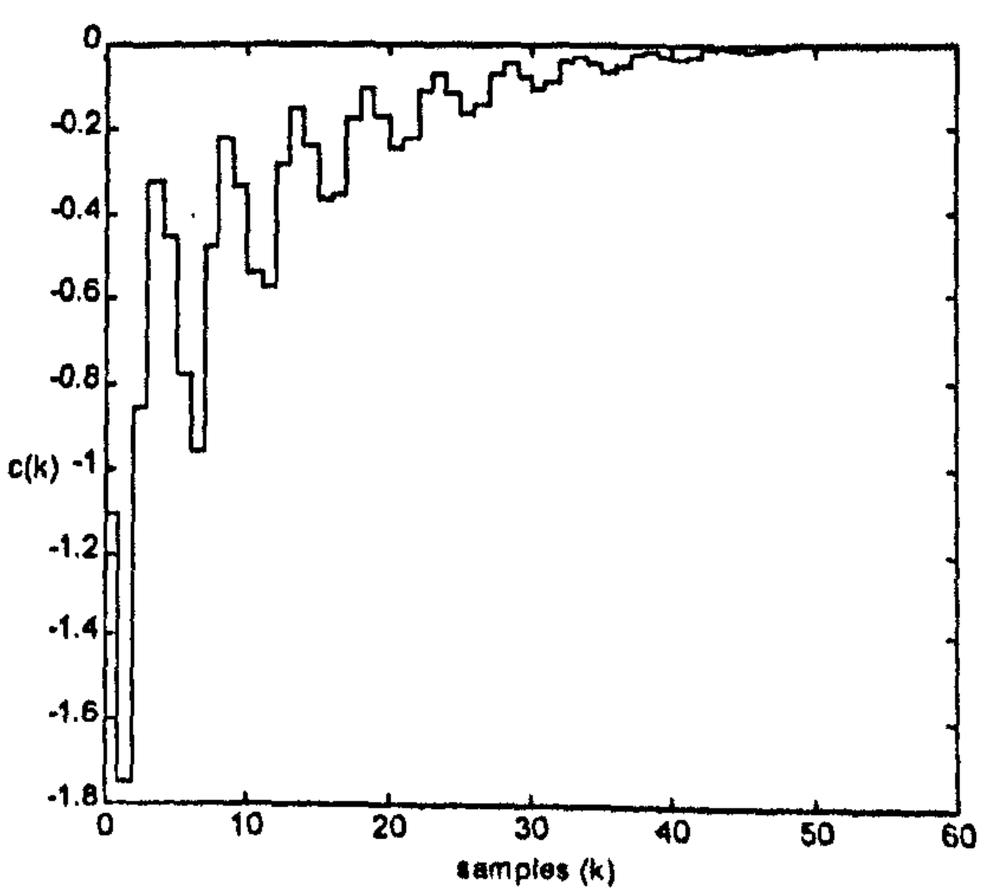

Figure 4.4.2.6:. Subsystem 3, final control signal

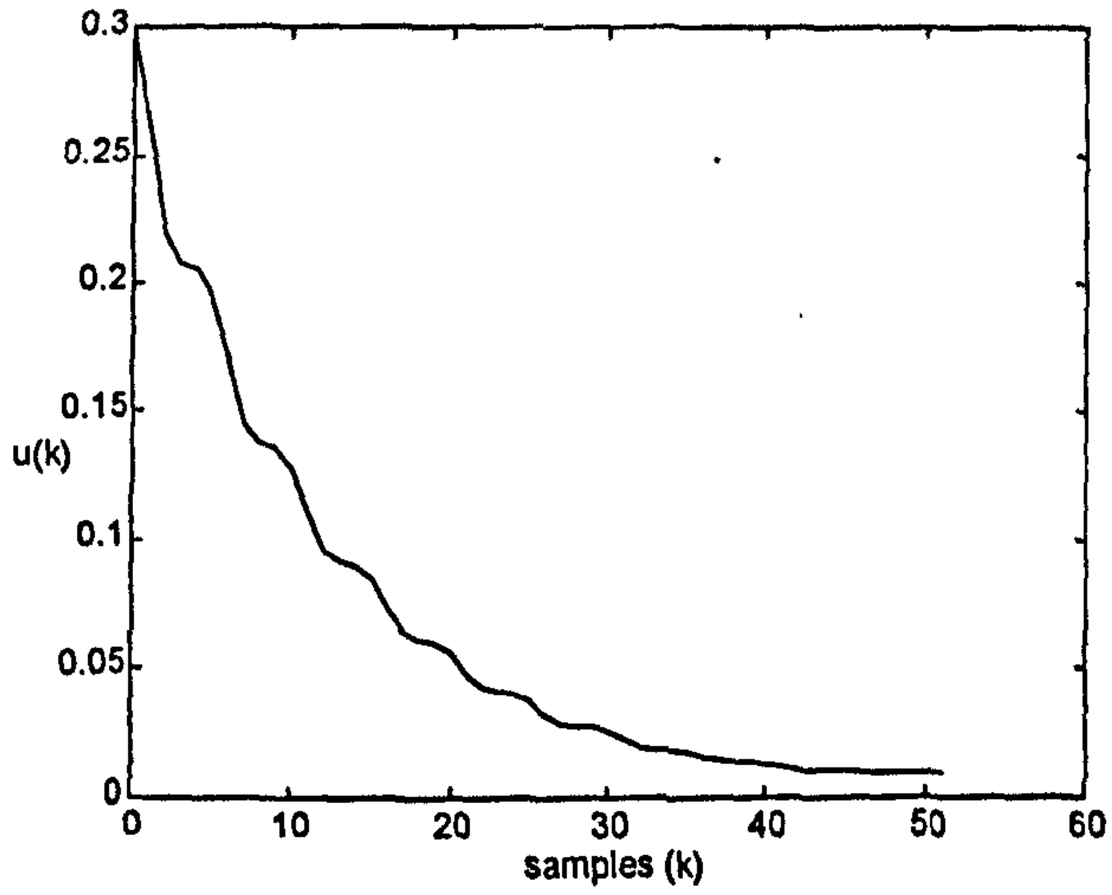

Figure 4.4.2.7 Subsystem 1, interaction vector

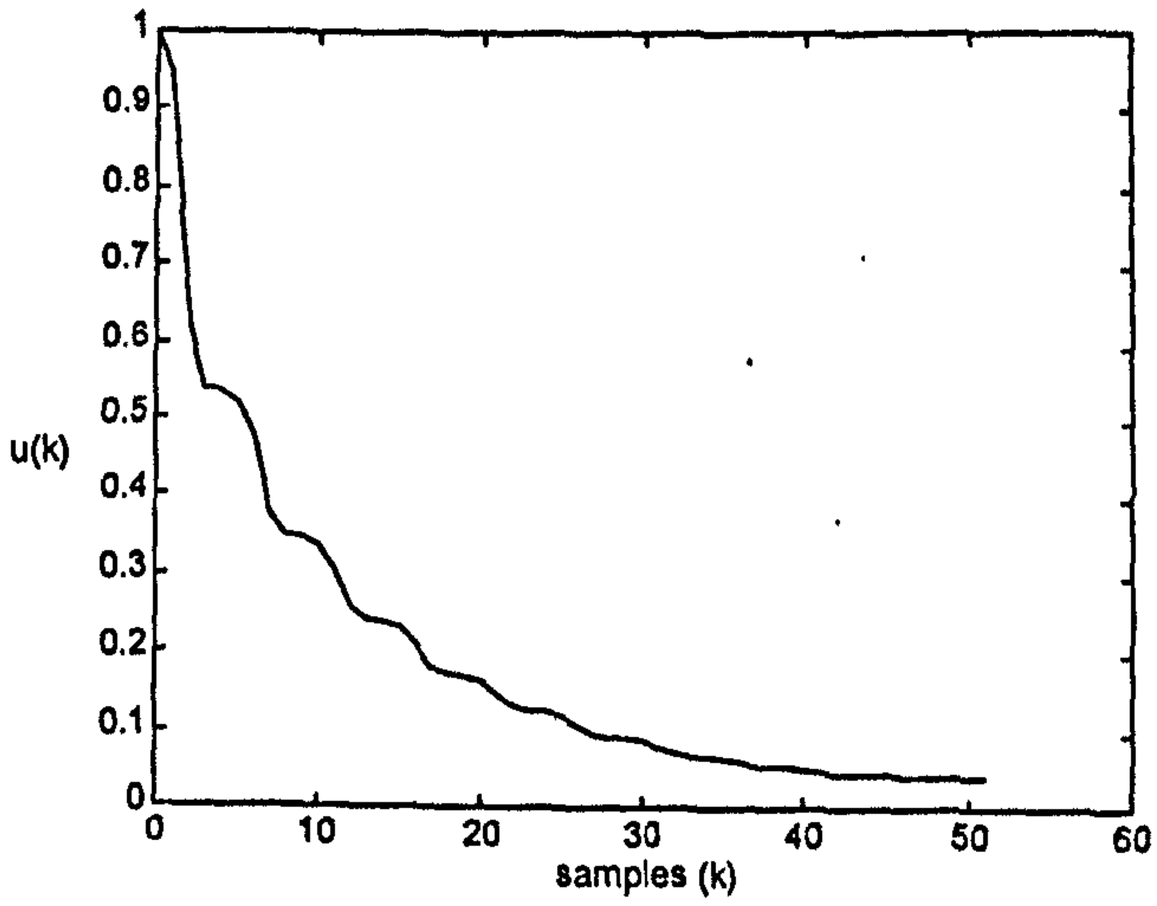

Figure 4.4.2.8 Subsystem 2, interaction vector

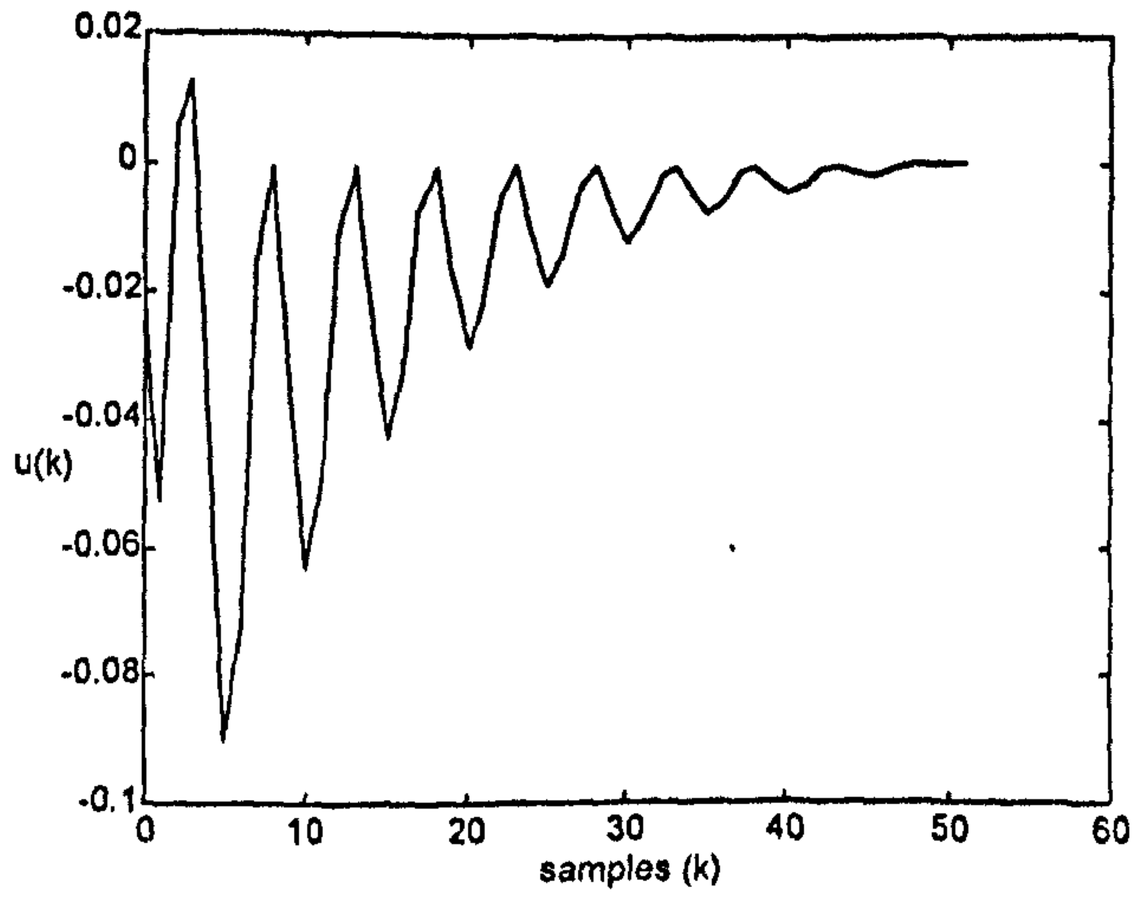

Figure 4.4.2.9 Subsystem 3, interaction vector

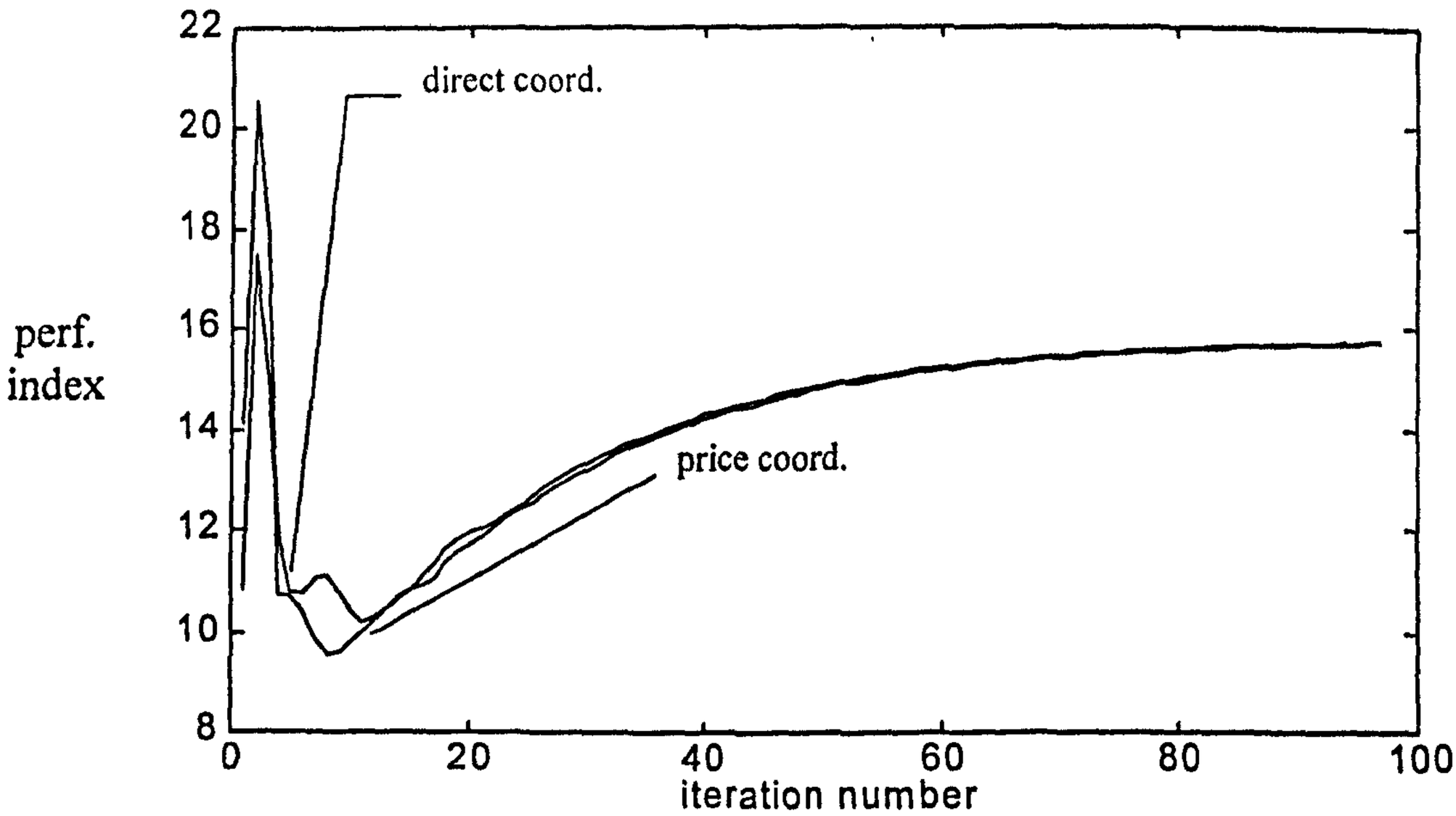

Figure 4.4.2.10 Convergence of performance index using single loop algorithms

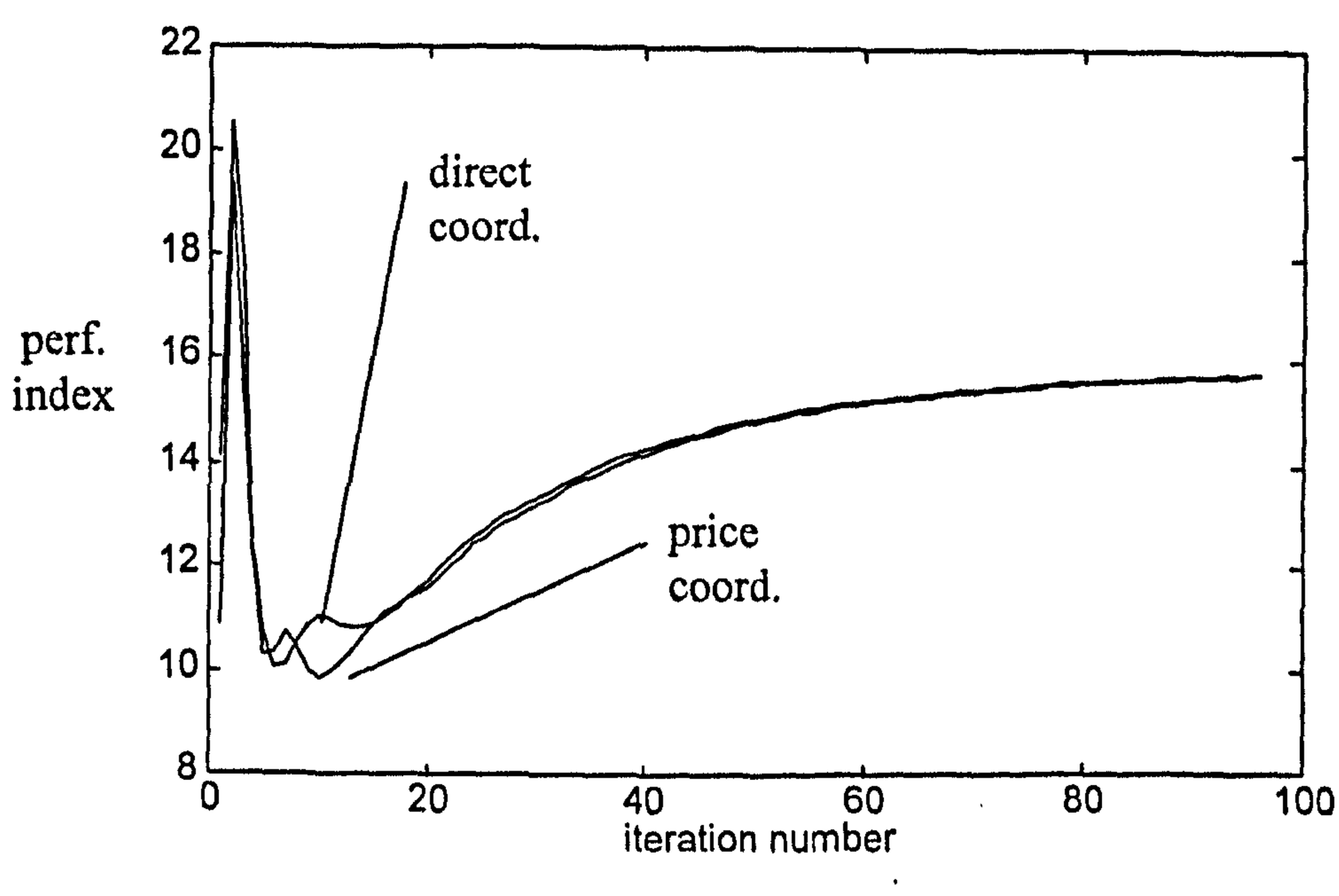

Figure 4.4.2.11 Convergence of performance index using double loop algorithms

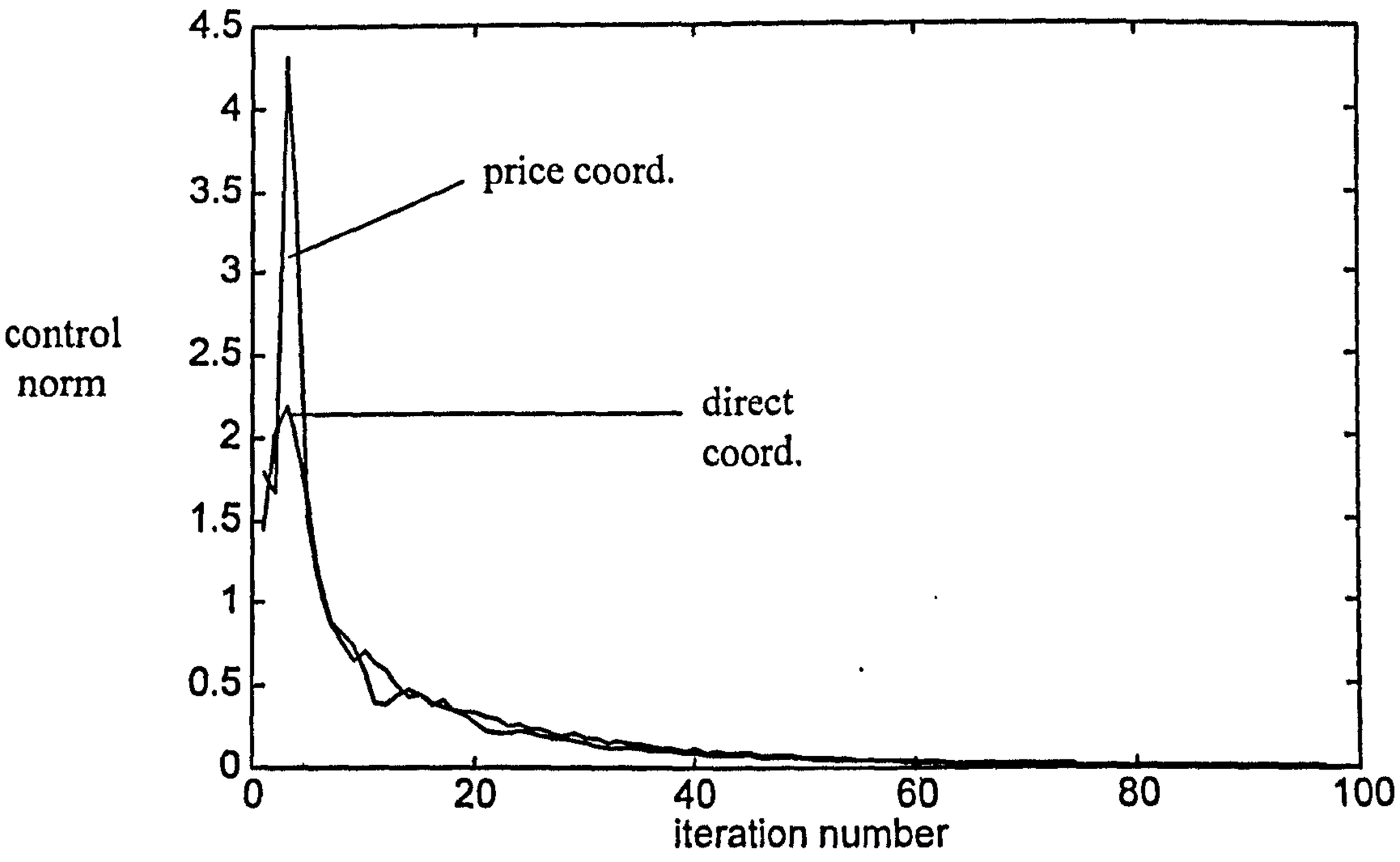

Figure 4.4.2.12 Convergence of control variation norm for single loop algorithms.

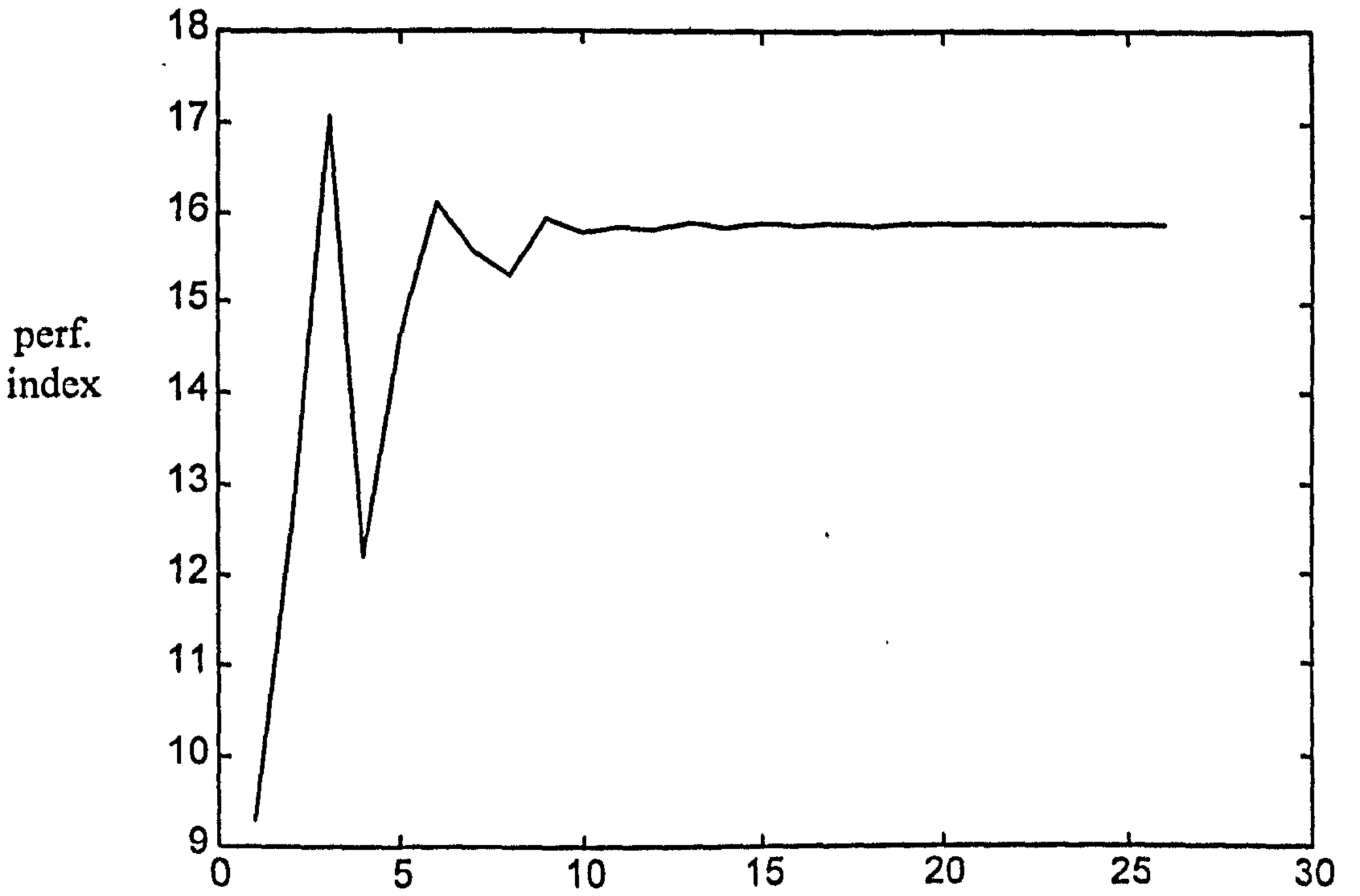

Figure 4.4.1.13 Convergence of real performance index of centralized DISOPE algorithm

4.5 SUMMARY

Four algorithms for solving hierarchical structures with model based interaction input have been presented, discussed and implemented in software using C++ programming Language. Two approaches have been used in implementation of each algorithm. In the price coordination approach, the interaction price is iterated using the dynamic version of price mechanism presented by Findeisen et al(1980) . The second approach, direct coordination, is similar to the interaction-prediction method (Singh, 1980; Jamshidi, 1983; Mahmoud et al, 1985) where the interaction price is calculated directly from the optimality conditions. However, in the proposed algorithm a more general version of interaction involving output function have been used . Simulation results indicate that algorithms using the direct coordination approach have a better convergence property when compared to that of the price coordination approach. This can be accounted by the extra computation required by iterations of the price updating mechanism . The price coordination approach however offers a greater flexibility in the choice of model-based output functions .In terms of techniques of implementation the double loop version of both approaches have slower rate of convergence than that of the single loop technique. The double loop procedure which utilizes the global structure of the interaction term is however more attractive in terms of the ease of software implementation and analysis. In the following chapter we proceed to prove the optimality of algorithms and show conditions under which the double loop version of the algorithms will converge.

CHAPTER 5

OPTIMALITY AND CONVERGENCE ANALYSIS FOR DOUBLE LOOP ALGORITHM FOR STRUCTURES WITH MODEL BASED INTERACTION MEASUREMENTS.

5.1 INTRODUCTION

In this chapter an analysis of optimality and convergence properties of double loop hierarchical DISOPE algorithms with a linear-quadratic model based problem described in Chapter 4 is presented. The optimality is established by showing that the global optimality conditions of the modified model based optimal control in the algorithms is equivalent to that of the global real optimal control problem (3.10). The convergence conditions of the double loop version of the algorithms, which are derived by using contraction mapping arguments (Kantorovich, Akilov, 1982), are obtained by applying similar procedures as presented initially by the work of Hassan and Singh (1976), and more recently by Becerra and Roberts (1996).

Initially to simplify notation, we will need the following definitions.

Define

$$f_{c_i}^*(x_i,c_i,u_i,k)\equiv\frac{\partial}{\partial c_i(k)}f_i^*(x_i(k),c_i(k),u_i(k),k)$$

$$f_{x_i}^*(x_i,c_i,u_i,k)\equiv\frac{\partial}{\partial x_i(k)}f_i^*(x_i(k),c_i(k),u_i(k),k)$$

$$f_{u_i}^*(x_i,c_i,u_i,k)\equiv\frac{\partial}{\partial u_i(k)}f_i^*(x_i(k),c_i(k),u_i(k),k)$$

$$q_{c_i}^*(x_i,c_i,u_i,k)\equiv\frac{\partial}{\partial c_i(k)}q_i^*(x_i(k),c_i(k),u_i(k),k)$$

$$q_{x_i}^*(x_i,c_i,u_i,k)\equiv\frac{\partial}{\partial x_i(k)}q_i^*(x_i(k),c_i(k),u_i(k),k)$$

$$q_{u_i}^*(x_i,c_i,u_i,k)\equiv\frac{\partial}{\partial u_i(k)}q_i^*(x_i(k),c_i(k),u_i(k),k)$$

$$K_{c_i}^*(x_i,c_i,k)\equiv\frac{\partial}{\partial c_i(k)}K_i^*(x_i(k),c_i(k),k)$$

$$K^*_{x_i}(x_i,c_i,k) \equiv \frac{\partial}{\partial x_i(k)} K^*_i(x_i(k),c_i(k),k) \tag{5.1}$$

and specify the optimal solution, that is the solution of global ROP (3.10) , as

$c^{op}(k), x^{op}(k), p^{op}(k)$ and $u^{op}(k)$; k∈[1,..N].

We also need to make the following definitions.

Define

$$\begin{aligned} A &= diag[A_1,..,A_{N_s}] \\ B &= diag[B_1,..,B_{N_s}] \\ D &= diag[D_1,..,D_{N_s}] \\ \mathcal{J} &= diag[\mathcal{J}_1,..,\mathcal{J}_{N_s}] \\ \mathcal{K} &= diag[\mathcal{K}_1,..,\mathcal{K}_{N_s}] \\ \mathcal{L} &= diag[\mathcal{L}_1,..,\mathcal{L}_{N_s}] \end{aligned} \tag{5.2}$$

where A, B and D are global matrices representing a linear model of global f^* and $\mathcal{J}, \mathcal{K}$ and $\mathcal{L}$ are matrices representing a linear model of global output function K^*.

Furthermore we define

$$\begin{aligned} Q &= diag[Q_1,..,Q_{N_s}] \\ R &= diag[R_1,..,R_{N_s}] \\ S &= diag[S_1,..,S_{N_s}] \\ \Phi &= diag[\Phi_1,..,\Phi_{N_s}] \end{aligned} \tag{5.3}$$

where Q, R S and Φ are global symmetric weighting matrices of appropriate dimension of the model based optimal control problem.

We will denote the single loop price coordination algorithm by PC1, the double loop version by PC2, the single loop direct coordination algorithm by DC1 and the double loop version by DC2.

In the next section we will established the optimality of the algorithm presented in chapter 4.

5.2. OPTIMALITY

The following theorems follow naturally from problem formulation and derivation of the price coordination and direct coordination hierarchical DISOPE algorithm for solving the structure with model based interaction input, with a linear model and quadratic performance criterion. The theorems are defined in conjunction with definition (3.1) which define the existence and uniqueness of the optimal solution of ROP and definition (3.2) which define the existence of the appropriate derivatives.

By considering the algorithm at the global level , theorem 5.1 establishes the optimality of algorithms PC1 and PC2 presented previously.

Theorem 5.1:

Under Assumptions 3.1 and 3.2, and assuming convergence, the converged solution of the price coordination algorithm PC1 and PC2 with a linear model and quadratic performance index satisfies the optimality conditions of the global real optimal control problem defined by (3.10).

Proof:

To prove the above theorem we consider the equivalent global optimality conditions of ROP (3.10) and MMOP1 define by (4.1):

From (3.14) -(3.21), at the optimal solution of ROP the following conditions are satisfied

$\nabla_{c(k)}\mathrm{H}^{*} = 0$ that is

$$[q^{*}_{c(k)}(x^{op},c^{op},u^{op},k)]^{T} + [f^{*}_{c(k)}(x^{op},c^{op},u^{op},k)]^{T} p^{op}(k+1) - [K^{*}_{c(k)}(x^{op},c^{op})]H^{T} l^{op}(k) = 0 \qquad (5.4)$$

$\nabla_{x(k)}\mathrm{H}^{*} - p(k) = 0$ that is

$$[q^*_{x(k)}(x^{op},c^{op},u^{op},k)]^T+[f^*_{x(k)}(x^{op},c^{op},u^{op},k)]^T p_{op}(k+1)$$
$$-[K^*_{x(k)}(x^{op},c^{op})]H^T l^{op}(k)-p^{op}(k)=0 \tag{5.5}$$

$\nabla_{u(k)}\mathrm{H}^*=0$ that is

$$[q^*_{u(k)}(x^{op},c^{op},u^{op},k)]^T+[f^*_{u(k)}(x^{op},c^{op},u^{op},k)]^T p^{op}(k+1)+l^{op}(k)=0 \tag{5.6}$$

with

$$x^{op}(k+1)=f^*(x^{op},c^{op},u^{op},k);\ \mathrm{k}\in[0,\mathrm{N\text{-}1}] \tag{5.7}$$

$$u^{op}(k)=HK^*(x^{op},c^{op}) \tag{5.8}$$

$$\begin{aligned} &x^{op}(0)=x_0;\quad x^{op}_t(N)=0,\quad t\in[1,q]\\ &p^{op}_t(N)=\Phi\bar{x}^0(N),\quad t\in[q+1,n]\end{aligned} \tag{5.9}$$

where $\bar{x}^0(N)=[x^{op}_{q+1}(N)\ldots x^{op}_n(N)]^T$

We require the converge solution , $u^c(k),x^c(k),u^c(k),p^c(k);\ k\in[1,N]$, of the modified model based optimal control problem (4.1). Noting that the convexification terms can be ignored, since at convergence $v(k)=c(k), z(k)=x(k)$ and $w(k)=u(k)$, this solution will satisfy the Hamiltonian

$$\begin{aligned} H^c(.)=&\tfrac{1}{2}[x^c(k)^T Qx^c(k)+c^c(k)^T Rc^c(k)+u^c(k)^T Su^c(k)]\\ &+\gamma^c(k)-\lambda^c(k)^T x^c(k)-\beta^c(k)^T c^c(k)+\zeta^c(k)^T u^c(k)\\ &+p^c(k+1)H^T[Ax^c(k)+Bc^c(k)+Du^c(k)+\alpha^c(k)]\\ &+l^c(k)^T[u^c(k)-H(\mathcal{J}x^c(k)+\mathcal{K}c^c(k)+\mathcal{L}u^c(k)+\theta^c(k)]\end{aligned} \tag{5.10}$$

and optimality conditions

$\nabla_{c(k)}\mathrm{H}^c(.)=0$

$$\Rightarrow Rc^c(k)+B^T p^c(k+1)-\mathcal{K}^T H^T l^c(k)-\beta^c(k)=0 \tag{5.11}$$

$\nabla_{x(k)}\mathrm{H}^c(.)-p^c(k)=0$

$$\Rightarrow Qx^c(k)+A^T p^c(k+1)-\mathcal{J}^T H^T l^c(k)-\lambda^c(k)=p^c(k) \tag{5.12}$$

$$\nabla_{u(k)} \mathrm{H}^c(.) = 0$$

$$\Rightarrow Su^c(k) + D^T p^c(k+1) - \mathcal{L}^T H^T l^c(k) - \zeta(k) - l(k) = 0 \tag{5.13}$$

together with

$$x^c(k+1) = Ax^c(k) + Bc^c(k) + Du^c(k) + \alpha^c(k) \tag{5.14}$$

where , from (4.29)

$$\alpha^c(k) = f^*(x^c(k), c^c(k), u^c(k), k) - Ax^c(k) - Bc^c(k) - Du^c(k)$$

$$\theta^c(k) = K^*(x^c(k), c^c(k)) - \mathcal{J}x^c(k) - \mathcal{K}c^c(k) - \mathcal{L}u^c(k) \tag{5.15}$$

From the modifier equations (4.28)

$$\begin{aligned}
\lambda^c(k) &= [A - f_x^*(x^c(k), c^c(k), u^c(k), k]^T p^c(k+1) + Qx^c(k) - q_x^*(x^c, c^c, u^c, k)^T \\
&\quad + K_x^*(x^c, c^c) H^T l^c(k) - \mathcal{J}^T H^T l^c(k) \\
\beta^c(k) &= [B - f_c^*(x^c(k), c^c(k), u^c(k), k]^T p^c(k+1) + Rc^c(k) - q_c^*(x^c, c^c, u^c, k)^T \\
&\quad + K_c^*(x^c, c^c) H^T l^c(k) - \mathcal{K}^T H^T l^c(k) \\
\zeta^c(k) &= [D - f_u^*(x^c(k), c^c(k), u^c(k), k]^T p^c(k+1) + Su^c(k) - q_u^*(x^c, c^c, u^c, k)^T \\
&\quad - \mathcal{L}^T H^T l^c(k)
\end{aligned} \tag{5.16}$$

and from the price updating mechanism

$$\begin{aligned}
l^{c,s+1}(k) &= l^{c,s}(k) + \varepsilon_l [u^c(k) \\
&\quad - H(\mathcal{J}x^c(k) + \mathcal{K}c^c(k) + \mathcal{L}u^c(k) + \theta^c(k)]
\end{aligned} \tag{5.17}$$

Eliminating $\lambda^c(k), \beta^c(k)$ and $\gamma^c(k)$ from (5.11),(5.12) and (5.13) using modifiers equation (5.16), it can be readily be seen that equation (5.11) becomes

$$\begin{aligned}
&[q_{c(k)}^*(x^c, c^c, u^c, k)]^T + [f_{c(k)}^*(x^c, c^c, u^c, k)]^T p^c(k+1) \\
&\quad - [K_{c(k)}^*(x^c, c^c)] H^T l^c(k) = 0
\end{aligned} \tag{5.18}$$

equation (5.12) becomes

$$\begin{aligned}
&[q_{x(k)}^*(x^c, c^c, u^c, k)]^T + [f_{x(k)}^*(x^c, c^c, u^c, k)]^T p^c(k+1) \\
&\quad - [K_{x(k)}^*(x^c, c^c)] H^T l^c(k) - p^c(k) = 0
\end{aligned} \tag{5.19}$$

and equation (5.13) is now

$$[q^{*}_{u(k)}(x^c,c^c,u^c,k)]^T+[f^{*}_{u(k)}(x^c,c^c,u^c,k)]^T p^c(k+1)+l^c(k)=0 \quad (5.20)$$

Substituting (5.15) in (5.14) produces

$$x^c(k+1)=f^{*}(x^c,c^c,u^c,k)\,;\,\mathrm{k}\in[0,\,\mathrm{N\text{-}1}] \quad (5.21)$$

with boundary conditions

$$\begin{aligned} &x^c(0)=x_0; \quad x^c_t(N)=0, \quad t\in[1,q] \\ &p^c_t(N)=\Phi\bar{x}^0(N), \quad t\in[q+1,n] \end{aligned} \quad (5.22)$$

where $\bar{x}^0(N)=[x^c_{q+1}(N)\ldots x^c_n(N)]^T$

Eliminating $\theta^c(k)$ from (5.17) using (5.15), and noting that at convergence the interaction price $l^{c,s+1}(k)=l^{c,s+1}(k)$ we obtain the following equation

$$0=\varepsilon_l[u^c(k)-H(K^{*}(x^c,c^c))] \quad (5.23)$$

Since scalar $\varepsilon_l>0$, then equation (5.23) reduces to

$$u^c(k)=H(K^{*}(x^c,c^c)) \quad (5.24)$$

Comparing (5.1)-(5.9) with (5.18)-(5.22) and (5.24), it is clear that the two set of optimality conditions are in agreement. Furthermore from assumption 3.1

$$x^c(k)=x^{op}(k),\ \ c^c(k)=c^{op}(k),\ u^c(k)=u^{op}(k),\ p^c(k)=p^{op}(k);\ k\in[0,N] \quad (5.25)$$

Q.E.D

The following theorem establishes the optimality of the direct coordination approach. Repeating the previous procedure, if we consider both algorithms at global level the two different techniques of implementation will generate the same method. Thus theorem 5.2 applies to the double loop and single loop implementation of the direct coordination method.

Theorem 5.2

Under Assumptions 3.1 and 3.2, and assuming convergence, the converged solution of the direct coordination algorithms DC1 and DC2 with a linear model and quadratic

performance index satisfies the optimality conditions of the global real optimal control problem defined by (3.10).

Proof:

To prove the theorem we will show that the optimality conditions of ROP given by (3.10) are equivalent to the converged solution of MMOP defined by (4.33) produced by the direct coordination approach.

The equivalent optimality conditions of (3.10) are given by equations (5.4)-(5.9).

We require the converged solution , $u^c(k), x^c(k), u^c(k), p^c(k);\ k \in [1, N]$, of the modified model based optimal control problem (4.33). Noting that the convexification terms can be ignored, since at convergence $v(k)=c(k)$, $z(k)=x(k)$ and $w(k)=u(k)$, this solution will satisfy the Hamiltonian

$$\begin{aligned} H^c(.) = \tfrac{1}{2}[x^c(k)^T Q x^c(k) + c^c(k)^T R c^c(k) + u^c(k)^T S u^c(k)] \\ + \gamma^c(k) - \lambda^c(k)^T x^c(k) - \beta^c(k)^T c^c(k) + \zeta^c(k)^T u^c(k) \\ + p^c(k+1) H^T [A x^c(k) + B c^c(k) + D u^c(k) + \alpha^c(k)] \\ + l^c(k)^T [u^c(k) - H(\mathcal{J} x^c(k) + \mathcal{K} c^c(k) + \theta^c(k)] \end{aligned} \tag{5.26}$$

and optimality conditions (5.11), (5.12) and

$$\nabla_{u(k)} \mathrm{H}^c(.) = 0$$

$$\Rightarrow S u^c(k) + D^T p^c(k+1) - \zeta(k) - l(k) = 0 \tag{5.27}$$

together with model dynamic equation (5.14) .From (4.29) and (4.62), we have

$$\alpha^c(k) = f^*(x^c(k), c^c(k), u^c(k), k) - A x^c(k) - B c^c(k) - D u^c(k) \tag{5.28}$$

$$\theta^c(k) = K^*(x^c(k), c^c(k)) - \mathcal{J} x^c(k) - \mathcal{K} c^c(k) \tag{5.29}$$

In addition to the above we have used the following optimality condition to calculate $u^c(k)$, that is;

$$u^c(k) = H(\mathcal{J} x^c(k) + \mathcal{K} c^c(k) + \theta^c(k)) \tag{5.30}$$

From (4.28) and (4.62) we have the following equations

$$\begin{aligned} \lambda^c(k) = [A - f_x^*(x^c(k), c^c(k), u^c(k), k]^T p^c(k+1) + Q x^c(k) - q_x^*(x^c, c^c, u^c, k)^T \\ + K_x^*(x^c, c^c) H^T l^c(k) - \mathcal{J}^T H^T l^c(k) \end{aligned}$$

$$\beta^c(k) = [B - f_c^*(x^c(k), c^c(k), u^c(k), k]^T p^c(k+1) + Rc^c(k) - q_c^*(x^c, c^c, u^c, k)^T$$
$$+ K_c^*(x^c, c^c) H^T l^c(k) - \mathcal{K}^T H^T l^c(k)$$
$$\zeta^c(k) = [D - f_u^*(x^c(k), c^c(k), u^c(k), k]^T p^c(k+1) + Su^c(k) - q_u^*(x^c, c^c, u^c, k)^T \quad (5.31)$$

Eliminating $\lambda^c(k)$, $\beta^c(k)$ and $\gamma^c(k)$ from (5.11),(5.12) and (5.27) using modifiers equation (5.31), produces (5.18) ,(5.19) and

$$[q_{u(k)}^*(x^c, c^c, u^c, k)]^T + [f_{u(k)}^*(x^c, c^c, u^c, k)]^T p^c(k+1) + l^c(k) = 0 \quad (5.32)$$

Eliminating $\alpha^c(k)$ in (5.14) using (5.29) we obtain (5.21) with boundary conditions (5.22).

Substituting (5.29) in equation (5.30) gives

$$u^c(k) = H(K^*(x^c, c^c)) \quad (5.33)$$

Comparing (5.1)-(5.9) with (5.18),(5.19),(5.32),(5,21),(5.22) and (5.34), it is clear that the two set of optimality conditions are in agreement. Furthermore from assumption 3.1

$$x^c(k) = x^{op}(k), \quad c^c(k) = c^{op}(k), \quad u^c(k) = u^{op}(k), \quad p^c(k) = p^{op}(k); \quad k \in [0, N] \quad (5.34)$$

Q.E.D

5.3 THE ALGORITHM MAPPING OF THE DOUBLE LOOP PRICE COORDINATION ALGORITHM PC2

For simplicity, we will consider the special case of no terminal conditions and $\Phi = 0$.

First, we derive the algorithm mapping of the inner loop of algorithm PC2.

Let s denote the iteration number of the inner loop and p denote the corresponding iteration number of the outer loop.

The inner loop:

The inner loop constitutes iterations in the ith decomposed modified model based optimal control problem ($MMOP_i$)

We need to established the transition from iteration s to iteration s+1 of the inner loop of the double loop price coordination algorithm PC2 with a linear model and quadratic performance criterion defined by (4.1) in terms of difference equations.

From (4.28) we obtain the following expressions representing the computation of modifiers.

$$\lambda_i^s(k)=[A_i - f_{x_i}^*(x_i^s,c_i^s,u_i^s),k]^T p_i^s(k+1)+Q_i\, x_i^s(k)-q_{x_i}^*(x_i^s,c_i^s,u_i^s,k)^T$$
$$+K_{x_i}^*(x_i^s,c_i^s)\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)-\mathcal{J}_i^T\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)$$

$$\beta_i^s(k)=[B_i - f_{c_i}^*(x_i^s,c_i^s,u_i^s),k]^T p_i^s(k+1)+R_i\, c_i^s(k)-q_{c_i}^*(x_i^s,c_i^s,u_i^s,k)^T$$
$$+K_{c_i}^*(x_i^s,c_i^s)\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)-\mathcal{K}_i^T\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) \qquad (5.35)$$

$$\zeta_i^s(k)=[D_i - f_{u_i}^*(x_i^s,c_i^s,u_i^s),k]^T p_i^s(k+1)+S_i\, u_i^s(k)-q_{u_i}^*(x_i^s,c_i^s,u_i^s,k)^T$$
$$-\mathcal{L}_i^T\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)$$

where $l_j^p(k)$ is prescribed by computation in the outer loop.

From equation (4.29) the calculation of parameters can be expressed as

$$\alpha_i^s(k)=f_i^*(x_i^s,c_i^s,u_i^s,k)-A_i x_i^s(k)-B_i c_i^s(k)-D_i\, u_i^s(k)$$
$$\theta_i^s(k)=K_i^*(x_i^s,c_i^s)-\mathcal{J}_i x_i^s(k)-\mathcal{K}_i\, c_i^s(k)-\mathcal{L}_i u_i^s(k) \qquad (5.36)$$

The solution of the $MMOP_i$ defined by (4.3) can be represented by

$$\hat{c}_i^s(k)=\overline{R}_i^{-1}[-B_i^T p_i^s(k+1)-\beta_i^s(k)-\mathcal{K}_i^T\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)+r_1 c_i^s(k)]$$

$$\hat{x}_i^s(k+1)=A_i\hat{x}_i^s(k)+B_i\hat{c}_i^s(k)+D_i\hat{u}_i^s(k)+\alpha_i^s(k)$$

$$\hat{p}_i^s(k)=\overline{Q}_i\hat{x}_i^s(k)+A\hat{p}_i^s(k+1)-\lambda_i^s(k)-\mathcal{J}_i^T\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)+r_2 x_i^s(k); \qquad (5.37)$$

$$\hat{p}_i^s(N)=0$$

$$\hat{u}_i^s(k)=\overline{S}_i^{-1}[-D_i^T p_i^s(k+1)-\zeta_i^s(k)-\mathcal{L}_i^T\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)+l+r_1 u_i^s(k)]$$

The control ,state, interaction and costate updating mechanism defined by (3.60) are expressed as

$$\begin{aligned} x_i^{s+1}(k) &= x_i^s(k) + \varepsilon_x(\hat{x}_i^s(k) - x_i^s(k)) \\ c_i^{s+1}(k) &= c_i^s(k) + \varepsilon_c(\hat{c}_i^s(k) - c_i^s(k)) \\ u_i^{s+1}(k) &= u_i^s(k) + \varepsilon_u(\hat{u}_i^s(k) - u_i^s(k)) \end{aligned} \tag{5.38}$$

$$p_i^{s+1}(k) = p_i^s(k) + \varepsilon_p(\hat{p}_i^s(k) - p_i^s(k))$$

Equations (5.35)-(5.38) are defined for $k \in [1, N-1]$ and $\hat{c}_i(k) = v_i(k)$, $\hat{x}_i(k) = z_i(k)$, $\hat{u}_i(k) = w_i(k)$ and $\hat{p}_i(k) = p_i(k)$.

Notice that in (5.37) we can write

$$\hat{p}_i^s(k+1) = -A_i^{-T}\overline{Q}_i\hat{x}_i^s(k) + A_i^T\hat{p}_i^s(k) + A_i^{-T}\lambda_i^s(k) + \mathcal{J}_i^T\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) + r_2 A_i^{-T} x_i^s(k);$$

$$\hat{p}_i^s(N) = 0 \tag{5.39}$$

Substituting (5.35), (5.36) in (5.37) and applying (5.39), we can write

$$\begin{bmatrix} \hat{x}_i^s(k+1) \\ \hat{p}_i^s(k+1) \end{bmatrix} = \mathbf{E}_i \begin{bmatrix} \hat{x}_i^s(k) \\ \hat{p}_i^s(k) \end{bmatrix} + \mathbf{M}_{1,i}\mathbf{z}_i^s(k) + g_{1,i}(\mathbf{z}_i^s(k)) \tag{5.40}$$

where $\mathbf{z}_i^s(k) = [c_i^s(k)^T \quad x_i^s(k)^T \quad u_i^s(k)^T \quad p_i^s(k+1)^T]^T$ and

$$\mathbf{E}_{1,i} = \begin{bmatrix} A_i & -B_i\overline{R}_i^{-1}B_i^T - D_i\overline{S}_i^{-1}D_i^{-1} \\ -A_i^{-T}\overline{Q}_i & A_i \end{bmatrix} \tag{5.41}$$

$$\mathbf{M}_{1,i} = \begin{bmatrix} r_1 B_i R_i^{-1} & 0_{n_i} & r_3 D_i \overline{S}_i^{-1} & 0_{n_i} \\ 0_{n_i,m_i} & r_2 A_i^{-T} & 0_{n_i,m_i} & 0_{n_i} \end{bmatrix} \tag{5.42}$$

$$g_{1,i}(\mathbf{z}_i^s(k)) = \begin{bmatrix} g_{11,i}(\mathbf{z}_i^s(k)) \\ g_{12,i}(\mathbf{z}_i^s(k)) \end{bmatrix} \tag{5.43}$$

with

$$g_{11,i}(\mathbf{z}_i^S(k)) = B_i \bar{R}_i^{-1}[B_i - f_{c_i}^*(x_i^S, c_i^S, u_i^S), k]^T p_i^S(k+1)$$

$$+B_i \bar{R}_i^{-1}[R_i\, c_i^S(k) - q_{c_i}^*(x_i^S, c_i^S, u_i^S, k)^T]$$

$$-B_i \bar{R}_i^{-1}[\mathcal{K}_i - K_{c_i}^*(x_i^S, c_i^S)]^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k))$$

$$+B_i \bar{R}_i^{-1} \mathcal{K}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) + D_i \bar{S}_i^{-1} l_i^p(k)$$

$$+D_i \bar{S}_i^{-1}[D_i - f_{u_i}^*(x_i^S, c_i^S, u_i^S), k]^T p_i^S(k+1)$$

$$+D_i \bar{S}_i^{-1}[S_i\, u_i^S(k) - q_{u_i}^*(x_i^S, c_i^S, u_i^S, k)]^T$$

$$+f_i^*(x_i^S, c_i^S, u_i^S, k) - A_i x_i^S(k) - B_i c_i^S(k) - D_i\, u_i^S(k)$$

(5.44)

$$g_{12,i}(\mathbf{z}_i^S(k)) = A_i^{-T}[A_i - f_{x_i}^*(x_i^S, c_i^S, u_i^S), k]^T p_i^S(k+1)$$

$$+A_i^{-T}[Q_i\, x_i^S(k) - q_{x_i}^*(x_i^S, c_i^S, u_i^S, k)^T] \qquad (5.45)$$

$$-A_i^{-T}[\mathcal{J}_i - K_{x_i}^*(x_i^S, c_i^S)]^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) + A_i^{-T} \mathcal{J}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)$$

It should be noted that, in (5.40), $\mathbf{E}_i$ is a transition matrix, $\mathbf{H}_{1,i}$ contains solely convexification terms (i.e $\mathbf{H}_{1,i} = 0$ if $r_1 = r_2 = r_3 = 0$), and $g_{12,i}(\mathbf{z}_i^S(k))$ represents the model-reality differences.

Equation (5.40) can be written as

$$\begin{bmatrix} \hat{x}_i^S(k) \\ \hat{p}_i^S(k) \end{bmatrix} = \mathbf{E}_i^k \begin{bmatrix} x_{io} \\ \hat{p}_i(0) \end{bmatrix} + \sum_{j=0}^{k-1} \mathbf{E}_i^j [\mathbf{M}_{1,i} \mathbf{z}_1^S(k-1-j) + g_{1,i}(\mathbf{z}_i^S(k-1-j)); \qquad (5.46)$$

$$\hat{p}_i^S(N) = 0$$

Writing $\phi(k) = \mathbf{E}_{1,i}^k$,at $k = N$, the final costate is given by

$$\hat{p}_i^S(N) = \phi_{21,i}(N) x_{io} + \phi_{22,i}(N) \hat{p}_i^S(0)$$

$$+ \sum_{j=0}^{k-1} \phi_{2,i}(j)[\mathbf{M}_{1,i} \mathbf{z}_1^S(N-1-j) + g_{1,i}(\mathbf{z}_i^S(N-1-j)) \qquad (5.47)$$

with

$$\mathbf{E}_i^N = \phi_i(N) = \begin{bmatrix} \phi_{11,i}(N) & \phi_{12,i}(N) \\ \phi_{21,i}(N) & \phi_{22,i}(N) \end{bmatrix} ; \phi_{2,i}(j) = \begin{bmatrix} \phi_{21,i}(j) & \phi_{22,i}(j) \end{bmatrix} \quad (5.48)$$

Hence, the initial costate can be expressed in the form

$$\begin{aligned} \hat{p}_i^S(0) &= \phi_{22,i}(N)^{-1}\phi_{21,i}(N)x_{io} \\ &\quad -\phi_{22,i}(N)^{-1}\sum_{j=0}^{k-1}\phi_{2,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_i^S(N-1-j) + g_{1,i}(\mathbf{z}_i^S(N-1-j)) \end{aligned} \quad (5.49)$$

Substituting (5.49) in (5.46), we obtain

$$\begin{aligned} \begin{bmatrix} \hat{x}_i^S(k) \\ \hat{p}_i^S(k) \end{bmatrix} &= \begin{bmatrix} \mu_{x_i}(N,k) \\ \mu_{p_i}(N,k) \end{bmatrix} x_{io} \\ &\quad -\begin{bmatrix} \phi_{12,i}(k) \\ \phi_{22,i}(k) \end{bmatrix}\phi_{22,i}(N)^{-1}\sum_{j=0}^{N-1}\phi_{2,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_i^S(N-1-j) + g_{1,i}(\mathbf{z}_i^S(N-1-j))] \\ &\quad +\sum_{j=0}^{k-1}\phi_i(j)[\mathbf{M}_{1,i}\mathbf{z}_i^S(k-1-j) + g_{1,i}(\mathbf{z}_i^S(k-1-j))] \end{aligned} \quad (5.50)$$

where

$$\begin{aligned} \mu_{x_i}(N,k) &= \phi_{11,i}(k) - \phi_{12,i}(k)\phi_{22,i}(N)^{-1}\phi_{21,i}(N) \\ \mu_{p_i}(N,k) &= \phi_{21,i}(k) - \phi_{22,i}(k)\phi_{22,i}(N)^{-1}\phi_{21,i}(N) \end{aligned} \quad (5.51)$$

Using (5.35) and (5.37) to eliminate $\hat{p}_i^S(k+1)$ and $\beta_i^S(k)$ from the optimal control estimate $\hat{c}_i^S(k)$ gives

$$\begin{aligned} \hat{c}_i^S(k) &= -\overline{R}_i^{-1}B_i^T A_i^{-T}\overline{Q}_i\mu_{x_i}(N,k)x_{io} \\ &\quad -\overline{R}_i^{-1}B_i^T A_i^{-T}\overline{Q}_i\phi_{12,i}(k)\phi_{22,i}(N)^{-1}\sum_{j=0}^{N-1}\phi_{2,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_i^S(N-1-j) + g_{1,i}(\mathbf{z}_i^S(N-1-j))] \\ &\quad +\overline{R}_i^{-1}B_i^T A_i^{-T}\overline{Q}_i\sum_{j=0}^{k-1}\phi_i(j)[\mathbf{M}_{1,i}\mathbf{z}_i^S(k-1-j) + g_{1,i}(\mathbf{z}_i^S(k-1-j))] \\ &\quad -\overline{R}_i^{-1}B_i^T A_i^{-T}\mu_{p_i}(N,k)x_{io} \end{aligned}$$

$$-\overline{R}_i^{-1}B_i^T A_i^{-T}\phi_{22,i}(k)\phi_{22,i}(N)^{-1}\sum_{j=0}^{N-1}\phi_{2,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_i^S(N-1-j)+g_{1,i}(\mathbf{z}_i^S(N-1-j))]$$

$$+\overline{R}_i^{-1}B_i^T A_i^{-T}\sum_{j=0}^{k-1}\phi_i(j)[\mathbf{M}_{1,i}\mathbf{z}_i^S(k-1-j)+g_{1,i}(\mathbf{z}_i^S(k-1-j))]$$

$$-r_2\overline{R}_i^{-1}B_i^T A_i^{-T}x_i^S(k)+r_1\overline{R}_i^{-1}c_i^S(k)+\overline{R}_i^{-1}\mathcal{K}_i^T\sum_{j=1}^{N_S}H_{ij}^T l_j^p(k)$$

$$+\overline{R}_i^{-1}[B_i - f_{c_i}^*(x_i^S,c_i^S,u_i^S),k]^T p_i^S(k+1)$$

$$+\overline{R}_i^{-1}R_i\, c_i^S(k)-\overline{R}_i^{-1}q_{c_i}^*(x_i^S,c_i^S,u_i^S,k)^T$$

$$-\overline{R}_i^{-1}[\mathcal{K}_i - K_{c_i}^*(x_i^S,c_i^S)]^T\sum_{j=1}^{N_S}H_{ij}^T l_j^p(k))+\overline{R}_i^{-1}\mathcal{J}_i^T\sum_{j=1}^{N_S}H_{ij}^T l_j^p(k)$$

$$-\overline{R}_i^{-1}B_i^T A_i^{-T}[A_i - f_{x_i}^*(x_i^S,c_i^S,u_i^S),k]^T p_i^S(k+1)$$

$$+\overline{R}_i^{-1}B_i^T A_i^{-T}Q_i\, x_i^S(k)-\overline{R}_i^{-1}B_i^T A_i^{-T}q_{x_i}^*(x_i^S,c_i^S,u_i^S,k)^T$$

$$+\overline{R}_i^{-1}B_i^T A_i^{-T}[\mathcal{J}_i - K_{x_i}^*(x_i^S,c_i^S)]^T\sum_{j=1}^{N_S}H_{ij}^T l_j^p(k)$$

$$-\overline{R}_i^{-1}B_i^T A_i^{-T}\mathcal{J}_i^T\sum_{j=1}^{N_S}H_{ij}^T l_j^p(k) \tag{5.51}$$

Repeating the previous procedure, we can eliminate $\hat{p}_i^S(k+1)$ and $\zeta_i^S(k)$ from the optimal interaction estimate $\hat{u}_i^S(k)$ to obtain

$$\ddot{u}_i^S(k)=-\overline{S}_i^{-1}D_i^T A_i^{-T}\overline{Q}_i\mu_{x_i}(N,k)x_{io}$$

$$-\overline{S}_i^{-1}D_i^T A_i^{-T}\overline{Q}_i\phi_{12,i}(k)\phi_{22,i}(N)^{-1}\sum_{j=0}^{N-1}\phi_{2,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_i^S(N-1-j)+g_{1,i}(\mathbf{z}_i^S(N-1-j))]$$

$$+\overline{S}_i^{-1}D_i^T A_i^{-T}\overline{Q}_i\sum_{j=0}^{k-1}\phi_i(j)[\mathbf{M}_{1,i}\mathbf{z}_i^S(k-1-j)+g_{1,i}(\mathbf{z}_i^S(k-1-j))]$$

$$-\overline{S}_i^{-1}D_i^T A_i^{-T}\mu_{p_i}(N,k)x_{io}$$

$$-\bar{S}_i^{-1} D_i^T A_i^{-T} \phi_{22,i}(k)\phi_{22,i}(N)^{-1} \sum_{j=0}^{N-1} \phi_{2,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_i^s(N-1-j) + g_{1,i}(\mathbf{z}_i^s(N-1-j))]$$

$$+\bar{S}_i^{-1} D_i^T A_i^{-T} \sum_{j=0}^{k-1} \phi_i(j)[\mathbf{M}_{1,i}\mathbf{z}_i^s(k-1-j) + g_{1,i}(\mathbf{z}_i^s(k-1-j))]$$

$$-r_2 \bar{S}_i^{-1} D_i^T A_i^{-T} x_i^s(k)s - \bar{S}_i^{-1} D_i^T A_i^{-T} \mathcal{J}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) - \bar{S}_i^{-1} l_i^p(k)$$

$$+r_3 S_i^{-1} u_i^s(k) + \bar{S}_i^{-1} \mathcal{L}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) + \bar{S}_i^{-1}[D_i - f_{u_i}^*(x_i^s, c_i^s, u_i^s, k)] p_i^s(k+1)$$

$$+\bar{S}_i^{-1} S_i\, u_i^s(k) - \bar{S}_i^{-1} q_{u_i}^*(x_i^s, c_i^s, u_i^s, k)^T - \bar{S}_i^{-1} \mathcal{L}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)$$

$$-\bar{S}_i^{-1} D_i^T A_i^{-T}[A_i - f_{x_i}^*(x_i^s, c_i^s, u_i^s, k)]^T p_i^s(k+1)$$

$$-\bar{S}_i^{-1} D_i^T A_i^{-T} Q_i\, x_i^s(k) + \bar{S}_i^{-1} D_i^T A_i^{-T} q_{x_i}^*(x_i^s, c_i^s, u_i^s, k)^T$$

$$+\bar{S}_i^{-1} D_i^T [\mathcal{J}_i - K_{x_i}^*(x_i^s, c_i^s)]^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) - \bar{S}_i^{-1} D_i^T \mathcal{J}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) \tag{5.52}$$

Combining (5.50),(5.51) and (5.52) produces

$$\begin{aligned}\hat{\mathbf{z}}_i^s(k) &= \mu_i(N,k) x_{io} \\ &\quad -\eta_i(k)\phi_{22,i}(N)^{-1} \sum_{j=0}^{N-1} \phi_{2,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_i^s(N-1-j) + g_{1,i}(\mathbf{z}_i^s(N-1-j))] \\ &\quad +\sum_{j=0}^{k-1} \psi_{1,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_i^s(k-1-j) + g_{1,i}(\mathbf{z}_i^s(k-1-j))] \\ &\quad +\mathbf{M}_{2,i}\mathbf{z}_i^s(k) + g_{2,i}(\mathbf{z}_i^s(k))\end{aligned} \tag{5.53}$$

where

$$\mu_i(N,k) = \begin{bmatrix} \mu_{c_i}(N,k) \\ \mu_{x_i}(N,k) \\ \mu_{u_i}(N,k) \\ \mu_{p_i}(N,k) \end{bmatrix} \quad \text{with} \quad \begin{aligned} \mu_{c_i}(N,k) &= \bar{R}_i^{-1} B_i^T A_i^{-T}[\bar{Q}\mu_{x_i}(N,k) - \mu_{p_i}(N,k)] \\ \mu_{u_i}(N,k) &= \bar{S}_i^{-1} D_i^T A_i^{-T}[\bar{Q}\mu_{x_i}(N,k) - \mu_{p_i}(N,k)] \end{aligned} \tag{5.54}$$

and

$$\eta_i(k) = \begin{bmatrix} \overline{R}_i^{-1} B_i^T A_i^{-T} [\overline{Q}\phi_{12,i}(k) - \phi_{22,i}(k)] \\ \phi_{12,i}(k) \\ \overline{S}_i^{-1} D_i^T A_i^{-T} [\overline{Q}\phi_{12,i}(k) - \phi_{22,i}(k)] \\ \phi_{22,i}(k) \end{bmatrix} \quad ; \quad (5.55)$$

$$\psi_{1_i}(j) = \begin{bmatrix} \overline{R}_i^{-1} B_i^T A_i^{-T} [\overline{Q}\phi_{1,i}(k) - \phi_{2,i}(k)] \\ \phi_{1,i}(k) \\ \overline{S}_i^{-1} D_i^T A_i^{-T} [\overline{Q}\phi_{1,i}(k) - \phi_{2,i}(k)] \\ \phi_{2,i}(k) \end{bmatrix} \quad (5.56)$$

$$\mathrm{M}_{2,i} = \begin{bmatrix} r_1 \overline{R}_i^{-1} & -r_2 \overline{R}_i^{-1} B_i^T A_i^{-T} & 0_{r_i,r_i} & 0_{n_i} \\ 0_{n_i,m_i} & 0_{n_i} & 0_{n_i,r_i} & 0_{n_i} \\ 0_{r_i,m_i} & -r_2 \overline{S}_i^{-1} D_i^T A_i^{-T} & r_3 \overline{S}_i^{-1} & 0_{r_i,n_i} \\ 0_{n_i,m_i} & 0_{n_i} & 0_{n_i,r_i} & 0_{n_i} \end{bmatrix} \quad (5.57)$$

and

$$g_{2,i}(\mathbf{z}_i^s(k)) = \begin{bmatrix} g_{21,i}(\mathbf{z}_i^s(k)) \\ 0 \\ g_{22,i}(\mathbf{z}_i^s(k)) \\ 0 \end{bmatrix} \quad (5.58)$$

with

$$\begin{aligned} g_{21,i}(\mathbf{z}_i^s(k)) = & \\ & \overline{R}_i^{-1}[[B_i - f_{c_i}^*(x_i^s, c_i^s, u_i^s), k]^T - B_i^T A_i^{-T}[A_i - f_{x_i}^*(x_i^s, c_i^s, u_i^s), k]^T] p_i^s(k+1) \\ & + \overline{R}_i^{-1}[R_i\, c_i^s(k) - q_{c_i}^*(x_i^s, c_i^s, u_i^s, k)^T] + \overline{R}_i^{-1} \mathcal{K}_i^T \sum_{j=1}^{N_s^s} H_{ij}^T l_j^p(k) \\ & + \overline{R}_i^{-1} B_i^T A_i^{-T}[Q_i\, x_i^s(k) - q_{x_i}^*(x_i^s, c_i^s, u_i^s, k)^T] \\ & - \overline{R}_i^{-1}[[\mathcal{K}_i - K_{c_i}^*(x_i^s, c_i^s)]^T + B_i^T A_i^{-T}[\mathcal{J}_i - K_{x_i}^*(x_i^s, c_i^s)]^T] \sum_{j=1}^{N_s^s} H_{ij}^T l_j^p(k) \end{aligned} \quad (5.59)$$

$$
\begin{aligned}
g_{23,i}(\mathbf{z}_i^s(k)) = & \\
& +\bar{S}_i^{-1}[[D_i - f_{u_i}^*(x_i^s, c_i^s, u_i^s), k]^T - D_i^T A_i^{-T}[A_i - f_{x_i}^*(x_i^s, c_i^s, u_i^s), k]^T] p_i^s(k+1) \\
& -\bar{S}_i^{-1} D_i^T A_i^{-T} \mathcal{K}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) + \bar{S}_i^{-1}[S_i u_i^s(k) - q_{u_i}^*(x_i^s, c_i^s, u_i^s, k)^T] \\
& -\bar{S}_i^{-1} D_i^T A_i^{-T}[Q_i x_i^s(k) - q_{x_i}^*(x_i^s, c_i^s, u_i^s, k)^T] \\
& -\bar{S}_i^{-1} D_i^T A_i^{-T}[\mathcal{J}_i - K_{x_i}^*(x_i^s, c_i^s)]^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) - \bar{S}_i^{-1} l_i^p(k)
\end{aligned}
\tag{5.60}
$$

From (3.46)-(3.48) and (5.38), the updating mechanism can be written as

$$
\mathbf{z}_i^{s+1}(k) = \varepsilon \hat{\mathbf{z}}_i^s(k) + [\mathbf{I}_{2n+m+r} - \varepsilon] \mathbf{z}_i^s(k) \tag{5.61}
$$

with

$$
\varepsilon_{zi} = \begin{bmatrix}
\varepsilon_c \mathbf{I}_{m_i} & 0_{m_i, n_i} & 0_{m_i, r_i} & 0_{m_i, n_i} \\
0_{n_i, m_i} & \varepsilon_x \mathbf{I}_{n_i} & 0_{n_i, r_i} & 0_{n_i} \\
0_{r_i, m_i} & 0_{r_i, n_i} & \varepsilon_u \mathbf{I}_{r_i} & 0_{r_i, n_i} \\
0_{n_i, m_i} & 0_{n_i} & 0_{n_i, r_i} & \varepsilon_p \mathbf{I}_i
\end{bmatrix} \tag{5.62}
$$

Combining (5.61) and (5.53) produces

$$
\begin{aligned}
\hat{\mathbf{z}}_i^{s+1}(k) = & \varepsilon_{zi} \mu_i(N,k) x_{io} \\
& -\varepsilon_{zi} \eta_i(k) \phi_{22,i}(N)^{-1} \sum_{j=0}^{N-1} \phi_i(N-1-j)[\mathbf{M}_{1,i} \mathbf{z}_i^s(j) + g_{1,i}(\mathbf{z}_i^s(j))] \\
& +\varepsilon_{zi} \sum_{j=0}^{k-1} \psi(k-1-j)[\mathbf{M}_{1,i} \mathbf{z}_i^s(k) + g_{1,i}(\mathbf{z}_i^s(k))] \\
& +[\varepsilon_{zi} \mathbf{M}_{2,i} + \mathbf{I}_{2n_i + m_i + r_i} - \varepsilon] \mathbf{z}_i^s(k) + \varepsilon_{zi} g_{2,i}(\mathbf{z}_i^s(k))
\end{aligned}
\tag{5.63}
$$

The algorithm mapping of the inner loop showing how $\mathbf{z}_i^s(k)$, $k \in [0,N]$ is updated from iteration s to iteration $s+1$ is obtained by the following simplification of (5.63), which can expressed as

$$
\begin{aligned}
\hat{\mathbf{z}}_i^{s+1}(k) = & \varepsilon_{zi} \mu_i(N,k) x_{io} \\
& +\varepsilon_{zi} \sum_{j=0}^{N-1} \Theta_{1i}(N,k,j)[\mathbf{M}_{1,i} \mathbf{z}_i^s(j) + g_{1,i}(\mathbf{z}_i^s(j))] \\
& +[\varepsilon_{zi} \mathbf{M}_{2,i} + \mathbf{I}_{2n_i + m_i + r_i} - \varepsilon_{zi}] \mathbf{z}_i^s(k) + \varepsilon_{zi} g_{2,i}(\mathbf{z}_i^s(k))
\end{aligned}
\tag{5.64}
$$

where

$$\Theta_{1i}(N,k,j)=\begin{cases}\psi_i(k-1-j)-\eta_i(k)\phi_{22,i}(N)\phi_{2,i}(N-1-j); & j\in[0,k-1]\\ \eta_i(k)\phi_{22,i}(N)^{-1}\phi_{2,i}(N-1-j); & j\in[k,N-1]\end{cases} \tag{5.65}$$

We now proceed to derive the algorithm mapping of the outer loop.

The outer loop

From algorithm 4.2.2 we note that the outer loop uses global_information available from the N_S subsystems from the inner loop. Let p denote the iteration number of the outer loop. The outer loop consists of the price updating mechanism (4.32) that is:

$$l^{p+1}(k+1)=l^{p}(k)+\varepsilon_l(\hat{u}^s(k)-H(\mathcal{J}\hat{x}^s(k)+\mathcal{K}\hat{c}^s(k)+\mathcal{L}\hat{u}^s(k)+\theta^s(k)) \tag{5.66}$$

which can be simplified as

$$l^{p+1}(k+1)=l^{p}(k)+\varepsilon_l\mathrm{T}_1\hat{\mathrm{y}}^s(k)-\varepsilon_l H\theta^s(k) \tag{5.67}$$

$$\text{with } \mathrm{T}_1=\begin{bmatrix}H\mathcal{K} & 0_{r,m} & 0_r\\ 0_{r,n} & H\mathcal{J} & 0_r\\ 0_{r,n} & 0_{r,m} & H\mathcal{L}-\mathrm{I}_r\end{bmatrix} \tag{5.68}$$

and

$$\begin{aligned}\hat{\mathrm{y}}^s(k)&=[\hat{c}^s(k)^T\ \ \hat{x}^s(k)^T\ \ \hat{u}^s(k)^T]^T\\ \mathrm{z}^s(k)&=[c^s(k)^T\ \ x^s(k)^T\ \ u^s(k)^T\ \ p^s(k+1)^T]^T\end{aligned} \tag{5.69}$$

$\hat{\mathrm{y}}^s(k)$ is the optimum estimates of global control, state and interaction (defined in equation (3.4a)) as prescribed by the inner loop.

Using global definitions (3.4a), (3.24a) ,(5.2) and (5.3), and from equation (5.53) an expression for $\hat{\mathrm{y}}^s(k)$ can be written as

$$\hat{\mathbf{y}}_i^S(k) = \mu_{\mathbf{y}}(N,k)x_o$$

$$- \quad -\eta_{\mathbf{y}}(k)\phi_{22}(N)^{-1}\sum_{j=0}^{N-1}\phi_2(j)[\mathbf{M}_3\mathbf{z}^S(N-1-j)+g_3(\mathbf{z}^S(N-1-j))]$$

$$-\eta_{\mathbf{y}}(k)\phi_{22}(N)^{-1}\sum_{j=0}^{N-1}\phi_2(j)[g_4(l^P(N-1-j))]$$

$$+\sum_{j=0}^{k-1}\psi_2(j)[\mathbf{M}_3\mathbf{z}^S(k-1-j)+g_3(\mathbf{z}^S(k-1-j))]+\sum_{j=0}^{k-1}\psi_2(j)[g_4(l^S(k-1-j))]$$

$$+\mathbf{M}_4\mathbf{z}^S(k)+g_5(\mathbf{z}^S(k))+g_6(l^P(k)) \tag{5.70}$$

where

$$\mu_{\mathbf{y}}(N,k)=\begin{bmatrix}\mu_c\ (N,k)\\ \mu_x\ (N,k)\\ \mu_u\ (N,k)\end{bmatrix}; \qquad \eta_{\mathbf{y}}(k)=\begin{bmatrix}\overline{R}^{-1}B^TA^{-T}[\overline{Q}\phi_{12}(k)-\phi_{22}(k)]\\ \phi_{12}(k)\\ \overline{S}^{-1}D^TA^{-T}[\overline{Q}\phi_{12}(k)-\phi_{22}(k)]\end{bmatrix} \tag{5.71}$$

with $\phi_{(.)}$ denotes the corresponding global version of $\phi_{(.),i}$

$$\mathbf{M}_3=\begin{bmatrix}r_1B^T\overline{R}^{-1} & 0_n & r_3D^T\overline{S}^{-1} & 0_n\\ 0_{n,m} & r_2A^{-T} & 0_{n,r} & 0_n\end{bmatrix}; \tag{5.72}$$

$$\psi_2(j)=\begin{bmatrix}\overline{R}^{-1}B^TA^{-T}[\overline{Q}\phi_1(k)-\phi_2(k)]\\ \phi_1(k)\\ \overline{S}^{-1}D^TA^{-T}[\overline{Q}\phi_1(k)-\phi_2(k)]\\ \phi_2(k)\end{bmatrix} \tag{5.73}$$

$$\mathbf{M}_4=\begin{bmatrix}r_1\overline{R}^{-1} & -r_1\overline{R}^{-1}B^TA^{-T} & 0_{m,r} & 0_{m,n}\\ 0n,m & 0n & 0_{n,r} & 0_n\\ 0r,m & -r_3\overline{S}^{-1}D^TA^{-T} & r_3\overline{S}^{-1} & 0_{r,n}\end{bmatrix} \tag{5.74}$$

$$g_3(\mathbf{z}^S(k))=\begin{bmatrix}g_{31}(\mathbf{z}^S(k))\\ g_{32}(\mathbf{z}^S(k))\end{bmatrix} \tag{5.75}$$

with

$$g_{31}(\mathbf{z}^S(k)) = B\bar{R}^{-1}[B - f_c^*(x^S, c^S, u^S), k]^T p^S(k+1)$$
$$+B\bar{R}^{-1}[Rc^S(k) - q_c^*(x^S, c^S, u^S, k)^T]$$
$$+ D\bar{S}^{-1}[D - f_u^*(x^S, c^S, u^S), k]^T p^S(k+1) \quad (5.76)$$
$$+D\bar{S}^{-1}[Su^S(k) - q_u^*(x^S, c^S, u^S, k)^T$$
$$+f^*(x^S, c^S, u^S, k) - Ax^S(k) - Bc^S(k) - Du^S(k)$$

$$g_{32}(\mathbf{z}^S(k)) = A^{-T}[A - f_x^*(x^S, c^S, u^S), k]^T p^S(k+1)$$
$$+A^{-T}[Qx^S(k) - q_x^*(x^S, c^S, u^S, k)^T] \quad (5.77)$$

$$g_4(l^P(k)) = \begin{bmatrix} g_{41}(l^P(k)) \\ g_{42}(l^P(k)) \end{bmatrix} \quad (5.78)$$

with

$$g_{41}(l^P(k)) = -B\bar{R}^{-1}[\mathcal{X} - K_c^*(x^S, c^S)]^T H^T l^P(k)$$
$$+B\bar{R}^{-1}\mathcal{X}^T H^T l^P(k) + D\bar{S}^{-1} l^P(k) \quad (5.79)$$

$$g_{42}(l^P(k)) = K_x^*(x^S, c^S)^T H^T l^P(k) \quad (5.80)$$

$$g_5(\mathbf{z}^S(k)) = \begin{bmatrix} g_{51}(\mathbf{z}^S(k)) \\ 0 \\ g_{53}(\mathbf{z}^S(k)) \\ 0 \end{bmatrix} \quad (5.81)$$

with

$$g_{51}(\mathbf{z}^S(k)) = \bar{R}^{-1}[[B - f_c^*(x^S, c^S, u^S), k]^T$$
$$-B^T A^{-T}[A - f_x^*(x^S, c^S, u^S), k]^T]p^S(k+1)$$
$$+\bar{R}^{-1}[Rc^S(k) - q_c^*(x^S, c^S, u^S, k)^T] \quad (5.82)$$
$$+\bar{R}^{-1}B^T A^{-T}[Qx^S(k) - q_x^*(x^S, c^S, u^S, k)^T]$$

$$g_{53,i}(\mathbf{z}_i^S(k)) = \bar{S}_i^{-1}[[D - f_u^*(x^S, c^S, u^S), k]^T$$
$$-D^T A^{-T}[A - f_x^*(x^S, c^S, u^S), k]^T]p^S(k+1)$$
$$+\bar{S}^{-1}[Su^S(k) - q_u^*(x^S, c^S, u^S, k)^T] \tag{5.83}$$
$$-\bar{S}^{-1}D^T A^{-T}[Qx^S(k) - q_x^*(x^S, c^S, u^S, k)^T]$$

$$g_6(l^S(k)) = \begin{bmatrix} g_{61}(l^S(k)) \\ 0 \\ g_{63}(l^S(k)) \\ 0 \end{bmatrix} \tag{5.84}$$

with

$$g_{61}(l^S(k)) = \bar{R}^{-1}[K_c^*(x^S, c^S)]^T - B^T A^{-T}[\mathcal{J} - K_x^*(x^S, c^S)]^T]H^T l^P(k) \tag{5.85}$$

$$g_{63}(l^S(k)) = -\bar{S}^{-1}D^T A^T \mathcal{K}^T H^T l^P(k)$$
$$-\bar{S}^{-1}D^T A^T[\mathcal{J} - K_x^*(x^S, c^S)]^T H^T l^P(k) - \bar{S}^{-1} l^P(k) \tag{5.86}$$

Equation (5.70) described the global output from the optimization of N_s subsystems in the lower level of the hierarchical algorithm PC2 to the outer loop. Note that the terms involving the interaction price $l(k)$ are isolated from the others in order to facilitate further derivation of the algorithm mapping of the outer loop .

Matrices $\mathbf{M}_3$ and $\mathbf{M}_4$ consist of convexification terms r_1 ,r_2 and r_3. $g_3(\mathbf{z}^S(k))$ and $g_6(\mathbf{z}^S(k))$ represents the global model-reality differences in the global performance criterion and global plant dynamics. $g_4(l^P(k))$ and $g_6(l^P(k))$ are made up of model-reality differences in the global output functions and coefficients of $l(k)$.

Expression (5.70) can be further simplified by use of the following definition.

Define

$$W(N,k,j) = \mu_{\mathbf{y}}(N,k)x_o$$

$$-\eta_{\mathbf{y}}(k)\phi_{22}(N)^{-1}\sum_{j=0}^{N-1}\phi_2(j)[\mathbf{M}_3\mathbf{z}^S(N-1-j)+g_{3i}(\mathbf{z}^S(N-1-j))]$$

$$+\sum_{j=0}^{k-1}\psi(j)[\mathbf{M}_3\mathbf{z}^S(k-1-j)+g_3(\mathbf{z}^S(k-1-j))]$$

$$+\mathbf{M}_4\mathbf{z}^S(k)+g_5(\mathbf{z}^S(k))+\varepsilon_l H\theta^S(k)$$

(5.87)

Definition (5.87) isolates terms prescribed by the inner loop that remain constant in the outer loop iteration. Using (5.87) in (5.70) and applying it to (5.67) produces the algorithm mapping of the outer loop which describes how $l(k)$, k∈[0,N] is updated from iteration p to iteration p+1;

$$l^{p+1}(k+1) = g_7(l^p(k)) + \varepsilon_l \mathbf{T} W(N,k,j) + \varepsilon_l \mathbf{T}\sum_{j=0}^{N-1}\Theta_2(N,k,j)[g_4 l^p(j)] \quad (5.88)$$

where

$$\Theta_2(N,k,j) = \begin{cases} \psi_2(k-1-j) - \eta_{\mathbf{y}}(k)\phi_{22}(N)\phi_2(N-1-j); & j\in[0,k-1] \\ \eta_{\mathbf{y}}(k)\phi_{22}(N)^{-1}\phi_2(N-1-j); & j\in[k,N-1] \end{cases} \quad (5.89)$$

and $\quad g_7(l^p(k)) = [\mathbf{I}_{m+n+r}l^p(k) - \varepsilon_l \mathbf{T} g_6(l^p(k))] \quad (5.90)$

5.3.1 Convergence of Double Loop Price Coordination algorithm PC2.

Convergence behaviour is investigated by considering successive iterations of the algorithm in the inner loop and the outer loop, as defined by the mappings (5.64) and (5.88) respectively.

First, we consider the convergence of the inner loop. We need to determine conditions such that

$$\|\mathbf{z}_i^{S+1}(k) - \mathbf{z}_i^S(k)\| \le \|\mathbf{z}_i^S(k) - \mathbf{z}_i^{S-1}(k)\| \quad (5.91)$$

where

$$
\begin{aligned}
\|\mathbf{z}_i(k)\| &= \sup_k \|\mathbf{z}_i(k)\|_q ; \quad k \in [0, N] \\
&= \sup_k (|\mathbf{z}_{1,i}|^q + \ldots + |\mathbf{z}_{n,i}|^q)^{1/q}; \quad q \in [1, \infty]; \quad k \in [0, N]
\end{aligned} \tag{5.92}
$$

The following additional assumption is required.

Assumption 5.1:

The functions $g_{1,i}(\mathbf{z}_i(k))$ and $g_{2,i}(\mathbf{z}_i(k))$ defined by (5.43) and (5.58) are Lipshitz continuous for all $\mathbf{z}_i(k)$, $k \in [0, N]$, with Lipschitz constants $h_{1,i}$ and $h_{2,i}$ respectively. That is

$$
\begin{aligned}
&\|g_{1,i}(\mathbf{z}_i^s(k)) - g_{1,i}(\mathbf{z}_i^{s-1}(k))\| \le h_{1,i} \|\mathbf{z}_i^s(k) - \mathbf{z}_i^{s-1}(k)\| \\
&\|g_{21,i}(\mathbf{z}_i^s(k)) - g_{21,i}(\mathbf{z}_i^{s-1}(k))\| \le h_{2,i} \|\mathbf{z}_i^s(k) - \mathbf{z}_i^{s-1}(k)\| \\
&\|g_{23,i}(\mathbf{z}_i^s(k)) - g_{23,i}(\mathbf{z}_i^{s-1}(k))\| \le h_{3,i} \|\mathbf{z}_i^s(k) - \mathbf{z}_i^{s-1}(k)\|
\end{aligned} \tag{5.93}
$$

The following theorem presents the local convergence conditions of the inner loop.

Theorem 5.3:

A sufficient condition for algorithm mapping (5.64) to satisfy (5.91) for every inner iteration s > 1 is given by the expression

$$
(\sigma_{1,i}(N) + h_{1,i}\sigma_{2,i}(N))N + \varepsilon_c h_{2,i} + \varepsilon_u h_{3,i} + \|\varepsilon \mathbf{M}_{2,i} + \mathbf{I}_{2n_i + m_i + r_i} - \varepsilon\| < 1 \tag{5.94}
$$

where $h_{1,i}$, $h_{2,i}$ and $h_{3,i}$ are defined in (5.93), ε and $\mathbf{M}_{2,i}$ are defined by (5.62) and (5.57); and

$$
\begin{aligned}
\|\sigma_{1,i}(N) &= \sup_{k \in [0, N-1]} \sup_{j \in [0, N-1]} \|\varepsilon \Theta_i(N, k, j) \mathbf{M}_{1,i}\| \\
\|\sigma_{2,i}(N) &= \sup_{k \in [0, N-1]} \sup_{j \in [0, N-1]} \|\varepsilon \Theta_i(N, k, j)\|
\end{aligned} \tag{5.95}
$$

with $\Theta_i(N, k, j)$ and $\mathbf{M}_{1,i}$ defined by (5.65) and (5.42).

Proof:

Consider two successive iterations of (5.64) . Taking the norm we obtain

$$\|\mathbf{z}_i^{s+1}(k)-\mathbf{z}_i^{s}(k)\| \le \|\varepsilon \sum_{j=0}^{N-1}\Theta_i(N,k,j)[\mathbf{M}_{1,i}(\mathbf{z}_i^{s}(j)-\mathbf{z}_i^{s-1}(j))$$
$$+g_{1,i}(\mathbf{z}_i^{s}(j))-g_{1,i}(\mathbf{z}_i^{s-1}(j))] \qquad (5.96)$$
$$+[\varepsilon\mathbf{M}_{2,i}+\mathbf{I}_{2n_i+m_i+r_i}-\varepsilon](\mathbf{z}_i^{s}(k)-\mathbf{z}_i^{s-1}(k))$$
$$+\varepsilon[g_{2,i}(\mathbf{z}_i^{s}(k))-g_{2,i}(\mathbf{z}_i^{s-1}(k))]\|$$

The above can be written as

$$\|\mathbf{z}_i^{s+1}(k)-\mathbf{z}_i^{s}(k)\| \le \|\varepsilon \sum_{j=0}^{N-1}\Theta_i(N,k,j)\mathbf{M}_{1,i}(\mathbf{z}_i^{s}(j)-\mathbf{z}_i^{s-1}(j))\|$$
$$+\| \sum_{j=0}^{N-1}\varepsilon\Theta_i(N,k,j)\, g_{1,i}(\mathbf{z}_i^{s}(j))-g_{1,i}(\mathbf{z}_i^{s-1}(j))\| \qquad (5.97)$$
$$+\|\varepsilon\mathbf{M}_{2,i}+\mathbf{I}_{2n_i+m_i+r_i}-\varepsilon\|(\mathbf{z}_i^{s}(k)-\mathbf{z}_i^{s-1}(k))$$
$$+\|\varepsilon[g_{2,i}(\mathbf{z}_i^{s}(k))-g_{2,i}(\mathbf{z}_i^{s-1}(k))]\|$$

From (5.59) and (5.60) and (5.62) we have

$$\varepsilon\, g_{2,i}(\mathbf{z}_i(k)) = \varepsilon_c\, g_{21,i}(\mathbf{z}_i(k)) + \varepsilon_x\, g_{23,i}(\mathbf{z}_i(k)) \qquad (5.98)$$

Then, using (5.93) and (5.95), (5.97) gives

$$\|\mathbf{z}_i^{s+1}(k)-\mathbf{z}_i^{s}(k)\| \le \{(\sigma_{1,i}(N)+h_{1,i}\sigma_{2,i}(N))N+\|\varepsilon\mathbf{M}_{2,i}+\mathbf{I}_{2n_i+m_i+r_i}-\varepsilon\|$$
$$+\varepsilon_c h_{2,i}+\varepsilon_u h_{3,i}\}\,\|\mathbf{z}_i^{s}(k)-\mathbf{z}_i^{s-1}(k)\| \qquad (5.99)$$

Hence the inner loop iteration will converge if

$$(\sigma_{1,i}(N)+h_{1,i}\sigma_{2,i}(N))N+\varepsilon_c h_{2,i}+\varepsilon_u h_{3,i}+\|\varepsilon\mathbf{M}_{2,i}+\mathbf{I}_{2n_i+m_i+r_i}-\varepsilon\|<1 \qquad (5.100)$$

Q.E.D

Convergence property of the outer loop is studied by considering successive iterations of the algorithm mapping defined by (5.91) and determining conditions such that

$$\|l^{p+1}(k)-l^{p}(k)\| \le \|l^{p}(k)-l^{p-1}(k)\| \qquad (5.101)$$

where

$$\|l(k)\| = \sup_k \|l(k)\|_q ; \quad k \in [0, N]$$
$$= \sup_k (|l_1|^q + \ldots + |l_r|^q)^{1/q}; \quad q \in [1, \infty]; \quad k \in [0, N] \tag{5.102}$$

The following additional assumptions is needed.

Assumption 5.2:

The functions $g_4(l(k))$ and $g_7(l(k))$ defined by (5.78) and (5.91) are Lipshitz continuous for all $l(k)$, $k \in [0, N]$, with Lipschitz constants h_3 and h_4 respectively. That is

$$\|g_4(l^p(k)) - g_4(l^{p-1}(k))\| \le h_3 \|l^p(k)) - l^{p-1}(k)\|$$
$$\|g_7(l^p(k)) - g_7(l^{p-1}(k))\| \le h_4 \|l^p(k)) - l^{p-1}(k)\| \tag{5.103}$$

The following theorem presents the global convergence conditions of the outer loop.

Theorem 5.4

A sufficient condition for algorithm mapping (5.88) to satisfy (5.101) for every outer loop iteration s > 1 is given by the expression

$$\|(h_4 + h_3\sigma_3(N))N\| < 1 \tag{5.104}$$

where h_3 ,and h_4 are defined in (5.103) and

$$\sigma_3(N) = \sup_{k \in [0, N-1]} \sup_{j \in [0, N-1]} \|\varepsilon_l \mathrm{T} \Theta_2(N, k, j)\| \tag{5.105}$$

with $\Theta_2(N, k, j)$ and T defined by (5.90) and (5.68).

Proof:

Consider two successive iterations of (5.88) . Taking the norm we obtain

$$\|l^{p+1}(k) - l^p(k)\| \le \|g_7(l^p(k)) - g_7(l^{p-1}(k)) + \varepsilon_l \mathrm{T} \sum_{j=0}^{N-1} \Theta_2(N, k, j)[g_4 l^p(j) - g_4 l^{p-1}(j)]\| \tag{5.106}$$

which can be expressed as

$$\|l^{p+1}(k)-l^{p}(k)\|\leq\|g_7(l^{p}(k))-g_7(l^{p-1}(k))\| + \|\sum_{j=0}^{N-1}\varepsilon_l \mathrm{T}\Theta_2(N,k,j)[g_4 l^{p}(j)-g_4 l^{p-1}(j)]\| \quad (5.107)$$

Then, using (5.101) and (5.103) gives

$$\|l^{p+1}(k)-l^{p}(k)\|\leq(h_4+h_3\sigma_3(N))N\|l^{p}(j)-l^{p-1}(j)\| \quad (5.108)$$

and, hence the iterations will contract asymptotically according to (5.101) if

$$\|(h_4+h_3\sigma_3(N))N\|<1 \quad (5.109)$$

Q.E.D

Conditions (5.94) and (5.104) show that the convergence of algorithm PC2 may depend on model-reality differences, the length optimization horizon, the choice of stepsizes for price and variables updates, and convexification factors as defined by matrices $\mathbf{M}_{1,i}$ and $\mathbf{M}_{2,i}$ in equations (5.42) and (5.57).

In the inner loop, the absence of model-reality differences means equations (5.44), (5.45), (5.59) and (5.60) are reduced to terms containing coefficients of the price $l_i^p(k)$ which are considered as constants in the inner loop. As a result , condition (5.94) reduces to

$$(\sigma_{1,i}(N))N+\|\varepsilon\mathbf{M}_{2,i}+\mathbf{I}_{2n_i+m_i+r_i}-\varepsilon\|<1 \quad (5.94a)$$

By inspecting (5.83), (5.84), (5.86) and (5.87) in the outer loop, it is observed that, the equations are also reduced to terms containing coefficients of the price $l(k)$. Inspecting conditions (5.94a) and (5.104) in conjunction with (5.90a), it is observed that the absence of model-reality differences will relax the sufficient convergence condition of PC2 to that which depends on the choice the price updating stepsize, the length of optimization horizon ,value of $\sigma_{1,i}$ (N) and the stepsize of variable updates .

Notice also conditions (5.94) and (5.104) also imply that increasing the length of optimization horizon (or increasing number of samples) may decrease the rate of convergence of the algorithm.

It is important to note that equations (5.94) and (5.104) are sufficient conditions only. It is not necessary to satisfy this condition for the algorithm to converge. However, the iterations in the inner loop and the outer loop are guaranteed to contract according to

(5.91) and (5.101) respectively, if the conditions are satisfied. This procedure demonstrates that convergence of the double loop price coordination algorithm PC2 is possible.

5.4 THE ALGORITHM MAPPING OF DOUBLE LOOP DIRECT COORDINATION ALGORITHM DC2

For simplicity we repeat the procedure in Section 5.2 by considering the special case of no terminal conditions $\Phi = 0$. First we derive the algorithm mapping of the inner loop of algorithm DC2. Let s denote the number of inner loop iterations and p denotes the corresponding outer loop iteration number.

The inner loop:

The inner loop constitutes iterations in the ith decomposed modified model based optimal control problem ($MMOP_i$)

We need to established the transition from iteration s to iteration s+1 of the inner loop of the double loop direct coordination algorithm DC2 with a linear model and quadratic performance criterion defined by (4.33) in terms of difference equations. From (4.28) we obtain the following expression representing the computation of modifiers.

$$\lambda_i^s(k)=[A_i - f_{x_i}^*(x_i^s,c_i^s,u_i^p),k]^T p_i^s(k+1)+Q_i\, x_i^s(k)-q_{x_i}^*(x_i^s,c_i^s,u_i^p,k)^T$$
$$+K_{x_i}^*(x_i^s,c_i^s)\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)-\mathcal{J}_i^T\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)$$

$$\beta_i^s(k)=[B_i - f_{c_i}^*(x_i^s,c_i^s,u_i^p),k]^T p_i^s(k+1)+R_i\, c_i^s(k)-q_{c_i}^*(x_i^s,c_i^s,u_i^p,k)^T$$
$$+K_{c_i}^*(x_i^s,c_i^s)\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)-\mathcal{K}_i^T\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)$$

$$\zeta_i^s(k)=[D_i - f_{u_i}^*(x_i^s,c_i^s,u_i^p),k]^T p_i^s(k+1)+S_i\, u_i^p(k)-q_{u_i}^*(x_i^s,c_i^s,u_i^p,k)^T \qquad (5.110)$$

where $l_j^p(k)$ is prescribed by computation in the outer loop.

Calculation of parameter $\alpha_i(k)$ remains as in (5.36). Parameter $\theta_i(k)$ is given by

$$\theta_i^S(k) = K_i^*(x_i^S, c_i^S) - \mathcal{J}_i x_i^S(k) - \mathcal{K}_i\, c_i^S(k) \tag{5.111}$$

The solution of the $MMOP_i$ defined by (4.33) can be represented by

$$\begin{aligned}
\hat{c}_i^S(k) &= \overline{R}_i^{-1}[-B_i^T p_i^S(k+1) + \beta_i^S(k) + \mathcal{K}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) + r_{1i} c_i^S(k)] \\
\hat{x}_i^S(k+1) &= A_i \hat{x}_i^S(k) + B_i \hat{c}_i^S(k) + D_i \hat{u}_i^p(k) + \alpha_i^S(k) \\
\hat{p}_i^S(k) &= \overline{Q}_i \hat{x}_i^S(k) + A\hat{p}_i^S(k+1) - \lambda_i^S(k) - \mathcal{J}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) - r_{2i} x_i^S(k); \\
\hat{p}_i^S(N) &= 0
\end{aligned} \tag{5.112}$$

where $u_j^p(k)$ is prescribed by computation in the outer loop.

The control ,state, interaction and costate updating mechanism defined by (3.60) is expressed as

$$\begin{aligned}
x_i^{S+1}(k) &= x_i^S(k) + \varepsilon_x(\hat{x}_i^S(k) - x_i^S(k)) \\
c_i^{S+1}(k) &= c_i^S(k) + \varepsilon_c(\hat{c}_i^S(k) - c_i^S(k)) \\
p_i^{S+1}(k) &= p_i^S(k) + \varepsilon_p(\hat{p}_i^S(k) - p_i^S(k))
\end{aligned} \tag{5.113}$$

Equations (5.110)-(5.113) are defined for k∈[1,N-1] and $\hat{c}_i(k) = v_i(k), \hat{x}_i(k) = z_i(k), \hat{u}_i(k) = w_i(k)$ and $\hat{p}_i(k) = p_i(k)$.

Notice that in (5.37) we can write

$$\begin{aligned}
\hat{p}_i^S(k+1) &= -A_i^{-T}\overline{Q}_i \hat{x}_i^S(k) + A_i^T \hat{p}_i^S(k) + A_i^{-T}\lambda_i^S(k) + \mathcal{K}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) + r_{2i} A_i^{-T} x_i^S(k) \\
\hat{p}_i^S(N) &= 0
\end{aligned} \tag{5.114}$$

Substituting (5.110),(5.111) in (5.112) and applying (5.114), we can write

$$\begin{bmatrix} \hat{x}_i^S(k+1) \\ \hat{p}_i^S(k+1) \end{bmatrix} = \mathbf{E}_{2i} \begin{bmatrix} \hat{x}_i^S(k) \\ \hat{p}_i^S(k) \end{bmatrix} + \mathbf{M}_{5,i}\mathbf{w}_i^S(k) + g_{8,i}(\mathbf{w}_i^S(k)) \tag{5.115}$$

where $\mathbf{w}_i^S(k) = [c_i^S(k)^T \quad x_i^S(k)^T \quad p_i^S(k+1)^T]$ and

$$\mathbf{E}_{2i} = \begin{bmatrix} A_i & -B_i \overline{R}_i^{-1} B_i^T \\ -A_i^{-T} \overline{Q}_i & A_i \end{bmatrix} \tag{5.116}$$

$$\mathbf{M}_{5,i} = \begin{bmatrix} r_{1i} B_i R_i^{-1} & 0_{n_i} & 0_{n_i} \\ 0_{n_i, m_i} & r_{2i} A_i^{-T} & 0_{n_i} \end{bmatrix} \tag{5.117}$$

$$g_{8,i}(\mathbf{w}_i^s(k)) = \begin{bmatrix} g_{81,i}(\mathbf{w}_i^s(k)) \\ g_{82,i}(\mathbf{w}_i^s(k)) \end{bmatrix} \tag{5.118}$$

with

$$\begin{aligned} g_{81,i}(\mathbf{w}_i^s(k)) = {} & B_i \overline{R}_i^{-1} [B_i - f_{c_i}^*(x_i^s, c_i^s, u_i^p), k]^T p_i^s(k+1) \\ & + B_i \overline{R}_i^{-1} [R_i c_i^s(k) - q_{c_i}^*(x_i^s, c_i^s, u_i^p, k)^T] \\ & + B_i \overline{R}_i^{-1} [\mathcal{X}_i - K_{c_i}^*(x_i^s, c_i^s)]^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)) \\ & + B_i \overline{R}_i^{-1} \mathcal{X}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) \\ & + f_i^*(x_i^s, c_i^s, u_i^p, k) - A_i x_i^s(k) - B_i c_i^s(k) \end{aligned} \tag{5.119}$$

$$\begin{aligned} g_{82,i}(\mathbf{w}_i^s(k)) = {} & A_i^{-T} [A_i - f_{x_i}^*(x_i^s, c_i^s, u_i^p), k]^T p_i^s(k+1) \\ & + A_i^{-T} [Q_i x_i^s(k) - q_{x_i}^*(x_i^s, c_i^s, u_i^p, k)^T] \\ & - A_i^{-T} [\mathcal{J}_i - K_{x_i}^*(x_i^s, c_i^s)]^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) \\ & + A_i^{-T} \mathcal{J}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) \end{aligned} \tag{5.120}$$

Notice that in (5.115), $\mathbf{E}_{2,i}$ is a transition matrix, $\mathbf{M}_{5,i}$ contains solely convexification terms (i.e $\mathbf{M}_{5,i} = 0$ if $r_{1i} = r_{2i} = 0$), and $g_{8,i}(\mathbf{w}_i^s(k))$ represents the model-reality differences.

Equation (5.115) can be written as

$$\begin{bmatrix} \hat{x}_i^s(k) \\ \hat{p}_i^s(k) \end{bmatrix} = \mathbf{E}_{2,i}^k \begin{bmatrix} x_{io} \\ \hat{p}_i(0) \end{bmatrix} + \sum_{j=0}^{k-1} \mathbf{E}_{2,i}^j [\mathbf{M}_{5,i} \mathbf{w}_i^s(k-1-j) + g_{8,i}(\mathbf{w}_i^s(k-1-j))];$$

$$\hat{p}_i^s(N) = 0 \tag{5.121}$$

Writing $\varphi_i(k) = \mathrm{E}_{2,i}^k$, the initial costate can be written in the form

$$\hat{p}_i^s(0) = \varphi_{22,i}(N)^{-1}\varphi_{21,i}(N)x_{io}$$
$$-\varphi_{22,i}(N)^{-1}\sum_{j=0}^{k-1}\varphi_{2,i}(j)[\mathbf{M}_{5,i}\mathbf{w}_i^s(N-1-j) + g_{8,i}(\mathbf{w}_i^s(N-1-j)) \tag{5.122}$$

Using (5.122) in (5.121) produces

$$\begin{bmatrix}\hat{x}_i^s(k)\\ \hat{p}_i^s(k)\end{bmatrix} = \begin{bmatrix}\mu_{x_i}(N,k)\\ \mu_{p_i}(N,k)\end{bmatrix}x_{io}$$
$$-\begin{bmatrix}\varphi_{12,i}(k)\\ \varphi_{22,i}(k)\end{bmatrix}\varphi_{22,i}(N)^{-1}\sum_{j=0}^{N-1}\varphi_{2,i}(j)[\mathbf{M}_{5,i}\mathbf{w}_i^s(N-1-j) + g_{8,i}(\mathbf{w}_i^s(N-1-j))]$$
$$+\sum_{j=0}^{k-1}\varphi_i(j)[\mathbf{M}_{5,i}\mathbf{w}_i^s(k-1-j) + g_{8,i}(\mathbf{w}_i^s(k-1-j))] \tag{5.123}$$

where

$$\mu_{x_i}(N,k) = \varphi_{11,i}(k) - \varphi_{12,i}(k)\varphi_{22,i}(N)^{-1}\varphi_{21,i}(N)$$
$$\mu_{p_i}(N,k) = \varphi_{21,i}(k) - \varphi_{22,i}(k)\varphi_{22,i}(N)^{-1}\varphi_{21,i}(N) \tag{5.124}$$

Using (5.35) and (5.112) to eliminate $\hat{p}_i^s(k+1)$ and $\beta_i^s(k)$ from the optimal control estimate $\hat{c}_i^s(k)$ gives

$$\hat{c}_i^s(k) = -\bar{R}_i^{-1}B_i^T A_i^{-T}\bar{Q}_i\mu_{x_i}(N,k)x_{io}$$
$$-\bar{R}_i^{-1}B_i^T A_i^{-T}\bar{Q}_i\varphi_{12,i}(k)\varphi_{22,i}(N)^{-1}\sum_{j=0}^{N-1}\varphi_{2,i}(j)[\mathbf{M}_{5,i}\mathbf{w}_i^s(N-1-j)$$
$$+g_{8,i}(\mathbf{w}_i^s(N-1-j))]$$
$$+\bar{R}_i^{-1}B_i^T A_i^{-T}\bar{Q}_i\sum_{j=0}^{k-1}\varphi_{1,i}(j)[\mathbf{M}_{5,i}\mathbf{w}_i^s(k-1-j)+g_{8,i}(\mathbf{w}_i^s(k-1-j))]$$
$$-\bar{R}_i^{-1}B_i^T A_i^{-T}\mu_{p_i}(N,k)x_{io}$$
$$-\bar{R}_i^{-1}B_i^T A_i^{-T}\mu_{p_i}(N,k)x_{io}$$
$$-\bar{R}_i^{-1}B_i^T A_i^{-T}\varphi_{22,i}(k)\varphi_{22,i}(N)^{-1}\sum_{j=0}^{N-1}\varphi_{2,i}(j)[\mathbf{M}_{5,i}\mathbf{w}_i^s(N-1-j)$$

$$+g_{8,i}(\mathbf{w}_i^s(N-1-j))]$$

$$+\bar{R}_i^{-1}B_i^T A_i^{-T}\sum_{j=0}^{k-1}\varphi_{1,i}(j)[\mathbf{M}_{5,i}\mathbf{w}_1^s(k-1-j)+g_{8,i}(\mathbf{w}_i^s(k-1-j))]$$

$$-r_{2i}\bar{R}_i^{-1}B_i^T A_i^{-T}x_i^s(k)+\bar{R}_i^{-1}B_i^T A_i^{-T}\mathcal{K}_i^T\sum_{j=1}^{N_s}H_{ij}^T l_j^p(k)$$

$$+r_{1i}\bar{R}_i^{-1}c_i^s(k)+\bar{R}_i^{-1}\mathcal{J}_i^T\sum_{j=1}^{N_s}H_{ij}^T l_j^p(k)$$

$$+\bar{R}_i^{-1}[B_i-f_{c_i}^*(x_i^s,c_i^s,u_i^p),k]^T p_i^s(k+1)$$

$$+\bar{R}_i^{-1}R_i\,c_i^s(k)-\bar{R}_i^{-1}q_{c_i}^*(x_i^s,c_i^s,u_i^p,k)^T$$

$$+\bar{R}_i^{-1}[\mathcal{K}_i-K_{c_i}^*(x_i^s,c_i^s)]^T\sum_{j=1}^{N_s}H_{ij}^T l_j^p(k))$$

$$-\bar{R}_i^{-1}B_i^T A_i^{-T}[A_i-f_{x_i}^*(x_i^s,c_i^s,u_i^p),k]^T p_i^s(k+1)$$

$$+\bar{R}_i^{-1}B_i^T A_i^{-T}Q_i\,x_i^s(k)-\bar{R}_i^{-1}B_i^T A_i^{-T}q_{x_i}^*(x_i^s,c_i^s,u_i^p,k)^T$$

$$+\bar{R}_i^{-1}B_i^T A_i^{-T}[\mathcal{J}_i-K_{x_i}^*(x_i^s,c_i^s)]^T\sum_{j=1}^{N_s}H_{ij}^T l_j^p(k) \qquad (5.125)$$

Combining (5.123) and (5.125) produces

$$\hat{\mathbf{w}}_1^s(k)=\mu_{2i}(N,k)x_{io}$$

$$-\eta_{2i}\varphi_{22,i}(N)^{-1}\sum_{j=0}^{N-1}\varphi_{2,i}(j)[\mathbf{M}_{5,i}\mathbf{w}_1^s(N-1-j)$$

$$+g_{8,i}(\mathbf{w}_i^s(N-1-j))] \qquad (5.126)$$

$$+\sum_{j=0}^{k-1}\psi_{2,i}(j)[\mathbf{M}_{5,i}\mathbf{w}_1^s(k-1-j)+g_{8,i}(\mathbf{w}_i^s(k-1-j))]$$

$$+\mathbf{M}_6\mathbf{w}_i^s(k)+g_{9,i}\mathbf{w}_i^s(k)$$

where

$$\mu_{2i}(N,k)=\begin{bmatrix}\mu_{c_i}(N,k)\\ \mu_{x_i}(N,k)\\ \mu_{p_i}(N,k)\end{bmatrix} \tag{5.127}$$

with $\mu_{c_i}(N,k)=-\overline{R}_i^{-1}B_i^T A_i^{-T}[\overline{Q}_i\mu_{x_i}(N,k)-\mu_{p_i}(N,k)]$

$$\eta_{2i}(N,k)=\begin{bmatrix}-\overline{R}_i^{-1}B_i^T A_i^{-T}[\overline{Q}_i\varphi_{12,i}(k)-\varphi_{22,i}(k)]\\ \varphi_{12,i}(k)\\ \varphi_{22,i}(k)\end{bmatrix} \tag{5.128}$$

$$\psi_{2,i}(j)=\begin{bmatrix}-\overline{R}_i^{-1}B_i^T A_i^{-T}[\overline{Q}_i\varphi_{1,i}(j)-\varphi_{22,i}(j)]\\ \varphi_{1,i}(j)\\ \varphi_{2,i}(j)\end{bmatrix} \tag{5.129}$$

$$\mathbf{M}_{6,i}=\begin{bmatrix}r_{1i}\overline{R}_i^{-1} & -r_{2i}\overline{R}_i^{-1}B_i^T A_i^{-T} & 0_{m_i,n_i}\\ 0_{n_i,m_i} & 0_{n_i} & 0_{n_i}\end{bmatrix} \tag{5.130}$$

and

$$g_{9,i}(\mathrm{w}_i^S(k))=\begin{bmatrix}g_{91,i}(\mathrm{w}_i^S(k))\\ 0\\ 0\end{bmatrix} \tag{5.131}$$

where

$$\begin{aligned}g_{91,i}(\mathrm{w}_i^S(k))=&\overline{R}_i^{-1}[[B_i-f_{c_i}^*(x_i^S,c_i^S,u_i^P),k]^T\\ &-B_i^T A_i^{-T}[A_i-f_{x_i}^*(x_i^S,c_i^S,u_i^P),k]^T]p_i^S(k+1)\\ &+\overline{R}_i^{-1}B_i^T A_i^{-T}\mathcal{J}_i^T\sum_{j=1}^{N_s}H_{ij}^T l_j^P(k)+\overline{R}_i^{-1}\mathcal{K}_i^T\sum_{j=1}^{N_s}H_{ij}^T l_j^P(k)\\ &+\overline{R}_i^{-1}[R_i\,c_i^S(k)-q_{c_i}^*(x_i^S,c_i^S,u_i^P,k)^T]\\ &+\overline{R}_i^{-1}B_i^T A_i^{-T}[Q_i\,x_i^S(k)-q_{x_i}^*(x_i^S,c_i^S,u_i^P,k)^T]\\ &+\overline{R}_i^{-1}[[\mathcal{K}_i-K_{c_i}^*(x_i^S,c_i^S)]^T\\ &+B_i^T A_i^{-T}[\mathcal{J}_i-K_{x_i}^*(x_i^S,c_i^S)]^T]\sum_{j=1}^{N_s}H_{ij}^T l_j^P(k)\end{aligned} \tag{5.132}$$

From (3.46)-(3.48) and (5.38), the updating mechanism can be written as

$$\mathbf{w}_i^{s+1}(k) = \varepsilon \hat{\mathbf{w}}_i^s(k) + [\mathbf{I}_{2n+m} - \varepsilon]\mathbf{w}_i^s(k) \tag{5.133}$$

with

$$\varepsilon_2 = \begin{bmatrix} \varepsilon_c \mathbf{I}_{m_i} & 0_{m_i,n_i} & 0_{m_i,n_i} \\ 0_{n_i,m_i} & \varepsilon_x \mathbf{I}_{n_i} & 0_{m_i,n_i} \\ 0_{n_i,m_i} & 0_{n_i} & \varepsilon_p \mathbf{I}_{n_i} \end{bmatrix} \tag{5.134}$$

Combining (5.126) and (5.133) , produces

$$\begin{aligned} \mathbf{w}_i^{s+1}(k) = {} & \varepsilon_2 \mu_{2i}(N,k) x_{io} \\ & - \varepsilon_2 \eta_{2i}(k) \varphi_{22,i}(N)^{-1} \sum_{j=0}^{N-1} \varphi_{2,i}(j)[\mathbf{M}_{5,i}\mathbf{w}_i^s(N-1-j) \\ & + g_{8,i}(\mathbf{w}_i^s(N-1-j))] \\ & + \varepsilon_2 \sum_{j=0}^{k-1} \psi_{2,i}(j)[\mathbf{M}_{5,i}\mathbf{w}_i^s(k-1-j) + g_{8,i}(\mathbf{w}_i^s(k-1-j))] \\ & + [\varepsilon_2 \mathbf{M}_{6,i} + \mathbf{I}_{2n_i+m_i} - \varepsilon_2]\mathbf{w}_i^s(k) + \varepsilon_2 g_{9,i}\mathbf{w}_i^s(k) \end{aligned} \tag{5.135}$$

Hence the algorithm mapping, showing how $\mathbf{w}_i(k)$, $k \in [0,N]$, is updated from iteration s to iteration s+1 in the inner loop, is given by

$$\begin{aligned} \mathbf{w}_i^{s+1}(k) = {} & \varepsilon_2 \mu_{2i}(N,k) x_{io} \\ & - \varepsilon_2 \sum_{j=0}^{N-1} \Theta_{3,i}(N,k,j)[\mathbf{M}_{5,i}\mathbf{w}_i^s(j) + g_{8,i}(\mathbf{w}_i^s(j))] \\ & + [\varepsilon_2 \mathbf{M}_{6,i} + \mathbf{I}_{2n_i+m_i} - \varepsilon_2]\mathbf{w}_i^s(k) + \varepsilon_2 g_{9,i}\mathbf{w}_i^s(k) \end{aligned} \tag{5.136}$$

where

$$\Theta_{3i}(N,k,j) = \begin{cases} \psi_{2,i}(k-1-j) - \eta_{2i}(k)\varphi_{22,i}(N)\varphi_{2,i}(N-1-j); & j \in [0,k-1] \\ -\eta_{2,i}(k)\phi\varphi_{22,i}(N)^{-1}\phi\varphi_{2,i}(N-1-j); & j \in [k,N-1] \end{cases} \tag{5.137}$$

Next we derive the algorithm mapping of the outer loop.

The outer loop:

From algorithm 4.3.2 we note that computation in the outer loop utilize the global information available from the N_s subsystems in the inner loop. Let p denote the number of iteration in the outer loop.

The outer loop consist of the following steps. The interconnection price is given by equation (4.63), that is

$$l^p(k) = \bar{\zeta}^s(k) - \bar{S}u^p(k) - D^T p^s(k+1); \qquad k \in [1, N-1] \qquad (5.138)$$

The interaction vector is computed directly from the interconnection model which is given by

$$\hat{u}^p(k) = H(\mathcal{J}\hat{x}^s(k) + \mathcal{K}\hat{c}^s(k) + \theta^p(k)) \qquad \mathrm{k} \in [1, \mathrm{N}-1] \qquad (5.139)$$

Finally the interaction term is updated after each iteration using the relaxation formula given by (4.64), that is

$$u^{p+1}(k) = u^p(k) + \varepsilon_u(\hat{u}^p(k) - u^p(k)) \qquad (5.140)$$

where $[.]^s$ denotes variables prescribed by the inner loop optimization unit.

Using (5.139) in (5.140) we eliminate $\hat{u}^p(k)$ to produce

$$u^{p+1}(k) = (1-\varepsilon_u)u^p(k) + \varepsilon_u \mathrm{T}_2 \hat{\mathrm{v}}(k) - \varepsilon_u H\theta^s(k) \qquad (5.141)$$

where

$$\mathrm{T}_2 = \begin{bmatrix} H\mathcal{K} & 0_{r,m} \\ 0_{r,n} & H\mathcal{J} \end{bmatrix} \qquad \text{and} \qquad \hat{\mathrm{v}}^s(k) = [\hat{c}^s(k)^T \ \hat{x}^s(k)^T]^T \qquad (5.142)$$

Using global definitions (3.4a), (3.24a), (5.2) and (5.3), and from inspection of the local optimization output equation (5.126) an expression for the global variable $\hat{\mathrm{v}}^s(k)$ can be written as

$$\begin{aligned}
\hat{\mathrm{v}}_1^s(k) = {} & \mu_{\mathrm{v}}(N,k)x_o \\
& - \eta_{\mathrm{v}}\varphi_{22}(N)^{-1} \sum_{j=0}^{N-1} \varphi_2(j)[\mathrm{M}_7 \mathrm{w}^s(N-1-j) + g_{10}(\mathrm{w}^s(N-1-j))] \\
& - \eta_{\mathrm{v}}\varphi_{22}(N)^{-1} \sum_{j=0}^{N-1} \varphi_2(j) g_{11}(u^p(N-1-j))] \\
& + \sum_{j=0}^{k-1} \psi_2(j)[\mathrm{M}_7 \mathrm{w}^s(k-1-j) + g_{10}(\mathrm{w}^s(k-1-j))] \\
& + \sum_{j=0}^{k-1} \psi_2(j) g_{10}(u^p(k-1-j))] + \mathrm{M}_8(\mathrm{w}^s(k)) + g_{12}(\mathrm{w}^s(k)) + g_{13}(u^p(k))
\end{aligned} \qquad (5.143)$$

where $\mathbf{w}^S(k) = [c^S(k)^T \quad x^S(k)^T \quad p^S(k+1)^T]$

and

$$\mu_{\mathbf{v}}(N,k) = \begin{bmatrix} \mu_c(N,k) \\ \mu_x(N,k) \end{bmatrix}; \quad \eta_{\mathbf{v}}(k) = = \begin{bmatrix} \overline{R}^{-1} B^T A^{-T} [\overline{Q} \varphi_{12}(k) - \varphi_{22}(k)] \\ \varphi_{12}(k) \end{bmatrix} \quad (5.144)$$

with $\varphi_{(.)}$ denoting the corresponding global version of $\varphi_{(.),i}$

$$\mathbf{M}_7 = \begin{bmatrix} r_1 B^T \overline{R}^{-1} & 0_n & 0_n \\ 0_{n,m} & r_2 A^{-T} & 0_n \end{bmatrix}; \quad (5.145)$$

$$\psi_2(j) = \begin{bmatrix} \overline{R}^{-1} B^T A^{-T} [\overline{Q} \varphi_1(k) - \varphi_2(k)] \\ \varphi_1(k) \\ \varphi_2(k)] \end{bmatrix} \quad (5.146)$$

$$\mathbf{M}_8 = \begin{bmatrix} r_1 \overline{R}^{-1} & -r_1 \overline{R}^{-1} B^T A^{-T} & 0_{m,n} \\ 0n,m & 0_n & 0_n \\ 0n,m & 0_n & 0_n \end{bmatrix} \quad (5.147)$$

$$g_{10}(\mathbf{w}^S(k)) = \begin{bmatrix} g_{10,1}(\mathbf{w}^S(k)) \\ g_{10,2}(\mathbf{w}^S(k)) \end{bmatrix} \quad (5.148)$$

with

$$g_{10,1}(\mathbf{w}^S(k)) = B\overline{R}^{-1}[\mathcal{K} - K_c^*(x^S, c^S)]^T H^T [\zeta^S(k) - D^T p^S(k+1)] \quad (5.149)$$

$$g_{10,2}(\mathbf{w}^S(k)) = [A^{-T} \mathcal{J} - A^{-T} K_x^*(x^S, c^S)]^T H^T [\zeta^S(k) - D^T p^S(k+1)] \quad (5.150)$$

$$g_{11}(u^P(k)) = \begin{bmatrix} g_{11,1}(u^P(k)) \\ g_{11,2}(y^P(k)) \end{bmatrix} \quad (5.151)$$

with

$$\begin{aligned} g_{11,1}(u^P(k)) &= B\overline{R}^{-1}[B - f_c^*(x^S, c^S, u^P), k]^T p^S(k+1) \\ &+ B\overline{R}^{-1}[Rc^S(k) - q_c^*(x^S, c^S, u^P, k)^T] \\ &+ f^*(x^S, c^S, u^P, k) - Ax^S(k) - Bc^S(k) - Du^P(k) \\ &- B\overline{R}^{-1}[\mathcal{K} - K_c^*(x^S, c^S)]^T H^T S u^P(k) + Du^P(k) \end{aligned} \quad (5.152)$$

$$g_{11,2}(u^P(k)) = A^{-T}[A - f_x^*(x^S, c^S, u^P), k]^T p^S(k+1)$$

$$+A^{-T}[Qx^S(k) - q_x^*(x^S, c^S, u^P, k)^T] \tag{5.153}$$

$$+A^{-T}[\mathcal{J} - K_x^*(x^S, c^S)]^T H^T S u^P(k)$$

$$g_{12,}(\mathrm{w}^S(k)) = \begin{bmatrix} g_{12,1}(\mathrm{w}^S(k)) \\ 0 \\ 0 \end{bmatrix} \tag{5.154}$$

with

$$g_{12,1}(\mathrm{w}^S(k)) = \overline{R}^{-1}[[\mathcal{K} - K_c^*(x^S, c^S)]^T \tag{5.155}$$

$$+B^T A^{-T}[\mathcal{J} - K_x^*(x^S, c^S)]^T]H^T[\zeta^S(k) - D^T p^S(k+1)^P(k)]$$

$$g_{13,}(u^S(k)) = \begin{bmatrix} g_{13,1}(u^S(k)) \\ 0 \\ 0 \end{bmatrix} \tag{5.156}$$

with

$$g_{13,1}(u^S(k)) = \overline{R}^{-1}[[B - f_c^*(x^S, c^S, u^P), k]^T$$

$$-B^T A^{-T}[A - f_x^*(x^S, c^S, u^P), k]^T]p^S(k+1)$$

$$+\overline{R}^{-1}[Rc^S(k) - q_c^*(x^S, c^S, u^P, k)^T]$$

$$+\overline{R}^{-1}B^T A^{-T}[Qx^S(k) - q_x^*(x^S, c^S, u^P, k)^T]$$

$$+\overline{R}^{-1}[[\mathcal{K} - K_c^*(x^S, c^S)]^T + B^T A^{-T}[\mathcal{J} - K_x^*(x^S, c^S)]^T]Su^P(k) \tag{5.157}$$

Equation (5.143) represents the global output from the optimization of N_S subsystems in the lower level of the hierarchical algorithm DC2 to the outer loop. Note that the terms involving the interaction price *l(k)* is eliminated using (5.138). Terms containing the interaction vector *u(k)* are isolated from the others in order to facilitate further derivation of algorithm mapping of the outer loop .

Matrices $\mathbf{M}_7$ and $\mathbf{M}_8$ consist of convexification terms r_{1i} and r_{2i} . $g_{10}(\mathrm{w}^S(k))$ and $g_{12}(\mathrm{w}^S(k))$ represents the global model-reality differences in the global performance criterion and global plant dynamics. $g_{12}(u^P(k))$ and $g_{13}(l^P(k))$ are

made up of model-reality differences in the global output functions and coefficients of $u(k)$.

Our aim is to express the iteration in the outer loop in terms of interaction variable $u(k)$ by substituting (5.143) into (5.141). To achieve this we make the following definition which will be used to simplify the resulting equation.

Define

$$W_{\mathbf{v}}(N,k,j) = \mu_{\mathbf{v}}(N,k)x_o$$
$$-\eta_{\mathbf{v}}\varphi_{22}(N)^{-1}\sum_{j=0}^{N-1}\varphi_2(j)[\mathbf{M}_7\mathbf{w}^s(N-1-j)+g_{10}(\mathbf{w}^s(N-1-j))]$$
$$+\sum_{j=0}^{k-1}\psi_2(j)[\mathbf{M}_7\mathbf{w}^s(k-1-j)+g_{10}(\mathbf{w}^s(k-1-j))]$$
$$+g_{12}(\mathbf{w}^s(k))+g_{13}(u^p(k))+\varepsilon_u H\theta^s(k) \qquad (5.158)$$

Definition (5.158) contains terms prescribed by the outer loop that remain constant throughout the computation in the outer loop. Using (5.158) in (5.143) and applying to (5.141), produces the algorithm mapping of the inner loop which describes how $u(k),\ k\in[0,N]$ is updated from iteration p to iteration p+1 in the outer loop;

$$u^{p+1}(k+1) = (1-\varepsilon_u)(u^p(k))+\varepsilon_u\mathrm{T}_2W_{\mathbf{v}}(N,k,j)$$
$$+\varepsilon_u\mathrm{T}_2\sum_{j=0}^{N-1}\Theta_3(N,k,j)[g_{11}u^p(j)]+\varepsilon_u\mathrm{T}_2g_{13}(u^p(k)) \qquad (5.159)$$

where

$$\Theta_3(N,k,j) = \begin{cases}\psi_2(k-1-j)-\eta_{\mathbf{v}}(k)\varphi_{22}(N)\phi_{2,}(N-1-j); & j\in[0,k-1]\\ -\eta_{\mathbf{v}}(k)\varphi_{22}(N)^{-1}\varphi_2(N-1-j); & j\in[k,N-1]\end{cases} \qquad (5.160)$$

5.4.1 Convergence of Double Loop Price Coordination algorithm DC2.

The convergence property is investigated by considering successive iterations of the algorithm in the inner loop and the outer loop, as defined by the mapping (5.136) and (5.159) respectively.

Initially we consider the convergence of the inner loop. We need to determine conditions such that

$$\|\mathbf{w}_i^{s+1}(k) - \mathbf{w}_i^{s}(k)\| \le \|\mathbf{w}_i^{s}(k) - \mathbf{w}_i^{s-1}(k)\| \tag{5.161}$$

where

$$\begin{aligned} \|\mathbf{w}_i(k)\| &= \sup_k \|\mathbf{w}_i(k)\|_q ; \quad k \in [0, N] \\ &= \sup_k (|\mathbf{w}_{1,i}|^q + \ldots + |\mathbf{w}_{n,i}|^q)^{1/q}; \quad q \in [1, \infty]; \quad k \in [0, N] \end{aligned} \tag{5.162}$$

The following additional assumption is required.

Assumption 5.3:

The functions $g_{8,i}(\mathbf{w}_i(k))$ and $g_{91,i}(\mathbf{w}_i(k))$ defined by (5.118) and (5.132) are Lipschitz continuous for all $\mathbf{w}_i(k)$, $k \in [0, N]$, with Lipschitz constants $h_{5,i}$ and $h_{6,i}$ respectively. That is

$$\begin{aligned} \|g_{8,i}(\mathbf{w}_i^{s}(k)) - g_{8,i}(\mathbf{w}_i^{s-1}(k))\| &\le h_{5,i} \|\mathbf{w}_i^{s}(k) - \mathbf{w}_i^{s-1}(k)\| \\ \|g_{91,i}(\mathbf{w}_i^{s}(k)) - g_{91,i}(\mathbf{w}_i^{s-1}(k))\| &\le h_{6,i} \|\mathbf{w}_i^{s}(k) - \mathbf{w}_i^{s-1}(k)\| \end{aligned} \tag{5.163}$$

The following theorem presents the local convergence conditions of the inner loop.

Theorem 5.5:

A sufficient condition for algorithm mapping (5.136) to satisfy (5.159) for every inner iteration s > 1 is given by the expression

$$(\sigma_{4,i}(N) + h_{5,i}\sigma_{5,i}(N))N + \varepsilon_c h_{6,i} + \|\varepsilon_2 \mathbf{M}_{6,i} + \mathbf{I}_{2n_i + m_i} - \varepsilon_2\| < 1 \tag{5.164}$$

where $h_{5,i}$ and $h_{6,i}$ are defined in (5.163), ε and $\mathbf{M}_{6,i}$ are defined by (5.134) and (5.130); and

$$\begin{aligned} \sigma_{4,i}(N) &= \sup_{k \in [0, N-1]} \sup_{j \in [0, N-1]} \|\varepsilon_2 \Theta_{2i}(N, k, j)\mathbf{M}_{1,i}\| \\ \sigma_{5,i}(N) &= \sup_{k \in [0, N-1]} \sup_{j \in [0, N-1]} \|\varepsilon_2 \Theta_{2i}(N, k, j)\| \end{aligned} \tag{5.165}$$

with $\Theta_i(N, k, j)$ and $\mathbf{M}_{1,i}$ defined by (5.65) and (5.42).

Proof:

Consider two successive iterations of (5.136) . Taking the norm we obtain

$$\|\mathbf{w}_i^{s+1}(k)-\mathbf{w}_i^{s}(k)\| \le \varepsilon_2 \sum_{j=0}^{N-1} \Theta_{3,i}(N,k,j)[\mathbf{M}_{5,i}(\mathbf{w}_i^{s}(j)-\mathbf{w}_i^{s-1}(j)) + g_{8,i}(\mathbf{w}_i^{s}(j))-g_{8,i}(\mathbf{w}_i^{s-1}(j))] + [\varepsilon_2\mathbf{M}_{6,i}+\mathbf{I}_{2n_i+m_i}-\varepsilon_2](\mathbf{w}_i^{s}(k)-\mathbf{w}_i^{s-1}(k)) + \varepsilon_2[g_{9,i}\mathbf{w}_i^{s}(k)-g_{9,i}\mathbf{w}_i^{s}(k)] \quad (5.166)$$

The above can be written as

$$\|\mathbf{w}_i^{s+1}(k)-\mathbf{w}_i^{s}(k)\| \le \|\varepsilon_2 \sum_{j=0}^{N-1} \Theta_{3,i}(N,k,j)\mathbf{M}_{5,i}(\mathbf{w}_i^{s}(j)-\mathbf{w}_i^{s-1}(j))\| + \|\varepsilon_2 \sum_{j=0}^{N-1} \Theta_{3,i}(N,k,j)[g_{8,i}(\mathbf{w}_i^{s}(j))-g_{8,i}(\mathbf{w}_i^{s-1}(j))]\| + \|\varepsilon_2\mathbf{M}_{6,i}+\mathbf{I}_{2n_i+m_i}-\varepsilon_2\|(\mathbf{w}_i^{s}(k)-\mathbf{w}_i^{s-1}(k)) + \|\varepsilon_2[g_{9,i}\mathbf{w}_i^{s}(k)-g_{9,i}\mathbf{w}_i^{s}(k)]\| \quad (5.167)$$

From (5.129) and (5.160) we have

$$\varepsilon g_{9,i}(\mathbf{w}_i(k)) = \varepsilon_c g_{91,i}(\mathbf{w}_i(k)) \quad (5.168)$$

Then, using (5.163) and (5.166), (5.167) gives

$$\|\mathbf{w}_i^{s+1}(k)-\mathbf{w}_i^{s}(k)\| \le \{(\sigma_{4,i}(N)+h_{5,i}\sigma_{5,i}(N))N+\varepsilon_c h_{6,i} + \|\varepsilon_2\mathbf{M}_{6,i}+\mathbf{I}_{2n_i+m_i}-\varepsilon_2\|\}\|(\mathbf{w}_i^{s}(k)-\mathbf{w}_i^{s-1}(k))\| \quad (5.169)$$

Hence the inner loop iterations will converge if

$$(\sigma_{4,i}(N)+h_{5,i}\sigma_{5,i}(N))N+\varepsilon_c h_{6,i}+\|\varepsilon_2\mathbf{M}_{6,i}+\mathbf{I}_{2n_i+m_i}-\varepsilon_2\|<1 \quad (5.170)$$

Q.E.D

Notice that the convergence condition of the inner loop is similar to that of the centralize Discrete DISOPE algorithm given by Becerra and Roberts (1996). This is because in the inner loop, interaction variable *u(k)* and interaction price *l(k)* are prescribed from the outer loop which remain constant throughout the inner iteration. Hence the inner loop has a similar iterative mapping to that of a centralized discrete DISOPE.

Next, we derive the sufficient condition for convergence of the outer loop by considering successive iterations of the algorithm mapping defined by (5.159) and determining conditions such that

$$\|u^{p+1}(k)-u^{p}(k)\|\leq\|u^{p}(k)-u^{p-1}(k)\| \tag{5.171}$$

where

$$\begin{aligned}\|u(k)\| &= \sup_{k}\|u(k)\|_{q}\,; \quad k\in[0,N]\\ &= \sup_{k}(|u_1|^{q}+\ldots+|u_r|^{q})^{1/q}; \quad q\in[1,\infty];\quad k\in[0,N]\end{aligned} \tag{5.172}$$

The following additional assumption is needed.

Assumption 5.2:

The functions $g_{11,}(u(k))$ and $g_{13,1}(l(k))$ defined by (5.151) and (5.157) are Lipschitz continuous for all $l(k)$, $k\in[0,N]$, with Lipschitz constants h_7 and h_8 respectively. That is

$$\begin{aligned}&\|g_{11,}(\mathbf{w}^{s}(k))-g_{11,}(\mathbf{w}^{s-1}(k))\|\leq h_7\,\|\mathbf{w}^{s}(k)-\mathbf{w}^{s-1}(k)\|\\ &\|g_{13,1}(\mathbf{w}^{s}(k))-g_{13,1}(\mathbf{w}^{s-1}(k))\|\leq h_8\,\|\mathbf{w}^{s}(k)-\mathbf{w}^{s-1}(k)\|\end{aligned} \tag{5.173}$$

The following theorem presents the global convergence conditions of the outer loop.

Theorem 5.6

A sufficient condition for algorithm mapping (5.159) to satisfy (5.171) for every outer loop iteration s > 1 is given by the expression

$$(\varepsilon_u\|\mathbf{T}_2\|h_8+h_7\sigma_6(N))N<\varepsilon_u \tag{5.174}$$

where h_7 and h_8 are defined in (5.173) and

$$\sigma_6(N)=\sup_{k\in[0,N-1]}\ \sup_{j\in[0,N-1]}\|\varepsilon_u\mathbf{T}_2\,\Theta_3(N,k,j)\| \tag{5.175}$$

with $\Theta_3(N,k,j)$ and $\mathbf{T}_2$ defined by (5.161) and (5.142).

Proof:

Consider two successive iterations of (5.159) . Taking the norm we obtain

$$\|u^{p+1}(k)-u^{p}(k)\|\leq\|(1-\varepsilon_u)(u^{p}(k)-u^{p-1}(k))$$
$$+\varepsilon_u \mathrm{T}_2 \sum_{j=0}^{N-1}\Theta_3(N,k,j)[g_{11}(u^{p}(j))-g_{11}(u^{p-1}(j))] \quad (5.176)$$
$$+\varepsilon_u \mathrm{T}_2[g_{13}(u^{p}(k))-g_{13}(u^{p-1}(k))]\|$$

which can be expressed as

$$\|u^{p+1}(k)-u^{p}(k)\|\leq\|(1-\varepsilon_u)(u^{p}(k)-u^{p-1}(k))\|$$
$$+\|\varepsilon_u \mathrm{T}_2 \sum_{j=0}^{N-1}\Theta_3(N,k,j)[g_{11}(u^{p}(j))-g_{11}(u^{p-1}(j))]\|$$
$$+\|\varepsilon_u \mathrm{T}_2[g_{13}(u^{p}(k))-g_{13}(u^{p-1}(k))]\| \quad (5.177)$$

Then, using (5.173) gives

$$\|u^{p+1}(k)-u^{p}(k)\|\leq \{1-\varepsilon_u+\varepsilon_u\|\mathrm{T}_2\|h_8$$
$$+h_7\sigma_6(N))N\}\|u^{p}(k)-u^{p-1}(k)\| \quad (5.178)$$

and, hence the iterations will contract asymptotically according to (5.101) if

$$1-\varepsilon_u+\varepsilon_u\|\mathrm{T}_2\|h_8+h_7\sigma_6(N))N<1 \quad (5.179)$$

That is
$$\varepsilon_u\|\mathrm{T}_2\|h_8+h_7\sigma_6(N))N<\varepsilon_u \quad (5.180)$$

Q.E.D

Note that if we choose the default stepsize for interaction update (i.e. ε_u=1) then equation (5.174) is reduce to

$$\|\mathrm{T}_2\|h_8+h_7\sigma_6(N))N<1 \quad (5.181)$$

Conditions (5.164) and (5.174) show that the convergence of algorithm DC2 may depend on model-reality differences, the length optimization horizon, the choice of stepsizes for price and variables updates, and convexification factors as defined by matrices $\mathbf{M}_{5,j}$ and $\mathbf{M}_{6,j}$ in equations (5.117) and (5.130).

In the inner loop, the absence of model-reality differences means equations (5.119), (5.120) and (5.132) are reduced to terms containing coefficients of the price $l_l^p(k)$. As a consequence of this, condition (5.164) reduces to

$$(\sigma_{4,i}(N))N+\|\varepsilon_2 \mathbf{M}_{6,i}+\mathbf{I}_{2n_i+m_i}-\varepsilon_2\|<1 \tag{5.182}$$

In the outer loop ,in absence of model-reality differences sufficient condition (5.181) is simplified to the following,

$$h_7\sigma_6(N))N<1 \tag{5.183}$$

This implies that the absence of model-reality difference will relax the convergence conditions of algorithm DC2 and may result in a faster speed of convergence.

From (5.164) and (5.174) it can be concluded that reducing the length of optimization horizon (or the number of samples) may also improve the rate of convergence in algorithm DC2.

It is important to note that conditions (5.164) and (5.174) are sufficient conditions only. It is not necessary to satisfy them for the algorithm to converge. However, the iterations in the inner loop and the outer loop are guaranteed to contract according to (5.161) and (5.171) respectively, if the conditions are satisfied. This procedure demonstrates that convergence of the double loop direct coordination algorithm DC2 is possible.

5.5 SUMMARY

In this chapter we have established the optimality of hierarchical algorithms for solving structure with model based interaction input. We have also derived the algorithm mapping of the double loop version of both the price coordination and direct coordination approach. From the algorithm mapping we have shown that it is possible to derive the sufficient convergence condition of the respective algorithms. Although the conditions derived are conservative in the sense they do not have to be satisfied in order for algorithms to converge, they point the way for further theoretical investigation into the convergence property of the algorithm. For instance from simulation results in chapter 4, it would appear that convexification factor r_1 ,convexification factor associated with control signal *c(k)*, have the greatest influence on convergence of the four algorithms presented. Note that at present the convergence condition of the single loop algorithms PC1 and DC1 is still intractable because the presence of the interaction term and interaction price multiplier in a single loop structure complicates the derivation of the algorithm mapping. In the double loop

structure the interaction term and the interaction price multiplier is conveniently decoupled from the inner loop and can be treated as constant, thus making it possible to derive the algorithm mapping using method described in this chapter.

In the following chapter we present implementations of algorithms for solving systems with different level of utilization of real interaction input in the hierarchical structure which was proposed earlier in chapter 3.

CHAPTER 6

ALGORITHMS FOR STRUCTURES WITH REAL INTERACTION MEASUREMENTS

6.1 INTRODUCTION

In this chapter we develop algorithms for solving optimal control problem of structures with real interaction measurements. The structures which have been described in Chapter 3, are dynamic analogies of hierarchical ISOPE structures with input-output information feedback (Brdys and Roberts, 1986). They have two different levels of utilization of real interaction measurements. A real interaction measurement is defined as the actual real system measurement given by the output function $y(k) = HK^{*}(x(k),c(k))$. Initially ,we consider the structure given by fig. 3.5.1., where the real interaction measurement is utilized in the parameter estimation step of the local optimization unit. Next we consider the third structure proposed in chapter 3, illustrated by fig. 3.6.1. In this structure the real interaction measurement is incorporated in both the parameter estimation step and as part of the interconnection equation itself. In contrast to algorithms described in chapter 4, where models of output function are utilized , the algorithms presented here use actual measurements from the reality to solve the equivalent modified model based optimal control problem. To derive the algorithms for both structures, we will use familiar procedures , that is , assuming that the model based problems are represented by linear quadratic approximations of the real optimal control problems. These in turn are solved using a standard method (Lewis and Syrmos,1995; Bryson and Ho, 1980) using two approaches, price coordination and direct coordination. Implementing the resulting algorithms in single loop and double loop techniques produces a total of eight algorithms which are summarized below.

1. Algorithms for structure with real interaction measurements in parameter estimation.

   PC3-Single Loop Price Coordination

   PC4-Double Loop Price Coordination

   DC3-Single Loop Direct Coordination

   DC4-Double Loop Price Coordination

2. Algorithms for structure with real input measurements in interaction and parameter estimation.

PC5-Single Loop Price Coordination

PC6-Double Loop Price Coordination

DC5-Single Loop Direct Coordination

DC6-Double Loop Price Coordination

The algorithms are implemented in software using the C++ programming language and simulated using earlier examples to test its efficiency and convergence property. The optimality of respective algorithms will also be established.

6.2 ALGORITHMS FOR STRUCTURES WITH REAL INTERACTION INPUT IN PARAMETER ESTIMATION

In this section we derive the algorithms for optimal control of hierarchical structures with real input in parameter estimation (see figure 3.5.1) using the price coordination and direct coordination approaches.

6.2.1 PRICE COORDINATION APPROACH PROBLEM FORMULATION AND SOLUTION

Consider the optimal control problem defined by $MMOP2_i$ in Chapter 3, Section 3.5. Augmenting the performance index with variable augmentation produces the following equivalent form $MMOP2_i$'.

$MMOP2_i$':

$$\min_{c(k),u(k)} Q_i = \phi(\bar{x}(N)) + \sum_{k=1}^{N-1} \{q_i(x_i(k),c_i(k),u_i(k),\gamma_i(k))$$

$$+l_i(k)^T u_i(k) - \sum_{j=1}^{N_s} l_j^T H_{ji} F_i(x_i(k),c_i(k),u_i(k),\theta_i(k))$$

$$-\lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)$$

$$+\frac{1}{2} r_{1i} \|c_i(k) - v_i(k)\|^2 + \frac{1}{2} r_{2i} \|x_i(k) - z_i(k)\|^2 \}$$

subject to

$$x_i(k+1) = f_i(x_i(k),c_i(k),u_i(k),\alpha_i(k)); \ k \in [1,N-1]$$

$$x_i(0) = x_{i0}$$

$$x_{i,t}(N) = 0; t \in [1,q]$$

$$\bar{x}_i(N) = [x_{i,q+1}(N), \ldots\ldots, x_{i,n}(N)]^T$$

where r_{1i} and r_{2i} are given scalar convexification factors. Notice that at the end of the iterations, $c(k)=v(k)$ and $x(k)=z(k)$ so that at this stage the augmentation terms and their derivatives are zero, so having no effect in the real optimality of the solution. Notice also in this structure we do not need a separation variable for the interaction term $u(k)$.

MMOP2$_i$' can be chosen as a linear quadratic approximation of the ROP$_i$ where there are standard procedures for its solution(Lewis and Syrmos, 1995, Bryson and Ho, 1980). This provides computational advantage. Assume now

$q_i(x_i(k), c_i(k), u_i(k), c_i(k)), \gamma_i(k)) =$

$$\frac{1}{2}x_i(k)^T Q_i x_i(k) + \frac{1}{2}c_i(k)^T R_i c_i(k) + \frac{1}{2}u_i(k)^T S_i u_i(k) + \gamma_i(k)$$

$f_i(x_i(k), c_i(k), u_i(k), \alpha_i(k)) =$

$$A_i x_i(k) + B_i c_i(k) + D_i u_i(k) + \alpha_i(k)$$

$F_i(x_i(k), c_i(k), u_i(k), \theta_i(k)) =$

$$\mathcal{J}_i x_i(k) + \mathcal{K}_i c_i(k) + \mathcal{L}_i u_i(k) + \theta_i(k)$$

$$\phi_i(\bar{x}_i(N)) = \frac{1}{2}x_i(N)^T \Phi_i x_i(N); \qquad k \in [0,N] \tag{6.1}$$

where $\Phi_i \geq 0, Q_i \geq 0, R_i > 0$ and $S_i \geq 0$ are weighting matrices of appropriate dimensions, A_i, B_i and D_i are matrices which represent a linear model of $f_i^*(.)$, $\mathcal{J}_i, \mathcal{K}_i$ and $\mathcal{L}_i$ are matrices which represent a linear model of the output function $F_i(.)$. The corresponding augmented Hamiltonian function is:

$$\mathrm{H}(.) = \frac{1}{2}x_i(k)^T Q_i x_i(k) + \frac{1}{2}c_i(k)^T R_i c_i(k) + \frac{1}{2}u_i(k)^T S_i u_i(k) + \gamma_i(k)$$

$$+ p_i(k+1)^T [A_i x_i(k) + B_i c_i(k) + D_i u_i(k) + \alpha_i(k)]$$

$$+ l_i(k)^T u_i(k) - \sum_{j=1}^{N_S} l_j^T(k) H_{ji} [\mathcal{J}_i x_i(k) + \mathcal{K}_i c_i(k) + \mathcal{L}_i u_i(k) + \theta_i(k)]$$

$$-\lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)$$

$$+\frac{1}{2} r_{1i} \|c_i(k) - v_i(k)\|^2 + \frac{1}{2} r_{2i} \|x_i(k) - z_i(k)\|^2, \quad \text{k}\in[0,\text{N-1}] \tag{6.2}$$

Applying the model based optimality conditions (3.63), (3.64) and (3.65) and (3.72) for the ith subsystem, we obtain the control law:

$$c_i(k) = -\overline{R}_i^{-1}(B_i^T p_i(k+1) - \overline{\beta}_i(k)); \qquad k \in [0, N-1] \tag{6.3}$$

the optimum interaction vector:

$$u_i(k) = -S_i^{-1}(D_i^T p_i(k+1) + \bar{l}_i(k)); \qquad k \in [0, N-1] \tag{6.4}$$

where

$$\bar{l}_i(k) = l_i(k) + \mathcal{L}_i^T \sum_{j=1}^{N_s} H_{ji}^T l_j(k) \tag{6.5}$$

and, in addition ,the following two-point boundary value problem (TPBVP):

$$x_i(k+1) = A_i x_i(k) - B_i \overline{R}_i^{-1}(B_i^T p_i(k+1) - \overline{\beta}_i(k))$$

$$-D_i \overline{S}_i^{-1}(D_i^T p_i(k+1) + \bar{l}_i(k)) + \alpha_i(k) \tag{6.6}$$

$$p_i(k) = \overline{Q}_i x_i(k) + A_i^T p_i(k+1) - \lambda_i(k) \tag{6.7}$$

with boundary conditions:

$$x_i(0) = x_{io}; \qquad x_{i,t}(N) = 0; \quad t \in [1, q_i]$$

$$\overline{p}_i(N) = \Phi_i \overline{x}_i(N) \tag{6.8}$$

where the augmented weighting matrices $\overline{R}_i$ and $\overline{Q}_i$ are given by:

$$\overline{R}_i = \overline{R}_i + r_{1i} I_{m_i} \tag{6.9}$$

$$\overline{Q}_i = Q_i + r_{2i} I_{n_i} \tag{6.10}$$

and the augmented multipliers $\beta_i(k)$ and $\lambda_i(k)$ are expressed as:

$$\overline{\beta}_i(k) = \beta_i(k) + \mathcal{K}_i^T \sum_{j=1}^{N_s} H_{ji}^T l_j(k) + r_{1i} v_i(k) \tag{6.11}$$

$$\overline{\lambda}_i(k) = \lambda_i(k) + \mathcal{J}_i^T \sum_{j=1}^{N_s} H_{ji}^T l_j(k) + r_{2i} z_i(k) \tag{6.12}$$

Using equations (3.81) and (3.82) the linear quadratic formulation enables the multipliers $\beta_i(k)$ and $\lambda_i(k)$, k∈[0,N-1] to be written as

$$\lambda_i(k) = -\left[\frac{\partial f_i^*(.)}{\partial z_i(k)} - A_i\right]^T \hat{p}_i(k+1) - \left[\frac{\partial q_i^T(.)}{\partial z_i(k)} - Q_i z_i(k)\right]^T + \left[\frac{\partial K_i^*(.)}{\partial z_i(k)} - \mathcal{J}_i\right]^T H_i^T l_i(k)$$

$$\beta_i(k) = -\left[\frac{\partial f_i^*(.)}{\partial v_i(k)} - B_i\right]^T \hat{p}_i(k+1) - \left[\frac{\partial q_i^T(.)}{\partial v_i(k)} - R_i v_i(k)\right]^T + \left[\frac{\partial K_i^*(.)}{\partial v_i(k)} - \mathcal{K}_i\right]^T H_i^T l_i(k)$$

k∈[0,N-1] (6.13)

The calculation of parameters $\alpha_i(k)$ and $\theta_i(k)$ are given by equation (3.83), that is;

$$\alpha_i(k) = f_i^*(z_i(k), v_i(k), H_i K^*(x_i(k), c_i(k))) - A_i z_i(k) - B_i v_i(k) - D_i H_i K^*(x_i(k), c_i(k)) \quad (6.14)$$

$$\theta_i(k) = K_i^*(z_i(k), v_i(k)) - \mathcal{J}_i z_i(k) - \mathcal{K}_i v_i(k) - \mathcal{L}_i H_i K^*(x_i(k), c_i(k));$$

k∈[0,N-1] (6.15)

Note that it is not necessary to calculate $\gamma_i(k)$. Notice also (6.14) is the main difference between the current and the previous structures (see fig. 3.5.1 and fig. 3.4.1 for comparison). Here the interaction term in the parameter estimation step is calculated directly from the real output function $y(k) = K^*(x(k), c(k))$ instead of its model.

The price multiplier vector remains as in the previous structure and is computed using the formula defined by (3.8), that is

$$l_i^{s+1}(k+1) = l_i^s(k) + \varepsilon_l[u_i(k) - \sum_{j=1}^{N_s} H_{ij}(\mathcal{J}_j x_j(k) + \mathcal{K}_j c_j(k) + \mathcal{L}_j u_j(k) + \theta_j(k))] \quad (6.16)$$

It is observed that the structure of TPBVP described (6.6) and (6.7) with boundary conditions (6.8) is identical to that of TPBVP given by (4.4) and (4.7). However (6.6) and (6.7) have no terms of $w_i(k)$ and $\zeta_i(k)$, the parameters $\alpha_i(k)$ and $\theta_i(k)$ are computed using real interaction measurements. Repeating the procedure described in

chapter 4, section 4.2, the solution of TPBVP can be obtained using the sweep method (Lewis and Syrmos, 1995, Bryson and Ho, 1975). The key is to assume the relationship between state and costate as

$$p_i(k) = V_i(k)x_i(k) + E_i(k)v_i + h_i(k); \qquad k \in [0, N]$$

where $V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}$; $E_i(N) = \begin{bmatrix} I_{q_i} \\ 0 \end{bmatrix}$; $h_i(N) = 0$

$$v_i = [p_{i,1}(N) \ldots . p_{i,q}(N)]^T \tag{6.17}$$

where $V_i(k)$ is an $n_i \times n_i$ matrix, $E_i(k)$ is $n_i \times q_i$ matrix , $h_i(k) \in \Re^{n_i}$ and $v_i \in \Re^{q_i}$. The procedure (see Appendix A for derivation) can be summarized as follows:

Procedure 6.2.1 : Solution of MMOP2$_i$ for structure with real interaction measurement in Parameter Estimation

Step a: From data solve backwards from $k = N$ to $k = 0$ the following set of difference equations.

$$V_i(k) = \overline{Q}_i + A_i V_i(k+1)[I_n + B_i \overline{R}_i B_i^T V_i(k+1) + D_i S_i D_i^T V_i(k+1)]^{-1} A_i$$

$$k \in [0, N-1], \qquad V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}; \tag{6.18}$$

$$E_i(k) = [A_i^T - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i S_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet (B_i \overline{R}_i^{-1} B_i + D_i S_i^{-1} D_i)] E_i(k+1)$$

$$k \in [0, N-1]; \qquad E_i(N) = \begin{bmatrix} I_{q_i} \\ 0 \end{bmatrix} \tag{6.19}$$

$$W_i(k) = W_i(k+1) - E_i(k+1)^T [I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i S_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet (B_i \overline{R}_i^{-1} B_i^T + D_i S_i^{-1} D_i^T)] E_i(k+1)$$

$$W(N) = 0; \qquad k \in [0, N-1] \tag{6.20}$$

Step b: Calculate $G_i(k)$ and $M_i(k)$ using the following equations:

$$G_i(k) = \overline{R}_i^{-1} B_i^T V_i(k+1)[I_n + B_i \overline{R}_i B_i^T V_i(k+1) + D_i S_i D_i^T V_i(k+1)]^{-1} A_i$$

$$M_i(k) = \overline{S}_i^{-1} D_i^T V_i(k+1)[I_n + B_i \overline{R}_i B_i^T V_i(k+1) + D_i \overline{S}_i D_i^T V_i(k+1)]^{-1} A_i \tag{6.21}$$

Step c: Solve the following difference equation backwards to obtain $h_i(k)$ and $\pi_i(k)$:

$$h_i(k) = [A_i^T - A_i^T V_i(k+1)[I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1) + D_i S_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$
$$(B_i \bar{R}_i^{-1} B_i + D_i S_i^{-1} D_i)]h_i(k+1) - \bar{\lambda}_i(k)$$
$$+A_i^T V_i(k+1)[I_n + B_i \bar{R}_i^T B_i^T V_i(k+1) + D_i S_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$
$$(B_i \bar{R}_i^{-1} \beta_i(k) - D_i S_i^{-1} \bar{l}_i(k) + \alpha_i(k));$$
$$\text{where} \quad h_i(N) = 0; \qquad k \in [0, N-1] \tag{6.22}$$

$$\pi_i(k) = \pi_i(k+1) + E_i(k+1)^T [I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1) + D_i S_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$
$$[-(B_i \bar{R}_i^{-1} B_i + D_i \bar{S}_i^{-1} D_i)]h_i(k+1) + B_i \bar{R}_i^T \beta_i(k) - D_i S_i^{-1} \bar{l}_i(k) + \alpha_i(k)]$$
$$\text{where } \pi_i(N) = 0; \qquad k \in [0, N-1] \tag{6.23}$$

Step d: Calculate $\tilde{p}_i(N)$, $g_i(k)$ and $m_i(k)$ from the following equations:

$$\tilde{p}_i(N) = v_i = -W_i(0)[E_i(0)^T x_{io} + \pi_i(0)]$$

$$g_i(k) = -\bar{R}_i^{-1} B_i^T V_i(k+1)[I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1) + D_i S_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$
$$[B_i \bar{R}_i^{-1} \beta_i(k) - D_i S_i^{-1} \bar{l}_i(k)) + \alpha_i(k)$$
$$-(B_i \bar{R}_i^{-1} B_i^T + D_i S_i^{-1} D_i^T)](E_i(k+1)v + h_i(k+1))$$
$$+E_i(k+1)\tilde{p}_i(N) + h_i(k+1)] + \bar{R}_i^{-1} \beta_i(k) \qquad k \in [0, N-1]$$
$$\tag{6.24}$$

$$m_i(k) = -S_i^{-1} D_i^T V_i(k+1)[I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1) + D_i S_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$
$$[(B_i \bar{R}_i^{-1} \beta_i(k) - D_i S_i^{-1} \bar{l}_i(k)) + \alpha_i(k)$$
$$-(B_i \bar{R}_i^{-1} B_i^T + D_i S_i^{-1} D_i^T)](E_i(k+1)v + h_i(k+1))$$
$$+E_i(k+1)\tilde{p}_i(N) + h_i(k+1)] - \bar{S}_i^{-1} \bar{l}_i(k)) \qquad k \in [0, N-1]$$
$$\tag{6.25}$$

Step e: Solve the following equation to obtain the state $x_i(k)$, $k \in [0, N]$

$$x_i(k+1) = (A_i - B_i G_i(k) - D_i M_i(k))x_i(k) + B_i g_i(k) + D_i m_i(k) + \alpha_i(k);$$
$$x_i(0) = x_{io}, \qquad x_i(N) = 0; \; t \in [1, q]; \quad k \in [0, N-1] \tag{6.26}$$

Step f: Calculate the costate $p_i(k)$, $k \in [0, N]$ from:

$$p_i(k) = V_i(k)x_i(k) + E_i(k)v_i + h_i(k); \qquad k \in [0, N] \tag{6.27}$$

Step g: Calculate the new control $c_i(k)$ and the new interaction $u_i(k)$, $k \in [0, N-1]$ from the following equations:

$$c_i(k) = -G_i(k)x_i(k) + g_i(k) \tag{6.28}$$

$$u_i(k) = -M_i(k)x_i(k) + m_i(k) \tag{6.29}$$

The previous analysis enables us to formulate the following two algorithms using the single loop and double loop techniques.

6.2.1.1 Single Loop Technique PC3

A single loop technique is obtained by iterating all the coordinating variables in equations (6.13) and (6.16) in the upper level simultaneously with the local optimization units defined by $MMOP2_i$ in the lower level. The lower level is made up of N_s local optimization units, each of which contains the $MMOP2_i$ unit, the parameter estimation unit and the variable update unit. Each local optimization unit is independent of other local optimization units, thus making this algorithm suitable for application of parallel processing.

In comparison to algorithm PC1 (for hierarchical structures with model based interaction input), the present algorithm has less variables to be iterated since there are no interaction separation variable $w_i(k)$ and the interaction modifier vector $\zeta_i(k)$. This is achieved at the expense of including the real interaction input in the parameter estimation step. Simulation examples in section 6.4 will show that the algorithm can have faster speed of convergence in comparison to algorithm PC1 described in Chapter 4.

The single loop algorithm can be summarized as follows:

Algorithm 6.2.1.1: Single Loop Price Coordination Hierarchical DISOPE for systems real interaction input in parameter estimation.

Data A_i, B_i, D_i, Q_i, R_i, S_i, x_{i0}, Φ_i, N, N_s, r_{1i}, r_{2i}, ϵ_c, ϵ_x, ϵ_p, ϵ_l and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Step 0. At level 2, choose the initial values of coordinating variables $\lambda_i^0(k), \beta_i^0(k)$ and $l_i^0(k)$. At level 1, let $\alpha_i^0(k) = \theta_i^0(k) = 0$, compute a nominal solution (using procedure 6.2.1) or choose a nominal solution for the ith local optimal control problem. Set iteration number $s=0$ and $v_i^0(k) = c_i^0(k)$, k∈[0.N-1]. $z_i^0(k) = x_i^0(k)$, $\bar{p}_i^0(k) = p_i^0(k)$, k∈[0,N] and $u_i^0(k) = H_i F(z_i^0(k), v_i^0(k))$ k∈[0,N-1] . Send them to level 2.

Step 1: At level two, calculate coordinating variables $\lambda_i^s(k)$, $\beta_i^s(k)$ and $l_i^s(k)$, k∈[1,N] using equations (6.13) and (6.16). Augment them according to (6.11), (6.12) and (6.5). Send them to level 1.

Step 2: At level 1, calculate $\alpha_i^s(k)$ and $\theta_i^s(k)$ from (6.14) and (6.15). Send them to the ith local optimal control problem and to level 1. This is called the parameter estimation step.

Step 3: At level 1, solve the MMOP2$_i$ using procedure 6.2.1. Note that if s=0, perform Step a in procedure 6.2.1, store the results for use in subsequent iterations. Otherwise start at Step b in the same procedure.

Step 4: At level 1, update the variables

$$z_i^{s+1}(k) = z_i^s(k) + \varepsilon_z(x_i^s(k) - z_i^s(k)),\ k \in [0, N]$$

$$v_i^{s+1}(k) = v_i^s(k) + \varepsilon_v(c_i^s(k) - v_i^s(k)),\ k \in [0, N-1]$$

$$\hat{p}_i^{s+1}(k) = \hat{p}_i^s(k) + \varepsilon_p(p_i^s(k) - \hat{p}_i^s(k)),\ k \in [0, N]$$

Send them to level 2.

Step 5: At level 2, convergence of coordinating variables is checked.

If $v^{s+1}(k) = v^s(k)$, $z^{s+1}(k) = z^s(k)$ and $u^{s+1}(k) = HF(z^s(k), c^s(k))$

. within a defined tolerance, stop, otherwise set s = s + 1 go to step 1. The whole process is repeated.

6.2.1.1 Double Loop Technique PC4

A similar reasoning which was given in Chapter 4, is utilized at present to develop a double loop price coordination algorithm PC4. The technique evolved from the need to take advantage of the global structure of the interconnection term defined by equation (3.73). The technique involves an iterative procedure of solving modifier equations (6.13) in an inner loop for a prescribed value of price $l(k)$. The task of the outer loop is to evaluate the global price $l(k)$ such that equation (3.73) is satisfied. This is equivalent to solving the following equation

$$u(l(k)) = HF(x(l(k)), c(l(k)), u(l(k)), \theta(x(l(k)), (c(l(k)), (u(l(k)))) \quad k \in [0, N-1] \tag{6.30}$$

given that $x(l(k))$, $c(l(k))$ and $u(l(k))$ are solutions of the inner loop problem under a prescribed *l(k)*. The linear quadratic model based problem $F(.)$ is defined by the appropriate equation in (6.1). The strategy for updating the price is given by a global version of (6.16) that is:

$$l^{s+1}(k+1) = l^{s}(k) + \varepsilon_l(u(k) - H(\mathcal{J}x(k) + \mathcal{K}c(k) + \mathcal{L}u(k) + \theta(k)) \tag{6.31}$$

The inner loop consists of two levels similar to the structure described in the Algorithm 6.2.1.1. On the upper level we have modifier equations as defined by (6.13). The lower level is made up of N_s local optimization units. Each local unit consists of a $MMOP2_i$ unit, a parameter estimation unit and a variable update unit. The local optimization unit can be solved independently ,thus making this technique suitable for application of parallel processing methods. The software implementation of the double loop technique is much easier because there is no need to separate the interacting components for each subsystem at the coordinator level. The resulting algorithm can have a faster speed of convergence over the equivalent double loop technique PC2 as illustrated by simulation examples in Section 6.4. This could be explained by the fewer number of iterating variables required in solving the hierarchical structure with real interaction input in parameter estimation.

The procedure for implementing the algorithm is summarized as follows:

Algorithm 6.2.1.2 : Double Loop Price Coordination Hierarchical DISOPE for systems real interaction input in parameter estimation.

Data: A_l, B_l, D_l, Q_l, R_l, S_l, x_{lo}, Φ_l, N, N_s, r_{1l}, r_{2l}, ϵ_c, ϵ_x, ϵ_p, ϵ_l and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Inner Loop:

Step 0. At level 2, choose the initial values of coordinating variables $\lambda_i^0(k)$ and $\beta_i^0(k)$. At level 1, let $\alpha_i^0(k) = \theta_i^0(k) = 0$, compute a nominal

solution (using procedure 6.2.1) or choose a nominal solution for the ith local optimal control problem. Set iteration number $s=0$ and $v_i^0(k)=c_i^0(k)$, k∈[0.N-1]. $z_i^0(k)=x_i^0(k)$, $\bar{p}_i^0(k)=p_i^0(k)$, k∈[0,N] and $u_i^0(k)=H_iF(z_i^0(k),v_i^0(k))$, k∈[0,N-1]. Send them to level 2.

Step 1. At level 2, calculate coordinating variables $\lambda_i^s(k)$ and $\beta_i^s(k)$, k∈[1,N] using equations (6.13) and (6.16). Augment them according to (6.11), (6.12) and (6.5). Send them to level 1.

Step 2. At level 1, calculate $\alpha_i^s(k)$ and $\theta_i^s(k)$ from (6.14) and (6.15). Send them to the ith local optimal control problem and to level 1. This is called the parameter estimation step.

Step 3. At level 1, solve the MMOP2$_i$ using procedure 6.2.1. Note that if s=0, perform Step a in procedure 6.2.1, and store the results for use in subsequent iterations. Otherwise start at Step b in the same procedure.

Step 4: At level 1, update the variables

$$z_i^{s+1}(k)=z_i^s(k)+\varepsilon_z(x_i^s(k)-z_i^s(k)),\ k\in[0,N]$$

$$v_i^{s+1}(k)=v_i^s(k)+\varepsilon_v(c_i^s(k)-v_i^s(k)),\ k\in[0,N-1]$$

$$\hat{p}_i^{s+1}(k)=\hat{p}_i^s(k)+\varepsilon_p(p_i^s(k)-\hat{p}_i^s(k)),\ k\in[0,N]$$

Send them to level 2.

Step 5: At level 2, convergence of coordinating variables is checked.

If $v^{s+1}(k)=v^s(k)$, $z^{s+1}(k)=z^s(k)$ and $u^{s+1}(k)=HF(z^s(k),c^s(k))$ within a defined tolerance, stop, otherwise set s = s + 1 and go to step 1. The whole process is then repeated.

Outer Loop:

Step 0: Set p = 0, choose a nominal value for $l^0(k)$. Send it to the inner loop

Step 1: Using prescribed global values of $x(k)$, $c(k)$, $u(k)$ and $\theta(k)$ from the inner loop calculate the price using equation (6.31). Send it to the inner loop.

Set p = p +1.

Step 2: Test for convergence using equation (6.30) . If within a defined tolerance, stop, otherwise repeat step 1.

6.2.2 DIRECT COORDINATION METHOD PROBLEM FORMULATION AND SOLUTION APPROACH

In this section we develop the hierarchical algorithms for solving EOP2 (see Chapter 3, Section 3.5) using methods derived directly from the optimality conditions. The approach we use is similar to that of the interaction-prediction approach (Mahmoud, Hassan and Darwish, 1985), however in the present problem a different and more general interaction structure is adopted. In this procedure the interaction price multiplier is calculated directly from the optimality condition and the interaction vector is computed using the interaction equation as given by optimality conditions (3.66) and (3.74) respectively. As a result , MMOP2$_i$' in Section 6.2.1 is now defined as a minimization problem with respect to the control vector only. Augmenting the performance index with variable convexification terms to aid convergence, MMOP2$_i$' is modified to the following equivalent augmented optimal control problem: MMOP2$_i$'':

$$\min_{c(k)} \mathbf{Q}_i = \phi(\bar{x}(N)) + \sum_{k=1}^{N-1} \{q_i(x_i(k), c_i(k), u_i(k), \gamma_i(k))$$

$$+ l_i(k)^T u_i(k) - \sum_{j=1}^{N_s} l_j^T(k) H_{ji} F_i(x_i(k), c_i(k), u_i(k), \theta_i(k))$$

$$- \lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)$$

$$+ \frac{1}{2} r_{1i} \|c_i(k) - v_i(k)\|^2 + \frac{1}{2} r_{2i} \|x_i(k) - z_i(k)\|^2 \}$$

subject to

$$x_i(k+1) = f_i(x_i(k), c_i(k), u_i(k), \alpha_i(k)); \quad k \in [1, N-1]$$

$$x_i(0) = x_{i0}$$

$$x_{i,t}(N) = 0; t \in [1, q]$$

$$\bar{x}_i(N) = [x_{i,q+1}(N), \ldots\ldots, x_{i,n}(N)]^T$$

where r_{1i} and r_{2i} and are given scalar convexification factors.

To solve MMOP2$_i$'' using hierarchical DISOPE techniques we choose the model to be a linear quadratic approximation of ROP for which there are standard procedures for its solution (Lewis and Syrmos, 1995). The parameters $\alpha_i(k)$, $\gamma_i(k)$ and $\theta_i(k)$, k$\in$[0,N-1] can be chosen as shift parameters.

Define

$$q_i(x_i(k),c_i(k),u_i(k),c_i(k)),\gamma_i(k))=$$

$$\frac{1}{2}x_i(k)^T Q_i x_i(k)+\frac{1}{2}c_i(k)^T R_i c_i(k)+\frac{1}{2}u_i(k)^T S_i u_i(k)+\gamma_i(k)$$

$$f_i(x_i(k),c_i(k),u_i(k),\alpha_i(k))=$$

$$A_i x_i(k)+B_i c_i(k)+D_i u_i(k)+\alpha_i(k)$$

$$F_i(x_i(k),c_i(k),u_i(k),\theta_i(k))=$$

$$\mathcal{J}_i x_i(k)+\mathcal{K}_i c_i(k)+\theta_i(k)$$

$$\phi_i(\bar{x}_i(N))=\frac{1}{2}x_i(N)^T \Phi_i x_i(N);\qquad \text{k}\in[0,\text{N}] \tag{6.32}$$

where $\Phi_i \geq 0, Q_i \geq 0, R_i > 0$ and $S_i \geq 0$ are weighting matrices of appropriate dimensions, A_i, B_i and D_i are matrices which represent a linear model of $f_i^*(.)$, $\mathcal{J}_i, \mathcal{K}_i$ and $\mathcal{L}_i$ are matrices which represent a linear model of the output function.

The corresponding augmented Hamiltonian function is:

$$\begin{aligned}\mathbf{H}(.)=&\frac{1}{2}x_i(k)^T Q_i x_i(k)+\frac{1}{2}c_i(k)^T R_i c_i(k)+\frac{1}{2}u_i(k)^T S_i u_i(k)+\gamma_i(k)\\&+p_i(k+1)^T[A_i x_i(k)+B_i c_i(k)+D_i u_i(k)+\alpha_i(k)]\\&+l_i(k)^T u_i(k)-\sum_{j=1}^{N_s} l_j^T(k)H_{ji}[\mathcal{J}_i x_i(k)+\mathcal{K}_i c_i(k)+\theta_i(k)]\\&-\lambda_i(k)^T x_i(k)-\beta_i(k)^T c_i(k)\\&+\frac{1}{2}r_{1i}\|c_i(k)-v_i(k)\|^2+\frac{1}{2}r_{2i}\|x_i(k)-z_i(k)\|^2,\ \text{k}\in[0,\text{N-1}]\end{aligned} \tag{6.33}$$

Using the model based optimality condition (3.64) for the ith subsystem, produces the control law:

$$c_i(k) = -\overline{R}_i^{-1}(B_i^T p_i(k+1) - \overline{\beta}_i(k)); \qquad k \in [0, N-1] \tag{6.34}$$

From optimality condition (3.66) , the interaction price vector $l_i(k)$ can be expressed as:

$$l_i(k) = -S_i u_i(k) - D_i^T p_i(k+1); \qquad k \in [0, N-1] \tag{6.35}$$

From optimality conditions (3.65) and (3.73),we obtain the following TPBVP:

$$x_i(k+1) = A_i x_i(k) - B_i \overline{R}_i^{-1}(B_i^T p_i(k+1) - \overline{\beta}_i(k)) + D_i u_i(k) + \alpha_i(k) \tag{6.36}$$

$$p_i(k) = \overline{Q}_i x_i(k) + A_i^T p_i(k+1) - \lambda_i(k) \tag{6.37}$$

with boundary conditions:

$$x_i(0) = x_{io}; \qquad x_{i,t}(N) = 0; \quad t \in [1, q_i] \tag{6.38}$$

$$\overline{p}_i(N) = \Phi_i \overline{x}_i(N)$$

In addition, we use optimality condition (3.74) to calculate the interaction vector $u_i(k)$ which can be written as:

$$u_i(k) = \sum_{j=1}^{N_s} H_{ij} F_j(x_i(k), c_i(k), u_i(k), \theta_i(k))$$

$$= \sum_{j=1}^{N_s} H_{ij}(\mathcal{J}_j x_j(k) + \mathcal{K}_j c_j(k) + \theta_j(k)); \quad k \in [0, N-1] \tag{6.39}$$

The augmented weighting matrices $\overline{R}_i$ and $\overline{Q}_i$ are given by:

$$\overline{R}_i = \overline{R}_i + r_{1i} I_{m_i} \tag{6.40}$$

$$\overline{Q}_i = Q_i + r_{2i} I_{n_i} \tag{6.41}$$

and the augmented multipliers $\overline{\beta}_i(k)$ and $\overline{\lambda}_i(k)$ are expressed as:

$$\overline{\beta}_i(k) = \beta_i(k) + \mathcal{K}_i^T \sum_{j=1}^{N_s} H_{ji}^T l_j(k) + r_{1i} v_i(k) \tag{6.42}$$

$$\overline{\lambda}_i(k) = \lambda_i(k) + \mathcal{J}_i^T \sum_{j=1}^{N_s} H_{ji}^T l_j(k) + r_{2i} z_i(k) \tag{6.43}$$

The computation of multipliers $\beta_i(k)$ and $\lambda_i(k)$ remains unchanged and is given by equation (6.13). The calculation of model-reality parameters $\alpha_i(k)$ and $\theta_i(k)$ are defined by equations (6.14) and (6.15) respectively.

The TPBVP defined by (6.36) ,(6.37) and (6.38) can be solved using Ricatti equation method (Bryson and Ho, 1975; Lewis and Syrmos, 1995). The key is to assume the relationship between state and costate as

$$p_i(k) = V_i(k)x_i(k) + E_i(k)\nu_i + h_i(k); \qquad k \in [0, N]$$

where $V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}$; $E_i(N) = \begin{bmatrix} I_{q_i} \\ 0 \end{bmatrix}$; $h_i(N) = 0$

$$\nu_i = [p_{i,1}(N) \ldots . p_{i,q}(N)]^T \tag{6.44}$$

where $V_i(k)$ is an $n_i \times n_i$ matrix, $E_i(k)$ is $n_i \times q_i$ matrix , $h_i(k) \in \Re^{n_i}$ and $\nu_i \in \Re^{q_i}$. The procedure (see Appendix B for derivation) can be summarized as follows:

Procedure 6.2.2 : Solution of MMOP2$_i$ for structure with real interaction measurement in Parameter Estimation_using direct coordination approach

---

Step a: From data solve backwards from $k = N$ to $k = 0$ the following set of difference equations.

$$V_i(k) = \overline{Q}_i + A_i V_i(k+1)[A_i - B_i G_i(k)]$$

$$k \in [0, N-1], \qquad V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}; \tag{6.45}$$

where $G_i(k) = [\overline{R}_i + B_i^T V_i(k+1) B_i]^{-1} B_i^T V_i(k+1) A_i$ (6.46)

$$E_i(k) = [A_i - B_i G_i(k)]^T E_i(k+1)$$

$$k \in [0, N-1]; \qquad E_i(N) = \begin{bmatrix} 0 \\ I_{q_i} \end{bmatrix} \tag{6.47}$$

$$W_i(k) = W_i(k+1) - E_i(k+1)^T [I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1)]^{-1}$$
$$B_i \overline{R}_i^{-1} B_i^T] E_i(k+1)$$
$$W(N) = 0; \qquad k \in [0, N-1] \tag{6.48}$$

Step b: Solve the following difference equation backwards to obtain $h_i(k)$ and $\pi_i(k)$ with prescribed interaction $u_i(k)$ from (6.39) :

$$h_i(k) = (A_i - B_i G_i(k))^T h_i(k+1) + (A_i - B_i G_i(k+1))^T V_i(k+1)(D_i u_i(k) + \alpha_i(k))$$
$$-\overline{\lambda}_i(k) + G_i(k)^T \overline{\beta}_i(k); \qquad h_i(N) = 0; \quad k \in [0, N-1] \tag{6.49}$$

$$\pi_i(k) = \pi_i(k+1) + E_i(k+1)^T [I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1)]^{-1} \bullet$$

$$[(B_i \overline{R}_i^T \beta_i(k) - B_i \overline{R}_i^{-1} B_i) h_i(k+1) + D_i u_i(k) + \alpha_i(k)]$$

where $\pi_i(N) = 0; \quad k \in [0, N-1]$ (6.50)

Step c: Calculate $\tilde{p}_i(N)$ and $g_i(k)$ from the following equations:

$$\tilde{p}_i(N) = \nu_i = -W_i(0)[E_i(0)^T x_{io} + \pi_i(0)] \tag{6.51}$$

$$g_i(k) = [\overline{R}_i + B_i^T V_i(k+1) B_i]^{-1}[-B_i^T V_i(k+1)(D_i u_i(k+1) + \alpha_i(k))$$
$$-B_i^T E_i(k+1)\tilde{p}_i(N) - B_i^T h_i(k+1) + \beta_i(k) \tag{6.52}$$

Step d: Solve the following equation to obtain the state $x_i(k)$, $k \in [0, N]$

$$x_i(k+1) = (A_i - B_i G_i(k)) x_i(k) + B_i g_i(k) + D_i u_i(k) + \alpha_i(k);$$
$$x_i(0) = x_{io}, \qquad x_i(N) = 0; \; t \in [1, q]; \quad k \in [0, N-1] \tag{6.53}$$

Step f: Calculate the costate $p_i(k)$, $k \in [0, N]$ from:

$$p_i(k) = V_i(k) x_i(k) + E_i(k) \nu_i + h_i(k); \qquad k \in [0, N] \tag{6.54}$$

Step g: Calculate the new control $c_i(k)$ $k \in [0, N-1]$ from the following equations:

$$c_i(k) = -G_i(k) x_i(k) + g_i(k) \tag{6.55}$$

The above analysis gives rise to the following two algorithms using the single loop and double loop techniques.

6.2.2.1 Single Loop Technique DC3

In this section a single loop implementation of a direct coordination algorithm for hierarchical systems with real input in parameter estimation is presented. The algorithm has the same structure as Algorithm 4.3.1 (DC1). The upper level is made up of equation (6.35) which calculates the interaction price vector $l_i(k)$, the computation of modifiers $\lambda_i(k)$ and $\beta_i(k)$ given by (6.13) and the calculation of interaction vector defined by (6.39). The lower level consists of N_s units of local optimization problems. Each of them is made up of a $MMOP2_i$ unit, a parameter estimation unit and a variable update unit. A single iterative unit is obtained by iterating the variables in the coordinator simultaneously with the local optimization units in the lower level. The algorithm has fewer variables to iterate when compare to the corresponding algorithm DC1 for systems with model based interaction input. As a result algorithm DC3 can have a better speed of convergence in comparison to DC1. This is illustrated by the simulation examples described in section 6.4.

Algorithm DC3 discussed above can be summarized as follows:

Algorithm 6.2.2.1: Single Loop Direct Coordination Hierarchical DISOPE for system with real interaction input in parameter estimation.

Data A_i, B_i, D_i, Q_i, R_i, S_i, x_{io}, Φ_i, N, N_s, r_{1i}, r_{2i}, ϵ_c, ϵ_x, ϵ_p and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Step 0: At level 2, choose the initial values of coordinating variables $\lambda_i^0(k), \beta_i^0(k)$ and $l_i^0(k)$. At level 1, let $\alpha_i^0(k) = \theta_i^0(k) = 0$, compute a nominal solution (using procedure 6.2.2) or choose a nominal solution for the ith local optimal control problem. Set iteration number $s=0$ and $v_i^0(k)=c_i^0(k)$, $k\in[0.N\text{-}1]$. $z_i^0(k) = x_i^0(k)$, $\bar{p}_i^0(k) = p_i^0(k)$, k∈[0,N] and $u_i^0(k) = H_i F(z_i^0(k), v_i^0(k))$,k∈[0,N-1] . Send them to level 2.

Step 1: At level 2, calculate interconnection price $l_i^s(k)$ from equation (6.35). Then compute modifiers β_i^s(k) and $\lambda_i^s(k)$ from equation (6.13). Augment them according to (6.42) and (6.43). Calculate the interaction vector u_i^s from equation (6.39). Send them to level 1.

Step 2: At level 1, calculate $\alpha_i^s(k)$ and $\theta_i^s(k)$ from equations (6.14) and (6.15). These are send to level 2 and the ith local optimal control unit.

Step 3: At level 1, solve the MMOP2$_i$ using procedure 6.2.2. Note that if s=0, perform Step a in procedure 6.2.2, store the results for use in subsequent iterations. Otherwise start at Step b in the same procedure.

Step 4: At level 1, update the variables

$$z_i^{s+1}(k) = z_i^s(k) + \varepsilon_z(x_i^s(k) - z_i^s(k)),\ k \in[0,N]$$

$$v_i^{s+1}(k) = v_i^s(k) + \varepsilon_v(c_i^s(k) - v_i^s(k)),\ k \in[0,N-1]$$

$$\hat{p}_i^{s+1}(k) = \hat{p}_i^s(k) + \varepsilon_p(p_i^s(k) - \hat{p}_i^s(k)),\ k \in[0,N]$$

Send them to level 2.

Step 5: At level 2, convergence of coordinating variables is checked.

If $v^{s+1}(k) = v^s(k)$, $z^{s+1}(k) = z^s(k)$ and $u^{s+1}(k) = HF(z^s(k), c^s(k))$

within a defined tolerance, stop, otherwise set s = s + 1 and go to step 1. The whole process is then repeated.

6.2.2.2 Double Loop Technique DC4

In this implementation, we utilize the global nature of the price and the interacting term by calculating them in an outer loop. These consists of calculation of the interaction term defined by a global version of (6.39), which can be expressed as

$$u(k) = H(\mathcal{J}x(k) + \mathcal{K}c(k) + \theta(k)) \qquad k \in [0, N-1] \tag{6.56}$$

where the global control $c(k)$, global state $x(k)$ and model reality θ(k) are provided by the inner loop. The interconnection price $l(k)$ is computed from the global equivalent of equation (6.35), that is

$$l(k) = -Su(k) - D^T p(k+1); \qquad k \in [0, N-1] \tag{6.57}$$

where $p(k)$ are given by the inner loop. In contrast to algorithm 4.3.2 (DC2), we do not have to include an updating formula for $u(k)$ because in this structure , the interaction term is calculated directly in (6.56) instead of its separation variable $w(k)$. As consequence, the outer loop iteration of the current hierarchical structure (EOP3) using the double loop technique with direct coordination approach is much simpler than that of algorithm DC2. This is illustrated in the simulation examples in Section 6.4, where a faster speed of convergence is observed.

The double loop technique described above can be summarized as follows:

Algorithm 6.2.2.2: Double Loop Direct Coordination Hierarchical DISOPE for system with real interaction input in parameter estimation.

Data: A_i, B_i, D_i, Q_i, R_i, S_i, x_{io}, Φ_i, N, N_s, r_{1i}, r_{2i}, ϵ_c, ϵ_x, ϵ_p and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Inner Loop

Step 0: At level 2, choose the initial values of coordinating variables $\lambda_i^0(k)$ and $\beta_i^0(k)$. At level 1, let $\alpha_i^0(k) = \theta_i^0(k) = 0$, compute (using

procedure 6.2.2) or choose a nominal solution for the ith local optimal control problem. Set iteration number $s = 0$ and $v_i^0(k) = c_i^0(k)$, k∈[0.N-1]. $z_i^0(k) = x_i^0(k)$ and $\bar{p}_i^0(k) = p_i^0(k)$, k∈[0,N]. Send them to level 2.

Step 1: At level two, calculate modifier vectors $\beta_i^s(k)$ and $\lambda_i^s(k)$, k∈[1,N] using equations (6.13) with $l_i^p(k)$ prescribed by the outer loop. Augment them according to (6.42) and (6.43). Send them to the local optimization problem in level 1.

Step 2: At level 1, calculate $\alpha_i^s(k)$ and $\theta_i^s(k)$ from (6.14) and (6.15). Send them to the ith local optimal control problem and to level 1. This is called the parameter estimation step.

Step 3: Perform the same process as in Step 3 of algorithm 6.2.2.1. Send the results to the outer loop.

Step 4: Perform Step 4-Step 5 of algorithm 6.2.2.1

Outer Loop

Step 0: Set p = 0, choose a nominal value for $l^0(k)$ and for $u^0(k)$. Send it to the inner loop.

Step 1: Calculate $u^p(k)$ from interaction equation (6.56) using prescribed values of $x^s(k)$, $c^s(k)$ and $\theta^s(k)$ from the inner loop. Then compute the price interaction $l^p(k)$ from (6.57).

Step 2: Test for convergence using equation (6.56) . If $u^{p+1}(k) = HF^p(.)$ within a defined tolerance, stop, otherwise repeat step 1.

Note that the inner loop and the outer loop run simultaneously.

6.2.3 OPTIMALITY OF ALGORITHMS FOR STRUCTURES WITH REAL INTERACTION INPUT IN PARAMETER ESTIMATION.

In this section we established the optimality of algorithms PC3, PC4, DC3 and DC4 which were derived and described in the previous sections. We use a similar approach utilized earlier in Chapter 5 by considering the global optimality conditions of each algorithm. This is done by showing that the conditions are equivalent to that of the global real optimal control problem defined by (3.10). The theorems are defined in conjunction with definition (3.1) which define the existence and uniqueness of the optimal solution of ROP and definition (3.2) which define the existence of the appropriate derivatives.

First we consider algorithms PC3 and PC4. At the global level, the optimality condition of the two algorithms are equivalent since they are derived by applying the same approach in a similar structure defined by EOP3 (see Chapter 3, section 3.5). Hence they can be considered jointly.

The following theorem establishes the optimality of PC3 and PC4.

Theorem 6.1:

Under Assumptions 3.1 and 3.2, and assuming convergence, the converged solution of the price coordination algorithms PC3 and PC4 with a linear model and quadratic performance index satisfies the optimality conditions of the global real optimal control problem defined by (3.10).

Proof:

The proof of is similar to that of theorem 5.1 and is described in Appendix B.

Using an identical reasoning as given previously, we can established the optimality of algorithm DC3 and DC4 jointly using the following theorem.

Theorem 6.2

Under Assumptions 3.1 and 3.2, and assuming convergence, the converged solution of the direct coordination algorithms DC3 and DC4 with a linear model and quadratic

performance index satisfies the optimality conditions of the global real optimal control problem defined by (3.10).

Proof:

The proof is similar to that of theorem 5.2 and is described in Appendix C.

6.3 ALGORITHMS FOR STRUCTURES WITH REAL INPUT IN INTERACTION AND PARAMETER ESTIMATION

In this section we derive the algorithms for optimal control of the hierarchical structure with real input in interaction and parameter estimation (see figure 3.6.1 in Chapter 3) using the price coordination and direct coordination approaches. The structure utilizes real output $y(k) = HK^*(x(k), c(k))$ as the interaction input instead of the model output function $y(k) = HF(x(k), c(k), u(k), \theta(k))$. As a result of this, there is no need to calculate the model reality parameter $\theta(k)$. The parameter estimation for $\alpha(k)$ remains as in Section 6.2. As in the previous section we will use the two approaches in deriving the algorithms. Implementing the algorithms using the single loop and double loop techniques will result in four algorithms which are denoted as PC5, PC6, DC5 and DC6.

Initially we will consider the price coordination approach.

6.3.1 PRICE COORDINATION METHOD PROBLEM FORMULATION AND SOLUTION APPROACH

Consider the optimal control problem defined by MMOP3$_i$ in Chapter 3, section 3.6. Augmenting the performance index with variable augmentation produces the following equivalent form MMOP3$_i$'.

MMOP3$_i$':

$$\min_{c(k),u(k)} Q_i = \phi(\bar{x}(N)) + \sum_{k=1}^{N-1} \{q_i(x_i(k), c_i(k), u_i(k), \gamma_i(k))$$

$$+ l_i(k)^T u_i(k) - \sum_{j=1}^{N_s} l_j^T H_{ji} K_i^*(x_i(k), c_i(k))$$

$$- \lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)$$

$$+\frac{1}{2}r_{1i}\|c_i(k)-v_i(k)\|^2+\frac{1}{2}r_{2i}\|x_i(k)-z_i(k)\|^2\}$$

subject to

$$x_i(k+1)=f_i(x_i(k),c_i(k),u_i(k),\alpha_i(k));\ \text{k}\in[1,\text{N-1}]$$

$$x_i(0)=x_{i0}$$

$$x_{i,t}(N)=0;t\in[1,q]$$

$$\bar{x}_i(N)=[x_{i,q+1}(N),.....,x_{i,n}(N)]^T$$

where r_{1i} and r_{2i} are given scalar convexification factors. Notice that at the end of the iterations, $c(k)=v(k)$ and $x(k)=z(k)$ so that at this stage the augmentation terms and their derivatives are zero, so having no effect in the real optimality of the solution. Assume now

$q_i(x_i(k),c_i(k),u_i(k),c_i(k)),\gamma_i(k))=$

$$\frac{1}{2}x_i(k)^T Q_i x_i(k)+\frac{1}{2}c_i(k)^T R_i c_i(k)+\frac{1}{2}u_i(k)^T S_i u_i(k)+\gamma_i(k)$$

$f_i(x_i(k),c_i(k),u_i(k),\alpha_i(k))=$

$$A_i x_i(k)+B_i c_i(k)+D_i u_i(k)+\alpha_i(k)$$

$$\phi_i(\bar{x}_i(N))=\frac{1}{2}x_i(N)^T\Phi_i x_i(N);\qquad \text{k}\in[0,\text{N}] \tag{6.58}$$

where $\Phi_i\geq 0, Q_i\geq 0, R_i>0$ and $S_i\geq 0$ are weighting matrices of appropriate dimensions, A_i, B_i and D_i are matrices which represent a linear model of $f_i^*(.)$. The parameters $\alpha_i(k)$, $\gamma_i(k)$ and $\theta_i(k)$, $\text{k}\in[0,\text{N-1}]$ can be chosen as shift parameters. Notice that in contrast to previous structures, the real output function $y(k)=K^*(x(k),c(k))$ is not approximated by a model. The corresponding Hamiltonian is defined as:

$$\mathbf{H}(.)=\frac{1}{2}x_i(k)^T Q_i x_i(k)+\frac{1}{2}c_i(k)^T R_i c_i(k)+\frac{1}{2}u_i(k)^T S_i u_i(k)+\gamma_i(k)$$

$$+p_i(k+1)^T[A_i x_i(k)+B_i c_i(k)+D_i u_i(k)+\alpha_i(k)]$$

$$+l_i(k)^T u_i(k)-\sum_{j=1}^{N_s} l_j^T(k)H_{ji}K_i^*(x_i(k),c_i(k))$$

$$-\lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)$$

$$+\frac{1}{2} r_{1i} \|c_i(k) - v_i(k)\|^2 + \frac{1}{2} r_{2i} \|x_i(k) - z_i(k)\|^2, \quad k \in [0, N-1] \tag{6.59}$$

Applying model based optimality conditions (3.88), (3.89), (3.90) and (3.96) for the ith subsystem, we obtain the control law:

$$c_i(k) = -\overline{R}_i^{-1}(B_i^T p_i(k+1) - \overline{\beta}_i(k)); \qquad k \in [0, N-1] \tag{6.60}$$

the estimated optimum interaction vector:

$$u_i(k) = -S_i^{-1}(D_i^T p_i(k+1) + l_i(k)); \qquad k \in [0, N-1] \tag{6.61}$$

and, in addition ,the following two-point boundary value problem (TPBVP):

$$x_i(k+1) = A_i x_i(k) - B_i \overline{R}_i^{-1}(B_i^T p_i(k+1) - \overline{\beta}_i(k))$$

$$-D_i \overline{S}_i^{-1}(D_i^T p_i(k+1) + l_i(k)) + \alpha_i(k) \tag{6.62}$$

$$p_i(k) = \overline{Q}_i x_i(k) + A_i^T p_i(k+1) - \overline{\lambda}(k) \tag{6.63}$$

with boundary conditions:

$$x_i(0) = x_{io}; \qquad x_{i,t}(N) = 0; \quad t \in [1, q_i]$$

$$\overline{p}_i(N) = \Phi_i \overline{x}_i(N) \tag{6.64}$$

where the augmented weighting matrices $\overline{R}_i$ and $\overline{Q}_i$ are given by:

$$\overline{R}_i = \overline{R}_i + r_{1i} I_{m_i} \tag{6.65}$$

$$\overline{Q}_i = Q_i + r_{2i} I_{n_i} \tag{6.66}$$

and the augmented multipliers $\overline{\beta}_i(k)$ and $\overline{\lambda}_i(k)$ are expressed as:

$$\overline{\beta}_i(k) = \beta_i(k) + \sum_{j=1}^{N_s} \frac{\partial^T K_i^*(x_i(k), c_i(k))}{\partial c_i(k)} H_{ji}^T l_j(k) + r_{1i} v_i(k) \tag{6.67}$$

$$\overline{\lambda}_i(k) = \lambda_i(k) + \sum_{j=1}^{N_s} \frac{\partial^T K_i^*(x_i(k), c_i(k))}{\partial x_i(k)} H_{ji}^T l_j(k) + r_{2i} z_i(k) \tag{6.68}$$

Notice that the augmentation contains derivatives $K_i^*(.)$ w.r.t. $x_i(k)$ and $c_i(k)$.

The modifiers are given by equations (3.104) and (3.105), which can be written as:

$$\lambda_i(k) = -\left[\frac{\partial f_i^*(.)}{\partial z_i(k)} - A_i\right]^T \hat{p}_i(k+1) - \left[\frac{\partial^T q_i^*(.)}{\partial z_i(k)} - Q_i z_i(k)\right]^T$$

$$\beta_i(k) = -\left[\frac{\partial f_i^*(.)}{\partial v_i(k)} - B_i\right]^T \hat{p}_i(k+1) - \left[\frac{\partial^T q_i^*(.)}{\partial v_i(k)} - R_i v_i(k)\right]^T$$

$$k \in [0, N-1] \tag{6.69}$$

Observe that the modifier equations are simpler in comparison to previous algorithms.

The calculation of parameters $\alpha_i(k)$ are given by equation (3.98), that is;

$$\begin{aligned}\alpha_i(k) = f_i^*(z_i(k), v_i(k), H_i K^*(x_i(k), c_i(k)) \\ - A_i z_i(k) - B_i v_i(k) - D_i H_i K^*(x_i(k), c_i(k))\end{aligned} \tag{6.70}$$

It is not necessary to calculate $\gamma_i(k)$

The interaction price is computed from (3.106), which can be expressed as

$$l_i^{s+1}(k+1) = l_i^s(k) + \varepsilon_l[u_i(k) - \sum_{j=1}^{N_s} H_{ij} K_j^*(x_j(k), c_j(k))] \tag{6.71}$$

It is observed that the structure of TPBVP described (6.62) and (6.63) with boundary conditions (6.64) is identical to that of TPBVP given by (6.6) and (6.7). The solution of the TPBVP is obtained using the sweep method. (Lewis and Syrmos, 1995; Bryson and Ho, 1975). The key is to assume the relationship between costate and state as

$$p_i(k) = V_i(k) x_i(k) + E_i(k) v_i + h_i(k); \qquad k \in [0, N]$$

where $$V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}; \; E_i(N) = \begin{bmatrix} I_{q_i} \\ 0 \end{bmatrix}; h_i(N) = 0$$

$$v_i = [p_{i,1}(N) .... p_{i,q}(N)]^T \tag{6.72}$$

where $V_i(k)$ is an $n_i \times n_i$ matrix, $E_i(k)$ is $n_i \times q_i$ matrix , $h_i(k) \in \Re^{n_i}$ and $v_i \in \Re^{q_i}$. Procedure 6.2.1 can be used to obtain the solution of MMOP3$_i$ with the following modifications.

1. The modifiers are calculated using (6.69) and are augmented using (6.67) and (6.68).
2. The interaction price multiplier is not augmented.

The above analysis leads to the formulation of algorithms PC5 and PC6.

6.3.1.1 Single Loop Technique PC5

Algorithm PC5 has a similar hierarchical structure to algorithm PC3. The main differences are a simpler equation structure for calculation of modifiers and the absence of model reality parameter $\theta_i(k)$.This is achieved at the cost of incorporating real measurements in parameter estimation and interaction input.

The single loop algorithm can be summarized as follows:

Algorithm 6.3.1.1: Single Loop Price Coordination Hierarchical DISOPE for systems with real input in interaction and parameter estimation.

Data: A_i, B_i, D_i, Q_i, R_i, S_i, x_{io}, Φ_i, N, N_s, r_{1i}, r_{2i}, ϵ_c, ϵ_x, ϵ_p, ϵ_l and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Step 0: At level 2, choose the initial values of coordinating variables $\lambda_i^0(k), \beta_i^0(k)$ and $l_i^0(k)$. At level 1, let $\alpha_i^0(k) = \theta_i^0(k) = 0$, choose or compute a nominal solution (using procedure 6.2.1) a nominal solution for the ith local optimal control problem. Set iteration number $s = 0$ and $v_i^0(k) = c_i^0(k)$, k∈[0.N-1].

$z_i^0(k) = x_i^0(k)$, $\bar{p}_i^0(k) = p_i^0(k)$, k∈[0,N] and $u_i^0(k) = 0$;. Send them to level 2.

Step 1: At level two, calculate coordinating variables $\lambda_i^s(k)$, $\beta_i^s(k)$ and $l_i^s(k)$, k∈[1,N] using equations (6.69) and (6.71). Augment them according to (6.67) and (6.68). Send them to level 1.

Step 2: At level 1, calculate $\alpha_i^s(k)$ from (6.70) . Send them to the ith local optimal control problem and to level 1. This is called the parameter estimation step.

Step 3: At level 1, solve MMOP3$_i$ using the modified procedure 6.2.1. Note that if s=0, perform Step a in procedure 6.2.1, store the results for use in subsequent iterations. Otherwise start at Step b.

Step 4: At level 1, update the variables

$$z_i^{s+1}(k) = z_i^s(k) + \varepsilon_z(x_i^s(k) - z_i^s(k)), \quad k \in [0, N]$$

$$v_i^{s+1}(k) = v_i^s(k) + \varepsilon_v(c_i^s(k) - v_i^s(k)) \ , \ k \in [0, N-1]$$

$$\hat{p}_i^{s+1}(k) = \hat{p}_i^s(k) + \varepsilon_p(p_i^s(k) - \hat{p}_i^s(k)) \ , \ k \in [0, N]$$

Send them to level 2.

Step 5: At level 2, convergence of coordinating variables is checked. If $v^{s+1}(k) = v^s(k)$, $z^{s+1}(k) = z^s(k)$ and $u^{s+1}(k) = HK^*(z^s(k), c^s(k))$ within a defined tolerance, stop, otherwise set s = s + 1 go to step 1. The whole process is repeated.

6.3.1.2: Double Loop Technique PC6

Algorithm PC6 forms the same double loop structure as algorithm PC4. The inner loop is made up of a modifiers calculation in the upper level and local optimization units in the lower level. The composition of local optimization units is similar to algorithm PC4. The function of the outer loop is to evaluate the global price $l(k)$ such that the following equation is satisfied

$$u(l(k)) = HK^*(x(l(k)), c(l(k))) \quad k \in [0, N-1] \tag{6.73}$$

given that $x(l(k))$, $c(l(k))$ and $u(l(k))$ are solutions of the inner loop problem under a prescribed $l(k)$. The price updating mechanism is given as a global equivalent of (6.71) which can be expressed as

$$l^{p+1}(k+1) = l^p(k) + \varepsilon_l[u(k) - HK^*(x(k), c(k))] \tag{6.74}$$

The double loop technique PC6 can be summarized as follows:

Algorithm 6.3.1.2: Double Loop Price Coordination Hierarchical DISOPE for systems with real input in interaction and parameter estimation.

Data: A_i, B_i, D_i, Q_i, R_i, S_i, x_{io}, Φ_i, N, N_s, r_{1i}, r_{2i}, ϵ_c, ϵ_x, ϵ_p, ϵ_l and means for calculating $f_i^*(.), K_i^*(.)$ and $\vec{q_i^*}(.)$.

Inner Loop:

Step 0: At level 2, choose the initial values of coordinating variables

$\lambda_i^0(k)$ and $\beta_i^0(k)$. At level 1, let $\alpha_i^0(k) = \theta_i^0(k) = 0$, choose or compute a nominal solution (using procedure 6.2.1) for the ith local optimal control problem. Set iteration number $s = 0$ and $v_i^0(k) = c_i^0(k)$, k∈[0.N-1].

$z_i^0(k) = x_i^0(k)$, $\bar{p}_i^0(k) = p_i^0(k)$, k∈[0,N] and $u_i^0(k) = 0$ Send them to level 2.

Step 1: At level two, calculate coordinating variables $\lambda_i^s(k)$ and $\beta_i^s(k)$, k∈[1,N] using equations (6.69) and (6.71). Augment them according to (6.67) and (6.68). Send them to level 1.

Step 2: At level 1, calculate $\alpha_i^s(k)$ from (6.70) . Send it to the ith local optimal control problem and to level 1. This is called the parameter estimation step.

Step 3: Repeat step 3 of algorithm 6.3.1.1. Send the result to the outer loop.

Step 4: Repeat step 4 and 5 of algorithm 6.3.1.1.

Outer Loop:

Step 0: Set p = 0, choose a nominal value for $l^0(k)$. Send it to the inner loop

Step 1: Using prescribed global values of $x(k)$, $c(k)$, $u(k)$ and $\theta(k)$ from the inner loop calculate the price using equation (6.74). Send it to the inner loop.
Set p = p +1.

Step 2: Test for convergence using equation (6.73) . If $l^{p+1}(k) = l^p(k)$ within a defined tolerance, stop, otherwise repeat step 1.

6.3.2 DIRECT COORDINATION APPROACH PROBLEM FORMULATION AND SOLUTION APPROACH

In this section, we develop algorithms DC5 and DC6 using direct coordination for the hierarchical structure defined by EOP3. As in all the direct coordination algorithms described previously , the procedure involves calculating the interaction price directly from optimality condition (3.90). The interaction term is calculated from optimality condition (3.97) which gives the interconnection equation. The procedure transforms MMOP3$_i$' in Section 6.3.1 into a minimization problem with respect to the control vector. In order to aid convergence the performance index in MMOP3$_i$ is augmented with variable convexification terms. Hence, MMOP3$_i$ becomes: MMOP3$_i$'':

$$\min_{c(k)} \mathbf{Q}_i = \phi(\bar{x}(N)) + \sum_{k=1}^{N-1} \{q_i(x_i(k), c_i(k), u_i(k), \gamma_i(k))$$

$$+ l_i(k)^T u_i(k) - \sum_{j=1}^{N_s} l_j^T(k) H_{ji} K_i^*(x_i(k), c_i(k))$$

$$- \lambda_i(k)^T x_i(k) - \beta_i(k)^T c_i(k)$$

$$+ \frac{1}{2} r_{1i} \| c_i(k) - v_i(k) \|^2 + \frac{1}{2} r_{2i} \| x_i(k) - z_i(k) \|^2 \}$$

subject to

$$x_i(k+1) = f_i(x_i(k), c_i(k), u_i(k), \alpha_i(k)); \ k \in [1, N-1]$$

$$x_i(0) = x_{i0}$$

$$x_{i,t}(N) = 0; t \in [1, q]$$

$$\bar{x}_i(N) = [x_{i,q+1}(N), \ldots\ldots, x_{i,n}(N)]^T$$

where r_{1i} and r_{2i} and are given scalar convexification factors.

To solve MMOP3$_i$'' using hierarchical DISOPE techniques we choose the model to be a linear quadratic approximation of ROP for which there are standard procedures for its solution (Lewis and Symos, 1995). The parameters $\alpha_i(k)$ and $\gamma_i(k)$, k$\in$[0,N-1] can be chosen as shift parameters.

Define

$$q_i(x_i(k),c_i(k),u_i(k),c_i(k)),\gamma_i(k))=$$

$$\frac{1}{2}x_i(k)^T Q_i x_i(k)+\frac{1}{2}c_i(k)^T R_i c_i(k)+\frac{1}{2}u_i(k)^T S_i u_i(k)+\gamma_i(k)$$

$$f_i(x_i(k),c_i(k),u_i(k),\alpha_i(k))=$$

$$A_i x_i(k)+B_i c_i(k)+D_i u_i(k)+\alpha_i(k)$$

$$\phi_i(\bar{x}_i(N))=\frac{1}{2}x_i(N)^T \Phi_i x_i(N); \qquad k\in[0,N] \tag{6.75}$$

where $\Phi_i \geq 0, Q_i \geq 0, R_i > 0$ and $S_i \geq 0$ are weighting matrices of appropriate dimensions, A_i, B_i and D_i are matrices which represent a linear model of $f_i^*(.)$. In this structure we utilize real output function $y(k)=K^*(x(k),c(k))$ instead of its model in the computation.

The corresponding augmented Hamiltonian function is:

$$\mathbf{H}(.)=\frac{1}{2}x_i(k)^T Q_i x_i(k)+\frac{1}{2}c_i(k)^T R_i c_i(k)+\frac{1}{2}u_i(k)^T S_i u_i(k)+\gamma_i(k)$$

$$+p_i(k+1)^T[A_i x_i(k)+B_i c_i(k)+D_i u_i(k)+\alpha_i(k)]$$

$$+l_i(k)^T u_i(k)-\sum_{j=1}^{N_s} l_j^T(k) H_{ji} K_i^*(x_i(k),c_i(k))$$

$$-\lambda_i(k)^T x_i(k)-\beta_i(k)^T c_i(k)$$

$$+\frac{1}{2}r_{1i}\|c_i(k)-v_i(k)\|^2+\frac{1}{2}r_{2i}\|x_i(k)-z_i(k)\|^2, \quad k\in[0,N\text{-}1] \tag{6.76}$$

Applying model based optimality conditions (3.87) for the ith subsystem, we obtain the control law:

$$c_i(k)=-\bar{R}_i^{-1}(B_i^T p_i(k+1)-\bar{\beta}_i(k)); \qquad k\in[0,N-1] \tag{6.77}$$

Interaction price $l_i(k)$ is given by optimality condition (3.89) which can written as:

$$l_i(k)=-S_i u_i(k)-D_i^T p_i(k+1); \qquad k\in[0,N-1] \tag{6.78}$$

To calculate the interaction vector we use optimality condition (3.96) , that is:

$$u_i(k)=\sum_{j=1}^{N_s} H_{ij} K_j^*(x_j(k),c_j(k)); \qquad k\in[0,N-1] \tag{6.79}$$

From optimality conditions (3.88) and (3.94), the following TPBVP is obtained:

$$x_i(k+1) = A_i x_i(k) - B_i \overline{R}_i^{-1}(B_i^T p_i(k+1) - \overline{\beta}_i(k)) + D_i u_i(k) + \alpha_i(k) \tag{6.80}$$

$$p_i(k) = \overline{Q}_i x_i(k) + A_i^T p_i(k+1) - \overline{\lambda}_i(k) \tag{6.81}$$

with boundary conditions:

$$x_i(0) = x_{io}; \qquad x_{i,t}(N) = 0; \quad t \in [1, q_i] \tag{6.82}$$

$$\overline{p}_i(N) = \Phi_i \overline{x}_i(N)$$

The augmented weighting matrices $\overline{R}_i$ and $\overline{Q}_i$ are given by:

$$\overline{R}_i = \overline{R}_i + r_{1i} I_{m_i} \tag{6.83}$$

$$\overline{Q}_i = Q_i + r_{2i} I_{n_i} \tag{6.84}$$

and the augmented multipliers $\overline{\beta}_i(k)$ and $\overline{\lambda}_i(k)$ are expressed as:

$$\overline{\beta}_i(k) = \beta_i(k) + \sum_{j=1}^{N_s} \frac{\partial^T K_i^*(x_i(k), c_i(k))}{\partial c_i(k)} H_{ji}^T l_j(k) + r_{1i} v_i(k) \tag{6.85}$$

$$\overline{\lambda}_i(k) = \lambda_i(k) + \sum_{j=1}^{N_s} \frac{\partial^T K_i^*(x_i(k), c_i(k))}{\partial x_i(k)} H_{ji}^T l_j(k) + r_{2i} z_i(k) \tag{6.86}$$

The modifiers are defined by optimality conditions (3.90) and (3.91), and are given by equation (6.69). The parameter estimation for $\alpha_i(k)$ remains the same as in the price coordination approach and is defined by equation (6.70).

The TPBVP defined by (6.80) ,(6.81) and (6.82) can be solved using a Ricatti equation method (Bryson and Ho, 1975; Lewis and Syrmos, 1995). The key is to assume the relationship between state and costate as

$$p_i(k) = V_i(k) x_i(k) + E_i(k) v_i + h_i(k); \qquad k \in [0, N]$$

where $V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}$; $E_i(N) = \begin{bmatrix} I_{q_i} \\ 0 \end{bmatrix}$; $h_i(N) = 0$

$$v_i = [p_{i,1}(N) \ldots . p_{i,q}(N)]^T \tag{6.87}$$

where $V_i(k)$ is an $n_i \times n_i$ matrix, $E_i(k)$ is $n_i \times q_i$ matrix , $h_i(k) \in \Re^{n_i}$ and $v_i \in \Re^{q_i}$. Procedure 6.2.2 can be used to obtain the solution of MMOP3$_i$ defined by TPBVP (6.80-6.82), with the following modifications:

a. The modifiers $\beta_i(k)$ and $\lambda_i(k)$ are computed using (6.69) and augmented according

to (6.85) and (6.86).

b. The interaction price $l_i(k)$ is not augmented.

c. The interaction vector $u_i(k)$ is computed from (6.79).

From the above analysis, we can derive direct coordination algorithms DC5 and DC6.

6.3.2.1 Single Loop Technique DC5

Algorithm DC5 has the same two level structure as described in algorithm DC3. However, it has a simpler formula for calculation of modifiers $\beta_i(k)$ and $\lambda_i(k)$. The parameter estimation is made simpler by having to compute a single model reality parameter, $\alpha_i(k)$. This is achieved at the cost of including real output measurements in the computation of the local optimal control and parameter estimation problems. A summary of single loop direct coordination algorithm DC5 is given below.

Algorithm 6.3.2.1: Single Loop Direct Coordination Hierarchical DISOPE for system with real input in interaction and parameter estimation measurements.

Data: A_i, B_i, D_i, Q_i, R_i, S_i, x_{io}, Φ_i, N, N_s, r_{1i}, r_{2i}, ϵ_c, ϵ_x, ϵ_p, ϵ_l and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Step 0: At level 2, choose the initial values of coordinating variables $\lambda_i^0(k), \beta_i^0(k)$ and $l_i^0(k)$. At level 1, let $\alpha_i^0(k) = 0$, choose or compute a nominal solution (using procedure 6.2.2) for the ith local optimal control problem. Set iteration number $s = 0$ and $v_i^0(k) = c_i^0(k)$, k∈[0.N-1].

$z_i^0(k) = x_i^0(k)$, $\bar{p}_i^0(k) = p_i^0(k)$, k∈[0,N] and $u_i^0(k) = 0$;. Send them to level 2.

Step 1: At level two, calculate coordinating variables $\lambda_i^s(k)$, $\beta_i^s(k)$ and $l_i^s(k)$, k∈[1,N] using equations (6.69) and (6.78). Augment the modifiers according to (6.85) and (6.86). Send them to level 1.

Step 2: At level 1, calculate $\alpha_i^s(k)$ from (6.70) . Send them to the ith local optimal

control problem and to level 1. This is called the parameter estimation step.

Step 3: At level 1, solve $MMOP3_i$ using the modified procedure 6.2.2. Note that if s=0, perform Step a in procedure 6.2.2, store the results for use in subsequent iterations. Otherwise start at Step b.

Step 4: At level 1, update the variables

$$z_i^{s+1}(k) = z_i^s(k) + \varepsilon_z(x_i^s(k) - z_i^s(k)),\ \ k \in [0, N]$$

$$v_i^{s+1}(k) = v_i^s(k) + \varepsilon_v(c_i^s(k) - v_i^s(k))\ ,\ \ k \in [0, N-1]$$

$$\hat{p}_i^{s+1}(k) = \hat{p}_i^s(k) + \varepsilon_p(p_i^s(k) - \hat{p}_i^s(k))\ ,\ \ k \in [0, N]$$

Send them to level 2.

Step 5: At level 2, convergence of coordinating variables is checked. If

$$v^{s+1}(k) = v^s(k),\ \ z^{s+1}(k) = z^s(k) \text{ and } u^{s+1}(k) = HK^*(z^s(k), c^s(k))$$

within a defined tolerance, stop, otherwise set s = s + 1 go to step 1. The whole process is repeated.

6.3.2.1 Double Loop Technique DC6

In the double loop implementation we use the global nature of the interaction price and the interaction equation by computing them in the outer loop. These made up of the global version of interconnection equation (6.79) which can be expressed as:

$$u(k) = HK^*(x(k), c(k)), \qquad k \in [0, N-1] \tag{6.88}$$

where the global control $c(k)$ and global state $x(k)$ are provided by the inner loop. The interconnection price $l(k)$ is computed from the global equivalent of equation (6.78), that is

$$l(k) = -Su(k) - D^T p(k+1); \qquad k \in [0, N-1] \tag{6.89}$$

where $p(k)$ is prescribed by the inner loop. Algorithm DC6 has all the advantages of DC5 plus a double loop structure which is much easier to implement. This is because we do not have to separate the individual components of the interacting variables for each subsystems.

The procedure described above can be summarized as follows:

Algorithm 6.3.2.2: Double Loop Direct Coordination Hierarchical DISOPE for system with real input in interaction and parameter estimation measurements.

Data: A_i, B_i, D_i, Q_i, R_i, S_i, x_{io}, Φ_i, N, N_s, r_{1l}, r_{2l}, ϵ_c, ϵ_x, ϵ_p, ϵ_l and means for calculating $f_i^*(.), K_i^*(.)$ and $q_i^*(.)$.

Inner Loop

Step 0: At level 2, choose the initial values of coordinating variables $\lambda_i^0(k)$ and $\beta_i^0(k)$. At level 1, let $\alpha_i^0(k) = 0$, choose or compute a nominal solution (using procedure 6.2.2) for the ith local optimal control problem. Set iteration number $s = 0$ and $v_i^0(k) = c_i^0(k)$, k∈[0.N-1].

$z_i^0(k) = x_i^0(k)$, $\bar{p}_i^0(k) = p_i^0(k)$, k∈[0,N] . Send them to level 2.

Step 1: At level two, calculate coordinating variables $\lambda_i^s(k)$ and $\beta_i^s(k)$, k∈[1,N] using equations (6.69) and (6.78). Augment the modifiers according to (6.85) and (6.86). Send them to level 1.

Step 2: At level 1, calculate $\alpha_i^s(k)$ from (6.70) . Send them to the ith local optimal control problem and to level 1. This is called the parameter estimation step.

Step 3: At level 1, solve the $MMOP3_i$ using the modified procedure 6.2.2. Note that if s=0, perform Step a in procedure 6.2.2, store the results for use in subsequent iterations. Otherwise start at Step b.

Step 4: Perform step 4 and step 5 of algorithm 6.3.2.1

Outer Loop

Step 0: Set p = 0, choose a nominal value for $l^0(k)$. Send it to the inner loop

Step 1: Calculate $u^p(k)$ from interaction equation (6.79) using prescribed values of $x^s(k)$, $c^s(k)$ and $p^s(k)$ from the inner loop. Then compute the price interaction $l^p(k)$ from (6.78).

Step 2: Test for convergence using equation (6.56) . If $l^{p+1}(k) = l^{p}(k)$ within a defined tolerance, stop, otherwise repeat step 1.

6.3.3 OPTIMALITY OF ALGORITHMS FOR STRUCTURES WITH REAL INPUT IN INTERACTION AND PARAMETER ESTIMATION.

The optimality of algorithms PC5, PC6, DC5 and DC6 will be established by the following theorems. We use a similar approach used earlier in section 6.2.4 by considering the global optimality conditions of each algorithm. The optimality is established by showing that the optimality conditions are equivalent to that of the global real optimal control problem defined by (3.10). The theorems are defined in conjunction with definition (3.1) which define the existence and uniqueness of the optimal solution of ROP and definition (3.2) which define the existence of the appropriate derivatives.

We initially consider algorithms PC5 and PC6. At the global level, the optimality condition of the two algorithms are essentially the same because they are derived by applying the same approach in a similar structure defined by EOP3 (see Chapter 3, section 3.5). Hence they can be considered jointly.

The following theorem establishes the optimality of PC5 and PC6.

Theorem 6.3:

Under Assumptions 3.1 and 3.2, and assuming convergence, the converged solution of the price coordination algorithm PC5 and PC6 with a linear model and quadratic performance index satisfies the optimality conditions of the global real optimal control problem defined by (3.10).

Proof:

The proof of is similar to that of theorem 5.1 and is described in Appendix D.

Applying a similar reasoning as given previously, we can establish the optimality of direct coordination algorithms DC5 and DC6 together in the following theorem.

Theorem 6.4

Under Assumptions 3.1 and 3.2, and assuming convergence, the converged solution of the price coordination algorithm DC3 and DC4 with a linear model and quadratic performance index satisfies the optimality conditions of the global real optimal control problem defined by (3.10).

Proof:

The proof is similar to that of theorem 5.2 and is described in Appendix E.

6.4 SIMULATION EXAMPLE

All eight algorithms described earlier in this chapter were implemented in C++ programming language using object oriented and modular techniques. In the implementation we make use of DMatrix class type and operators developed by Becerra (1995), which optimizes operation involving matrix structures.

The convergence of the appropriate vectors in each algorithm is verified by comparing the norm equation (4.62) with a given tolerance. Simulation will be carried out using the same examples described earlier in Chapter 4, section 4.5. This is done to enable us to compare the performance of all the algorithms in this chapter in relation to those presented in Chapter 4. To ascertain if the solution achieved at the end of each iteration is the correct optimal solution of the ROP, we test that the solution satisfies the optimality conditions of ROP as defined by equations (3.14), (3.15) and (3.16). Throughout the simulation it is assumed that the values of derivatives with respect to $v(k)$, and $z(k)$ of $q^*(.)$, $f^*(.)$ and $K^*(.)$ are available.

It should be noted that the choice of parameters settings in the examples is made by trial and error. The choices shown are by no means exhaustive. Settings which result in non convergence are not shown in the table of results. Only a sample of parameter settings for each algorithm is given for the purpose of illustration. The number of iterations used to compare the performance of the algorithms is defined as the number of times the global performance index is evaluated.

Example 6.4.1

Example 4.4.1. is repeated here. The example is made up of an interconnected system containing three nonlinear subsystems. The tolerance specified for convergence is set at ε_T=0.01. The ROP is as follows:

ROP:

$$\min_{u(k),c(k)} \frac{1}{2}\sum_{i=1}^{3}\sum_{k=0}^{51}[x_i(k)^T Q_i^* x_i(k)+c_i(k)^T R_i^* c_i(k)+u_i(k)^T S_i^* u_i(k)].$$

where

$$Q_1^* = diag(0.5,0.5);\ R_1^* = 0.1;\ S_1^* = diag(0.5,0.5)$$

$$Q_2^* = diag(0.5,0.5);\ R_2^* = 0.1;\ S_2^* = diag(0.5,0.5)$$

$$Q_3^* = diag(0.5,0.5,0.5);\ R_3^* = 0.1;\ S_3^* = diag(0.5,0.5,0.5)$$

subject to:

Subsystem 1:

$$\begin{aligned}
x_{11}(k+1) &= 0.125x_{11}(k)+0.005x_{12}(k)+0.0025u_{11}(k)+0.0125u_{12}(k)\\
&\quad +0.0025c_{11}(k)+0.125x_{11}(k)x_{12}(k)\\
x_{12}(k+1) &= 0.05x_{12}(k)-0.0125u_{11}(k)+0.005u_{12}(k)-0.0125c_{11}(k)\\
&\quad +0.025u_{11}(k)u_{12}(k)\\
y_{11}(k) &= x_{11}(k)+0.05c_{11}(k)+0.025x_{11}(k)c_{11}(k)\\
y_{21}(k) &= x_{21}(k)\\
x_{11}(0) &= 1.0 \quad x_{12}(0) = 0.8
\end{aligned}$$

Subsystem 2:

$$\begin{aligned}
x_{21}(k+1) &= -0.0625x_{21}(k)+0.0125u_{21}(k)+0.0125u_{22}(k)+0.005c_{11}(k)\\
&\quad +0.05x_{21}(k)^3\\
x_{22}(k+1) &= -0.0125x_{21}(k)+0.975x_{22}(k)+0.005u_{21}(k)+0.005c_{21}(k)\\
&\quad +0.025u_{21}(k)x_{22}(k)\\
y_{21}(k) &= x_{21}(k)+0.05c_{21}(k)\\
y_{22}(k) &= x_{22}(k)\\
x_{21}(0) &= 0.5\ , x_{22}(0) = 0.6
\end{aligned}$$

Subsystem 3:

$$x_{31}(k+1) = -0.00425u_{32}(k) + 0.975x_{31}(k) + 0.025x_{33}(k) + 0.025x_{31}(k)x_{33}(k)$$
$$x_{32}(k+1) = 0.0025u_{31}(k) + 0.025u_{33}(k) + 0.9875x_{32}(k)$$
$$x_{33}(k+1) = 0.01u_{31}(k) - 0.025u_{32}(k) - 0.025x_{31}(k) + 0.975x_{33}(k) + 0.0025c_{31}(k) + 0.005x_{31}(k)x_{32}(k)$$
$$y_{31}(k) = x_{31}(k) + 0.0125c_{31}(k) + \sin(x_{31}(k))$$
$$y_{32}(k) = x_{32}(k)$$
$$y_{33}(k) = x_{33}(k)$$
$$x_{31}(0) = 1.5 \quad x_{32}(k) = 1.0 \quad x_{33}(k) = 1.2$$

MOP:

$$\min_{u(k),c(k)} \frac{1}{2} \sum_{i=1}^{3} \sum_{k=0}^{51} [x_i(k)^T Q_i x_i(k) + c_i(k)^T R_i c_i(k) + u_i(k)^T S_i u_i(k) + \gamma_i(k)]$$

where

$$Q_1 = diag(0.4, 0.4) \; ; \quad R_1 = 0.08 \; ; \; S_1 = diag(0.4, 0.4)$$

$$Q_2 = diag(0.4, 0.4) \; ; \quad R_2 = 0.08 \; ; \; S_2 = diag(0.4, 0.4)$$

$$Q_3 = diag(0.4, 0.4, 0.4) \; ; \quad R_3 = 0.08 \; ; \; S_3 = diag(0.4, 0.4)$$

subject to:

Subsystem 1:

$$x_1(k+1) = \begin{bmatrix} 0.125 & 0.005 \\ 0 & 0.05 \end{bmatrix} x_1(k) + \begin{bmatrix} 0.0025 \\ -0.0125 \end{bmatrix} c_1(k) + \begin{bmatrix} 0.0025 & 0.0125 \\ -0.0125 & 0.005 \end{bmatrix} u_1(k) + \alpha_1(k)$$

$$y_1(k) = \theta_1(k)$$
$$x_{11}(0) = 1.0 \quad x_{12}(0) = 0.8$$

Subsystem 2:

$$x_2(k+1) = \begin{bmatrix} -0.625 & 0 \\ -0.0125 & 0.975 \end{bmatrix} x_2(k) + \begin{bmatrix} 0.005 \\ 0.005 \end{bmatrix} c_2(k) + \begin{bmatrix} 0.0125 & 0.0125 \\ 0 & 0.005 \end{bmatrix} u_2(k) + \alpha_2(k)$$
$$y_2(k) = \theta_2(k)$$
$$x_{21}(0) = 0.5 \; , x_{22}(0) = 0.6$$

Subsystem 3:

$$x_3(k+1)=\begin{bmatrix}0.975 & 0 & 0.025\\ 0 & 0.9875 & 0\\ -0.025 & 0 & 0.975\end{bmatrix}x_3(k)+\begin{bmatrix}0\\ 0\\ 0.0025\end{bmatrix}c_3(k)$$
$$+\begin{bmatrix}0 & -0.00425 & 0\\ 0.0025 & 0.0 & 0.025\\ 0.01 & -0.025 & 0\end{bmatrix}u_3(k)+\alpha_3(k)$$

$$y_3(k)=\theta_3(k)$$

$$x_{31}(0)=1.5,\ x_{32}(0)=1.0,\ x_{33}(0)=1.2$$

The interconnection matrix H is given as

$$H=\begin{bmatrix}0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\end{bmatrix}$$

To test the effectiveness of the hierarchical algorithms in dealing with model reality differences the above choice of MOP is made. The MOP plant dynamics is a linear approximation of the ROP. The MOP performance measure is deliberately chosen to be 80% of the ROP performance index. Notice that the MOP output function y(k) is assumed to be zero in the case of algorithm for systems with real input in parameter estimation (i.e algorithms PC3, PC4, DC3 and DC4).

Tables 6.4.1.1 shows the simulation results for algorithms PC3, PC4, DC3 and DC4. Simulation results of algorithms for systems with real input in interaction and parameter estimation (algorithms PC5, PC6, DC5 and DC6) are shown in table 6.4.1.2. The final states, final control signals and interaction vectors are similar to those given by figures 4.5.1.1-4.5.1.9, therefore they are not included among the figures shown on the following pages. Entries marked * indicate the 'best' choice among the samples.

Figure 6.4.1.1 compares the convergence behaviour in terms of performance index of algorithms PC3, PC4, DC3 and DC4. The convergence of the price coordination

algorithms, PC3 and PC4, shows an increasing pattern towards the final performance index value. This is because in the iterations, the interconnection constraint defined by optimality condition (3.96), is not satisfied until the price update mechanism (6.16) has sufficiently converged. This is in contrast with the direct coordination algorithms, where the interaction variable and the price are computed in the second level using values of variables prescribed in the previous iteration which results in satisfaction of interconnection constraint in each iteration. In general direct coordination algorithms performed better than price coordination algorithms in this example.

From table 6.4.1.1 it is observed that using convexification scalar r_2 improves the speed of convergence of algorithms PC3 and PC4. When ε_1=0.15 and r_2=2.0, algorithm PC3 converges after 17 iterations. The effect of setting r_2=2.0, is illustrated by figures 6.4.1.3 and 6.4.1.4. It is noted that changing the stepsize ε_v from 1.0 to 0.85 does not improve the convergence speed of algorithm PC3 further. However in the direct coordination algorithms DC3 , changing the stepsize from 1.0 to 0.85 improves the performance of the algorithm. From the table it can be seen that changing r_2=0.85 to 1.0 does not improve the convergence speed of algorithm DC4. Single loop algorithm DC3 converges faster than the double loop implementation DC4. This is due to the extra computation involved in the outer loop of DC4. However there is no significant difference in terms of convergence pattern of the norms of interaction, states and control signals. This property can be observed by comparing figures 6.4.1.5 and 6.4.1.6. This could be explained by the fact that the inner loop of DC4 use the same $MMOP_i$ solution structure as that of DC3.

We now consider table 6.4.1.2 which shows the performance of algorithms for structures with real input in interaction and parameter estimation. Algorithms based on direct coordination have better speed of convergence when compared to the price coordination algorithms. This is displayed by figure 6.4.1.7. The price coordination algorithms show similar increasing convergence pattern as illustrated by figure 6.4.1.1. The same reasoning can be applied to explain the behaviour shown by algorithms PC5 and PC6. The single loop implementations PC5 and DC5, converged faster than the double loop implementations PC6 and DC6 respectively. Figure 6.4.1.8 shows the convergence of PC6 in terms of the norms of control signal ,state and interaction

vectors. From the table it is observed that convexification scalar r_2 influences the convergence of the price coordination algorithm. This is shown by figure 6.4.1.9 which illustrates that changing r_2 from 0.0 to 1.0 and stepsize ε_v from 1.0 to 0.85 improves the convergence of the control norm of PC6. The control signal norm changes from an oscillating converging pattern to that of a monotonic one. Figure 6.4.1.10 shows the effect of changing the stepsize ε_v from 1.0 to 0.85 on the convergence of the control norm for algorithm DC5. Note that at ε_v=0.85 the norm of the control signal converges faster.

Figure 6.4.1.11 compares the convergence of algorithms PC4 and PC6. Note that PC6, which uses real input in interaction and parameter estimation, converged faster than PC4 which uses real input in parameter estimation. This could be explained by the extra computation required in PC4 because it has an extra model reality parameter, $\theta(k)$, in its structure. Figure 6.4.1.12 compares the convergence of the control norm of algorithm DC4 and DC6 using the same parameter settings. Note that the control signal norm of DC6 which uses real input in interaction and parameter estimation converged at a faster rate than DC4, even though they finally converged equally after 11 iterations. Comparing results in table 6.4.1.1 and 6.4.1.2, it can be concluded, for this example, that the price coordination algorithms for systems with real input in interaction and parameter estimation PC5, PC6 performed better than PC3 and PC4 (for systems with real input in parameter estimation). However no significant difference is shown in terms of convergence speed for the direct coordination algorithms DC3, DC4 ,DC5 and DC6. Comparing the results with the centralized DISOPE algorithm shows that in this example the direct coordination algorithm compares favourably. This is despite the fact that the computation in each subsystem is carried sequentially. In general, it is expected that application of parallel processing methods will result in a better speed of convergence in all of the algorithms discussed above.

Algorithm	ε_l	r_1	r_2	ε_v	Number of iterations	CPU (s)	Final Performance Index
price cor. single loop	0.15	0	0	1.0	20	85	121.4894
PC3	0.15	0	1.0	1.0	19	71	121.4819
	0.2	0	1.0	1.0	21	75	121.4955
*	0.15	0	2.0	1.0	17	65	121.4878
	0.15	0.5	2.0	1.0	27	98	121.4816
	0.15	0	2.0	0.85	17	64	121.4843
price cor. double loop	0.4	0	0	1.0	22	79	121.4929
PC4 *	0.4	0	1.0	1.0	19	70	121.4817
	0.4	1.0	1.0	1.0	44	150	121.4856
	0.3	0	1.0	1.0	26	92	121.4901
	0.15	0	1.0	0.8	51	171	121.4773
direct cor. single loop	N/A	0	0	1.0	9	48	121.4885
DC3		1.0	0	1.0	44	170	121.4975
		0	1.0	1.0	9	38	121.4960
*		0	1	0.85	8	35	121.4951
		0	0	0.85	9	38	121.4854
direct cor. double loop	N/A	0	0	0.85	11	47	121.4841
DC4		0	0	1.0	11	48	121.4821
		0	1.0	1.0	44	169	121.4932
		0	1.0	1.0	11	48	121.4874
*		0	1.0	0.85	11	45	121.4974
Centralize DISOPE (Centralized Problem)	N/A	0	0	1.0	6	32	121.5032

Table 6.4.1.1 Algorithm performance for Example 6.4.1
(for systems with real interaction input in parameter estimation)

Algorithm	ε_l	r_1	r_2	ε_v	Number of iterations	CPU (s)	Final Performance Index
price cor. single loop	0.15	0	0	1.0	19	73	121.4878
PC5	0.1	0	0	1.0	26	97	121.4781
	0.15	0	1.0	1.0	16	58	121.4842
*	0.15	0	1.0	0.85	15	56	121.4854
	0.15	1.0	0	0.85	45	151	121.4876
	0.15	0	2.0	0.85	17	62	121.4953
price cor. double loop	0.4	0	0	1.0	19	73	121.5013
PC6	0.2	0	0	1.0	38	138	121.4953
	0.4	1.0	0	1.0	44	156	121.4823
	0.4	0	2.0	0.85	17	58	121.4962
*	0.4	0	1.0	0.85	16	56	121.4856
	0.3	0	1.0	0.85	26	96	121.4893
direct cor. single loop	N/A	0	0	1.0	10	40	121.5033
DC5		1.0	0	1.0	43	153	121.4943
		0.5	0.5	1.0	26	96	121.4879
		0	0	0.8	9	33	121.5012
		0	1.0	1.0	10	41	121.4988
*		0	0	0.85	9	33	121.4891
direct cor. double loop	N/A	0	0	1.0	11	43	121.4893
DC6		0	2.0	1.0	11	44	121.4970
		1.0	0	1.0	44	153	121.4873
		0.5	0	1.0	28	100	121.4870
*		0	0	0.85	10	34	121.4899
Centralize DISOPE (Centralized Problem)		0	0	1.0	6	32	121.5032

Table 6.4.1.2 Algorithms performance for Example 6.4.1
(for systems with real input in interaction and parameter estimation)

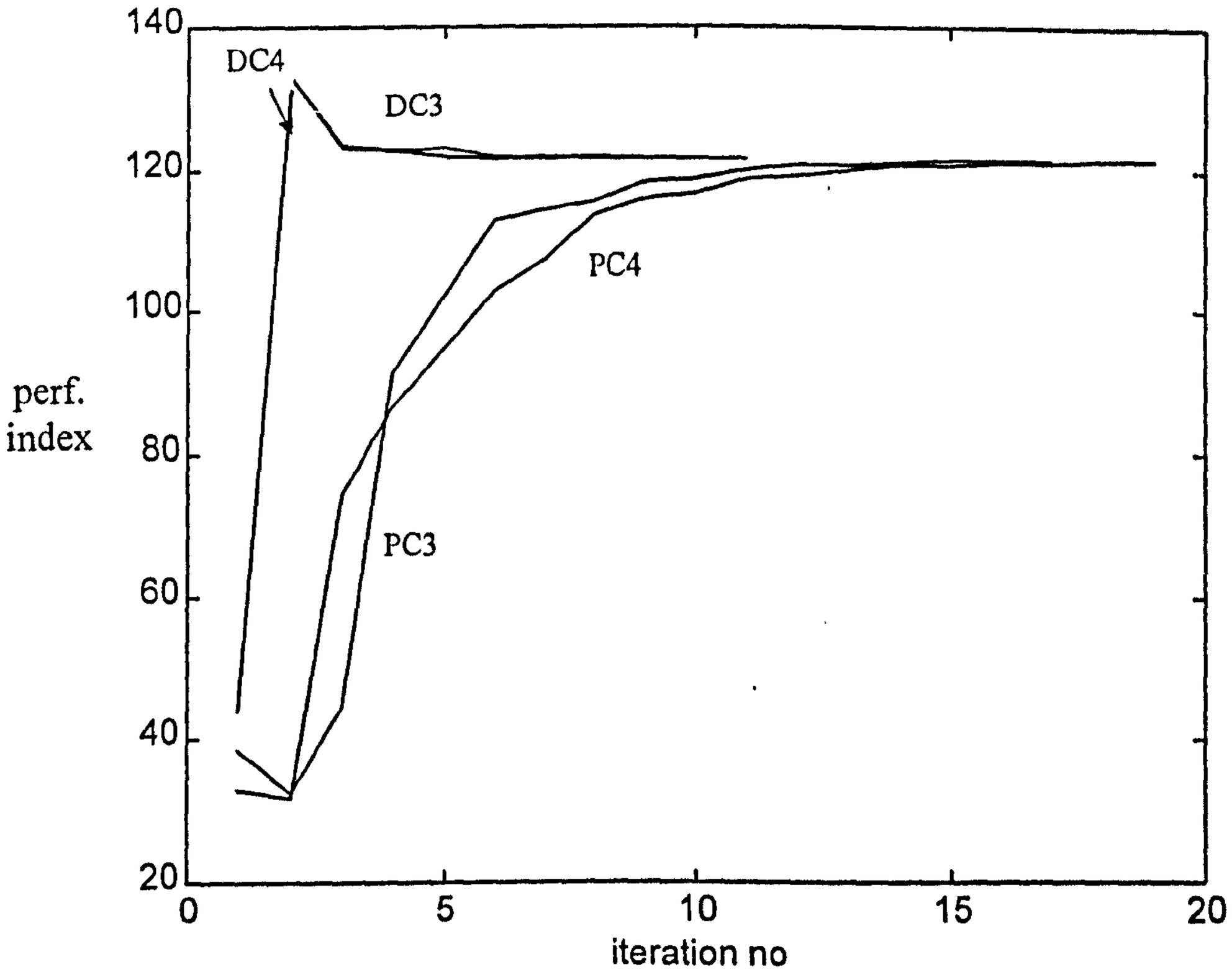

Fig. 6.4.1.1 Convergence of performance index for algorithms PC3, PC4, DC3 and DC4.

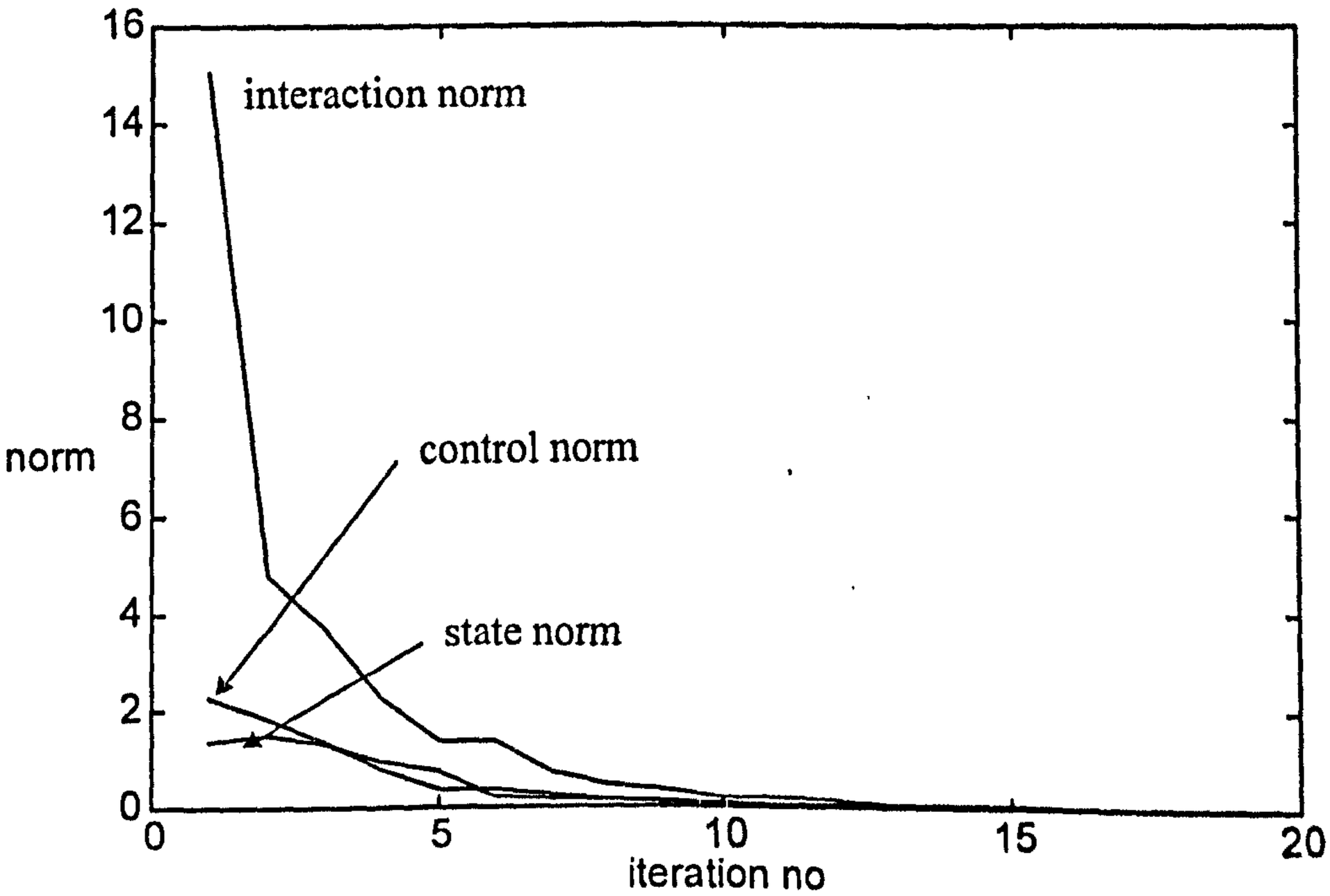

Fig.6.4.1.2 Convergence of state norm, control norm and interaction norm for algorithm PC4 with $\varepsilon_l = 0.4$, $r_1 = 0$, $r_2 = 1.0$, $\varepsilon_v = 1.0$

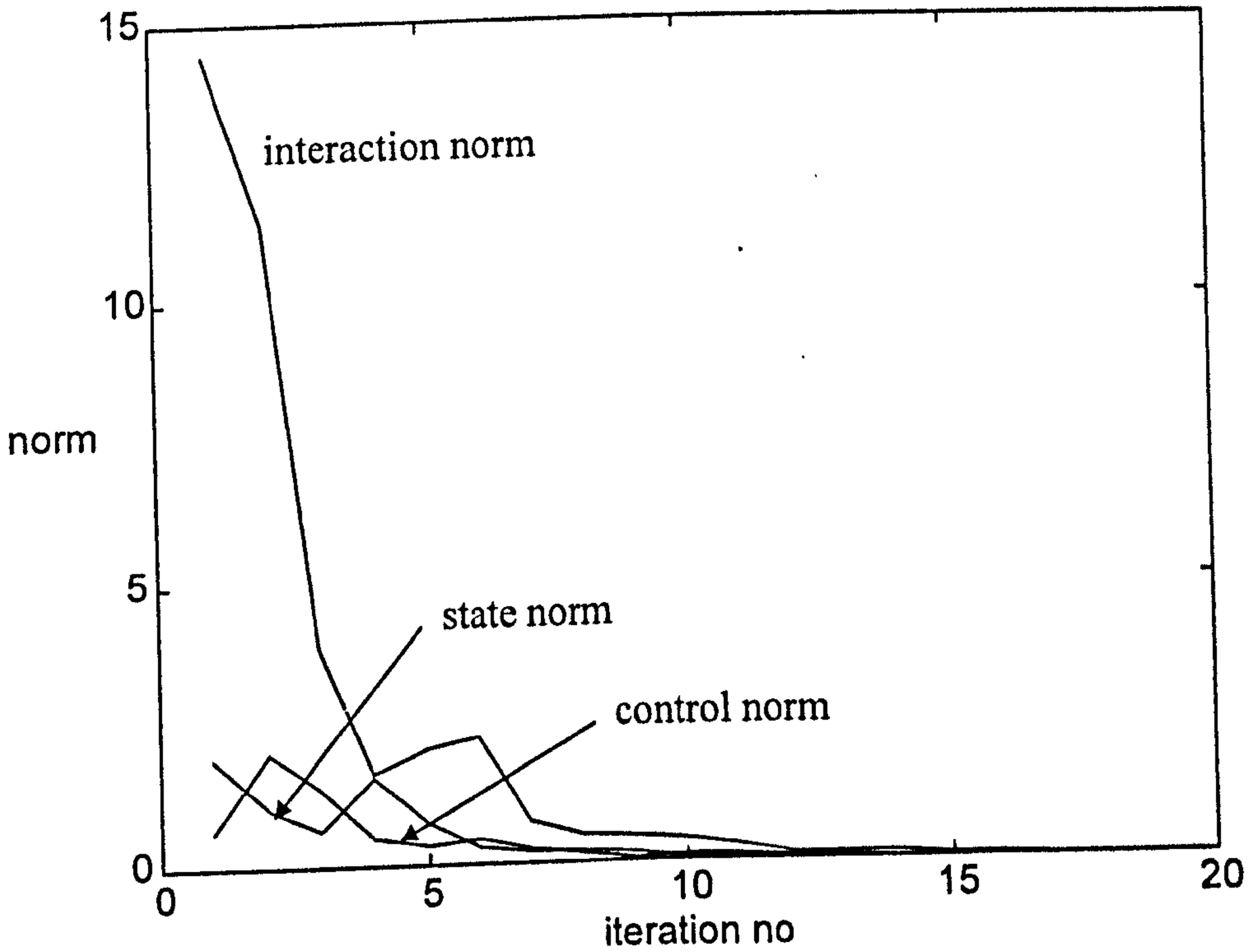

Fig.6.4.1.3 Convergence of state norm, control norm and interaction norm for algorithm PC3 with $\varepsilon_l = 0.15$, $r_1 = 0$, $r_2 = 0.0$, $\varepsilon_v = 1.0$

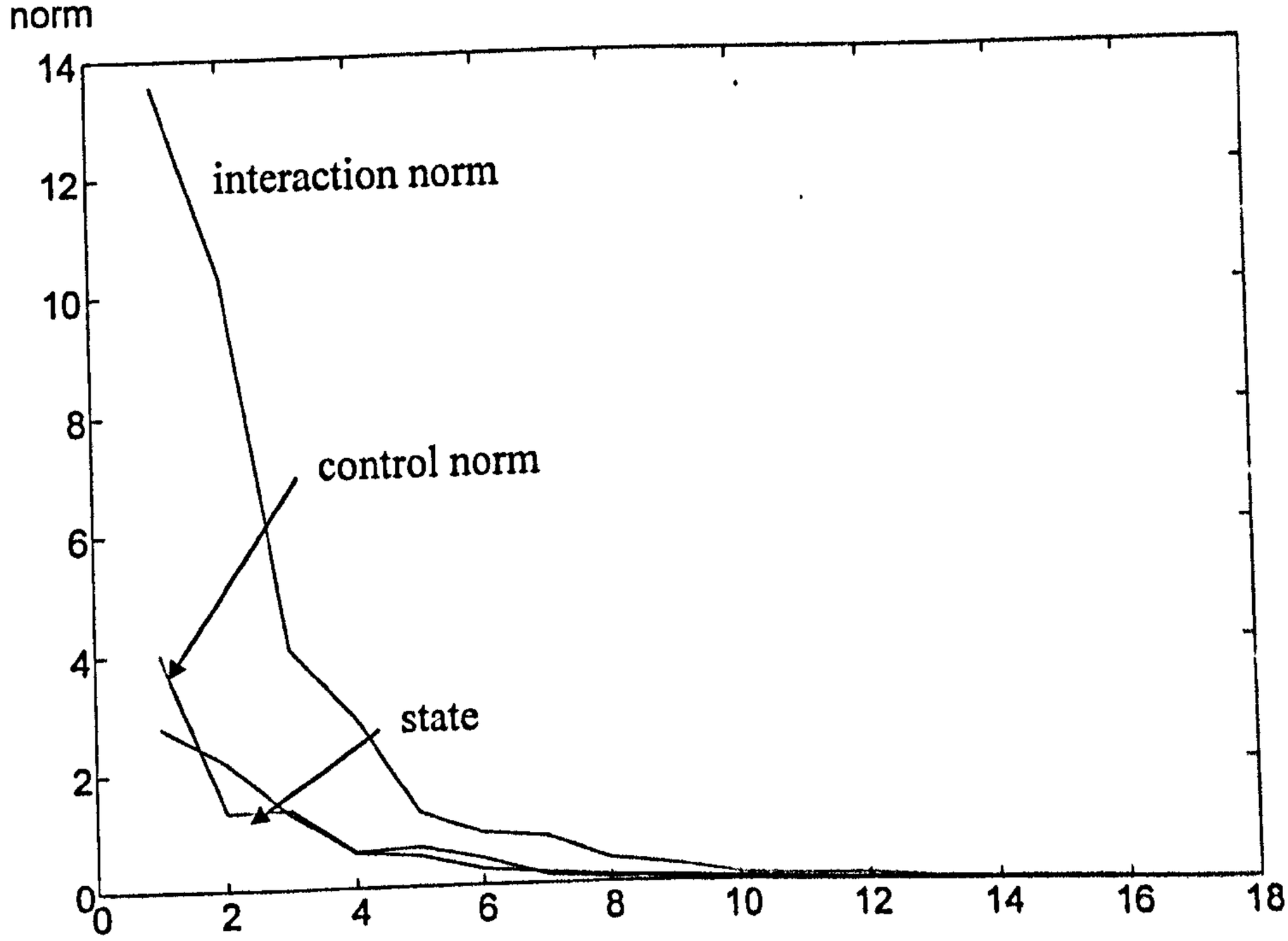

Fig.6.4.1.4 Convergence of state norm, control norm and interaction norm for algorithm PC3 with $\varepsilon_l = 0.15$, $r_1 = 0$, $r_2 = 2.0$, $\varepsilon_v = 1.0$

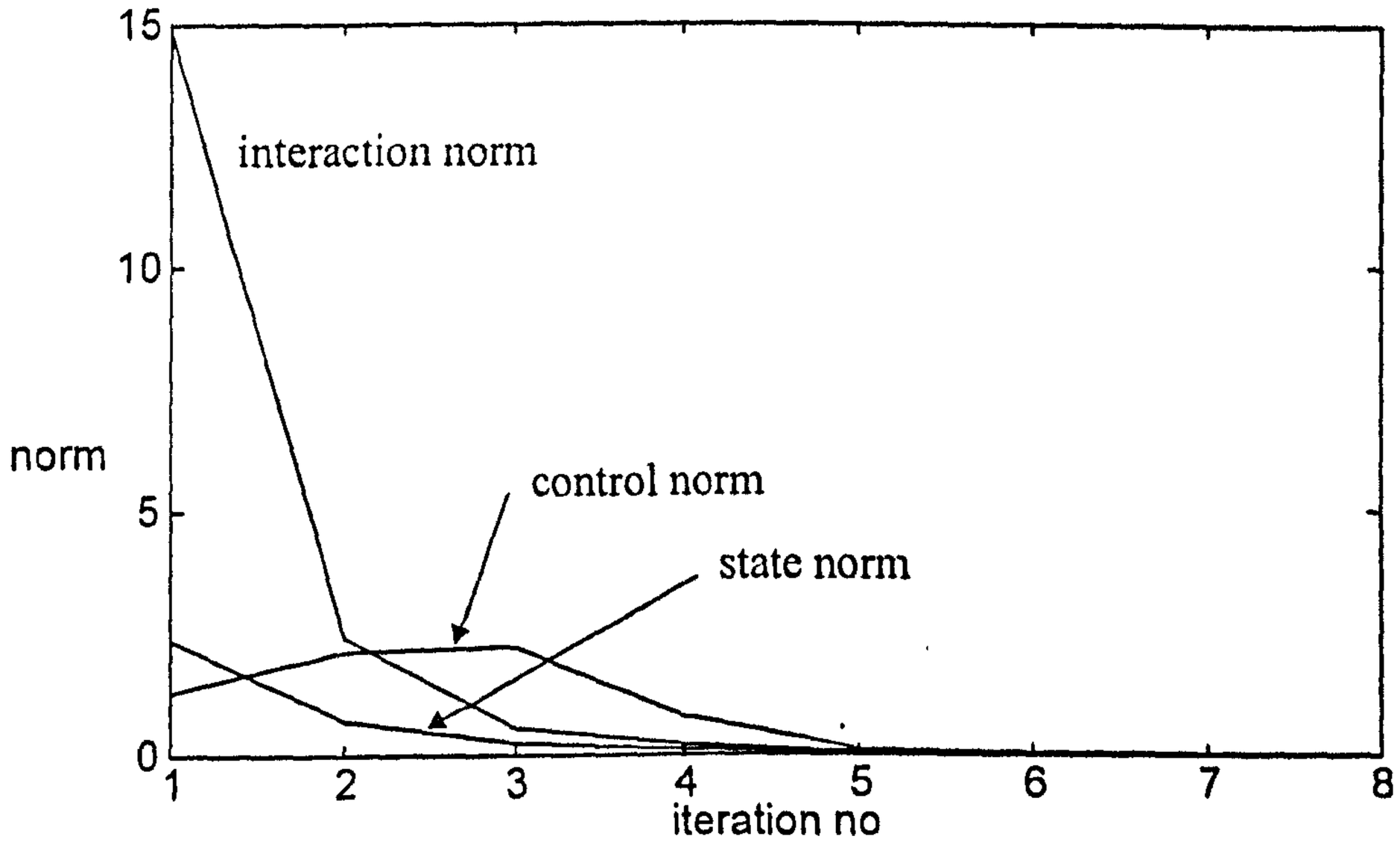

Fig.6.4.1.5 Convergence of state norm, control norm and interaction norm for algorithm DC3 with $r_1 = 0, r_2 = 1.0, \varepsilon_v = 0.85$

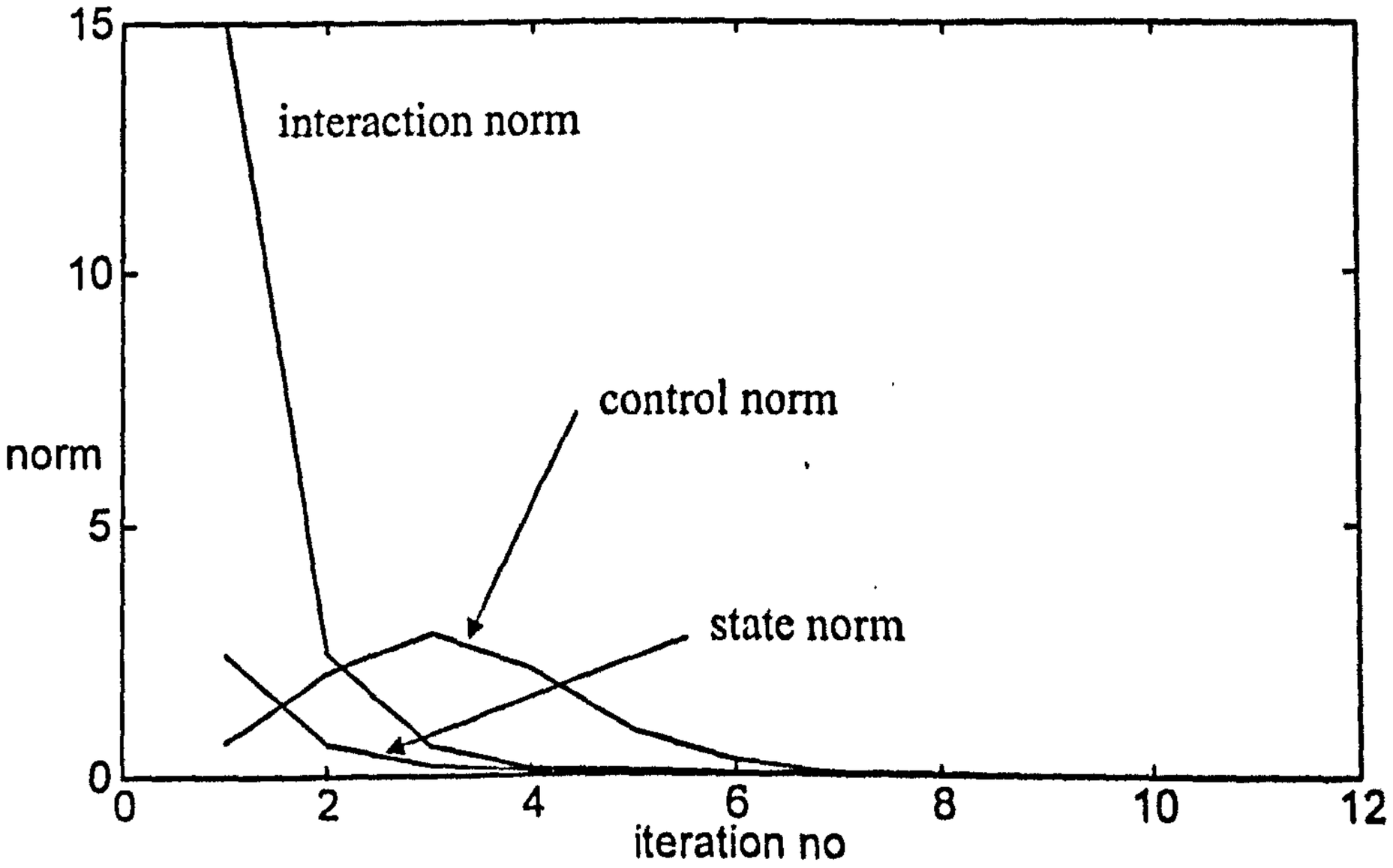

Fig.6.4.1.6 Convergence of state norm, control norm and interaction norm for algorithm DC4 with $r_1 = 0, r_2 = 1.0, \varepsilon_v = 0.85$

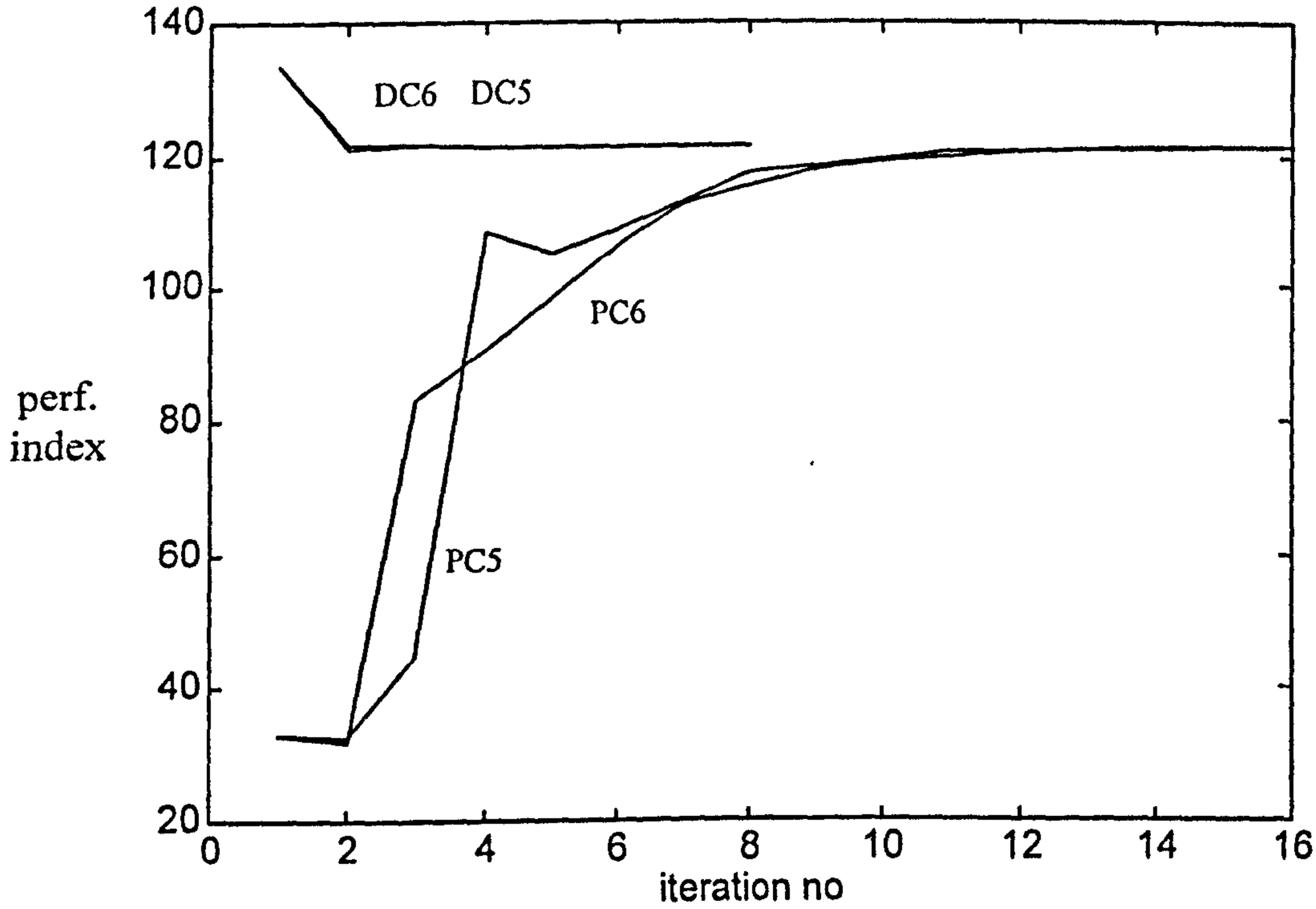

Fig. 6.4.1.7 Convergence of algorithms PC5, PC6, DC5 and DC6 at the 'best' tuning parameter setting

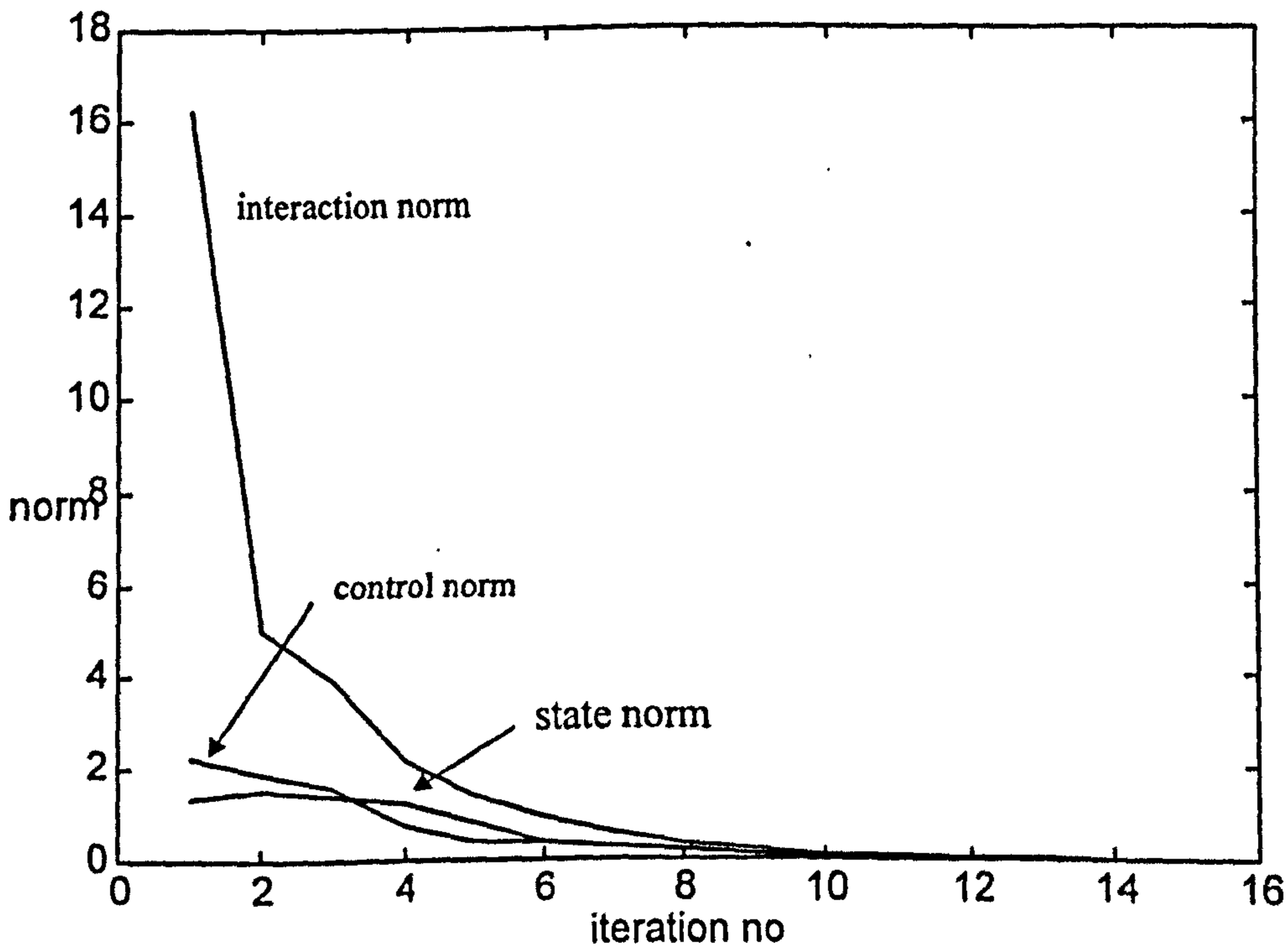

Fig.6.4.1.8 Convergence of state norm, control norm and interaction norm for algorithm PC6 with $\varepsilon_l = 0.15$, $r_1 = 0$, $r_2 = 1.0$, $\varepsilon_v = 0.85$

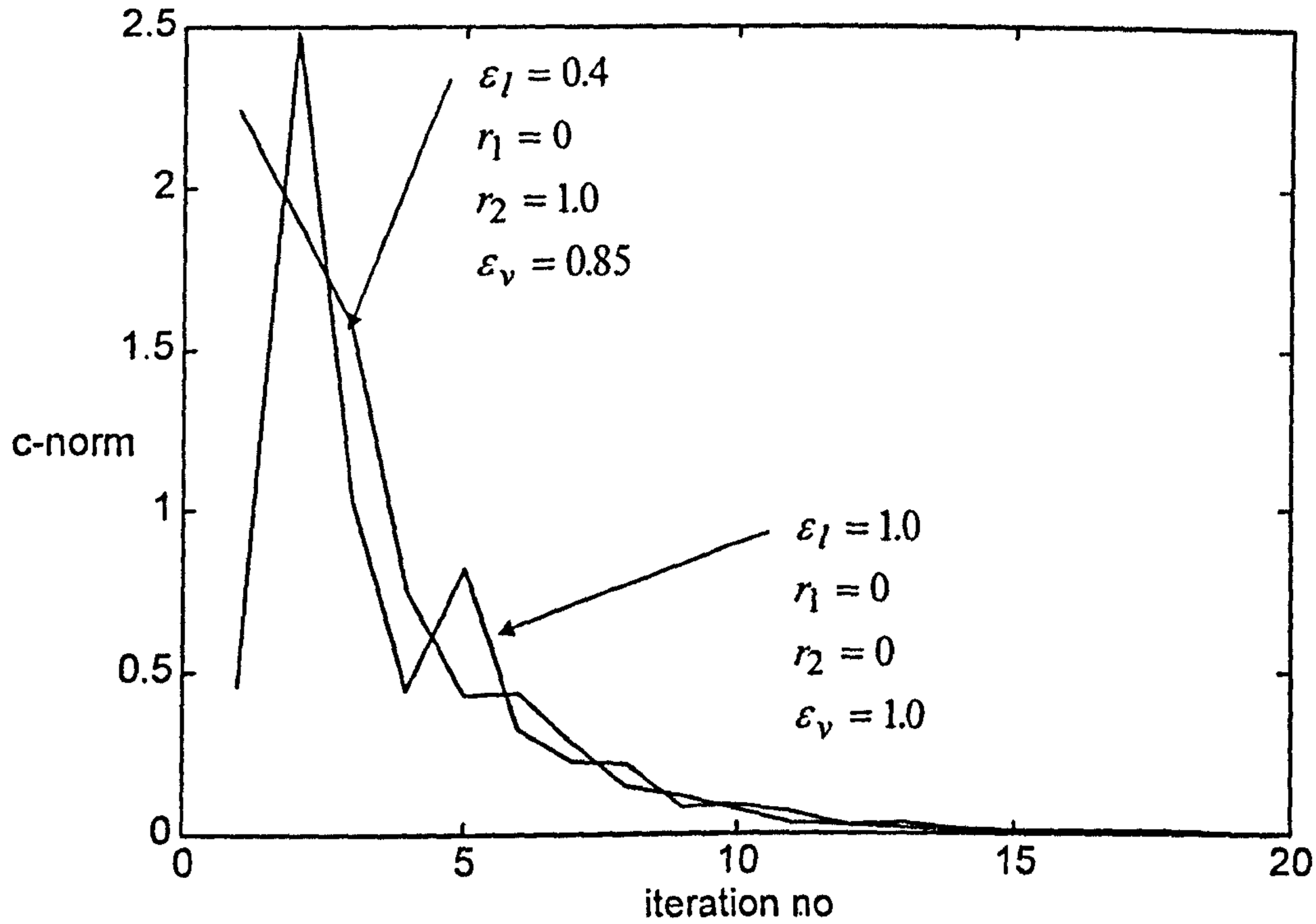

Fig.6.4.1.9. Convergence of control norm for algorithm PC6, with 2 different parameters choices

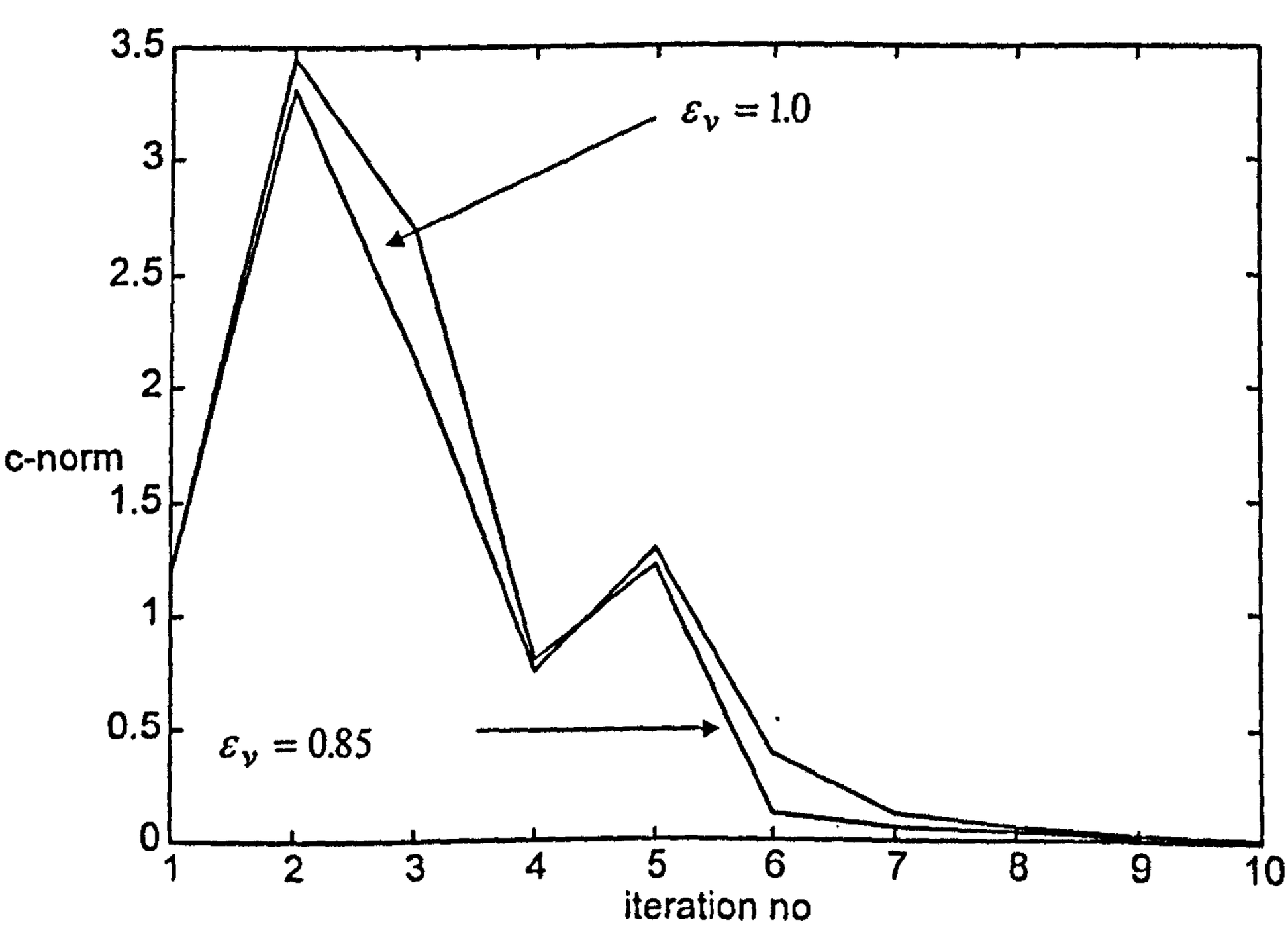

Fig.6.4.1.10 Convergence of control norm for algorithm DC5 with 2 different values of ε_v

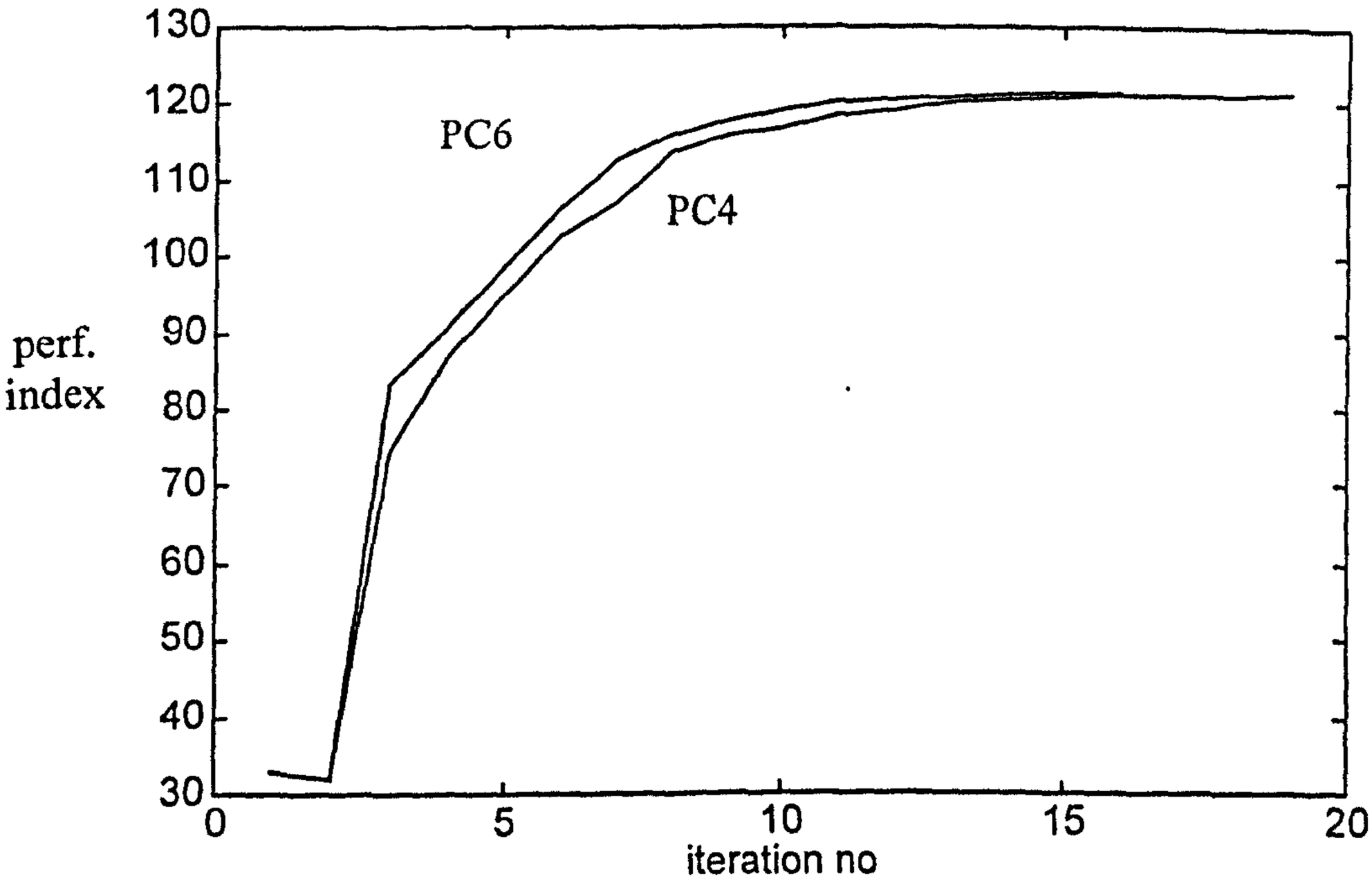

Fig.6.4.1.11 Convergence of performance index for PC4 and PC6 .

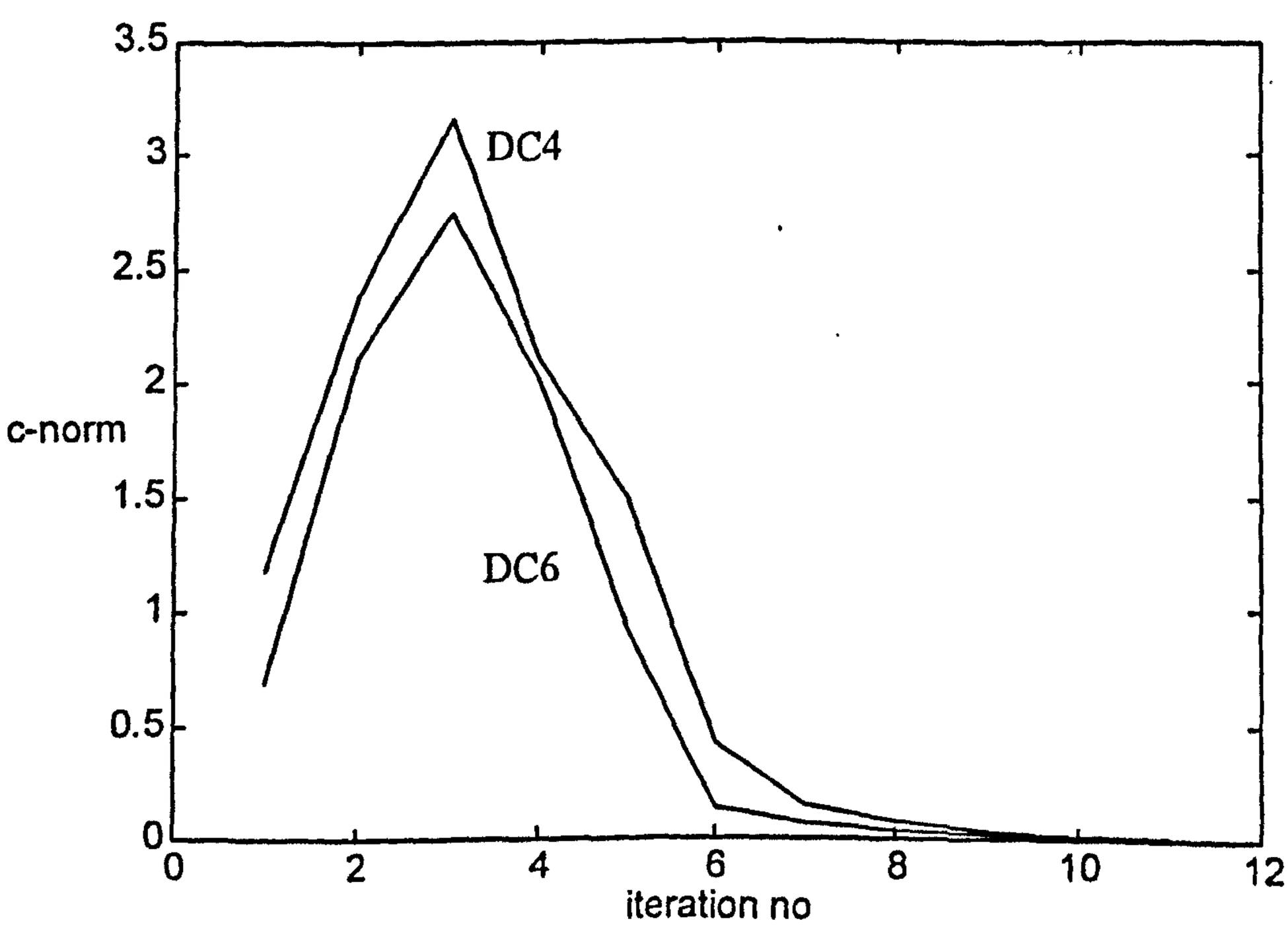

Fig. 6.4.1.12 Convergence of control norm of double loop algorithms DC4 and DC6 with $r_1 = r_2 = 0$ and $\varepsilon_v = 1.0$

Example 6.4.2

This example is similar to example 4.4.2 It is a modification of an example from Findeisen et al (1980), which is made of an interconnected system containing three non-linear subsystems. Note that the coefficient of the control and interaction vectors in the subsystems of the ROP are nonlinear. The example is tested on the eight different algorithms presented in this chapter. The tolerance specified for convergence is set at ε_T=0.01. $T = 0.05$.

ROP:

$$\min_{u(k),c(k)} \frac{1}{2} \sum_{i=1}^{3} \sum_{k=0}^{51} [x_i(k)^T Q_i^* x_i(k) + c_i(k)^T R_i^* c_i(k) + u_i(k)^T S_i^* u_i(k)].$$

where

$Q_1^* = 0.5$; $R_1^* = 0.1$; $S_1^* = 0.5$

$Q_2^* = 0.5$; $R_2^* = 0.1$; $S_2^* = 0.5$

$Q_3^* = 0.5$; $R_3^* = 0.1$; $S_3^* = 0.5$

subject to

subsystem 1:

$$x_{1,1}(k+1) = T(1+0.5\sin(\frac{2\pi k}{5})c_{1,1}(k)) + T(0.5+0.2\sin(\frac{2\pi k}{5}))u_{1,1}(k) + x_{1,1}(k)$$

$$y_{1,1}(k) = x_{1,1}(k)$$

$$x_{1,1}(0) = 1.0$$

subsystem 2:

$$x_{2,1}(k+1) = T(1+0.5\sin(\frac{2\pi k}{5})c_{2,1}(k)) + T(0.5+0.5\sin(\frac{2\pi k}{5}))u_{2,1}(k) + 0.5x_{2,1}(k)$$

$$y_{2,1}(k) = x_{2,1}(k) + c_{2,1}(k)$$

$$x_{2,1}(0) = 0.8$$

subsystem 3:

$$x_{3,1}(k+1) = T(1+0.5\sin(\frac{2\pi k}{5})c_{3,1}(k)) + T(0.5+0.5\sin(\frac{2\pi k}{5}))u_{3,1}(k) + x_{3,1}(k)$$

$$y_{3,1}(k) = x_{3,1}(k)$$

$$x_{3,1}(0) = 0.3$$

The interconnection matrix H is given as

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The model-based problem is an Linear Quadratic approximation of ROP. The output function $y_i(k)$ is deliberately chosen to be zero to test the convergence of the algorithms.

MOP:

$$\min_{u(k),c(k)} \frac{1}{2} \sum_{i=1}^{3} \sum_{k=0}^{51} [x_i(k)^T Q_i x_i(k) + c_i(k)^T R_i c_i(k) + u_i(k)^T S_i u_i(k) + \gamma(k)]$$

where

$Q_1 = 0.5$; $R_1 = 0.1$; $S_1 = 0.5$

$Q_2 = 0.5$; $R_2 = 0.1$; $S_2 = 0.5$

$Q_3 = 0.5$; $R_3 = 0.1$; $S_3 = 0.5$

subject to

subsystem 1:

$x_{1,1}(k+1) = 0.05c_{1,1}(k) + 0.025u_{1,1}(k) + x_{1,1}(k)$

$y_{1,1}(k) = 0$

$x_{1,1}(0) = 1.0$

subsystem 2:

$x_{2,1}(k+1) = 0.05c_{2,1}(k) + 0.025u_{2,1}(k) + 0.5x_{2,1}(k)$

$y_{2,1}(k) = 0$

$x_{2,1}(0) = 0.8$

subsystem 3:

$x_{3,1}(k+1) = 0.05c_{3,1}(k) + 0.025u_{3,1}(k) + x_{3,1}(k)$

$y_{3,1}(k) = 0$

$x_{3,1}(0) = 0.3$

The model output function is used in algorithms PC3. PC4, DC3 and DC4 only. In algorithms PC5, PC6, DC5 and DC6 the real output function as specified in the ROP is used.

Algorithm	ε_l	r_1	r_2	ε_v	Number of iterations	CPU (s)	Final Performance Index
price cor. single loop	0.05	0.9	0	0.85	111	266	15.8735
PC3	0.05	0.9	0	1.0	94	228	15.8736
	0.05	0.5	0	1.0	73	152	15.8497
	0.02	0.5	0	1.0	68	147	15.8487
	0.02	0	0	0.9	41	102	15.8901
	0.02	0	0.5	0.9	101	240	15.8975
*	0.023	0	0	0.9	36	89	15.8907
price cor. double loop	0.08	0	0	1.0	55	154	15.8905
PC4	0.08	1.0	0	1.0	101	234	15.8714
	0.15	0	0	1.0	50	119	15.8977
	0.07	0	1.0	0.9	36	87	15.8903
*	0.07	0	0	0.9	35	85	15.8903
direct cor. single loop	N/A	0.5	0	0.8	92	225	15.8826
DC3		1.0	0	0.9	112	264	15.8719
		0.6	0	0.9	84	206	15.8802
		0.5	0.5	1	104	255	15.8830
*		0.5	0	0.85	82	202	15.8838
direct cor. double loop	N/A	0.5	0	0.8	92	206	15.8816
DC4		0.6	0	0.85	89	204	15.8892
*		0.5	0	0.85	81	208	15.8823
		0.5	0.5	0.85	82	209	15.8793
		0.4	0	0.85	89	222	15.8802
Centralize DISOPE (Centralized Problem)	N/A	0.0	0.0	1.0	25	40	15.8891

Table 6.4.2.1 Algorithms performance for Example 6.4.2

(for systems with real interaction input in parameter estimation)

The results of the simulation for a set of sample tuning parameters settings are shown in tables 6.4.2.1 and 6.4.2.2. The optimum state, final control signals and interaction vectors are similar to those given by figures 4.4.2.1-4.4.2.9, therefore they are not included among the figures illustrated in this chapter. Entries marked * indicate the best choice of parameter settings from the samples. From table 6.4.2.1 , it is observed that ε_l and ε_v influence the rate of convergence for price coordination algorithms PC3 and PC4 in this example. There is no significant difference in speed of convergence when the best settings of each algorithm are compared. This is illustrated by fig. 6.4.2.1 and fig. 6.4.2.2. The change in direction in the convergence pattern of the performance index in algorithms DC3 and DC4 can be explained by the presence of model-reality differences in the model based problem. Note that in this example we have chosen a linear time invariant function to approximate a sinusoidal time-varying function as defined in MOP. This convergence pattern is also exhibited by the centralized DISOPE algorithm in solving the equivalent centralized problem as shown in figure 4.4.2.13 in Chapter 4. In algorithm PC3 the convexification scalar r_1 does increase the speed of convergence but only when $\varepsilon_l = 0.05$. In direct coordination algorithms DC3 and DC4 , r_1 influence the rate of convergence. There is no significant difference between the single loop (DC3) and double loop (DC4) implementation of the direct coordination algorithms. Notice also, for this example, the price coordination algorithms has a better overall convergence property than that of direct coordination (refer to figures 6.4.2.1 and 6.4.2.2). Figure 6.4.2.3 compares the convergence of the interaction norm with different values of parameter choices. Setting r_1.=0.5 , $\varepsilon_l = 0.05$ and $\varepsilon_v = 1.0$ results in an oscillating convergence pattern with a much slower tail end convergence. Convergence behaviour for the single loop algorithm DC3 with different parameter choices is shown by figure 6.4.2.4. Setting $r_2 = 0.5$ and $\varepsilon_v = 1.0$ improves the speed of convergence of the algorithm. Figures 6.4.2.5 and 6.4.2.6 illustrate the convergence pattern of the control norm and interaction norm of the double loop algorithms DC4 and PC4 with the best choice of parameter settings. Note the oscillating pattern of convergence displayed by algorithm DC4. This is in contrast to the smooth monotonic pattern displayed by the double

loop price coordination algorithm PC4. This may be explained by the stabilizing effect of ε_l in price coordination algorithms for this example.

We now consider table 6.4.2.2 which shows the performance of algorithms for systems with real input in interaction and parameter estimation. The algorithms utilized real values from the output function. In general it has been observed that algorithms for this systems have better convergence behaviour when compared to those that utilized output model (PC1-PC4 and DC1-DC4). This is because there is no need to calculate model-reality parameter $\theta(k)$ and to use interaction separation variable $w(k)$ in structures that utilized real output function. From the table it can be seen, that there is no significant difference between single loop and double loop implementations of the two approaches. This is also shown in figures 6.4.2.7 and 6.4.2.8. The choice of the price updating stepsize ε_l influences the convergence behaviour of the price coordination algorithms. Setting the convexification factor r_2 at 0.9 and ε_v at 0.85 gives the best performance among sample settings of algorithm PC5 (see fig. 6.4.2.9). This shows that the choice r_2 can influence the convergence of the algorithm using the price coordination approach. In the direct coordination algorithms DC5 and DC6 , the choice of r_1 influence the speed of convergence. This is consistent with the observation made in table 6.4.2.1 . Setting ε_v at 1.0 and r_1 at 0.3 gives the best performance among the samples shown for DC5 (see fig. 6.4.2.10) and DC6. Figures 6.4.2.11 and 6.4.2.12 compares the convergence of algorithms with real input in parameter estimation and algorithms with real input in interaction and parameter estimation. It is clearly illustrated that the utilization of real input improves the algorithms performance in term of speed of convergence.

Comparing the performance of the eight algorithms with that of the centralized DISOPE algorithm in this example, shows a favourable performance for the price coordination algorithms in spite of the fact that the subsystems are being solved sequentially. It is expected that application of parallel processing methods to the price coordination algorithms will result in a better performance.

Algorithm	ε_l	r_1	r_2	ε_v	Number of iterations	CPU (s)	Final Performance Index
price cor. single loop	0.05	0.9	0	0.85	111	233	15.8730
PC5	0.05	0	0.9	0.85	22	50	15.8911
	0.06	0	0.9	1.0	41	90	15.8916
	0.05	0	2.0	0.85	26	59	15.8920
*	0.05	0	0	1.0	24	55	15.8914
price cor. double loop	0.05	0	1.0	0.85	50	106	15.8901
PC6	0.1	0	1.0	0.85	27	60	15.8918
	0.1	0	0	0.85	26	58	15.8918
	0.15	0	0	0.85	38	81	15.8915
*	0.12	0	0	0.85	25	55	15.8950
	0.12	1.0	0	0.85	118	233	15.8796
direct cor. single loop	N/A	1.0	0	0.85	119	239	15.8768
DC5		0.5	2.0	0.85	82	169	15.8825
		0.5	1.0	0.85	82	169	15.8751
		0.5	0	1.0	70	142	15.8826
		0.4	0	1.0	63	129	15.8512
*		0.3	0	1.0	55	111	15.8735
direct cor. double loop	N/A	0.5	0	1.0	70	131	15.8265
DC6		0.4	0	1.0	63	124	15.8511
*		0.3	0	1.0	55	108	15.8734
		0.25	0	0.85	122	240	15.8922
		0.3	1.0	0.85	122	239	15.8921
Centralize DISOPE (Centralized Problem)	N/A	0.0	0.0	1.0	25	40	15.8891

Table 6.4.2.2 Algorithms performance for Example 6.4.2

(for systems with real input in interaction and parameter estimation)

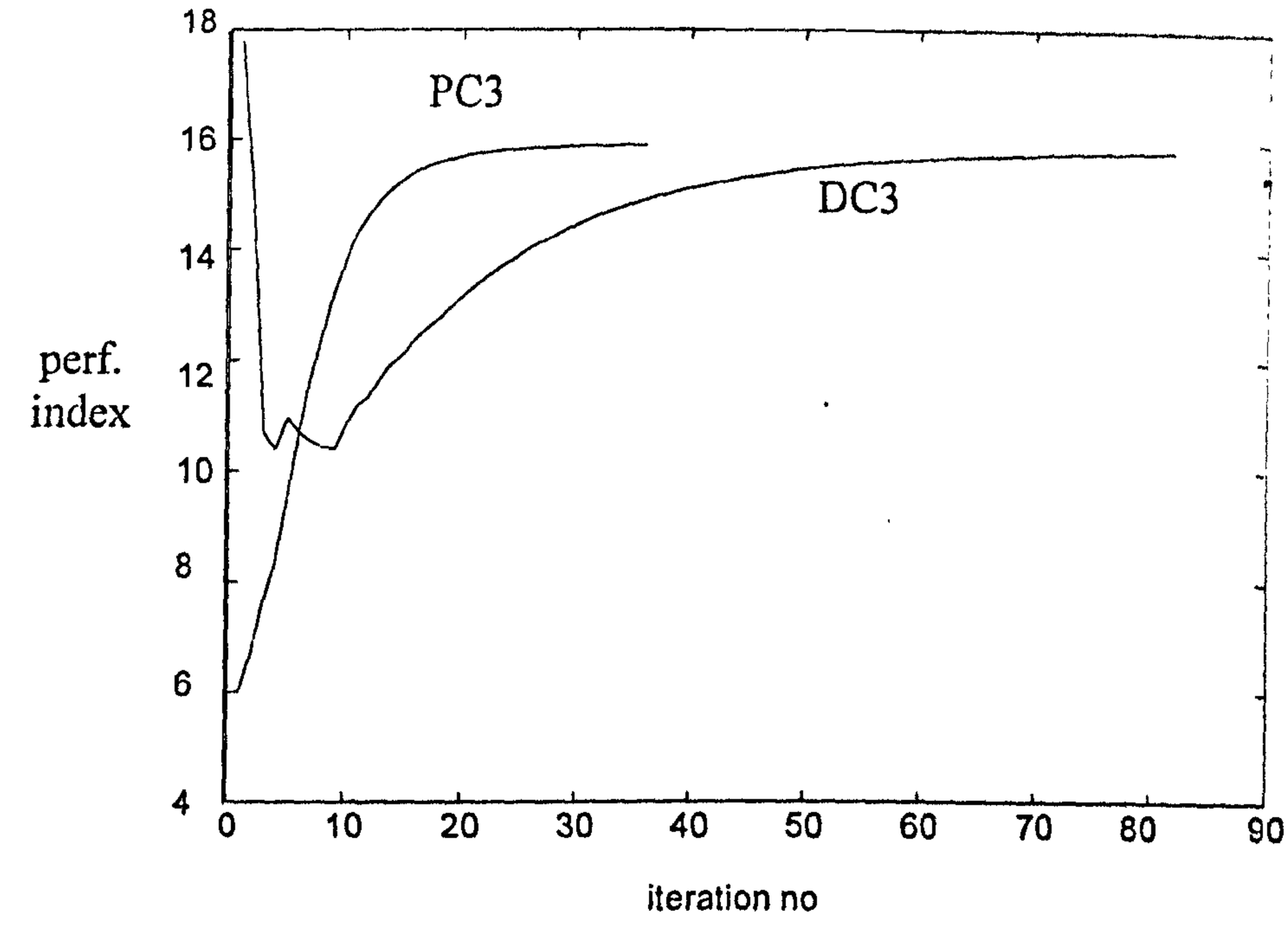

Fig. 6.4.2.1 Convergence of performance index for algorithm PC3 and DC3

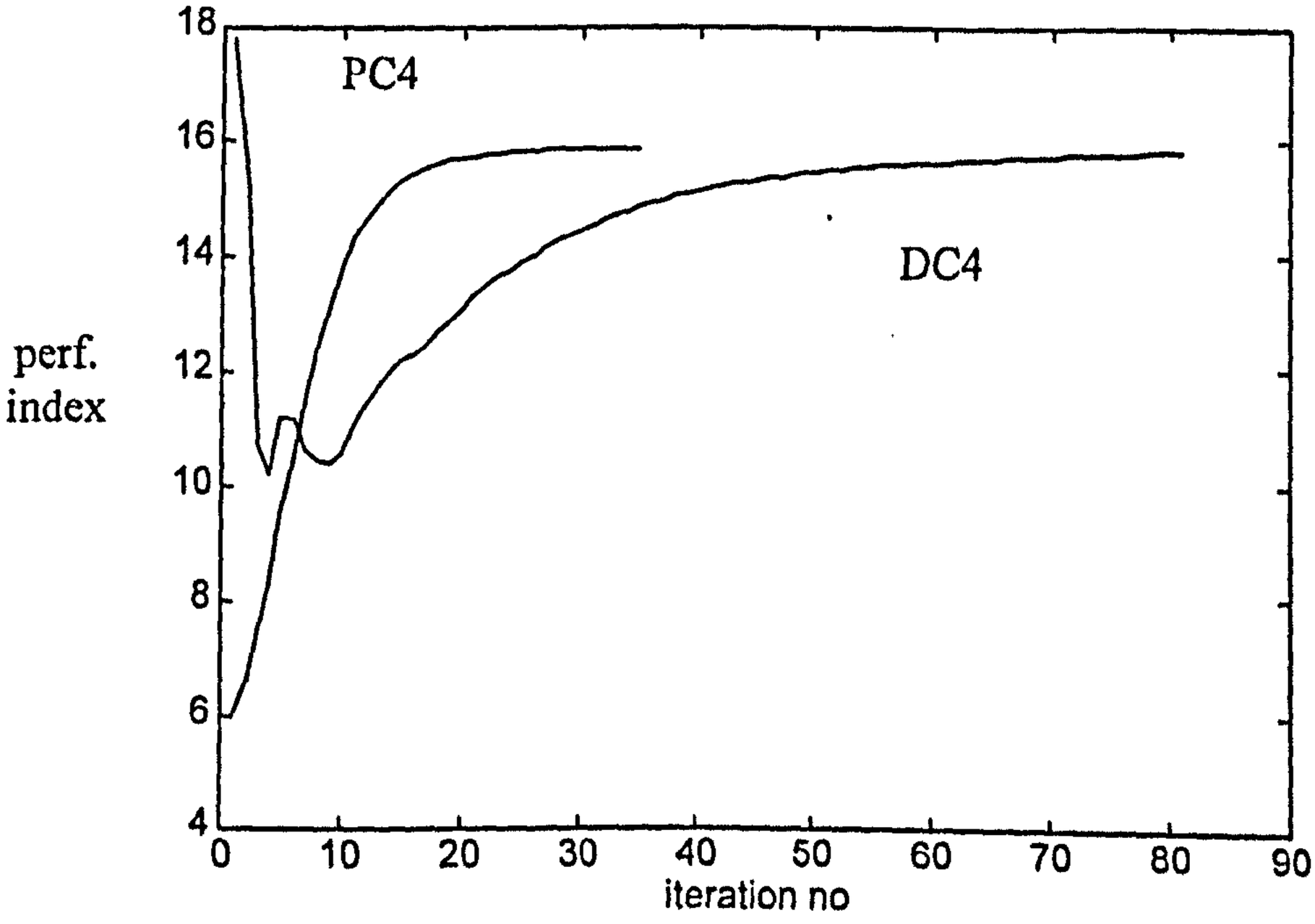

Fig. 6.4.2.2 Convergence of performance index for PC4 and DC4 .

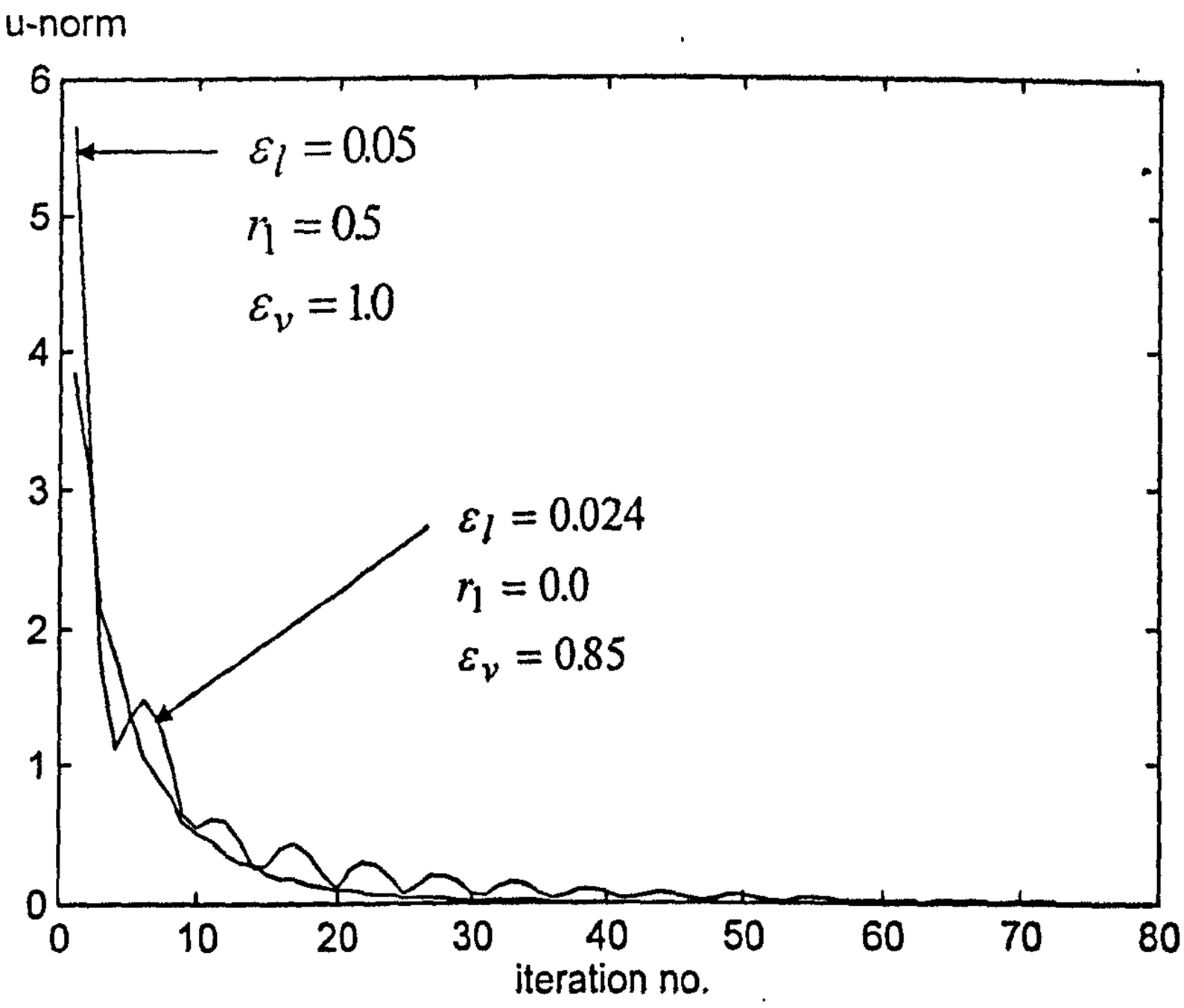

Fig.6.4.2.3 Convergence of interaction norm for algorithm PC3, with 2 different parameters choices

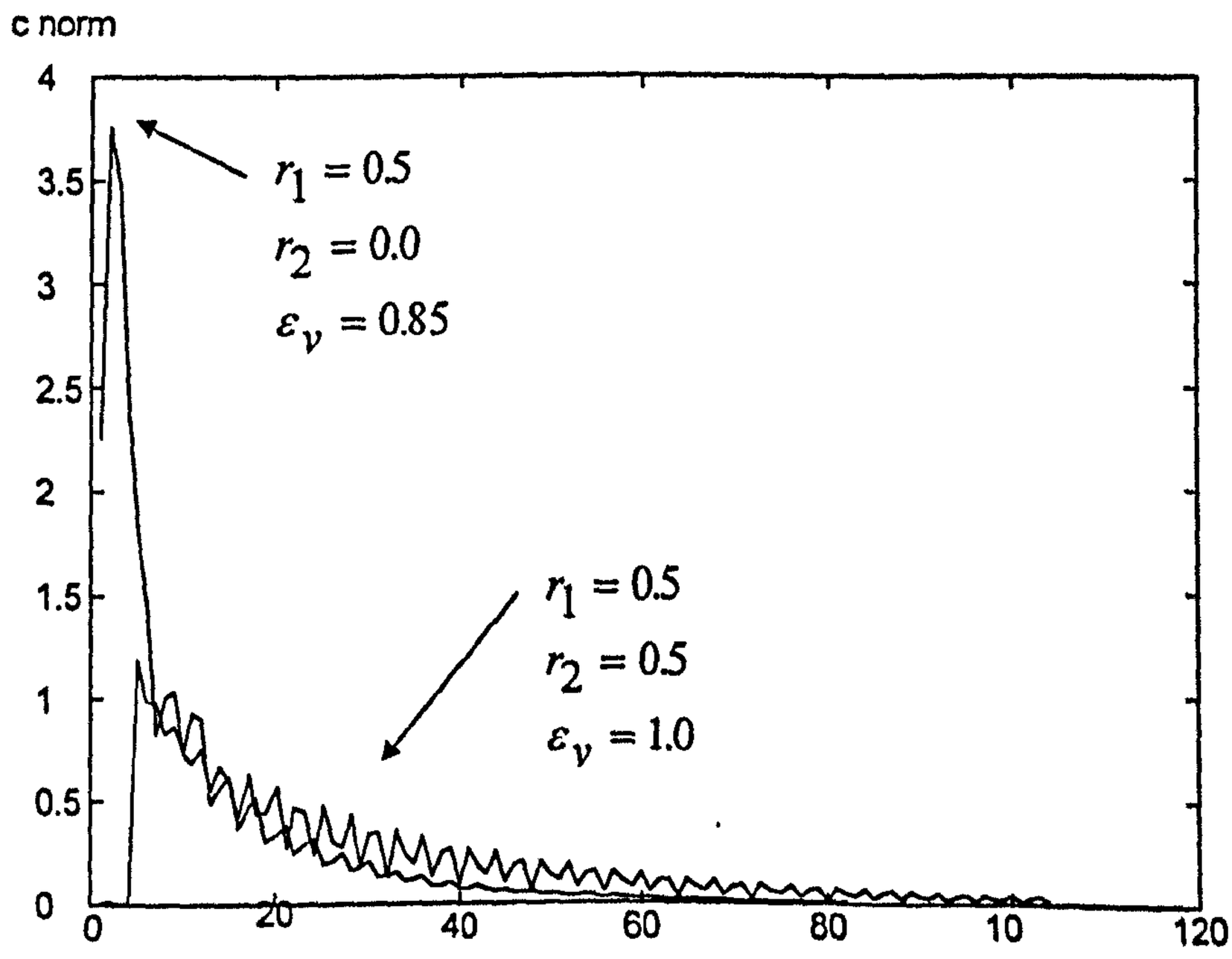

Fig.6.4.2.4 Convergence of control norm for algorithm DC3, with 2 different parameters choices

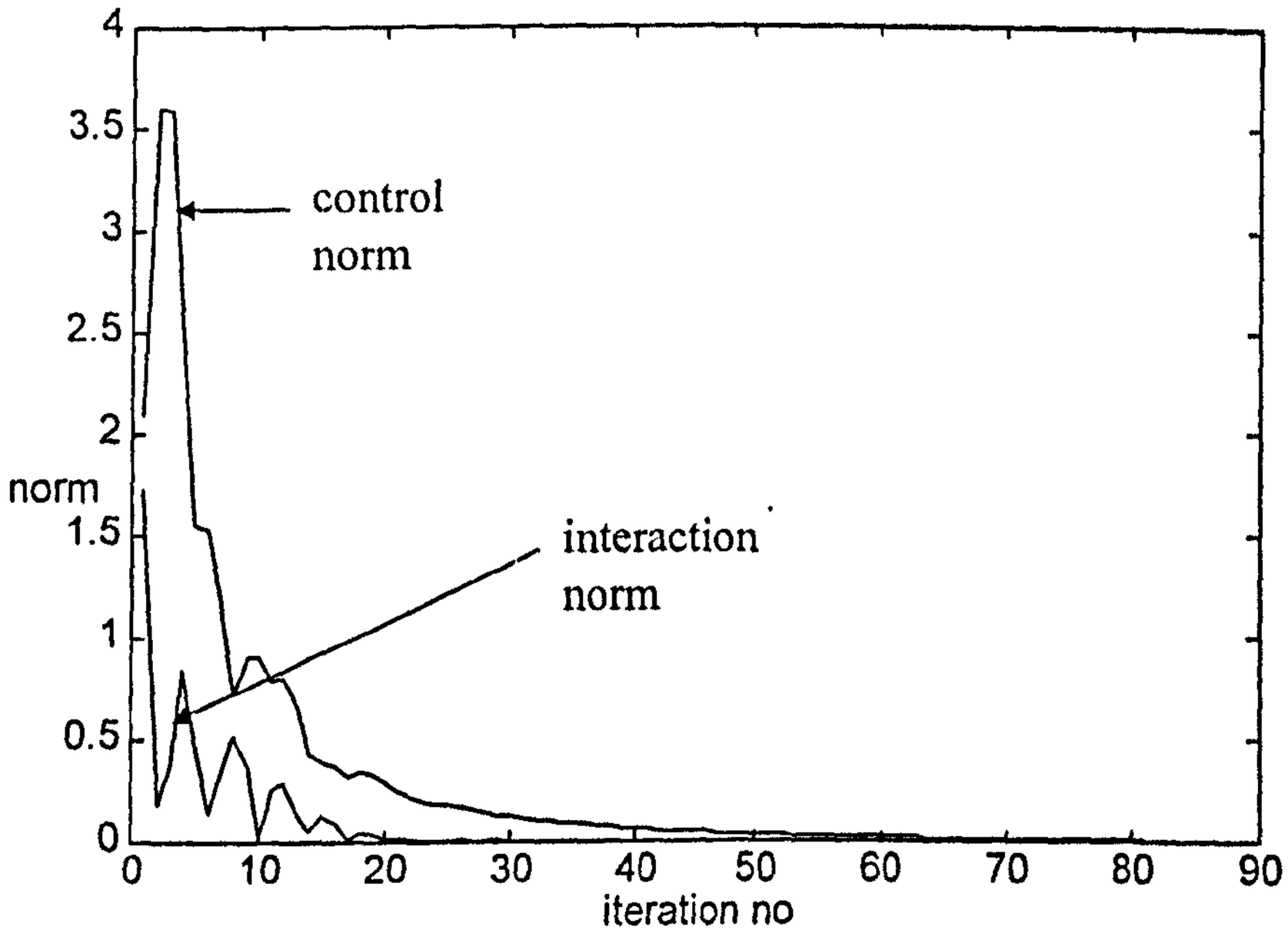

Fig.6.4.2.5 Convergence of control norm and interaction norm for algorithm DC4, with $r_1 = 0.5, r_2 = 0.0$ and $\varepsilon_v = 0.85$

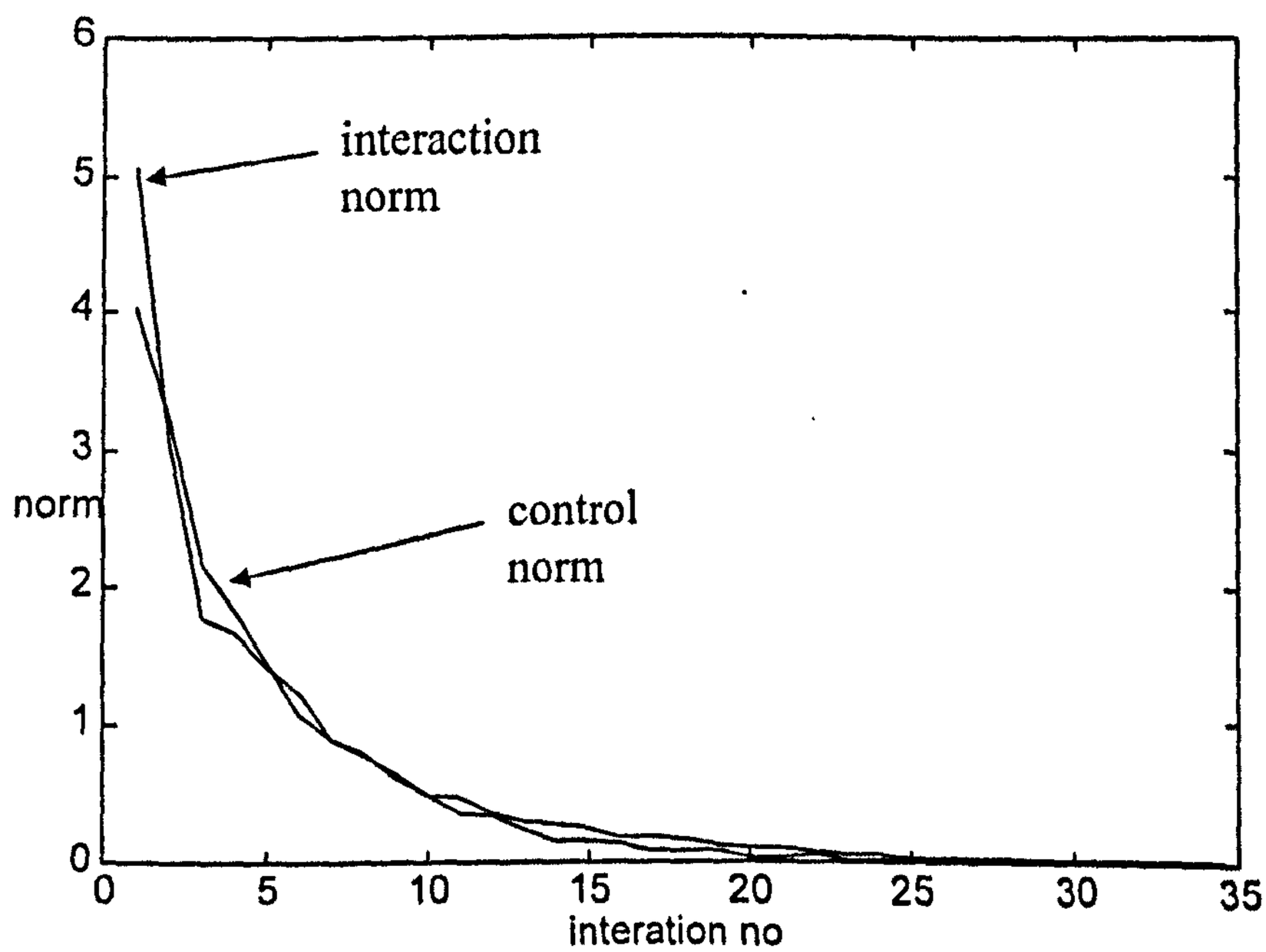

Fig.6.4.2.6 Convergence of control norm and interaction norm for algorithm PC4, with $r_1 = 0.0, r_2 = 0.0, \varepsilon_l = 0.07 \quad \varepsilon_v = 0.85$

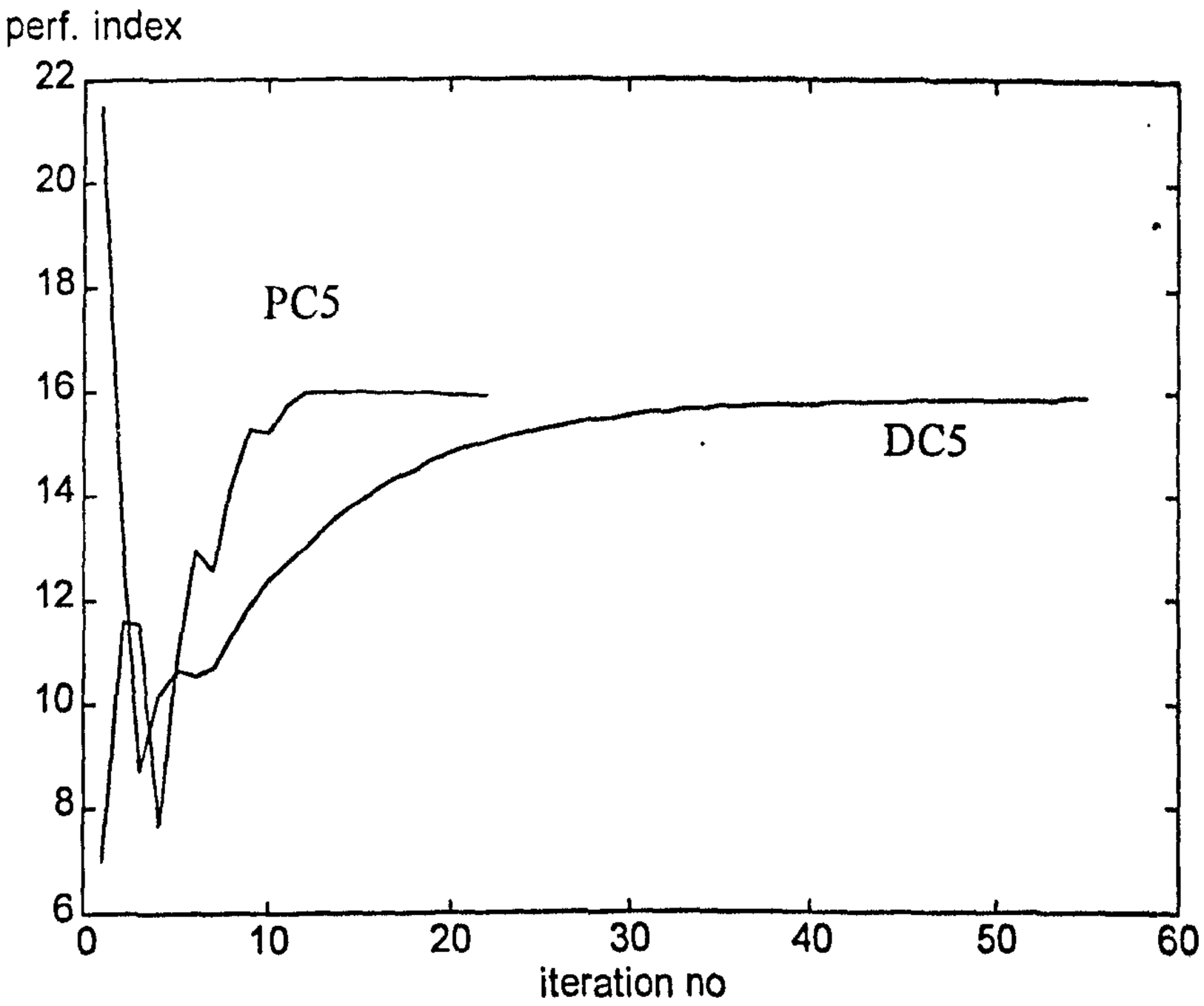

Fig. 6.4.2.7 Convergence of performance index for algorithms PC5 and DC5 .

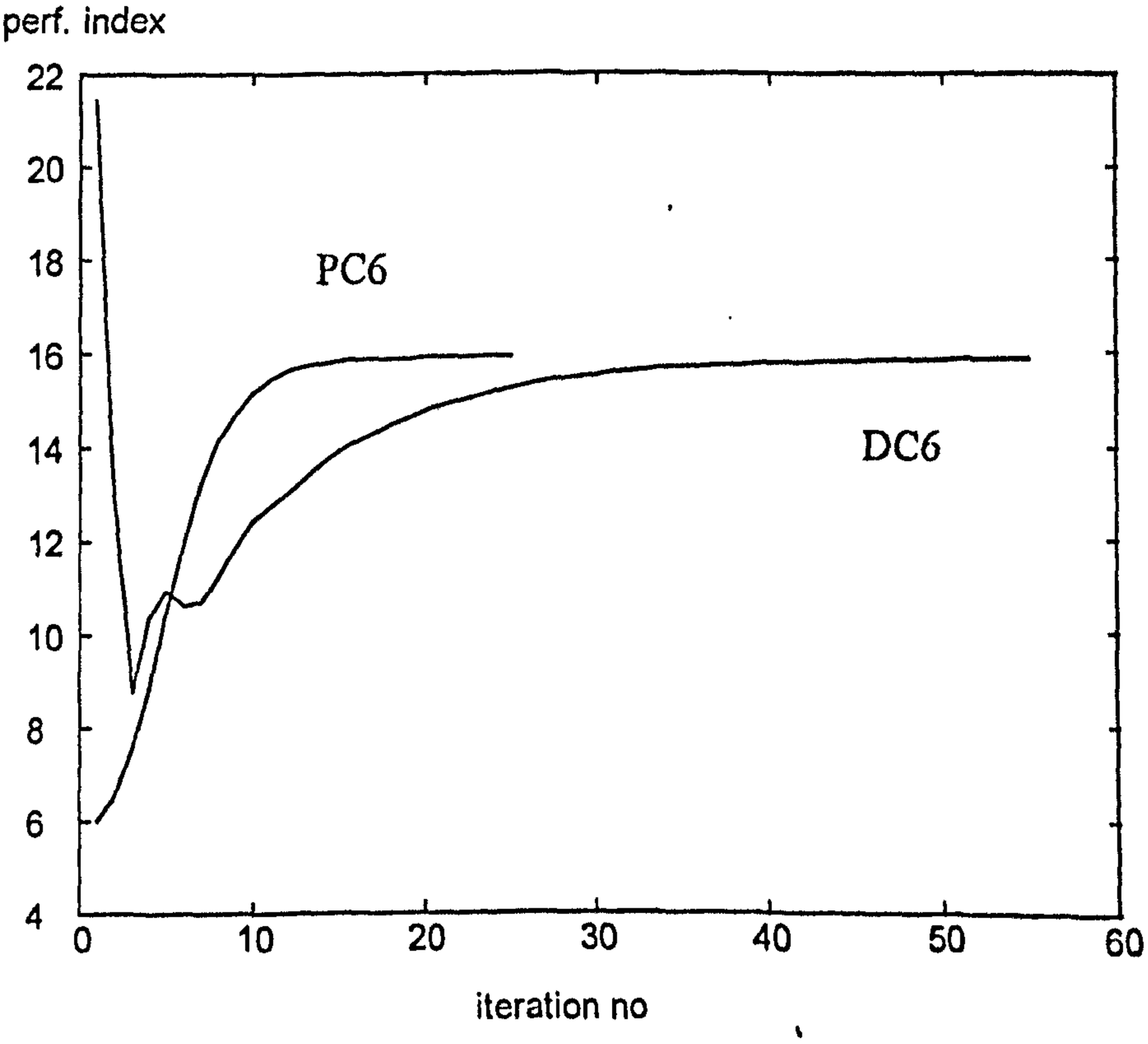

Fig. 6.4.2.8 Convergence of performance index for algorithms PC6 and DC6.

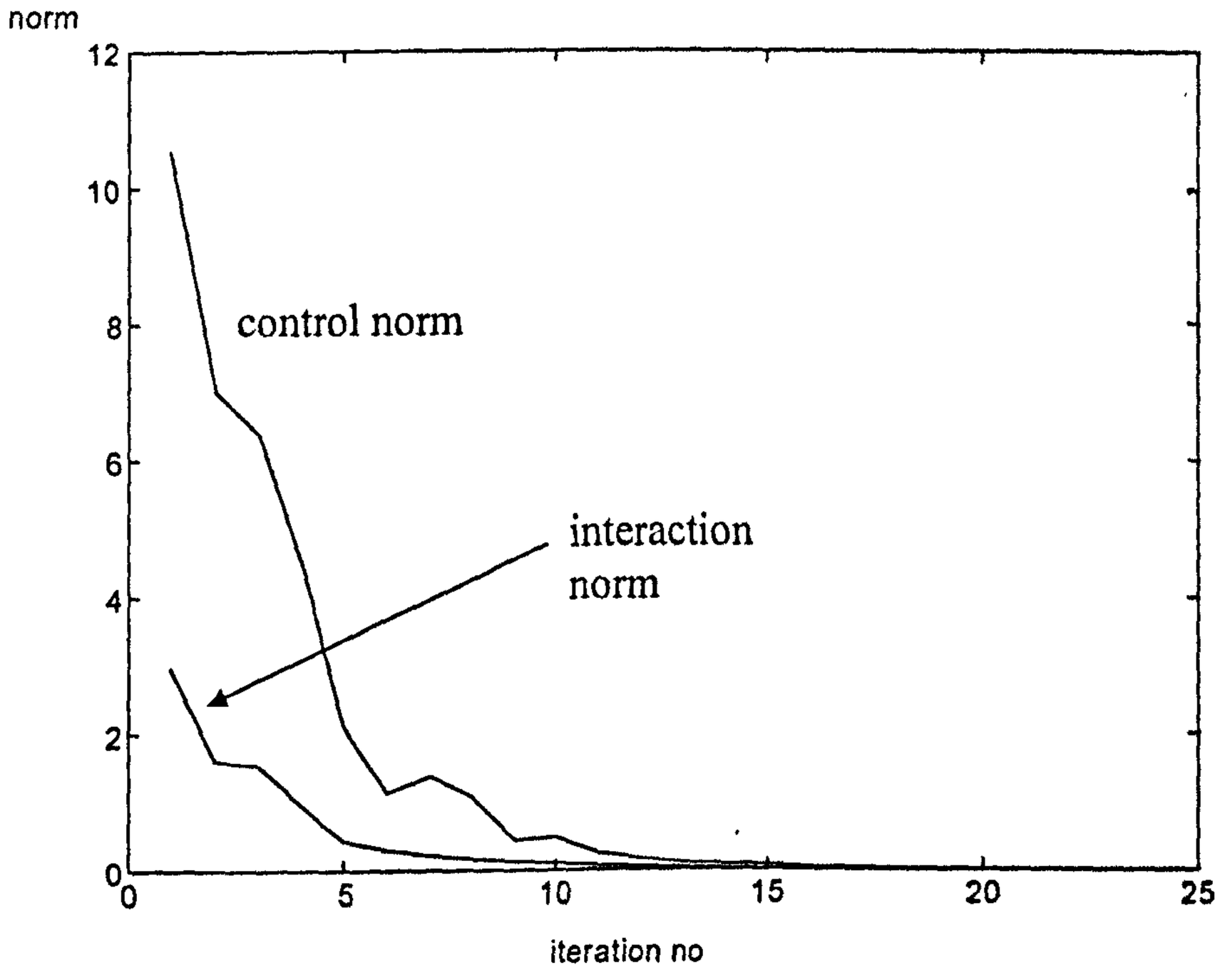

Fig.6.4.2.9 .Convergence of control norm and interaction norm for single loop algorithm PC5 with $\varepsilon_l = 0.05, r_1 = 0, r_2 = 0.9, \varepsilon_v = 0.85$.

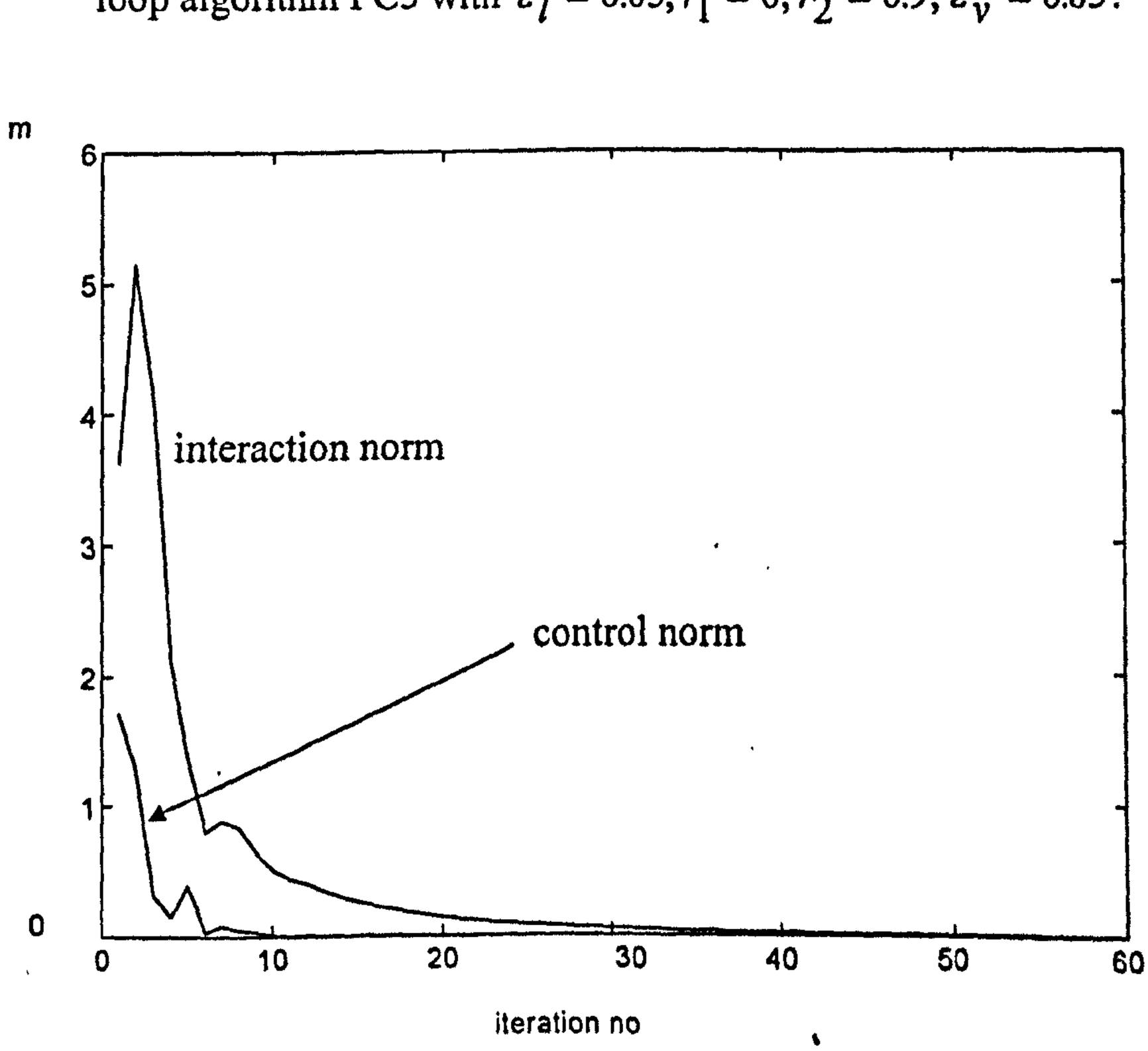

Fig.6.4.2.10 .Convergence of control norm and interaction norm for single loop algorithm DC5 with $r_1 = 0.3,\ r_2 = 0.0, \varepsilon_v = 1.0$.

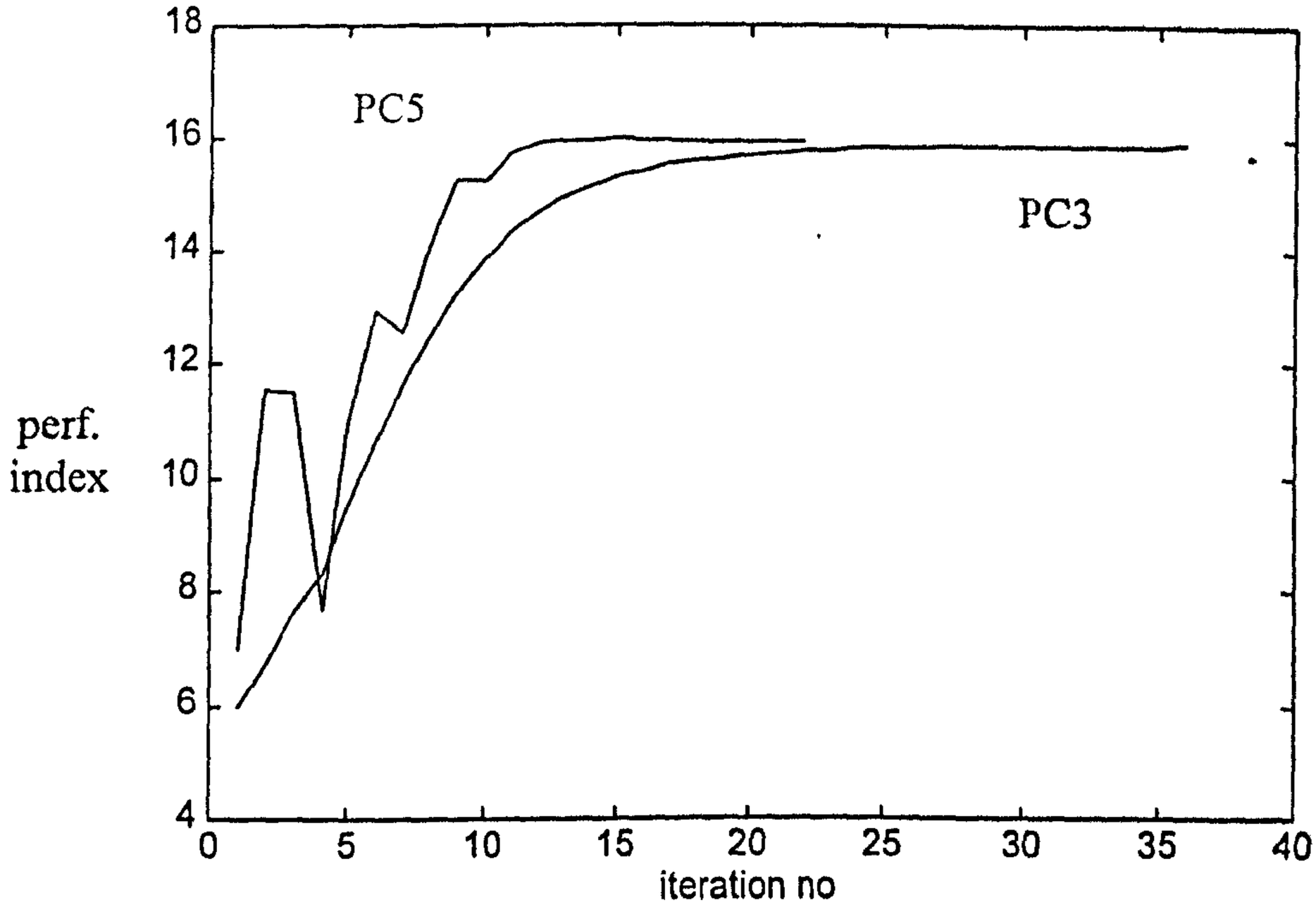

Fig.6.4.2.11 Convergence of performance index algorithms PC3 and PC5 .

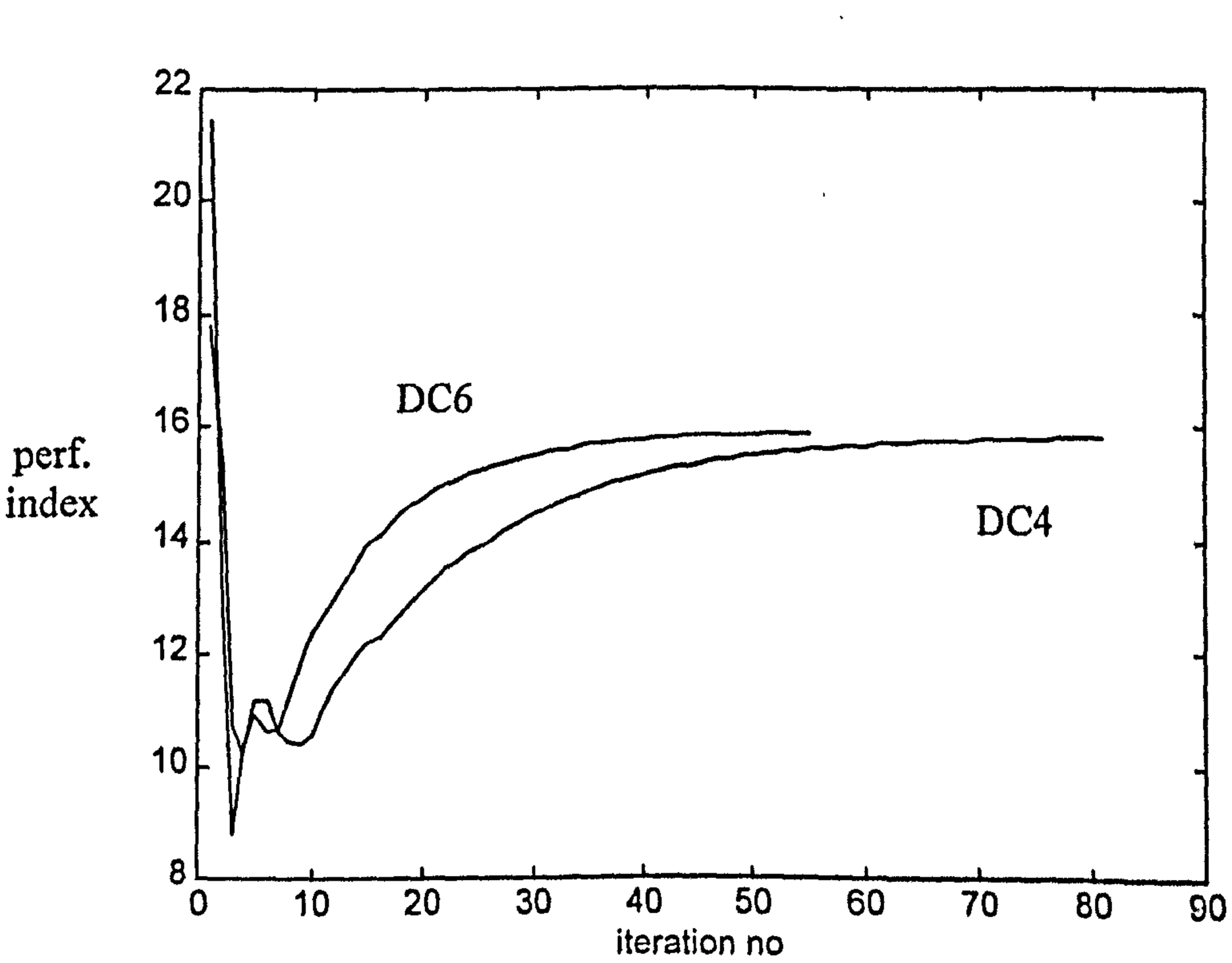

Fig.6.4.2.12. Convergence of algorithm DC4 and DC6 .

6.5 COMPARISON OF HIERARCHICAL DISOPE ALGORITHMS PC1-PC6 AND DC1-DC6

We will be comparing the efficiency of both approaches, price coordination approach and direct coordination approach, based on the simulation results of examples 4.4.1 (Chapter 4), 6.4.1 and 4.4.2 (Chapter 4), 6.4.2. The best results selected for each algorithm in two examples is tabulated in Tables 6.5.1 and 6.5.2 respectively. Comparison is made on the basis of iteration number made for the algorithm to converge.

From Table 6.5.1 , it is observed that for example 1, the algorithms based on structures with model based interaction input are the slowest to converge. In general for this example, direct coordination algorithms are more efficient in the sense that a faster speed of convergence is observed. It is also observed that the speed of convergence improves with an increase in the utilization of real interaction input in the structure. The double loop version is observed to converge slower than the single loop version except for the case of algorithms based on the structure with model based interaction input.

From Table 6.5.2 it is observed that the price coordination approach is more efficient in solving example 2 than the direct coordination approach. This may be explained by the stabilizing effect of price stepsize ε_l in price coordination algorithms for this example. The speed of convergence improves with an increase in the utilization of real interaction input in the structures. The algorithm based on the structure with real input in interaction and parameter estimation is observed to be most efficient in this sense. This can be explained the reduction of variables required in the iterative structure as a result of utilization of real inputs in interaction and parameter estimation.

Algorithm	Price Coordination Approach Number of iterations		Direct Coordination Approach Number of iterations	
	Single Loop	Double Loop	Single Loop	Double Loop
Structures with model based interaction input	25	20	11	12
Structure with real interaction input in parameter estimation	17	19	8	11
Structure with real input in interaction and parameter estimation	15	16	7	8

Table 6.5.1 Comparison of best results for Example 1.(Examples 4.41 and 6.4.1)

Algorithm	Price Coordination Approach Number of iterations		Direct Coordination Approach Number of iterations	
	Single Loop	Double Loop	Single Loop	Double Loop
Structures with model based interaction input	94	94	97	96
Structure with real interaction input in parameter estimation	36	35	82	81
Structure with real input in interaction and parameter estimation	22	25	55	55

Table 6.5.2 Comparison of best results for example 2(Examples 4.4.2 and 6.4.2)

6.6 SUMMARY

In this chapter we have developed eight algorithms for optimal control of structures with real input measurements. The algorithms can be classified into two groups. One group , PC3, PC4, DC3 and DC4 is based on a structure with real input in parameter estimation. The other group, PC5, PC6, DC5 and DC6 is based on a structure with real input in interaction and parameter estimations. The later group of algorithms have

less variables to iterate because their structure do not require calculation of model-reality parameter $\theta(k)$. Instead they utilize the real interaction measurement which results in inclusion of real output function $y(k) = K^*(x(k), c(k))$ in the solution of modified model based optimal control problem, $MMOP_i$. Simulation of the algorithms were carried on out examples. The first example is made up of nonlinear interconnected system consisting of three nonlinear subsystems. The dynamics of state is a mixture of fast and slow dynamics. In this example it is shown that direct coordination algorithms performed better than the price coordination algorithms .The second example is nonlinear example with time varying coefficients consisting of three subsystems. Results show that the price coordination algorithms performed better than the direct coordination algorithms in example 6.4.2.. From the simulation we can conclude that the performance of the algorithms depends on the structure of the problems they are solving. For sinusoidal nonlinear systems (example 6.4.2) it was observed that price coordination algorithms have better convergence property. Overall, it was observed that the algorithms for structures with real input measurement have a better convergence property when compared to those with model based interaction input presented earlier in chapter 4.

CHAPTER 7

CONCLUSIONS

7.1 CONCLUSIONS

The development of new hierarchical DISOPE algorithms for optimal control of interconnected nonlinear systems with a more general interaction structure and the study of their convergence properties has been the central subject of the research work described in this thesis. Four basic hierarchical optimal control structures are developed by taking into account the manner we incorporate and utilize real process measurements from interaction inputs in the model-based optimal control problems. The structures are of an iterative type and utilize real interaction input in the model when available. These structures are : structure with model based interaction input, structure with real interaction input in parameter estimation , structure with real input in interaction and parameter estimation and structure with total real interaction measurements. They are a dynamic analogy of hierarchical ISOPE structures derived by Brdys and Roberts (1986). The proposed structures are derived by extending the DISOPE technique to interconnected systems to take into account model-reality differences. These may have been deliberately introduced to facilitate the solution of complex intractable nonlinear optimal control problems or arise due to uncertainty in the model used for computation. Here, it is assumed that the derivatives of the reality functions and the models are piecewise continuous . It is also shown that the structures represent optimal methods in the sense that the converged solution will agree with the solution of original real optimal control problem (ROP). Of the four basic structures described, we used the first three to develop new hierarchical optimal control algorithms based on a linear quadratic approach.

Two approaches are used in the formulation of the hierarchical optimal control algorithms based on structures mentioned previously. They are the price coordination approach and the direct coordination approach.

In the price coordination approach the Lagrange multiplier associated with the interconnection constraint is calculated using an interaction balance updating

mechanism as suggested by Findeisen et al (1980). The optimal control problem becomes a minimization problem with respect to the interaction and the control terms. The estimated optimal interaction and the estimated optimal control are both calculated from the resulting minimization procedure. Algorithms using this approach do not satisfy the interconnection constraint in the solution of MOP in the iterations until the price mechanism has sufficiently converged. As a result the convergence of the global performance index of algorithms using this approach may exhibit an increasing pattern towards convergence as opposed to a decreasing pattern expected in a minimization problem.

In the direct coordination approach, the price associated with each interconnection constraint is calculated directly from optimality conditions. The optimal control problem constitutes a minimization problem with respect to the control term. In the iterations, the interaction vector and the interconnection price are computed in the second level using the previous estimated optimum variables resulting from the MOP solution in level one. In this procedure, the convergence of the global performance index may exhibit the normal decreasing pattern towards convergence as expected in a minimization problem.

Two techniques of implementation were used in the algorithms. In the single loop technique, for the price coordination approach, the coordinator is made up of the price updating mechanism and the model-reality modifiers computation. For the direct coordination approach, the coordinator is made up of price, interaction and model-reality modifiers computations. In the double loop technique, the global structure of interconnection constraints is exploited by creating an outer loop where global price and interaction computation can be made. In the price coordination approach, the price updating mechanism is computed in the outer loop while, in the direct coordination approach, both the interaction term and the interconnection price are computed in the outer loop. The double loop technique is much easier to implement in software because there is no need to separate the interacting components for each subsystem at the coordinator level. This separation of global component also facilitates the convergence analysis of the algorithm.

A total of twelve algorithms , PC1-PC6 and DC1-DC6 have been developed based on the three basic hierarchical structures using a linear quadratic model formulation. Simulation studies using two examples were carried out to test the effectiveness of the algorithms using a sample combination of tuning parameters. The efficiency of the algorithms is measured on the basis of the number of iterations needed for convergence. Simulation results show that algorithms based on a structure with model based interaction input (PC1, PC2, DC1 and DC2) were the slowest to converge. Algorithms based on a structure with real input in interaction and parameter estimation (PC5, PC6, DC5 and DC6) ,are observed to be most efficient. It is also observed that application of convexification terms improves the robustness of the algorithms. Direct coordination algorithms are observed to be most efficient for solving a time invariant nonlinear interconnected system as shown by Examples 4.4.1 and 6.4.1. For a time varying sinusoidal system such as described in Examples 4.4.2 and 6.4.2, algorithms based on price coordination approach are observed to be most efficient.

There is no significant difference between the single loop and double loop implementations . However the double loop implementation is preferable because of the ease of implementation and debugging in software.

The optimality of the twelve algorithms was established by showing that their optimality conditions at convergence agree with the solution of the real global optimal control problem (ROP). Initial convergence analysis was carried out to establish the sufficient conditions for convergence of the double loop algorithms. As mentioned previously, the double loop structure facilitates convergence analysis by the separation of the global component of the interconnection at the coordinator which reduces the mathematical complexity of the algorithms. The sufficient conditions show that the convergence properties may be influenced by the length of optimization horizon, updating mechanism stepsize, model-reality differences ,and convexification factors. It is observed that a longer optimization horizon and model-reality differences may cause divergence. Although the conditions derived are only sufficient and they do not necessarily need to be satisfied for convergence to occur, they point the way for

further theoretical investigation into the convergence properties of the algorithms. Convergence analysis of the single loop structure is difficult because of the mathematical complexities posed by the interconnecting components in the coordinator.

In summary, it may be said that the objectives stated in Chapter 1 have been achieved. Firstly, new hierarchical structures for optimal control of interconnected nonlinear systems have been developed using an analogous approach to that of ISOPE hierarchical structures (Brdys and Roberts, 1986). The nonlinear systems are assumed to be Frechet differentiable. Secondly, twelve hierarchical optimal control algorithms were developed based on a linear quadratic formulation and implemented in software using three of the four basic hierarchical structures. Simulations studies were carried out to test the effectiveness and efficiency of the algorithms using two different examples. Finally, the optimality of the structures and algorithms were established and theoretical investigation of convergence properties of the double algorithms were carried out. As a result, sufficient convergence conditions have been derived for the double loop hierarchical algorithms.

7.2 SUGGESTIONS FOR FURTHER RESEARCH

The subject of the research is concerned with the development and implementation of new hierarchical DISOPE algorithms. However the algorithms described are still at their initial stage of development before full on-line implementation can be considered. In recent years, various extensions of the centralized DISOPE algorithms has been developed (Becerra, 1994; Roberts, 1995; Becerra and Roberts, 1996). The author considers that these developments should be extended to the hierarchical algorithms described in this thesis in order to realize their full potential. Some possible extension to the research work described in this thesis are described below.

(a) In addition to further convergence analysis, local stability studies need to be carried out to examine the stability of the DISOPE algorithms described in this thesis.

(b) Extensions of hierarchical DISOPE for handling control , state dependent

(c) Application of more efficient updating techniques for the updating mechanism in the price coordination approach to improve convergence behaviour

(d) The extension of hierarchical DISOPE approach for handling time-delays in the dynamics of the (in general nonlinear) real optimal control problem.

(e) Application of hierarchical DISOPE techniques to optimal predictive control and batch process optimization.

REFERENCES

Abdelwahed S.S., Hassan M.F. and Sultan M.A.(1995) Partially asychronous co-state prediction algorithm. *IEE Proceeedings -Control Theory Applications.*, Vol 142, No.2, pp88-94.

Augustine P. and Roberts P.D. (1993). On-line hierarchical steady-state optimising control of a power plant using the ISOPE with dynamic information. *Transactions of Institute Of Measurements and Control.* Vol. 15(4), pp170-177.

Amini-Largarni Z. (1990). Algorithm development and analysis for on-line optimising control of large-scale industrial processses. PhD Thesis: City University, Department of Electrical, Electronic and Information Engineering, U.K

Amini-Largarni Z., Lin J. and Roberts P.D. (1992). Single iterative algorithm with global feedback for integrated system optimization and parameter estimation of large scale industrial processes: optimality, convergence and simulation. *IEE Proceedings, Part D, Vol. 139, pp. 447-459.*

Anderson B.D.O and Moore J.B. (1989). *Optimal Control: Linear Quadratic Methods.* (London: Prentice Hall)

Becerra V.M. (1993). Dynamic Integrated System Optimization and Parameter Estimation for optimal control of processes with model-reality differences- C++ version. City University, Control Engineering Research Centre, London. Internal Report CERC/VB/120: March 1993.

Becerra V.M. (1994). Development and applications of novel optimal control algorithms. PhD Thesis: City University, Control Engineering Research Centre, Department of Electrical, Electronic and Information Engineering, U.K

Becerra V.M. and Roberts P.D. (1996). Dynamic integrated system optimization and parameter estimation for discrete time optimal control of nonlinear systems. International Journal Of Control, Vol.63 ,pp257-281.

Bellman R.E and Dreyfus S.E. (1962). *Applied Dynamic Programming.* (Princeton,New Jersey: Princeton University Press).

Brdys M. (1983) Hierarchical Optimizing control of steady-state large scale systems under model-reality differences of mixed type- A mutually interacting approach. In: *3rd IFAC Symposium on Large Scale System Theory and Applications Proceedings.* Warsaw, pp49-57.

Brdys M. and Roberts P.D. (1986). Optimal structures for steady-state adaptive optimizing control of large-scale industrial processes. *International Journal of Systems Science,* Vol. 17, pp. 1449-1474.

Brdys M., Ellis J.E. and Roberts P.D.(1987). Augmented Integrated System Optimization and Parameter Estimation technique: derivation, optimality and convergence. *IEE Proceedings.* Vol 14, Part D, No. 3,pp. 201-209.

Brdys M., Roberts P.D., Badi M.M., Kokkinos I.C and Abdullah N.,(1989). Double Loop Iterative Strategies for Hierarchical Control of Industrial Processes. *Automatica,* Vol. 25, No. 5, pp. 743-751.

Brdys M., Abdullah N. and Roberts P.D.(1990) Augmented model-based double iterative loop techniques for hierarchical control of complex industrial processes. *International Journal of Control,* Vol. 52, pp. 549-570.

Bryson A.E and Ho Y.C.(1975). *Aplied Optimal Control.* Revised Printing (New York: Hemisphere Publishing Corporation).

Cai X. and Zhou J.(1985) Optimization of Large-Scale Time-delay Systems: Two interaction prediction algorithms with convergence proofs and application. In: *IFAC Control and Technology for Development Proceedings,* Beijing, pp. 323-328.

Chen S, Brdys M. and Roberts P.D. (1986). An integrated system optimization and parameter estimation technique for hierarchical control of steady-state systems. *International Journal of System Science,* Vol. 17, pp 1209-1228.

Cohen G. (1978). Optimization by decomposition and coordination: A unified approach. *IEEE Transactions on Automatic Control.* Vol. 23, No.2, pp. 222-232.

Durbeck R.C. (1965). Principles for simplification of optimising control models. PhD Thesis: Case Western Reserve University, USA.

Findeisen W., Bailey F.N., Brdys M., Malinowski K., Tatjewski P and Wozniak A (1980). *Control and Coordination in Hierarchical Systems.* Wiley , Chichester..

Foord A.G. (1974). On-line optimisation of a petrochemical complex. PhD Thesis: University Of Cambridge, U.K.

Hassan M. and Singh M.G. (1976). Optimization of nonlinear systems using a new two level method. *Automatica.* Vol.12, pp.359-363.

Hendawy Z.M. and Roberts P.D. (1989). A price correction mechanism using an augmented interaction balance method with feedback of large scale industrial processes: derivation and applicability. *Advanved Modelling and Simulation.* No.16, pp. 53-64.

Jamshidi M. (1983). *Large-Scale Systems: Modelling and Control.* (New York: Elsevier Science Publishing).

Kantorovich L and Akilov G. (1964). *Functional Analysis in Normed Spaces.* Pergamon Press, Oxford.

Kirk D.E. (1970). *Optimal Control Theory.*(Englewood, Cliffs: Prentice-Hall)

Lapidus L. and Luus R. (1967). *Optimal Control of Engineering Processes.* (Waltham, Massachusetts: Ginn/Blaisdell).

Leigh J.R. (1992). *Applied Digital Control: Theory, Design and Implementation.* 2nd Edition. (New York: Prentice Hall).

Lewis F.L. (1986). *Optimal Estimation.* (New York: John Wiley & Sons).

Lewis F.L. and Syrmos V.L.(1995). *Optimal Control* . 2nd Edition. (New York: John Wiley & Sons).

Lin J., Hendawy Z.M. and Roberts P.D. (1988). A modified integrated system optimisation angd parameter estimation technique for hierarchical control of steady-state systems. *Advanved Modelling and Simulation.* No.12, pp. 43-52.

Lin J., Kambhampati C. and Roberts P.D. (1989). A new two model approach for optimising control of large-scale steady state systems. *IMA Journal of Mathematical Control and Informations.* Vol 6, pp 39-61.

Mahmoud M.S. (1977). Multilevel systems control and applications. A survey. *IEEE Transactions on Systems, Man and Cybernatics,* SMC-7,pp125-143.

Mahmoud M.S., Hassan M.F and Singh M. G. (1980). A new hierarchical approach to the joint problem of systems identification and optimisation. *Large Scale Systems.* Vol.1. pp: 159-60.

Mahmoud M.S., Hassan M.F and Darwish M.G. (1985). *Large-Scale Control Systems, Theories and Techniques* (New York:Dekker).

Mayne D.Q. and Michalska H. (1990). Receding Horizon Control Of nonlinear systems. *IEEE Tansactions on Automatic Control.* Vol.35. No. 7, pp. 814-824.

Mesarovic M.D., Macko D. and Takahara Y. (1970). *Theory of Hierarchical Multilevel Systems.* (New York: Academic Press).

Michalska H, Ellis J.E. and Roberts P.D. (1985). Joint coordination method for the steady-state control of large scale systems. *International Journal of System Science*, Vol. 16, pp 605-618.

Mohd_Ismail B.A.A.(1994). On hierarchical structures for optimal control of large scale dynamical systems. City University, Control Engineering Research Centre, London. Internal Report CERC/BAAMI/133: August 1994.

Mohd_Ismail B.A.A. and Roberts P.D.(1995) On hierarchical structures for optimal control of interconnected dynamical systems. Paper presented at: *IFAC/IFORS/IMACS Symposium: Large Scale Systems: Theory and Applications*.London.UK: July 11-13, pp.203-207.

Papageorgiu M. and Schmidt G. (1980). On the hierarchical solution of nonlinear optimal control problems. *Large Scale Systems*. Vol.1, pp. 265-271.

Pontryagin L.S., Boltyanskii R.V., Gamkrelidze R.V. and Mishchenko E.F. (1962). *The Mathematical Theory Of Optimal Processes*. (New York: Willey Interscience).

Press W.H., Flannery B.P., Teukolsky S.A. and Vetterling W.T. (1992). *Numerical Recipes In C*. Second Edition. (Cambridge: Cambridge University Press).

Roberts P.D. (1979). An algorithm for steady state system optimissation and parameter estimation. *International Journal of Systems Science*, Vol. 10, pp.719-734.

Roberts P.D. (1988). On the structure of on-line hierarchical control algorithms. In: *IMACS 1988, 12th World Congress on Scientific Computation Proceedings*. Paris, France. Vol.3, 1988, pp.648-650.

Roberts P.D. (1992). Optimal control of nonlinear systems with model-reality differences. In: *31st. Conference on Decision and Control Proceedings*, pp257-258. Tucson, Arizona: December 16-18, 1992.

Roberts P.D.(1993a). An algorithm for optimal control of nonlinear systems with model-reality differences. In: *IFAC World Congress 1993 Proceedings*. Vol. 8, pp.407-412. Sydney, Australia: August 1993.

Roberts P.D. (1993b). Convergence analysis of the DISOPE algorithm. Unpublished manuscript.

Roberts P.D. (1994). Unit memory repetitive processes and iterative optimal control algorithms. in: *Control 94, IEE Conference Publication No. 389*, pp. 454-459. Coventry, UK: 21-24 March 1994.

Roberts P.D. (1995). Coping with model-reality differences in industrial process optimisation- A review of integrated system optimization and parameter estimation (ISOPE). *Computers in Industry* . Vol. 26, pp. 281-290.

Roberts P.D., Wan B. and Lin J. (1992). Steady state hierarchical control of large scale industrial processes: a survey. Plenary paper presented at: *IFAC/IFORS/IMACS Symposium: Large Scale Systems: Theory and Applications*. Beijing, P.R. China: August 23-25, pp.1-10.

Rockafellar R.T. (1974). Augmented Lagrange multiplier functions and duality in nonconvex programming. *SIAM Journal of Control*. Vol. 12, pp. 268-285.

Rubio J.E. (1971). *Theory of Linear Systems*. (New York: Acedemic Press).

Sage A.P. and White C.C. (1977). *Optimum Systems Control*. Second Edition. (Englewood Cliffs, New Jersey: Prentice Hall)

Schoeffler J.D. (1971). Static multilevel systems, in Wismer D.A., ed., *Optimization Methods of Large-Scale Systems*. (New York: McGraw-Hill).

Singh M.G. (1980). *Dynamical Hierarachical Control*. (Amsterdam: North-Holland Publishing Company).

Singh M.G. and Titli A.(1978). *Systems: Decomposition, Optimisation and Control*. (Oxford: Pergamon Press).

Tamura H. and Yoshikawa T. (1990). *Large-Scale Systems Control and Decision Making*. (New York: Dekker)

Tatjewsky P., Abdullah N. and Roberts P.D.(1988). Comparison of some algorithms for hierarchical steady-state optimising control of interconnected industrial processes. *IEE International Conference on Control '88 Proceedings*, Oxford, UK. pp. 257-531.

Tzafestas S.G. and Hassan M.F.(1986). Complex large scale systems methodologies in conjuction with modern computer technology. *Control- Theory and Advanced Technology*, Vol. 2, No. 2, pp105-130.

Tzafestas S.G. and Kapsiotis G.(1995). A unified hierarchical control algorithm for constrained non-linear multi-delay interconnected systems: Application to production planning. *Control- Theory and Advanced Technology*, Vol. 10, No. 4, pp1737-1761.

Teo K.L., Goh C.J. and Wong K.H.(1991). *A Unified Computational Approach to Optimal Control Problems*. (London: Longman Scientific & Technical).

APPENDIX

A. DERIVATION OF PROCEDURE 6.2.1

Here we want to solve the following discrete-time two-point boundary value problem (TPBVP):

$$x_i(k+1) = A_i x_i(k) - B_i \bar{R}_i^{-1}(B_i^T p_i(k+1) - \bar{\beta}_i(k))$$
$$-D_i \bar{S}_i^{-1}(D_i^T p_i(k+1) + \bar{I}_i(k)) + \alpha_i(k) \qquad \text{(A.1)}$$

$$p_i(k) = \bar{Q}_i x_i(k) + A_i^T p_i(k+1) - \bar{\lambda}_i(k) \qquad \text{(A.2)}$$

with boundary conditions:

$$x_i(0) = x_{io}; \qquad x_{i,t}(N) = 0; \quad t \in [1, q_i]$$

$$\bar{p}_i(N) = \Phi_i \bar{x}_i(N) \qquad \text{(A.3)}$$

where all the quantities are defined in Chapter 6 and the control law and the optimum interaction vector are given by:

$$c_i(k) = -\bar{R}_i^{-1}(B_i^T p_i(k+1) - \bar{\beta}_i(k)); \qquad k \in [0, N-1] \qquad \text{(A.4)}$$

and

$$u_i(k) = -S_i^{-1}(D_i^T p_i(k+1) + \bar{I}_i(k)); \qquad k \in [0, N-1] \qquad \text{(A.5)}$$

respectively.

The solution of this linear TPBVP can be obtained using the sweep method (Lewis, and Syrmos, 1995; Bryson and Ho, 1975). The key is to assume the relationship between state and costate as

$$p_i(k) = V_i(k) x_i(k) + E_i(k) v_i + h_i(k); \qquad k \in [0, N] \qquad \text{(A.6)}$$

where $V_i(k)$ is an $n_i \times n_i$ matrix, $E_i(k)$ is $n_i \times q_i$ matrix , $h_i(k) \in \Re^{n_i}$ and $v_i \in \Re^{q_i}$.

Substituting this in (A.1) and grouping terms results in the following:

$$x_i(k+1) = [I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1) + D_i \bar{S}_i^{-1} D_i^T V_i(k+1)]^{-1} [A_i x_i(k) + B_i R_i^{-1} \bar{\beta}_i(k)$$
$$-D_i \bar{S}_i^{-1} \bar{I}_i(k)) + \alpha_i(k) - (B_i \bar{R}_i^{-1} B_i^T + D_i \bar{S}_i^{-1} D_i^T) E_i(k+1) v_i$$
$$-(B_i \bar{R}_i^{-1} B_i^T + D_i \bar{S}_i^{-1} D_i^T) h_i(k+1)] \qquad \text{(A.7)}$$

Substituting (A.6) in (A.2) gives

$$V_i(k)x_i(k) + E_i(k)\nu_i + h_i(k)$$

$$= \overline{Q}_i x_i(k) + A_i^T V_i(k+1)x_i(k+1) + A_i^T E_i(k+1)\nu + A_i^T h_i(k+1) - \lambda_i(k) \qquad \text{(A.8)}$$

Substituting (4.9) into (4.10) and grouping terms

$$[-V_i(k) + \overline{Q}_i + A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} A_i]x_i(k)$$
$$+[A_i^T E_i(k+1) - E_i(k+1) - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$
$$(B_i \overline{R}_i^{-1} B_i^T + D_i \overline{S}_i^{-1} D_i^T)E_i(k+1)]\nu_i$$
$$+[A_i - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$
$$(B_i \overline{R}_i^{-1} B_i^T + D_i \overline{S}_i^{-1} D_i^T)]h_i(k+1) - h_i(k) + \overline{\lambda}_i(k)$$
$$+A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1}(B_i \overline{R}_i^{-1} \beta_i(k)$$
$$-D_i \overline{S}_i^{-1} \overline{l}_i(k) + \alpha_i(k)] = 0; \qquad \text{(A.9)}$$

Equating coefficients to zero in (A.9) results in the following set of difference equations, which can be solved backwards from terminal conditions shown.

$$V_i(k) = \overline{Q}_i + A_i V_i(k+1)[I_n + B_i \overline{R}_i B_i^T V_i(k+1) + D_i \overline{S}_i D_i^T V_i(k+1)]^{-1} A_i$$

$$k \in [0, N-1], \qquad V_i(N) = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_i \end{bmatrix}; \qquad \text{(A.10)}$$

$$E_i(k) = [A_i^T - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$
$$(B_i \overline{R}_i^{-1} B_i + D_i \overline{S}_i^{-1} D_i)]E_i(k+1)$$

$$k \in [0, N-1]; \qquad E_i(N) = \begin{bmatrix} I_{q_i} \\ 0 \end{bmatrix} \qquad \text{(A.11)}$$

$$h_i(k) = [A_i^T - A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^{-1} B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$
$$(B_i \overline{R}_i^{-1} B_i + D_i \overline{S}_i^{-1} D_i)]h_i(k+1) - \overline{\lambda}_i(k)$$
$$+A_i^T V_i(k+1)[I_n + B_i \overline{R}_i^T B_i^T V_i(k+1) + D_i \overline{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$
$$(B_i \overline{R}_i^{-1} \beta_i(k) - D_i \overline{S}_i^{-1} \overline{l}_i(k) + \alpha_i(k);$$
$$\text{where} \qquad h_i(N) = 0; \qquad k \in [0, N-1] \qquad \text{(A.12)}$$

Let us assume the (assumed fixed) terminal constraints function as

$$\Psi_i(k) = E_i(k)^T x_i(k) + W_i(k)\nu_i + \pi_i(k); \qquad k \in [0, N]$$

where $W_i(k)$ is a $q_i \times q_j$ matrix and $\pi_i(k) \in \Re^{q_j}$ and

$$\Psi_i(k) = [x_{i,1}(N).....x_{i,q}(N)]^T = [0....0]^T$$

then $$\Psi_i(k+1) = E_i(k+1)^T x_i(k+1) + W_i(k+1)\nu_i + \pi_i(k+1); \qquad \text{(A.13)}$$

Using (A.7) and equating $\Psi_i(k)$ and $\Psi_i(k+1)$ in (A.13) results in a repeat of (A.11) plus the following difference equations, which may be solved backwards.

$$W_i(k) = W_i(k+1) - E_i(k+1)^T [I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1) + D_i \bar{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$

$$(B_i \bar{R}_i^{-1} B_i^T + D_i \bar{S}_i^{-1} D_i^T)] E_i(k+1)$$

$$W(N) = 0; \qquad k \in [0, N-1] \qquad \text{(A.14)}$$

$$\pi_i(k) = \pi_i(k+1) + E_i(k+1)^T [I_n + B_i \bar{R}_i^{-1} B_i^T V_i(k+1) + D_i \bar{S}_i^{-1} D_i^T V_i(k+1)]^{-1} \bullet$$

$$[-(B_i \bar{R}_i^{-1} B_i + D_i \bar{S}_i^{-1} D_i)] h_i(k+1) + (B_i \bar{R}_i^T \beta_i(k) - D_i \bar{S}_i^{-1} \bar{l}_i(k) + \alpha_i(k))]$$

where $\pi_i(N) = 0; \qquad k \in [0, N-1] \qquad \text{(A.15)}$

We need to find an expression for multiplier ν_i. then from the terminal constraint function (A.13), we obtain the following

$$\nu_i = W_i(k)[E_i(k)^T x_i(k) - \pi_i(k)] \qquad \text{(A.16)}$$

for k∈[0,N]. Noticing that $W_i(N)$ is ill conditioned

$$\tilde{p}_i(N) = \nu_i = -W_i(0)[E_i(0)^T x_{io} + \pi_i(0)] \qquad \text{(A.17)}$$

Hence $\tilde{p}_i(N)$ can be obtained once $E_i(k)$,$W_i(k)$ and $\pi_i(k)$, $k \in [0, N]$. Thus from (A.6) we obtained the following

$$p_i(k) = V_i(k)x_i(k) + E_i(k)\tilde{p}_i(N) + h_i(k); \qquad k \in [0, N] \qquad \text{(A.18)}$$

Substituting (A.18) in (A.4) gives

$$c_i(k) = -\bar{R}_i^{-1} B_i^T [V_i(k+1)x_i(k+1) + E_i(k+1)\tilde{p}_i(N) + h_i(k+1) + \bar{R}_i^{-1}\bar{\beta}_i(k));$$
$$k \in [0, N-1] \qquad \text{(A.19)}$$

which can be expressed as

$$c_i(k) = -G_i(k)x_i(k) + g_i(k) \qquad \text{(A.20)}$$

where $G_i(k) = \bar{R}_i^{-1} B_i^T V_i(k+1)[I_n + B_i \bar{R}_i B_i^T V_i(k+1) + D_i \bar{S}_i D_i^T V_i(k+1)]^{-1} A_i$

and

$$g_i(k) = -\bar{R}_i^{-1} B_i^T V_i(k+1)[I_n + B_i \bar{R}_i B_i^T V_i(k+1) + D_i \bar{S}_i D_i^T V_i(k+1)]^{-1} \bullet$$

$$[(B_i\bar{R}_i^T\beta_i(k)-D_i\bar{S}_i^{-1}\bar{I}_i(k))+\alpha_i(k)$$

$$-(B_i\bar{R}_i^{-1}B_i^T+D_i\bar{S}_i^{-1}D_i^T)](E_i(k+1)v+h_i(k+1))$$

$$+E_i(k+1)\tilde{p}_i(N)+h_i(k+1)]+\bar{R}_i^T\beta_i(k) \qquad k\in[0,N-1] \qquad \text{(A.21)}$$

The interaction term is given by

$$u_i(k)=-\bar{S}_i^{-1}D_i^T[V_i(k+1)x_i(k+1)+E_i(k+1)\tilde{p}_i(N)+h_i(k+1)]$$

$$-\bar{S}_i^{-1}\bar{I}_i(k)) \qquad k\in[0,N-1] \qquad \text{(A.22)}$$

If we define

$$M_i(k)=\bar{S}_i^{-1}D_i^TV_i(k+1)[I_n+B_i\bar{R}_iB_i^TV_i(k+1)+D_i\bar{S}_iD_i^TV_i(k+1)]^{-1}A_i$$

and

$$m_i(k)=-\bar{S}_i^{-1}D_i^TV_i(k+1)[I_n+B_i\bar{R}_iB_i^TV_i(k+1)+D_i\bar{S}_iD_i^TV_i(k+1)]^{-1}\bullet$$

$$[(B_i\bar{R}_i^T\beta_i(k)-D_i\bar{S}_i^{-1}\bar{I}_i(k))+\alpha_i(k)$$

$$-(B_i\bar{R}_i^{-1}B_i^T+D_i\bar{S}_i^{-1}D_i^T)](E_i(k+1)v+h_i(k+1))$$

$$+E_i(k+1)\tilde{p}_i(N)+h_i(k+1)]-\bar{S}_i^{-1}\bar{I}_i(k)) \qquad k\in[0,N-1] \qquad \text{(A.23)}$$

then (A.22) can written as

$$u_i(k)=-M_i(k)x_i(k)+m_i(k) \qquad \text{(A.24)}$$

Thus from (4.22) and (4.26), the state equation can be expressed as

$$x_i(k+1)=(A_i-B_iG_i(k)-D_iM_i(k))x_i(k)+B_ig_i(k)+D_im_i(k)+\alpha_i(k);$$

$$x_i(0)=x_{io}, \qquad x_i(N)=0;\ t\in[1,q]; \quad k\in[0,N-1] \qquad \text{(A.25)}$$

A straightforward reasoning on dependence of different variables involved, gives rise to Procedure 6.2.1.

B. PROOF OF THEOREM 6.1

To prove the above theorem we consider the equivalent global optimality conditions of ROP (3.10) and MMOP2' define by (6.1):

From (3.14) -(3.21), at the optimal solution of ROP the following conditions are satisfied

$\nabla_{c(k)}\mathbf{H}^*=0$ that is

$$[q^*_{c(k)}(x^{op},c^{op},u^{op},k)]^T+[f^*_{c(k)}(x^{op},c^{op},u^{op},k)]^T p^{op}(k+1)$$
$$-[K^*_{c(k)}(x^{op},c^{op})]H^T l^{op}(k)=0 \qquad \text{(B.1)}$$

$\nabla_{x(k)}\mathbf{H}^*-p(k)=0$ that is

$$[q^*_{x(k)}(x^{op},c^{op},u^{op},k)]^T+[f^*_{x(k)}(x^{op},c^{op},u^{op},k)]^T p_{op}(k+1)$$
$$-[K^*_{x(k)}(x^{op},c^{op})]H^T l^{op}(k)-p^{op}(k)=0 \qquad \text{(B.2)}$$

$\nabla_{u(k)}\mathbf{H}^*=0$ that is

$$[q^*_{u(k)}(x^{op},c^{op},u^{op},k)]^T+[f^*_{u(k)}(x^{op},c^{op},u^{op},k)]^T p^{op}(k+1)+l^{op}(k)=0 \qquad \text{(B.3)}$$

with

$$x^{op}(k+1)=f^*(x^{op},c^{op},u^{op},k);\ \mathrm{k}\in[0,\ \mathrm{N\text{-}1}] \qquad \text{(B.4)}$$

$$u^{op}(k)=HK^*(x^{op},c^{op}) \qquad \text{(B.5)}$$

$$x^{op}(0)=x_0;\quad x^{op}_t(N)=0,\quad t\in[1,q]$$
$$p^{op}_t(N)=\Phi\bar{x}^0(N),\quad t\in[q+1,n] \qquad \text{(B.6)}$$

where $\bar{x}^0(N)=[x^{op}_{q+1}(N)\ldots x^{op}_n(N)]^T$

We require the converged solution , $u^c(k),x^c(k),u^c(k),p^c(k);\ k\in[1,N]$, of the modified model based optimal control problem (6.1). Noting that the convexification terms can be ignored, since at convergence $v(k)=c(k)$, $z(k)=x(k)$ and $w(k)=u(k)$, this solution will satisfy the Hamiltonian

$$\begin{aligned}\mathbf{H}^c(.)=&\tfrac{1}{2}[x^c(k)^T Qx^c(k)+c^c(k)^T Rc^c(k)+u^c(k)^T Su^c(k)]\\&+\gamma^c(k)-\lambda^c(k)^T x^c(k)-\beta^c(k)^T c^c(k)\\&+p^c(k+1)H^T[Ax^c(k)+Bc^c(k)+Du^c(k)+\alpha^c(k)]\\&+l^c(k)^T[u^c(k)-H(\mathcal{A}x^c(k)+\mathcal{K}c^c(k)+\mathcal{L}u^c(k)+\mathcal{O}^c(k)]\end{aligned} \qquad \text{(B.7)}$$

and optimality conditions

$\nabla_{c(k)}\mathbf{H}^c(.)=0$

$$\Rightarrow Rc^c(k)+B^T p^c(k+1)-\mathcal{K}^T H^T l^c(k)-\beta^c(k)=0 \tag{B.8}$$

$\nabla_{x(k)}\mathbf{H}^c(.)-p^c(k)=0$

$$\Rightarrow Qx^c(k)+A^T p^c(k+1)-\mathcal{J}^T H^T l^c(k)-\lambda^c(k)=p^c(k) \tag{B.9}$$

$\nabla_{u(k)}\mathbf{H}^c(.)=0$,which can be expressed as

$$\Rightarrow Su^c(k)+D^T p^c(k+1)-\frac{\partial^T}{\partial u^c(k)}[u^c(k)-H(\mathcal{J}x^c(k)+\mathcal{K}c^c(k)+\mathcal{L}u^c(k)]l^c(k)=0 \tag{B.10}$$

together with

$$x^c(k+1)=Ax^c(k)+Bc^c(k)+Du^c(k)+\alpha^c(k) \tag{B.11}$$

where , from (4.29)

$$\alpha^c(k)=f^*(x^c(k),c^c(k),HK^*(x^c(k),c^c(k),k) - Ax^c(k)-Bc^c(k)-DHK^*(x^c(k),c^c(k)$$

$$\theta^c(k)=K^*(x^c(k),c^c(k))-\mathcal{J}x^c(k)-\mathcal{K}c^c(k)-\mathcal{L}HK^*(x^c(k),c^c(k) \tag{B.12}$$

From the modifier equations (6.13)

$$\begin{aligned}\lambda^c(k)&=[A-f_x^*(x^c(k),c^c(k),u^c(k),k]^T p^c(k+1)+Qx^c(k)-q_x^*(x^c,c^c,u^c,k)^T\\&\quad+K_x^*(x^c,c^c)H^T l^c(k)-\mathcal{J}^T H^T l^c(k)\\ \beta^c(k)&=[B-f_c^*(x^c(k),c^c(k),u^c(k),k]^T p^c(k+1)+Rc^c(k)-q_c^*(x^c,c^c,u^c,k)^T\\&\quad+K_c^*(x^c,c^c)H^T l^c(k)-\mathcal{K}^T H^T l^c(k)\end{aligned} \tag{B.13}$$

and from the price updating mechanism

$$\begin{aligned}l^{c,s+1}(k)&=l^{c,s}(k)+\varepsilon_l[u^c(k)\\&\quad-H(\mathcal{J}x^c(k)+\mathcal{K}c^c(k)+\mathcal{L}u^c(k)+\theta^c(k)]\end{aligned} \tag{B.14}$$

Eliminating $\lambda^c(k)$ and $\beta^c(k)$ from (B.8) and (B.9) using modifiers equation (B.13), it can be readily be seen that equation (B.8) becomes

$$[q^{*}_{c(k)}(x^c,c^c,u^c,k)]^T+[f^{*}_{c(k)}(x^c,c^c,u^c,k)]^T p^c(k+1) \\ -[K^{*}_{c(k)}(x^c,c^c)]H^T l^c(k)=0 \qquad \text{(B.15)}$$

equation (B.9) becomes

$$[q^{*}_{x(k)}(x^c,c^c,u^c,k)]^T+[f^{*}_{x(k)}(x^c,c^c,u^c,k)]^T p^c(k+1) \\ -[K^{*}_{x(k)}(x^c,c^c)]H^T l^c(k)-p^c(k)=0 \qquad \text{(B.16)}$$

From (B.12) , it can be concluded that there is no model-reality differences with respect to $u^c(k)$, therefore it can assumed that $q_{u(k)}(.)=q^{*}_{u(k)}(.)$ and $f_{u(k)}(.)=f^{*}_{u(k)}(.)$, .Since at optimum we have $K^{*}(.)=F(.)$,then (B.10) reduces

$$[q^{*}_{u(k)}(x^c,c^c,u^c,k)]^T+[f^{*}_{u(k)}(x^c,c^c,u^c,k)]^T p^c(k+1)+l^c(k)=0 \qquad \text{(B.17)}$$

Substituting (B.12) in (B.11) produces

$$x^c(k+1)=f^{*}(x^c,c^c,u^c,k)\,;\ \mathrm{k}\in[0,\ \mathrm{N\text{-}1}] \qquad \text{(B.18)}$$

with boundary conditions

$$x^c(0)=x_0;\quad x^c_t(N)=0,\quad t\in[1,q] \\ p^c_t(N)=\Phi\bar{x}^0(N),\quad t\in[q+1,n] \qquad \text{(B.19)}$$

where $\bar{x}^0(N)=[x^c_{q+1}(N)\ldots x^c_n(N)]^T$

Eliminating $\theta^c(k)$ from (B.14) using (B.12), and noting that at convergence the interaction price $l^{c,s+1}(k)=l^{c,s+1}(k)$ we obtain the following equation

$$0=\varepsilon_l[u^c(k)-H(K^{*}(x^c,c^c))] \qquad \text{(B.20)}$$

Since scalar $\varepsilon_l>0$, then equation (5.23) reduces to

$$u^c(k)=H(K^{*}(x^c,c^c)) \qquad \text{(B.21)}$$

Comparing (B.1)-(B.6) with (B.15)-(B.19) and (B.21), it is clear that the two set of optimality conditions are in agreement. Furthermore from assumption 3.1

$$x^c(k)=x^{op}(k),\quad c^c(k)=c^{op}(k),\quad u^c(k)=u^{op}(k),\quad p^c(k)=p^{op}(k);\quad k\in[0,N]$$

(B.22)

Q.E.D

C. PROOF OF THEOREM 6.2

To prove the theorem we will show that the optimality conditions of ROP given by (3.10) are equivalent to the converged solution of MMOP2'' defined by (6.32) produced by the direct coordination approach.

The equivalent optimality conditions of (3.10) are given by equations (B.1)-(B.6).

We require the converged solution , $u^c(k), x^c(k), u^c(k), p^c(k);\ k \in [1, N]$, of the modified model based optimal control problem (6.32). Noting that the convexification terms can be ignored, since at convergence $v(k)=c(k)$, $z(k)=x(k)$ and $w(k)=u(k)$, this solution will satisfy the Hamiltonian

$$\begin{aligned}\mathbf{H}^c(.) = \tfrac{1}{2}[x^c(k)^T Q x^c(k) + c^c(k)^T R c^c(k) + u^c(k)^T S u^c(k)] \\ + \gamma^c(k) - \lambda^c(k)^T x^c(k) - \beta^c(k)^T c^c(k) \\ + p^c(k+1) H^T [A x^c(k) + B c^c(k) + D u^c(k) + \alpha^c(k)] \\ + l^c(k)^T [u^c(k) - H(\mathcal{J} x^c(k) + \mathcal{K} c^c(k) + \theta^c(k)]\end{aligned} \tag{C.1}$$

and optimality conditions (B.8), (B.9) and

$$\nabla_{u(k)} \mathbf{H}^c(.) = 0$$

$$\Rightarrow S u^c(k) + D^T p^c(k+1) + l(k) = 0 \tag{C.2}$$

together with model dynamic equation (B.11). From (6.14) and (6.15), we have

$$\begin{aligned}\alpha^c(k) = f^*(x^c(k), c^c(k), HK^*(x^c(k), c^c(k), k) \\ - A x^c(k) - B c^c(k) - D H K^*(x^c(k), c^c(k)\end{aligned} \tag{C.3}$$

$$\theta^c(k) = K^*(x^c(k), c^c(k)) - \mathcal{J} x^c(k) - \mathcal{K} c^c(k) \tag{C.4}$$

In addition to the above we have used the following optimality condition to calculate $u^c(k)$, that is;

$$u^c(k) = H(\mathcal{J} x^c(k) + \mathcal{K} c^c(k) + \theta^c(k)) \tag{C.5}$$

From (6.13) we have the following equations

$$\begin{aligned}\lambda^c(k) = [A - f_x^*(x^c(k), c^c(k), u^c(k), k]^T p^c(k+1) + Q x^c(k) - q_x^*(x^c, c^c, u^c, k)^T \\ + K_x^*(x^c, c^c) H^T l^c(k) - \mathcal{J}^T H^T l^c(k)\end{aligned}$$

$$\beta^c(k) = [B - f_c^*(x^c(k), c^c(k), u^c(k), k]^T p^c(k+1) + Rc^c(k) - q_c^*(x^c, c^c, u^c, k)^T + K_c^*(x^c, c^c) H^T l^c(k) - \mathcal{X}^T H^T l^c(k)$$

(C.6)

Eliminating $\lambda^c(k)$ and $\beta^c(k)$ from (B.8) and (B.9) using modifiers equation (C.6), produces (B.15) and (B.16) .

From (B.12) , it can be concluded that there is no model-reality differences with respect to $u^c(k)$, therefore it can assumed that $q_{u(k)}(.) = q_{u(k)}^*(.)$ and $f_{u(k)}(.) = f_{u(k)}^*(.)$, therefore (C.2) can be expressed as

$$[q_{u(k)}^*(x^c, c^c, u^c, k)]^T + [f_{u(k)}^*(x^c, c^c, u^c, k)]^T p^c(k+1) + l^c(k) = 0 \quad \text{(C.7)}$$

Eliminating $\alpha^c(k)$ in (B.11) using (C.3) we obtain (B.18) with boundary conditions (B.19).

Substituting (C.4) in equation (C.5) gives

$$u^c(k) = H(K^*(x^c, c^c)) \quad \text{(C.8)}$$

Comparing (B.1)-(B.6) with (B.15), (B.16), (C.7), (C.8), (B.18) and (B.19), it is clear that the two set of optimality conditions are in agreement. Furthermore from assumption 3.1,

$$x^c(k) = x^{op}(k), \quad c^c(k) = c^{op}(k), \quad u^c(k) = u^{op}(k), \quad p^c(k) = p^{op}(k); \quad k \in [0, N]$$

(C.9)

Q.E.D

D. PROOF OF THEOREM 6.3

To prove the theorem we will show that the optimality conditions of ROP given by (3.10) are equivalent to the converged solution of MMOP3' defined by (6.58) produced by the direct coordination approach.

The equivalent optimality conditions of (3.10) are given by equations (B.1)-(B.6).

We require the converged solution , $u^c(k), x^c(k), u^c(k), p^c(k); \; k \in [1, N]$, of the modified model based optimal control problem (6.58). Noting that the convexification terms can be ignored, since at convergence $v(k) = c(k)$ and $z(k) = x(k)$ this solution will satisfy the Hamiltonian

$$\begin{aligned}\mathbf{H}^c(.) = \tfrac{1}{2}[x^c(k)^T Q x^c(k) + c^c(k)^T R c^c(k) + u^c(k)^T S u^c(k)] \\ +\gamma^c(k) - \lambda^c(k)^T x^c(k) - \beta^c(k)^T c^c(k) \\ +p^c(k+1)^T [Ax^c(k) + Bc^c(k) + Du^c(k) + \alpha^c(k)] \cdot \\ +l^c(k)^T [u^c(k) - HK^*(x^c(k), c^c(k)]\end{aligned} \tag{D.1}$$

and optimality conditions

$$\nabla_{c(k)} \mathbf{H}^c(.) = 0$$

$$\Rightarrow Rc^c(k) + B^T p^c(k+1) - K^{*T}_{c^c(k)} H^T l^c(k) - \beta^c(k) = 0 \tag{D.2}$$

$$\nabla_{x(k)} \mathbf{H}^c(.) - p^c(k) = 0$$

$$\Rightarrow Qx^c(k) + A^T p^c(k+1) - K^{*T}_{x^c(k)} H^T l^c(k) - \lambda^c(k) = p^c(k) \tag{D.3}$$

$$\nabla_{u(k)} \mathbf{H}^c(.) = 0 \ ,$$

$$\Rightarrow Su^c(k) + D^T p^c(k+1) + l^c(k) = 0 \tag{D.4}$$

together with

$$x^c(k+1) = Ax^c(k) + Bc^c(k) + Du^c(k) + \alpha^c(k) \tag{D.5}$$

where , from (6.70)

$$\begin{aligned}\alpha^c(k) = f^*(x^c(k), c^c(k), HK^*(x^c(k), c^c(k), k) \\ - Ax^c(k) - Bc^c(k) - DHK^*(x^c(k), c^c(k))\end{aligned} \tag{D.6}$$

From the modifier equations (6.69)

$$\begin{aligned}\lambda^c(k) = [A - f_x^*(x^c(k), c^c(k), u^c(k), k]^T p^c(k+1) + Qx^c(k) - q_x^*(x^c, c^c, u^c, k)^T \\ \beta^c(k) = [B - f_c^*(x^c(k), c^c(k), u^c(k), k]^T p^c(k+1) + Rc^c(k) - q_c^*(x^c, c^c, u^c, k)^T\end{aligned} \tag{D.7}$$

and from the price updating mechanism

$$l^{c,s+1}(k) = l^{c,s}(k) + \varepsilon_l [u^c(k) - HK^*(x^c(k), c^c(k))] \tag{D.8}$$

Eliminating $\lambda^c(k)$ and $\beta^c(k)$ from (D.2) and (D.3) using modifiers equation (D.7), it can be readily be seen that equation (D.2) becomes

$$[q^*_{c(k)}(x^c,c^c,u^c,k)]^T+[f^*_{c(k)}(x^c,c^c,u^c,k)]^T p^c(k+1) -[K^*_{c(k)}(x^c,c^c)]^T H^T l^c(k)=0 \quad \text{(D.9)}$$

equation (B.9) becomes

$$[q^*_{x(k)}(x^c,c^c,u^c,k)]^T+[f^*_{x(k)}(x^c,c^c,u^c,k)]^T p^c(k+1) -[K^*_{x(k)}(x^c,c^c)]^T H^T l^c(k)-p^c(k)=0 \quad \text{(D.10)}$$

From (D.6) , it can be concluded that there is no model-reality differences with respect to $u^c(k)$, therefore it can assumed that $q_{u(k)}(.)=q^*_{u(k)}(.)$ and $f_{u(k)}(.)=f^*_{u(k)}(.)$, then (D.4) is equivalent to

$$[q^*_{u(k)}(x^c,c^c,u^c,k)]^T+[f^*_{u(k)}(x^c,c^c,u^c,k)]^T p^c(k+1)+l^c(k)=0 \quad \text{(D.11)}$$

Substituting (D.6) in (D.5) produces

$$x^c(k+1)=f^*(x^c,c^c,u^c,k)\,;\ \mathrm{k}\in[0,\ \mathrm{N\text{-}1}] \quad \text{(D.12)}$$

with boundary conditions

$$x^c(0)=x_0;\quad x^c_t(N)=0,\quad t\in[1,q] \qquad p^c_t(N)=\Phi\bar{x}^0(N),\quad t\in[q+1,n] \quad \text{(D.13)}$$

where $\bar{x}^0(N)=[x^c_{q+1}(N)\ldots x^c_n(N)]^T$

Noting that at convergence the interaction price $l^{c,s+1}(k)=l^{c,s+1}(k)$ we obtain the following equation

$$0=\varepsilon_l[u^c(k)-H(K^*(x^c,c^c))] \quad \text{(D.14)}$$

Since scalar $\varepsilon_l>0$, then equation (D.14) reduces to

$$u^c(k)=H(K^*(x^c,c^c)) \quad \text{(D.15)}$$

Comparing (B.1)-(B.6) with (D.9)-(D.13) and (D.14), it is clear that the two set of optimality conditions are in agreement. Furthermore from assumption 3.1

$$x^c(k)=x^{op}(k),\ \ c^c(k)=c^{op}(k),\ u^c(k)=u^{op}(k),\ p^c(k)=p^{op}(k);\ k\in[0,N] \quad \text{(D.16)}$$

Q.E.D

E. PROOF OF THEOREM 6.4

Theorem 6.4 can be proven using the same definitions as in appendix D and following the similar reasonings as in appendix C.

F. CONVERGENCE OF ALGORITHM PC4

For simplicity, we will consider the special case of no terminal conditions and $\Phi = 0$.

First, we derive the algorithm mapping of the inner loop of algorithm PC4.

Let s denote the iteration number of the inner loop and p denote the corresponding iteration number of the outer loop.

The inner loop:

The inner loop constitutes iterations in the ith decomposed modified model based optimal control problem ($MMOP2_i$)

We need to established the transition from iteration s to iteration s+1 of the inner loop of the double loop price coordination algorithm PC4 with a linear model and quadratic performance criterion defined by (6.1) in terms of difference equations.

From (6.13) we obtain the following expressions representing the computation of modifiers.

$$\lambda_i^s(k)=[A_i - f_{x_i}^*(x_i^s,c_i^s,u_i^s),k]^T p_i^s(k+1)+Q_i\, x_i^s(k)-q_{x_i}^*(x_i^s,c_i^s,u_i^s,k)^T$$
$$+K_{x_i}^*(x_i^s,c_i^s)\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)-\mathcal{J}_i^T\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)$$

$$\beta_i^s(k)=[B_i - f_{c_i}^*(x_i^s,c_i^s,u_i^s),k]^T p_i^s(k+1)+R_i\, c_i^s(k)-q_{c_i}^*(x_i^s,c_i^s,u_i^s,k)^T$$
$$+K_{c_i}^*(x_i^s,c_i^s)\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)-\mathcal{K}_i^T\sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) \qquad \text{(F.1)}$$

where $l_j^p(k)$ is prescribed by computation in the outer loop.

From equation (4.29) the calculation of parameters can be expressed as

$$\alpha_i^S(k)=f_i^*(x_i^S,c_i^S,u_i^S,k)-A_i x_i^S(k)-B_i c_i^S(k)-D_i\ H_i K^*(x_i^s(k),c_i^s(k))$$

$$\theta_i^S(k)=K_i^*(x_i^S,c_i^S)-\mathcal{J}_i x_i^S(k)-\mathcal{K}_i\ c_i^S(k)-\mathcal{L}_i H_i K^*(x_i^s(k),c_i^s(k))$$

(F.2)

The solution of the MMOP2$_i$ defined by (6.1) can be represented by

$$\hat{c}_i^S(k)=\overline{R}_i^{-1}[-B_i^T p_i^S(k+1)+\beta_i^S(k)+\mathcal{K}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)+r_1 c_i^S(k)]$$

$$\hat{x}_i^S(k+1)=A_i\hat{x}_i^S(k)+B_i\hat{c}_i^S(k)+D_i\hat{u}_i^S(k)+\alpha_i^S(k)$$

$$\hat{p}_i^S(k)=\overline{Q}_i\hat{x}_i^S(k)+A\hat{p}_i^S(k+1)-\lambda_i^S(k)-\mathcal{J}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)+r_2 x_i^S(k); \qquad \text{(F.3)}$$

$$\hat{p}_i^S(N)=0$$

$$\hat{u}_i^S(k)=\overline{S}_i^{-1}[-D_i^T p_i^S(k+1)+l_i^P(k)+\mathcal{L}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k).+]$$

The control ,state, interaction and costate updating mechanism defined by (3.61) are expressed as

$$x_i^{s+1}(k)=x_i^S(k)+\varepsilon_x(\hat{x}_i^S(k)-x_i^S(k))$$

$$c_i^{s+1}(k)=c_i^S(k)+\varepsilon_c(\hat{c}_i^S(k)-c_i^S(k)) \qquad \text{(F.4)}$$

$$p_i^{s+1}(k)=p_i^S(k)+\varepsilon_p(\hat{p}_i^S(k)-p_i^S(k))$$

Equations (F1)-(F4) are defined for k∈[1,N-1] and

$\hat{c}(k)=v(k)$, $\hat{x}(k)=z(k)$ and $\hat{p}(k)=p(k)$.

Notice that in (5.38) we can write

$$\hat{p}_i^S(k+1)=-A_i^{-T}\overline{Q}_i\hat{x}_i^S(k)+A_i^T\hat{p}_i^S(k)+A_i^{-T}\lambda_i^S(k)+\mathcal{J}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)-r_2 A_i^{-T}x_i^S(k);$$

$$\hat{p}_i^S(N)=0$$

(F.5)

Substituting (F.1), (F.2) in (F.3) and applying (F.5), we can write

$$\begin{bmatrix}\hat{x}_i^S(k+1)\\ \hat{p}_i^S(k+1)\end{bmatrix}=\mathbf{E}_i\begin{bmatrix}\hat{x}_i^S(k)\\ \hat{p}_i^S(k)\end{bmatrix}+\mathbf{M}_{1,i}\mathbf{z}_i^S(k)+g_{1,i}(\mathbf{z}_i^S(k)) \qquad \text{(F.6)}$$

where $\mathbf{z}_i^S(k)=[c_i^S(k)^T \;\; x_i^S(k)^T \;\; p_i^S(k+1)^T]^T$ and

$$\mathbf{E}_{1,i}=\begin{bmatrix} A_i & -B_i\overline{R}_i^{-1}B_i^T-D_i\overline{S}_i^{-1}D_i^{-1} \\ -A_i^{-T}\overline{Q}_i & A_i \end{bmatrix} \quad . \tag{F.7}$$

$$\mathbf{M}_{1,i}=\begin{bmatrix} r_1B_iR_i^{-1} & 0_{n_i} & 0_{n_i} \\ 0_{n_i,m_i} & r_2A_i^{-T} & 0_{n_i} \end{bmatrix} \tag{F.8}$$

$$g_{1,i}(\mathbf{z}_i^S(k))=\begin{bmatrix} g_{11,i}(\mathbf{z}_i^S(k)) \\ g_{12,i}(\mathbf{z}_i^S(k)) \end{bmatrix} \tag{F.9}$$

with

$$\begin{aligned} g_{11,i}(\mathbf{z}_i^S(k))=&B_i\overline{R}_i^{-1}[B_i\;-f_{c_i}^*(x_i^S,c_i^S,u_i^S),k]^T\,p_i^S(k+1) \\ &+B_i\overline{R}_i^{-1}[R_i\,c_i^S(k)-q_{c_i}^*(x_i^S,c_i^S,u_i^S,k)^T] \\ &-B_i\overline{R}_i^{-1}[\mathcal{K}_i\;-K_{c_i}^*(x_i^S,c_i^S)]^T\sum_{j=1}^{N_S}H_{ij}^Tl_j^P(k)) \\ &+B_i\overline{R}_i^{-1}\mathcal{K}_i^T\sum_{j=1}^{N_S}H_{ij}^Tl_j^P(k)+D_i\overline{S}_i^{-1}l_i^P(k) \\ &+f_i^*(x_i^S,c_i^S,u_i^S,k)-A_ix_i^S(k)-B_ic_i^S(k)-D_i\,u_i^S(k) \end{aligned} \tag{F.10}$$

$$g_{12,i}(\mathbf{z}_i^S(k))=A_i^{-T}[A_i\;-f_{x_i}^*(x_i^S,c_i^S,u_i^S),k]^T\,p_i^S(k+1)$$

$$+A_i^{-T}[Q_i\,x_i^S(k)-q_{x_i}^*(x_i^S,c_i^S,u_i^S,k)^T] \tag{F.11}$$

$$-A_i^{-T}[\mathcal{J}_i\;-K_{x_i}^*(x_i^S,c_i^S)]^T\sum_{j=1}^{N_S}H_{ij}^Tl_j^P(k)+A_i^{-T}\mathcal{J}_i^T\sum_{j=1}^{N_S}H_{ij}^Tl_j^P(k)$$

It should be noted that, in (F.6), $\mathbf{E}_i$ is a transition matrix, $\mathbf{H}_{1,i}$ contains solely convexification terms (i.e $\mathbf{H}_{1,i}=0$ if $r_1=r_2=0$), and $g_{12,i}(\mathbf{z}_i^S(k))$ represents the model-reality differences.

Equation (F.6) can be written as

$$\begin{bmatrix}\hat{x}_i^S(k)\\ \hat{p}_i^S(k)\end{bmatrix} = \mathbf{E}_i^k\begin{bmatrix}x_{io}\\ \hat{p}_i(0)\end{bmatrix} + \sum_{j=0}^{k-1}\mathbf{E}_i^j[\mathbf{M}_{1,i}\mathbf{z}_1^S(k-1-j) + g_{1,i}(\mathbf{z}_i^S(k-1-j)); \tag{F.12}$$

$$\hat{p}_i^S(N) = 0$$

Writing $\phi(k) = \mathbf{E}_{1,i}^k$,at $k = N$, the final costate is given by

$$\hat{p}_i^S(N) = \phi_{21,i}(N)x_{io} + \phi_{22,i}(N)\hat{p}_i^S(0)$$
$$+\sum_{j=0}^{k-1}\phi_{2,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_1^S(N-1-j) + g_{1,i}(\mathbf{z}_i^S(N-1-j)) \tag{F.13}$$

with

$$\mathbf{E}_i^N = \phi_i(N) = \begin{bmatrix}\phi_{11,i}(N) & \phi_{12,i}(N)\\ \phi_{21,i}(N) & \phi_{22,i}(N)\end{bmatrix};\phi_{2,i}(j) = \begin{bmatrix}\phi_{21,i}(j) & \phi_{22,i}(j)\end{bmatrix} \tag{F.14}$$

Hence, the initial costate can be expressed in the form

$$\hat{p}_i^S(0) = \phi_{22,i}(N)^{-1}\phi_{21,i}(N)x_{io}$$
$$-\phi_{22,i}(N)^{-1}\sum_{j=0}^{k-1}\phi_{2,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_1^S(N-1-j) + g_{1,i}(\mathbf{z}_i^S(N-1-j))$$

(F.15)

Substituting (5.50) in (5.47), we obtain

$$\begin{bmatrix}\hat{x}_i^S(k)\\ \hat{p}_i^S(k)\end{bmatrix} = \begin{bmatrix}\mu_{x_i}(N,k)\\ \mu_{p_i}(N,k)\end{bmatrix}x_{io}$$
$$-\begin{bmatrix}\phi_{12,j}(k)\\ \phi_{22,j}(k)\end{bmatrix}\phi_{22,j}(N)^{-1}\sum_{j=0}^{N-1}\phi_{2,j}(j)[\mathbf{M}_{1,i}\mathbf{z}_1^S(N-1-j) + g_{1,i}(\mathbf{z}_i^S(N-1-j))]$$
$$+\sum_{j=0}^{k-1}\phi_i(j)[\mathbf{M}_{1,i}\mathbf{z}_1^S(k-1-j) + g_{1,i}(\mathbf{z}_i^S(k-1-j))] \tag{F.16}$$

where

$$\mu_{x_i}(N,k) = \phi_{11,i}(k) - \phi_{12,i}(k)\phi_{22,i}(N)^{-1}\phi_{21,i}(N)$$
$$\mu_{p_i}(N,k) = \phi_{21,i}(k) - \phi_{22,i}(k)\phi_{22,i}(N)^{-1}\phi_{21,i}(N) \tag{F.17}$$

Using (F.1) and (F.3) to eliminate $\hat{p}_i^S(k+1)$ and $\beta_i^S(k)$ from the optimal control estimate $\hat{c}_i^S(k)$ gives

$$\hat{c}_i^S(k) = -\overline{R}_i^{-1} B_i^T A_i^{-T} \overline{Q}_i \mu_{x_i}(N,k) x_{io}$$

$$-\overline{R}_i^{-1} B_i^T A_i^{-T} \overline{Q}_i \phi_{12,i}(k) \phi_{22,i}(N)^{-1} \sum_{j=0}^{N-1} \phi_{2,i}(j) [\mathbf{M}_{1,i} \mathbf{z}_i^S(N-1-j) + g_{1,i}(\mathbf{z}_i^S(N-1-j))]$$

$$+\overline{R}_i^{-1} B_i^T A_i^{-T} \overline{Q}_i \sum_{j=0}^{k-1} \phi_i(j) [\mathbf{M}_{1,i} \mathbf{z}_i^S(k-1-j) + g_{1,i}(\mathbf{z}_i^S(k-1-j))]$$

$$-\overline{R}_i^{-1} B_i^T A_i^{-T} \mu_{p_i}(N,k) x_{io}$$

$$-\overline{R}_i^{-1} B_i^T A_i^{-T} \phi_{22,i}(k) \phi_{22,i}(N)^{-1} \sum_{j=0}^{N-1} \phi_{2,i}(j) [\mathbf{M}_{1,i} \mathbf{z}_i^S(N-1-j) + g_{1,i}(\mathbf{z}_i^S(N-1-j))]$$

$$+\overline{R}_i^{-1} B_i^T A_i^{-T} \sum_{j=0}^{k-1} \phi_i(j) [\mathbf{M}_{1,i} \mathbf{z}_i^S(k-1-j) + g_{1,i}(\mathbf{z}_i^S(k-1-j))]$$

$$-r_2 \overline{R}_i^{-1} B_i^T A_i^{-T} x_i^S(k) + r_1 \overline{R}_i^{-1} c_i^S(k) + \overline{R}_i^{-1} \mathcal{K}_i^T \sum_{j=1}^{N_S} H_{ij}^T l_j^p(k)$$

$$+\overline{R}_i^{-1} [B_i - f_{c_i}^*(x_i^S, c_i^S, u_i^S), k]^T p_i^S(k+1)$$

$$+\overline{R}_i^{-1} R_i c_i^S(k) - \overline{R}_i^{-1} q_{c_i}^*(x_i^S, c_i^S, u_i^S, k)^T$$

$$-\overline{R}_i^{-1} [\mathcal{K}_i - K_{c_i}^*(x_i^S, c_i^S)]^T \sum_{j=1}^{N_S} H_{ij}^T l_j^p(k)) + \overline{R}_i^{-1} \mathcal{J}_i^T \sum_{j=1}^{N_S} H_{ij}^T l_j^p(k)$$

$$-\overline{R}_i^{-1} B_i^T A_i^{-T} [A_i - f_{x_i}^*(x_i^S, c_i^S, u_i^S), k]^T p_i^S(k+1)$$

$$+\overline{R}_i^{-1} B_i^T A_i^{-T} Q_i x_i^S(k) - \overline{R}_i^{-1} B_i^T A_i^{-T} q_{x_i}^*(x_i^S, c_i^S, u_i^S, k)^T$$

$$+\overline{R}_i^{-1} B_i^T A_i^{-T} [\mathcal{J}_i - K_{x_i}^*(x_i^S, c_i^S)]^T \sum_{j=1}^{N_S} H_{ij}^T l_j^p(k)$$

$$-\overline{R}_i^{-1} B_i^T A_i^{-T} \mathcal{J}_i^T \sum_{j=1}^{N_S} H_{ij}^T l_j^p(k) \qquad \text{(F.18)}$$

Combining (F.17) and (F.18) produces

$$\hat{\mathbf{z}}_i^S(k) = \mu_i(N,k)x_{io}$$
$$-\eta_i(k)\phi_{22,i}(N)^{-1}\sum_{j=0}^{N-1}\phi_{2,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_i^S(N-1-j)+g_{1,i}(\mathbf{z}_i^S(N-1-j))]$$
$$+\sum_{j=0}^{k-1}\psi_{1,i}(j)[\mathbf{M}_{1,i}\mathbf{z}_i^S(k-1-j)+g_{1,i}(\mathbf{z}_i^S(k-1-j))] \qquad \text{(F.19)}$$
$$+\mathbf{M}_{2,i}\mathbf{z}_i^S(k)+g_{2,i}(\mathbf{z}_i^S(k))$$

where

$$\mu_i(N,k)=\begin{bmatrix}\mu_{c_i}(N,k)\\ \mu_{x_i}(N,k)\\ \mu_{p_i}(N,k)\end{bmatrix} \quad \text{with} \quad \mu_{c_i}(N,k)=\bar{R}_i^{-1}B_i^T A_i^{-T}[\bar{Q}\mu_{x_i}(N,k)-\mu_{p_i}(N,k)] \qquad \text{(F.20)}$$

and
$$\eta_i(k)=\begin{bmatrix}\bar{R}_i^{-1}B_i^T A_i^{-T}[\bar{Q}\phi_{12,i}(k)-\phi_{22,i}(k)]\\ \phi_{12,i}(k)\\ \phi_{22,i}(k)\end{bmatrix} \quad ; \qquad \text{(F.21)}$$

$$\psi_{1i}(j)=\begin{bmatrix}\bar{R}_i^{-1}B_i^T A_i^{-T}[\bar{Q}\phi_{1,i}(k)-\phi_{2,i}(k)]\\ \phi_{1,i}(k)\\ \phi_{2,i}(k)\end{bmatrix} \qquad \text{(F.22)}$$

$$\mathbf{M}_{2,i}=\begin{bmatrix}r_1\bar{R}_i^{-1} & -r_2\bar{R}_i^{-1}B_i^T A_i^{-T} & 0_{n_i}\\ 0_{n_i,m_i} & 0_{n_i} & 0_{n_i}\\ 0_{n_i,m_i} & 0_{n_i} & 0_{n_i}\end{bmatrix} \qquad \text{(F.23)}$$

and
$$g_{2,i}(\mathbf{z}_i^S(k))=\begin{bmatrix}g_{21,i}(\mathbf{z}_i^S(k))\\ 0\\ 0\end{bmatrix} \qquad \text{(F.24)}$$

with

$$g_{21,i}(\mathbf{z}_i^s(k))=$$

$$\bar{R}_i^{-1}[[B_i - f_{c_i}^*(x_i^s,c_i^s,u_i^s),k]^T - B_i^T A_i^{-T}[A_i - f_{x_i}^*(x_i^s,c_i^s,u_i^s),k]^T]p_i^s(k+1)$$

$$+\bar{R}_i^{-1}[R_i\, c_i^s(k) - q_{c_i}^*(x_i^s,c_i^s,u_i^s,k)^T] + \bar{R}_i^{-1}\mathcal{K}_i^T \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k)$$

$$+\bar{R}_i^{-1}B_i^T A_i^{-T}[Q_i\, x_i^s(k) - q_{x_i}^*(x_i^s,c_i^s,u_i^s,k)^T]$$

$$-\bar{R}_i^{-1}[[\mathcal{K}_i - K_{c_i}^*(x_i^s,c_i^s)]^T + B_i^T A_i^{-T}[\mathcal{J}_i - K_{x_i}^*(x_i^s,c_i^s)]^T] \sum_{j=1}^{N_s} H_{ij}^T l_j^p(k) \tag{F.25}$$

From (3.46)-(3.48) and (5.38), the updating mechanism can be written as

$$\mathbf{z}_i^{s+1}(k) = \varepsilon \hat{\mathbf{z}}_i^s(k) + [\mathbf{I}_{2n+m} - \varepsilon]\mathbf{z}_i^s(k) \tag{F.26}$$

with

$$\varepsilon_{\mathbf{z}i} = \begin{bmatrix} \varepsilon_c \mathbf{I}_{m_i} & 0_{m_i,n_i} & 0_{m_i,n_i} \\ 0_{n_i,m_i} & \varepsilon_x \mathbf{I}_{n_i} & 0_{n_i} \\ 0_{n_i,m_i} & 0_{n_i} & \varepsilon_p \mathbf{I}_i \end{bmatrix} \tag{F.27}$$

Combining (F.19) and (F.26) produces

$$\begin{aligned} \hat{\mathbf{z}}_i^{s+1}(k) = {} & \varepsilon_{\mathbf{z}i}\,\mu_i(N,k)x_{io} \\ & -\varepsilon_{\mathbf{z}i}\,\eta_i(k)\phi_{22,i}(N)^{-1} \sum_{j=0}^{N-1} \phi_i(N-1-j)[\mathbf{M}_{1,i}\mathbf{z}_i^s(j) + g_{1,i}(\mathbf{z}_i^s(j))] \\ & +\varepsilon_{\mathbf{z}i} \sum_{j=0}^{k-1} \psi(k-1-j)[\mathbf{M}_{1,i}\mathbf{z}_i^s(k) + g_{1,i}(\mathbf{z}_i^s(k))] \\ & +[\varepsilon_{\mathbf{z}i}\mathbf{M}_{2,i} + \mathbf{I}_{2n_i+m_i} - \varepsilon]\mathbf{z}_i^s(k) + \varepsilon_{\mathbf{z}i}\, g_{2,i}(\mathbf{z}_i^s(k)) \end{aligned} \tag{F.28}$$

The algorithm mapping of the inner loop showing how $\mathbf{z}_i^s(k)$, k $\in$[0,N] is updated from iteration s to iteration s+1 is obtained by the following simplification of (F.28), which can expressed as

$$\begin{aligned} \hat{\mathbf{z}}_i^{s+1}(k) = {} & \varepsilon_{\mathbf{z}i}\,\mu_i(N,k)x_{io} \\ & +\varepsilon_{\mathbf{z}i} \sum_{j=0}^{N-1} \Theta_{1i}(N,k,j)[\mathbf{M}_{1,i}\mathbf{z}_i^s(j) + g_{1,i}(\mathbf{z}_i^s(j))] \\ & +[\varepsilon_{\mathbf{z}i}\mathbf{M}_{2,i} + \mathbf{I}_{2n_i+m_i} - \varepsilon_{\mathbf{z}i}]\mathbf{z}_i^s(k) + \varepsilon_{\mathbf{z}i}\, g_{2,i}(\mathbf{z}_i^s(k)) \end{aligned} \tag{F.29}$$

where

$$\Theta_{1i}(N,k,j)=\begin{cases}\psi_i(k-1-j)-\eta_i(k)\phi_{22,i}(N)\phi_{2,i}(N-1-j); & j\in[0,k-1]\\ \eta_i(k)\phi_{22,i}(N)^{-1}\phi_{2,i}(N-1-j); & j\in[k,N-1]\end{cases}$$

(F.30)

We now proceed to derive the algorithm mapping of the outer loop.

The outer loop

From Algorithm 6.2.2 we note that the outer loop uses global_information available from the N_S subsystems from the inner loop. Let p denote the iteration number of the outer loop. The outer loop consists of the price updating mechanism (6.16) that is:

$$l^{p+1}(k+1)=l^{p}(k)+\varepsilon_l(\hat{u}^s(k)-H(\mathcal{J}\hat{x}^s(k)+\mathcal{K}\hat{c}^s(k)+\mathcal{L}\hat{u}^s(k)+\theta^s(k))$$

(F.31)

which can be simplified as

$$l^{p+1}(k+1)=l^{p}(k)+\varepsilon_l \mathrm{T}_1\hat{\mathbf{y}}^s(k)-\varepsilon_l H\theta^s(k) \qquad \text{(F.32)}$$

$$\text{with } \mathrm{T}_1=\begin{bmatrix}H\mathcal{K} & 0_{r,m} & 0_r\\ 0_{r,n} & H\mathcal{J} & 0_r\\ 0_{r,n} & 0_{r,m} & H\mathcal{L}-\mathbf{I}_r\end{bmatrix} \qquad \text{(F.33)}$$

and

$$\hat{\mathbf{y}}^s(k)=[\hat{c}^s(k)^T\ \hat{x}^s(k)^T\ \hat{u}^s(k)^T]^T$$
$$\mathbf{z}^s(k)=[c^s(k)^T\ x^s(k)^T\ \ p^s(k+1)^T]^T \qquad \text{(F.34)}$$

$\hat{\mathbf{y}}^s(k)$ is the optimum estimates of global control, state and interaction (defined in equation (3.4a)) as prescribed by the inner loop.

Using global definitions (3.4a), (3.24a) ,(5.2) and (5.3), and from equation (F.19) an expression for $\hat{\mathbf{y}}^s(k)$ can be written as

$$\hat{\mathbf{y}}_i^S(k) = \mu_{\mathbf{y}}(N,k)x_o$$

$$-\eta_{\mathbf{y}}(k)\phi_{22}(N)^{-1}\sum_{j=0}^{N-1}\phi_2(j)[\mathbf{M}_3\mathbf{z}^{\mathbf{S}}(N-1-j)+g_3(\mathbf{z}^S(N-1-j))]$$

$$-\eta_{\mathbf{y}}(k)\phi_{22}(N)^{-1}\sum_{j=0}^{N-1}\phi_2(j)[g_4(l^P(N-1-j))]$$

$$+\sum_{j=0}^{k-1}\psi_2(j)[\mathbf{M}_3\mathbf{z}^{\mathbf{S}}(k-1-j)+g_3(\mathbf{z}^S(k-1-j))]+\sum_{j=0}^{k-1}\psi_2(j)[g_4(l^S(k-1-j))]$$

$$+\mathbf{M}_4\mathbf{z}^{\mathbf{S}}(k)+g_5(\mathbf{z}^S(k))+g_6(l^P(k)) \tag{F.35}$$

where

$$\mu_{\mathbf{y}}(N,k)=\begin{bmatrix}\mu_c\ (N,k)\\ \mu_x\ (N,k)\\ \mu_u\ (N,k)\end{bmatrix};\qquad \eta_{\mathbf{y}}(k)==\begin{bmatrix}\overline{R}^{-1}B^TA^{-T}[\overline{Q}\phi_{12}(k)-\phi_{22}(k)]\\ \phi_{12}(k)\\ \overline{S}^{-1}D^TA^{-T}[\overline{Q}\phi_{12}(k)-\phi_{22}(k)]\end{bmatrix} \tag{F.36}$$

with $\phi_{(.)}$ denotes the corresponding global version of $\phi_{(.),i}$

$$\mathbf{M}_3=\begin{bmatrix}r_1B^T\overline{R}^{-1} & 0_n & 0_{n,r}\\ 0_{n,m} & r_2A^{-T} & 0_{n,r}\end{bmatrix}; \tag{F.37}$$

$$\psi_2(j)=\begin{bmatrix}\overline{R}^{-1}B^TA^{-T}[\overline{Q}\phi_1(k)-\phi_2(k)]\\ \phi_1(k)\\ \overline{S}^{-1}D^TA^{-T}[\overline{Q}\phi_1(k)-\phi_2(k)]\end{bmatrix} \tag{F.38}$$

$$\mathbf{M}_4=\begin{bmatrix}r_1\overline{R}^{-1} & -r_1\overline{R}^{-1}B^TA^{-T} & 0_{m,n}\\ 0_{n,m} & 0_n & 0_n\\ 0_{n,m} & 0_n & 0_n\end{bmatrix} \tag{F.39}$$

$$g_3(\mathbf{z}^S(k))=\begin{bmatrix}g_{31}(\mathbf{z}^S(k))\\ g_{32}(\mathbf{z}^S(k))\end{bmatrix} \tag{F.40}$$

with

$$g_{31}(\mathbf{z}^S(k)) = B\bar{R}^{-1}[B - f_c^*(x^S, c^S, u^S), k]^T p^S(k+1)$$
$$+B\bar{R}^{-1}[Rc^S(k) - q_c^*(x^S, c^S, u^S, k)^T] \quad \text{(F.41)}$$
$$+f^*(x^S, c^S, u^S, k) - Ax^S(k) - Bc^S(k) - Du^S(k)$$

$$g_{32}(\mathbf{z}^S(k)) = A^{-T}[A - f_x^*(x^S, c^S, u^S), k]^T p^S(k+1) \quad \text{(F.42)}$$
$$+A^{-T}[Qx^S(k) - q_x^*(x^S, c^S, u^S, k)^T]$$

$$g_4(l^P(k)) = \begin{bmatrix} g_{41}(l^P(k)) \\ g_{42}(l^P(k)) \end{bmatrix} \quad \text{(F.43)}$$

with

$$g_{41}(l^P(k)) = -B\bar{R}^{-1}[\mathcal{K} - K_c^*(x^S, c^S)]^T H^T l^P(k)$$
$$+B\bar{R}^{-1}\mathcal{K}^T H^T l^P(k) + D\bar{S}^{-1} l^P(k) \quad \text{(F.44)}$$

$$g_{42}(l^P(k)) = K_x^*(x^S, c^S)^T H^T l^P(k) \quad \text{(F.45)}$$

$$g_5(\mathbf{z}^S(k)) = \begin{bmatrix} g_{51}(\mathbf{z}^S(k)) \\ 0 \\ g_{53}(\mathbf{z}^S(k)) \end{bmatrix} \quad \text{(F.46)}$$

with

$$g_{51}(\mathbf{z}^S(k)) = \bar{R}^{-1}[[B - f_c^*(x^S, c^S, u^S), k]^T$$
$$-B^T A^{-T}[A - f_x^*(x^S, c^S, u^S), k]^T] p^S(k+1) \quad \text{(F.47)}$$
$$+\bar{R}^{-1}[Rc^S(k) - q_c^*(x^S, c^S, u^S, k)^T]$$
$$+\bar{R}^{-1}B^T A^{-T}[Qx^S(k) - q_x^*(x^S, c^S, u^S, k)^T]$$

$$g_{53,i}(\mathbf{z}_i^S(k)) = -\bar{S}_i^{-1}D^T A^{-T}[A - f_x^*(x^S, c^S, u^S), k]^T p^S(k+1)$$
$$-\bar{S}^{-1}D^T A^{-T}[Qx^S(k) - q_x^*(x^S, c^S, u^S, k)^T] \quad \text{(F.48)}$$

$$g_6(l^S(k))=\begin{bmatrix} g_{61}(l^S(k)) \\ 0 \\ g_{63}(l^S(k)) \end{bmatrix} \tag{F.49}$$

with

$$g_{61}(l^S(k))=\overline{R}^{-1}[K_c^*(x^S,c^S)]^T-B^TA^{-T}[\mathcal{J}-K_x^*(x^S,c^S)]^T]H^Tl^p(k) \tag{F.50}$$

$$\begin{aligned} g_{63}(l^S(k))&=-\overline{S}^{-1}D^TA^T\mathcal{K}^TH^Tl^p(k) \\ &-\overline{S}^{-1}D^TA^T[\mathcal{J}-K_x^*(x^S,c^S)]^TH^Tl^p(k)-\overline{S}^{-1}l^p(k) \end{aligned} \tag{F.51}$$

Equation (F.35) described the global output from the optimization of N_S subsystems in the lower level of the hierarchical algorithm PC2 to the outer loop. Note that the terms involving the interaction price $l(k)$ are isolated from the others in order to facilitate further derivation of the algorithm mapping of the outer loop .

Matrices $\mathbf{M}_3$ and $\mathbf{M}_4$ consist of convexification terms r_1 , r_2 and r_3. $g_3(\mathbf{z}^S(k))$ and $g_6(\mathbf{z}^S(k))$ represents the global model-reality differences in the global performance criterion and global plant dynamics. $g_4(l^p(k))$ and $g_6(l^p(k))$ are made up of model-reality differences in the global output functions and coefficients of $l(k)$.

Expression (F.35) can be further simplified by use of the following definition.

Define

$$\begin{aligned} W(N,k,j) &= \mu_{\mathbf{y}}(N,k)x_o \\ &-\eta_{\mathbf{y}}(k)\phi_{22}(N)^{-1}\sum_{j=0}^{N-1}\phi_2(j)[\mathbf{M}_3\mathbf{z}^S(N-1-j)+g_{3i}(\mathbf{z}^S(N-1-j))] \\ &+\sum_{j=0}^{k-1}\psi(j)[\mathbf{M}_3\mathbf{z}^S(k-1-j)+g_3(\mathbf{z}^S(k-1-j))] \\ &+\mathbf{M}_4\mathbf{z}^S(k)+g_5(\mathbf{z}^S(k))+\varepsilon_l H\theta^S(k) \end{aligned} \tag{F.52}$$

Definition (F.52) isolates terms prescribed by the inner loop that remain constant in the outer loop iteration. Using (F.52) in (F.35) and applying it to (F.32) produces the

algorithm mapping of the outer loop which describes how $l(k)$, k∈[0,N] is updated from iteration p to iteration p+1;

$$l^{p+1}(k+1) = g_7(l^p(k)) + \varepsilon_l \mathbf{T} W(N,k,j) + \varepsilon_l \mathbf{T} \sum_{j=0}^{N-1} \Theta_2(N,k,j)[g_4 l^p(j)] \quad \text{(F.53)}$$

where

$$\Theta_2(N,k,j) = \begin{cases} \psi_2(k-1-j) - \eta_{\mathbf{y}}(k)\phi_{22}(N)\phi_2(N-1-j); & j \in [0,k-1] \\ \eta_{\mathbf{y}}(k)\phi_{22}(N)^{-1}\phi_2(N-1-j); & j \in [k,N-1] \end{cases} \quad \text{(F.54)}$$

and $g_7(l^p(k)) = [\mathbf{I}_{m+n+r} l^p(k) - \varepsilon_l \mathbf{T} g_6(l^p(k))]$ (F.55)

Convergence behaviour is investigated by considering successive iterations of the algorithm in the inner loop and the outer loop, as defined by the mappings (F.29) and (F.53) respectively.

First, we consider the convergence of the inner loop. We need to determine conditions such that condition (5.91) is satisfied.

The following additional assumption is required.

Assumption F.1:

The functions $g_{1,i}(\mathbf{z}_i(k))$ and $g_{2,i}(z_i(k))$ defined by (F.9) and (F.24) are Lipshitz continuous for all $\mathbf{z}_i(k)$, $k \in [0,N]$, with Lipschitz constants $h_{1,i}$ and $h_{2,i}$ respectively. That is

$$\begin{aligned} \|g_{1,i}(\mathbf{z}_i^s(k)) - g_{1,i}(\mathbf{z}_i^{s-1}(k))\| &\le h_{1,i} \|\mathbf{z}_i^s(k) - \mathbf{z}_i^{s-1}(k)\| \\ \|g_{21,i}(\mathbf{z}_i^s(k)) - g_{21,i}(\mathbf{z}_i^{s-1}(k))\| &\le h_{2,i} \|\mathbf{z}_i^s(k) - \mathbf{z}_i^{s-1}(k)\| \end{aligned} \quad \text{(F.56)}$$

The following theorem presents the local convergence conditions of the inner loop.

Theorem F.3:

A sufficient condition for algorithm mapping (F.29) to satisfy (5.91) for every inner iteration s > 1 is given by the expression

$$(\sigma_{1,i}(N) + h_{1,i}\sigma_{2,i}(N))N + \varepsilon_c h_{2,i} + \|\varepsilon \mathbf{M}_{2,i} + \mathbf{I}_{2n_i+m_i} - \varepsilon\| < 1 \quad \text{(F.57)}$$

where $h_{1,i}$ and $h_{2,i}$ are defined in (F.56), ε and $\mathbf{M}_{2,i}$ are defined by (F.27) and (F.23) and

$$\|\sigma_{1,i}(N) = \sup_{k\in[0,N-1]} \sup_{j\in[0,N-1]} \|\varepsilon\Theta_i(N,k,j)\mathbf{M}_{1,i}\|$$
$$\|\sigma_{2,i}(N) = \sup_{k\in[0,N-1]} \sup_{j\in[0,N-1]} \|\varepsilon\Theta_i(N,k,j)\| \qquad \text{(F.58)}$$

with $\Theta_i(N,k,j)$ and $\mathbf{M}_{1,i}$ defined by (F.30) and (F.8).

Proof:

Condition (F.57) is similar in structure to condition (5.94), therefore Theorem F.3 can be proven using similar procedure as shown in the proof of Theorem 5.3.

Convergence property of the outer loop is studied by considering successive iterations of the algorithm mapping defined by (F.53) and determining conditions such that condition

$$\|l^{p+1}(k) - l^{p}(k)\| \le \|l^{p}(k) - l^{p-1}(k)\| \qquad \text{(F.59)}$$

where

$$\|l(k)\| = \sup_k \|l(k)\|_q; \quad k\in[0,N]$$
$$= \sup_k(|l_1|^q + \ldots + |l_r|^q)^{1/q}; \quad q\in[1,\infty]; \quad k\in[0,N] \qquad \text{(F.60)}$$

The following additional assumptions is needed.

Assumption F.2:

The functions $g_4(l(k))$ and $g_7(l(k))$ defined by (F.43) and (F.55) are Lipshitz continuous for all $l(k)$, $k\in[0,N]$, with Lipschitz constants h_3 and h_4 respectively. That is

$$\|g_4(l^p(k)) - g_4(l^{p-1}(k))\| \le h_3 \|l^p(k)) - l^{p-1}(k)\|$$
$$\|g_7(l^p(k)) - g_7(l^{p-1}(k))\| \le h_4 \|l^p(k)) - l^{p-1}(k)\| \qquad \text{(F.61)}$$

The following theorem presents the global convergence conditions of the outer loop.

Theorem F.4

A sufficient condition for algorithm mapping (F.54) to satisfy (F.59) for every outer loop iteration s > 1 is given by the expression

$$(h_4 + h_3\sigma_3(N))N\| < 1 \qquad \text{(F.62)}$$

where h_3 , h_4 and h_5 are defined in (5.101) and

$$\|\sigma_3(N) = \sup_{k \in [0,N-1]} \sup_{j \in [0,N-1]} \|\varepsilon_l \mathbf{T} \Theta_2(N,k,j)\| \tag{F.63}$$

with $\Theta_2(N,k,j)$ and $\mathbf{T}$ defined by (F.54) and (F.33).

Proof:

Theorem F.4 can be proven using similar contraction mapping arguments as provided by the proof of Theorem 5.4.