$\mathcal{PT}$-symmetric Deformations of the Korteweg-de Vries Equation

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ABSTRACT: We propose a new family of complex $\mathcal{PT}$-symmetric extensions of the Korteweg-de Vries equation. The deformed equations can be associated to a sequence of non-Hermitian Hamiltonians. The first charges related to the conservation of mass, momentum and energy are constructed. We investigate solitary wave solutions of the equation of motion for various boundary conditions.

1. Introduction

$\mathcal{PT}$-symmetry has served as a very fruitful guiding principle to identify potentially interesting non-Hermitian Hamiltonians, which may constitute physically relevant non-dissipative systems. The interest in these type of configurations has started with a numerical observation made in [1], where it was found that the Hamiltonian

$$H = p^2 - g(iz)^{N+1} \tag{1.1}$$

possess a real, positive and discrete eigenvalue spectrum for integers $N \geq 1$ with coupling constant $g \in \mathbb{R}^+$, despite it being non-Hermitian $H \neq H^\dagger$ and unbounded from below, for $N = 4n - 1$ with $n \in \mathbb{N}$. The virtue of $\mathcal{PT}$-symmetry results from the fact that whenever the Hamiltonian and the wavefunctions are left invariant under a $\mathcal{PT}$-transformation the eigenvalues are guaranteed to be real. However, the anti-linear nature of the $\mathcal{PT}$-operator is responsible for the fact that such a guarantee can not be provided by the $\mathcal{PT}$-symmetry of the Hamiltonian alone [2, 3]. Unlike as for linear operators, for the $\mathcal{PT}$-operator its two dimensional representation can be realized, in which case one speaks of broken PT-symmetry. One is then in a situation in which the corresponding wavefunctions are not $\mathcal{PT}$-symmetric and the eigenvalues occur in complex conjugate pairs. Nonetheless, even though $\mathcal{PT}$-symmetry of the Hamiltonian can not guarantee the reality of the spectrum, it pre-selects a subclass of promising non-dissipative systems. For recent results and a review see for instance [4, 5, 6].

A couple of month ago Bender, Brody, Chen and Furlan [7] have applied the above principle to identify interesting extensions of the Korteweg-de Vries (KdV) equation [8]

$$u_t + uu_x + u_{xxx} = 0. \tag{1.2}$$

The scaling properties of this equation for $x \to \alpha x, t \to \beta t, u \to \gamma u$ are well known, e.g. [9]

$$\frac{\gamma}{\beta}u_t + \frac{\gamma^2}{\alpha}uu_x + \frac{\gamma}{\alpha^3}u_{xxx} = 0, \tag{1.3}$$

and it has been remarked already at least thirty years ago that the KdV equation remains invariant under a $\mathcal{PT}$-transformation $t \to -t, x \to -x$, see for instance p. 414 in [9]. This is of course just the particular case $\alpha = \beta = -\gamma = -1$. However, this property has only been exploited in the above mentioned spirit in [7], where the KdV equation has been extended to the complex domain in a $\mathcal{PT}$-symmetric manner

$$u_t - iu(iu_x)^\varepsilon + u_{xxx} = 0 \qquad \varepsilon \in \mathbb{R}. \tag{1.4}$$

One may think of equation (1.4) as being obtained from (1.2) by a scale invariant deformation

$$u_x \to -\hat{q}(qu_x)^\varepsilon \qquad \varepsilon \in \mathbb{R}, \tag{1.5}$$

of the second term. When the deformation parameters scale as $q \to \alpha/\gamma q$, $\hat{q} \to \gamma/\alpha\hat{q}$, equation (1.4) has the same behaviour under scaling as (1.3) for all values of ε. The special case $q = \hat{q} = i$ yields a $\mathcal{PT}$-symmetric expression for $\alpha = \beta = -\gamma = -1$. Intriguingly, the equation (1.4) were found to possess interesting solitary wave solutions and two conserved charges were also constructed.

2. A new $\mathcal{PT}$-symmetric deformation of the KdV equation

It should be mentioned that complex extensions of the KdV equation have been studied before, see e.g. [10, 11, 12] and in passing even some special cases of equation (1.4) have been dealt with for instance in [13]. However, only few properties have been studies for the latter and $\mathcal{PT}$-symmetry has not been adopted as a guiding principle. Motivated by the interesting findings in [7] and the usefulness of $\mathcal{PT}$-symmetric complex deformations in other contexts, see e.g. [4, 5, 6], we extend here its application. We suggest that instead of deforming the second term in (1.3), by the same principle one may equally well deform the last term or possibly all terms. We shall demonstrate that the former case possesses some advantageous features when compared with the previously outlined deformation.

Let us start by using the same $\mathcal{PT}$-symmetric deformation principle

$$u_x \to -i(iu_x)^\varepsilon \qquad \varepsilon \in \mathbb{R} \tag{2.1}$$

as employed in [7], albeit now for the last term. This amounts to replacing the third derivative as

$$u_{xxx} \to i\varepsilon(iu_x)^{\varepsilon-2}\left[(\varepsilon-1)u_{xx}^2 + u_x u_{xxx}\right]. \tag{2.2}$$

In this way, simply applying (2.2) to (1.2), we obtain a new $\mathcal{PT}$-symmetric deformation of the KdV-equation

$$u_t + uu_x + i\varepsilon(\varepsilon - 1)(iu_x)^{\varepsilon-2}\, u_{xx}^2 + \varepsilon(iu_x)^{\varepsilon-1}u_{xxx} = 0. \tag{2.3}$$

At first sight the deformation (2.3) appears to be far less appealing than the deformation (1.4). In the latter the effect of the deformation was simply that the nonlinear term of the KdV-equation has become somewhat more nonlinear, whereas in (2.3) we have replaced the linear term by two highly nonlinear terms. Nonetheless, as a trade off the deformation (2.3) has some very attractive features, which are not present in (1.4). For instance, having a physical application in mind we expect the deformed equation to be at least Galilean invariant just like its undeformed counterpart (1.2). This property is lost in (1.4), but instead (2.3) is Galilean invariant, as it remains invariant under the transformation

$$x \to x - ct, \qquad t \to t, \qquad u \to u + c, \tag{2.4}$$

where c is the velocity of the moving reference frame. Furthermore, it was difficult to construct conserved quantities for (1.4). Only two charges could be constructed so far and in addition they turned out to be complicated infinite series. We shall now demonstrate that this task is surprisingly simple for (2.3), despite its high degree of nonlinearity by relating it to a Hamiltonian formulation, which seems also impossible for (1.4) as it appears to be a non-Hamiltonian dynamical system.

3. $\mathcal{PT}$-symmetric deformations from a Hamiltonian formalism

As we remarked, the $\mathcal{PT}$-symmetry analysis, which led to (1.4), was carried out directly for the equation of motion. Recalling that (1.1) was obtained as a deformation of the standard harmonic oscillator and that this principle has been applied to various Hamiltonian systems, it appears highly desirable to perform deformations for the KdV system also on the level of a Hamiltonian. This will enable us to relate these systems to the arguments, which allow statements about the reality of the spectrum by utilizing $\mathcal{PT}$-symmetry as outlined in the introduction. In the equation of motion this property enters more indirectly and it is less clear which kind of conclusions can be drawn from the symmetry property.

It is well known for a long time that the KdV-equation can be formulated as a Hamiltonian system [14, 15, 16, 17]. Thus in this spirit and in more direct analogy to the construction of (1.1), we propose to study the new non-Hermitian Hamiltonian density

$$\mathcal{H} = u^3 - \frac{1}{1+\varepsilon}(iu_x)^{\varepsilon+1} \qquad \varepsilon \in \mathbb{R}. \tag{3.1}$$

For $\varepsilon \to 1$ we recover the standard Hamiltonian density for the KdV-equation. Clearly $\mathcal{H}$ in (3.1) is $\mathcal{PT}$-symmetric, since it remains invariant under the transformation: $t \to -t, x \to -x, i \to -i$ and $u \to u$. Similarly as in the standard quantum mechanical setting, outlined in the introduction, $\mathcal{PT}$-symmetry can be utilized to ensure the reality of the energy E, which follows trivially with $\mathcal{H}(u(x)) = \mathcal{H}^\dagger(u(-x))$

$$E = \int_{-a}^{a} \mathcal{H}(u(x))dx = -\int_{a}^{-a} \mathcal{H}(u(-x))dx = \int_{-a}^{a} \mathcal{H}^\dagger(u(x))dx = E^\dagger. \tag{3.2}$$

Let us now derive the corresponding equation of motion by invoking the variational principle for the Hamiltonian $H(u) = \int \mathcal{H} dx$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\delta H(u)}{\delta u}\right) = \frac{\partial}{\partial x}\left(\frac{\delta \int \mathcal{H} dx}{\delta u}\right) = \frac{\partial}{\partial x}\left(\sum\nolimits_{n=0}^{\infty} (-1)^n \frac{d^n}{dx^n}\frac{\partial \mathcal{H}}{\partial u_{nx}}\right). \tag{3.3}$$

Evaluating (3.3) for $\mathcal{H}$ in (3.1) yields

$$u_t + (-3u^2 + \varepsilon (iu_x)^{\varepsilon-1}\ u_{xx})_x = 0, \tag{3.4}$$

or when not written as a conservation law

$$u_t - 6uu_x + i\varepsilon(\varepsilon-1)(iu_x)^{\varepsilon-2}\ u_{xx}^2 + \varepsilon(iu_x)^{\varepsilon-1} u_{xxx} - \kappa = 0, \tag{3.5}$$

where κ is a constant. Note that (3.5) is almost (2.3), but corresponds to a deformation of the scaled KdV equation (1.3), with $\alpha = \beta = 1$, $\gamma = -6$ and $\kappa = 0$, which, depending on the context, is also frequently used in the literature for convenience.

3.1 Integrals of motion and conserved quantities

Having seen how to obtain the $\mathcal{PT}$-symmetrically deformed KdV equation (2.3), or more precisely its scaled version (3.5), from a Hamiltonian principle, we shall demonstrate next that is has further interesting properties, which are absent in the deformation (1.4). As mentioned, for (1.4) the authors of [7] could only construct the two first conserved quantities in form of complicated infinite sums. Here we find instead that for (3.5) these quantities can be computed in a straightforward manner. Assuming to have a conserved quantity of the form $\mathcal{I}^{(n)} = \int \mathcal{T}^{(n)} dx$, all we have to verify is whether its Poisson bracket with the Hamiltonian is vanishing, see e.g. [14]. Viewing $\mathcal{I}^{(n)}(u)$ and $H(u)$ as functionals of u we have by definition

$$\frac{d\mathcal{I}^{(n)}}{dt} = \int \frac{\delta \mathcal{T}^{(n)}}{\delta u}\frac{\partial u}{\partial t} dx = \int \frac{\delta \mathcal{T}^{(n)}}{\delta u}\left(\frac{\delta H}{\delta t}\right)_x dx =: \left\{\mathcal{I}^{(n)}, H\right\}. \tag{3.6}$$

Let us now employ (3.6) to establish that

$$\mathcal{I}^{(1)} = \int u dx, \qquad \mathcal{I}^{(2)} = \int u^2 dx \qquad \text{and} \qquad \mathcal{I}^{(3)} = H(u), \tag{3.7}$$

are indeed preserved under an evolution in time. We find that these quantities are conserved when we invoke as standard boundary condition the non-compact or compact case for $u, u_x, \ldots$, that is being either vanishing at infinity or periodic in space, respectively. This is easily seen by computing

$$\frac{d\mathcal{I}^{(1)}}{dt} = \left\{\mathcal{I}^{(1)}, H\right\} = \int (3u^2 - \varepsilon(iu_x)^{\varepsilon-1}\ u_{xx})_x dx = 0, \tag{3.8}$$

$$\frac{d\mathcal{I}^{(2)}}{dt} = \left\{\mathcal{I}^{(2)}, H\right\} = \int \left(4u^3 - \frac{2\varepsilon}{1+\varepsilon}(iu_x)^{\varepsilon+1} - 2\varepsilon u(iu_x)^{\varepsilon-1}u_{xx}\right)_x dx = 0, \tag{3.9}$$

$$\frac{d\mathcal{I}^{(3)}}{dt} = \left\{\mathcal{I}^{(3)}, H\right\} = -\left\{H, \mathcal{I}^{(3)}\right\} = 0. \tag{3.10}$$

The last conservation law follows trivially from the anti-symmetry property of the Poisson brackets. We can also be more explicit and compute the corresponding flux. Constructing vanishing Poisson bracket amounts to seeking solutions of the conservation law

$$\mathcal{T}_t^{(n)} + \mathcal{X}_x^{(n)} = 0, \tag{3.11}$$

with $-\mathcal{X}^{(n)}$ being the nth flux and $\mathcal{T}^{(n)}$ being the nth conserved density. Then $\mathcal{I}^{(n)} = \int \mathcal{T}^{(n)} dx$ is a conserved charge provided the appropriate boundary conditions hold. The case $n = 1$ corresponds to the equation of motion itself as can be read off directly from (3.4). For the case $n = 2$ we may re-write (3.9) as a conservation law in the form

$$\left(u^2\right)_t + \left(\frac{2\varepsilon}{1+\varepsilon}(iu_x)^{\varepsilon+1} + 2\varepsilon u(iu_x)^{\varepsilon-1}u_{xx} - 4u^3\right)_x = 0. \tag{3.12}$$

We can also be more concrete about $\mathcal{T}^{(3)} = \mathcal{H}$ and compute the associated flux

$$\mathcal{X}^{(3)} = (\frac{\varepsilon^2}{2} - \varepsilon)(iu_x)^{2\varepsilon-2}u_{xx}^2 + 3\left(\varepsilon u u_{xx} - 2u_x^2\right)u(iu_x)^{\varepsilon-1} - i\varepsilon(iu_x)^{2\varepsilon-1}u_{xxx} - \frac{9}{2}u^4, \tag{3.13}$$

thus confirming (3.10). At this stage it is not clear whether there exist higher conserved quantities. However, we suspect that similarly as for most cases of the modified KdV equations and the generalized KdV equations only three charges exist. We recall that the equation $u_t + u^p u_x + u_{qx} = 0$ is only integrable, i.e. possesses an infinite amount of conserved quantities, for the cases $q = 3, p = 1, 2$; $q = 1, p \in \mathbb{N}$ and $q \in \mathbb{N}, p = 1$, see e.g. [9].

3.2 Solutions of the equations of motion

We shall now construct solutions of the equations of motion (3.5). One may expect to find a rich variety of different types of solutions similarly as for the standard KdV equation. Over the years several methods have been developed to find such solutions ranging from minimizing the sum of the conserved charges [18], the inverse scattering method [19], Hirota's bilinearization method [20], etc. Some methods demand as a prerequisite the model to be integrable. As this feature is not guaranteed for the model at hand, in fact the conjecture is that the model is not integrable, our aim is here just to obtain a first impression in order to indicate that the above family of equations deserve further attention. Following a simple procedure which has turned out to be useful for the standard KdV equation, we may integrate (3.5) directly by assuming the solution to be a steady progressing wave

$$u(x,t) = w(kx - \omega t) = v(x - ct), \tag{3.14}$$

with $c = \omega/k$. Substituting (3.14) into the equation of motion (3.5) yields after some straightforward manipulations

$$v_x^{(n)} = e^{\frac{i\pi(4n+3\varepsilon+1)}{2(1+\varepsilon)}}\left[\frac{\varepsilon+1}{\varepsilon}(v^3 + \frac{c}{2}v^2 + \kappa v + \hat{\kappa})\right]^{\frac{1}{\varepsilon+1}}, \tag{3.15}$$

with $\hat{\kappa}$ being an additional constant of integration and n labeling the various branches of the function. Separating variables then yields

$$x - ct = e^{\frac{i\pi(4n+\varepsilon-1)}{2(1+\varepsilon)}}\left(\frac{\varepsilon}{\varepsilon+1}\right)^{\frac{1}{\varepsilon+1}}\int\frac{dv}{(v^3 + \frac{c}{2}v^2 + \kappa v + \hat{\kappa})^{1/(\varepsilon+1)}}. \tag{3.16}$$

Apart from computing the integral in (3.16), the main problem is here that we need to solve the equation for v in order to obtain $v(x-ct)$. This is only possible in very few exceptional cases, but the knowledge of the inverse function $(x-ct)(v)$ in some domain will be valuable as it provides the information about the kind of general behaviour which is possible. For convenience we choose the dispersion relation and the constants of integration to be parameterized as

$$c = 4k^2(2-m), \quad \kappa = 4k^4(1-m) \quad \text{and} \quad \hat{\kappa} = 0. \tag{3.17}$$

This choice is guided by the known solutions for $\varepsilon = 1$ and leads naturally to three qualitatively different cases.

3.3 Analogues of the cnoidal solution

Let us first recall how to solve equation (3.16) for the case $\varepsilon = 1$, which should result into an elliptic integral as we integrate the inverse of the square root of a cubic polynomial. With the choice of constants (3.17) we may bring (3.16) into the usual form of an elliptic integral

$$kx - \omega t = \pm \frac{k}{\sqrt{2}} \int_{-2k^2}^{w} \frac{dt}{\sqrt{t^3 + 2k^2(2-m)t^2 + 4k^4(1-m)t}} = \pm \int_{0}^{\phi(w)} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{3.18}$$

with $\phi(w) = \arcsin\sqrt{(1+w/2k^2)/m}$. From (3.18) we deduce therefore that $w(kx-ct)$ becomes the well known cnoidal solution for the KdV equation

$$u(x,t) = -2k^2 \,\mathrm{dn}^2(kx - \omega t|m), \tag{3.19}$$

with dn being a Jacobian elliptic function depending on the parameter $m \in [0,1]$, see e.g. [21] for notation and properties. As (3.19) indicates for $\varepsilon = 1$, the cases $m = 0, 1$ are special in general. For generic values of ε we evaluate (3.16) with the parameterization (3.17) to

$$\begin{aligned} x - ct = {} & e^{\frac{i\pi(4n+\varepsilon-1)}{2(1+\varepsilon)}} \left(\frac{v(1+\varepsilon)}{\varepsilon}\right)^{\frac{\varepsilon}{1+\varepsilon}} \left(\frac{1}{4k^2(1-m)}\right)^{\frac{1}{1+\varepsilon}} \\ & \times F_1\left(\frac{\varepsilon}{1+\varepsilon}; \frac{1}{1+\varepsilon}, \frac{1}{1+\varepsilon}; \frac{1+2\varepsilon}{1+\varepsilon}; \frac{-v}{2k^2(1-m)}; \frac{-v}{2k^2}\right). \end{aligned} \tag{3.20}$$

Here F_1 is the Appell hypergeometric function defined via a double infinite sum as

$$F_1(\alpha;\beta,\beta';\gamma;x;y) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{n+m}(\beta)_n(\beta')_m}{n!m!(\gamma)_{n+m}} x^n y^m \tag{3.21}$$

with $(\alpha)_n := \prod_{k=1}^{n}(\alpha + k - 1)$. Since we can not solve (3.20) for v let us plot $(x-ct)$ as a function of v and search for real solutions.

We depict our findings in figure 1. For $\varepsilon = 1$ we recognize the cnoidal solution (3.19). For clarity we did not indicate the vanishing imaginary part in this case. For the other values of ε we find always two different types of solutions. The first resembles qualitatively the cnoidal solution and is either real for $v \in [-1/2k^2, 0]$ or $v \in [0, 1/2k^2]$. In figure 1 we present $\varepsilon = 3$; $n = 2, 4$; $k = 1/\sqrt{2}$ for the former case and $\varepsilon = 5$; $n = 2, 5$; $k = i/\sqrt{2}$ for the latter. The second type is more similar to the $\tan^2$ solution for $\varepsilon = 1$ to be discussed in the next section. These solution are real either for $v \in (-\infty, 0]$ or $v \in [0, \infty)$. In figure 1 the former case is illustrated by $\varepsilon = 3$; $n = 2, 4$; $k = i/\sqrt{2}$ and the latter by $\varepsilon = 5$; $n = 2, 5$; $k = 1/\sqrt{2}$.

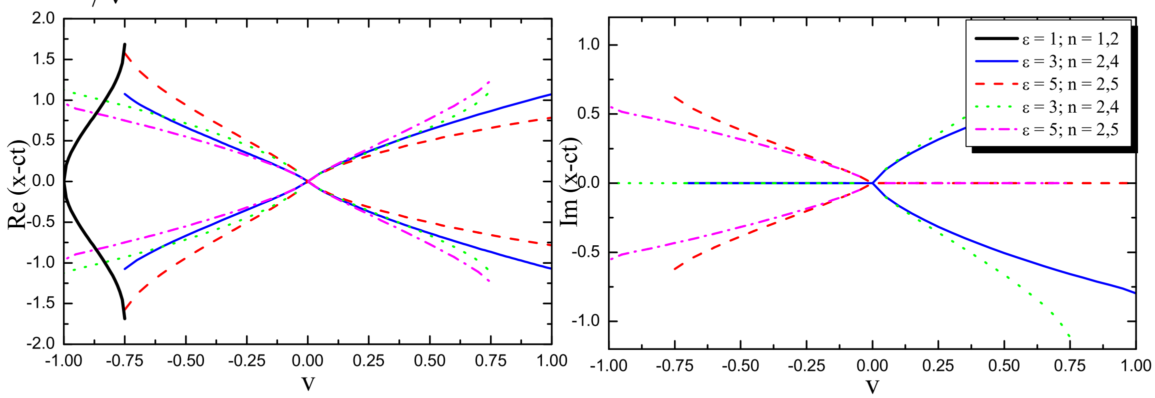

Figure 1: (x-ct) as a function of v for $\varepsilon = 1, 3, 5$ for some particular branches.

3.4 Analogues of the $\tan^2$ solution

Next we consider the limit $m \to 0$. Keeping the parameterization (3.17) we make the further convenient choice $k = \pm 1/\sqrt{2}$, similarly as in the previous section, which amounts now to the boundary condition $\kappa = 1$. Then the Appell hypergeometric function F_1 reduces to the Gauss hypergeometric function ${}_2F_1$ defined as

$$ {}_2F_1(\alpha;\beta;\gamma;x) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(\gamma)_n} x^n = F_1(\alpha;\beta/2,\beta/2;\gamma;x;x). \tag{3.22} $$

Using furthermore the identity

$$ {}_2F_1(\alpha;2\beta;2\alpha+\beta;x) = \alpha x^{-\alpha} B_x(\alpha, 1-2\beta), \tag{3.23} $$

where $B_z(\alpha,\beta)$ is the incomplete beta function

$$ B_z(\alpha,\beta) = \int_0^z t^{\alpha-1}(1-t)^{\beta-1}, \tag{3.24} $$

we obtain the simpler expression

$$ x - 4t = e^{\frac{i\pi(4n+\varepsilon-1)}{2(1+\varepsilon)}} \left(\frac{\varepsilon}{\varepsilon+1}\right)^{\frac{1}{\varepsilon+1}} B_{-v}\left(\frac{\varepsilon}{\varepsilon+1}, \frac{\varepsilon-1}{\varepsilon+1}\right). \tag{3.25} $$

For $\varepsilon = 1$ (3.25) reduces further to $x - 4t = \sqrt{2}\arctan(\pm\sqrt{v})$, which may be solved for v, such that we obtain $u(x,t) = \tan^2[(x-4t)/\sqrt{2}]$ as a solution for the standard KdV equation. For generic values of ε we depict (3.25) for various values of the parameters in figure 2. For $\varepsilon = 1$ we perceive the real solution $x - 4t = \sqrt{2}\arctan(\pm\sqrt{v})$ in panel (a). A qualitatively similar type of solution is obtained for instance for some branches for $\varepsilon = 5$ as is seen also in panel (a). Panel (b) confirms that for $v > 0$ this solution is real. (The solid line is on top of the dashed line) Interesting qualitatively different types of solutions are obtained for instance for some branches for $\varepsilon = 3, 11$. We observe from panel (c) that these solutions are very reminiscent of the one soliton solution, to be discussed in the next section, albeit with the fundamental difference that they are not vanishing asymptotically for large $(x - ct)$. This is seen simply by using the property $B_1(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$ of the incomplete beta function. For $v \to 1$ we obtain in (3.25) the definite values $e^{\frac{i\pi(4n+\varepsilon-1)}{2(1+\varepsilon)}} \left(\frac{\varepsilon}{\varepsilon+1}\right)^{1/(\varepsilon+1)} \Gamma(\frac{\varepsilon}{\varepsilon+1})\Gamma(\frac{\varepsilon-1}{\varepsilon+1})/\Gamma(\frac{2\varepsilon-1}{\varepsilon+1})$. This limit is finite for the parameter range except for $\varepsilon = 1$, when $\lim_{x\to 0}\Gamma(x) \to \infty$. In this case we obtain a purely complex one soliton solution as can also be seen clearly in panel (b). Having Galilean invariance for our equations, we may also move this function as $v \to v + 1$, such that the tails are located at $v = 0$ rather than $v = -1$, which is a more familiar setting.

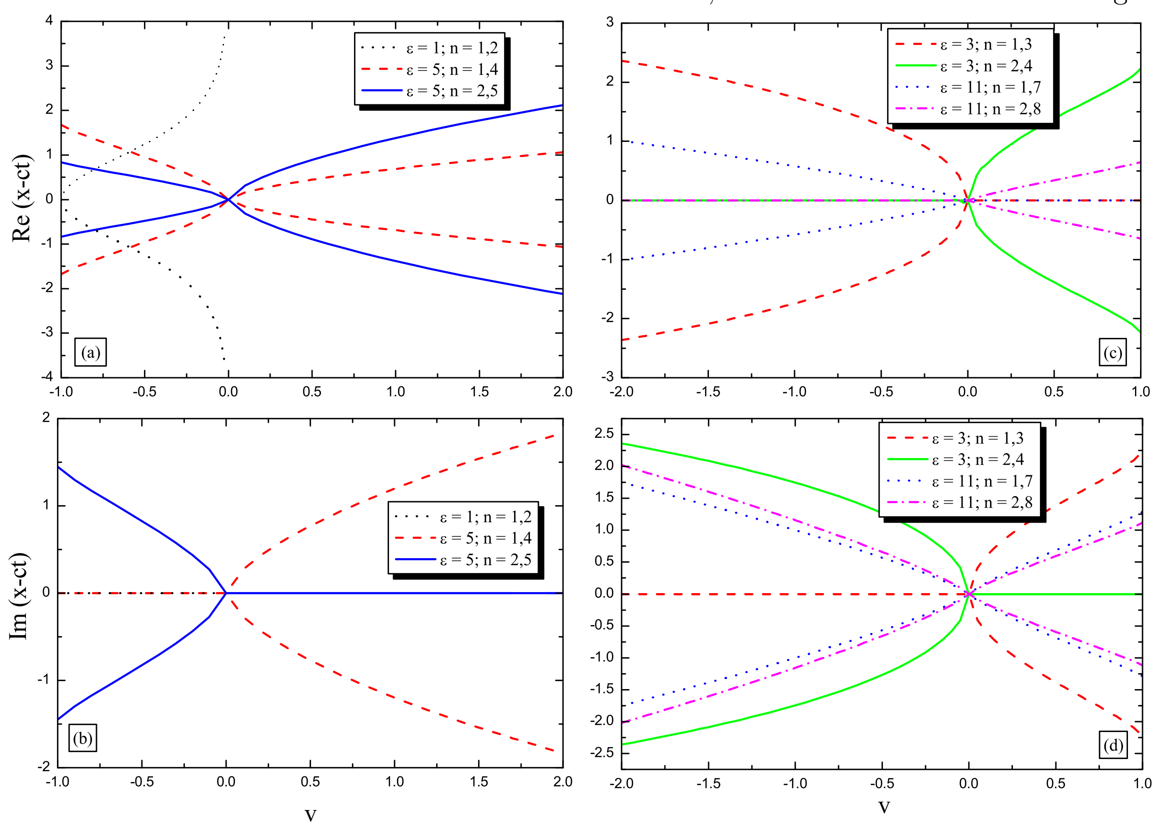

Figure 2: (x-ct) as a function of v for $\varepsilon = 1, 3, 5, 11$ for some particular branches and m=0.

3.5 Analogues of the one soliton solution

Next we take the limit $m \to 1$ corresponding to the special case $\kappa = 0$, which implements vanishing boundary conditions. Indeed, adopting the parameterization (3.17) the limit $m \to 1$ in (3.19) for $\varepsilon = 1$ yields the asymptotically vanishing single soliton solution

$u(x,t) = -2k^2 \operatorname{sech}^2(kx - \omega t)$. Taking this limit in (3.20) for generic values of ε gives

$$x - ct = e^{\frac{i\pi(4n+\varepsilon-1)}{2(1+\varepsilon)}} \left(\frac{\varepsilon}{\varepsilon+1}\right)^{\frac{1}{\varepsilon+1}} (2k^2)^{\frac{1+\varepsilon}{2-\varepsilon}} \, B_{-\frac{v}{2k^2}} \left(\frac{\varepsilon-1}{\varepsilon+1}, \frac{\varepsilon}{\varepsilon+1}\right). \tag{3.26}$$

We depict this function for various values of the parameters in figure 3. The famous one soliton solution is clearly visible for $\varepsilon = 1$. In the other cases we obtain again two qualitatively different types of real solutions. One type being real in the finite ranges $v \in [-1/2k^2, 0]$ and $v \in [0, 1/2k^2]$ exemplified by $\varepsilon = 5$; $n = 1, 4$; $k = 1/\sqrt{2}$ and $\varepsilon = 3$; $n = 2, 4$; $k = i/\sqrt{2}$, respectively. The other type is real in the ranges for $v \in (-\infty, 0]$ and $v \in [0, \infty)$, which we illustrated in figure 3 by $\varepsilon = 3$; $n = 1, 3$; $k = i/\sqrt{2}$ and $\varepsilon = 5$; $n = 2, 5$; $k = 1/\sqrt{2}$, respectively.

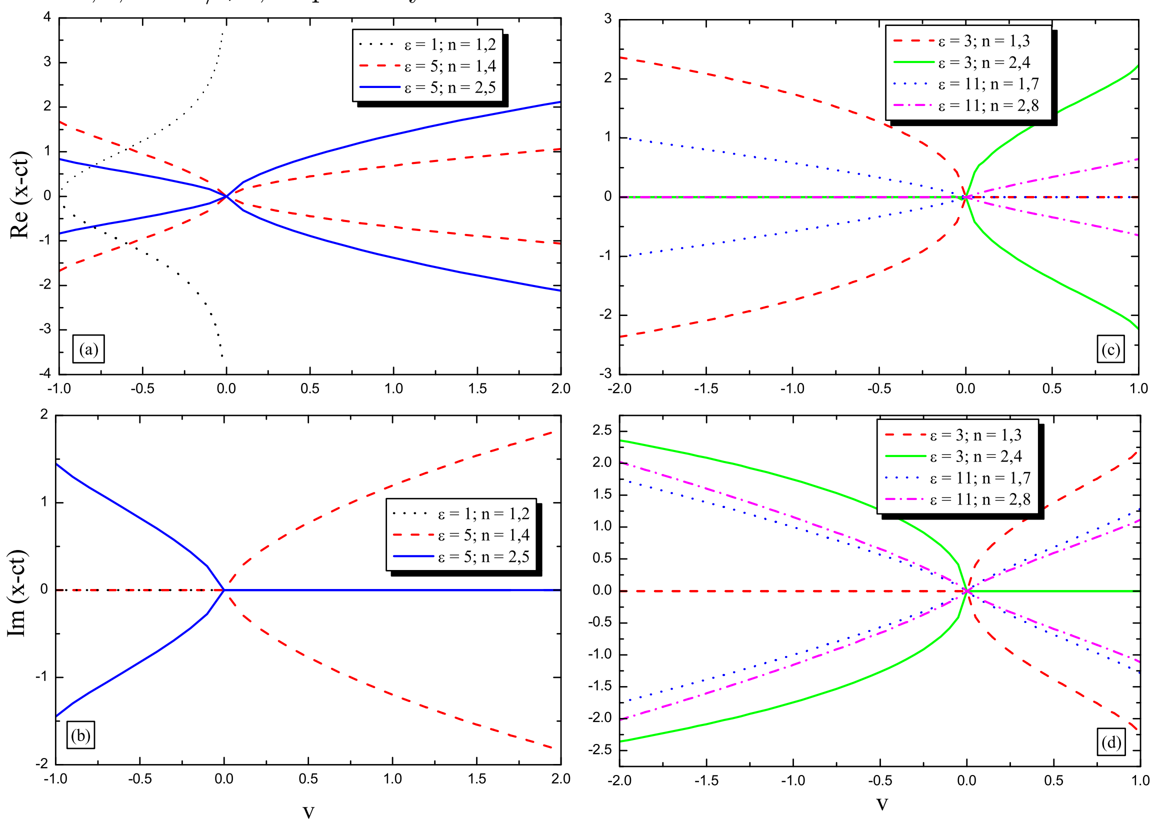

Figure 3: (x-ct) as a function of v for $\varepsilon = 1, 5$ and $\varepsilon = 3, 11$ with $k = \pm 1/\sqrt{2}$ and $k = \pm i/\sqrt{2}$, respectively, for some particular branches with m=1.

4. Conclusions

Alternatively to [7], we proposed a new $\mathcal{PT}$-symmetric complex deformed version of the KdV equation. The suggested deformation allows for a simple non-Hermitian Hamiltonian formulation involving a Hamiltonian density very reminiscent to the prototype $\mathcal{PT}$-symmetrically complex deformed quantum mechanical system (1.1). The model (3.1) is Galilean invariant and three charges, related to the conservation of mass, momentum and energy, together with their conservation laws, were constructed. We demonstrated that there exist steady progressing wave solutions for these models and identified analogues to the cnoidal and $\tan^2$ solution. However, we did not find asymptotically vanishing analogues to the one soliton solution.

Clearly there are many important questions left to be answered. It would be interesting to establish that there exist three and only three charges for the proposed deformation. Besides solving the equations more explicitly it will be natural to seek for solutions on some rays in the complex plane. It will be straightforward to extend these considerations to the modified KdV and the generalized KdV equations. We shall leave these issues for future investigations [22].

Acknowledgments. I am grateful to D.C. Brody for bringing reference [7] to my attention. Discussions with C. Figueira de Morisson Faria are gratefully acknowledged.

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