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Inefficient Reallocation, Loss Aversion and Prospect Theory

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Abstract

The paper shows that bounded rationality, in the form of limited knowledge of utility, is an explanation for common stylized facts of prospect theory like loss aversion, *status quo* bias and non-linear probability weighting. Locally limited utility knowledge is considered within a classical demand model framework, suggesting that costs of inefficient search for optimal consumption will produce a value function that obeys the loss aversion axiom of Tversky and Kahneman (1991). Moreover, since this adjustment happens over time, new predictions are made that explain why the *status quo* bias is reinforced over time. This search can also describe the behavior of a consumer facing an uncertain future wealth level. The search cost justifies non-linear forms of probability weighting. The effects that have been observed in experiments will follow as a consequence.

KEYWORDS: *status quo* bias, reference dependence, loss aversion, cost of choice, search costs, probability weighting, transaction costs, bounded rationality.

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1 Introduction and Literature

Experimental evidence will often show that people are not entirely rational, at least according to classical assumptions on what it means to be rational. By irrational, it is usually

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meant that we observe a person acting in what we believe is not her self-interest, implicitly assuming what her self-interest is. In classical theory, it is not always possible to resolve such contradictions by the usual methods: a better description of the environment, or a more sophisticated description of the preferences of the person observed. This is important because, however we might think of people, we do tend to consider the world as governed by logically consistent rules, and this includes the world of human biology. So then why are biological mechanisms creating a being that behaves irrationally? This question is central to a more refined view of *homo economicus*, and the obvious answer is that perfect brains are hard to grow. I approach behavioral economic questions from the point of view that what we call irrational behavior fits into two categories: (a) classically rational behaviors that are considered irrational only because the environment or preferences are not described correctly, and (b) behaviors based on heuristics that are quasi-rational in the "average" environment, but which can lead to suboptimal choices in the setting observed. Heuristic behaviors can be learned or hardwired, but they are fundamentally related to limitations of reasoning capacities. The test of rationality, with its common meaning, is that every behavior must make sense from an economic maximizing perspective if certain limitations on reasoning abilities are assumed.

In many contexts, an individual's choice is influenced by the *status quo*. This means that choices are reference dependent. The seminal paper of Kahneman and Tversky (1979) proposed "prospect theory," and a new direction of research developed, commingling theoretical economics and psychological research. This theory proposes *ad hoc* deviations from expected utility theory, which are used to explain countless experimental observations that are hard to account for in classical terms: all else being equal, gains are discounted more than losses, small changes are discounted more than large changes, speed-ups in a sequence of payments are preferred to delays, improving sequences of returns are preferred over declining sequences (Loewenstein, 1988), gains with small odds are overweighted, and decisions are biased towards certainty. Confirming evidence can be found in, e.g., Bateman et al. (1997), who propose an experimental set-up to contrast the predictions of reference dependent utilities with those of classic Hicksian theory. Many other papers study individual deviations from classic theory, e.g. Thaler (1981), Loewenstein (1988), Loewenstein and Sicherman (1991), etc.

In a more recent treatment, Tversky and Kahneman (1991) present a set of axioms that are used to generalize the deterministic part of old prospect theory, to explain a large set of experimental effects related to reference dependence. Let the choice set $X = \{x, y, z, r, s, ...\}$

be isomorphic to $\mathbb{R}^2_+$, and for $x = (x_1, x_2)$, $x_1, x_2 \geq 0$ are the consumption values of goods 1 and 2. If $\geq_r$ denotes the preference structure related to r, then:

- (A_0) $\forall\, r \in X :\ \geq_r$ is complete, transitive and continuous; moreover $\{x \geq_r y \,\wedge\, x \neq y\} \Rightarrow x >_r y$.
- (A_1) $\forall\, r,s,x,y \in X :$ Let $x_1 \geq r_1 > s_1 = y_1,\, y_2 > x_2,$ and $r_2 = s_2$. Then we have that $x =_s y \Rightarrow x >_r y$.
- (A_2) $\forall\, t,s,x,y \in X :$ Let $x_1 > y_1, y_2 > x_2, s_2 = t_2, y_1 \geq s_1 \geq t_1 \vee t_1 \geq s_1 \geq x_1$. Then $x =_s y \Rightarrow y \geq_t x$.

The indices can be reversed throughout, and the domain is 2–dimensional for simplicity. Assumption A_0 is the standard collection of axioms to insure that there exists a strictly increasing continuous utility representation U_r for the preference structure $\geq_r$ given at the reference point r. Classical theory follows if the subscript r is superfluous; i.e., if any reference would generate the same preferences. Assumption A_1 is what recreates *loss aversion*. It implies an asymmetry between gains and losses in terms of the absolute value of the utility change. A_2 is the *diminishing sensitivity* assumption. It says that the absolute value of the change in utility is smaller if the consumption shift happens further from the origin, all else being equal.

It is instructive to consider prospect theory in the one dimensional case, as formulated by Kahneman and Tversky (1979). In the following, consider the reference to be at the origin. The three assumptions for the value function over wealth changes, as expressed by Bowman et al. (1999), are:

- (P_0) $\mathcal{V}(x)$ is continuous, strictly increasing, and $\mathcal{V}(0) = 0$.
- (P_1) Let $y > x > 0$. Then $\mathcal{V}(y) + \mathcal{V}(-y) < \mathcal{V}(x) + \mathcal{V}(-x)$.
- (P_2) $\mathcal{V}(x)$ is strictly concave for $x > 0$ and strictly convex for $x < 0$.

The features of the value function implied by these assumptions can be easily seen in Figure 1, as given by Kahneman and Tversky (1979) or Tversky and Kahneman (1991). Assumption P_1 is the loss aversion assumption, and it implies that a shift from $x > 0$ to $y > 0$ in wealth will produce a lower utility change than a shift from $-x$ to $-y$, all in absolute value. Assumption P_2 is the diminishing sensitivity assumption, and it makes the impact of equal changes of wealth decreasing in the distance to the origin, which is the reference point.

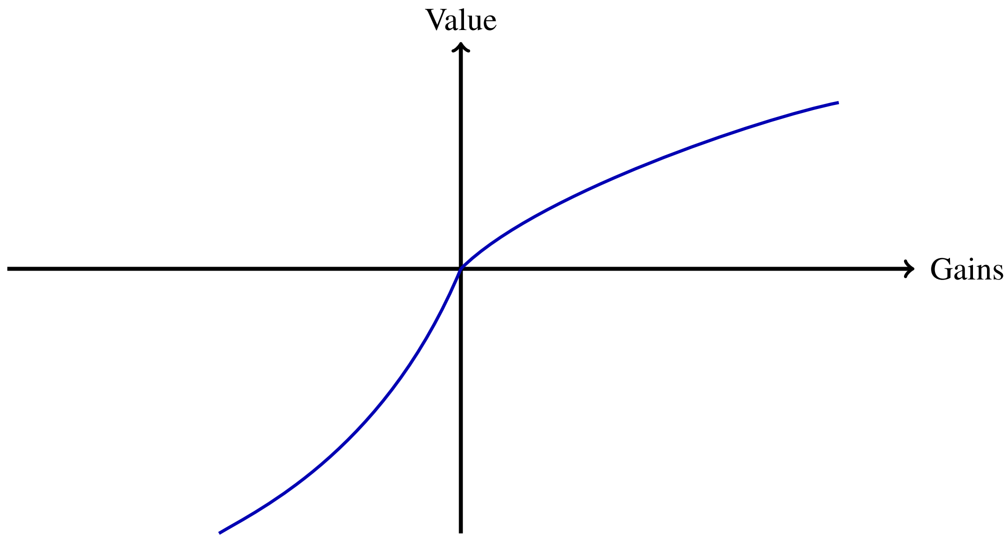


Figure 1: Value function over wealth changes.

This paper studies the impact of the inefficiency of adjustment to optimal consumption because of search, in a classical setting and with bounded rationality. Search in the consumption space is necessary since the consumer has incomplete knowledge of his preferences, which will be the assumed limitation. The goal is to model the intuitive notion that it would be hard for a consumer to compare wildly different lifestyles, and in practice he will be more likely to consider small changes to his current consumption behavior. Formally, it is assumed that the consumer has only local knowledge of his utility, and he also knows the rate at which his utility can be improved by marginal changes in his consumption vector. An adjustment in consumption must happen every time there is a wealth shift or, equivalently, a shift in the endowment with a certain good, which is the settings of most experiments. It will be shown that, with relatively weak assumptions, slow or costly search generates loss aversion, as formulated in assumption A_1 in Tversky and Kahneman (1991), or as given by P_1 in Kahneman and Tversky (1979). The main idea is that finding optimal consumption bundles is slow and therefore inefficient, and any shock to the wealth level or to a consumption endowment requires a new optimization of consumption.

Utility losses appear because the allocation is suboptimal during search, and an alternative approach is to make search itself costly, either in terms of lost time, or directly. Both positive and negative changes in wealth or endowment require reallocation, and this means that lost utility will be magnified, while gained utility will be reduced. This effect is the main feature of prospect theory – loss aversion: in absolute values, gains are valued less that losses for equal changes in wealth of endowment. Moreover, the search process can be considered in situations when the consumer faces shocks in wealth or endowment in the form of lotteries. A consumer with a heuristic decision making process that takes search losses into account when planning future allocations will show effects like subcertainty, the overweighting of small probabilities, and subadditivity of the probability weighting function, which are described in Kahneman and Tverky (1979). An interesting observation leading to comparative statics can also be made. The concavity of the instantaneous utility function, a classic assumption leading to risk aversion, can be shown to determine how uncertain gains and losses influence the search process. As we would expect, a more risk averse consumer in the classical sense will search towards lower cost allocations when facing a lottery. We can conclude that a more risk averse consumer is also more pessimistic.

The idea of costly change is not completely new. Samuelson and Zeckhauser (1988) mention that *status quo* bias could be generated by the cost of thinking, as well as by transaction costs and other psychological effects. In this setting, adjustments are costly because thinking about a better consumption decision leads to a utility loss over time and in Appendix B a setting in which changing the consumption vector costs time is considered.

One important question for any new model is whether it offers testable implications. Here, costly reallocation of resources happens over time. Having the incurred cost depend on time suggests an experimental set-up in which the *status quo* can be cemented by letting time pass before a new option is presented. The expected effect is that the more time passes with a *status quo*, the more the loss aversion effect will be accentuated, thereby providing an experimentally testable implication of the theory. There is already evidence in support of this interpretation. In Strahilevitz and Loewenstein (1998), the authors look in depth at the effect of the history of ownership on the valuation of an object. They find that, in addition to an instantaneous effect, a longer history of ownership of an object will also increase its valuation, thus the value lost in an exchange will be higher than the value gained in the beginning. Another way of looking at this is to say that increased duration of ownership will make the object more a part of the *status quo*, or more a part of his planned consumption – having an object, or wealth level, for a longer period of time should make it more "comfortable" for the consumer. This is similar to thinking of the *status quo* as a plan, or an expectation for future consumption.

Other attempts have been made to place prospect theory within a framework of classical assumptions. The work of Kőszegi and Rabin (2006, 2009) develops a model for the formation of the reference point, which is set as the probabilistic belief that the person held in the recent past about the possible outcome. This set-up can be used to explain laboratory observations like the endowment effect from the more basic gain-loss bias. Because expectations determine the reference, the theory predicts that traders or merchants will not be affected by the endowment effect, because their beliefs about future outcomes take into account the fast shifts in the consumption vector. The idea is closely related in interpretation to the one in the present paper. A trader will not search for a new optimal consumption bundle, given a shift in wealth or allocation, and this is because the trader will expect the *status quo* to be temporary.

Another detailed attempt at explaining some parts of prospect theory in classic terms is in the working paper of Rick Harbaugh (2009). The main idea is that winning or losing gambles will signal the skill of the player. Thus, a gamble with some probability for loss, given a perceived default payoff, will signal low skill, while a gamble with a probability of win above the perceived default will signal high skill. Therefore the framing of a trade matters, and loss aversion is the result of avoiding the loss of a gamble. Probability weighting is another consequence, since a high gain with a small probability will also be a strong signal for skill, while losing an almost-sure bet is a strong signal of lack of skill. This approach is superior in explaining framing effects, but is not very useful in settings with little uncertainty.

The paper has six sections. The second lays the theory used to model search, proposes a general search mechanism in continuous time and proves central results. The third uses the theory to justify loss aversion by inefficient allocation during search, and justifies the loss aversion axiom A_1. Section four looks at search when the consumer faces a lottery shock to wealth and shows that risk aversion is related to pessimism in planning. Section five considers a functional form example to show how observations related to the probability weighting function of prospect theory can be recreated. The last section discusses an experimental method that can be used to test the model for confirmation and to contrast its implications to those of existing theory, and concludes. Appendix A develops the search method and useful results, Appendix B constructs an alternative theory that models costly search as lost utility from lost time, and Appendix C contains proofs.

2 Theory

Consider an individual with a rational, continuous and locally nonsatiated preference relation, in a deterministic continuous time setting. The consumer chooses between consumption bundles $x(t) = (x_1(t), ..., x_n(t))$, based on his utility function $U : \mathbb{R}^n_+ \to \mathbb{R}$, $U(x(t)) = U(x_1(t), ..., x_n(t))$, which obeys the usual assumptions of strict quasiconcavity and continuity. $i \in \{1,..,n\}$ indexes the goods in the consumption bundle, and $t \in (0,\infty)$ denotes the time. The consumer starts with a consumption vector plan $x(0) = (x_1(0),...,x_n(0))$, and with lifetime wealth $W(0)$. $p = (p_1,...,p_n)$ is a positive and constant market price vector, known to the consumer, and β is the intertemporal discount factor. The discounting formula is calibrated such that we can write equations with a familiar discount factor $\beta \in (0,1)$. For that, use equations like:

$$NPV(\text{consumption } x(t) \text{ between times } \tau_1 \text{ and } \tau_2) := \frac{\ln\beta}{\beta-1}\int_{\tau_1}^{\tau_2} e^{t\ln\beta}\, p\cdot x(t)dt.$$

$$\left[= \frac{\beta^{\tau_1}-\beta^{\tau_2}}{1-\beta} p\cdot x \text{ if, say, } x(t) \text{ is constant}\right]$$

Say that $x(0)$ is the optimal consumption vector for the instantaneous utility maximization problem, given the budget $p\cdot x(0)$. An important question is if the consumer, starting with total (lifetime) wealth $W(0)$, is satisfied with continuing to consume $x(0)$ in the future. E.g., if β is a small number, it could be that the consumer would want to use his wealth sooner. Optimal behavior depends on whether β is smaller or larger than $1/(1+r)$, where r is the real interest rate. The life-cycle model (Hall, 1978) argues that, in an economy with many identical consumers, the two values are equal and the consumer uses a constant fraction of his lifetime wealth each period. It is more complicated when we involve economic growth or realistic behavioral models (Shefrin, 1988) and, in either case, consumer heterogeneity would require a more general approach. However, to avoid unnecessary complications, we assume that $\beta = 1/(1+r)$. To support the assumption, consider that the average person, who faces the decisions that we consider on time scales significantly shorter than his lifetime, would be well represented by a stable habitual consumption vector. The consumer will have access to simple banking, in which he can save his wealth for later consumption. In Section 4 we'll consider lotteries, so contingent claims cannot be available, at least for the scope of the decisions considered, because our risk averse consumer couldn't distinguish between lotteries and their expected value.

Now, we deviate significantly from classical assumptions by limiting the knowledge of our consumer. The only utility he can know is the utility at the point of his current chosen consumption plan. He can consider changing his consumption $x(0)$, and as soon as he does, he can know his new utility level. The only information the consumer has, besides $x(0)$ and $U(x(0))$, is the rate at which utility can be increased on any of the dimensions, $\nabla U(x(0))$. Therefore, the problem the consumer faces is similar to a simple optimization problem in which we're computationally limited to knowing only the linear approximation of an unknown function. In addition, the utility of 0 consumption, $U(0,...,0) =: \underline{U}$, is also always known.

Assume that at the start the consumer has one bundle in mind, $x(0)$, which is optimal. The plan to consume it forever employs all the initial financial resources the consumer has, $W(0) = p \cdot x(0)/(1-\beta)$. The constant consumption decision is optimal. However, if $W(0) \neq p \cdot x(0)/(1-\beta)$, no change means that the consumer will either go bankrupt at some point, or has unused resources.

The consumer's rational behavior is limited by the local nature of the knowledge of his preferences. From his initial choice $x(0)$, he has the opportunity to search for lifetime utility improvements by small changes. The search method utilized is steepest descent in its continuous time limit. Appendix A describes it and some of the results in more detail. This means that $x(t)$ will change along $\nabla \mathcal{U}(x)$, where $\mathcal{U}(x)$ is the objective function. Let's assume the consumer starts at the optimum, and there is a $t = 0$ wealth shock such that $W(0) > p \cdot x(0)/(1-\beta)$. He suddenly has access to more wealth than before, so the plan of consuming $x(t) = x(0)$ forever is suboptimal. It then makes sense to look for a new long run optimal allocation that employs more wealth, which is a step in his search process. If the search would converge instantly, we would need to consider only a simple constrained optimization problem:

$$\max_x U(x), \quad \text{s.t. } \frac{p \cdot x}{1-\beta} \leq W. \tag{1}$$

If the initial shock in wealth is such that $W(0) < p \cdot x(0)/(1-\beta)$, the initial consumption plan is that the consumer will have $x(0)$ for as long as his wealth $W(0)$ will cover, and then switch to the utility level of no consumption expenditure, $\underline{U}$. As long as the marginal utility gain from spending is higher at $(0,...,0)$ than at any $x \neq (0,...,0)$, the initial consumption plan is suboptimal, which means that the consumer will want to search for allocations of lower utility in the beginning, and postpone the moment when he runs out of wealth. With instant search, the problem he solves is the following:

$$\max_x U(x) \int_0^{\tau(x)} \beta^t dt + \underline{U} \int_{\tau(x)}^{\infty} \beta^t dt, \quad s.t. \ \frac{1-\beta^{\tau(x)}}{1-\beta} p \cdot x \leq W. \tag{2}$$

τ is the moment in time when the consumer switches from consumption bundle x to 0. The lifetime budget constraint will bind, so x determines τ, because the consumer will stop x only because he's out of resources. In the continuous time version of the maximization problem, it doesn't matter how low the value $W(0)$ becomes, since there is a small enough duration τ for which the wealth W will be sufficient to consume the initial allocation $x(0)$.

Now let's consider how the search works when the convergence towards the optimum is not instant. The main problem is that, as search proceeds, the wealth level will also change according to the spending on intermediate points, which changes the objective function for the future. This means that, in choosing the next step in the search, the consumer has to also consider the effect of the search step on the future-self's search process. Fortunately, the consumer has only local knowledge of his utility, which greatly simplifies the problem. The next step depends only on the momentary knowledge and wealth level.

A way to formally state the consumer's bounded rationality constraint is to have his prior on $U(x)$ as:

$$U(x) = \begin{cases} U(x(0)) + \nabla U(x(0)) \cdot (x - x(0)), & x \in \mathcal{B}(x(0), \varepsilon|\nabla U(x(0))|), \\ \underline{U}, & x \in \mathbb{R}^n_+ / \mathcal{B}(x(0), \varepsilon|\nabla U(x(0))|), \end{cases} \tag{3}$$

for some small ε. If the search step takes time δ_τ, then $\varepsilon|\nabla U(x(0))|/\delta_\tau$ is the rate of movement in the search process, assumed constant. Rather than interpreting this as bad *a priori* knowledge, we can think of it as the consumer strongly disliking changes larger than $\varepsilon|\nabla U(x(0))|$ in his consumption bundle because, say, he doesn't have any certainty on his utility if he moves too fast and without a clear indication of improvement, $|\nabla U(x(0))|$.

Proposition 1. *Assume that the boundedly rational consumer knows only his current consumption vector, his current utility level and gradient, as well as the utility of no consumption. Given a line search method for the maximum, at any point in time the optimal next step is given by the method applied to problems (1) or (2), where x is the momentary consumption vector and W is the momentary wealth level.*

For proofs, see Appendix C. The entire search process is simply the successions of search steps described in the proposition.

At any point, the search direction depends only on the current consumption x, the direction of the gradient $\nabla U(x)$ and the price vector p.

Proposition 2. *If the lifetime budget constraint doesn't bind and doesn't hold with equality, the search moves in the direction of $\nabla U(x)$. If the constraint binds, the search moves in the direction $\nabla U(x) + [U(x) - \underline{U}] \cdot \frac{-p}{p \cdot x}$.*

When the lifetime budget constraint doesn't bind, search moves in the direction $\nabla U(x)$, towards higher spending allocations. It's interesting to consider whether the direction of search when the lifetime budget constraint binds is towards lower spending per period.

Proposition 3. *Let the utility function $U(x)$ be strictly concave. When the lifetime budget constraint binds, the consumer searches towards allocations with lower spending. If the constraint doesn't bind, but holds with equality, the direction of movement is along the budget constraint.*

An important question is why we need to introduce concavity to generate the result. Intuitively, the corollary proves a smoothing result. When the consumer has less wealth than it would be needed to sustain the level of consumption x forever, consumption smoothing implies that he will look for a cheaper allocation, and we know that smoothing requires assumptions on the intertemporal utility trade-off, or simply that the one-period utility is concave. Therefore, the search steps have to be towards lower allocations only if we have come kind of concavity generating risk aversion. Strict concavity is not be the weakest condition required, but it is sufficient.

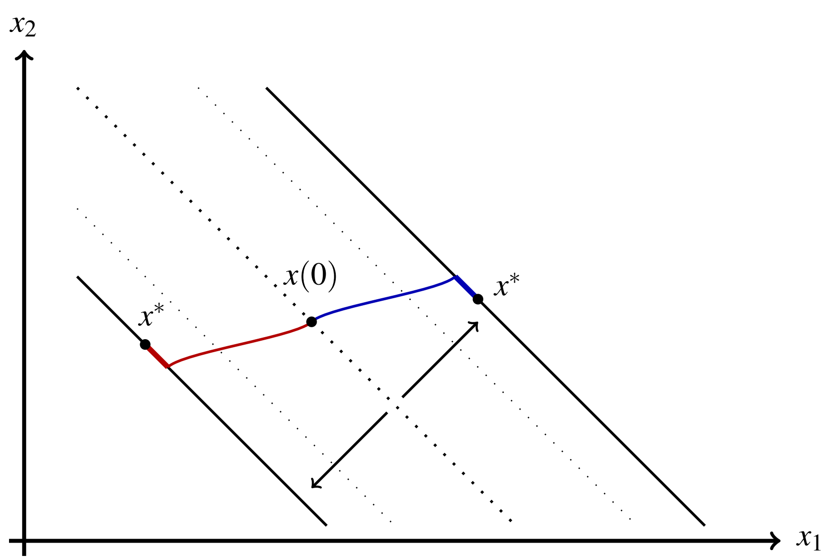


Figure 2: A search path in 2 dimensions.

For each step in the search algorithm proposed, we have a marginal change in the consumption bundle x, and a change in lifetime wealth remaining. Lifetime wealth is reduced by the instantaneous consumption and increased by interest:

$$W'(t) = (-\ln\beta)\left[-\frac{p\cdot x(t)}{1-\beta} + W(t)\right]. \tag{4}$$

The intermediate allocations generate a path $x(t)$ in the allocation space, and the lifetime wealths a function $W(t)$. The last important question left is whether the search method proposed reaches a new optimum, i.e., if $x(t)$ given by the algorithm proposed will converge to a point x^* and $W(t)$ to W^*, where $W^* = p\cdot x^*/(1-\beta)$.

Proposition 4. *Let the search process be a continuous steepest descent algorithm, where the wealth bound is updated after each step. If $U(x)$ is strictly concave, the search for the optimal allocation and wealth converges, and the limit wealth is exactly enough to sustain the limit allocation forever.*

Because the focus is explaining observations in experimental conditions, we can ignore the cases where the the consumer goes bankrupt before he reaches a new optimal allocation. It is also conceivable that wealth is so large that search won't stop and spending per time can increase forever, because of interest. While this can be sensible for some consumers, we ignore it and focus on the cases with convergence. It is also intuitive that these circumstances are likelier to happen if the search is slow.

3 Implications of Slow Search

In this section, we show how a loss aversion effect can form from a slow adjustment of the consumption bundle.

3.1 Efficiency Loss in Search

First, let's look at the consumer's lifetime utility change from a wealth shock, as the search rate is increased or decreased. For that, we compare with a situation with instant search and one with no search. Consider a sudden change in wealth ΔW at $t = 0$. Assume that the optimum consumption level reached with instant search gives instantaneous utility OU. The lifetime utility gain from $\Delta W > 0$ after instant search is $OU - U(x(0))/(1-\beta)$. If there is no search, or the search is very slow, the gain in lifetime utility is 0. Slow search gives an intermediate lifetime utility gain:

$$\frac{\ln\beta}{\beta-1}\int_0^\infty [U(x(t)) - U(x(0))]\beta^t dt \in \left(0, \frac{OU - U(x(0))}{1-\beta}\right).$$

If $\Delta W < 0$, the lifetime utility loss after instant search is $\frac{U(x(0))-OU}{1-\beta}$, while with no search, the utility loss is

$$\frac{U(x(0))}{1-\beta} - \frac{\ln\beta}{\beta-1}\left[\int_0^\infty [U(x(t)) - U(x(0))]\beta^t dt + \int_0^\infty \underline{U}\beta^t dt\right] = \\ = \frac{U(x(0)) - \underline{U}}{1-\beta}\frac{\Delta W}{W}.$$

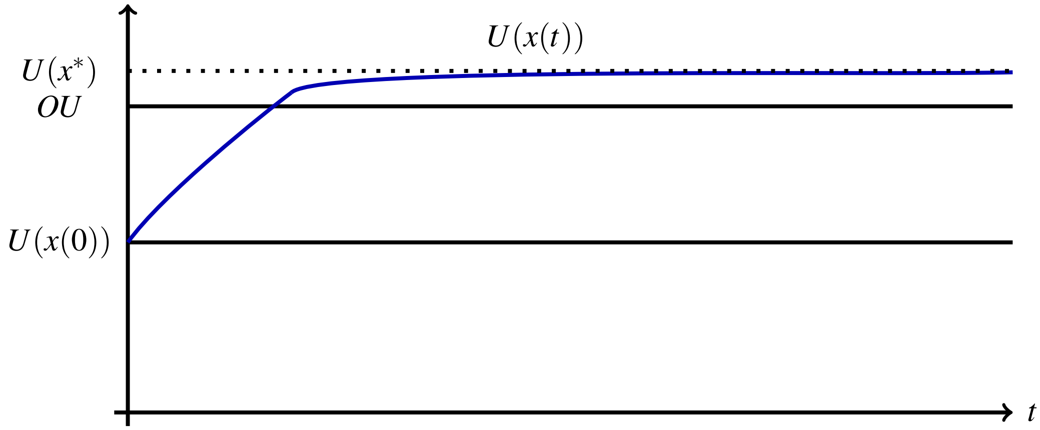


Figure 3: Instantaneous utility after $\Delta W > 0$.

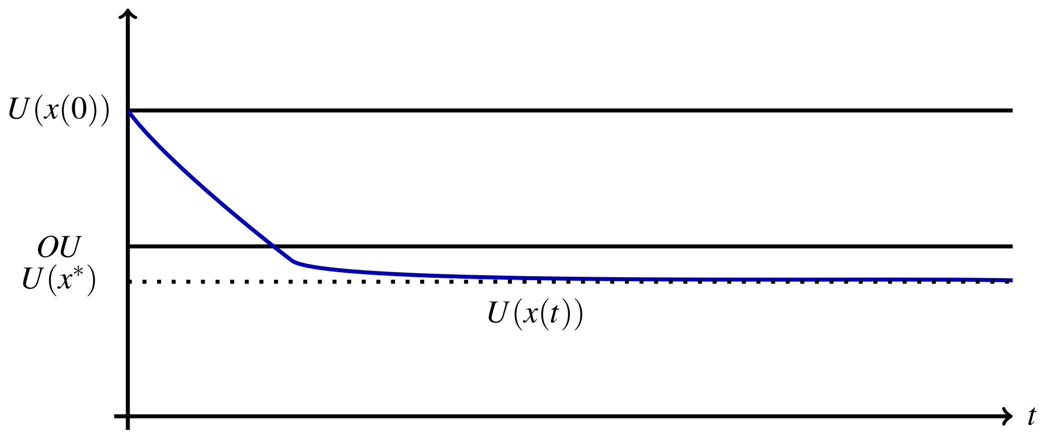


Figure 4: Instantaneous utility after $\Delta W < 0$.

This time, slow search gives an intermediate lifetime utility loss:

$$\frac{\ln \beta}{\beta - 1} \int_0^\infty [U(x(0)) - U(x(t))] \beta^t dt \in \left(\frac{U(x(0)) - OU}{1-\beta}, \frac{U(x(0)) - \underline{U}}{1-\beta} \cdot \frac{\Delta W}{W} \right).$$

Now plot the lifetime utility changes, as a function of ΔW. The ranges obtained show that the change in utility has the properties highlighted in Figure 1. Lifetime utility is lost for a wealth shock ΔW because of the inefficiency of the allocation during search. This way the utility of a gain is decreased while the dis-utility of a loss is magnified, creating an effect of loss aversion.

In Figure 5, both the upper and the lower bounds on the lifetime utility change can be reached, depending on the speed of the search process. We have normalized $\big(OU - U(x(0))\big)/(1-\beta)$ to linearity for exposition. The usual concave shape supports the same conclusion. Observe that we cannot say if the derivative of the lifetime utility function has a discontinuity at 0, which is usually assumed for the value function in prospect theory. If the cost function is assumed *ad hoc*, we only need for a kink that its relative size doesn't go to 0 as the change ΔW goes to 0. Here, we have modeled the cost as lost efficiency of allocations with concave utilities. Small shocks to wealth will lead to quadratically decreasing efficiency losses, since the utility function is locally flat. However, we focus on heuristic decision making, so it is plausible that experiments with small shocks will induce behavior that is optimal for important (large) shocks. It can also be that small decisions introduce direct time costs or other effects, leading again to *status quo* bias and relatively high change costs.

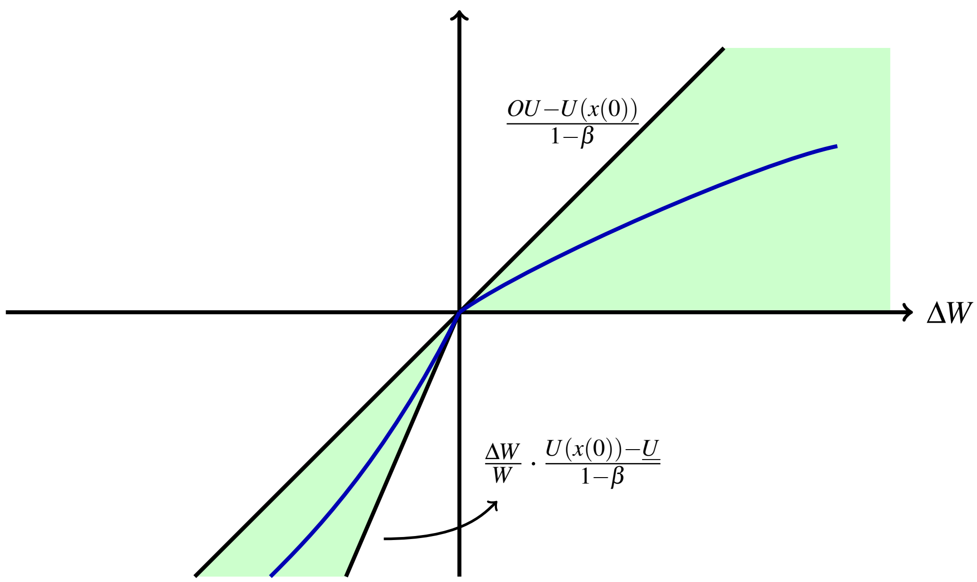


Figure 5: Lifetime utility change ranges.

3.2 Costly Adjustments

Search for consumption leads to inefficiency, or is costly in some other way. For example, in Appendix B, a different set-up introduces cost as lost time. It is not necessary to propose any specific search mechanism to make this point. Costly search will generate a reference dependent value function that obeys the loss aversion axiom. To consider the payoff of a new wealth level or endowment vector, the consumer will heuristically estimate the cost of switching to it. For this section, we only need to assume a cost function, $C : \mathbb{R}_+ \to \mathbb{R}_+$, that is differentiable and strictly increasing, with $C(0) = 0$ and $C'(l) > 0$ for $l > 0$.

For any value function $\mathcal{U} : \mathbb{R}^n_+ \to \mathbb{R}$ which describes the consumer's choices absent search, define the reference dependent utility over consumption changes v from the reference r, $V_r(v) := \mathcal{U}(r+v) - \mathcal{U}(r) - C(\|v\|)$, for all possible reference consumptions. $\|\cdot\|$ can be almost any p-norm, e.g., the euclidean distance, or, in light of our previous discussion, the seminorm that gives the money value of the change, $|p \cdot v|$. The following proposition verifies that this reference dependent utility satisfies the loss aversion assumption A_1.

Proposition 5. *Let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable, with $C(0) = 0$ and $C'(l) > 0$ for all $l > 0$, and let $\mathcal{U}(x) : \mathbb{R}^n_+ \to \mathbb{R}$ of class $\mathcal{C}^1$, strictly quasiconcave. Define $V_r(x-r) := \mathcal{U}(x) - \mathcal{U}(r) - C(\|x-r\|)$, for any $r \in \mathbb{R}^n_+$, where the norm is any d-norm $\|\cdot\|_d$, $1 \leq d < \infty$, or the seminorm $|p\cdot(x-r)|$. Then the preference structure given by $V_r(x-r)$ satisfies assumption A_1, for any pair of indices $i \neq j$.*

The *status quo* has increased value because the search cost is sunk cost. Loss aversion also implies *status quo* bias, so costly search will generate the two experimental effects.

4 Searching with Lotteries

Slow search has other implications, besides loss aversion, that relate to prospect theory. In this section we consider how the consumer, given his limited knowledge, will evaluate lotteries. We will assume that he is an expected utility maximizer, and that he knows the probabilities of an uncertain event which will change his lifetime wealth level. For simplicity, we assume that the lottery is resolved after his initial search has enough time to converge. This is relevant for many practical situations in which the risk is evaluated, because uncertainties rarely have immediate resolutions. It is intuitive that his heuristic decision making will also apply in decisions that involve lotteries that are resolved instantly.

The next question is if we can describe the path of the allocation $x(t)$ when the consumer faces an initial lottery. Let the consumer start from an optimal initial point $x(0)$, and consider a shock to wealth in the form of a lottery, $(\alpha, W_1(0); 1-\alpha, W_2(0))$, where $W_1(0) < p\cdot x(0)/(1-\beta) < W_2(0)$. $W_j(0)$ is the lifetime wealth if contingency j is realized. As the consumer spends $p\cdot x(t)$ per unit of time, we know that $W_j(t)$ will also change, but the interpretation is the same. Because the initial point is optimal, we must have that $p||\nabla U(x(0))$. Then, using the general result from Proposition 6, $\exists \alpha_c \in (0,1)$ such that

$$\frac{1-\alpha_c}{1-\beta}\nabla U(x(0)) + \frac{\alpha_c W_1}{p\cdot x(0)}\left[\nabla U(x(0)) - (U(x(0)) - \underline{U})\frac{p}{p\cdot x(0)}\right] = 0,$$

because we know that $\nabla U(x(0))$ and $\nabla U(x(0)) - (U(x(0)) - \underline{U})p/p\cdot x(0)$ are vectors pointing in opposite directions. In this case, the odds of the gamble are such that there is no need to search for a better allocation, momentarily. However, we know that

$$W_1'(0) = (-\ln\beta)\left[-\frac{p\cdot x(0)}{1-\beta} + W_1(0)\right] < 0.$$

This will reduce the effect of the low wealth outcome on the direction of search, so the gradient of the objective changes and we start moving towards higher cost allocations. If the probability of the low wealth event $\alpha < \alpha_c$, the search is going to move towards higher spending allocations, and *vice versa*. The following result gives the search direction at any point for a general lottery function, assuming that all $p \cdot x/(1-\beta) \leq W_j$ either bind or don't hold with equality.

Proposition 6. *Let the consumer have current consumption choice x, and say that he faces a lottery* $(\alpha_1, W_1; ...; \alpha_l, W_l)$*, expressed in terms of the lifetime wealths for each contingency. Let* $W_1 \leq ... \leq W_k < p \cdot x/(1-\beta) < W_{k+1} \leq ... \leq W_l$*, and* $\alpha_1 + ... + \alpha_l = 1$*. The direction of his search will be given by*

$$\frac{\alpha_{k+1} + ... + \alpha_l}{1-\beta} \nabla U(x) + \frac{\alpha_1 W_1 + ... + \alpha_k W_k}{p \cdot x} \left[\nabla U(x) - (U(x) - \underline{U}) \frac{p}{p \cdot x} \right].$$

Consider the simple lottery again. It cannot be that searching stops at any x^* such that $W_1(t) < W^* := p \cdot x^*/(1-\beta) < W_2(t)$. That is because $W_1(t)$ will be decreasing, and the gradient of the objective function will start pointing towards higher cost allocations. However, without more assumptions on the shape of $U(x)$, we cannot say how the two "forces" will balance out as consumption changes. It is sensible that, if $U(x)$ is sufficiently concave, the bundle $x(t)$ can stay somewhere such that the spending rate will not converge towards any $W_j(t)$. As before, it is also possible that the search will not converge because the values $W_j(t)$ are also changing very fast, but we ignore such unusual behavior assuming that the search is fast enough.

The following result gives us convergence for simple lotteries and $U(x)$ that is not very concave, or equivalently for small enough lottery payoffs. Under the conditions of the proposition, the movement is towards higher cost consumption for small α, or lower cost for large α.

Proposition 7. *If the consumer faces a lottery in the distant future* $(\alpha, W_1; 1-\alpha, W_2)$*, with* $W_1 < p \cdot x(0)/(1-\beta) < W_2$*, and the instantaneous utility function has* $0 \succ \nabla^2 U(x) \succeq -MI$*, then for any subdomain* $D \in \mathbb{R}^n_+$*,* $\exists\, 0 < \alpha_l < \alpha_h < 1$ *for M small enough such that for* $\alpha \in (0, \alpha_l] \cup [\alpha_h, 1)$ *the search converges,* $x(t) \to x^*$*, and* $W^* = W_1(t)$ *for* $\alpha \leq \alpha_h$*,* $W^* = W_1(t)$ *for* $\alpha \geq \alpha_h$*, where* $W^* := p \cdot x^*/(1-\beta)$*, and* W_j *is the lifetime utility for outcome j.*

For $\alpha > \alpha_h$, the consumer behaves as if "planing" for the worse outcome, and will reach a point x^* which can be consumed without bankruptcy if the worse outcome happens. For

$\alpha < \alpha_l$, the consumer behaves as if he's ignoring the worse outcome, and reaches a point where he consumes at a rate that is sustainable only with the high wealth outcome. We will say that the consumer plans for a certain outcome if his search is dominated by it, leading him to incur the cost of switching his consumption to match the outcome's lifetime wealth. For $\alpha \in (\alpha_l, \alpha_h)$, it is possible that the search will never converge. When the lottery will be resolved, the consumer will have a consumption that is in between what can be afforded under the two wealth outcomes. This behavior is suggestive of hedging. Searching for an optimal allocation leads to utility loss for the initial search, as well as in the future, if the lottery will be resolved opposite to the plan. The behavior suggests a bias towards certainty, since more certain outcomes are less likely to lead to future search costs.

We see here a behavior that suggests the consumer heads towards a discounted consumption level approximated by the expectation of the lottery faced. This is reminiscent of the work of Kőszegi and Rabin (2006), which assume that "the reference point is fully determined by the expectations a person held in the recent past."

We can make another interesting observation leading to comparative statics. Propositions 6, 7 show that planning is influenced by the concavity of the instantaneous utility function. All else being equal, at any point the consumer is more likely to plan for the worse outcome if his instantaneous utility function is more concave, i.e., if $|\nabla U(x) - (U(x) - \underline{U})p/(p \cdot x)|$ is higher. This confirms a classic assumption: a consumer that will be more likely to plan for the worse outcome, i.e., more pessimistic, is a consumer with a more concave utility function, so more risk averse. Also note that, in the limit of no concavity of $U(x)$, $|\nabla U(x) - (U(x) - \underline{U})p/(p \cdot x)| \to 0$ (Proposition 3), and the consumer will act in such a way as to make sure he consumes all possible resources if the high wealth contingency is realized. Low concavity leads to a behavior suggestive of optimism.

Corollary 8. *Let $U^a(x)$, $U^b(x)$ be two instantaneous utility functions for consumers a and b, which have current consumptions x_0. If $\nabla U^a(x_0) = \nabla U^b(x_0)$ and $0 \succ \nabla^2 U^a(x) \succ \nabla^2 U^b(x)$ for all x, then consumer a will increase his spending more than consumer b.*

5 Evaluating Lotteries

In this section we will assume that the consumer incurs search costs according to $C(\cdot)$, and that he always plans for the likeliest outcome when facing a lifetime wealth shock in the form of a lottery. We want to see what experimental evidence summarized by prospect theory can be explained this way. To keep things simpler, we'll assume a high intertemporal discount factor β, so that the utility cost of a future consumption adjustment is approximately the same as the cost of a current adjustment. For evaluating specific numerical lotteries, we pick the examples in Kahneman and Tversky (1979), and we use specific examples of utility and cost functions over wealth changes.

Let $V_r(\Delta W) := \mathcal{U}(\Delta W) - C(|\Delta W|)$ be our reference dependent value function for changes ΔW. $\mathcal{U}(\Delta W)$ is the lifetime utility gain from ΔW without any search costs. We make the usual concavity assumption for it. Furthermore, assume $C(|\Delta W|)/|\mathcal{U}(\Delta W)|$ is decreasing when $|\Delta W|$ grows. This means that the relative utility cost of search is smaller for higher wealth changes. In our discussion of search, the loss was created by the inefficient allocation of resources.

First, let's consider the following example for a value function, $V_r(\Delta W)$, where

$$\mathcal{U}(\Delta W) := \frac{\Delta W}{10^5} - \frac{1}{5}\left(\frac{\Delta W}{10^5}\right)^2, \; C(|\Delta W|) := \frac{1}{2}\ln\left(1 + \frac{|\Delta W|}{10^5}\right).$$

The utility gain with no search is a quadratic approximation, and the cost function is much more concave than the quadratic, so that search costs become relatively less significant for large shocks in wealth. There is a $1/2$ factor in front, so that the cost isn't overwhelming for small $|\Delta W|$.

The Allais paradox, as presented by Kahneman and Tversky (1979), is that in experiments gamble B is preferred to gamble A, and C to D, by a large majority of subjects:

A : 2,500 with probability 0.33, 2.400 with probability 0.66, 0 with probability 0.01,

B : 2,400 with certainty,

C : 2,500 with probability 0.33, 0 with probability 0.67,

D : 2,400 with probability 0.34, 0 with probability 0.66.

This means that a number of subjects make decisions that violate the substitution axiom. This result is interpreted to imply subcertainty for the probability weighting function in prospect theory, i.e, $\pi(p) + \pi(1-p) < 1$ for $p \in (0,1)$. However, if we evaluate the lotteries with our value function example:

$$V_r(\text{"}A\text{"}) = \mathcal{U}(2400)\cdot 0.66 + \mathcal{U}(2500)\cdot 0.33 - C(2400) - C(100)\cdot 0.33 - C(2400)\cdot 0.01 = 0.01183,$$
$$V_r(\text{"}B\text{"}) = \mathcal{U}(2400) - C(2400) = 0.01202,$$
$$V_r(\text{"}C\text{"}) = (\mathcal{U}(2500) - C(2500))\cdot 0.33 = 0.004134,$$
$$V_r(\text{"}D\text{"}) = (\mathcal{U}(2400) - C(2400))\cdot 0.34 = 0.004089.$$

Lottery A leads to a lower gain than B. Although the expected value of $\mathcal{U}$ is higher for A, the small possibility of a loss increases search costs. C has a higher gain than D, which comes from the higher expected $\mathcal{U}$ gain, since search costs are similar.

A similar story can justify the subadditivity property in probability waiting (Kahneman and Tversky, 1979). Lottery F is experimentally preferred to E, and G to H:

$$E: \quad 3,000 \text{ with probability } 0.002, \qquad F: \quad 6,000 \text{ with probability } 0.001,$$
$$\quad 0 \text{ with probability } 0.998, \qquad\qquad 0 \text{ with probability } 0.999,$$

$$G: \quad -3,000 \text{ with probability } 0.002, \qquad H: \quad -6,000 \text{ with probability } 0.001,$$
$$\quad 0 \text{ with probability } 0.998, \qquad\qquad 0 \text{ with probability } 0.999.$$

$$V_r(\text{"}E\text{"}) = (\mathcal{U}(3000) - C(3000))\cdot 0.002 = 0.00003008,$$
$$V_r(\text{"}F\text{"}) = (\mathcal{U}(6000) - C(6000))\cdot 0.001 = 0.00003015,$$
$$V_r(\text{"}G\text{"}) = (\mathcal{U}(-3000) - C(3000))\cdot 0.002 = -0.00008992,$$
$$V_r(\text{"}H\text{"}) = (\mathcal{U}(-6000) - C(6000))\cdot 0.001 = -0.00008985.$$

Finally, consider the effect of small probability overweighting, $\pi(p) > p$, for small p. In the following, I is preferred to J, and L to K in experiments:

$$I: \quad 5,000 \text{ with probability } 0.001, \qquad J: \quad 5 \text{ with certainty},$$
$$\quad 0 \text{ with probability } 0.999,$$

$$K: \quad -5,000 \text{ with probability } 0.001, \qquad L: \quad -5 \text{ with certainty}.$$
$$\quad 0 \text{ with probability } 0.999,$$

$$V_r(\text{"}I\text{"}) = (\mathcal{U}(5000) - C(5000)) \cdot 0.001 = 0.00002510,$$
$$V_r(\text{"}J\text{"}) = \mathcal{U}(5) - C(5) = 0.00002500,$$
$$V_r(\text{"}K\text{"}) = (\mathcal{U}(-5000) - C(5000)) \cdot 0.001 = -0.00007490,$$
$$V_r(\text{"}L\text{"}) = \mathcal{U}(-5) - C(5) = -0.00007500.$$

When comparing lotteries I and J, the cost of search is much less relative to the change in utility for the high wealth shock, and with our functions this effects overcomes the utility concavity effect. Subproportionality, which is defined as

$$\frac{\pi(pq)}{\pi(p)} \leq \frac{\pi(pqr)}{\pi(pr)},$$

for $0 < p, q, r \leq 1$, is also partially supported. Our example function has reproduced the experimental observations that justify the shape of the probability weighting function in prospect theory.

6 Discussion and Conclusion

We have proposed a behavioral model of consumer optimization which leads to inefficient search for optimal allocations. We have modeled the utility losses as an effect of search, once the consumer is faced with a shock to wealth or endowment. This cost can justify loss aversion as observed in experiments. A reference dependent value function that incorporates a cost of search satisfies the loss aversion axiom of Tversky and Kahneman (1991), and the equivalent loss aversion assumption of prospect theory in Kahneman and Tversky (1979). After a lottery shock, the consumer will plan for the likelier outcome, and this will lead to observed features of the probability weighting function in prospect theory. In addition, new predictions can be derived, which can serve to contrast this theoretical set-up from other models. For once, the issue of time in the setting of the reference point is crucial. An experiment that looks at the strength of the deviations from classical Hicksian demand in terms of the historical duration of a *status quo*, and tries to separate this effect from other potentially valid ones, can be used to confirm.

There is important supporting evidence for the effect of time on loss aversion in the experimental work of Strahilevitz and Loewenstein (1998). In four related studies, the authors try to separate the effect of ownership history and conclude that, in addition to an endowment effect component that seems to set in almost instantly, the duration of ownership will also increase the valuation of a given object. In the present framework, the two components of this effect can be explained. The initial effect, which sets in almost immediately, is the reflection of the fact that only a small investment in looking for the optimum at the beginning can have a large effect. Because the rate at which the optimum is approached decreases to zero, we must have an efficiency gain that comes only after a while, as the optimum is approached asymptotically. This gives the later duration-dependent component that the authors observe.

To support the theory, a mechanism that generates costly search is constructed, with some basic assumptions about an individual's search behavior. The search mechanism is proposed to be the heuristic equivalent of steepest descent, which is an intuitive solution in convex maximization problems. The individual is said to behave as if he would be performing such an algorithm, starting from an initial condition – here the *status quo*. This is a bounded rationality setting, where the limitation is that individuals have no *a priori* knowledge of utility values for consumption bundles they do not hold, unless they choose them during search.

The theoretical work of Kőszegi and Rabin (2006, 2009) is further suggesting of the importance of beliefs about future realizations on the choice of *status quo*. If beliefs on the future are important, then it's not a great leap to assume that planning is important. One of the explanations provided by their model – that traders and merchants are not influenced by the endowment effect because they don't hold updated beliefs based on the market traffic – is also paralleled by an explanation within the framework of the current paper: an agent who doesn't expect to hold on to some bundle of goods will not try to find an optimal level of consumption after every trade. Furthermore, the later explanation has the advantage that it replaces an *ad hoc* assumption – expectation as *status quo* – with an underlining mechanism.

A valuable extension to the current work would be to distinguish which features of a reduced functional form for the consumer's value function, together with search planning, are crucial for recreating subcertainty, the overweighting of small probabilities, and subadditivity, effects which justify the probability weighting function of prospect theory.

Appendix A: The Search Method

In order to link search to consumption choice, a model of search is proposed in this section. The theory sections in Appendix B uses an assumption that can give the time rate of the improvement,

$$\|x(t)-x^*\| = e^{-tr}\|x(0)-x^*\| \Leftrightarrow (\|x(t)-x^*\|)'(t) = -r\cdot\|x(t)-x^*\|. \quad (5)$$

In this section, steepest descent is proposed and discussed, but such a result can be achieved by more than one mechanism. For now, let's consider search in discrete time. If $f(x)$ is $\mathscr{C}^1$, bounded from above, the domain is bounded, and the gradient is Lipschitz continuous with some constant N, we know that the steepest descent algorithm will converge for step sizes $0<\varepsilon<1/N$ (Ruszczyński, 5.3.2, 2006). Maximum lifetime utility is bounded from above in our set-up, since we always assume that lifetime wealth is bounded in every contingency, and by the same argument so is the domain of search. A stronger result can be obtained with more assumptions on $f(x)$.

Proposition 9. *Let $f(x)$ be a twice continuously differentiable function over consumption x, which does not include a measure of time. It is assumed that $-mI \succeq \nabla^2 f(x) \succeq -MI$, $\forall x \in D$, the domain, and $0<m\leq M$. Furthermore D is convex. (This implies that the Hessian is negative definite on the domain D, $\nabla^2 f(x) \preceq 0$.) A unique maximum $x^* \in D$ must exist, which is assumed to be an interior point. The search generated by the steepest descent algorithm, starting from x_0 given, with a step length $\varepsilon \in (0,2/M)$, generates a sequence $\{x_t\} \to x^*$, and the following results hold:*

$$(a)\ \|x_t - x^*\| \leq q^t\|x_0-x^*\|,$$
$$(b)\ f(x^*)-f(x_t) \leq \frac{M}{m} qmax^{2t}[f(x^*)-f(x_0)],$$

where $qmax := \max[|1-\varepsilon m|, |1-\varepsilon M|] < 1$.

When $\|x_{t+1}-x^*\|/\|x_t-x^*\| = const. < 1$, the search is said to converge Q-linearly. For specially constructed f, we can get near the upper bound in the limit, so the upper bound cannot be improved (Cartis et al., 2009). As for a lower bound on the search convergence rate, it is obvious that, given the search process, a step length and a starting point, specific examples can be constructed such that the first steps will coincide with a maximum. But we can try to consider how fast the search method works for generic quadratic examples, which will approximate well any f with $-mI \succeq \nabla^2 f(x) \succeq -MI$, locally. For quadratics, Akaike (1959) shows that the steepest descent method with directional minimization, which is superior to constant step length, will statistically converge only Q-linearly, unless the first search direction happens to be an eigenvalue of the Hessian. This confirms the observation that in practical use steepest descent is not better than Q-linear. To show that the search speed is limited to Q-linear, we consider steepest descent with constant step length in the best case scenario: where $f(x)$ is quadratic and the initial search direction is an eigenvector of $\nabla^2 f(x)$.

Proposition 10. *Consider a quadratic $f(x)$ with non-degenerate Hessian $-mI \succeq \nabla^2 f \succeq -MI$, optimized by steepest descent with constant step length $\varepsilon \in (0, 2/M)$. Let x_0 be the starting point, and $\nabla f(x_0)$ an eigenvector of $\nabla^2 f(x_0)$. Then the search method will proceed only along the direction of $\nabla f(x_0)$, and the following hold:*

$$(a)\ \|x_t - x^*\| \geq qmin^t \|x_0 - x^*\|,$$
$$(b)\ f(x^*) - f(x_t) \geq \frac{m}{M} qmin^{2t} [f(x^*) - f(x_0)],$$

where $qmin := \min\{|1-\varepsilon m|, ..., |1-\varepsilon M|\} < 1$.

We can conclude that generally our search process has a $Q-linear$ convergence speed. Now we consider what happens in the continuous time limit to the search process.

Lemma 11. *Consider the steepest descent search with step length inversely proportional to the time interval, and the familiar assumptions on $f(x)$. In the continuous time limit, the upper and lower bounds on the convergence of the objective function and the argument are exponentials, and the search converges Q-linearly to the optimum.*

We have assumed the step length to be inversely proportional to the time of one search iteration. In terms of computational costs, the step length doesn't make a difference. We can, however, justify it with economic intuition, assuming that larger consumption changes take more time. The assumption is needed because we want to consider a continuous time description of individual consumption, even if the search behavior is essentially discrete. Another way to view $x(t)$ is as a continuous approximation of a messy, discrete consumption pattern, in the same way classic period by period aggregation is a discrete approximation. Then the continuous steepest descent is an approximation of a real discrete process.

From Lemma 11, assumption (5) holds if we take r to satisfy the upper and lower convergence speed bounds. Since we derive stylized results, we only need that the search process converges Q-linearly in Appendix B.

Appendix B: Alternate Theoretical Set-up

Consider an individual with a rational, continuous and locally nonsatiated preference relation, implying some utility function $\mathcal{U} : \mathbb{R}_+^{n+1} \to \mathbb{R}$, who is making a one-time consumption decision over bundles $(x_1,...,x_n,t) = (x,t)$, where the last component represents time, or leisure. Furthermore, $\mathcal{U}$ is strictly quasiconcave and of class $\mathcal{C}^1$. Assume that his utility is additively separable in time: $\mathcal{U}(x,t) = U(x) + T(t)$. It must follow that $U \in \mathcal{C}^1$, strictly quasiconcave. The consumption vectors $x = (x_1,...,x_n)$ are expressed in units of money for simplicity, so the price is normalized to $(1,...,1)$. Leisure and every good will optimally be consumed in some amount because:

$$T'(t) \xrightarrow{t\to 0} \infty;\ \forall i : \forall v_1,...,v_{i-1},v_{i+1},...,v_n : \frac{\partial U(v_1,...,v_{i-1},x_i,v_{i+1},...,v_n)}{\partial x_i} \xrightarrow{x_i\to 0} \infty\,.$$

So, for simplicity, the solution to the classical utility maximization problem is an interior solution for nonzero wealth. Because total utility is additively separable, the individual solves a classical utility maximization problem if the leisure consumption is fixed:

$$\arg\max_x U(x) \quad s.t. \quad \sum_{i=1}^{n} x_i \le B, \quad B \ge 0. \tag{6}$$

Let $\delta(r,x) \equiv \sum_i (r_i - x_i)$ be the wealth shift measure in units of money from the initial value $B(r) \equiv \sum_i r_i$, given some change in endowment $r \to x$. In general, x will be suboptimal for the wealth value $B+\delta$. Let $\Delta(x,x^*) \equiv \|x - x^*\|$ be the distance in consumption bundle space to the new optimum $x^*(\delta)$ that solves (6) for $B+\delta$.

Given any deviation from the reference point, the individual will incur a cost over time at a constant rate while adjusting to the new optimal consumption allocation. To be specific, begin by setting $x(t)$ as the temporary consumption decision at search time t. In the following, the setting is deterministic and the search path $x(t)$ is continuous. Search starts at $t = 0$ and will continue if the payoff from reaching a more efficient allocation offsets the cost incurred. Let $t_s \ge 0$ be the stopping time. To simplify notation, say $\Delta(t) = \Delta(x(t),x^*)$, and assume it is differentiable, and:

$$-\frac{\Delta'(t)}{\Delta(t)} \equiv r > 0, \text{ so } \Rightarrow t_s = \int_0^{t_s} \mathrm{d}t = \frac{\log\Delta(0) - \log\Delta(t_s)}{r}. \tag{7}$$

The interpretation of this assumption is that reducing the misallocation by a small percentage happens in the same small period of time, i.e., the log-rate at which the distance to optimal allocation is reduced is constant. This assumption will be justified with a mechanism for costly search in section 6, but now it is an assumption. We are dealing with a one-time consumption allocation problem and time discounting is ignored; implicitly, the consumption aggregation period is assumed much larger than the search time.

An endowment change $r \to r+v$ gives a change in wealth $\delta(r,r+v) = \sum_i v_i$. For an arbitrary v with $\sum_i v_i = 1$, consider changes of the form $r \to r+\delta v$, $\delta \in (-\sum_i r_i, \infty)$.

The following proposition shows that the optimal utility value for the new wealth level is continuous and differentiable in δ, that $U(r+\delta v)$ will deviate as $o(\delta)$ from this optimal utility, and that the optimal demand function is continuous in δ.

Proposition 12. *Let $r \in \mathbb{R}^n_+$, and $\Phi(\delta) \equiv U(x^*(\delta)) = \max_x \{U(x) \mid \sum_i x_i - (B(r)+\delta) = 0\}$. Let $x^*(\delta)$ and r be the solutions for the classical utility maximization problem (6) with budgets $B(r)+\delta$ and $B(r)$. Then $\Phi(\delta)$ is continuous and differentiable. Furthermore, for all v such that $\sum_i v_i = 1$, $\Phi(\delta) - U(r+\delta v) = o(\delta)$, where $\frac{o(\delta)}{|\delta|} \xrightarrow{\delta \to 0} 0$, and $x^*(\delta)$ is continuous in δ.*

So $x^*(\delta)$ is a continuous curve on the domain, that intersects any expenditure boundary once. Without other assumptions on the utility function, we can only establish that $\Delta(r+\delta v, x^*(\delta))$ will be bounded by a positive continuous function passing through the origin:

$$\Delta(r+\delta v, x^*(\delta)) = \|x^*(\delta) - (r+\delta v)\| \le \|x^*(\delta) - r\| + |\delta| \cdot \|v\|. \tag{8}$$

This, however, does not say anything about the lower bound of $\Delta(r+\delta v, x^*(\delta))$. We can do more if we employ stronger assumptions on U and a result by Debreu (1972, 1976), which gives conditions for the invertibility of the demand function $x^*(\delta)$. Specifically, the Gaussian curvature of the hypersurface of constant utility $\{x \mid U(x) = U(x^*)\}$ has to be non-zero at x^*, i.e., the level curves have no *flat* parts, in a quadratic sense. Observe that the Hessian condition of Proposition 9 is stronger than and implies that the Gaussian curvature is strictly greater than zero.

Proposition 13. *For all $\delta \in (-B, \infty)$, let $x^*(\delta)$ be the demand induced by the classical utility maximization problem of U with wealth $B+\delta$. If $\kappa(x^*(\delta)) \neq 0$ is the Gaussian curvature of $\{x \mid U(x) = U(x^*(\delta))\}$ at $x^*(\delta)$, then $x^*(\cdot)$ is differentiable at δ, and $\frac{\partial x_k^*}{\partial \delta} = -\frac{x_k^*}{B+\delta}\left(\sum_j \varepsilon_{kj}\right)$, $\forall k$, where ε_{kj} is the price elasticity of demand for good k and price j.*

If $x^*(\delta)$ is differentiable, then so is $\Delta(r+\delta v, x^*(\delta))$. Therefore, for small δ, $\Delta(r+\delta v, x^*(\delta))$ can be approximated by a linear function, which is nonzero for a.e. v:

$$\Delta(r+\delta v, x^*(\delta)) \approx |\delta| \cdot \left\| \left(\frac{\partial x_1^*}{\partial \delta} - v_1, \ldots, \frac{\partial x_n^*}{\partial \delta} - v_n \right) \right\|. \tag{9}$$

The upper and lower bounds for the linear approximation at small δ values are given by:

$$|\delta| \cdot \left| \left\| \left(\frac{\partial x_1^*}{\partial \delta}, \ldots, \frac{\partial x_n^*}{\partial \delta} \right) \right\| - \| (v_1, \ldots, v_n) \| \right| \lessapprox \Delta(r+\delta v, x^*(\delta))$$
$$\lessapprox |\delta| \cdot \left\| \left(\frac{\partial x_1^*}{\partial \delta}, \ldots, \frac{\partial x_n^*}{\partial \delta} \right) \right\| + |\delta| \cdot \| (v_1, \ldots, v_n) \|.$$

The slope is positive and bounded if the price elasticities are bounded locally on the demand curve, which is a reasonable assumption far away from extreme allocations. In the following, it will be assumed that $\Delta(r+\delta v, x^*(\delta))$ is monotonic and strictly increasing in δ. The assumption is reasonable if ε_{kj} can be taken to change at a small enough rate, which is true if each consumption component x_i is the aggregate of many goods. Alternatively, there can be very many x_i that can have elasticities which change rapidly, but for which $\|(\frac{\partial x_1^*}{\partial \delta} - v_1, ..., \frac{\partial x_n^*}{\partial \delta} - v_n)\|$ still changes at a rate smaller than one, because of the averaging effect.

Effective spending of resources means that the temporary consumption choice $x(t)$ is close to the budget frontier, as optimal allocations should use all the available wealth. The difference to the maximum in the utility value will be $o(x^* - x(t))$, because the budget frontier is tangent to the level curve:

$$\begin{aligned} &U(x) = U(x^*) + \sum_{i=1}^{n} \frac{\partial U}{\partial x_i}(x_i - x_i^*) - o(x^* - x), \\ &\frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial x_j} \text{ and } \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i^* \Rightarrow U(x) - U(x^*) = -o(x^* - x) \leq -o(\|x^* - x\|). \end{aligned} \tag{10}$$

Define the utility centered around the reference point on the budget frontier:

$$\forall x^* \in \mathbb{R}_+^n: \text{ let } U_x: \left\{ x \,\middle|\, \sum_i x_i = \sum_i x_i^* \right\} \to \mathbb{R}\,,\; U_{x^*}(x - x^*) \equiv U(x) - U(x^*). \tag{11}$$

The differential $DU_{x^*}(x^* - x)$ becomes the $\mathbf{0}$ map at $x^* = x$ and it's continuous, and T is strictly increasing in leisure, so:

$$\begin{aligned} \exists \Delta(t_s) > 0 \text{ s.t. } &\left| \frac{\mathrm{d}U_{x^*}(\alpha(x^* - x))}{\mathrm{d}\alpha} \right| \leq \left| \frac{\mathrm{d}T(t - \frac{1}{r}\log(\alpha\|x - x^*\|) + \frac{1}{r}\log\Delta(0))}{\mathrm{d}\alpha} \right| = \\ &\frac{T'(t - \frac{1}{r}\log(\alpha\|x - x^*\|) + \frac{1}{r}\log\Delta(0))}{r\alpha}, \text{ if } \|x - x^*\| \leq \Delta(t_s). \end{aligned} \tag{12}$$

Therefore the search stops at $\Delta(t_s)$. We can find bounds on the final utility in terms of δ and $\Delta(t_s)$:

$$\Phi(\delta) + T(t) \geq \mathcal{U}(x(t_s), t - t_s) = \begin{cases} \Phi(\delta) - o(\Delta(t_s)) + T\left(t - \dfrac{\log\frac{\Delta(0)}{\Delta(t_s)}}{r}\right), & \Delta(0) > \Delta(t_s) \\ \Phi(\delta) - o(\Delta(0)) + T(t), & \Delta(0) \leq \Delta(t_s). \end{cases} \tag{13}$$

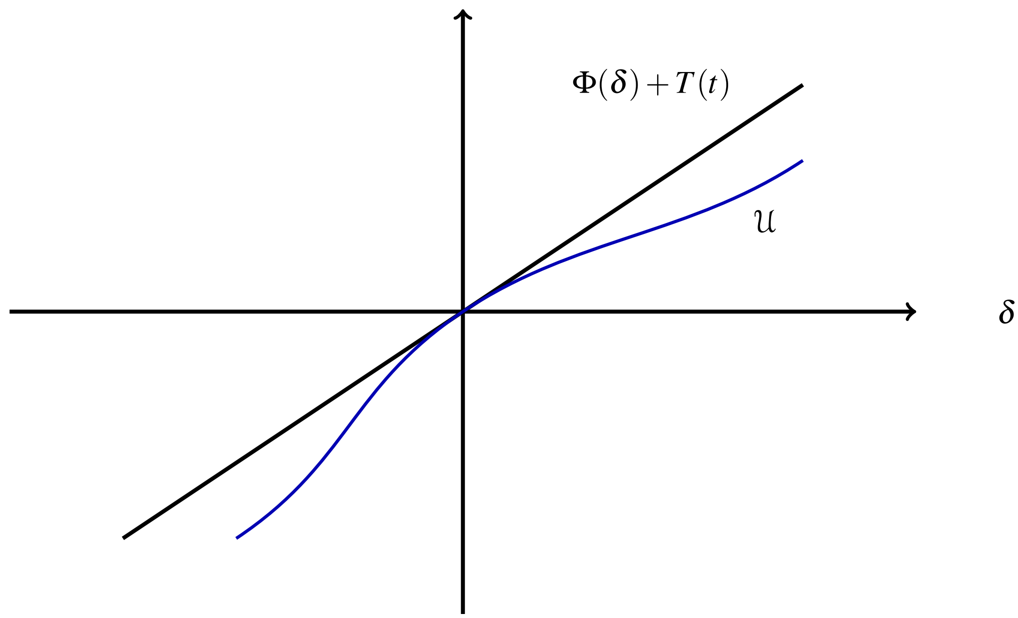


Figure 6: $\mathcal{U}$ deviates from the linear shape of $\Phi(\delta)+T$.

Because the budget frontiers intersect the demand function $x^*(\delta)$ only once, $\Phi(\delta)$ can be scaled arbitrarily by any monotonic transformation. For simplicity and convenience, let $\Phi(\delta)$ be linear.$\mathcal{U}(r,t)=\Phi(0)+T(t)$ is the utility at the *status quo*, and $\Phi(\delta)+T(t)$ is the utility in absence of search costs, for a given δ. If δ is small enough, we can simplify things further by making $T(t)\equiv k_1 t$ and $\Delta(0)=\Delta(r+\delta v,x^*(\delta))\equiv k_2|\delta|$ linear. Consider the effect of the adjustment cost on the slope, for positive and negative δ, in Figure 6.

$$\Phi(\delta)+T(t)\geq\mathcal{U}(x(t_s),t-t_s)=\begin{cases}\Phi(\delta)-o(\Delta(t_s))+T(t)-\dfrac{k_1}{r}\log\dfrac{k_2|\delta|}{\Delta(t_s)}, & k_2|\delta|>\Delta(t_s)\\[2ex] \Phi(\delta)-o(k_2|\delta|)+T(t), & k_2|\delta|\leq\Delta(t_s).\end{cases}\tag{14}$$

The absolute value of the slope is decreased for positive δ values and increased for negative ones. This gives the loss aversion effect. Compare the shape of the function with the one in Figure 1. The cost function $C(\cdot)$ used before would be given by:

$$C(\|\delta v\|)=\begin{cases}o(\Delta(t_s))+\dfrac{k_1}{r}\log\dfrac{k_2|\delta|}{\Delta(t_s)}, & k_2|\delta|>\Delta(t_s)\\[2ex] o(k_2|\delta|), & k_2|\delta|\leq\Delta(t_s).\end{cases}\tag{15}$$

This graph requires that we talk about risk aversion. In the original Kahneman and Tversky (1979) treatment, one of the assumptions is that utility of money is concave, to account for risk aversion. The cardinality of the utility function itself is linked to the basic test for risk aversion, which is assumptions (P_1). Here, we deal with choice over a set of goods, with utility arbitrary up to monotonic transformations. To be able to connect it to the one dimensional utility over money only, we need to fixate the cardinal values of U. The natural choice is to set U such that $\Phi(\delta)$ satisfies the usual one-dimensional utility assumptions, including concavity for risk aversion, but here assuming a linear $\Phi(\delta)$ would not change the results.

Appendix C: Proofs

Proposition 1. *Assume that the boundedly rational consumer knows only his current consumption vector, his current utility level and gradient, as well as the utility of no consumption. Given a line search method for the maximum, at any point in time the optimal next step is given by the method applied to problems (1) or (2), where x is the momentary consumption vector and W is the momentary wealth level.*

Proof. Consider the search process at a moment in time. Let the current consumption vector be $x(0)$ and current wealth $W(0)$. Besides $U(x(0))$, the consumer knows the linear approximation $U(x) \approx L(x) := U(x(0)) + \nabla U(x(0)) \cdot (x - x(0))$. The consumer's problem is to find the best move from $x(0)$ that will maximize his lifetime utility, which means he should potentially consider the effect of his next step on the search process in the future. However, even as he anticipates future knowledge and future search, he doesn't have access to the utility levels of choices outside of $\mathscr{B}(x(0),\varepsilon)$, so, in the absence of other information, his best move is the same as if he would be doing a one step search. Therefore, he will maximize the objective function derived from problems (1):

$$\max_x \mathcal{U}(x) = \max_x \{L(x) + \lambda\,[(1-\beta)W^* - p\cdot x]\},$$

or (2):

$$\max_x \mathcal{U}(x) = \max_x \left\{ L(x) - (L(x) - \underline{U}) \left(1 + \frac{W}{p\cdot x}(\beta - 1)\right)\right\},$$

s.t. $x \in \mathcal{B}(x(0), \varepsilon|\nabla U(x(0))|)$. The best point to move to is $x(0) + \varepsilon \nabla \mathcal{U}(x(0))$, in the direction of steepest descent. This maximization will be one step in the search algorithm. As $\varepsilon, \delta_\tau \to 0$, we get a continuous movement along the line of steepest descent. □

Proposition 2. *If the lifetime budget constraint doesn't bind and doesn't hold with equality, the search moves in the direction of* $\nabla U(x)$. *If the constraint binds, the search moves in the direction* $\nabla U(x) + [U(x) - \underline{U}] \cdot \frac{-p}{p\cdot x}$.

Proof. The maximization problem is

$$\max_x U(x)\int_0^{\tau(x)} e^{t\ln\beta}dt + \underline{U}\int_{\tau(x)}^{\infty} e^{t\ln\beta}dt,$$

$$s.t. \frac{1-\beta^{\tau(x)}}{1-\beta} p\cdot x \leq W.$$

If the budget constraint doesn't bind, we have that $\tau(x)=\infty$, and the local maximization problem is trivial because we are away from the boundary. That is, in the Lagrangian we can ignore the constraint. Then the objective maximizing direction is given by $\nabla U(x)$. When the lifetime budget constrain binds, we have that

$$1-\frac{W}{p\cdot x}(1-\beta)=\beta^{\tau(x)},$$

so we can rewrite the maximization problem as

$$\max_x \mathcal{U}(x) := -\left(U(x)-\underline{U}\right)\left[1-\frac{W}{p\cdot x}(1-\beta)\right]+U(x) = \left(U(x)-\underline{U}\right)\frac{W(1-\beta)}{p\cdot x}+\underline{U}.$$

This gives us

$$\nabla\mathcal{U}(x)=\frac{W}{p\cdot x}(1-\beta)\left[\nabla U+\left(U(x)-\underline{U}\right)\cdot\frac{-p}{p\cdot x}\right].$$

□

Proposition 3. *Let the utility function $U(x)$ be strictly concave. When the lifetime budget constraint binds, the consumer searches towards allocations with lower spending. If the constraint doesn't bind, but holds with equality, the direction of movement is along the budget constraint.*

Proof. By the multivariate mean value theorem, we have that $U(x)-\underline{U}=U(x)-U(0)=\nabla U((1-c)x)\cdot x$, for some value $c\in(0,1)$. Therefore, we can write the projection on p of the expression that shows the direction of search like

$$\frac{p}{|p|}\cdot\left[\nabla U(x)+\left(U(x)-\underline{U}\right)\cdot\frac{-p}{p\cdot x}\right]=\nabla U\cdot\frac{p}{|p|}-\left(U(x)-\underline{U}\right)\frac{|p|}{p\cdot x}=$$

$$=\nabla U(x)\cdot\frac{p}{|p|}-\nabla U((1-c)x)\cdot x\frac{|p|}{p\cdot x}.$$

Multiply by $\frac{p\cdot x}{|p|}$ to get

$$\left(\nabla U(x)\cdot\frac{p}{|p|}\right)\left(x\cdot\frac{p}{|p|}\right)-\nabla U((1-c)x)\cdot x \leq \nabla U(x)\cdot x-\nabla U((1-c)x)\cdot x,$$

because the first term is positive. $\nabla U(x) \cdot x - \nabla U((1-c)x) \cdot x$ is negative since U is concave, therefore the projection is negative, which is what we required.

Now let's consider the case when the lifetime budget holds with equality, $p \cdot x/(1-\beta) = W$, but doesn't bind. We'll prove that the best marginal change direction has to be perpendicular on p. *W.l.o.g.* consider a small change in x by $a+b$, where $a \cdot p = 0$, $b \cdot p = |b||p| > 0$. If b increases spending per unit of time, then lifetime utility can be increased by a change of consumption of $-b$, by $\nabla \mathcal{U}(x) \cdot (-b)$, since we have proved that $\nabla \mathcal{U}(x)$ is directed towards allocations of lower spending when the spending rate exceeds W. This means that a change of a is better than a change of $a+b$. Similarly, if b decreases spending per unit of time, then we can increase lifetime utility by moving back by $\nabla U(x) \cdot (-b)$, because $\nabla U(x)$ points towards allocations of higher spending.

□

Proposition 4. *Let the search process be a continuous steepest descent algorithm, where the wealth bound is updated after each step. If $U(x)$ is strictly concave, the search for the optimal allocation and wealth converges, and the limit wealth is exactly enough to sustain the limit allocation forever.*

Proof.

$$W'(t) = (-\ln\beta)\left[-\frac{p \cdot x(t)}{1-\beta} + W(t)\right].$$

If the lifetime wealth constraint doesn't bind and doesn't hold with equality, the direction of movement is $\nabla U(x)$. Because preferences are assumed non-satiated, as the algorithm proceeds we have that consumption per unit of time, $p \cdot x(t)$, is increased. Moreover, as spending increases, current lifetime utility is also increased. As mentioned, we ignore the case of unbounded growth of consumption, when the new initial wealth is so large that it exceeds the net present cost of the path of consumption growing infinitely. In other words,

$$\frac{p \cdot x(t)}{1-\beta} \to W(t),$$

and

$$\frac{p \cdot x(t)}{1-\beta} = W(t) \Rightarrow W'(t) = 0,$$

so the wealth level converges, $W(t) \to W^* > 0$. Search when the lifetime wealth constraint holds with equality is discussed last.

Now assume that the lifetime wealth constraint binds, and begin by incorporating the constraint into (2). We then have the following generic objective function that we consider for a step in the algorithm:

$$\max_x \left\{ -\left(U(x) - \underline{U}\right)\left[1 - \frac{W}{p\cdot x}(1-\beta)\right] + U(x) \right\}.$$

In the steepest descent algorithm, a step in the search should weakly increase the value of the objective function. (See Theorems 5.1, 5.3 in Ruszczyński (2006).) This function is the discounted lifetime utility of the consumer, where he stops searching and consumes x as long as he can afford it. Therefore, each step in the search increases lifetime utility. Moreover, lifetime wealth is decreased after each step, since the consumer spends more than he can afford in the long run. From equation (4):

$$\frac{1}{1-\beta} p\cdot x(t) \geq W \Rightarrow \frac{\ln\beta}{\beta-1} p\cdot x(t) \geq W(-\ln\beta) \Rightarrow W'(t) \leq 0.$$

Since lifetime wealth is bounded from below by 0 and $W'(t)$ is monotonic, $W(t) \to W^*$. As mentioned, we ignore the case where the consumer searches so slowly that he goes bankrupt before he reaches a new optimum, i.e., $W^* > 0$.

When the lifetime budget constraint holds with equality, the consumer solves

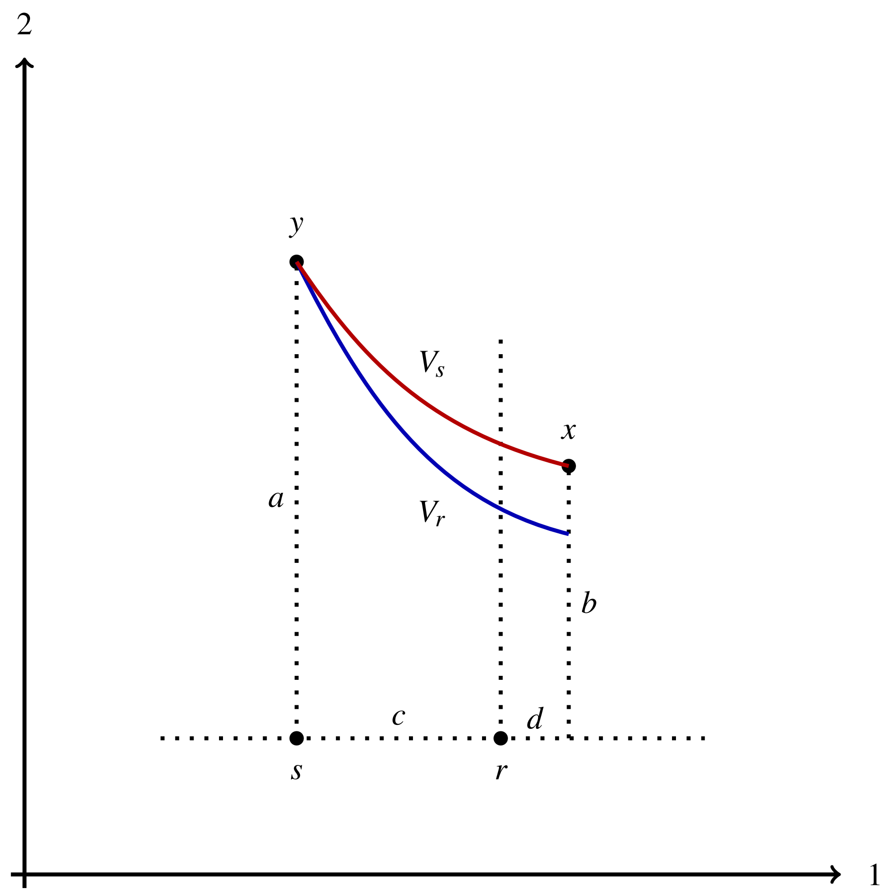
$$\max_x \mathcal{U}(x) := \max_x \left\{ U(x) + \lambda\left[(1-\beta)W^* - p\cdot x\right] \right\},$$

by Proposition 3. With $U(x)$ strictly concave, steepest descent will find the unique maximum, x^*, and $W^* = p\cdot x^*/(1-\beta)$. □

Proposition 5. *Let $C : \mathbb{R}_+ \to \mathbb{R}_+$ be differentiable, with $C(0) = 0$ and $C'(l) > 0$ for all $l > 0$, and let $\mathcal{U}(x) : \mathbb{R}^n_+ \to \mathbb{R}$ of class $\mathcal{C}^1$, strictly quasiconcave. Define $V_r(x-r) := \mathcal{U}(x) - \mathcal{U}(r) - C(\|x-r\|)$, for any $r \in \mathbb{R}^n_+$, where the norm is any d-norm $\|\cdot\|_d$, $1 \leq d < \infty$, or the seminorm $|p\cdot(x-r)|$. Then the preference structure given by $V_r(x-r)$ satisfies assumption A_1, for any pair of indices $i \neq j$.*

Proof. Let $r,s,x,y \in \mathbb{R}^n_+$, and $i = 1, j = 2$, *w.l.o.g.* In addition, $x_1 \geq r_1 > s_1 = y_1$, $y_2 > x_2$, $r_2 = s_2$, and $x =_s y$. Define $a \equiv y_2 - s_2$, $b \equiv x_2 - s_2$, $c \equiv r_1 - s_1 > 0$ and $d \equiv x_1 - r_1 \geq 0$. Then $x =_s y$ means:

$$V_s(x-s) = V_s(c+d,b,0,\dots,0) = \mathcal{U}(x) - \mathcal{U}(s) - C\big(\|(c+d,b,0,\dots,0)\|\big) =$$
$$V_s(y-s) = V_s(0,a,0,\dots,0) = \mathcal{U}(y) - \mathcal{U}(s) - C\big(\|(0,a,0,\dots,0)\|\big).$$



With respect to the reference r, the value functions are:

$$\begin{aligned} V_r(x-r) &= V_r(d,b,0,\dots,0) = \mathcal{U}(x) - \mathcal{U}(r) - C\big(\|(d,b,0,\dots,0)\|\big), \text{ and} \\ V_r(y-r) &= V_r(-c,a,0,\dots,0) = \mathcal{U}(y) - \mathcal{U}(r) - C\big(\|(-c,a,0,\dots,0)\|\big), \\ \Rightarrow V_r(x-r) - V_r(y-r) &= \mathcal{U}(x) - \mathcal{U}(y) - C\big(\|(d,b,0,\dots,0)\|\big) + C\big(\|(-c,a,0,\dots,0)\|\big). \end{aligned}$$

So, solving for $\mathcal{U}(x) - \mathcal{U}(y)$ in the first equation:

$$\begin{aligned} V_r(x-r) - V_r(y-r) &= +C\big(\|(c+d,b,0,\dots,0)\|\big) + C\big(\|(-c,a,0,\dots,0)\|\big) - \\ &\quad - C\big(\|(d,b,0,\dots,0)\|\big) \qquad - C\big(\|(0,a,0,\dots,0)\|\big) \\ &> 0. \end{aligned}$$

The last inequality follows from the fact that, for $p < \infty$, $c > 0$, C strictly increasing, and:

$$\begin{aligned} \|(0,a,0,\dots,0)\| &< \|(-c,a,0,\dots,0)\|, \\ \|(d,b,0,\dots,0)\| &< \|(c+d,b,0,\dots,0)\|. \end{aligned}$$

Therefore $x >_r y$, which was required. $\square$

Proposition 6. *Let the consumer have current consumption choice x, and say that he faces a lottery* $(\alpha_1, W_1; ...; \alpha_l, W_l)$*, expressed in terms of the lifetime wealths for each contingency. Let* $W_1 \leq ... \leq W_k < \frac{p \cdot x}{1-\beta} < W_{k+1} \leq ... \leq W_l$*, and* $\alpha_1 + ... + \alpha_l = 1$*. The direction of his search will be given by*

$$\frac{\alpha_{k+1} + ... + \alpha_l}{1-\beta} \nabla U(x) + \frac{\alpha_1 W_1 + ... + \alpha_k W_k}{p \cdot x} \left[\nabla U(x) - (U(x) - \underline{U}) \frac{p}{p \cdot x} \right].$$

Proof. We have argued that the consumer, having restricted knowledge, behaves at any point as if he's maximizing his lifetime utility given his current consumption plan. For this, write the maximization problem for the consumer:

$$\max_x \frac{\ln \beta}{\beta - 1} \left\{ \sum_j \alpha_j \left[U(x) \int_0^{\tau_j(x)} \beta^t dt + \underline{U} \int_{\tau_j(x)}^{\infty} \beta^t dt \right] \right\}$$

$$\text{s.t. } \forall j \in \{1, ..., k\} : \frac{1 - \beta^{\tau_j(x)}}{1-\beta} p \cdot x = W_j, \text{ and } \tau_{k+1} = ... = \tau_n = \infty.$$

It's easy to solve the constraints for $\beta^{\tau_j(x)}$ and introduce them into the objective function, and we get

$$\max_x \left\{ \sum_{j=1}^{k} \frac{\alpha_j W_j}{p \cdot x} \left[U(x) - (U(x) - \underline{U}) \left(1 + \frac{W_j}{p \cdot x} (\beta - 1) \right) \right] + U(x) \sum_{j=k+1}^{l} \alpha_j \right\}.$$

The gradient of this new objective function is the formula given in the proposition. □

Proposition 7. *If the consumer faces a lottery in the distant future* $(\alpha, W_1; 1-\alpha, W_2)$*, with* $W_1 < p \cdot x(0)/(1-\beta) < W_2$*, and the instantaneous utility function has* $0 \succ \nabla^2 U(x) \succeq -MI$*, then for any subdomain* $D \in \mathbb{R}^n_+$*,* $\exists\, 0 < \alpha_l < \alpha_h < 1$ *for M small enough such that for* $\alpha \in (0, \alpha_l] \cup [\alpha_h, 1)$ *the search converges,* $x(t) \to x^*$*, and* $W^* = W_1(t)$ *for* $\alpha \leq \alpha_h$*,* $W^* = W_1(t)$ *for* $\alpha \geq \alpha_h$*, where* $W^* := p \cdot x^*/(1-\beta)$*, and* W_j *is the lifetime utility for outcome j.*

Proof. First, let's consider $W_1 < p \cdot x/(1-\beta) < W_2$. At any x, the direction of search is given by

$$\frac{1-\alpha}{1-\beta} \nabla U(x) + \frac{\alpha W_1}{p \cdot x} \left[\nabla U(x) - (U(x) - \underline{U}) \frac{p}{p \cdot x} \right]. \tag{16}$$

Let's rewrite the expression for component i, by dividing by the positive scalar $\frac{\alpha(1-\beta)W_1}{p \cdot x}$ and regrouping:

$$U_i(x) \left(1 + \frac{1-\alpha}{\alpha} \frac{p \cdot x}{(1-\beta)W_1} \right) - (U(x) - \underline{U}) \frac{p_i}{p \cdot x}.$$

Now consider a small change in x_j by δ_j, happening in the time δ_τ, and write the marginal change in the component i of the search direction vector:

$$U_{ij}(x)\delta_j\left(1+\frac{1-\alpha}{\alpha}\frac{p\cdot x}{(1-\beta)W_1}\right)+U_i(x)\frac{1-\alpha}{\alpha}\frac{p_j\delta_j}{(1-\beta)W_1}+ \tag{17}$$

$$+U_i(x)\frac{1-\alpha}{\alpha}\frac{p\cdot x}{(1-\beta)W_1^2}\left(-W_1'\delta_\tau\right)-U_i(x)\frac{p_j\delta_j}{p\cdot x}-(U(x)-\underline{U})\frac{-p_ip_j\delta_j}{(p\cdot x)^2}.$$

For small enough maximum curvatures M, the negative first term is dominated by the other. Observe that the 2nd, 3rd and 5th terms are strictly positive ($W_1'<0$!), and the 4th is negative. We know that $(1-\beta)W_1 < p\cdot x$, so for any $\alpha < 1/2$ we have that the second term is bigger in absolute value than the 4th. Given a small M, $\exists\alpha_l^{ij}>0$ such that the expression is positive on D for x, and define $\alpha_l := \min_{i,j}\{\alpha_l^{ij}\} > 0$. We can conclude that, for any $0<\alpha\leq\alpha_l$, a marginal movement towards higher spending will not lead to a reversal of direction.

Similarly, we can discuss high α values. Group the last two terms together by factoring $-p_j\delta_j/p\cdot x$, and observe that the expression is negative, by Proposition 3. Terms 2 and 3 can be grouped together into a positive expression as well, with a factor $U_i(x)(1-\alpha)/\alpha$ in front. $U_i(x)>0$, so $\exists\alpha_h^{ij}<1$ such that the expression is negative for $\alpha>\alpha_h^{ij}$ and $x\in D$. This means that, for $\alpha>\alpha_h := \max_{i,j}\{\alpha_h^{ij}\}<1$, a marginal movement towards lower spending will not lead to a reversal of direction.

The above deals with the case when the budget constraint for the low wealth contingency binds, but the other one doesn't bind or hold with equality. As usual, we exclude the cases when the search is not fast enough and $W_1\to 0$, or when the consumer increases his consumption forever: $W_1,W_2\to\infty$ and $p\cdot x\to\infty$. The search moves either toward lower or higher spending, until the spending level $p\cdot x(t)$ reaches either $W_1(t)$ or $W_2(t)$ respectively. At that point, the respective contingent wealths converge, and maximization problem changes.

Now, let's assume that $W_1(t)\to W^*>0$, so the current consumption level $x(t)$ is such that $p\cdot x(t)/(1-\beta)=W^*$. Then the claim is that the search will move perpendicular to p, similarly to the situation in Proposition 3. The argument is similar and tedious; from (16), note that moving away from the plane $p\cdot x(t)$ means that utility is lost proportional to

$$\frac{\alpha}{1-\beta}\left[\nabla U(x)-(U(x)-\underline{U})\frac{p}{p\cdot x}\right],$$

and gained proportional to $(1-\alpha)\nabla U(x)/(1-\beta)$. If that were positive, we wouldn't have moved towards the lower spending. So the movement has to be along the plane. If

$W_2(t) \to W^*$, then a similar argument shows that the search will then move along the plane $p \cdot x(t)$, because

$$\frac{\alpha W_1}{p \cdot x} \left[\nabla U(x) - (U(x) - \underline{U}) \frac{p}{p \cdot x} \right]$$

is decreasing from W_1. In this stage, the search is a simple optimization problem, and we get $x(t) \to x^*$ and $p \cdot x^* = W^*(1-\beta)$. □

Corollary 8. *Let $U^a(x)$, $U^b(x)$ be two instantaneous utility functions for consumers a and b, which have current consumptions x_0. If $\nabla U^a(x_0) = \nabla U^b(x_0)$ and $0 \succ \nabla^2 U^a(x) \succ \nabla^2 U^b(x)$ for all x, then consumer a will increase his spending more than consumer b.*

Proof.

$$\begin{aligned}
& U^j(x_0) - U^j(0) = \int_0^1 \nabla U^j(\theta x_0) \cdot x_0 d\theta = \\
& = \int_0^1 \left[\nabla U^j(x_0) - \int_0^1 \nabla^2 U^j(x_0 - \rho(1-\theta)x_0) \cdot x_0 d\rho \right] \cdot x_0 d\theta = \\
& = \int_0^1 \nabla U^j(x_0) \cdot x_0 d\theta - \int_0^1 \int_0^1 x_0^T \cdot \nabla^2 U^j(x_0 - \rho(1-\theta)x_0) \cdot x_0 d\rho d\theta.
\end{aligned}$$

We know that, for strictly concave utilities, $\nabla^2 U^j(x)$ are negative definite, so we can conclude that $U^a(x_0) - \underline{U}^a < U^b(x_0) - \underline{U}^b$. From (16), we see that the gradient of the objective of b minus that of a is $p/(p \cdot x)(U^a(x_0) - U^b(x_0) - \underline{U}^a + \underline{U}^b) < 0$, which is directed towards lower cost consumptions.

□

Proposition 9. *Let $f(x)$ be a twice continuously differentiable function over consumption x, which does not include a measure of time. It is assumed that $-mI \succeq \nabla^2 f(x) \succeq -MI$, $\forall x \in D$, the domain, and $0 < m \leq M$. Furthermore D is convex. (This implies that the Hessian is negative definite on the domain D, $\nabla^2 f(x) \preceq 0$.) A unique maximum $x^* \in D$ must exist, which is assumed to be an interior point. The search generated by the steepest descent algorithm, starting from x_0 given, with a step length $\varepsilon \in (0, 2/M)$, generates a sequence $\{x_t\} \to x^*$, and the following results hold:*

$$\begin{aligned}
& (a)\ \|x_t - x^*\| \leq q^t \|x_0 - x^*\|, \\
& (b)\ f(x^*) - f(x_t) \leq \frac{M}{m} qmax^{2t} [f(x^*) - f(x_0)],
\end{aligned}$$

where $qmax := \max[|1 - \varepsilon m|, |1 - \varepsilon M|] < 1$.

Proof. Apply, for example, Theorem 5.5 in Ruszczyński (2006) to obtain (a). The Hessian condition $-mI \succeq \nabla^2 f(x) \succeq -MI$, and the fact that $\nabla f(x^*) = 0$ give the following inequalities:

$$\frac{m}{2}\|x-x^*\|^2 \leq f(x^*) - f(x) \leq \frac{M}{2}\|x-x^*\|^2, \ \forall x \in D.$$

Using (a), this implies:

$$f(x^*) - f(x_t) \leq \frac{M}{2}\|x_t - x^*\|^2 \leq \frac{M}{2}q^{2t}\|x_0 - x^*\|^2 \leq \frac{M}{m}q^{2t}[f(x^*) - f(x_0)].$$

□

Proposition 10. *Consider a quadratic $f(x)$ with non-degenerate Hessian $-mI \succeq \nabla^2 f \succeq -MI$, optimized by steepest descent with constant step length $\varepsilon \in (0, 2/M)$. Let x_0 be the starting point, and $\nabla f(x_0)$ an eigenvector of $\nabla^2 f(x_0)$. Then the search method will proceed only along the direction of $\nabla f(x_0)$, and the following hold:*

$$(a)\ \|x_t - x^*\| \geq qmin^t \|x_0 - x^*\|,$$
$$(b)\ f(x^*) - f(x_t) \geq \frac{m}{M} qmin^{2t}[f(x^*) - f(x_0)],$$

where $qmin := \min\{|1-\varepsilon m|, \ldots, |1-\varepsilon M|\} < 1$.

Proof. For steepest descent with constant step length, $x^{t+1} = x^t + \varepsilon \cdot \nabla f(x^t)$, so

$$\nabla f(x_{t+1}) = \nabla f(x_t) + (x_{t+1} - x_t) \cdot \nabla f(x_t) = \nabla f(x_t) + \varepsilon \nabla^2 f(x_t) \nabla f(x_t) =$$
$$= \nabla f(x_t)(1 - \varepsilon K),$$

where $K \in [m, M]$ is the eigenvalue of $\nabla f(x^t)$. After each step, the gradient has the same direction, which means that all search steps are on the same direction. Therefore, we can analyze the search in one dimension. The section of f on one dimension will be a 1-dimensional quadratic function, and we'll parametrize the points with $y \in \mathbb{R}$. Let y^* be the maximum, and y_0 the starting point. Steepest descent gives:

$$y_{t+1} - y_t = \varepsilon K(y_t - y^*) \Rightarrow y_n = (1-\varepsilon K)^t y_0 + \varepsilon K \frac{(1-\varepsilon K)^t - 1}{1 - \varepsilon K - 1}$$
$$\Rightarrow y_t - y^* = (1-\varepsilon K)^n (y_0 - y^*) \Rightarrow \|x_t - x^*\| = (1-\varepsilon K)^t \|x_0 - x^*\|.$$

The condition for convergence is $\varepsilon \in (0, 2/K)$. From this, the fastest speed of convergence is reached for the direction with the eigenvalue that minimizes $|1-\varepsilon K|$, so $qmin := \min\{|1-\varepsilon m|, \ldots, |1-\varepsilon M|\}$. As in Proposition 9,

$$\frac{m}{2}\|x-x^*\|^2 \leq f(x^*) - f(x) \leq \frac{M}{2}\|x-x^*\|^2, \ \forall x \in D.$$

This means that

$$f(x^*) - f(x_t) \geq \frac{m}{2}\|x_t - x^*\|^2 \geq \frac{m}{2} qmin^{2t}\|x_0 - x^*\|^2 \geq \frac{m}{M} qmin^{2t}[f(x^*) - f(x_0)].$$

□

Lemma 11. *Consider the steepest descent search with step length inversely proportional to the time interval, and the familiar assumptions on $f(x)$. In the continuous time limit, the upper and lower bounds on the convergence of the objective function and the argument are exponentials, and the search converges Q-linearly to the optimum.*

Proof. Let s be the size of the time period, $s\varepsilon$ the step length in the search, and T a total finite amount of time. The number of time periods is $t := T/s$. Define $q := 1 - s\varepsilon K$, where K is such that q^t can describe either the upper or lower bounds on the search speed. Now consider the bonds in the limit when the search period goes to 0:

$$\lim_{s\to 0} q^t = \lim_{s\to 0}(1 - s\varepsilon K)^{T/s} = (e^{\varepsilon K})^{-T}.$$

We know that $(1-\frac{1}{n})^n$, where $n := \frac{1}{s\varepsilon K}$, is monotonically increasing in n, so as s decreases, the upper bond on $\|x_t - x^*\|$ after a finite period of time T is reduced. Also, for arbitrarily small time periods s, the search process will converge to the same limit, and it is Q-linear. The same arguments apply to $[f(x^*) - f(x_t)]$. □

Proposition 12. *Let $r \in \mathbb{R}^n_+$, and $\Phi(\delta) \equiv U(x^*(\delta)) = \max_x \{U(x) \mid \sum_i x_i - (B(r) + \delta) = 0\}$. Let $x^*(\delta)$ and r be the solutions for the classical utility maximization problem (6) with budgets $B(r)+\delta$ and $B(r)$. Then $\Phi(\delta)$ is continuous and differentiable. Furthermore, for all v such that $\sum_i v_i = 1$, $\Phi(\delta) - U(r+\delta v) = o(\delta)$, where $\frac{o(\delta)}{|\delta|} \xrightarrow{\delta\to 0} 0$, and $x^*(\delta)$ is continuous in δ.*

Proof. By Proposition 2.13(b) of Kreps (1990), $\Phi(\delta)$ is continuous. $U(x)$ is differentiable so, by the envelope theorem, $\Phi(\delta)$ is differentiable also. Moreover, from the envelope theorem:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\delta} \equiv \lambda = \left.\frac{\partial U}{\partial x_i}\right|_{x=x^*}, \ \forall i, \text{ and } \frac{\mathrm{d}U(r+\delta v)}{\mathrm{d}\delta} = \sum_i v_i \left.\frac{\partial U}{\partial x_i}\right|_{x=r+\delta v}.$$

Therefore:

$$\left.\frac{\mathrm{d}(\Phi(\delta) - U(r+\delta v))}{\mathrm{d}\delta}\right|_{\delta=0} = \left.\frac{\partial U}{\partial x_i}\right|_{x=r} - \sum_i v_i \left.\frac{\partial U}{\partial x_i}\right|_{x=r} = \lambda(1 - \sum_i v_i) = 0.$$

Applying Taylor's Theorem gives $\Phi(\delta) - U(r+\delta v) = o(\delta)$. To show that $x^*(\delta)$ is continuous, let $\delta_n \to \delta$, with $x^*(\delta_n) \nrightarrow x^*(\delta)$. But $\Phi(\delta_n) \to \Phi(\delta)$, since Φ is continuous.

Therefore, $U(x^*(\delta)) = \Phi(\delta) = \lim_n \Phi(\delta_n) = \lim_n U(x^*(\delta_n)) = U(\lim_n x^*(\delta_n))$, where the last equality follows from the continuity of U. Moreover:

$$\sum_{i=1}^{n} x_i^*(\delta_n) - (B+\delta_n) = 0 \Rightarrow \sum_{i=1}^{n} \lim_n x_i^*(\delta_n) - (B+\delta) = 0$$
$$\Rightarrow \lim_n x^*(\delta_n) \in \arg\max_x \left\{ U(x) \;\middle|\; \sum_{i=1}^{n} x_i - (B+\delta) = 0 \right\},$$

because $\lim_n x^*(\delta_n)$ solves the utility maximization problem. Since U is strictly quasiconcave, the solution to the maximization problem must be unique, so $\lim_n x^*(\delta_n) = x^*(\delta)$, so $x^*(\delta)$ is continuous in δ. □

Proposition 13. *For all $\delta \in (-B, \infty)$, let $x^*(\delta)$ be the demand induced by the classical utility maximization problem of U with wealth $B+\delta$. If $\kappa(x^*(\delta)) \neq 0$ is the Gaussian curvature of $\{x \mid U(x) = U(x^*(\delta))\}$ at $x^*(\delta)$, then $x^*(\cdot)$ is differentiable at δ, and $\frac{\partial x_k^*}{\partial \delta} = -\frac{x_k^*}{B+\delta}\left(\sum_j \varepsilon_{kj}\right)$, $\forall k$, where ε_{kj} is the price elasticity of demand for good k and price j.*

Proof. Define the general demand function $f(p,W)$ given some arbitrary nonzero price vector p and wealth W. By Debreu (1972, 1976), the Jacobian of f at x is given by:

$$J(f) = \frac{\|\nabla U\|}{\frac{\partial U}{\partial x_n}\kappa(x)}, \text{ if } \kappa(x) \neq 0.$$

Therefore $x^*(\delta) \equiv f((1,...,1), B+\delta)$ is differentiable as a function of δ. Because the demand function is homogeneous of degree zero, we can write:

$$\sum_{i=1}^{n} \frac{\partial x_k^*}{\partial p_i} p_i + \frac{\partial x_k^*}{\partial \delta}(B+\delta) = 0 \Rightarrow \frac{\partial x_k^*}{\partial \delta} = -\frac{x_k^*}{B+\delta}\left(\sum_j \varepsilon_{kj}\right), \forall k.$$

□

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