ON BRAUER ALGEBRA SIMPLE MODULES OVER THE COMPLEX FIELD

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Abstract. This paper gives two results on the simple modules for the Brauer algebra $B_n(\delta)$ over the complex field. First we describe the module structure of the restriction of all simple $B_n(\delta)$-modules to $B_{n-1}(\delta)$. Second we give a new geometrical interpretation of Ram and Wenzl's construction of bases for 'δ-permissible' simple modules.

1. Introduction

1.1. Classical Schur-Weyl duality relates the representations of the general linear group and the symmetric group via commuting actions on tensor space. The Brauer algebra was introduced by Brauer in 1937 to play the role of the symmetric group when one replaces the general linear group by the orthogonal or symplectic group. For any non-negative integer n, any commutative ring $\mathtt{k}$, and any $\delta \in \mathtt{k}$, we can define the Brauer algebra $B_n(\delta)$ as being the $\mathtt{k}$-algebra with basis all pair partitions of $2n$. We can represent these basis elements as diagrams (so-called Brauer diagrams) having $2n$ vertices arranged in 2 rows of n vertices each, such that each vertex is linked to precisely one other vertex. The multiplication is then given by concatenation, removing all closed loops, and scalar multiplication by δ^k where k is the number of closed loops removed. It's easy to see that $B_n(\delta)$ is generated by the set $\{\sigma_i, e_i \,:\, 1 \leq i \leq n-1\}$ where σ_i and e_i are given in Figure 1.

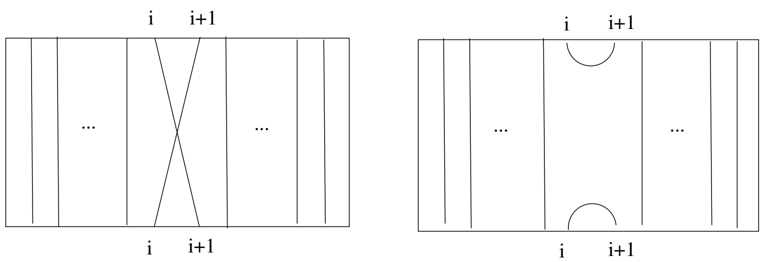

Figure 1

The symmetric group algebra $\mathtt{k}\Sigma_n$ appears naturally as the subalgebra of $B_n(\delta)$ generated by the σ_i's. Note that $\mathtt{k}\Sigma_n$ also occurs as a quotient of $B_n(\delta)$ as explained below. This turns out to be very helpful in studying the representation theory of $B_n(\delta)$.

Assume for a moment that δ is a unit. Consider the idempotent given by $e = \frac{1}{\delta}e_{n-1}$. Then it is easy to see that

$$eB_n(\delta)e \cong B_{n-2}(\delta) \quad \text{and} \quad B_n(\delta)/B_n(\delta)eB_n(\delta) \cong \mathtt{k}\Sigma_n. \tag{1}$$

Now fix $\mathtt{k} = \mathbb{C}$ and recall that the simple $\mathbb{C}\Sigma_n$-modules are indexed by partitions of n, that is for each partition λ we have a (simple) Specht module S^λ. Using (1) we can easily deduce by induction on n that the simple modules for $B_n(\delta)$ are indexed by the set Λ_n of partitions of $n, n-2, n-4, \ldots$. For each $\lambda \in \Lambda_n$, we denote the corresponding simple module by $L_n(\lambda)$.

When $B_n(\delta)$ is semisimple, the simple modules can be constructed explicitely by 'inflating' (or 'globalising') the corresponding Specht module, see for example [8]. However the algebra $B_n(\delta)$ is not always semisimple. In 1988, Wenzl showed in [17] that if $B_n(\delta)$ is not semisimple then $\delta \in \mathbb{Z}$, and in 2005, Rui gave an explicit criterion for semisimplicity in [16].

In this paper, we study the simple modules when $B_n(\delta)$ is not semisimple. So we will assume that $\delta \in \mathbb{Z}$. For the moment we will also assume that $\delta \neq 0$. In this case, $B_n(\delta)$ is a quasi-hereditary algebra with respect to the opposite order to the one given by the size of partitions. (In fact, we will work with a refinement of this order, see Section 2.2). In particular, the indecomposable projective modules $P_n(\lambda)$ $(\lambda \in \Lambda_n)$ have a filtration by standard modules $\Delta_n(\lambda)$ $(\lambda \in \Lambda_n)$. The standard modules can be constructed explicitly (as inflation of Specht modules, as in the semisimple case) and we have surjective homomorphisms

$$P_n(\lambda) \twoheadrightarrow \Delta_n(\lambda) \twoheadrightarrow L_n(\lambda)$$

for each $\lambda \in \Lambda_n$. Now the decomposition matrix $D_{\lambda\mu} = [\Delta_n(\mu) : L_n(\lambda)]$ has been determined by the second author in [14] and its inverse is given in [2]. This gives a closed form for the dimension of the simple modules (although the coefficients of $(D_{\lambda\mu})^{-1}$ are not easy to compute in practice).

1.2. We have natural embeddings of the Brauer algebras

$$B_{n-1}(\delta) \hookrightarrow B_n(\delta) \hookrightarrow B_{n+1}(\delta),$$

defined by adding two vertices to each Brauer diagram, one at the end of each row, and connecting these new vertices by an edge. So we have corresponding restriction functors $\mathrm{res}_n : B_n(\delta)\text{-mod} \to B_{n-1}(\delta)\text{-mod}$ and induction functors $\mathrm{ind}_n : B_n(\delta)\text{-mod} \to B_{n+1}(\delta)\text{-mod}$. For partitions λ and μ, we write $\lambda \rhd \mu$ (resp. $\lambda \lhd \mu$) if λ is obtained from μ by adding (resp. removing) a box to its Young diagram. From [6] we have exact sequences

$$0 \to \oplus_{\mu\lhd\lambda}\Delta_{n-1}(\mu) \to \mathrm{res}_n\,\Delta_n(\lambda) \to \oplus_{\mu\rhd\lambda}\Delta_{n-1}(\mu) \to 0, \quad \text{and} \tag{2}$$

$$0 \to \oplus_{\mu\lhd\lambda}\Delta_{n+1}(\mu) \to \mathrm{ind}_n\,\Delta_n(\lambda) \to \oplus_{\mu\rhd\lambda}\Delta_{n+1}(\mu) \to 0 \tag{3}$$

where we define $\Delta_{n-1}(\mu) = 0$ when $\mu \notin \Lambda_{n-1}$.

The first objective of this paper is to describe the corresponding result for all simple modules. More precisely, we describe completely the module structure of $\mathrm{res}_n\,L_n(\lambda)$ for all $\lambda \in \Lambda_n$ and all non-negative integers n. The corresponding problem for the modular representations of the symmetric group has attracted a lot of interests and many important results have been proved in this case, see for example [10] and references therein. However a complete solution is yet to be found in this case.

1.3. Recall that the Young graph $\mathcal{Y}$ has, as vertex set, the set Λ of all partitions, and two partitions λ and μ are connected by an edge if $\lambda \lhd \mu$ or $\lambda \rhd \mu$. Leduc and Ram constructed in [11] bases for the standard modules for the Brauer algebra for generic values of δ in terms of walks on $\mathcal{Y}$. Their construction relies on complex combinatorial objects such as the King polynomials (first introduced in [7]). These bases do not specialise to $\delta \in \mathbb{Z}$

(except in very low rank). However, it follows implicitly from [15] that the truncation of these representations to certain 'δ- permissible up-down tableaux' gives bases for the 'δ-permissible' simple modules.

More recently [4] introduced a geometric characterisation of the representation theory of the Brauer algebra. It turns out that the combinatorics used in [15] and [11] can be explained in a uniform and natural way in this geometrical context. In particular, we obtain a striking characterisation of the roots of the King polynomials.

Motivated by this, the second objective of this paper is to recast the contruction of [11] in the geometrical setting. This provides a unification of the classical and modern approaches, but is also done with a view to treating arbitrary simple modules (and other characteristics) in further work.

1.4. Structure of the paper. In Section 2, we recall and extend the necessary setup from [4] for the geometrical interpretation of the representation theory of the Brauer algebra $B_n(\delta)$. In Section 3, we recall the construction of weight diagrams and cap diagrams associated to every partition λ and integer δ introduced in [14] and [2]. We develop some of their properties and recall how these can be used to describe the blocks and the decomposition numbers for $B_n(\delta)$. In Section 4 we give a complete description of the module structure of the restriction from $B_n(\delta)$ to $B_{n-1}(\delta)$ of every simple module in terms of cap diagrams. We start Section 5 by recalling the representations constructed by Leduc and Ram for the generic Brauer algebra. We then give a geometric interpretation of the combinatorics used in their construction and deduce, by specialisation and truncation, explicit bases for an important class of simple modules.

2. Geometrical setting

2.1. Euclidean space and reflection groups. Consider the space $\mathbb{R}^{\mathbb{N}}$ consisting of all (possibly infinite) $\mathbb{R}$-linear combination of the symbols ϵ_i ($i \in \mathbb{N}$). For each $x = \sum_{i\in\mathbb{N}} x_i\epsilon_i$, write $x = (x_1, x_2, x_3, \ldots)$. The inner product on finitary elements in $\mathbb{R}^{\mathbb{N}}$ is given by $\langle \epsilon_i, \epsilon_j\rangle = \delta_{ij}$. Now define W to be the infinite reflection group on $\mathbb{R}^{\mathbb{N}}$ of type D generated by the reflections $(i, j)_\pm$ ($i < j \in \mathbb{N}$) where

$$(i, j)_\pm : (\ldots, x_i, \ldots, x_j, \ldots) \mapsto (\ldots, \pm x_j, \ldots, \pm x_i, \ldots).$$

Define W_+ to be the subgroup generated by $(i, j)_+$ ($i < j \in \mathbb{N}$). So W_+ is the infinite reflection group on $\mathbb{R}^{\mathbb{N}}$ of type A. The group W (resp. W_+) defines a set $\mathbb{H}$ (resp. $\mathbb{H}_+$) of hyperplanes corresponding to the reflections $(i, j)_\pm$ (resp. $(i, j)_+$) on $\mathbb{R}^{\mathbb{N}}$. We define the *degree of singularity* of an element $x \in \mathbb{R}^{\mathbb{N}}$, denoted by $\deg(x)$, to be the number of hyperplanes in $\mathbb{H}$ containing x, that is the number of pairs of entries x_i, x_j ($i < j$) satisfying $x_i = \pm x_j$. The set of hyperplanes $\mathbb{H}$ (resp. $\mathbb{H}_+$) subdivide $\mathbb{R}^{\mathbb{N}}$ into so-called W-alcoves, (resp. W_+-alcoves), see [9]. Define the element $\rho \in \mathbb{R}^{\mathbb{N}}$ by

$$\rho = (0, -1, -2, -3, \ldots).$$

Now define the *dominant chamber* X_+ to be the W_+-alcove containing ρ, and the *fundamental alcove* to be the W-alcove containing ρ.

2.2. Embedding of the Young graph. Recall that the Young graph $\mathcal{Y}$ has vertex set the set $\Lambda = \cup_{n\geq 0}\Lambda_n$ of all partitions and has an edge between two partitions λ and μ if $\lambda \triangleright \mu$ or $\lambda \triangleleft \mu$.

Proposition 2.2.1. *Let $\lambda \in \Lambda_n$ and $\delta \in \mathbb{Z}$. The dimension of $\Delta_n(\lambda)$ is given by the number of walks of length n starting at $\emptyset$ and ending at λ.*

Proof. This follows from (2) by induction on n. $\square$

For each $\delta \in \mathbb{Z}$, we will now define an embedding of the graph $\mathcal{Y}$ into $\mathbb{R}^{\mathbb{N}}$. This embedding is the key to all the geometrical tools for Brauer algebra representation theory.

Define $\mathcal{Z}$ as the graph with vertex set $\mathbb{R}^{\mathbb{N}}$ and an edge (x, x') whenever $x - x' = \pm\epsilon_i$ for some i. For $x \in \mathbb{R}^{\mathbb{N}}$ define $\mathcal{Z}(x)$ as the connected component of $\mathcal{Z}$ containing x. Define $\mathcal{Z}_+$ as the subgraph of $\mathcal{Z}$ on vertices in the dominant chamber X_+. Define $\mathcal{Z}_+(x)$ as the connected component of $\mathcal{Z}_+$ containing x. A walk on $\mathcal{Z}_+$ is called a dominant walk.

For each partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ (where $\lambda_i = 0$ for all $i >> 0$), consider the transpose partition $\lambda^T = (\lambda_1^T, \lambda_2^T, \lambda_3^T, \ldots)$. For each $\delta \in \mathbb{Z}$, define $\rho_\delta \in \mathbb{R}^{\mathbb{N}}$ by

$$\rho_\delta = (-\frac{\delta}{2}, -\frac{\delta}{2} - 1, -\frac{\delta}{2} - 2, -\frac{\delta}{2} - 3, \ldots) = -\frac{\delta}{2}(1,1,1,...) + \rho.$$

Now define the embedding $e_\delta : \mathcal{Y} \to \mathcal{Z}$ by setting for each vertex $\lambda \in \Lambda$,

$$e_\delta(\lambda) = \lambda^T + \rho_\delta. \tag{4}$$

Note that $e_\delta(\lambda) \in X_+$ for all $\lambda \in \Lambda$ and all $\delta \in \mathbb{Z}$. In fact we have the following important observation.

Lemma 2.2.2. *For every $\delta \in \mathbb{Z}$ the map $e_\delta : \mathcal{Y} \to \mathcal{Z}_+(\rho_\delta)$ is a graph isomorphism.*

Using Lemma 2.2.2 we can rephrase Proposition 2.2.1 as follows.

Proposition 2.2.3. *Fix $\delta \in \mathbb{Z}$. Points $x \in \mathbb{R}^{\mathbb{N}}$ reachable by dominant walks on $\mathcal{Z}$ of length n from ρ_δ index the standard modules of $B_n(\delta)$. Moreover the number of dominant walks on $\mathcal{Z}$ from ρ_δ to x gives the dimension of the corresponding standard module.*

2.3. Representations of Temperley-Lieb algebras. To explain our geometrical programme in this paper we mention an analogous situation in Lie theory — specifically the representation theory of the Temperley–Lieb algebra $TL_n(\delta)$ (this is, via Schur–Weyl duality, simply the sl_2 case of a wider sl_N phenomenon). We refer the reader to [13, Section 12] and references therein for more details.

For sl_2 one should replace $\mathbb{R}^{\mathbb{N}}$ with $\mathbb{R}$, replace $\mathcal{Z}$ with the corresponding graph (whose connected components are simply chains of vertices) and W and W_+ with the reflections groups of type affine-A_1 and A_1 respectively, acting on $\mathbb{R}$. In Figure 2, we see different sets of walks on a connected component of $\mathcal{Z}$. The hyperplanes or walls in $\mathbb{H}$ are denoted by solid thick lines in these pictures. There is, in principle, a representation for each choice of position of the A_1-wall. The relative position of the first affine wall depends on δ and on the ground field. Figure 2(a), shows all walks from the origin to a given point, which form a basis for a module (isomorphic to a Young module in this case) when the A_1-wall is in generic position. Figure 2(b) shows the basis of dominant walks for a Temperley–Lieb

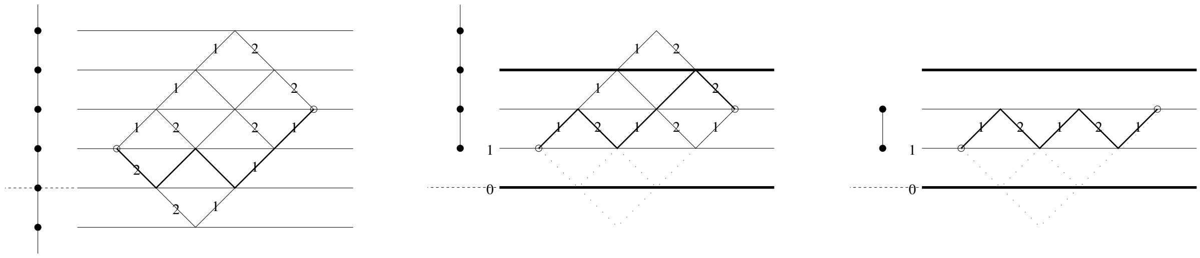

FIGURE 2. (a), (b), (c) Walks on the $\mathcal{Z}$-graph in type-A_1.

Specht module, obtained when the A_1-wall is in the 'natural' position. Figure 2(c) then shows the subset of walks restricted to regular points, which we shall call *restricted walks* (in this example there is only a single such walk), giving a basis for the simple head of the Specht module for a suitable δ. (Indeed bases for arbitrary Temperley–Lieb simples can be described using a refinement of the same technology.)

Furthermore, the only off-diagonal entries in the 'unitary' representations of $TL_n(\delta)$ generators corresponding to these walk bases are between (two) walks differing at a single point. The difference is a reflection of the point in a certain hyperplane in $\mathbb{R}$. The mixing depends on the 'height' of the hyperplane (i.e. the vertical axis in Figure 2), and vanishes at certain heights corresponding to affine walls (this is the 'quantisation' of Young's famous hook-length orthogonal form, expressed geometrically) — this explains why, in this case, restricted walks decouple from the rest.

This example is nothing more than an analogy for us, since there is no corresponding piece of Lie theory underlying our case. Nonetheless we will see in Section 5 that the features described above with the W_+-wall in the natural position also hold here. But first we need to describe the appropriate analogue of 'restricted walks'.

2.4. δ-regularity and the δ-restricted walks. By analogy with the Temperley-Lieb case, we might consider walks restricted to regular points. Note however that the degree of singularity of ρ_δ is given by

$$\deg(\rho_\delta) = \begin{cases} 0 & \text{if } \delta \geq 0 \\ -m & \text{if } \delta = 2m < 0 \text{ or } 2m+1 < 0. \end{cases}$$

So we will first rescale the notion of regularity in a homogeneous way.

Definition 2.4.1. *(i) For $x \in \mathbb{R}^{\mathbb{N}}$, we say that x is δ-regular if* $\deg(x) = \deg(\rho_\delta)$*, and that x is δ-singular if* $\deg(x) > \deg(\rho_\delta)$.
(ii) For a partition $\lambda \in \Lambda$ we define the δ-degree of singularity of λ, denoted by $\deg_\delta(\lambda)$*, by*

$$\deg_\delta(\lambda) = \deg(e_\delta(\lambda)).$$

(iii) We say that λ is δ-regular (resp. δ-singular) if $e_\delta(\lambda)$ is δ-regular (resp. δ-singular).

Now we can define the restricted region in a homogeneous way as follows.

Definition 2.4.2. *(i) Define the δ-restricted graph $\mathcal{Z}_\delta$ to be the maximal connected subgraph of $\mathcal{Z}_+$ containing ρ_δ such that all vertices are δ-regular.*
(ii) We define $\mathcal{Y}_\delta$ to be the inverse image of $\mathcal{Z}_\delta$ under the map e_δ, and we set A_δ to be the

vertex set of $\mathcal{Y}_\delta$.
(iii) We call walks on $\mathcal{Z}_\delta$*, or* $\mathcal{Y}_\delta$*,* δ*-restricted walks.*

Remark 2.4.3. (i) Note that the set A_δ corresponds precisely to the set of δ-permissible partitions defined in [17].
(ii) Note also that for $\delta \geq 0$ the set A_δ corresponds precisely to the intersection of the vertex set of $\mathcal{Z}(\rho_\delta)$ with the fundamental alcove. This can be seen from the explicit description of A_δ given in Proposition 5.4.1.

3. Weight diagrams, cap diagrams and decomposition numbers

3.1. Weight diagrams and blocks. Fix $\delta \in \mathbb{Z}$. In this section we recall the construction of the weight diagram x_λ associated to any partition λ given in [2].
Recall from Section 2.2 that $e_\delta(\lambda)$ is a strictly decreasing sequence in $\mathbb{Z}$ for δ even , and in $\frac{1}{2} + \mathbb{Z}$ for δ odd. The weight diagram x_λ has vertices indexed by $\mathbb{N}_0$ if δ is even or by $\mathbb{N} - \frac{1}{2}$ if δ is odd. Each vertex will be labelled with one of the symbols $\circ$, $\times$, $\vee$, $\wedge$. For $e_\delta(\lambda) = (x_1, x_2, x_3, \ldots)$ define

$$I_\wedge(e_\delta(\lambda)) = \{x_i : x_i > 0\} \quad \text{and} \quad I_\vee(e_\delta(\lambda)) = \{|x_i| : x_i < 0\}.$$

Now vertex n in the weight diagram x_λ is labelled by

$$\begin{cases} \circ & \text{if } n \notin I_\vee(e_\delta(\lambda)) \cup I_\wedge(e_\delta(\lambda)) \\ \times & \text{if } n \in I_\vee(e_\delta(\lambda)) \cap I_\wedge(e_\delta(\lambda)) \\ \vee & \text{if } n \in I_\vee(e_\delta(\lambda)) \setminus I_\wedge(e_\delta(\lambda)) \\ \wedge & \text{if } n \in I_\wedge(e_\delta(\lambda)) \setminus I_\vee(e_\delta(\lambda)) \end{cases}$$

Moreover, if $x_i = 0$ for some i then we label vertex 0 by either $\wedge$ or $\vee$ (this choice will not affect what follows). Otherwise we label vertex 0 by a $\circ$. (Note that this case only happens when δ is even).

Example 3.1.1. We have

$$e_\delta(\emptyset) = (-\frac{\delta}{2}, -\frac{\delta}{2} - 1, -\frac{\delta}{2} - 2, -\frac{\delta}{2} - 3, -\frac{\delta}{2} - 4, \ldots).$$

Write $\delta = 2m$ or $2m + 1$ for some $m \in \mathbb{Z}$. If $m \geq 0$ then $x_\emptyset$ has the first m vertices labelled by $\circ$ and all remaining vertices labelled by $\vee$. If $m < 0$ and δ is odd then $x_\emptyset$ has the first $-m$ vertices labelled by $\times$ and all remaining vertices labelled by $\vee$. Finally if $m < 0$ and δ is even then the first vertex is labelled by $\vee$ (or $\wedge$), the next $-m$ vertices are labelled by $\times$ and all remaining vertices are labelled by $\vee$. The picture is given in Figure 3.

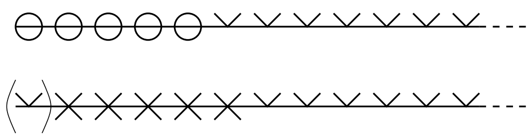

FIGURE 3. The weight diagram $x_\emptyset$ for $\delta = 2m$ or $2m + 1$ with $m = 5$ or $m = -5$

It is immediate to see that the δ-degree of singularity of λ is equal to the number of $\times$'s in x_λ. Note also that the weight diagram x_λ is labelled by $\vee$ for all vertices $n >> 0$.

It was shown in [4] that two simple $B_n(\delta)$-modules $L_n(\lambda)$ and $L_n(\mu)$ are in the same block if and only if $e_\delta(\lambda) \in We_\delta(\mu)$. Now it's easy to see that this is equivalent to saying that x_λ is obtained from x_μ by repeatedly applying one of the following operations:
- swapping a $\vee$ and a $\wedge$,
- replacing two $\vee$'s (resp. $\wedge$'s) by two $\wedge$'s (resp. $\vee$'s).

For $\lambda \in \Lambda$ we define $\mathcal{B}(\lambda)$ to be the set of partitions in the W-orbit of λ (via the embedding e_δ). Then we have

$$\mathcal{B}(\lambda) = \cup_m \mathcal{B}_m(\lambda)$$

where $\mathcal{B}_m(\lambda)$ is the block of $B_m(\delta)$ containing λ and the union is taken over all m such that $\lambda \in \Lambda_m$.

With δ still fixed, we now refine the order on Λ given by the size of partitions to get the following **partial order** $\leq$. We set $\lambda < \mu$ if x_μ is obtained from x_λ by swapping a $\vee$ and a $\wedge$ so that the $\wedge$ moves to the right, or if x_μ contains a pair of $\wedge$'s instead of a corresponding pair of $\vee$'s in x_λ, and extending by transitivity. Note that if $\lambda \leq \mu$ then we have $|\lambda| \leq |\mu|$ and $\mu \in \mathcal{B}(\lambda)$.

3.2. Some properties of the weight diagrams. Here we give some properties of the weight diagrams which will be used in sections 4 and 5.

Lemma 3.2.1. *Let $\lambda, \mu \in \Lambda$ with $\mu \rhd \lambda$. Then we have either* $\deg_\delta(\mu) = \deg_\delta(\lambda)$ *or* $\deg_\delta(\mu) = \deg_\delta(\lambda) \pm 1$. *Moreover, the labels of the weight diagrams x_λ and x_μ are equal everywhere except in at most two adjacent vertices, where we have one of the following configurations.*

***Case I**: $\deg_\delta(\mu) = \deg_\delta(\lambda)$ and either*
(i) $x_\lambda = \times\vee$, $x_\mu = \vee\times$, or *(ii) $x_\lambda = \wedge\times$, $x_\mu = \times\wedge$, or*
(iii) $x_\lambda = \circ\vee$, $x_\mu = \vee\circ$, or *(iv) $x_\lambda = \wedge\circ$, $x_\mu = \circ\wedge$, or*
(v) δ is odd, and vertex $\frac{1}{2}$ is labelled by $\vee$ in x_λ and by $\wedge$ in x_μ.

***Case II**: $\deg_\delta(\mu) = \deg_\delta(\lambda) + 1$ and either*
(vi) $x_\lambda = \wedge\vee$, $x_\mu = \circ\times$, or *(vii) $x_\lambda = \wedge\vee$, $x_\mu = \times\circ$.*

***Case III**: $\deg_\delta(\mu) = \deg_\delta(\lambda) - 1$ and either*
(viii) $x_\lambda = \circ\times$, $x_\mu = \vee\wedge$, or *(ix) $x_\lambda = \times\circ$, $x_\mu = \vee\wedge$.*

Proof. This follows directly from the definition of the weight diagram. □

The next lemma will play a key role in what follows.

Lemma 3.2.2. *Let $\delta = 2m$ or $2m + 1$ for some $m \in \mathbb{Z}$ and let λ be any partition. Then we have*

$$\#(\circ \text{ in } x_\lambda) - \#(\times \text{ in } x_\lambda) = m.$$

Proof. We will prove this by induction on the size of λ. The result is clearly true for $\lambda = \emptyset$ by Example 3.1.1. Now suppose that the result holds for a partition λ and let μ be a partition obtained by adding one box to λ. Then it is easy to observe from Lemma 3.2.1 that the result holds for μ. □

We now explain two ways of recovering the (Young diagram of) the partition λ from its weight diagram x_λ.

First ignore all the $\circ$'s (but not their positions). Now read all the symbols below the line successively from right to left, then all the symbols above the line successively from left to right as illustrated in Figure 4.

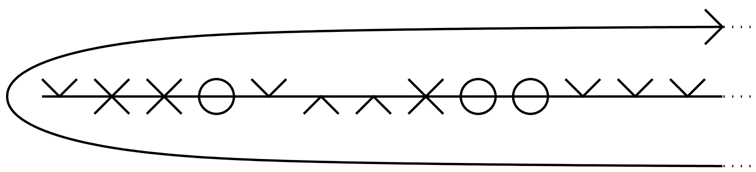

FIGURE 4

Now it's easy to see that the Young diagram of the partition λ can be drawn as follows. At each entry, if there is no symbol go one step up, and if there is a symbol then go one step to the right. Note that the weight diagram x_λ ends with infinitely many $\vee$'s. So we always start with infinitely many steps up (the left edge of the quadrant in which λ lives) and end with infinitely many steps to the right (the top edge of the quadrant in which λ lives), as expected. Note also that, for δ even, the entry indexed by 0 should only be read once as a step up if it is labeled by $\circ$ and a step to the right otherwise.

Alternatively, one could read the entries from right to left above the line first and then from left to right below the line, as illustrated in Figure 5.

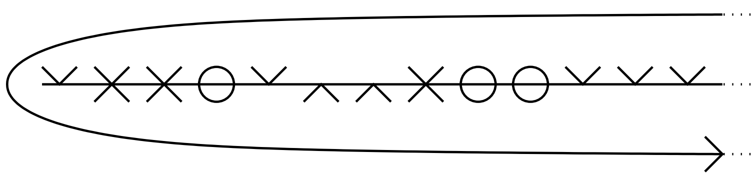

FIGURE 5

In this case, the Young diagram can be drawn by going one step to the left if there is a symbol and one step down otherwise.

The partition corresponding to the weight diagram in Figure 4 and 5 is given by $\lambda = (10, 10, 9, 9, 8, 5, 3, 3)$, see Figure 6.

Proposition 3.2.3. *Let $\delta \in \mathbb{Z}$ and $\lambda \in \Lambda$.*
(i) There is a bijection between the set of $\times$'s in x_λ and the set of all pairs $(i > j) \in \mathbb{N}^2$ satisfying

$$\lambda_i^T + \lambda_j^T - i - j + 2 = \delta.$$

Moreover, given a pair $(i > j)$ and a $\times$ in position n in x_λ it corresponds to under this bijection, we have that $(i, j) \in [\lambda]$ if and only if

$$\#(\times\text{'s left of } n) - \#(\circ\text{'s left of } n) < \left\{ \begin{array}{ll} m-1 & \text{if } \delta = 2m \\ m & \text{if } \delta = 2m+1 \end{array} \right. \tag{5}$$

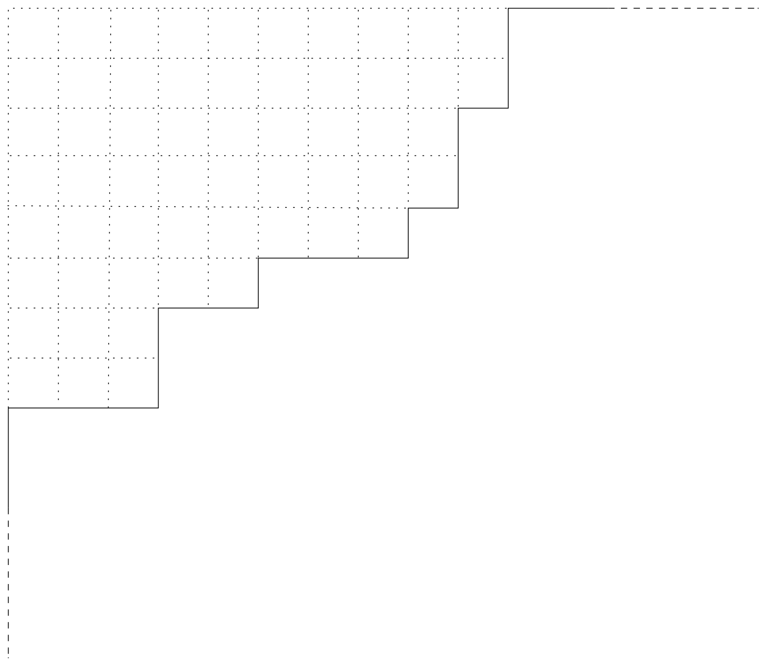

FIGURE 6

(ii) There is a bijection between the set of $\circ$*'s in* x_λ *and the set of all pairs* $(i \leq j) \in \mathbb{N}^2$ *satisfying*

$$-\lambda_i - \lambda_j + i + j = \delta.$$

Moreover, given a pair $(i \leq j)$ *and a* $\circ$ *in position* n *in* x_λ *it corresponds to under this bijection, we have that* $(i,j) \in [\lambda]$ *if and only if*

$$\#(\times\text{'s left of } n) - \#(\circ\text{'s left of } n) > \begin{cases} m-1 & \text{if } \delta = 2m \\ m & \text{if } \delta = 2m+1 \end{cases} \tag{6}$$

Proof. (i) The first part follows from the definition of x_λ.
Now we have $(i,j) \in [\lambda]$ if and only if $i \leq \lambda_j^T$, that is $\lambda_j^T - i \geq 0$.

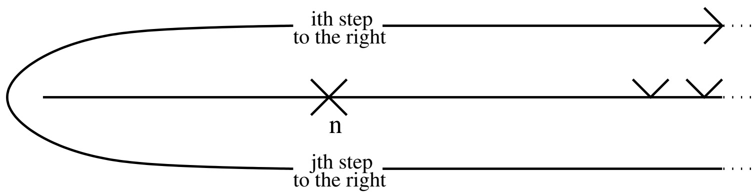

FIGURE 7

Reading the partition from the weight diagram as in the Figure 7 we obtain

$$\begin{aligned} \lambda_j^T &= \#(\text{steps up after } j) \\ &= \#(\circ\text{'s left of } n) + \#(\vee\text{'s left of } n) + \#(\circ\text{'s}) + \#(\wedge\text{'s}) - \delta_{\delta,2m}, \qquad \text{and} \\ i &= \#(\text{steps to the right up to (and including) } i) \\ &= \#(\times\text{'s}) + \#(\wedge\text{'s}) + \#(\vee\text{'s left of } n) + \#(\times\text{'s left of } n) + 1. \end{aligned}$$

So we have

$$\begin{aligned}\lambda_j^T - i &= \#(\circ\text{'s}) - \#(\times\text{'s}) + \#(\circ\text{'s left of } n) - \#(\times\text{'s left of } n) - 1 - \delta_{\delta,2m} \\ &= m + \#(\circ\text{'s left of } n) - \#(\times\text{'s left of } n) - 1 - \delta_{\delta,2m} \qquad \text{using Lemma 3.2.2.}\end{aligned}$$

Hence we have that $\lambda_j^T - i \geq 0$ if and only if

$$\#(\times\text{'s left of } n) - \#(\circ\text{'s left of } n) < m - \delta_{\delta,2m}$$

as required.

(ii) Suppose that n is a vertex labelled with $\circ$ in x_λ. Then reading the partition λ as in Figure 8, this corresponds to the i-th and j-th steps down. Note that when $n = 0$ or $\frac{1}{2}$ we have $i = j$.

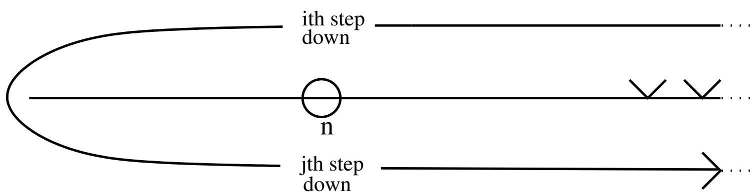

Figure 8

We want to show first that $n = \lambda_i - i + \frac{\delta}{2} = -(\lambda_j - j + \frac{\delta}{2})$. Now if we let k be the number of vertices strictly to the left of n, then we have that $k = n$ for δ even and $k = n - \frac{1}{2}$ for δ odd. Thus if we write $\delta = 2m$ or $2m+1$ for some $m \in \mathbb{Z}$ then this is equivalent to showing that

$$\lambda_i - i = k - m, \tag{7}$$

and

$$\lambda_j - j = \begin{cases} -k-m & \text{if } \delta = 2m \\ -k-m-1 & \text{if } \delta = 2m+1 \end{cases} \tag{8}$$

Now we have

$$\begin{aligned}\lambda_i &= \#(\text{steps to the left after the ith step down}) \\ &= \#(\times\text{'s left of } n) + \#(\vee\text{'s left of } n) + \#(\wedge\text{'s}) + \#(\times\text{'s}), \quad \text{and} \\ i &= \#(\text{steps down up to and including } i) \\ &= \#(\circ\text{'s right of } n) + \#(\wedge\text{'s right of } n) + 1\end{aligned}$$

So this gives

$$\begin{aligned}\lambda_i - i &= \#(\times\text{'s left of } n) + \#(\vee\text{'s left of } n) + \#(\wedge\text{'s left of } n) + \#(\circ\text{'s left of } n) \\ &\quad + \#(\times\text{'s }) - \#(\circ\text{'s right of } n) - \#(\wedge\text{'s right of } n) \\ &\quad + \#(\wedge\text{'s right of or at } n) - \#(\circ\text{'s left of } n) - 1 \\ &= k + \#(\times\text{'s}) - \#(\circ\text{'s}) \\ &= k - m \qquad \text{using Lemma 3.2.2.}\end{aligned}$$

This proves (7).
Similarly we have

$$\begin{aligned} \lambda_j &= \#(\text{steps to the left after the jth step down}) \\ &= \#(\times\text{'s right of } n) + \#(\wedge\text{'s right of } n) \quad \text{and} \\ j &= \#(\text{steps down up to and including } j) \\ &= \#(\circ\text{'s}) + \#(\wedge\text{'s }) + \#(\vee\text{'s left of } n) + \#(\circ\text{'s left of } n) + 1 - \delta_{\delta,2m} \end{aligned}$$

where the term -$\delta_{\delta,2m}$ cancels the double counting of the vertex indexed by 0 in the even δ case.
So this gives

$$\begin{aligned} \lambda_j - j &= -\#(\vee\text{'s left of } n) - \#(\circ\text{'s left of } n) - \#(\wedge\text{'s left of } n) - \#(\times\text{'s left of } n) \\ &\quad +\#(\times\text{'s }) - \#(\circ\text{'s }) - 1 + \delta_{\delta,2m} \\ &= \begin{cases} -k-m & \text{if } \delta = 2m \\ -k-m-1 & \text{if } \delta = 2m+1 \end{cases} \qquad \text{using Lemma 3.2.2} \end{aligned}$$

proving (8).
Moreover, we have $(i,j) \in [\lambda]$ if and only if $j \leq \lambda_i$, that is $\lambda_i - j \geq 0$. Now we get

$$\begin{aligned} \lambda_i - j &= \#(\times\text{'s}) - \#(\circ\text{'s}) + \#(\times\text{'s left of } n) - \#(\circ\text{'s left of } n) - \delta_{\delta,2m+1} \\ &= -m + \#(\times\text{'s left of } n) - \#(\circ\text{'s left of } n) - \delta_{\delta,2m+1} \qquad \text{using Lemma 3.2.2.} \end{aligned}$$

Hence we have that $\lambda_i - j \geq 0$ if and only if

$$\#(\times\text{'s left of } n) - \#(\circ\text{'s left of } n) > m - \delta_{\delta,2m}$$

as required. □

3.3. Cap diagrams and decomposition numbers. In this section we associate an oriented cap diagrams c_λ to any partition λ (and $\delta \in \mathbb{Z}$) as in [2]. Note that this is slight reformulation of the Temperley-Lieb half diagrams associated to λ given in [14], but we keep the information about the positions of the $\times$'s and $\circ$'s. A similar construction was also given in [12].

First, draw the vertices of x_λ on the horizontal edge of the NE quadrant of the plane. Now, in x_λ find a pair of vertices labelled $\vee$ and $\wedge$ in order from left to right that are neighbours in the sense that there are only separated by $\circ$s, $\times$s or vertices already joined by a cap. Join this pair of vertices together with a cap. Repeat this process until there are no more such $\vee$ $\wedge$ pairs. (This will occur after a finite number of steps.)

Ignoring all $\circ$s, $\times$s and vertices on a cap, we are left with a sequence of a finite number of $\wedge$s followed by an infinite number of $\vee$s. Starting from the leftmost $\wedge$, join each $\wedge$ to the next from the left which has not yet been used, by a cap touching the vertical boudary of the NE quadrant, without crossing any other caps. If there is a free $\wedge$ remaining at the end of this procedure, draw an infinite ray up from this vertex, and draw infinite rays from each of the remaining $\vee$s.

Examples of this construction are given in Figure 10.

Here we have drawn the 'curls' from [2] as caps touching the edge of the NE quadrant, as this is better suited for the combinatorics introduced in Section 3.

By [14], we have that the decomposition numbers $D_{\lambda\mu} = [\Delta_n(\mu) : L_n(\lambda)] = (P_n(\lambda) : \Delta_n(\mu))$ can be described using these cap diagrams as follows. Define the polynomial $d_{\lambda\mu}(q)$ by setting $d_{\lambda\mu}(q) \neq 0$ if and only if x_μ is obtained from x_λ by changing the labellings of the elements in some of the pairs of vertices joined by a cap in c_λ from $\vee\wedge$ to $\wedge\vee$ or from $\wedge\wedge$ to $\vee\vee$. In that case define $d_{\lambda\mu}(q) = q^k$ where k is the number of pairs whose labellings have been changed. Then we have

$$D_{\lambda\mu} = d_{\lambda\mu}(1). \tag{9}$$

4. Restriction of simple modules

Here we describe completely the module structure of the restriction of $L_n(\lambda)$ to $B_{n-1}(\delta)$ for any $\lambda \in \Lambda_n$. For $\lambda \in \Lambda$ we define $\mathrm{supp}(\lambda) = \{\mu \in \Lambda \mid \mu \rhd \lambda \text{ or } \mu \lhd \lambda\}$. Recall $\mathcal{B}(\mu)$ is the block containing $\mu \in \Lambda$ (as defined in Section 3.1). We define pr^μ to be the functor projecting onto $\mathcal{B}(\mu)$ and write res_n^μ for $\mathrm{pr}^\mu \circ \mathrm{res}_n$. We have that $\mathrm{res}_n L_n(\lambda)$ decomposes as

$$\mathrm{res}_n L_n(\lambda) = \bigoplus_{\mathcal{B}(\mu)} \mathrm{res}_n^\mu L_n(\lambda),$$

and using (2) the direct sum can be taken over all blocks $\mathcal{B}(\mu)$ with $\mathcal{B}(\mu) \cap \mathrm{supp}(\lambda) \neq \emptyset$. Thus it is enough to describe $\mathrm{res}_n^\mu L_n(\lambda)$ for each $\mu \in \mathrm{supp}(\lambda) \cap \Lambda_{n-1}$. We have three cases to consider depending on the relative degree of singularity of λ and μ as in Lemma 3.2.1.
Case I: $\deg_\delta(\mu) = \deg_\delta(\lambda)$.
Case II: $\deg_\delta(\mu) = \deg_\delta(\lambda) + 1$.
Case III: $\deg_\delta(\mu) = \deg_\delta(\lambda) - 1$.

Case I has been dealt with in [5]. We state the result here for completeness.

Proposition 4.0.1. [5]*(Proposition 4.1) If $\lambda \in \Lambda_n$ and $\mu \in \mathrm{supp}(\lambda) \cap \Lambda_{n-1}$ with $\deg_\delta(\mu) = \deg_\delta(\lambda)$ then we have*

$$\mathrm{res}_n^\mu L_n(\lambda) = L_{n-1}(\mu).$$

It will be convenient to define a new notation here for cases II and III. Suppose that $\lambda' \in \mathrm{supp}(\lambda)$ with $\deg_\delta(\lambda') = \deg_\delta(\lambda) + 1$. Then it's easy to see from Lemma 3.2.1 and the description of blocks given in Section 3.1 that

$$\mathrm{supp}(\lambda') \cap \mathcal{B}(\lambda) = \{\lambda^+, \lambda^-\}$$

with one of λ^+ or λ^- being equal to λ. We can also assume that $\lambda^+ > \lambda^-$. Moreover we have that the weight diagrams of λ', λ^+ and λ^- differ in precisely two adjacent vertices, say $i-1$ and i as depicted in Figure 9.

In Figures 10–18 we will always assume that the weight diagram of λ' has labels $\circ\ \times$ in positions $i-1$,i. The other case is exactly the same.

Using the above notation we can now state the result for Case II as follows.

Proposition 4.0.2. [5]*(Theorem 4.8) Let $\lambda', \lambda^+, \lambda^-$ be as above then we have*

$$\begin{aligned} &(i) \quad \mathrm{res}_n^{\lambda'} L_n(\lambda^+) = L_{n-1}(\lambda'), \quad \textit{and} \\ &(ii) \quad \mathrm{res}_n^{\lambda'} L_n(\lambda^-) = 0. \end{aligned}$$

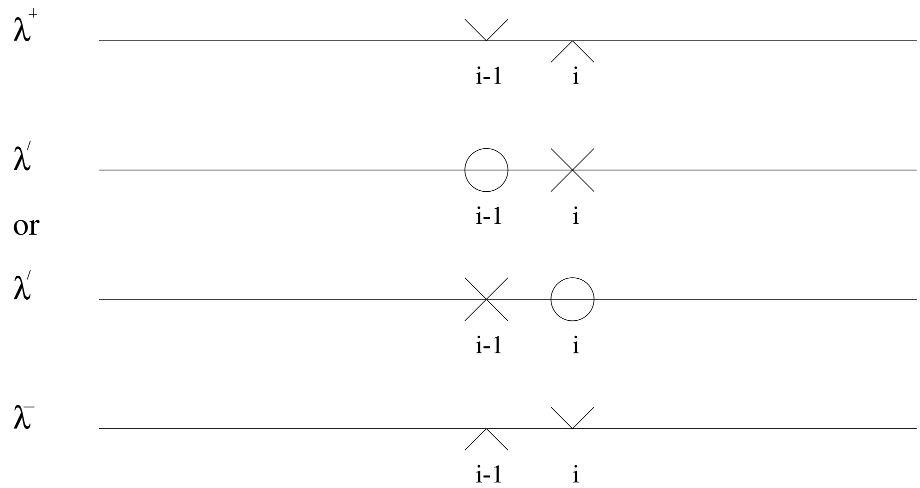

FIGURE 9. Weight diagrams corresponding to λ', λ^- and λ^+.

Keeping the same notation, for Case III we need to describe $\mathrm{res}_n^{\lambda^+} L_n(\lambda')$. (Note that as λ^+ and λ^- are in the same block we have $\mathrm{res}_n^{\lambda^-} L_n(\lambda') = \mathrm{res}_n^{\lambda^+} L_n(\lambda')$.) This is more complicated than the previous two cases. Indeed we will see that the number of composition factors can get arbitrarily large (as n varies).

Note that for any $\mu' \in \mathcal{B}(\lambda')$ we have $\mathrm{supp}(\mu') \cap \mathcal{B}(\lambda^+) = \{\mu^+, \mu^-\}$ and $\mathrm{supp}(\mu^\pm) \cap \mathcal{B}(\lambda') = \{\mu'\}$. The weight diagrams of μ', μ^+ and μ^- differ in precisely vertices $i-1$ and i and these two vertices are labelled as in λ', λ^+ and λ^- respectively.

We now recall a result from [14](proof of (7.7)) which will be needed in the proof of the next theorem.

Proposition 4.0.3. *Let $\lambda', \lambda^+, \lambda^-$ and μ', μ^+, μ^- be as above. Suppose $\mu' \in \Lambda_n$. Then we have*

$$\begin{aligned} &\mathrm{ind}_{n-1}^{\lambda'} P_{n-1}(\mu^-) \cong P_n(\mu'), \qquad \textit{and} \\ &\mathrm{ind}_{n-1}^{\lambda'} P_{n-1}(\mu^+) \cong 2P_n(\mu'). \end{aligned}$$

Note that the cap diagram associated to a partition splits the NE quadrant of the plane into open connected components, called chambers. We say that a vertex, a cap, or a ray belongs to a chamber C if it is in the closure of C. Note that each vertex labelled with $\times$ or $\circ$ belongs to precisely one chamber and each vertex labelled $\vee$ or $\wedge$ belongs to precisely two chambers. In $x_{\lambda'}$, the vertex i is labelled with $\times$ or $\circ$, so it belongs to a unique chamber C_i in the cap diagram $c_{\lambda'}$.

Now we define the subset $I(\lambda', \lambda^+)$ of the set of vertices of $c_{\lambda'}$ by setting $j \in I(\lambda', \lambda^+)$ if and only if j belongs to C_i and one of the following three possibilities holds

$$j > i \text{ and } j \text{ is labelled with } \vee, \tag{10}$$

$$j < i \text{ and } j \text{ is labelled with } \wedge, \text{ or} \tag{11}$$

$$j < i \text{ and } j \text{ is labelled with } \vee \text{ and it is either on a ray or connected to some } k > i. \tag{12}$$

Examples of all $j \in I(\lambda', \lambda^+)$ for various λ' are given in Figure 10.

For each $j \in I(\lambda', \lambda^+)$, define $\lambda'_{(j)}$ as follows. If j satisfies (10) above then $\lambda'_{(j)}$ is the partition whose weight diagram is obtained from $x_{\lambda'}$ by labelling vertex j with $\wedge$, and vertices $i-1$ and i with $\vee$, leaving everything else unchanged. If j satisfies case (11) above then $\lambda'_{(j)}$ is the partition whose weight diagram is obtained from $x_{\lambda'}$ by labelling vertex j with $\vee$, and vertices $i-1$ and i with $\wedge$, leaving everything else unchanged. Finally, if j satisfies case (12) then $\lambda'_{(j)}$ is the partition whose weight diagram is obtained from $x_{\lambda'}$ by labelling vertex j, i and $i-1$ with $\wedge$, leaving everything else unchanged.

Now define

$$\Lambda_{n-1}(\lambda', \lambda^+) = \{\lambda'_{(j)} \; : \; j \in I(\lambda', \lambda^+)\} \cap \Lambda_{n-1}.$$

Theorem 4.0.4. *Let $\lambda', \lambda^+, \lambda^-$ be as above.*
If $|\lambda'| = n$ then we have $\operatorname{res}_n^{\lambda^+} L_n(\lambda') = L_{n-1}(\lambda^-)$.
If $|\lambda'| < n$ then we have that $\operatorname{res}_n^{\lambda^+} L_n(\lambda')$ has simple head and simple socle isomorphic to $L_{n-1}(\lambda^+)$ and we have

$$\operatorname{rad}(\operatorname{res}_n^{\lambda^+} L_n(\lambda'))/\operatorname{soc}(\operatorname{res}_n^{\lambda^+} L_n(\lambda')) \;\cong\; L_{n-1}(\lambda^-) \oplus \bigoplus_{\mu \in \Lambda_{n-1}(\lambda', \lambda^+)} L_{n-1}(\mu)$$

Proof. If $|\lambda'| = n$ we have $L_n(\lambda') = \Delta_n(\lambda')$ and $L_{n-1}(\lambda^-) = \Delta_{n-1}(\lambda^-)$ so the result follows immediately from the exact sequence given in (2).
Now suppose that $|\lambda'| < n$. Let us start by finding the composition factors of this module. Note that for any $\mu \in \mathcal{B}(\lambda^+)$ we have

$$\begin{aligned} [\operatorname{res}_n^{\lambda^+} L_n(\lambda') : L_{n-1}(\mu)] &= \dim \operatorname{Hom}_{n-1}(P_{n-1}(\mu), \operatorname{res}_n^{\lambda^+} L_n(\lambda')) \\ &= \dim \operatorname{Hom}_n(\operatorname{ind}_{n-1}^{\lambda'} P_{n-1}(\mu), L_n(\lambda')). \end{aligned}$$

If $\operatorname{Hom}_n(\operatorname{ind}_{n-1}^{\lambda'} P_{n-1}(\mu), L_n(\lambda')) \neq 0$, then $P_n(\lambda')$ must be a summand of $\operatorname{ind}_{n-1}^{\lambda'} P_{n-1}(\mu)$. So we must have $(P_{n-1}(\mu) : \Delta_{n-1}(\lambda^+)) \neq 0$ or $(P_{n-1}(\mu) : \Delta_{n-1}(\lambda^-)) \neq 0$ using (3). The Δ-factors of the projective module $P_{n-1}(\mu)$ are given by (9). As $\mu \in \mathcal{B}(\lambda^+)$, using the block description on weight diagrams given in Section 2.1, we have that the vertices $i-1$ and i in x_μ are labelled by $\vee$ or $\wedge$. Note that we must also have either $\mu \geq \lambda^+$ or $\mu \geq \lambda^-$. We now have four cases to consider, depending on the labellings of $i-1$ and i.

Case A. In x_μ, vertices $i-1$ and i are labelled by $\vee$ and $\wedge$ respectively. Here μ is of the from μ^+ and so from Proposition 4.0.3 we have $\operatorname{ind}_{n-1}^{\lambda'} P_{n-1}(\mu) = 2P_n(\mu')$. Thus we must have $\mu = \lambda^+$ and we have

$$\dim \operatorname{Hom}_n(\operatorname{ind}_{n-1}^{\lambda'} P_{n-1}(\lambda^+), L_n(\lambda')) = 2.$$

Case B. In x_μ, vertices $i-1$ and i are labelled by $\wedge$ and $\vee$ respectively. Here μ is of the form μ^- and so, by Proposition 4.0.3 we have $\operatorname{ind}_{n-1}^{\lambda'} P_{n-1}(\mu) = P_n(\mu')$.Thus we must have $\mu = \lambda^-$ and we have

$$\dim \operatorname{Hom}_n(\operatorname{ind}_{n-1}^{\lambda'} P_{n-1}(\lambda^-), L_n(\lambda')) = 1.$$

Case C. In x_μ, both vertices $i-1$ and i are labelled by $\vee$. There are two subcases C(i) and C(ii) to consider here depending on the cap diagram c_μ. First consider the case C(i) as depicted in Figure 11.

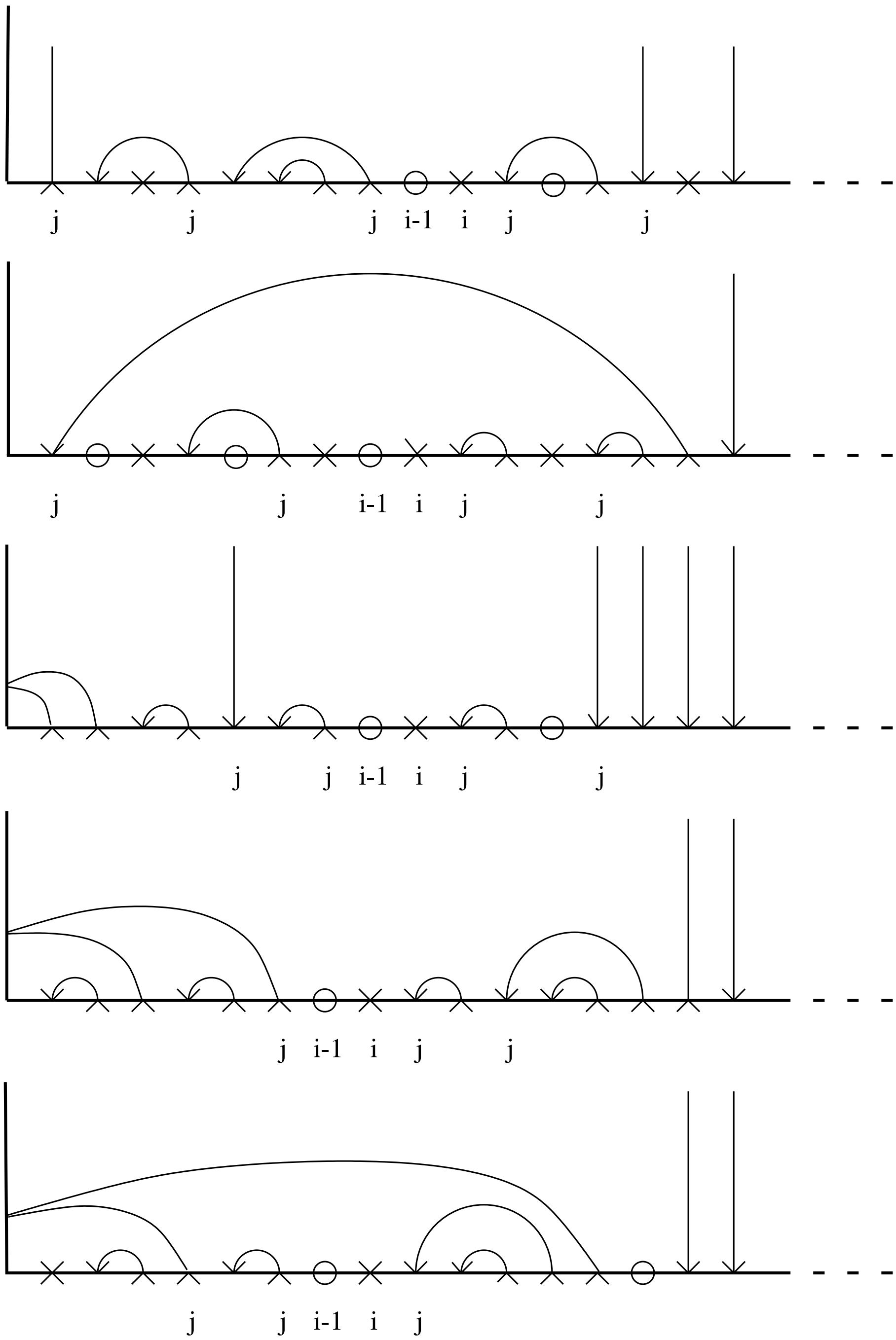

FIGURE 10. Examples of cap diagrams associated to λ' with the set of vertices $j \in I(\lambda', \lambda^+)$

Using (9) we have that $(P_{n-1}(\mu) : \Delta_{n-1}(\lambda^-)) = 0$ and if $(P_{n-1}(\mu) : \Delta_{n-1}(\lambda^+)) \neq 0$ then x_{λ^+} is obtained from x_μ by swapping the labelling of vertices i and j and possibly other pairs connected by a cap in c_μ. Now when we apply the functor $\mathrm{ind}_{n-1}^{\lambda'}$ to $P_{n-1}(\mu)$, it follows from (3) that all its Δ-factors corresponding to weight diagrams with the labels of i and j as in x_μ (namely $\vee$ and $\wedge$ resp.) will go to zero. Let η be the partition obtained from μ by

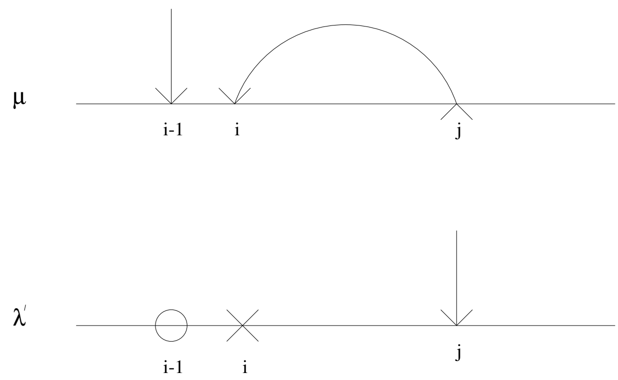

FIGURE 11. Case C(i)

swapping the labels on vertices i and j and leaving everything else unchanged. Note that η is of the form η^+ and it is the largest Δ-factor not annihilated by $\mathrm{ind}_{n-1}^{\lambda'}$. Now it's easy to see from (9) that there is a one-to-one correspondence between the Δ-factors of $P_{n-1}(\mu)$ with the labels of i and j swapped (that is, those not annihilated by the induction functor) and the Δ-factors of $P_n(\eta')$. This implies that

$$\mathrm{ind}_{n-1}^{\lambda'} P_{n-1}(\mu) = P_n(\eta').$$

Thus we must have $\eta' = \lambda'$ with λ' as in Figure 11, where vertices i and j are in the closure of the same chamber. Thus for each such μ we have $\dim \mathrm{Hom}_n(\mathrm{ind}_{n-1}^{\lambda'} P_{n-1}(\mu), L_n(\lambda')) = 1$.

Now consider the case C(ii) with μ as depicted in Figure 12.

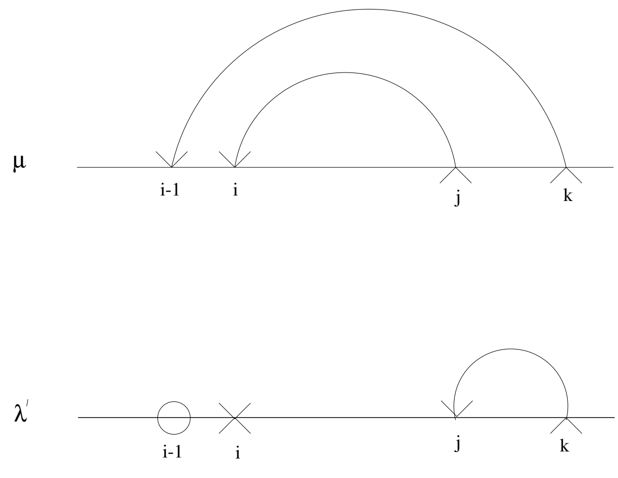

FIGURE 12. Case C(ii)

If $(P_{n-1}(\mu) : \Delta_{n-1}(\lambda^\pm)) \neq 0$ then $\lambda^\pm$ is obtained from μ by swapping the labelling of vertices i and j for λ^+ and of vertices $i-1$ and k for λ^- and possibly other pairs connected by a cap as well. Now when we apply the functor $\mathrm{ind}_{n-1}^{\lambda'}$ to $P_{n-1}(\mu)$, it follows from (3) that all its Δ-factors corresponding to diagrams with the labels of i and j and the labels of $i-1$ and k are either both swapped or both unchanged will go to zero. Let η be the partition whose weight diagram is obtained from x_μ by swapping the labels on vertices i and j and leaving everything else unchanged. Note that η is of the form η^+ and it is the largest Δ-factor not annihilated by $\mathrm{ind}_{n-1}^{\lambda'}$. Now it's easy to see that there is a one-to-one correspondence between the Δ-factors of $P_{n-1}(\mu)$ not annihilated by the induction functor and the Δ-factors of $P_n(\eta')$. This implies that

$$\mathrm{ind}_{n-1}^{\lambda'} P_{n-1}(\mu) = P_n(\eta').$$

Thus we must have $\eta' = \lambda'$ with λ' as depicted in Figure 12, where vertices i and j are in the closure of the same chamber. For each such μ we have $\dim \mathrm{Hom}_n(\mathrm{ind}_{n-1}^{\lambda'} P_{n-1}(\mu), L_n(\lambda')) = 1$.

We have seen that Case C covers all $j \in I(\lambda', \lambda^+)$ satisfying (10).

Case D. In x_μ both vertices $i-1$ and i are labelled by $\wedge$. This case splits into six subcases D(i)-(vi) as depicted in Figures 13-18. Using the same argument as in Case C, it is easy to show that in each case we have $\dim \mathrm{Hom}_n(\mathrm{ind}_{n-1}^{\lambda'} P_{n-1}(\mu), L_n(\lambda')) = 1$. Note that in all cases the vertices i and j in λ' must be in the same chamber otherwise x_μ wouldn't have the required cap diagram. We have that cases D(i)(iii)-(v) correspond to all vertices j satisfying (11), and Cases D(ii) and (vi) correspond to all vertices j satisfying (12).

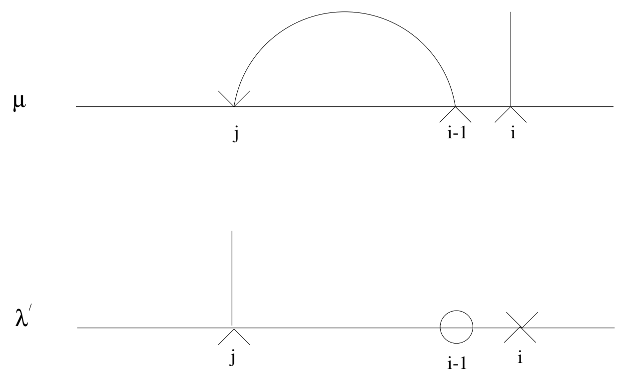

FIGURE 13. Case D(i)

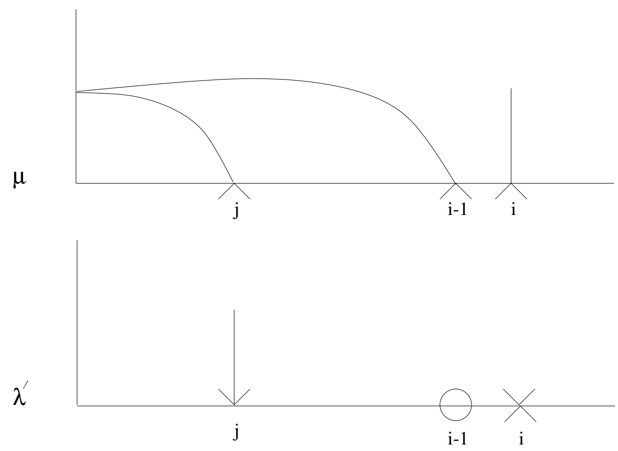

FIGURE 14. Case D(ii)

Finally, using the fact that all simple modules are self-dual, the restriction must have the required module structure. □

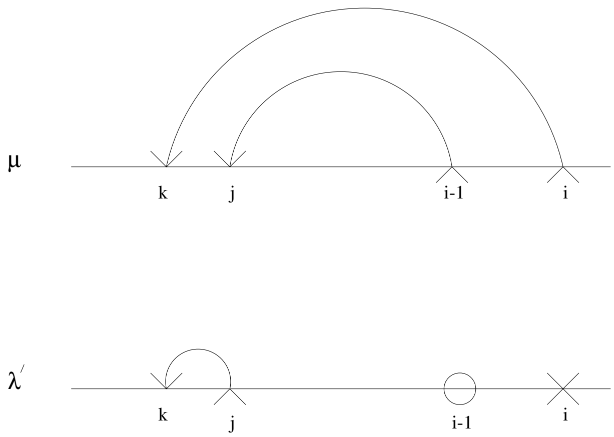

FIGURE 15. Case D(iii)

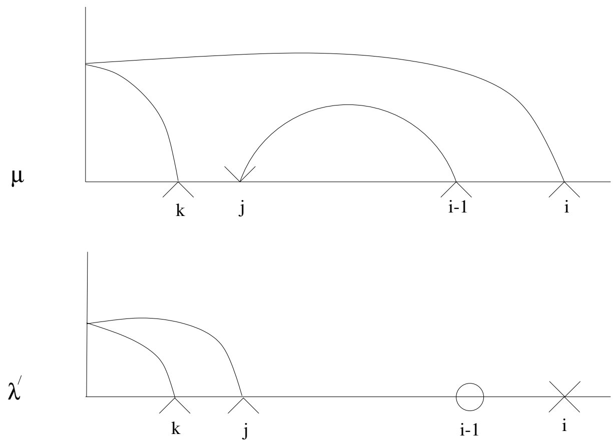

FIGURE 16. Case D(iv)

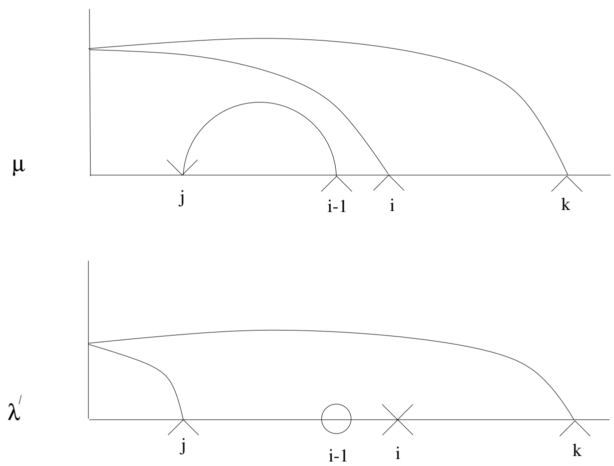

FIGURE 17. Case D(v)

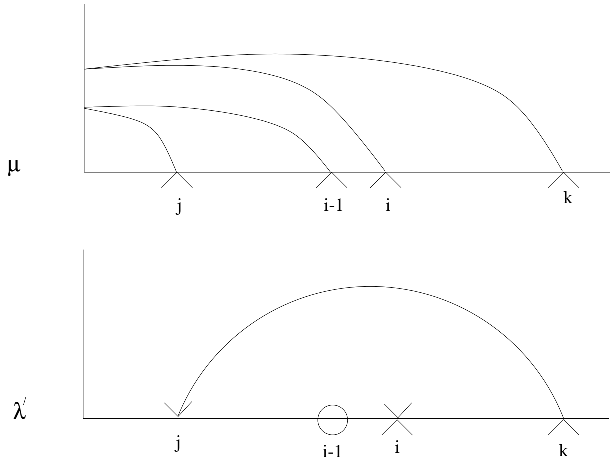

Figure 18. Case D(vi)

Corollary 4.0.5. *Let $\lambda \in A_\delta$. Then the dimension of $L_n(\lambda)$ is given by the number of walks on $\mathcal{Y}_\delta$ from $\emptyset$ to λ.*

Proof. This follows immediately by induction on n using Propositions 4.0.1 and 4.0.2. $\square$

5. Walk bases for simple modules

5.1. Leduc-Ram walk bases for generic simple modules. In this section we recall the construction given in [11] of walk bases for simple modules for the generic Brauer algebra $B_n(u)$, where u is an indeterminate and the algebra is defined over $\mathbb{C}(u)$. Their construction uses two combinatorial objects associated with partitions which we now recall. We start with the King polynomials, which were originally derived from Weyl's character formula in [7].

Let λ be a partition, and denote by $[\lambda]$ its Young diagram. For each box $(i,j) \in [\lambda]$ we define

$$d(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 1 & \text{if } i \leq j \\ -\lambda_i^T - \lambda_j^T + i + j - 1 & \text{if } i > j \end{cases}$$

We also write $h(i,j)$ for the usual hook length. We then define the King polynomial

$$P_\lambda(u) = \prod_{(i,j)\in[\lambda]} \frac{u - 1 + d(i,j)}{h(i,j)}.$$

For example, $P_\emptyset(u) = 1$, $P_{(1)}(u) = u$,

$$P_{(1^2)}(u) = \frac{u(u-1)}{2}, P_{(2)}(u) = \frac{(u+2)(u-1)}{2},$$

$$P_{(1^3)}(u) = \frac{u(u-2)(u-1)}{3!}, P_{(2,1)}(u) = \frac{(u+2)u(u-2)}{3}, P_{(3)}(u) = \frac{(u+4)u(u-1)}{3!},$$

$$P_{(1^4)}(u) = \frac{u(u-3)(u-2)(u-1)}{4!}, P_{(2,1^2)}(u) = \frac{(u+2)u(u-3)(u-1)}{4.2}, \ldots$$

We denote the set of all walks on the Young graph $\mathcal{Y}$ by Ω and the subset of all walks of length n starting at $\emptyset$ and ending at λ by $\Omega^n(\lambda)$. For a walk $S \in \Omega$, we write $S = (s(0), s(1), s(2), ...)$ where $s(m)$ is the m-th partition in the walk S. We then define $\Omega_m(S)$ to be the set of all walks T that differs from S in at most position m, that is $t(j) = s(j)$ for all $j \neq m$. If $T \in \Omega_m(S)$ we say that (S, T) form an m-diamond pair, and in this case we define the Brauer diamond $\diamond_m(S,T) \in \mathbb{Z}[u]$ by

$$\diamond_m(S,T) = \begin{cases} \pm(s(m+1)_k - k - t(m)_l + l) & \text{if } t(m) = s(m-1) \pm \epsilon_l \\ & \text{and } s(m+1) = s(m) \pm \epsilon_k \\ & \\ \pm(u + t(m)_l - l + s(m+1)_k - k) & \text{if } t(m) = s(m-1) \mp \epsilon_l \\ & \text{and } s(m+1) = s(m) \pm \epsilon_k \end{cases}$$

Theorem 5.1.1. [11]*(6.22) There is an action of the generic Brauer algebra $B_n(u)$ on the $\mathbb{C}(u)$-vector space Π^λ with basis $\Omega^n(\lambda)$ given by*

$$\sigma_m T = \sum_{S \in \Omega_m(T)} (\sigma_m(u))_{ST} S$$

$$e_m T = \sum_{S \in \Omega_m(T)} (e_m(u))_{ST} S$$

where

$$(\sigma_m(u))_{SS} = \begin{cases} \frac{1}{\diamond_m(S,S)} & \text{if } s(m-1) \neq s(m+1) \\ \frac{1}{\diamond_m(S,S)} \left(1 - \frac{P_{s(m)}(u)}{P_{s(m-1)}(u)}\right) & \text{otherwise} \end{cases}$$

and for $S \neq T$

$$(\sigma_m(u))_{ST} = \begin{cases} \sqrt{\frac{(\diamond_m(S,S)-1)(\diamond_m(S,S)+1)}{\diamond_m(S,S)^2}} & \text{if } s(m-1) \neq s(m+1) \\ -\frac{1}{\diamond_m(S,T)} \left(\frac{\sqrt{P_{s(m)}(u) P_{t(m)}(u)}}{P_{s(m-1)}(u)}\right) & \text{otherwise} \end{cases},$$

and similarly for any S, T

$$(e_m(u))_{ST} = \begin{cases} \frac{\sqrt{P_{s(m)}(u) P_{t(m)}(u)}}{P_{s(m-1)}(u)} & \text{if } s(m-1) = s(m+1) \\ 0 & \text{otherwise} \end{cases}$$

We will give a geometric interpretation of the King polynomials $P_\lambda(u)$ and the Brauer diamonds $\diamond_m(S,T)$ in the next two sections. This will allow us to define an action of the Brauer algebra $B_n(\delta)$ on the $\mathbb{C}$- span of all δ-restricted walks in $\Omega^n(\lambda)$.

5.2. A geometric interpretation of the roots of the King polynomials. Recall the definition of the δ-degree of singularity of a partition given in Definition 2.4.1.

Theorem 5.2.1. *Fix $\delta \in \mathbb{Z}$ and let $\lambda \in \Lambda$. Let $m_\delta(\lambda)$ be the multiplicity of δ as a root of the King polynomial $P_\lambda(u)$. Then we have*

$$m_\delta(\lambda) = \deg_\delta(\lambda) - \deg_\delta(\emptyset).$$

In particular, we have that $P_\lambda(\delta) \neq 0$ if and only if λ is δ-regular.

Proof. Write $a = \min\{\#(\circ\text{'s in } x_\lambda), \#(\times\text{'s in } x_\lambda)\}$ and let $\delta = 2m$ or $2m+1$ for some $m \in \mathbb{Z}$. Using Example 3.1.1 and Lemma 3.2.2 we have that for $m \geq 0$,

$$a = \#(\times\text{'s in } x_\lambda) = \deg_\delta(\lambda) = \deg_\delta(\lambda) - \deg_\delta(\emptyset)$$

as $\deg_\delta(\emptyset) = 0$; and for $m < 0$ we have

$$\begin{aligned} a &= \#(\circ\text{'s in } x_\lambda) \\ &= \#(\circ\text{'s in } x_\lambda) - \#(\times\text{'s in } x_\lambda) + \#(\times\text{'s in } x_\lambda) \\ &= m + \#(\times\text{'s in } x_\lambda) \\ &= \#(\times\text{'s in } x_\lambda) - \#(\times\text{'s in } x_\emptyset) \\ &= \deg_\delta(\lambda) - \deg_\delta(\emptyset). \end{aligned}$$

Thus it's enough to show that $m_\delta(\lambda) = a$.
Now by definition of $P_\lambda(u)$ and Proposition 3.2.3 we have that $m_\delta(\lambda)$ is precisely the number of $\times$'s in x_λ satisfying (5) added to the number of $\circ$'s in x_λ satisfying (6). We can represent the sequence of $\times$'s and $\circ$'s appearing in x_λ reading from left to right by a graph as follows. Start at $(0,0)$ and for each term in the sequence add $(1,0)$ if it is a $\circ$, or add $(0,1)$ if it is a $\times$. The graph is given in Figure 19 for $m \geq 0$ and in Figure 20 for $m < 0$.

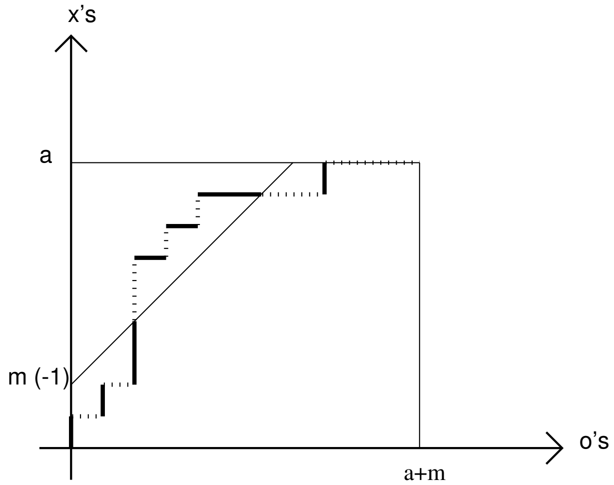

FIGURE 19. Graph representing the sequence of $\times$ and $\circ$ in x_λ for $m \geq 0$

Now observe that the admissibility conditions (5) and (6) can be rephrased as follows. A $\times$ (resp. $\circ$) satisfies (5) (resp. (6)) if and only if the corresponding step in the graph is below (resp. above) the line $y = x + m - \delta_{\delta,2m}$. Admissible (resp. non-admissible) steps are represented by solid lines (resp. dotted lines) in the graphs. It follows immediately that the total number of admissible $\times$'s and $\circ$'s is equal to a.

□

Remark 5.2.2. It was shown in [17, Corollary (3.5)] that $\lambda \in A_\delta$ if and only if $P_\mu(\delta) \neq 0$ for all $\mu \subseteq \lambda$. Theorem 5.2.1 strengthens this result to give a full characterisation of the singularities of the King polynomial in terms of the δ-degree of singularity.

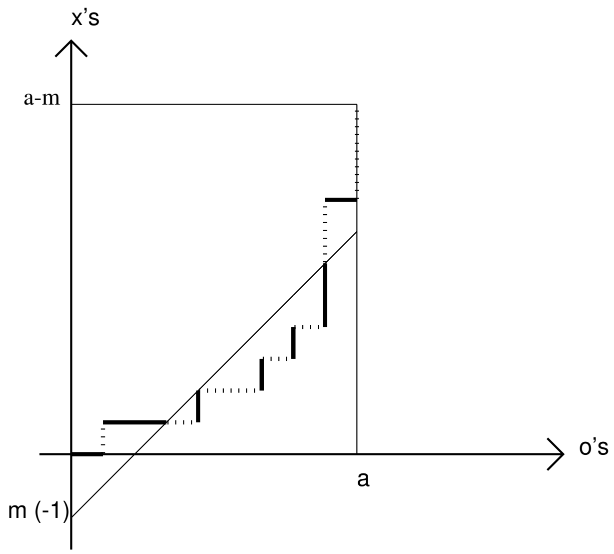

FIGURE 20. Graph representing the sequence of $\times$ and $\circ$ in x_λ for $m < 0$

5.3. A geometric interpretation of the Brauer diamonds. In this section we give a geometric interpretation of the Brauer diamonds when we specialise $u = \delta$.

Recall the isomorphism between the Young graph $\mathcal{Y}$ and $\mathcal{Z}_+(\rho_\delta)$ given in Section 2.2. Using this we will view walks on $\mathcal{Y}$ as walks in $\mathcal{Z}_+(\rho_\delta)$ where each edge is of the form $x \to x \pm \epsilon_i$ for some $x \in \mathbb{R}^{\mathbb{N}}$ and some $i \geq 1$.

Let (S,T) be an m-diamond pair. The Brauer diamond only depends on the $m-1$, m and $m+1$ steps in the walks, so we will write $S = (x(m-1), x(m), x(m+1))$ and $T = (x(m-1), y(m), x(m+1))$, where the $x(i)$'s and $y(i)$'s are in $\mathbb{R}^{\mathbb{N}}$.

Theorem 5.3.1. *The Brauer diamonds satisfy the following identities.*
Case 1. *If* $S = (x, x \pm \epsilon_i, x \pm \epsilon_i \pm \epsilon_j)$*,* $T = (x, x \pm \epsilon_j, x \pm \epsilon_i \pm \epsilon_j)$*,*
then we have for $i \neq j$

$$\diamond(S,T) = \diamond(T,S) = 0$$
$$\diamond(S,S) = -\diamond(T,T) = \langle x, \epsilon_i - \epsilon_j \rangle,$$

and for $i = j$ *we have*

$$\diamond(S,S) = -1.$$

Case 2. *If* $S = (x, x \pm \epsilon_i, x \pm \epsilon_i \mp \epsilon_j)$*,* $T = (x, x \mp \epsilon_j, x \pm \epsilon_i \mp \epsilon_j)$ *with* $i \neq j$*,*
then we have

$$\diamond(S,T) = \diamond(T,S) = 0$$
$$\diamond(S,S) = -\diamond(T,T) = \pm\langle x, \epsilon_i + \epsilon_j \rangle.$$

Case 3. *If* $S = (x, x + \alpha, x)$*,* $T = (x, x + \beta, x)$ *with* $\alpha, \beta \in \{\pm\epsilon_i \,:\, i \geq 1\}$*,*
then we have

$$\diamond(S,T) = \diamond(T,S) = \langle x, \alpha + \beta \rangle + 1$$

Proof. We will give a proof for (half of) Case 1, the other cases can be computed similarly. Going back to the original definition, we consider the diamond pair in $\mathcal{Y}$ given by $S = (\lambda, \lambda + \epsilon_k, \lambda + \epsilon_k + \epsilon_l)$, $T = (\lambda, \lambda + \epsilon_l, \lambda + \epsilon_k + \epsilon_l)$. Suppose that the box ϵ_k is added in column

i in the Young diagram (that is $\lambda_k+1=i$) and the box ϵ_l is added in column j in the Young diagram (that is $(\lambda+\epsilon_k)_l+1=j$). Note that for $S\neq T$ we must have $k\neq l$ and $i\neq j$. In this case we have

$$\diamond(S,T)=(\lambda_l+1)-l-(\lambda_l+1)+l=0,$$

$$\diamond(T,S)=(\lambda_k+1)-k-(\lambda_k+1)+k=0$$

and

$$\begin{aligned}\diamond(S,S) &= (\lambda_l+1)-l-(\lambda_k+1)+k\\ &= j-(\lambda'_j-1)-i+(\lambda'_i-1)\\ &= (\lambda'_i-\frac{\delta}{2}-i+1)-(\lambda'_j-\frac{\delta}{2}-j+1)\\ &= \langle e_\delta(\lambda),\epsilon_i-\epsilon_j\rangle\\ &= -\diamond(T,T).\end{aligned}$$

If $i=j$, then $l=k+1$ and we have

$$\diamond(S,S)=(\lambda_{k+1}+1)-(k+1)-(\lambda_k+1)+k=-1.$$

Finally, if $k=l$ then $j=i+1$ and we have

$$\begin{aligned}\diamond(S,S) &= (\lambda_k+2)-k-(\lambda_k+1)+k\\ &= 1\\ &= (\lambda'_i-\frac{\delta}{2}-i+1)-(\lambda'_{i+1}-\frac{\delta}{2}-(i+1)+1)\\ &= \langle e_\delta(\lambda),\epsilon_i-\epsilon_{i+1}\rangle.\end{aligned}$$

□

5.4. Walk bases for δ-restricted simple modules. Recall the definition of the set of δ-restricted partitions A_δ and the δ-restricted Young graph $\mathcal{Y}_\delta$ given in Section 2.2.

Note that the matrix entries defining the representation Π^λ of $B_n(u)$ given in Theorem 5.1.1 do not in general specialise to $u=\delta$. However we will show that, for any $\lambda\in A_\delta$, if we only consider the submatrices with entries labelled by δ-restricted walks, then these do specialise to give a representation of $B_n(\delta)$ which is isomorphic to $L_n(\lambda)$. (Note that these do not correspond to quotients or submodules for $B_n(u)$.)

We start by giving an explicit description of A_δ.

Proposition 5.4.1. *A partition λ belongs to A_δ if and only if one of the following conditions holds.*
(i) $\delta\geq 0$ and $\lambda_1^T+\lambda_2^T\leq\delta$.
(ii) $\delta=-2m$ (for some $m\in\mathbb{N}$) and $\lambda_1\leq m$.
(iii) $\delta=-2m+1$ (for some $m\in\mathbb{N}$) and $\lambda_1+\lambda_2\leq 2m+1$.

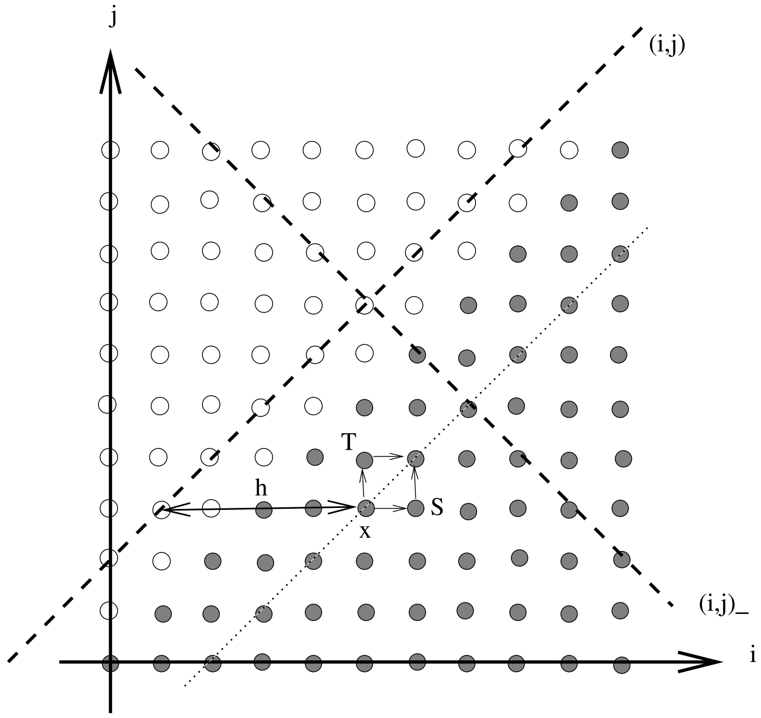

FIGURE 21. **Case 1**: $S = (x, x+\epsilon_i, x+\epsilon_i+\epsilon_j)$, $T = (x, x+\epsilon_j, x+\epsilon_i+\epsilon_j)$ with $i < j$ and $h = \langle x, \epsilon_i - \epsilon_j \rangle$. Projection of $\mathbb{R}^{\mathbb{N}}$ onto the ij-plane, showing the reflection hyperplane. Fibres containing partitions are shaded.

Proof. For $\delta = 2m$ or $2m+1$, the weight diagram $x_\emptyset$ consists of m $\circ$'s (for $m \geq 0$) or m $\times$ for $m < 0$ followed by infinitely many $\vee$'s (see Figure 3). Moreover, all possible configurations of translation equivalent weight diagrams are given in Lemma 3.2.1 (i)-(v). It follows that the weight diagrams corresponding to partitions in A_δ are precisely those having m $\circ$ (for $m \geq 0$) or m $\times$ (for $m < 0$) and with the other vertices either all labelled by $\vee$'s, or labelled by one $\wedge$ and infinitely many $\vee$'s, in that order. The result then follows from the end of Section 2.2 (see Figure 4 and 5). $\square$

Remark 5.4.2. Proposition 5.4.1 also follows by combining [17] (definition before Theorem (3.4) and Corollary (3.5)(b)) with Theorem 5.2.1.

Theorem 5.4.3. [15, Theorem 2.4(b)] *Let $\lambda \in A_\delta$. Then there is an action of $B_n(\delta)$ on the vector space spanned by all walks of length n on $\mathcal{Y}_\delta$ from $\emptyset$ to λ. This module is isomorphic to $L_n(\lambda)$.*

Proof. We consider the action of the generic Brauer algebra $B_n(u)$ on the Leduc-Ram representations Π^λ and claim that the truncation of this action to δ-restricted walks gives a well-defined representation by setting $u = \delta$. It suffices to show that:
(A) all matrix entries $(\sigma_m(u))_{ST}, (e_m(u))_{ST}$ where at least one of S or T are δ-restricted walks do not have a pole at $u = \delta$,
(B) the matrix entries $(\sigma_m(\delta))_{ST}, (e_m(\delta))_{ST}$, where precisely one of S or T is δ-restricted and the other is not, vanish.
(C) the submatrices $(\sigma_m(\delta))_{ST}$ and $(e_m(\delta))_{ST}$ formed by taking all δ-restricted walks S and

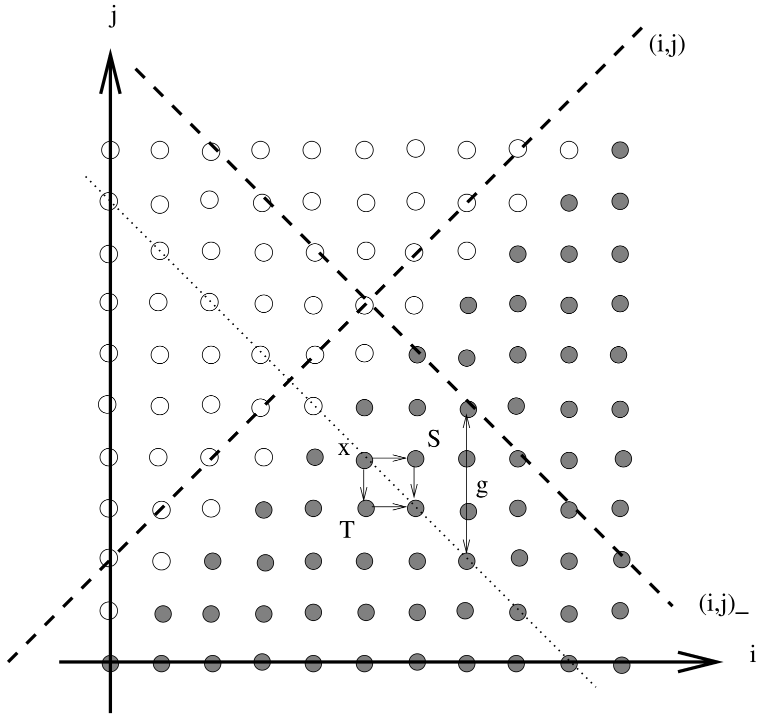

FIGURE 22. **Case 2**: $S = (x, x+\epsilon_i, x+\epsilon_i-\epsilon_j)$, $T = (x, x-\epsilon_j, x+\epsilon_i-\epsilon_j)$ with $i<j$ and $g = \langle x, \epsilon_i+\epsilon_j\rangle$

T satisfy the relations (R1)–(R9):

$$\left.\begin{array}{l}(R1)\ \sigma_i^2 = 1\\ (R2)\ e_i\sigma_i = \sigma_i e_i = e_i\\ (R3)\ e_i^2 = \delta e_i\end{array}\right\}\text{ for all } 1\le i\le n-1,$$

$$\left.\begin{array}{l}(R4)\ \sigma_i\sigma_j = \sigma_j\sigma_i\\ (R5)\ e_i\sigma_j = \sigma_j e_i\\ (R6)\ e_ie_j = e_je_i\end{array}\right\}\text{ for all } 1\le i,j\le n-1 \text{ with } |i-j|\ge 2,$$

$$\left.\begin{array}{l}(R7)\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j\\ (R8)\ \sigma_je_i\sigma_j = \sigma_ie_j\sigma_i\\ (R9)\ \sigma_j\sigma_ie_j = e_ie_j\end{array}\right\}\text{ for all } 1\le i,j\le n-1 \text{ with } |i-j|= 1$$

(which relations are known to define $B_n(\delta)$).

We start by proving (A) and (B).

Note that if $(\sigma_m(u))_{ST}$ or $(e_m(u))_{ST}$ are non-zero, then (S,T) is an m-diamond pair. So we will consider the three cases of m-diamond pairs given in Theorem 5.3.1.

Case 1. $S = (x, x\pm\epsilon_i, x\pm\epsilon_i\pm\epsilon_j)$, $T = (x, x\pm\epsilon_j, x\pm\epsilon_i\pm\epsilon_j)$.
First note that the submatrix of $(e_m(u))$ mixing between S and T is identically zero as $s(m-1)\neq s(m+1)$. So there is nothing to check here.
For $i=j$ we have $S=T$ and $(\sigma_m(\delta))_{SS} = -1$. For $i\neq j$, write $h = \langle x, \epsilon_i-\epsilon_j\rangle$. As x is strictly decreasing we have that $h\neq 0$. So, the submatrix of the matrix $(\sigma_m(\delta))$ mixing

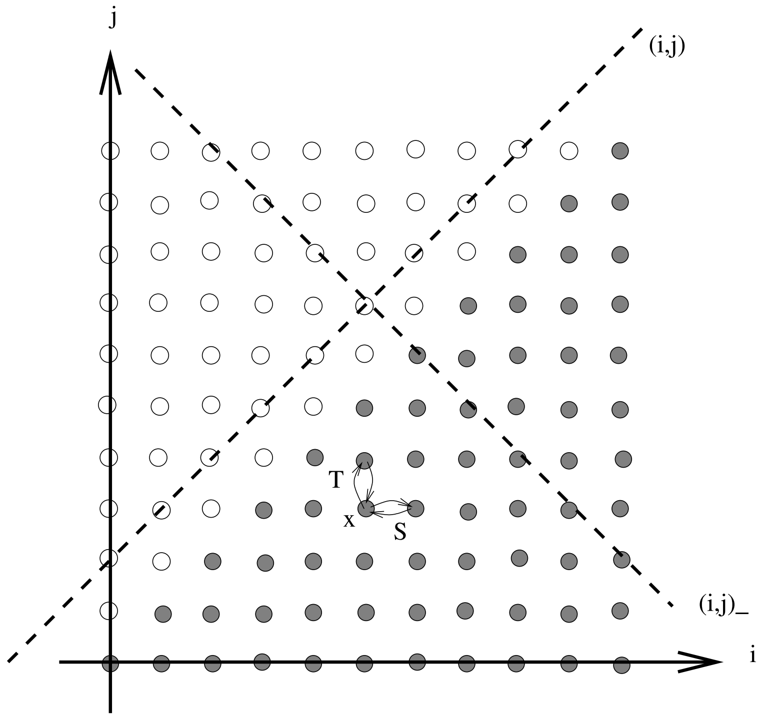

FIGURE 23. **Case 3**: $S = (x, x + \epsilon_i, x)$, $T = (x, x + \epsilon_j, x)$ with $i < j$.

between the walks S and T given by

$$\begin{pmatrix} \frac{1}{h} & \frac{\sqrt{h^2-1}}{|h|} \\ \frac{\sqrt{h^2-1}}{|h|} & -\frac{1}{h} \end{pmatrix}$$

is always well-defined, see Figure 21. This proves (A).
Observe that in this case T is δ-restricted if and only if S is δ-restricted (see Figure 21). Indeed, if S was not δ-restricted, then $\langle x \pm \epsilon_i, \epsilon_i + \epsilon_k \rangle$ would be zero for some k. But as $x \pm \epsilon_i \pm \epsilon_j$ is δ-regular, we must have $k = j$. But then we would have $\langle x \pm \epsilon_j, \epsilon_i + \epsilon_j \rangle = \langle x \pm \epsilon_i, \epsilon_i + \epsilon_j \rangle = 0$ which contradicts the fact that T is δ-restricted. So there is noting to check for (B) in this case.

Case 2. $S = (x, x \pm \epsilon_i, x \pm \epsilon_i \mp \epsilon_j)$, $T = (x, x \mp \epsilon_j, x \pm \epsilon_i \mp \epsilon_j)$ with $i \neq j$.
As in Case 1, we have that the submatrix of $(e_m(u))$ is identically zero in this case. Write $g = \langle x, \epsilon_i + \epsilon_j \rangle$. Then the submatrix of $(\sigma_m(\delta))$ mixing between the walks S and T is given by

$$\begin{pmatrix} \pm\frac{1}{g} & \frac{\sqrt{g^2-1}}{|g|} \\ \frac{\sqrt{g^2-1}}{|g|} & \mp\frac{1}{g} \end{pmatrix},$$

(see Figure 22). Now we claim that if T is δ-restricted, then we cannot have $g = 0$. Indeed, for T δ-restricted, we have that x, $x \mp \epsilon_j$ and $x \mp \epsilon_j \pm \epsilon_i$ all have the same degree of singularity. Now if $g = \langle x, \epsilon_i + \epsilon_j \rangle = 0$ then we have $x_j = -x_i$. But then, $(x \mp \epsilon_j)_j = -x_i \mp 1$ and as $x \mp \epsilon_j$ has the same degree of singularity as x we must have that $x \mp \epsilon_j$ has a coordinate equal to $x_i \pm 1$. Thus $x \mp \epsilon_j$ has both entries x_i and $x_i \pm 1$. But this would imply that $x \mp \epsilon_j \pm \epsilon_i$

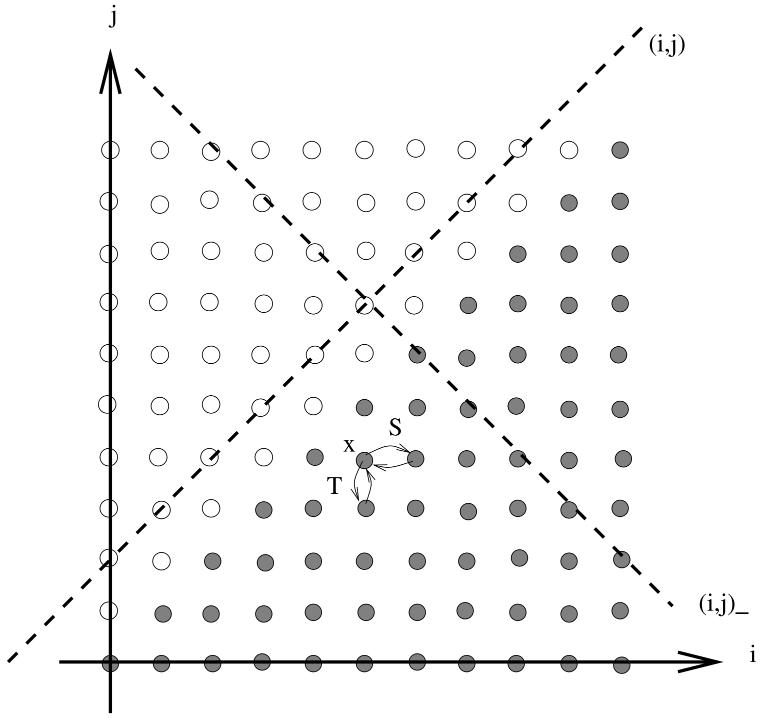

FIGURE 24. **Case 3**: $S = (x, x + \epsilon_i, x)$, $T = (x, x - \epsilon_j, x)$ with $i < j$.

is not strictly decreasing, which is a contradiction. Hence we have shown that g cannot be zero and the matrix entries are all well-defined, proving (A).

Now suppose that T is δ-restricted but S is not. So we have that x is δ-regular and $x \pm \epsilon_i$ is not. Thus we have that $x_i \pm 1 = -x_h$ for some h. But as $x \pm \epsilon_i \mp \epsilon_j$ is δ-regular, we have that $h = j$ and so $x_j = -x_i \mp 1$. This shows that $g = \langle x, \epsilon_i + \epsilon_j \rangle = \mp 1$ and hence $g^2 - 1 = 0$. This proves (B).

Case 3. $S = (x, x + \alpha, x)$, $T = (x, x + \beta, x)$ where $\alpha, \beta \in \{\pm\epsilon_i \,:\, i \geq 1\}$, see Figures 23 and 24. In this case the submatrix of $(\sigma_m(\delta))$ mixing between S and T is given by

$$\begin{pmatrix} \frac{1}{2\langle x,\alpha\rangle+1}\left(1 - \frac{P_{x+\alpha}(\delta)}{P_x(\delta)}\right) & \frac{-1}{\langle x,\alpha+\beta\rangle+1}\left(\frac{\sqrt{P_{x+\alpha}(\delta)P_{x+\beta}(\delta)}}{P_x(\delta)}\right) \\ \frac{-1}{\langle x,\alpha+\beta\rangle+1}\left(\frac{\sqrt{P_{x+\alpha}(\delta)P_{x+\beta}(\delta)}}{P_x(\delta)}\right) & \frac{1}{2\langle x,\beta\rangle+1}\left(1 - \frac{P_{x+\beta}(\delta)}{P_x(\delta)}\right) \end{pmatrix}$$

and the submatrix of (e_m) mixing between S and T is given by

$$\begin{pmatrix} \frac{|P_{x+\alpha}(\delta)|}{P_{x+\alpha}(\delta)} & \frac{\sqrt{P_{x+\alpha}(\delta)P_{x+\beta}(\delta)}}{P_x(\delta)} \\ \frac{\sqrt{P_{x+\alpha}(\delta)P_{x+\beta}(\delta)}}{P_x(\delta)} & \frac{|P_{x+\beta}(\delta)|}{P_{x+\beta}(\delta)} \end{pmatrix}.$$

First note that if T is δ-restricted, then using Theorem 5.2.1 we have that $P_x(\delta) \neq 0$. Thus the entries in the submatrix representing the action of e_m are all well-defined.

Now suppose that we had $\langle x, \alpha + \beta \rangle + 1 = 0$, that is $\langle x, \alpha + \beta \rangle = -1$. So we get $\langle x + \beta, \alpha + \beta \rangle = 0$. Now $\alpha + \beta = \pm(\epsilon_i \pm \epsilon_j)$ for some i, j and we can assume $i \neq j$ as $S \neq T$

and $\alpha+\beta \neq 0$. Moreover, as $x+\beta$ is strictly decreasing we cannot have $\alpha+\beta = \pm(\epsilon_i - \epsilon_j)$. Now suppose $\langle x+\beta, \epsilon_i+\epsilon_j\rangle = 0$, with $\alpha = \pm\epsilon_i$ and $\beta = \pm\epsilon_j$. As T is δ-restricted, we have that x and $x+\beta = x\pm\epsilon_j$ have the same degree of singularity. Thus x must have an entry equal to $x_i \pm 1$ (in position $i\pm 1$). But then $x+\alpha = x\pm\epsilon_i$ is not strictly decreasing, which is a contradiction. This proves that the off-diagonal entries of (σ_m) are well-defined.

Now, if T is δ-restricted but S is not then we have that the off diagonal entries in (σ_m) and (e_m) are all zero using Theorem 5.2.1. This proves (B).

Now we claim that the diagonal entries in (σ_m) are also well-defined. Observe that it is possible to have $2\langle x,\alpha\rangle+1=0$. However we claim that, as a polynomial in δ, $P_x(\delta)-P_{x+\alpha}(\delta)$ is divisible by $2\langle x,\alpha\rangle+1$. To see this, note that before specialisation, the matrix $(\sigma_m(u))$ gives a well-defined representation of $B_n(u)$ and so we have $(\sigma_m(u))^2 = I$ the identity matrix. In particular we have that

$$\sum_{T\in\Omega_m(S)} (\sigma_m(u))_{ST}(\sigma_m(u))_{TS} = 1.$$

So we have

$$(\sigma_m(u))^2_{SS} + \sum_{\substack{T\in\Omega_m(S)\\ T\neq S}} (\sigma_m(u))^2_{ST} = 1.$$

Now we have seen that $\lim_{u\to\delta}(\sigma_m(u))_{ST}$ for all $T\in\Omega_m(S)$ and $T\neq S$ exist and are finite. Thus we must have that $\lim_{u\to\delta}(\sigma_m(u))^2_{SS}$ exists and is finite. This means that $(\sigma_m(u))_{SS}$ is a rational function with no poles at $u=\delta$, proving our claim. This completes the proof of (A).

We now turn to (C). Note that it follows from (A) and (B) that the relations (R1)–(R6) are all satisfied. For example for the relation (R2) we have that for S, S' any δ-restricted walks

$$\sum_{T\in\Omega_m(S)} (e_m(u))_{ST}(\sigma_m(u))_{TS'} = \sum_{V\in\Omega_m(S)} (\sigma_m(u))_{SV}(e_m(u))_{VS'} = (e_m(u))_{SS'}.$$

Now note that if T (resp, V) is not δ-restricted then we know from (B) that the matrix entries specialise to 0 when setting $u=\delta$. Thus all terms involving non δ-restricted walks will vanish in the specialisation and hence the relation above will specialise when $u=\delta$ to give the required relation for $B_n(\delta)$. Exactly the same argument works for all the other relations (R1)–(R6) as these involve products of at most 2 generators.

For the relations (R7)–(R9) we also need to consider sums of products of the form

$$\sum_{T,U}(g_m(u))_{ST}(h_{m+1}(u))_{TU}(k_m(u))_{US'} \qquad \text{or} \qquad \sum_{VW}(g_{m+1}(u))_{SV}(h_m(u))_{VW}(k_{m+1}(u))_{WS'}$$

where S and S' are δ-restricted, $T\in\Omega_m(S)$ and $U\in\Omega_m(S')$ with $T\in\Omega_{m+1}(U)$, $V\in\Omega_{m+1}(S)$ and $W\in\Omega_{m+1}(S')$ with $V\in\Omega_m(W)$, and $g_m, h_m, k_m \in \{\sigma_m, e_m\}$. Note that this implies that the only step in the walks T and U which could be δ-singular is step m. Moreover, as the m-th step in T and U coincide, these two walks are either both δ-restricted, or both not δ-restricted. Similarly we have that the only step in the walks V and W which could be δ-singular is step $m+1$. Moreover, as the $m+1$-th step in V and W coincide, these two walks are either both δ-restricted or both not δ-restricted.

We need to show that, under these conditions, the matrix entries $(h_m(u))_{TU}$ and $(h_{m+1}(u))_{VW}$, do not have a pole at $u = \delta$. Then using (B) as above we would have that these relations would specialise when $u = \delta$ to give the required relations, that is, all the terms involving non δ-restricted walks would vanish in the specialisation.

Using the remark above, we can assume that none of T, U, V, W are δ-restricted. Note that as $t(m)$ (resp. $v(m+1)$) is δ-singular but $t(m+2) = s(m+2)$ (resp. $v(m-1) = s'(m-1)$) is δ-regular, we must have $t(m) \neq t(m+2)$ (resp $v(m-1) \neq v(m+1)$). This implies immediately that $(e_{m+1}(u))_{TU} = (e_m(u))_{VW} = 0$ and so we're done in this case. We now turn to $(\sigma_{m+1}(u))_{TU}$ and $(\sigma_m(u))_{VW}$. By the remark above, all we need to show is that $\diamond_{m+1}(T,T) \neq 0$ and $\diamond_m(V,V) \neq 0$ when we specialise to $u = \delta$. We use Theorem 5.3.1. Note that under our assumption Case 3 cannot happen. Moreover, in Case 1 the Brauer diamonds are independent of u and so there is nothing to prove there. In Case 2 we have

$$\diamond_{m+1}(T,T) = \pm\langle t(m), \epsilon_i + \epsilon_j\rangle = \pm\langle t(m+2), \epsilon_i + \epsilon_j\rangle \neq 0$$

in the specialisation, as $t(m+2)$ is δ-regular. Similarly, we have

$$\diamond_m(V,V) = \pm\langle v(m-1), \epsilon_i + \epsilon_j\rangle \neq 0$$

in the specialisation, as $v(m-1)$ is δ-regular. So we have proved (C).

It remains to show that this module is isomorphic to $L_n(\lambda)$, defined as the simple head of the standard module $\Delta_n(\lambda)$. Denote the representation of $B_n(\delta)$ on δ-restricted walks defined above by $\tilde{L}_n(\lambda)$. By looking at the action of the generators σ_m and e_m , we immediately see that

$$\mathrm{res}_n\, \tilde{L}_n(\lambda) \cong \bigoplus_{\lambda' \in \mathrm{supp}(\lambda)\cap A_\delta} \tilde{L}_{n-1}(\lambda')$$

We will prove by induction on n that $\tilde{L}_n(\lambda) \cong L_n(\lambda)$. If $n = 0$ then there is nothing to prove. Assume that the result holds for $n-1$. Let $\lambda' \in \mathrm{supp}(\lambda) \cap A_\delta$ (note that as $\delta \neq 0$, we have $\mathrm{supp}(\lambda) \cap A_\delta \neq 0$). Then we have

$$\begin{aligned}
\mathrm{Hom}_n(\Delta_n(\lambda), \tilde{L}_n(\lambda)) &\cong \mathrm{Hom}_n(\mathrm{ind}_{n-1}^{\lambda}\, \Delta_{n-1}(\lambda'), \tilde{L}_n(\lambda))\\
&\cong \mathrm{Hom}_{n-1}(\Delta_{n-1}(\lambda'), \mathrm{res}_n^{\lambda'}\, \tilde{L}_n(\lambda))\\
&\cong \mathrm{Hom}_{n-1}(\Delta_{n-1}(\lambda'), \mathrm{pr}^{\lambda'} \oplus_{\mu\in\mathrm{supp}(\lambda)\cap A_\delta} \tilde{L}_{n-1}(\mu))\\
&\cong \mathrm{Hom}_{n-1}(\Delta_{n-1}(\lambda'), \mathrm{pr}^{\lambda'} \oplus_{\mu\in\mathrm{supp}(\lambda)\cap A_\delta} L_{n-1}(\mu)) \quad \text{by induction}\\
&\cong \mathrm{Hom}_{n-1}(\Delta_{n-1}(\lambda'), L_{n-1}(\lambda'))\\
&= \mathbb{C}.
\end{aligned}$$

This shows that $\tilde{L}_n(\lambda)$ contains $L_n(\lambda)$ as a composition factor. But using Corollary 4.0.5, we have that $\dim L_n(\lambda) = \dim \tilde{L}_n(\lambda)$ and so we must have $\tilde{L}_n(\lambda) \cong L_n(\lambda)$.

□

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