D-Brane Gauge Theories from Toric Singularities and Toric Duality

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Abstract

Via partial resolution of Abelian orbifolds we present an algorithm for extracting a consistent set of gauge theory data for an arbitrary toric variety whose singularity a D-brane probes. As illustrative examples, we tabulate the matter content and superpotential for a D-brane living on the toric del Pezzo surfaces as well as the zeroth Hirzebruch surface. Moreover, we discuss the non-uniqueness of the general problem and present examples of vastly different theories whose moduli spaces are described by the same toric data. Our methods provide new tools for calculating gauge theories which flow to the same universality class in the IR. We shall call it "Toric Duality."

1   Introduction

The study of D-branes as probes of geometry and topology of space-time has by
now been of wide practice (cf. e.g. [1]). In particular, the analysis of the mod-
uli space of gauge theories, their matter content, superpotential and β-function, as
world-volume theories of D-branes sitting at geometrical singularities is still a widely
pursued topic. Since the pioneering work in [2], where the moduli and matter con-
tent of D-branes probing ALE spaces had been extensively investigated, much work
ensued. The primary focus on (Abelian) orbifold singularities of the type $\mathbb{C}^2/\mathbb{Z}_n$
was quickly generalised using McKay's Correspondence, to arbitrary (non-Abelian)
orbifold singularities $\mathbb{C}^2/(\Gamma \subset SU(2))$, i.e., to arbitrary ALE spaces, in [3].

Several directions followed. With the realisation [5, 6] that these singularities
provide various horizons, [2, 3] was quickly generalised to a treatment for arbitrary
finite subgroups $\Gamma \subset SU(N)$, i.e., to generic Gorenstein singularities, by [7]. The
case of $SU(3)$ was then promptly studied in [8, 9, 10] using this technique and a
generalised McKay-type Correspondence was proposed in [8, 11]. Meanwhile, via
T-duality transformations, certain orbifold singularities can be mapped to type II
brane-setups in the fashion of [12]. The relevant gauge theory data on the world
volume can thereby be conveniently read from configurations of NS-branes, D-brane
stacks as well as orientifold planes. For $\mathbb{C}^2$ orbifolds, the A and D series have been
thus treated [12, 13], whereas for $\mathbb{C}^3$ orbifolds, the Abelian case of $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ has been
solved by the brane box models [14, 15]. First examples of non-Abelian $\mathbb{C}^3$ orbifolds
have been addressed in [16] as well as recent works in [17].

Thus rests the status of orbifold theories. What we note in particular is that once
we specify the properties of the orbifold in terms of the algebraic properties of the
finite group, the gauge theory information is easily extracted. Of course, orbifolds
are a small subclass of algebro-geometric singularities. This is where we move on to
toric varieties. Inspired by the linear σ-model approach of [18], which provides a rich
structure of the moduli space, especially in connexion with various geometrical phases
of the theory, the programme of utilising toric methods to study the behaviour of the
gauge theory on D-branes which live on and hence resolve certain singularities was
initiated in [19]. In this light, toric methods provide a powerful tool for studying the
moduli space of the gauge theory. In treating the F-flatness and D-flatness conditions
for the SUSY vacuum in conjunction, these methods show how branches of the moduli
space and hence phases of the theory may be parametrised by the algebraic equations
of the toric variety. Recent developments in "brane diamonds," as an extension of
the brane box rules, have been providing great insight to such a wider class of toric
singularities, especially the generalised conifold, via blown-up versions of the standard

brane setups [20]. Indeed, with toric techniques much information could be extracted as we can actually analytically describe patches of the moduli space.

Now Abelian orbifolds have toric descriptions and the above methodolgy is thus immediately applicable thereto. While bearing in mind that though non-Abelian orbifolds have no toric descriptions, a single physical D-brane has been placed on various general toric singularities. Partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, such as the conifold and the suspended pinched point have been investigated in [21, 22] and brane setups giving the field theory contents are constructed by [23, 25, 24]. Groundwork for the next family, coming from the toric orbifold $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$, such as the del Pezzo surfaces and the zeroth Hirzebruch, has been laid in [26]. Essentially, given the gauge theory data on the D-brane world volume, the procedure of transforming this information (F and D terms) into toric data which parametrises the classical moduli space is by now well-established.

One task is therefore immediately apparent to us: how do we proceed in the reverse direction, i.e., *when we probe a toric singularity with a D-brane, how do we know the gauge theory on its world-volume?* We recall that in the case of orbifold theories, [7] devised a general method to extract the gauge theory data (matter content, superpotential etc.) from the geometry data (the characters of the finite group Γ), and *vice versa* given the geometry, brane-setups for example, conveniently allow us to read out the gauge theory data. The same is not true for toric singularities, and the second half of the above bi-directional convenience, namely, a general method which allows us to treat the inverse problem of extracting gauge theory data from toric data is yet pending, or at least not in circulation.

The reason for this shortcoming is, as we shall see later, that the problem is highly non-unique. It is thus the purpose of this writing to address this inverse problem: given the geometry data in terms of a toric diagram, how does one read out (at least one) gauge theory data in terms of the matter content and superpotential? We here present precisely this algorithm which takes the matrices encoding the singularity to the matrices encoding a good gauge theory of the D-brane which probes the said singularity.

The structure of the paper is as follows. In Section 2 we review the procedure of proceeding from the gauge theory data to the toric data, while establishing nomenclature. In Subsection 3.1, we demonstrate how to extract the matter content and F-terms from the charge matrix of the toric singularity. In Subsection 3.2, we exemplify our algorithm with the well-known suspended pinched point before presenting in detail in Subsection 3.3, the general algorithm of how to obtain the gauge theory information from the toric data by the method of partial resolutions. In Subsection 3.4, we show how to integrate back to obtain the actual superpotential once the F-

flatness equations are extracted from the toric data. Section 4 is then devoted to the illustration of our algorithm by tabulating the D-terms and F-terms of D-brane world volume theory on the toric del Pezzo surfaces and Hirzebruch zero. We finally discuss in Section 5, the non-uniqueness of the inverse problem and provide, through the studying of two types of ambiguities, ample examples of rather different gauge theories flowing to the same toric data. Discussions and future prospects are dealt with in Section 6.

2 The Forward Procedure: Extracting Toric Data From Gauge Theories

We shall here give a brief review of the procedures involved in going from gauge theory data on the D-brane to toric data of the singularity, using primarily the notation and concepts from [19]. In the course thereof special attention will be paid on how toric diagrams, SUSY fields and linear σ-models weave together.

A stack of n D-brane probes on algebraic singularities gives rise to SUSY gauge theories with product gauge groups resulting from the projection of the $U(n)$ theory on the original stack by the geometrical structure of the singularity. For orbifolds $\mathbb{C}^k/\Gamma$, we can use the structure of the finite group Γ to fabricate product $U(n_i)$ gauge groups [2, 3, 7]. For toric singularities, since we have only (Abelian) $U(1)$ toroidal actions, we are so far restricted to product $U(1)$ gauge groups[2]. In physical terms, we have *a single D-brane probe*. Extensive work has been done in [26, 19] to see how the geometrical structure of the variety can be thus probed and how the gauge theory moduli may be encoded. The subclass of toric singularities, namely Abelian orbifolds, has been investigated to great detail [2, 4, 19, 22, 26] and we shall make liberal usage of their properties throughout.

Now let us consider the world-volume theory on the D-brane probe on a toric singularity. Such a theory, as it is a SUSY gauge theory, is characterised by its matter content and interactions. The former is specified by quiver diagrams which in turn give rise to **D-term** equations; the latter is given by a superpotential, whose partial derivatives with respect to the various fields are the so-called **F-term** equations. F and D-flatness subsequently describe the (classical) moduli space of the theory. The basic idea is that the D-term equations together with the FI-parametres, in conjunction with the F-term equations, can be concatenated together into a matrix which gives the vectors forming the dual cone of the toric variety which the D-branes probe. We summarise the algorithm of obtaining the toric data from the gauge theory in the following, and to illuminate our abstraction and notation we will use the simple

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[2] Proposals toward generalisations to D-brane stacks have been made [26].

$$\mathbb{Z}_2 \ \text{x} \ \mathbb{Z}_2$$

Toric Diagram
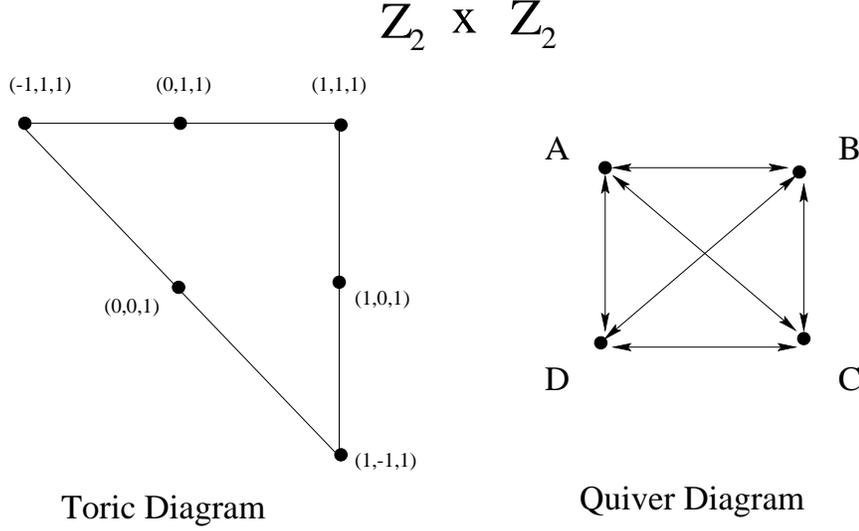
Quiver Diagram

Figure 1: The toric diagram for the singularity $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and the quiver diagram for the gauge theory living on a D-brane probing it. We have labelled the nodes of the toric diagram by columns of G_t and those of the quiver, with the gauge groups $U(1)_{\{A,B,C,D\}}$.

example of the Abelian orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ as given in Figure 1.

1. <u>Quivers and D-Terms:</u>

   (a) The bi-fundamental matter content of the gauge theory can be conveniently encoded into a **quiver diagram** $\mathcal{Q}$, which is simply the (possibly directed) graph whose **adjacency matrix** a_{ij} is precisely the matrix of the bi-fundamentals. In the case of an Abelian orbifold[3] prescribed by the group Γ, this diagram is the McKay Quiver (i.e., for the irreps R_i of Γ, a_{ij} is such that $R \otimes R_i = \oplus_j a_{ij} R_j$ for some fundamental representation R). We denote the set of nodes as $\mathcal{Q}_0 := \{v\}$ and the set of the edges, $\mathcal{Q}_1 := \{a\}$. We let the number of nodes be r; for Abelian orbifolds, $r = |\Gamma|$ (and for generic orbifolds r is the number of conjugacy classes of Γ). Also, we let the number of edges be m; this number depends on the number of supersymmetries which we have. The adjacency matrix (bi-fundamentals) is thus $r \times r$ and the gauge group is $\prod\limits_{j=1}^{r} SU(w_j)$. For our example of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $r = 4$, indexed as 4 gauge groups $U(1)_A \times U(1)_B \times U(1)_C \times U(1)_D$ corresponding to the 4 nodes, while $m = 4 \times 3 = 12$, corresponding to the 12 arrows in Figure 1. The adjacency matrix for the quiver is $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. Though for such simple examples as Abelian orbifolds and conifolds, brane

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[3]This is true for all orbifolds but of course only Abelian ones have known toric description.

setups and [7] specify the values of w_j as well as a_{ij} completely[4], there is yet no discussion in the literature of obtaining the matter content and gauge group for generic toric varieties in a direct and systematic manner and a partial purpose of this note is to present a solution thereof.

(b) From the $r \times r$ adjacency matrix, we construct a so-called $r \times m$ **incidence matrix** d for $\mathcal{Q}$; this matrix is defined as $d_{v,a} := \delta_{v,head(a)} - \delta_{v,tail(a)}$ for $v \in \mathcal{Q}_0$ and $a \in \mathcal{Q}_1$. Because each column of d must contain a 1, a -1 and the rest 0's by definition, one row of d is always redundant; this physically signifies the elimination of an overall trivial $U(1)$ corresponding to the COM motion of the branes. Therefore we delete a row of d to define the matrix Δ of dimensions $(r-1) \times m$; and we could always extract d from Δ by adding a row so as to force each column to sum to zero. This matrix Δ thus contains almost as much information as a_{ij} and once it is specified, the gauge group and matter content are also, with the exception that precise adjoints (those charged under the same gauge group factor and hence correspond to arrows that join a node to itself) are not manifest. For our example the 4×12 matrix d is as follows and Δ is the top 3 rows:

$$d = \left(\begin{array}{c|cccccccccccc} & X_{AD} & X_{BC} & X_{CB} & X_{DA} & X_{AB} & X_{BA} & X_{CD} & X_{DC} & X_{AC} & X_{BD} & X_{CA} & X_{DB} \\ \hline A & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ B & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ C & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 0 \\ D & 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1 \end{array} \right)$$

(c) The moment maps, arising in the sympletic-quotient language of the toric variety, are simply $\mu := d \cdot |x(a)|^2$ where $x(a)$ are the affine coordinates of the $\mathbb{C}^r$ for the torus $(\mathbb{C}^*)^r$ action. Physically, $x(a)$ are of course the bi-fundamentals in chiral multiplets (in our example they are $X_{ij \in \{A,B,C,D\}}$ as labelled above) and the D-term equations for each $U(1)$ group is [18]

$$D_i = -e^2 \left(\sum_a d_{ia}|x(a)|^2 - \zeta_i \right)$$

with ζ_i the FI-parametres. In matrix form we have $\Delta \cdot |x(a)|^2 = \vec{\zeta}$ and see that D-flatness gives precisely the moment map. These ζ-parametres will encode the resolution of the toric singularity as we shall shortly see.

2. Monomials and F-Terms:

(a) From the super-potential W of the SUSY gauge theory, one can write the F-Term equation as the system $\frac{\partial}{\partial X_j} W = 0$. The remarkable fact is that

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[4]For arbitrary orbifolds, $\sum_j w_i n_i = |\Gamma|$ where n_i are the dimensions of the irreps of Γ; for Abelian case, $n_i = 1$.

we could solve the said system of equations and express the m fields X_i in terms of $r + 2$ parametres v_j which can be summarised by a matrix K.

$$X_i = \prod_j v_j^{K_{ij}}, \qquad i = 1, 2, .., m; \quad j = 1, 2, .., r + 2 \qquad (2.1)$$

This matrix K of dimensions $m \times (r + 2)$ is the analogue of Δ in the sense that it encodes the F-terms and superpotential as Δ encodes the D-terms and the matter content. In the language of toric geometry K defines a cone[5] $\mathbf{M}_+$: a non-negative linear combination of m vectors $\vec{K}_i$ in an integral lattice $\mathbf{Z}^{r+2}$.

For our example, the superpotential is

$$W = X_{AC}X_{CD}X_{DA} - X_{AC}X_{CB}X_{BA} + X_{CA}X_{AB}X_{BC} - X_{CA}X_{AD}X_{DC}$$
$$+ X_{BD}X_{DC}X_{CB} - X_{BD}X_{DA}X_{AB} - X_{DB}X_{BC}X_{CD},$$

giving us 12 F-term equations and with the manifold of solutions parametrisable by $4 + 2$ new fields, whereby giving us the 12×6 matrix (we here show the transpose thereof, thus the horizontal direction corresponds to the original fields X_i and the vertical, v_j):

$$K^t = \left(\begin{array}{c|ccccccccccc}
 & X_{AC} & X_{BD} & X_{CA} & X_{DB} & X_{AB} & X_{BA} & X_{CD} & X_{DC} & X_{AD} & X_{BC} & X_{CB} & X_{DA} \\
\hline
v_1 = X_{AC} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
v_2 = X_{BD} & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
v_3 = X_{BA} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
v_4 = X_{CD} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\
v_5 = X_{AD} & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
v_6 = X_{CB} & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array} \right).$$

For example, the third column reads $X_{CA} = v_2 v_5^{-1} v_6$, i.e., $X_{AD}X_{CA} = X_{BD}X_{CB}$, which the the F-flatness condition $\frac{\partial W}{\partial X_{DC}=0}$. The details of obtaining W and K from each other are discussed in [19, 26] and Subsection 3.4.

(b) We let T be the space of (integral) vectors dual to K, i.e., $K \cdot T \geq 0$ for all entries; this gives an $(r + 2) \times c$ matrix for some positive integer c. Geometrically, this is the definition of a dual cone $\mathbf{N}_+$ composed of vectors $\vec{T}_i$ such that $\vec{K} \cdot \vec{T} \geq 0$. The physical meaning for doing so is that K may have negative entries which may give rise to unwanted singularities

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[5] We should be careful in this definition. Strictly speaking we have a lattice $\mathbf{M} = \mathbf{Z}^{r+2}$ with its dual lattice $\mathbf{N} \cong \mathbf{Z}^{r+2}$. Now let there be a set of $\mathbf{Z}_+$-independent vectors $\{\vec{k}_i\} \in \mathbf{M}$ and a cone is defined to be generated by these vectors as $\sigma := \{\sum_i a_i \vec{k}_i \mid a_i \in \mathbb{R}_{\geq 0}\}$; Our $\mathbf{M}_+$ should be $\mathbf{M} \cap \sigma$. In much of the literature $\mathbf{M}_+$ is taken to be simply $\mathbf{M}'_+ := \{\sum_i a_i \vec{k}_i \mid a_i \in \mathbf{Z}_{\geq 0}\}$ in which case we must make sure that any lattice point contained in $\mathbf{M}_+$ but not in $\mathbf{M}'_+$ must be counted as an independent generator and be added to the set of generators $\{\vec{k}_i\}$. After including all such points we would have $\mathbf{M}'_+ = \mathbf{M}_+$. Throughout our analyses, our cone defined by K as well the dual cone T will be constituted by such a complete set of generators.

and hence we define a new set of c fields p_i (*a priori* we do not know the number c and we present the standard algorithm of finding dual cones in the Appendix). Thus we reduce (2.1) further into

$$v_j = \prod_\alpha p_\alpha^{T_{j\alpha}} \tag{2.2}$$

whereby giving $X_i = \prod_j v_j^{K_{ij}} = \prod_\alpha p_\alpha^{\sum_j K_{ij}T_{j\alpha}}$ with $\sum_j K_{ij}T_{j\alpha} \geq 0$. For our $\mathbb{Z}_2 \times Z_2$ example, $c = 9$ and

$$T_{j\alpha} = \begin{pmatrix} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \\ \hline X_{AC} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ X_{BD} & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ X_{BA} & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ X_{CD} & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ X_{AD} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ X_{CB} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

(c) These new variables p_α are the matter fields in Witten's linear σ-model. How are these fields charged? We have written $r + 2$ fields v_j in terms of c fields p_α, and hence need $c - (r + 2)$ relations to reduce the independent variables. Such a reduction can be done via the introduction of the new gauge group $U(1)^{c-(r+2)}$ acting on the p_i's so as to give a new set of D-terms. The charges of these fields can be written as $Q_{k\alpha}$. The gauge invariance condition of v_i under $U(1)^{c-(r+2)}$, by (2.2), demands that the $(c - r - 2) \times c$ matrix Q is such that $\sum_\alpha T_{j\alpha}Q_{k\alpha} = 0$. This then defines for us our charge matrix Q which is the cokernel of T:

$$TQ^t = (T_{j\alpha})(Q_{k\alpha})^t = 0, \qquad j = 1, .., r+2; \quad \alpha = 1, .., c; \quad k = 1, .., (c-r-2)$$

For our example, the charge matrix is $(9 - 4 - 2) \times 9$ and one choice is

$$Q_{k\alpha} = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(d) In the linear σ-model language, the F-terms and D-terms can be treated in the same footing, i.e., as the D-terms (moment map) of the new fields p_α; with the crucial difference being that the former must be set exactly to zero[6] while the latter are to be resolved by arbitrary FI-parameters. Therefore in addition to finding the charge matrix Q for the new fields p_α coming from the original F-terms as done above, we must also find the corresponding charge matrix Q_D for the p_i coming from the original D-terms. We can find Q_D in two steps. Firstly, we know the charge

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[6]Strictly speaking, we could have an F-term set to a non-zero constant. An example of this situation could be when there is a term $a\phi + \phi\tilde{Q}Q$ in the superpotential for some chargeless field ϕ and charged fields $\tilde{Q}$ and Q. The F-term for ϕ reads $\tilde{Q}Q = -a$ and not 0. However, in our context ϕ behaves like an integration constant and for our purposes, F-terms are set exactly to zero.

matrix for X_i under $U(1)^{r-1}$, which is Δ. By (2.1), we transform the charges to that of the v_j's, by introducing an $(r-1) \times (r+2)$ matrix V so that $V \cdot K^t = \Delta$. To see this, let the charges of v_j be V_{lj} then by (2.1) we have $\Delta_{li} = \sum_j V_{lj} K_{ij} = V \cdot K^t$. A convenient V which does so for our $\mathbb{Z}_2 \times \mathbb{Z}_2$ example is $\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}_{(4-1) \times (4+2)}$. Secondly, we use (2.2) to transform the charges from v_j's to our final variables p_α's, which is done by introducing an $(r+2) \times c$ matrix $U_{j\alpha}$ so that $U \cdot T^t = \mathrm{Id}_{(r+2) \times (r+2)}$. In our example, one choice for U is $U_{j\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{(4+2) \times 9}$.

Therefore, combining the two steps, we obtain $Q_D = V \cdot U$ and in our example, $(V \cdot U)_{l\alpha} = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$.

3. Thus equipped with the information from the two sides: the F-terms and D-terms, and with the two required charge matrices Q and $V \cdot U$ obtained, finally we concatenate them to give a $(c-3) \times c$ matrix Q_t. The transpose of the kernel of Q_t, with (possible repeated columns) gives rise to a matrix G_t. The columns of this resulting G_t then define the vertices of the toric diagram describing the polynomial corresponding to the singularity on which we initially placed our D-branes. Once again for our example, $Q_t = \left(\begin{array}{ccccccccc|c} 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & \zeta_1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & \zeta_2 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & \zeta_3 \end{array} \right)$ and $G_t = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.

The columns of G_t, up to repetition, are precisely marked in the toric diagram for $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Figure 1.

Thus we have gone from the F-terms and the D-terms of the gauge theory to the nodes of the toric diagram. In accordance with [27], G_t gives the algebraic variety whose equation is given by the maximal ideal in the polynomial ring $\mathbb{C}[YZ, XYZ, Z, X^{-1}YZ, XY^{-1}Z, XZ]$ (the exponents (i, j, k) in $X^i Y^j Z^k$ are exactly the columns), which is $uvw = s^2$, upon defining $u = (YZ)(XYZ)^2(Z)(XZ)^2$; $v = (YZ)^2(Z)^2(X^{-1}YZ)^2$; $w = (Z)^2(XY^{-1}Z)(XZ)^2$ and $s = (YZ)^2(XYZ)(Z)^2(X^{-1}YZ)(XY^{-1}Z)(XZ)^2$; this is precisely $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. In physical terms this equation parametrises the moduli space obtained from the F and D flatness of the gauge theory.

We remark two issues here. In the case of there being no superpotential we could still define K-matrix. In this case, with there being no F-terms, we simply take K to be the identity. This gives $T = \mathrm{Id}$ and $Q = 0$. Furthermore U becomes Id and

$V = \Delta$, whereby making $Q_t = \Delta$ as expected because all information should now be contained in the D-terms. Moreover, we note that the very reason we can construct a K-matrix is that all of the equations in the F-terms we deal with are in the form $\prod_i X_i^{a_i} = \prod_j X_j^{b_j}$; this holds in general if every field X_i appears twice and precisely twice in the superpotential. More generic situations would so far transcend the limitations of toric techniques.

Schematically, our procedure presented above at length, what it means is as follows: we begin with two pieces of physical data: (1) matrix d from the quiver encoding the gauge groups and D-terms and (2) matrix K encoding the F-term equations. From these we extract the matrix G_t containing the toric data by the flow-chart:

$$
\begin{array}{ccccccc}
\text{Quiver} \to d & & \to & \Delta & & & \\
& & & \downarrow & & & \\
\text{F-Terms} \to K & \overset{V \cdot K^t = \Delta}{\to} & V & & & \\
\downarrow & & & \downarrow & & & \\
T = \text{Dual}(K) & \overset{U \cdot T^t = \text{Id}}{\to} & U & \to & VU & \\
\downarrow & & & & \downarrow & & \\
Q = [\text{Ker}(T)]^t & & \longrightarrow & Q_t = \begin{pmatrix} Q \\ VU \end{pmatrix} & \to & G_t = [\text{Ker}(Q_t)]^t
\end{array}
$$

3 The Inverse Procedure: Extracting Gauge Theory Information from Toric Data

As outlined above we see that wherever possible, the gauge theory of a D-brane probe on certain singularities such as Abelian orbifolds, conifolds, etc., can be conveniently encoded into the matrix Q_t which essentially concatenates the information contained in the D-terms and F-terms of the original gauge theory. The cokernel of this matrix is then a list of vectors which prescribes the toric diagram corresponding to the singularity. It is natural to question ourselves whether the converse could be done, i.e., whether given an arbitrary singularity which affords a toric description, we could obtain the gauge theory living on the D-brane which probes the said singularity. This is the inverse problem we projected to solve in the introduction.

3.1 Quiver Diagrams and F-terms from Toric Diagrams

Our result must be two-fold: first, we must be able to extract the D-terms, or in other words the quiver diagram which then gives the gauge group and matter content; second, we must extract the F-terms, which we can subsequently integrate back to give the superpotential. These two pieces of data then suffice to specify the gauge

theory. Essentially we wish to trace the arrows in the above flow-chart from G_t back to Δ and K. The general methodology seems straightforward:

1. Read the column-vectors describing the nodes of the given toric diagram, repeat the appropriate columns to obtain G_t and then set $Q_t = \mathrm{Coker}(G_t)$;

2. Separate the D-term $(V \cdot U)$ and F-term (Q_t) portions from Q_t;

3. From the definition of Q, we obtain[7] $T = \ker(Q)$.

4. Farka's Theorem [27] guarantees that the dual of a convex polytope remains convex whence we could invert and have $K = \mathrm{Dual}(T^t)$; Moreover the duality theorem gives that $\mathrm{Dual}(\mathrm{Dual}(K)) = K$, thereby facilitating the inverse procedure.

5. Definitions $U \cdot T^t = \mathrm{Id}$ and $V \cdot K^t = \Delta \Rightarrow (V \cdot U) \cdot (T^t \cdot K^t) = \Delta$.

We see therefore that once the appropriate Q_t has been found, the relations

$$K = \mathrm{Dual}(T^t) \qquad \Delta = (V \cdot U) \cdot (T^t \cdot K^t) \tag{3.3}$$

retrieve our desired K and Δ. The only setback of course is that the appropriate Q_t is NOT usually found. Two ambiguities are immediately apparent to us: (A) In step 1 above, there is really no way to know a priori which of the vectors we should repeat when writing into the G_t matrix; (B) In step 2, to separate the D-terms and the F-terms, i.e., which rows constitute Q and which constitute $V \cdot U$ within Q_t, seems arbitrary. We shall in the last section discuss these ambiguities in more detail and actually perceive it to be a matter of interest. Meanwhile, in light thereof, we must find an alternative, to find a canonical method which avoids such ambiguities and gives us a consistent gauge theory which has such well-behaved properties as having only bi-fundamentals etc.; this is where we appeal to partial resolutions.

Another reason for this canonical method is compelling. The astute reader may question as to how could we guarantee, in our mathematical excursion of performing the inverse procedure, that the gauge theory we obtain at the end of the day is one that still lives on the world-volume of a D-brane probe? Indeed, if we naïvely traced back the arrows in the flow-chart, bearing in mind the said ambiguities, we have no *a fortiori* guarantee that we have a brane theory at all. However, the method via partial resolution of Abelian orbifolds (which are themselves toric) does give us assurance. When we are careful in tuning the FI-parametres so as to stay inside cone-partitions of the space of these parametres (and avoid flop transitions) we do still have the

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[7]As mentioned before we must ensure that such a T be chosen with a complete set of $\mathbb{Z}_+$-independent generators;

resulting theory being physical [26]. Essentially this means that with prudence we tune the FI-parametres in the allowed domains from a parent orbifold theory, thereby giving a subsector theory which still lives on the D-brane probe and is well-behaved. Such tuning we shall practice in the following.

The virtues of this appeal to resolutions are thus twofold: not only do we avoid ambiguities, we are further endowed with physical theories. Let us thereby present this canonical mathod.

3.2   A Canonical Method: Partial Resolutions of Abelian Orbifolds

Our programme is standard [26]: theories on the Abelian orbifold singularity of the form $\mathbb{C}^k/\Gamma$ for $\Gamma(k,n) = \mathbb{Z}_n \times \mathbb{Z}_n \times ...\mathbb{Z}_n$ ($k-1$ times) are well studied. The complete information (and in particular the full Q_t matrix) for $\Gamma(k,n)$ is well known: $k=2$ is the elliptic model, $k=3$, the Brane Box, etc. In the toric context, $k=2$ has been analysed in great detail by [2], $k=3, n=2$ in e.g. [23, 25, 24], $k=3, n=3$ in [26]. Now we know that given any toric diagram of dimension k, we can embed it into such a $\Gamma(k,n)$-orbifold for some sufficiently large n; and we choose the smallest such n which suffices. This embedding is always possible because the toric diagram for the latter is the k-simplex of length n enclosing lattice points and any toric diagram, being a collection of lattice points, can be obtained therefrom via deletions of a subset of points. This procedure is known torically as **partial resolutions** of $\Gamma(k,n)$. The crux of our algorithm is that the deletions in the toric diagram corresponds to the turning-on of the FI-parametres, and which in turn induces a method to determine a Q_t matrix for our original singularity from that of $\Gamma(n,k)$.

We shall first turn to an illustrative example of the suspended pinched point singularity (SPP) and then move on to discuss generalities. The SPP and conifold as resolutions of $\Gamma(3,2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ have been extensively studied in [25]. The SPP, given by $xy = zw^2$, can be obtained from the $\Gamma(3,2)$ orbifold, $xyz = w^2$, by a single $\mathbb{P}^1$ blow-up. This is shown torically in Figure 2. Without further ado let us demonstrate our procedure.

1. <u>Embedding into $\mathbb{Z}_2 \times \mathbb{Z}_2$</u>: Given the toric diagram D of SPP, we recognise that it can be embedded minimally into the diagram D' of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Now information on D' is readily at hand [25], as presented in the previous section. Let us recapitulate:

$$Q'_t := \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & \zeta_1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & \zeta_2 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & \zeta_3 \end{pmatrix},$$
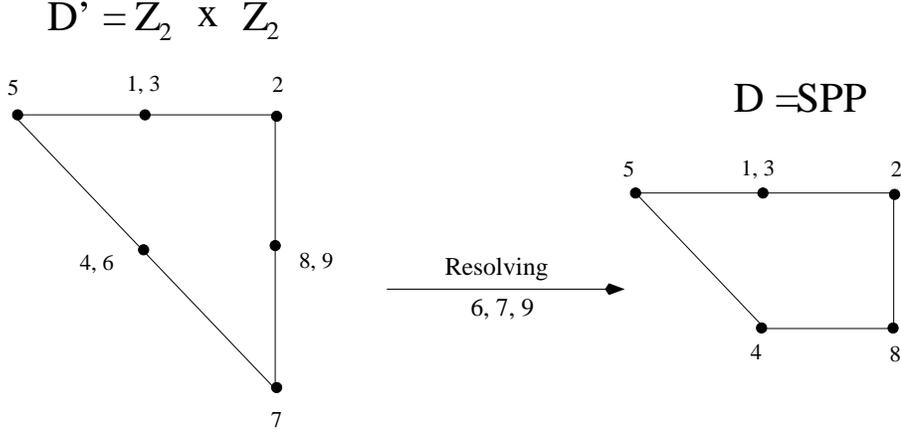
Figure 2: The toric diagram showing the resolution of the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ singularity to the suspended pinch point (SPP). The numbers i at the nodes refer to the i-th column of the matrix G_t and physically correspond to the fields p_i in the linear σ-model.

and

$$G'_t := \mathrm{coker}(Q'_t) = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

which is drawn in Figure 1. The fact that the last row of G_t has the same number (i.e., these three-vectors are all co-planar) ensures that D' is Calabi-Yau [1]. Incidentally, it would be very helpful for one to catalogue the list of Q_t matrices of $\Gamma(3, n)$ for $n = 2, 3...$ which would suffice for all local toric singularities of Calabi-Yau threefolds.

In the above definition of Q'_t we have included an extra column $(0, 0, 0, \zeta_1, \zeta_2, \zeta_3)$ so as to specify that the first three rows of Q'_t are F-terms (and hence exactly zero) while the last three rows are D-terms (and hence resolved by FI-parametres $\zeta_{1,2,3}$). We adhere to the notation in [25] and label the columns (linear σ-model fields) as $p_1...p_9$; this is shown in Figure 2.

2. <u>Determining the Fields to Resolve by Tuning ζ</u>: We note that if we turn on a single FI-parametre we would arrive at the SPP; this is the resolution of D' to D. The subtlety is that one may need to eliminate more than merely the 7th column as there is more than one field attributed to each node in the toric diagram and eliminating column 7 some other columns corresponding to the adjacent nodes (namely out of 4,6,8 and 9) may also be eliminated. We need a judicious choice of ζ for a consistent blowup. To do so we must solve for fields $p_{1,...,9}$ and tune the ζ-parametres such that at least p_7 acquires non-zero VEV (and whereby resolved). Recalling that the D-term equations are actually linear equations in the modulus-squared of the fields, we shall henceforth define $x_i := |p_i|^2$ and consider linear-systems therein.

Therefore we perform Gaussian row-reduction on Q' and solve all fields in terms of x_7 to give: $\vec{x} = \{x_1, x_2, x_1 + \zeta_2 + \zeta_3, \frac{2\,x_1 - x_2 + x_7 - \zeta_1 + \zeta_2}{2}, 2\,x_1 - x_2 + \zeta_2 + \zeta_3, \frac{2\,x_1 - x_2 + x_7 + \zeta_1 + \zeta_2 + 2\,\zeta_3}{2}, x_7, \frac{x_2 + x_7 - \zeta_1 - \zeta_2}{2}, \frac{x_2 + x_7 + \zeta_1 + \zeta_2}{2}\}$.

The nodes far away from p_7 are clearly unaffected by the resolution, thus the fields corresponding thereto continue to have zero VEV. This means we solve the above set of solutions $\vec{x}$ once again, setting $x_{5,1,3,2} = 0$, with $\zeta_{1,2,3}$ being the variables, giving upon back substitution, $\vec{x} = \{0, 0, 0, \frac{x_7 - \zeta_1 - \zeta_3}{2}, 0, \frac{x_7 + \zeta_1 + \zeta_3}{2}, x_7, \frac{x_7 - \zeta_1 + \zeta_3}{2}, \frac{x_7 + \zeta_1 - \zeta_3}{2}\}$. Now we have an arbitrary choice and we set $\zeta_3 = 0$ and $x_7 = \zeta_1$ to make p_4 and p_8 have zero VEV. This makes $p_{6,7,9}$ our candidate for fields to be resolved and seems perfectly reasonable observing Figure 2. The constraint on our choice is that all solutions must be ≥ 0 (since the x_i's are VEV-squared).

3. <u>Solving for G_t:</u> We are now clear what the resolution requires of us: in order to remove node p_7 from D' to give the SPP, we must also resolve 6, 7 and 9. Therefore we immediately obtain G_t by directly removing the said columns from G_t':

$$G_t := \mathrm{coker}(Q_t) = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_8 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

the columns of which give the toric diagram D of the SPP, as shown in Figure 2.

4. <u>Solving for Q_t:</u> Now we must perform linear combination on the rows of Q_t' to obtain Q_t so as to force columns 6, 7 and 9 zero. The following constraints must be born in mind. Because G_t has 6 columns and 3 rows and is in the null space of Q_t, which itself must have $9 - 3$ columns (having eliminated $p_{6,7,9}$), we must have $6 - 3 = 3$ rows for Q_t. Also, the row containing ζ_1 must be eliminated as this is precisely our resolution chosen above (we recall that the FI-parametres are such that $\zeta_{2,3} = 0$ and are hence unresolved, while $\zeta_1 > 0$ and must be removed from the D-terms for SPP).

We systematically proceed. Let there be variables $\{a_{i=1,...6}\}$ so that $y := \sum_i a_i \mathrm{row}_i(Q_t')$ is a row of Q_t. Then (a) the 6th, 7th and 9th columns of y must be set to 0 and moreover (b) with these columns removed y must be in the nullspace spanned by the rows of G_t. We note of course that since Q_t' was in the nullspace of G_t' initially, that the operation of row-combinations is closed within a nullspace, and that the columns to be set to 0 in Q_t' to give Q_t are precisely those removed in G_t' to give G_t, condition (a) automatically implies (b). This condition (a) translates to the equations $\{a_1 + a_6 = 0, -a_1 + a_2 - a_6 = 0, -a_2 + a_4 = 0\}$ which afford the solution $a_1 = -a_6; a_2 = a_4 = 0$. The fact that $a_4 = 0$ is comforting, because it eliminates the row containing ζ_1. We choose $a_1 = 1$. Furthermore we must keep row 5 as ζ_2 is yet unresolved (thereby setting

$a_5 = 1$). This already gives two of the 3 anticipated rows of Q_t: row_5 and row_1 - row_6. The remaining row must corresponds to an F-term since we have exhausted the D-terms, this we choose to be the only remaining variable: $a_3 = 1$. Consequently, we arrive at the matrix

$$
Q_t = \begin{pmatrix}
p_1 & p_2 & p_3 & p_4 & p_5 & p_8 & \\
1 & -1 & 1 & 0 & -1 & 0 & 0 \\
-1 & 1 & 0 & 1 & 0 & -1 & \zeta_2 \\
-1 & 0 & 0 & -1 & 1 & 1 & \zeta_3
\end{pmatrix}.
$$

5. <u>Obtaining K and Δ Matrices:</u> The hard work is now done. We now recognise from Q_t that $Q = (1, -1, 1, 0, -1, 0)$, giving

$$
T_{j\alpha} := \ker(Q) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}; \qquad
K^t := \text{Dual}(T^t) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1
\end{pmatrix}.
$$

Subsequently we obtain $T^t \cdot K^t = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$, which we do observe indeed

to have every entry positive semi-definite. Furthermore we recognise from Q_t that $V \cdot U = \left(\begin{smallmatrix} -1 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 & 1 \end{smallmatrix}\right)$, whence we obtain at last, using (3.3),

$$
\Delta = \begin{pmatrix}
-1 & 1 & 0 & 1 & -1 & 0 \\
1 & -1 & 1 & 0 & 0 & -1
\end{pmatrix}
\qquad \Rightarrow \qquad
d = \left(\begin{array}{c|cccccc}
 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \\
\hline
U(1)_A & -1 & 1 & 0 & 1 & -1 & 0 \\
U(1)_B & 1 & -1 & 1 & 0 & 0 & -1 \\
U(1)_C & 0 & 0 & -1 & -1 & 1 & 1
\end{array}\right),
$$

giving us the quiver diagram (included in Figure 3 for reference), matter content and gauge group of a D-brane probe on SPP in agreement with [25]. We shall show in the ensuing sections that the superpotential we extract has similar accordance.

3.3 The General Algorithm for the Inverse Problem

Having indulged ourselves in this illustrative example of the SPP, we proceed to outline the general methodology of obtaining the gauge theory data from the toric diagram.

1. <u>Embedding into $\mathbb{C}^k/(\mathbb{Z}_n)^{k-1}$:</u> We are given a toric diagram D describing an algebraic variety of complex dimension k (usually we are concerned with local Calabi-Yau singularities of $k = 2, 3$ so that branes living thereon give $\mathcal{N} = 2, 1$ gauge theories). We immediately observe that D could always be embedded into D', the toric diagram of the orbifold $\mathbb{C}^k/(\mathbb{Z}_n)^{k-1}$ for some sufficiently large integer n. The matrices Q'_t and G'_t for D' are standard. Moreover we know that the matrix G_t for our original variety D must be a submatrix of G'_t. Equipped with Q'_t and G'_t our task is to obtain Q_t; and as an additional check we could verify that Q_t is indeed in the nullspace of G_t.

$$d = \begin{pmatrix} \text{A:} & -1 & 1 & 0 & 1 & -1 & 0 \\ \text{B:} & 1 & -1 & 1 & 0 & 0 & -1 \\ \text{C:} & 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix}$$

$$a_{ij} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
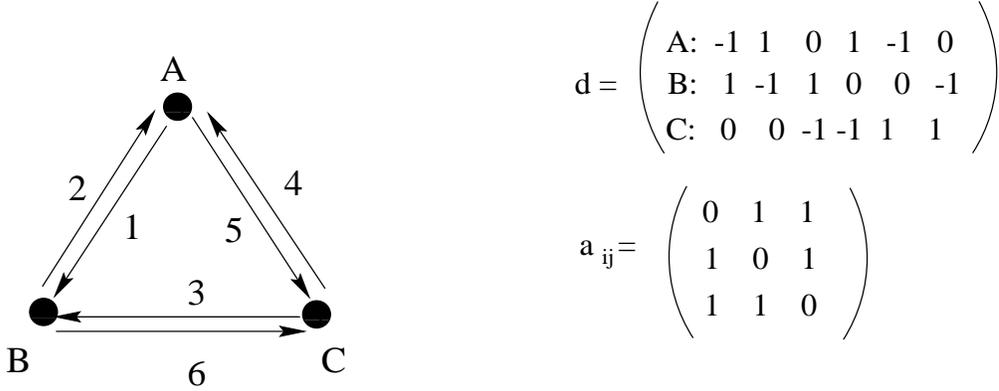
Figure 3: The quiver diagram showing the matter content of a D-brane probing the SPP singularity. We have not marked in the chargeless field ϕ (what in a non-Abelian theory would become an adjoint) because thus far the toric techniques do not yet know how to handle such adjoints.

2. <u>Determining the Fields to Resolve by Tuning ζ</u>: Q'_t is a $k \times a$ matrix[8] (because D' and D are dimension k) for some a; G'_t, being its nullspace, is thus $(a-k) \times a$. D is a partial resolution of D'. In the SPP example above, we performed a single resolution by turning on one FI-parametre, generically however, we could turn on as many ζ's as the embedding permits. Therefore we let G_t be $(a-k) \times (a-b)$ for some b which depends on the number of resolutions. Subsequently the Q_t we need is $(k - b) \times (a - b)$.

Now b is determined directly by examining D' and D; it is precisely the number of fields p associated to those nodes in D' we wish to eliminate to arrive at D. Exactly which b columns are to be eliminated is determined thus: we perform Gaussian row-reduction on Q'_t so as to solve the k linear-equations in a variables $x_i := |p_i|^2$, with F-terms set to 0 and D-terms to FI-parametres. The a variables are then expressed in terms of the ζ_i's and the set B of x_i's corresponding to the nodes which we definitely know will disappear as we resolve $D' \to D$. The subtlety is that in eliminating B, some other fields may also acquire non zero VEV and be eliminated; mathematically this means that $\text{Order}(B) < b$.

Now we make a judicious choice of which fields will remain and set them to zero and impose this further on the solution $x_{i=1,..,a} = x_i(\zeta_j; B)$ from above until $\text{Order}(B) = b$, i.e., until we have found all the fields we need to eliminate. We know this occurs and that our choice was correct when all $x_i \geq 0$ with those equaling 0 corresponding to fields we do not wish to eliminate as can be observed from the toric diagram. If not, we modify our initial choice and repeat until satisfaction. This procedure then determines the b columns which we wish

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[8]We henceforth understand that there is an extra column of zeroes and ζ's.

to eliminate from Q'_t.

3. <u>Solving for G_t and Q_t</u>: Knowing the fields to eliminate, we must thus perform linear combinations on the k rows of Q'_t to obtain the $k-b$ rows of Q_t based upon the two constraints that (1) the b columns must be all reduced to zero (and thus the nodes can be removed) and that (2) the $k-b$ rows (with b columns removed) are in the nullspace of G_t. As mentioned in our SPP example, condition (1) guarantees (2) automatically.

In other words, we need to solve for k variables $\{x_{i=1,..,k}\}$ such that

$$\sum_{i=1}^{k} x_i \, (Q'_t)_{ij} = 0 \qquad \text{for} \quad j = p_1, p_2, ...p_b \in B. \qquad (3.4)$$

Moreover, we immediately obtain G_t by eliminating the b columns from G'_t. Indeed, as discussed earlier, (3.4) implies that $\sum_{i=1}^{k} \sum_{j \neq p_{1...b}} x_i \, (Q'_t)_{ij} \, (G_t)_{mj} = 0$ for $m = 1, ..., a - k$ and hence guarantees that the Q_t we obtain is in the nullspace of G_t.

We could phrase equation (3.4) for x_i in matrix notation and directly evaluate

$$Q_t = NullSpace(W)^t \cdot \tilde{Q}'_t \qquad (3.5)$$

where $\tilde{Q}'_t$ is Q'_t with the appropriate columns $(p_{1...b})$ removed and W is the matrix constructed from the deleted columns.

4. <u>Obtaining the K Matrix (F-term)</u>: Having obtained the $(k-b) \times (a-b)$ matrix Q_t for the original variety D, we proceed with ease. Reading from the extraneous column of FI-parametres, we recognise matrices Q (corresponding to the rows that have zero in the extraneous column) and $V \cdot U$ (corresponding to those with combinations of the unresolved ζ's in the last column). We let $V \cdot U$ be $c \times (a - b)$ whereby making Q of dimension $(k - b - c) \times (a - b)$. The number c is easily read from the embedding of D into D' as the number of unresolved FI-parametres.

From Q, we compute the kernel T, a matrix of dimensions $(a - b) - (k - b - c) \times (a - b) = (a - k + c) \times (a - b)$ as well as the matrix K^t of dimensions $(a - k + c) \times d$ describing the dual cone to that spanned by the columns of T. The integer d is uniquely determined from the dimensions of T in accordance with the algorithm of finding dual cones presented in the Appendix. From these two matrices we compute $T^t \cdot K^t$, of dimension $(a - b) \times d$.

5. <u>Obtaining the Δ Matrix (D-term)</u>: Finally, we use (3.3) to compute $(V \cdot U) \cdot (T^t \cdot K^t)$, arriving at our desired matrix Δ of dimensions $c \times d$, the incidence

matrix of our quiver diagram. The number of gauge groups we have is therefore $c + 1$ and the number of bi-fundamentals, d.

Of course one may dispute that finding the kernel T of Q is highly non-unique as any basis change in the null-space would give an equally valid T. This is indeed so. However we note that it is really the combination $T^t \cdot K^t$ that we need. This is a dot-product in disguise, and by the very definition of the dual cone, this combination remains invariant under basis changes. Therefore this step of obtaining the quiver Δ from the charge matrix Q_t is a unique procedure.

3.4   Obtaining the Superpotential

Having noticed that the matter content can be conveniently obtained, we proceed to address the interactions, i.e., the F-terms, which require a little more care. The matrix K which our algorithm extracts encodes the F-term equations and must at least be such that they could be integrated back to a single function: the *superpotential.*

Reading the possible F-flatness equations from K is *ipso facto* straight-forward. The subtlety exists in how to find the right candidate among many different linear relations. As mentioned earlier, K has dimensions $m \times (r - 2)$ with m corresponding to the fields that will finally manifest in the superpotential, $r - 2$, the fields that solve them according to (2.1) and (2.2); of course, $m \geq r - 2$. Therefore we have $r - 2$ vectors in $\mathbb{Z}^m$, giving generically $m - r + 2$ linear relations among them. Say we have $\text{row}_1 + \text{row}_3 - \text{row}_7 = 0$, then we simply write down $X_1 X_3 = X_7$ as one of the candidate F-terms. In general, a relation $\sum_i a_i K_{ij} = 0$ with $a_i \in \mathbb{Z}$ implies an F-term $\prod_i X_i^{a_i} = 1$ in accordance with (2.1). Of course, to find all the linear relations, we simply find the $\mathbb{Z}$-nullspace of K^t of dimension $m - r + 2$.

Here a great ambiguity exists, as in our previous calculations of nullspaces: any linear combinations therewithin may suffice to give a new relation as a candidate F-term[9]. Thus educated guesses are called for in order to find the set of linear relations which may be most conveniently integrated back into the superpotential. Ideally, we wish this back-integration procedure to involve no extraneous fields (i.e., integration constants[10]) other than the m fields which appear in the K-matrix. Indeed, as we shall see, this wish may not always be granted and sometimes we must include new fields. In this case, the whole moduli space of the gauge theory will be larger than the one encoded by our toric data and the new fields parametrise new branches of

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[9]Indeed each linear relation gives a possible candidate and we seek the correct ones. For the sake of clarity we shall call candidates "relations" and reserve the term "F-term" for a successful candidate.

[10]By constants we really mean functions since we are dealing with systems of partial differential equations.

the moduli in the theory.

Let us return to the SPP example to enlighten ourselves before generalising. We recall from subsection 3.2, that $K = \begin{pmatrix} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \\ \hline v_1 & 1 & 0 & 0 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_4 & 0 & 0 & 1 & 1 & 0 & 0 \\ v_5 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$ from which we can read out only one relation $X_3 X_6 - X_4 X_5 = 0$ using the rule described in the paragraph above. Of course there can be only one relation because the nullspace of K^t is of dimension $6 - 5 = 1$.

Next we must calculate the charge under the gauge groups which this term carries. We must ensure that the superpotential, being a term in a Lagrangian, be a gauge invariant, i.e., carries no overall charge under Δ. From $d = \begin{pmatrix} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \\ \hline U(1)_A & -1 & 1 & 0 & 1 & -1 & 0 \\ U(1)_B & 1 & -1 & 1 & 0 & 0 & -1 \\ U(1)_C & 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix}$ we find the charge of $X_3 X_6$ to be $(q_A, q_B, q_C) = (0 + 0, 1 + (-1), (-1) + 1) = (0, 0, 0)$; of course by our very construction, $X_4 X_5$ has the same charge. Now we have two choices: (a) to try to write the superpotential using only the six fields; or (b) to include some new field ϕ which also has charge $(0, 0, 0)$. For (a) we can try the ansatz $W = X_1 X_2 (X_3 X_6 - X_4 X_5)$ which does give our F-term upon partial derivative with respect to X_1 or X_2. However, we would also have a new F-term $X_1 X_2 X_3 = 0$ by $\frac{\partial}{\partial X_6}$, which is inconsistent with our K since columns 1, 2 and 3 certainly do not add to 0.

This leaves us with option (b), i.e., $W = \phi(X_3 X_6 - X_4 X_5)$ say. In this case, when $\phi = 0$ we not only obtain our F-term, we need not even correct the matter content Δ. This branch of the moduli space is that of our original theory. However, when $\phi \neq 0$, we must have $X_3 = X_4 = X_5 = X_6 = 0$. Now the D-terms read $|X_1|^2 - |X_2|^2 = -\zeta_1 = \zeta_2$, so the moduli space is: $\{\phi \in \mathbb{C}, X_1 \in \mathbb{C}\}$ such that $\zeta_1 + \zeta_2 = 0$ for otherwise there would be no moduli at all. We see that we obtain another branch of moduli space. As remarked before, this is a general phenomenon when we include new fields: the whole moduli space will be larger than the one encoded by the toric data. As a check, we see that our example is exactly that given in [25], after the identification with their notation, $Y_{12} \to X_6, X_{24} \to X_3, Z_{23} \to X_1, Z_{32} \to X_2, Y_{34} \to X_4, X_{13} \to X_5, Z_{41} \to \phi$ and $(X_1 X_2 - \phi) \to \phi$. We note that if we were studying a non-Abelian extension to the toric theory, as by brane setups (e.g. [25]) or by stacks of probes (in progress from [26]), the chargeless field ϕ would manifest as an adjoint field thereby modifying our quiver diagram. Of course since the study of toric methods in physics is so far restricted to product $U(1)$ gauge groups, such complexities do not arise. To avoid confusion we shall henceforth mark only the bi-fundamentals in our quiver diagrams but will write the chargeless fields explicit in the superpotential.

Our agreement with the results of [25] is very reassuring. It gives an excellent

example demonstrating that our canonical resolution technique and the inverse algorithm do indeed, in response to what was posited earlier, give a theory living on a D-brane probing the SPP (T-dual to the setup in [25]). However, there is a subtle point we would like to mention. There exists an ambiguity in writing the superpotential when the chargeless field ϕ is involved. Our algorithm gives $W = \phi(X_3 X_6 - X_4 X_6)$ while [25] gives $W = (X_1 X_2 - \phi)(X_3 X_6 - X_4 X_6)$. Even though they have identical moduli, it is the latter which is used for the brane setup. Indeed, the toric methods by definition (in defining Δ from a_{ij}) do not handle chargeless fields and hence we have ambiguities. Fortunately our later examples will not involve such fields.

The above example of the SPP was a naïve one as we need only to accommodate a single F-term. We move on to a more complicated example. Suppose we are now given

$$
d = \begin{pmatrix}
 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \\
A & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\
B & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
C & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\
D & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0
\end{pmatrix}
\text{ and } K = \begin{pmatrix}
X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{pmatrix}.
$$

We shall see in the next section, that these arise for the del Pezzo 1 surface. Now the nullspace of K has dimension $10 - 6 = 4$, we could obtain a host of relations from various linear combinations in this space. One relation is obvious: $X_2 X_7 - X_3 X_6 = 0$. The charge it carries is $(q_A, q_B, q_C, q_D) = (0+0, -1+0, 0+1, 1+(-1)) = (0, -1, 1, 0)$ which cancels that of X_9. Hence $X_9(X_2 X_7 - X_3 X_6)$ could be a term in W. Now $\frac{\partial}{\partial X_2}$ thereof gives $X_7 X_9$ and from K we see that $X_7 X_9 - X_1 X_5 X_{10} = 0$, therefore, $-X_1 X_2 X_5 X_{10}$ could be another term in W. We repeat this procedure, generating new terms as we proceed and introducing new fields where necessary. We are fortunate that in this case we can actually reproduce all F-terms without recourse to artificial insertions of new fields: $W = X_2 X_7 X_9 - X_3 X_6 X_9 - X_4 X_8 X_7 - X_1 X_2 X_5 X_{10} + X_3 X_4 X_{10} + X_1 X_5 X_6 X_8$.

Enlightened by these examples, let us return to some remarks upon generalities. Making all the exponents of the fields positive, the F-terms can then be written as

$$
\prod_i X_i^{a_i} = \prod_j X_j^{b_j}, \tag{3.6}
$$

with $a_i, b_j \in \mathbb{Z}^+$. Indeed if we were to have another field X_k such that $k \notin \{i\}, \{j\}$ then the term $X_k \left(\prod_i X_i^{a_i} - \prod_j X_j^{b_j} \right)$, on the condition that X_k appears only this once, must be an additive term in the superpotential W. This is because the F-flatness condition $\frac{\partial W}{\partial X_k} = 0$ implies (3.6) immediately. Of course judicious observations are called for to (A) find appropriate relations (3.6) and (B) find X_k among our m fields. Indeed (B) may not even be possible and new fields may be forced to be introduced, whereby making the moduli space of the gauge theory larger than that encodable by the toric data.

In addition, we must ensure that each term in W be chargeless under the product

gauge groups. What this means for us is that for each of the terms $X_k \left(\prod_i X_i^{a_i} - \prod_j X_j^{b_j} \right)$ we must have $\mathrm{Charge}_s(X_k) + \sum_i a_i \mathrm{Charge}_s(X_i) = 0$ for $s = 1, .., r$ indexing through our r gauge group factors (we note that by our very construction, for each gauge group, the charges for $\prod_i X_i^{a_i}$ and for $\prod_j X_j^{b_j}$ are equal). If X_k in fact cannot be found among our m fields, it must be introduced as a new field ϕ with appropriate charge. Therefore with each such relation (3.6) read from K, we iteratively perform this said procedure, checking $\Delta_{sk} + \sum_i a_i \Delta_{si} = 0$ at each step, until a satisfactory superpotential is reached. The right choices throughout demands constant vigilance and astuteness.

4   An Illustrative Example: the Toric del Pezzo Surfaces

As the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ resolutions were studied in great detail in [25], we shall use the data from [26] to demonstrate the algorithm of finding the gauge theory from toric diagrams extensively presented in the previous section.

The toric diagram of the dual cone of the (parent) quotient singularity $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ as well as those of its resolution to the three toric del Pezzo surface are presented in Figure 4.

del Pezzo 1: Let us commence our analysis with the first toric del Pezzo surface[11]. From its toric diagram, we see that the minimal $\mathbb{Z}_n \times \mathbb{Z}_n$ toric diagram into which it embeds is $n = 3$. As a reference, the toric diagram for $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is given in Figure 4 and the quiver diagram, given later in the convenient brane-box form, in Figure 5. Luckily, the matrices Q_t' and G_t' for this Abelian quotient is given in [26].

---

[11]Now some may identify the toric diagram of del Pezzo 1 as given by nodes (using the notation in Figure 4) $(1, -1, 1)$, $(2, -1, 0)$, $(-1, 1, 1)$, $(0, 0, 1)$ and $(-1, 0, 2)$ instead of the one we have chosen in the convention of [26], with nodes $(0, -1, 2)$, $(0, 0, 1)$, $(-1, 1, 1)$, $(1, 0, 0)$ and $(0, 1, 0)$. But of course these two G_t matrices describe the same algebraic variety. The former corresponds to $\mathrm{Spec}\left(\mathbb{C}[XY^{-1}Z, X^2Y^{-1}, X^{-1}YZ, Z, X^{-1}Z^2] \right)$ while the latter corresponds to $\mathrm{Spec}\left(\mathbb{C}[Y^{-1}Z^2, Z, X^{-1}YZ, X, Y] \right)$. The observation that $(X^2Y^{-1}) = (X)(X^{-1}YZ)^{-1}(Z)$, $(XY^{-1}Z) = (X)(Y)^{-1}(Z)$ and $(X^{-1}Z^2) = (Y^{-1}Z^2)(Y)(X^{-1})$ for the generators of the polynomial ring gives the equivalence. In other words, there is an $SL(5, \mathbb{Z})$ transformation between the 5 nodes of the two toric diagrams.

Adding the extra column of FI-parametres we present these matrices below[12]:

$$
G'_t = \left(
\begin{array}{cccccccccccccccccccccccc}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} & p_{19} & p_{20} & p_{21} & p_{22} & p_{23} & p_{24} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 3 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1
\end{array}
\right. \cdots\cdots
$$

$$
\cdots\cdots \left.
\begin{array}{cccccccccccccccccc}
p_{25} & p_{26} & p_{27} & p_{28} & p_{29} & p_{30} & p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} & p_{38} & p_{39} & p_{40} & p_{41} & p_{42} \\
0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\right)
$$

---

[12]In [26], a canonical ordering was used; for our purposes we need not belabour this point and use their Q'_{total} as Q'_t. This is perfectly legitimate as long as we label the columns carefully, which we have done.

and

$$Q'_t =$$

Left block (columns p_1 through p_{24}):

p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}	p_{17}	p_{18}	p_{19}	p_{20}	p_{21}	p_{22}	p_{23}	p_{24}	
1	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	0	0	
0	1	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	0	0	
0	0	1	0	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	0	0	
0	0	0	1	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	0	0	
0	0	0	0	1	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	0	0	
0	0	0	0	0	1	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	0	0	
0	0	0	0	0	0	1	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	0	0	
0	0	0	0	0	0	0	1	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	1	0	0	−1	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	1	0	−1	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	−1	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	−1	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	−1	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	−1	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	−1	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	−1	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_1
0	1	−1	−1	2	−1	0	1	0	−1	0	0	0	0	−1	1	0	0	0	0	0	0	0	0	ζ_2
0	1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_3
1	−2	1	0	−2	2	0	−1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_4
−1	1	−1	1	0	−1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_5
0	0	0	1	−2	1	0	−1	0	1	0	0	0	0	0	−1	0	0	0	0	0	1	0	0	ζ_6
0	−1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_7
0	1	−1	0	2	−1	0	1	0	−1	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	ζ_8

......

Right block (columns p_{25} through p_{42}):

p_{25}	p_{26}	p_{27}	p_{28}	p_{29}	p_{30}	p_{31}	p_{32}	p_{33}	p_{34}	p_{35}	p_{36}	p_{37}	p_{38}	p_{39}	p_{40}	p_{41}	p_{42}	
−1	0	0	0	0	0	0	0	0	1	0	−1	0	0	0	1	−1	−1	1
−1	0	0	0	0	0	0	0	1	0	−1	0	0	0	1	−1	−1	1	
−1	0	0	0	0	0	0	−1	2	0	−1	0	0	0	2	−2	−2	2	
−1	0	0	0	0	0	0	−1	1	1	−1	0	0	0	2	−1	−2	1	
−2	0	0	0	0	0	0	−1	2	1	−2	0	0	0	3	−2	−2	2	
−2	0	0	0	0	0	0	0	2	0	−1	0	0	1	1	−1	−1	1	
−2	0	0	0	0	0	0	0	1	1	−1	0	0	0	1	−1	−1	1	
−1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	−1	0	1	
−1	0	0	0	0	0	0	0	0	1	−1	0	0	0	1	−1	0	1	
−1	0	0	0	0	0	0	−1	1	0	−1	0	0	0	2	−1	−1	2	
−1	0	0	0	0	0	0	−1	2	−1	0	0	0	0	1	−1	−1	2	
0	0	0	0	0	0	0	−1	1	0	0	0	0	0	1	−1	−1	1	
0	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	−1	1	
1	0	0	0	0	0	0	0	−1	0	0	0	0	0	−1	1	1	−1	
1	0	0	0	0	0	0	0	−1	−1	1	0	0	0	−1	2	0	−1	
1	0	0	0	0	0	0	−1	0	−1	1	0	0	0	−1	1	0	0	
1	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	
0	0	0	0	0	0	0	0	0	−1	1	0	0	0	−1	1	0	0	
0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	1	0	0	
−1	0	0	0	0	0	0	0	0	1	−1	0	0	0	1	−1	0	0	
−1	0	0	0	0	0	0	0	1	0	−1	0	0	0	1	0	−1	0	
−1	0	0	0	0	0	0	0	1	−1	0	0	0	0	0	0	0	0	
0	1	0	0	0	0	0	−1	0	0	−1	0	0	0	1	−1	0	1	
0	0	1	0	0	0	0	−1	1	−1	0	0	0	0	0	−1	0	1	
0	0	0	1	0	0	0	0	−1	1	−1	0	0	0	0	−1	1	0	
0	0	0	0	1	0	0	0	0	0	0	0	0	0	−1	−1	1	0	
0	0	0	0	0	1	0	−1	0	1	−1	0	0	0	1	−1	0	0	
0	0	0	0	0	0	1	−1	1	−1	0	0	0	0	1	−1	−1	1	
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	−1	−1	
0	0	0	0	0	0	0	0	0	0	0	0	1	0	−1	1	0	−1	
0	0	0	0	0	0	0	0	0	0	0	0	0	1	−1	0	1	−1	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_1
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_2
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_4
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_5
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_6
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_7
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	ζ_8

......

According to our algorithm, we must perform Gaussian row-reduction on Q'_t to solve for 42 variables x_i. When this is done we find that we can in fact express all variables in terms of 3 x_i's together with the 8 FI-parametres ζ_i. We choose these three x_i's to be $x_{10,29,36}$ corresponding to the 3 outer vertices which we know must be

13

8

7, 14, 17

37

del Pezzo 0

5   9

4

6, 7, 12, 14, 15, 18, 30

8

37   38

del Pezzo 3

Resolving
10, 36, 29, 38, 32
5, 20, 9, 11, 26, 4, 16, 23
27, 28, 41, 42, 39, 40
1, 2, 3, 6, 12, 15, 18, 19
21, 22, 24, 25, 30, 31, 33, 34, 35

Resolving
10, 13, 20, 11, 26, 16, 23
27, 28, 36, 41, 42, 39, 40, 29
1, 2, 3, 17, 19, 21, 22, 24, 25
31, 32, 33, 34, 35

10   *(-1, -1, 3)*

(0, -1, 2)   *(-1, 0, 2)*

5, 13, 20   9, 11, 26

Z_3 x Z_3

(1, -1, 1)   *(0, 0, 1)*   *(-1, 1, 1)*

4, 16, 23   8, 27, 28

1, 2, 3, 6, 7, 12, 14, 15
17, 18, 19, 21, 22, 24, 25
30, 31, 32, 33, 34, 35

36   29

(2, -1, 0)

37, 41, 42   38, 39, 40

(1, 0, 0)   *(0, 1, 0)*

(-1, 2, 0)

Resolving
10, 36, 29
5, 20, 9, 11, 26, 4, 16, 23
27, 28, 41, 42, 39, 40
1, 2, 3, 6, 12, 15, 18, 19
21, 22, 24, 25, 30, 31, 33, 34, 35

13

8

7, 14, 17, 32

37   38

del Pezzo 1

Resolving
10, 5, 13, 20, 21, 11, 26
16, 23, 27, 28, 41, 42
38, 39, 40, 29, 1, 2, 3
6, 12, 17, 19, 22, 24, 25
20, 31, 32, 33, 34, 35   36

Resolving
10, 5, 13, 20, 11, 26
16, 23, 27, 28, 41, 42
38, 39, 40, 29, 1, 2, 3
6, 12, 17, 19, 22, 24, 25
20, 31, 32, 33, 34, 35

9

4   8

7, 12, 14, 15, 18

37

Hirzebruch 0

9

4   8

7, 12, 14, 15, 18, 21

36   37

del Pezzo 2

Figure 4: The resolution of the Gorenstein singularity $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to the three toric del Pezzo surfaces as well as the zeroth Hirzebruch surface. We have labelled explicitly which columns (linear σ-model fields) are to be associated to each node in the toric diagrams and especially which columns are to be eliminated (fields acquiring non-zero VEV) in the various resolutions. Also, we have labelled the nodes of the parent toric diagram with the coordinates as given in the matrix G_t for $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$.

resolved in going from $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to del Pezzo 1.

Next we select the fields which must be kept and set them to zero in order to determine the range for ζ_i. Bearing in mind the toric diagrams from Figure 4, these fields we judiciously select to be: $p_{13,8,37,38}$. Setting $x_{13,8,37,38} = 0$ gives us the solution $\{\zeta_6 = 0; x_{29} = \zeta_7 = \zeta_3 = \zeta_1 - \zeta_5; x_{10} = \zeta_4 + \zeta_5 + \zeta_3; x_{36} = \zeta_7 - \zeta_8\}$, which upon back-substitution to the solutions x_i we obtained from Q'_t, gives zero for $x_{13,8,37,38}$ (which we have chosen by construction) as well as $x_{7,14,17,32}$; for all others we obtain positive values. This means precisely that all the other fields are to be eliminated and these 8 columns $\{$ 13, 8, 37, 38, 7,14,17,32 $\}$ are to be kept while the remaining 42-8=34 are to be eliminated from Q'_t upon row-reduction to give Q_t. In other words, we have found our set B to be $\{1,2,3,4,5,6,9,10,11,12,15,16,18,19,20,21,22,23,24,25, 26,27,28,29,30,31,33,34,35,36,39,40,41,42\}$ and thus according to (3.5) we immediately obtain

$$Q_t = \begin{pmatrix} p_7 & p_8 & p_{13} & p_{14} & p_{17} & p_{32} & p_{37} & p_{38} & \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \zeta_2 + \zeta_8 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & \zeta_6 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \zeta_1 + \zeta_3 + \zeta_5 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We note of course that 5 out of the 8 FI-parametres have been eliminated automatically; this is to be expected since in resolving $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to del Pezzo 1, we remove precisely 5 nodes. Obtaining the D-terms and F-terms is now straight-forward. Using (3.3) and re-inserting the last row we obtain the D-term equations (incidence matrix) to be

$$d = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

From this matrix we immediately observe that there are 4 gauge groups, i.e., $U(1)^4$ with 10 matter fields X_i which we have labelled in the matrix above. In an equivalent notation we rewrite d as the adjacency matrix of the quiver diagram (see Figure 5) for the gauge theory:

$$a_{ij} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 \end{pmatrix}.$$

The K-matrix we obtain to be:

$$K^t = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

which indicates that the original 10 fields X_i can be expressed in terms of 6. This was actually addressed in the previous section and we rewrite that pleasant superpotential

here:

$$W = X_2 X_7 X_9 - X_3 X_6 X_9 - X_4 X_8 X_7 - X_1 X_2 X_5 X_{10} + X_3 X_4 X_{10} + X_1 X_5 X_6 X_8.$$

<u>del Pezzo 2:</u> Having obtained the gauge theory for del Pezzo 1, we now repeat the above analysis for del Pezzo 2. Now we have the FI-parametres restricted as $\{p_{36} = \zeta_2 = 0; \zeta_3 = \zeta_4; x_{29} = \zeta_4 + \zeta_6; x_{10} = \zeta_1 + \zeta_4\}$, making the set to be eliminated as $B = \{$ 1, 2, 3, 5, 6, 10, 11, 13, 16, 17, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 38, 39, 40, 41, 42 $\}$. Whence, we obtain

$$Q_t = \left(\begin{array}{ccccccccccc|c}
p_4 & p_7 & p_8 & p_9 & p_{12} & p_{14} & p_{15} & p_{18} & p_{21} & p_{36} & p_{37} & \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \zeta_4 + \zeta_6 + \zeta_8 \\
1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \zeta_7 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta_1 + \zeta_3 + \zeta_5 \\
-1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & \zeta_2 \\
0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\
-1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),$$

and observe that 4 D-terms have been resolved, as 4 nodes have been eliminated from $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. From this we easily extract (see Figure 5)

$$d = \left(\begin{array}{ccccccccccccc}
X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} \\
-1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0
\end{array}\right);$$

moreover, we integrate the F-term matrices

$$K^t = \left(\begin{array}{ccccccccccccc}
X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right)$$

to obtain the superpotential

$$W = X_2 X_9 X_{11} - X_9 X_3 X_{10} - X_4 X_8 X_{11} - X_1 X_2 X_7 X_{13} + X_{13} X_3 X_6$$
$$- X_5 X_{12} X_6 + X_1 X_5 X_8 X_{10} + X_4 X_7 X_{12}.$$

<u>del Pezzo 3:</u> Finally, we shall proceed to treat del Pezzo 3. Here we have the range of the FI-parametres to be $\{\zeta_1 = \zeta_6 = \zeta_6 = 0; x_{29} = \zeta_3 = -\zeta_5; x_{10} = \zeta_4; \zeta_2 = x_{36}; \zeta_8 = -\zeta_2 - \zeta_{10}\}$, which gives the set B as $\{1, 2, 3, 10, 11, 13, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 39, 40, 41, 42\}$, and thus according to (3.5) we immediately obtain

$$Q_t = \left(\begin{array}{ccccccccccccc|c}
p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{12} & p_{14} & p_{15} & p_{18} & p_{30} & p_{37} & p_{38} & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \zeta_2 + \zeta_4 + \zeta_8 \\
1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \zeta_7 \\
-1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & \zeta_6 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta_3 + \zeta_5 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_1 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

We note indeed that 3 out of the 8 FI-parametres have been automatically resolved, as we have removed 3 nodes from the toric diagram for $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. The matter content (see Figure 5) is encoded in

$$d = \begin{pmatrix}
 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} & X_{14} \\
 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\
 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\
 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0
\end{pmatrix},$$

and from the F-terms

$$K^t = \begin{pmatrix}
 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} & X_{14} \\
 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

we integrate to obtain the superpotential

$$W = X_3 X_8 X_{13} - X_8 X_9 X_{11} - X_5 X_6 X_{13} - X_1 X_3 X_4 X_{10} X_{12}$$
$$+ X_7 X_9 X_{12} + X_1 X_2 X_5 X_{10} X_{11} + X_4 X_6 X_{14} - X_2 X_7 X_{14}.$$

Note that we have a quintic term in W; this is an interesting interaction indeed.

del Pezzo 0: Before proceeding further, let us attempt one more example, viz., the degenerate case of the del Pezzo 0 as shown in Figure 4. This time we note that the ranges for the FI-parametres are $\{\zeta_5 = -x_{29} + \zeta_6 - A; \zeta_6 = x_{29} - B; x_{29} = B + C; \zeta_8 = -x_{36} + B; x_{36} = B + C + D; x_{10} = A + E\}$ for some positive A, B, C, D and E, that $B = \{$ 1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42 $\}$ and whence the charge matrix is

$$Q_t = \begin{pmatrix}
p_7 & p_8 & p_{13} & p_{14} & p_{17} & p_{37} & \\
1 & 0 & 0 & 0 & -1 & 0 & \zeta_2 + \zeta_6 + \zeta_8 \\
-1 & 0 & 0 & 1 & 0 & 0 & \zeta_1 + \zeta_3 + \zeta_5 \\
-1 & 1 & 1 & -1 & -1 & 1 & 0
\end{pmatrix}.$$

We extract the matter content (see Figure 5) as $d = \begin{pmatrix}
 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \\
 & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 1 \\
 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\
 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0
\end{pmatrix},$

and the F-terms as $K^t = \begin{pmatrix}
 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \\
 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{pmatrix}$, and from the latter

we integrate to obtain the superpotential

$$W = X_1 X_4 X_9 - X_4 X_5 X_7 - X_2 X_3 X_9 - X_1 X_6 X_8 + X_2 X_5 X_8 + X_3 X_6 X_7.$$

Of course we immediately recognise the matter content (which gives a triangular quiver which we shall summarise below in Figure 5) as well as the superpotential

from equations (4.7-4.14) of [19]; it is simply the theory on the Abelian orbifold $\mathbb{C}^3/\mathbb{Z}_3$ with action $(\alpha \in \mathbb{Z}_3) : (z_1, z_2, z_3) \rightarrow (e^{\frac{2\pi i}{3}} z_1, e^{\frac{2\pi i}{3}} z_2, e^{\frac{2\pi i}{3}} z_3)$. Is our del Pezzo 0 then $\mathbb{C}^3/\mathbb{Z}_3$? We could easily check from the G_t matrix (which we recall is obtained from G'_t of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ by eliminating the columns corresponding to the set B):

$$G_t = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 \end{pmatrix}.$$

These columns (up to repeat) correspond to monomials $Z, X^{-1}YZ, Y^{-1}Z^2, X$ in the polynomial ring $\mathbb{C}[X, Y, Z]$. Therefore we need to find the spectrum (set of maximal ideals) of the ring $\mathbb{C}[Z, X^{-1}YZ, Y^{-1}Z^2, X]$, which is given by the minimal polynomial relation: $(X^{-1}YZ) \cdot (Y^{-1}Z^2) \cdot X = (Z)^3$. This means, upon defining $p = X^{-1}YZ; q = Y^{-1}Z^2; r = X$ and $s = Z$, our del Pezzo 0 is described by $pqr = s^3$ as an algebraic variety in $\mathbb{C}^4(p, q, r, s)$, which is precisely $\mathbb{C}^3/\mathbb{Z}_3$. Therefore we have actually come through a full circle in resolving $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to $\mathbb{C}^3/\mathbb{Z}_3$ and the validity of our algorithm survives this consistency check beautifully. Moreover, since we know that our gauge theory is exactly the one which lives on a D-brane probe on $\mathbb{C}^3/\mathbb{Z}_3$, this gives a good check for physicality: that our careful tuning of FI-parametres via canonical partial resolutions does give a physical D-brane theory at the end. We tabulate the matter content a_{ij} and the superpotential W for the del Pezzo surfaces below, and the quiver diagrams, in Figure 5.

	del Pezzo 1	del Pezzo 2	del Pezzo 3
Matter a_{ij} =	$\begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$
Superpotential W =	$X_2X_7X_9 - X_3X_6X_9$ $-X_4X_8X_7 - X_1X_2X_5X_{10}$ $+X_3X_4X_{10} + X_1X_5X_6X_8$	$X_2X_9X_{11} - X_9X_3X_{10}$ $-X_4X_8X_{11} - X_1X_2X_7X_{13}$ $+X_{13}X_3X_6 - X_5X_{12}X_6$ $+X_1X_5X_8X_{10} + X_4X_7X_{12}$	$X_3X_8X_{13} - X_8X_9X_{11}$ $-X_5X_6X_{13} - X_1X_3X_4X_{10}X_{12}$ $+X_7X_9X_{12} + X_1X_2X_5X_{10}X_{11}$ $+X_4X_6X_{14} - X_2X_7X_{14}$

	del Pezzo 0 $\cong \mathbb{C}^3/\mathbb{Z}_3$	Hirzebruch 0 $\cong \mathbb{P}^1 \times \mathbb{P}^1 := F_0 = E_1$
Matter a_{ij}	$\begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 3 \\ 3 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$
Superpotential W	$X_1X_4X_9 - X_4X_5X_7$ $-X_2X_3X_9 - X_1X_6X_8$ $+X_2X_5X_8 + X_3X_6X_7$	$X_1X_8X_{10} - X_3X_7X_{10}$ $-X_2X_8X_9 - X_1X_6X_{12}$ $+X_3X_6X_{11} + X_4X_7X_9$ $+X_2X_5X_{12} - X_4X_5X_{11}$

Upon comparing Figure 4 and Figure 5, we notice that as we go from del Pezzo 0 to 3, the number of points in the toric diagram increases from 4 to 7, and the number of gauge groups (nodes in the quiver) increases from 3 to 6. This is consistent with the observation for $\mathcal{N} = 1$ theories that the number of gauge groups equals the number of perimetre points (e.g., for del Pezzo 1, the four nodes 13, 8, 37 and 38) in the toric diagram. Moreover, as discussed in [28], the rank of the global symmetry group (E_i

del Pezzo 0

del Pezzo 1

Parent Theory

Z₃ x Z₃

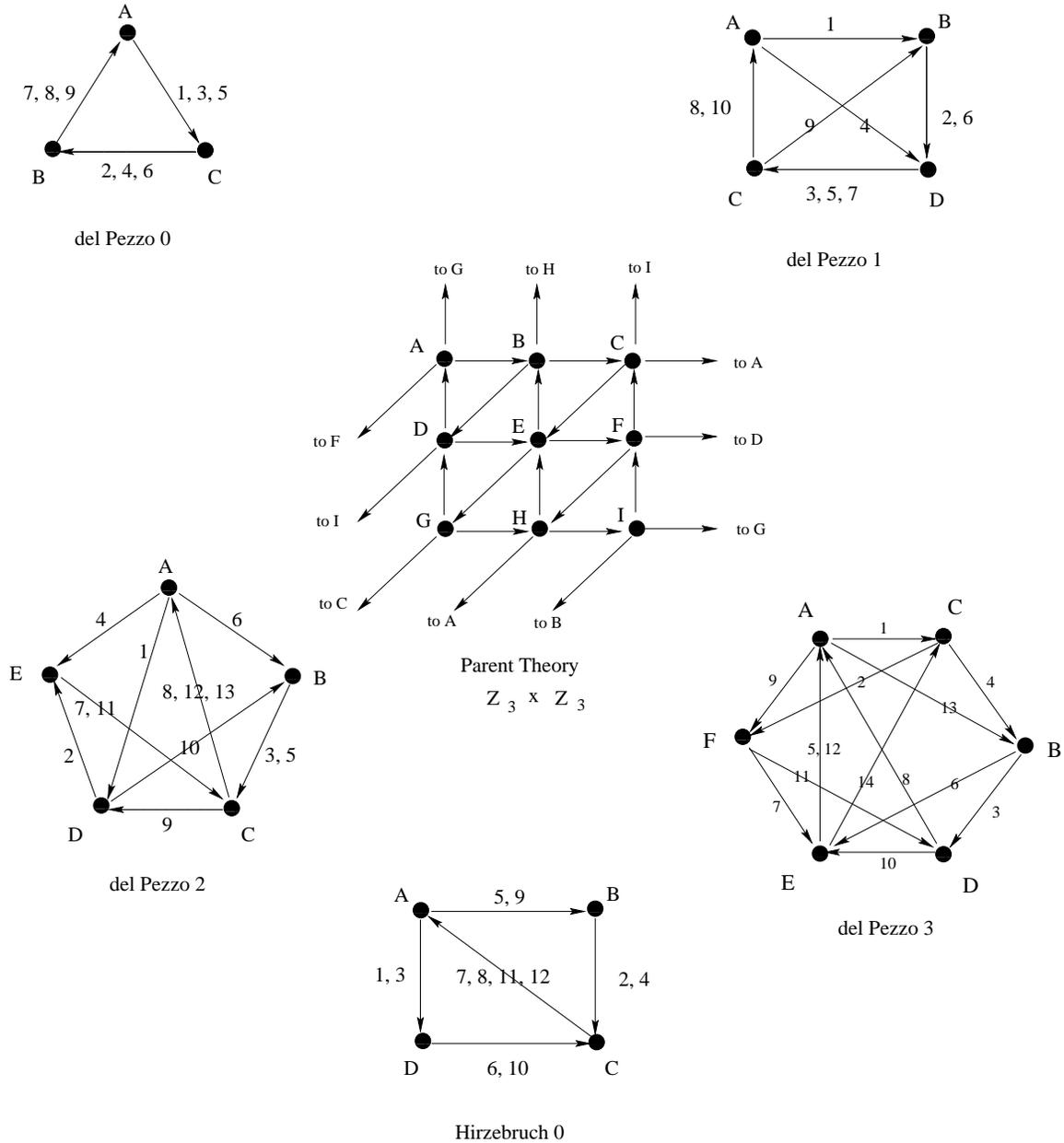del Pezzo 2

del Pezzo 3

Hirzebruch 0

Figure 5: The quiver diagrams for the matter content of the brane world-volume gauge theory on the 4 toric del Pezzo singularities as well as the zeroth Hirzebruch surface. We have specifically labelled the $U(1)$ gauge groups (A, B, ..) and the bi-fundamentals (1, 2, ..) in accordance with our conventions in presenting the various matrices Q_t, Δ and K. As a reference we have also included the quiver for the parent $\mathbb{Z}_3 \times \mathbb{Z}_3$ theory.

for del Pezzo i) which must exist for these theories equals the number of perimetre point minus 3; it would be an intereting check indeed to see how such a symmetry manifests itself in the quivers and superpotentials.

Hirzebruch 0: Let us indulge ourselves with one more example, namely the 0th Hirzebruch surface, or simply $\mathbb{P}^1 \times \mathbb{P}^1 := F_0 := E_1$. The toric diagram is drawn in Figure 4. Now the FI-parametres are $\{\zeta_4 = -x_{29}-x_{36}-\zeta_5-\zeta_8-A; \zeta_5 = -A-B; \zeta_7 = x_{10} + x_{29} + x_{36} + \zeta_8 - C; \zeta_8 = -x_{10} - x_{29} - x_{36} + D; D = A + B; C = A + B; A = x_{10} - E; x_{10} = E + F; x_{29} = B + G\}$ for positive A, B, C, D, E, F and G. Moreover, $B = \{$ 1, 2, 3, 5, 6, 10, 11, 13, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42 $\}$. We note that this can be obtained directly by partial resolution of fields 21 and 36 from del Pezzo 2 as is consistent with Figure 4. Therefrom we obtain the charge matrix

$$Q_t = \begin{pmatrix} p_4 & p_7 & p_8 & p_9 & p_{12} & p_{14} & p_{15} & p_{18} & p_{37} & \\ -1 & 2 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & \zeta_2 + \zeta_4 + \zeta_6 + \zeta_8 \\ 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & \zeta_7 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \zeta_1 + \zeta_3 + \zeta_5 \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

from which we have the matter content $d = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} \\ -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$

the quiver for which is presented in Figure 5. The F-terms are

$$K^t = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

from which we obtain

$$W = X_1 X_8 X_{10} - X_3 X_7 X_{10} - X_2 X_8 X_9 - X_1 X_6 X_{12} + X_3 X_6 X_{11} + X_4 X_7 X_9 + X_2 X_5 X_{12} - X_4 X_5 X_{11},$$

a perfectly acceptable superpotential with only cubic interactions. We include these results with our table above.

5   Uniqueness?

In our foregoing discussion we have constructed in detail an algorithm which calculates the matter content encoded by Δ and superpotential encoded in K, given the toric diagram of the singularity which the D-branes probe. As abovementioned, though this algorithm gives one solution for the quiver and the K-matrix once the matrix Q_t is determined, the general inverse process of going from toric data to gauge theory data,

is highly **non-unique** and a classification of all possible theories having the same toric description would be interesting[13]. Indeed, by the very structure of our algorithm, in immediately appealing to the partial resolution of gauge theories on $\mathbb{Z}_n \times \mathbb{Z}_n$ orbifolds which are well-studied, we have granted ourselves enough extraneous information to determine a unique Q_t and hence the ability to proceed with ease (this was the very reason for our devising the algorithm).

However, generically we do not have any such luxury. At the end of subsection 3.1, we have already mentioned two types of ambiguities in the inverse problem. Let us refresh our minds. They were (A) the **F-D ambiguity** which is the inability to decide, simply by observing the toric diagram, which rows of the charge matrix Q_t are D-terms and which are F-terms and (B) the **repetition ambiguity** which is the inability to decide which columns of G_t to repeat once having read the vectors from the toric diagram. Other ambiguities exist, such as in each time when we compute nullspaces, but we shall here discuss to how ambiguities (A) and (B) manifest themselves and provide examples of vastly different gauge theories having the same toric description. There is another point which we wish to emphasise: as mentioned at the end of subsection 3.1, the resolution method guarantees, upon careful tuning of the FI-parametres, that the resulting gauge theory does originate from the world-volume of a D-brane probe. Now of course, by taking liberties with experimentation of these ambiguities we are no longer protected by physicality and in general the theories no longer live on the D-brane. It would be a truly interesting exercise to check which of these different theories do.

F-D Ambiguity: First, we demonstrate type (A) by returning to our old friend the SPP whose charge matrix we had earlier presented. Now we write the same matrix without specifying the FI-parametres:

$$Q_t = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

We could apply the last steps of our algorithm to this matrix as follows.

(a) If we treat the first row as Q (the F-terms) and the second and third as $V \cdot U$ (the D-terms) we obtain the gauge theory as discussed in subsection 3.3 and in [25].

(b) If we treat the second row as Q and first with the third as $V \cdot U$, we obtain
$d = \begin{pmatrix} -1 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$ which is an exotic theory indeed with a field (p_5) charged under three gauge groups.

Let us digress a moment to address the stringency of the requirements upon

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[13]We thank R. Plesser for pointing this issue out to us.

matter contents. By the very nature of finite group representations, any orbifold theory must give rise to only adjoints and bi-fundamentals because its matter content is encodable by an adjacency matrix due to tensors of representations of finite groups. The corresponding incidence matrix d, has (a) only 0 and ± 1 entries specifying the particular bi-fundamentals and (b) has each column containing precisely one 1, one -1 and with the remaining entries 0. However more exotic matter contents could arise from more generic toric singularities, such as fields charged under 3 or more gauge group factors; these would then have d matrices with conditions (a) and (b) relaxed[14]. Such exotic quivers (if we could even call them quivers still) would give interesting enrichment to those well-classified families as discussed in [29].

Moreover we must check the anomaly cancellation conditions. These could be rather involved; even though for $U(1)$ theories they are a little simpler, we still need to check *trace anomalies* and *cubic anomalies*. In a trace-anomaly-free theory, for each node in the quiver, the number of incoming arrows must equal the number of outgoing (this is true for a $U(1)$ theory which is what toric varieties provide; for a discussion on this see e.g. [8]). In matrix language this means that each row of d must sum to 0.

Now for a theory with only bi-fundamental matter with ± 1 charges, since $(\pm 1)^3 = \pm 1$, the cubic is equal to the trace anamaly; therefore for these theories we need only check the above row-condition for d. For more exotic matter content, which we shall meet later, we do need to perform an independent cubic-anomaly check.

Now for the above d, the second row does not sum to zero and whence we do unfortunately have a problematic anomalous theory. Let us push on to see whether we have better luck in the following.

(c) Treating row 3 as the F-terms and the other two as the D-terms gives
$d = \begin{pmatrix} 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$ which has the same anomaly problem as the one above.

(d) Now let rows 1 and 2 as the F-terms and the 3rd, as the D-terms, we obtain
$d = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 1 & 1 \end{pmatrix}$, which is a perfectly reasonable matter content. Integrating $K = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$ gives the superpotential $W = \phi(X_1 X_2 X_5 - X_3 X_4)$ for some field ϕ of charge $(0,0)$ (which could be an adjoint for example; note

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[14]Note that we still require that each column sums to 0 so as to be able to factor out an overall $U(1)$.
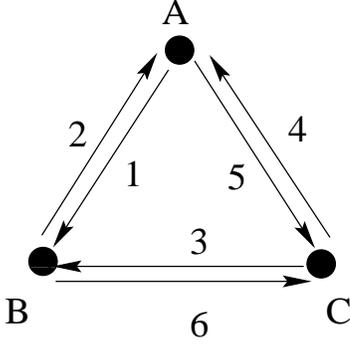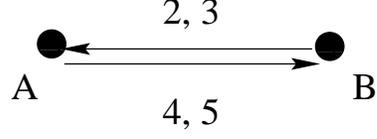
Theory (a)   Theory (d)

Figure 6: The vastly different matter contents of theories (a) and (d), both anomaly free and flow to the toric diagram of the suspended pinched point in the IR.

however that we can not use X_1 even though it has charge $(0,0)$ for otherwise the F-terms would be altered). This theory is perfectly legitimate. We compare the quiver diagrams of theories (a) (which we recall from Figure 3) and this present example in Figure 6. As a check, let us define the gauge invariant quantities: $a = X_2X_4$, $b = X_2X_5$, $c = X_3X_4$, $d = X_3X_5$ and $e = X_1$. Then we have the algebraic relations $ad = bc$ and $eb = c$, from which we immediately obtain $ad = eb^2$, precisely the equation for the SPP.

(e) As a permutation on the above, treating rows 1 and 3 as the F-terms gives a theory equivalent thereto.

(f) Furthermore, we could let rows 2 and 3 be Q giving us $d = \begin{pmatrix} 0 & 1 & -1 & -1 & -1 \\ 0 & -1 & 1 & 1 & 1 \end{pmatrix}$, but this again gives an anomalous matter content.

(g) Finally, though we cannot treat all rows as F-terms, we can however treat all of them as D-terms in which Q_t is simply Δ as remarked at the end of Section 2 before the flow chart. In this case we have the matter content $d = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$ which clearly is both trace-anomaly free (each row adds to zero) and cubic-anomaly-free (the cube-sum of the each row is also zero). The superpotential, by our very choice, is of course zero. Thus we have a perfectly legitimate theory without superpotential but with an exotic field (the first column) charged under 4 gauge groups.

We see therefore, from our list of examples above, that for the simple case of the SPP we have 3 rather different theories (a,d,g) with contrasting matter content and superpotential which share the same toric description.

Repetition Ambiguity: As a further illustration, let us give one example of type (B) ambiguity. First let us eliminate all repetitive columns from the G_t of SPP, giving us:

$$G_t = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

which is perfectly allowed and consistent with Figure 2. Of course many more possibilities for repeats are allowed and we could redo the following analyses for each of them. As the nullspace of our present choice of G_t, we find Q_t, and we choose, in light of the foregoing discussion, the first row to represent the D-term:

$$Q_t = \begin{pmatrix} -1 & 1 & -1 & 0 & 1 & \zeta \\ 1 & -2 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Thus equipped, we immediately retrieve, using our algorithm,

$$d = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ 1 & -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 \end{pmatrix} \quad K^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We see that d passes our anomaly test, with the same bi-fundamental matter content as theory (d). The superpotential can be read easily from K (since there is only one relation) as $W = \phi(X_5^2 - X_3 X_4)$. As a check, let us define the gauge invariant quantities: $a = X_1 X_2$, $b = X_1 X_4$, $c = X_3 X_2$, $d = X_3 X_4$ and $e = X_5$. These have among themselves the algebraic relations $ad = bc$ and $e^2 = d$, from which we immediately obtain $bc = ae^2$, the equation for the SPP. Hence we have yet another interesting anomaly free theory, which together with our theories (a), (d) and (g) above, shares the toric description of the SPP.

Finally, let us indulge in one more demonstration. Now let us treat both rows of our Q_t as D-terms, whereby giving a theory with no superpotential and the exotic matter content $d = \begin{pmatrix} -1 & 1 & -1 & 0 & 1 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & -1 \end{pmatrix}$ with a field (column 2) charged under 3 gauge groups. Indeed though the rows sum to 0 and trace-anomaly is avoided, the cube-sum of the second row gives $1^3 + 1^3 + (-2)^3 = -6$ and we do have a cubic anomaly.

In summary, we have an interesting phenomenon indeed! Taking so immediate an advantage of the ambiguities in the above has already produced quite a few examples of vastly different gauge theories flowing in the IR to the same universality class by having their moduli spaces identical. The vigilant reader may raise two issues. First, as mentioned earlier, one may take the pains to check whether these theories do indeed live on a D-brane. Necessary conditions such as that the theories may be obtained from an $\mathcal{N} = 4$ theory must be satisfied. Second, the matching of moduli spaces may not seem so strong since they are on a classical level. However, since we are dealing with product $U(1)$ gauge groups (which is what toric geometry is capable to dealing

with so far), the classical moduli receive no quantum corrections[15]. Therefore the matching of the moduli for these various theories do persist to the quantum regime, which hints at some kind of "duality" in the field theory. We shall call such a duality **toric duality**. It would be interesting to investigate how, with non-Abelian versions of the theory (either by brane setups or stacks of D-brane probes), this toric duality may be extended.

6 Conclusions and Prospects

The study of resolution of toric singularities by D-branes is by now standard. In the concatenation of the F-terms and D-terms from the world volume gauge theory of a single D-brane at the singularity, the moduli space could be captured by the algebraic data of the toric variety. However, unlike the orbifold theories, the inverse problem where specifying the structure of the singularity specifies the physical theory has not yet been addressed in detail.

We recognise that in contrast with D-brane probing orbifolds, where knowing the group structure and its space-time action uniquely dictates the matter content and superpotential, such flexibility is not shared by generic toric varieties due to the highly non-unique nature of the inverse problem. It has been the purpose and main content of the current writing to device an **algorithm** which constructs the matter content (the incidence matrix d) and the interaction (the F-term matrix K) of a well-behaved gauge theory given the toric diagram D of the singularity at hand.

By embedding D into the Abelian orbifold $\mathbb{C}^k/(\mathbb{Z}_n)^{k-1}$ and performing the standard partial resolution techniques, we have investigated how the induced action upon the charge matrices corresponding to the toric data of the latter gives us a convenient charge matrix for D and have constructed a programmatic methodology to extract the matter content and superpotential of *one* D-brane world volume gauge theory probing D. The theory we construct, having its origin from an orbifold, is nicely behaved in that it is anomaly free, with bi-fundamentals only and well-defined superpotentials. As illustrations we have tabulated the results for all the toric del Pezzo surfaces and the zeroth Hirzebruch surface.

Directions of further work are immediately clear to us. From the patterns emerging from del Pezzo surfaces 0 to 3, we could speculate the physics of higher (non-toric) del Pezzo cases. For example, we expect del Pezzo n to have $n + 3$ gauge groups. Moreover, we could attempt to fathom how our resolution techniques translate as Higgsing in brane setups, perhaps with recourse to diamonds, and realise the various theories on toric varieties as brane configurations.

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[15]We thank K. Intriligator for pointing this out.

Indeed, as mentioned, the inverse problem is highly non-unique; we could presumably attempt to classify all the different theories sharing the same toric singularity as their moduli space. In light of this, we have addressed two types of ambiguity: that in having multiple fields assigned to the same node in the toric diagram and that of distinguishing the F-terms and D-terms in the charge matrix. In particular we have turned this ambiguity to a matter of interest and have shown, using our algorithm, how vastly different theories, some with quite exotic matter content, may have the same toric description. This commonality would correspond to a duality wherein different gauge theories flow to the same universality class in the IR. We call this phenomenon **toric duality**. It would be interesting indeed how this duality may manifest itself as motions of branes in the corresponding setups. Without further ado however, let us pause here awhile and leave such investigations to forthcoming work.

Appendix: Finding the Dual Cone

Let us be given a convex polytope C, with the edges specifying the faces of which given by the matrix M whose columns are the vectors corresponding to these edges. Our task is to find the dual cone $\tilde{C}$ of C, or more precisely the matrix N such that

$$N^t \cdot M \geq 0 \qquad \text{for all entries.}$$

There is a standard algorithm, given in [27]. Let M be $n \times p$, i.e., there are p n-dimensional vectors spanning C. We note of course that $p \geq n$ for convexity. Out of the p vectors, we choose $n-1$. This gives us an $n \times (n-1)$ matrix of co-rank 1, whence we can extract a 1-dimensional null-space (as indeed the initial p vectors are all linearly independent) described by a single vector u.

Next we check the dot product of u with the remaining $p - (n-1)$ vectors. If all the dot products are positive we keep u, and if all are negative, we keep $-u$, otherwise we discard it.

We then select another $n-1$ vectors and repeat the above until all combinations are exhausted. The set of vectors we have kept, u's or $-u$'s then form the columns of N and span the dual cone $\tilde{C}$.

We note that this is a very computationally intensive algorithm, the number of steps of which depends on $\binom{p}{n-1}$ which grows exponentially.

A subtle point to remark. In light of what we discussed in a footnote in the paper on the difference between $\mathbf{M}_+ = \mathbf{M} \cap \sigma$ and $\mathbf{M}'_+$, here we have computed the dual of σ. We must ensure that $\mathbb{Z}_+$-independent lattice points inside the cones be not missed.

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