A FRAMEWORK FOR FUZZY TOPOLOGY WITH PARTICULAR REFERENCE TO SEQUENTIALITY AND COUNTABILITY

by

Fouad Khalil Mohannadi

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Department of Mathematics

The City University

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DECLARATION

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ABSTRACT

Pu and Liu's Q-theory is combined with Lowen's goodness criterion for fuzzy extensions to provide a framework for fuzzifying topology. This framework is used for the study of fuzzy countability properties and for the fuzzification of classical sequentiality. In extending classical notions to fuzzy theory care is taken to ensure that they are a special case of the emerging fuzzy concepts.

An examination of convergence in the sense of Pu and Liu in special fuzzy topological spaces demonstrates the advantage of Chang's definition of fuzzy topology, which is therefore adopted.

A new criterion (called excellence) for the suitability of the fuzzy extensions of classical topological properties is introduced. In addition to passing Lowen's goodness test, an excellent property is expected to behave, under fuzzy extensions of induction and coinduction, in a way resembling that of the original classical property under these constructions. Fuzzy second countability, quasi-first countability and fuzzy sequentiality are found to be excellent extensions of classical second countability, first countability and sequentiality respectively.

NOMENCLATURE

Most of the following classical symbols are not defined in the thesis. Note that some of them, such as "   ", "   " and "ϵ" will be used also, in a different sense, that involves fuzzy notions, as it will be clear in Part I.

$\mathbb{R}$	the real line
$\mathbb{N}$	the set of the natural numbers
I	the closed unit interval [0,1]
ϕ	the empty set
$<$, $\nless$	the relation "less than" on I and its negation
$\leqslant$, $\nleqslant$	the relation "less than or equal to" on I and its negation
$>$, $\ngtr$	the relation "greater than" on I and its negation
$\geqslant$, $\ngeqslant$	the relation "greater than or equal to" on I and its negation
min	minimum
max	maximum
inf	infimum
sup	supremum
l.u.b.	least upper bound

g.l.b.	greatest lower bound
$x \in X$	x is an element of X
$x \notin X$	x is not an element of X
$\{x \in X: \Psi(x)\}$	the set of all elements x in X satisfying the condition(s) $\Psi(x)$
$\{x\}$	the singleton set having the element x
$\{x_\alpha : \alpha \in \Omega\}$	the set of elements x_α indexed by an indexing set Ω
Z^c	the complement of the set Z
$Z-Z_1$	the set $\{x : x \in Z,\ x \notin Z_1\}$
$\subset$, $\not\subset$	the relation "is properly contained in" on a power set and its negation
$\subseteq$, $\not\subseteq$	the relation "is contained in" on a power set and its negation
$\supset$, $\not\supset$	the relation "properly contains" on a power set and its negation
$\supseteq$, $\not\supseteq$	the relation "contains" on a power set and its negation
$\bigcup\limits_{\alpha\in\Omega} Z_\alpha$, $\bigcup\limits_{Z\in C} Z$	the union of the family $\{Z_\alpha : \alpha \in \Omega\}$ (respectively C)
$\bigcap\limits_{\alpha\in\Omega} Z_\alpha$, $\bigcap\limits_{Z\in C} Z$	the intersection of the family $\{Z_\alpha : \alpha \in \Omega\}$ (respectively C)
$f : X \longrightarrow Y$	a function from X into Y
$\prod\limits_{\alpha\in\Omega} X_\alpha$	the cartesian product of the family $\{X_\alpha : \alpha \in \Omega\}$
$\prod\limits_{\alpha\in\Omega} x_\alpha$	an element of the cartesian product with x_α as its α-th coordinate

$f_\circ g$	the composition function f after g
$f(Z)$, $f^{-1}(Z')$	the image of Z and the inverse image of Z' under f
$f(x)$, $f^{-1}(y)$	the image of x and the inverse image of y under f
$f\|_X$	the restriction of the function f to the set X
i_d	the identity function
i_n	the inclusion function
(X, τ)	a set X together with a family τ of subsets of X (usually a topological space)
τ_o	the indiscrete topology
Δ	the discrete topology
τ_s	the Franklin topology of τ (see p.103)
τ_{cc}	the countable complement topology
τ_i	an induced topology
τ_c	a coinduced topology
$(I, 1^*)$	the closed unit interval I with its usual topology
1_r^*	the topology $\{(\lambda, 1] : \lambda \in I - \{1\}\} \bigcup \{I, \phi\}$ on I
$\langle x_n \rangle$	an infinite sequence of terms x_n
$\langle x_{n_k} \rangle$	a subsequence of $\langle x_n \rangle$
$x_n \xrightarrow{\tau} x$	the sequence $\langle x_n \rangle$ converges to x (with respect to the topology τ)

$x_n \overset{\tau}{\not\longrightarrow} x$	the negation of the statement $x_n \overset{\tau}{\longrightarrow} x$
C_1	first countable
C_2	second countable
iff, $\longleftrightarrow$	if and only if
(a) $\longrightarrow$ (b)	(a) implies (b)
$\square$	Q.E.D.

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Notations representing fuzzy concepts will be given as these concepts appear in the text. The following list of the most frequently used ones is intended to help the reader trace back the fuzzy notations to the place where they are first introduced.

		Item
$A^{-1}(\lambda,1]$	the strong λ-cut of a fuzzy set A	1.1.3
Supt A	the support of A	1.1.3
ext A	the (proper) extension of A	5.2.9
res A	the restriction of A	5.1.4
X_Z	the crisp set of support Z	1.1.5
Φ_X	the empty fuzzy set in X	1.1.6
F_X	the full set in X	1.1.6
η_λ	the constant set of value λ	1.1.7
$Z^{(\lambda)}$	the lower semicrisp set of support Z and value λ	1.1.8
$(x)_\lambda$	the fuzzy point of support x and value λ	1.2.1

q	the quasi-coincidence relation	1.3.1
$\mathcal{Q}_e(J)$	the system of Q-neighbourhood (relative to the fuzzy topology J) of the fuzzy point e	2.1.5
$\mathbf{V}_e(J)$	the system of neighbourhoods (relative to the fuzzy topology J) of the fuzzy point e	2.1.5
fts	fuzzy topological space	2.1.2
J_Δ	the discrete fuzzy topology	2.2.3
J_0	the indiscrete fuzzy topology	2.2.4
L_0	the indiscrete Lowen fuzzy topology	2.2.4
$J(\mathbf{b})$	the trivial fuzzy topology (with the values of its constant sets being in **b**)	2.2.4
$cr(\tau)$	the crisp fuzzy topology of a topology τ	2.2.2
$\omega(\tau)$	the natural fuzzy topology of τ	2.2.6
$\omega_\lambda(\tau)$	the λ-cut fuzzy topology of τ	2.2.17
$\iota(J)$	the initial topology of a fuzzy topology J	2.2.10
J_s	the Franklin fuzzy topology of J	3.3.5
C_1^f	fuzzy first countable	3.1.6
C_2^f	fuzzy second countable	3.1.1
Q-C_1	quasi-first countable	3.1.7
Sequential^f	fuzzy sequential	3.3.7
$\langle e_n \rangle q(N)A$	the fuzzy sequence $\langle e_n \rangle$ is eventually quasi-coincident to A	3.2.4
$e_n \xrightarrow{J} e$	the fuzzy sequence $\langle e_n \rangle$ converges to e (in the fuzzy topology J)	3.2.6
$e_n \not\xrightarrow{J} e$	the negation of the statement $e_n \xrightarrow{J} e$	3.2.6
$\iota[\mathbf{C}']$	the family of the initial topological spaces of a family **C'** of fuzzy topological spaces	5.3.2
$\omega[\mathbf{C}]$	the family of the natural fuzzy topological spaces of a family **C** of topological spaces	5.3.1

INTRODUCTION

The concept of fuzzy set was first introduced by Zadeh [58] in 1965 "to be used in dealing with classes" which "do not have precisely defined criteria of membership". In [58], he defined fuzzy sets in terms of functions from a set to the closed unit interval and introduced other basic related notions such as fuzzy union, intersection and complement, all of which have now become standard. These notions were explored in 1967 by Goguen [23] who extended the concept of fuzzy set by replacing the unit interval by any poset; thus leading to a definition of L-sets corresponding to a given lattice L. Much later, Brown [5] considered the case of a Boolean lattice. This was followed by De Luca and Termini [11] who, noticing that the closed unit interval is actually a Brouwerian lattice, considered the case of functions to these types of lattice.

No attempt, though, to formulate a theory of fuzzy topological spaces was made until 1968. Taking advantage of Zadeh's definition of fuzzy union and intersection, Chang [8] introduced the notion of a fuzzy topological space. In [8], Chang also defined fuzzy image and fuzzy inverse image under a function (which are now standard) and extended a number of properties of functions, such as continuity, to fuzzy topology. But he avoided completely the fuzzification of the notion of point, thus instead of describing neighbourhoods and sequences of fuzzy points, he talked about neighbourhoods and sequences of fuzzy <u>sets</u>.

After Chang, many mathematicians tried to formulate a reasonable definition for fuzzy point and its membership to a fuzzy set. In his first two papers in English on fuzzy topology

[54, 55], Wong also chose to avoid the idea of fuzzy point and fuzzy membership. But the third paper [56], which he published in 1974 was precisely dedicated to those concepts. Unfortunately, his definition of fuzzy membership, which was also adopted in 1977 by Christoph [9], turned out to be not a good choice. Furthermore some of the results obtained using this definition contain errors, as was shown later by Gottwald [25].

The problems encountered at this stage in the fuzzification of points prompted some authors, notably Hutton [e.g. 27] to adopt the so-called "pointless approach". Liu [32] observed that although such an approach has led to important results, especially in relation to the fuzzification of topological properties that are independent of the notion of point, its limitation is evident and there are instances when it is unavoidable to deal with the notion of point.

Choosing not to follow the "pointless approach" of Hutton, some authors, such as Sarkar [45, 46], Srivastava et al [47, 48], Deng [12, 13], and Bülbül [6,7] went for a modified version (which we term "proper membership definition") of Wong's fuzzy membership, while keeping his definition of fuzzy point. Unfortunately this still excludes classical points (the so called crisp points). Other authors, such as Gottwald [25], Ghanim et al [22] and Kerre [30] preferred, as Gottwald put it, to "use only the concept" fuzzy 'singleton' and "to speak of singletons instead of points".

Pu and Liu [43] observed that the difficulties encountered when trying to formulate a reasonable definition of fuzzy point and fuzzy membership are a consequence of "the limitation of the notion of traditional neighbourhood". To overcome such difficulties and to allow for a definition of fuzzy point that includes crisp points, they introduced an ingenious fuzzy relation, to which they gave the name "quasi-coincidence" (q-relation). This, together with their fuzzy membership relation, provides a fuzzy extension of classical membership. These two fuzzy extensions are connected in a way similar to that relating classical membership with its antithesis; that is a fuzzy point "belongs" to a fuzzy set if and only if it is not quasi-coincident to its complement. Moreover the two extensions coincide in the classical case. Pu and Liu's idea has also a bonus. By extending the concept of quasi-coincidence to a couple of fuzzy sets, they arrived at a fuzzy excluded middle principle that has the classical version as a special case, thus providing an answer to Muir's [42] complaint about the apparent failure of this law in fuzzy-set theory.

But most importantly, quasi-coincidence provides an insight into the way in which classical concepts and theories are to be fuzzified. Although an alternative fuzzification can be built on the "proper membership definition" of fuzzy point, Pu and Liu's approach has the advantage of having classical thoery as a special case. The introduction of quasi-coincidence and the ensuing Q-neighbourhoods, is, in our opinion, one of the most important steps in the development of fuzzy topology.

Another big step in the development of fuzzy topology is the invention of the so called "goodness criterion" by Lowen [35], which was the result of his recognition of the special place of the fuzzy topology defined by lower semicontinuous functions. We shall refer to such a fuzzy topology as "the natural fuzzy topology". Even before spelling out his goodness principle, Lowen

[34], in an earlier work, used it as a guide in the fuzzification of classical concepts. This idea has been used in one way or another by several authors, such as Martin [39], Srivastava et al [47, 48] and Bülbül [7] in studies covering different aspects of fuzzy topology.

The Q-theory of Pu and Liu and Lowen's goodness criterion have been noticed by many authors, and from time to time, one or the other was used as a tool in fuzzy research. However no advantage has yet been taken of the way in which each seems tailor-made for the other. A major aim of this thesis is to show how well the two notions reinforce each other.

In this study we build up a framework for fuzzy topology which combines both concepts and which we hope will form the standard one. We use it both for fuzzifying classical sequential spaces and for studying other already fuzzified notions such as countability properties, properties of functions and the basic topological constructions of induction and coinduction. In building this framework we concentrated on extending to fuzzy theory these topological concepts that are most relevant for our purpose. However, our general approach can serve as a guide for studying other fuzzy extensions.

The thesis is divided into six parts.

In the first part, we state our preliminary definitions concerning fuzzy sets and give all properties of fuzzy sets that are used in the thesis. Many of these properties have been proved as separate items by different authors. However we present them as consequences of the fact that the family of all fuzzy sets in a set X is a complete Brouwerian dually Brouwerian lattice. Some useful new notions, such as lower semicrisp sets are also introduced. We investigate as well the ideas of fuzzy point and fuzzy membership. Explaining thoroughly our reasons for rejecting both Wong's definitions of these concepts and the "proper membership definition", we stress the advantages of Pu and Liu's alternative notions which we adopt. The properties of fuzzy membership and quasi-coincidence given there are a combination of these stated in Pu and Liu [43] and others deduced by us. In this part and elsewhere we discuss other authors' alternatives to some of our definitions and also mention faulty statements. Such items will be marked by an asterisk.

Part II is dedicated to fuzzy topology. We propose fuzzy versions of some known classical assertions concerning open sets and bases and examine the notion of fuzzy subbase. We also study some special examples of fuzzy topological spaces. We devote much attention to the natural fuzzy topological space and use the notion of lower semicrisp set to find a base for it. We discuss other related concepts, such as the initial and the Martin topologies, and construct a useful type of fuzzy topological space, which will be used later for providing counterexamples.

In part III we take advantage of Pu and Liu's fuzzy convergence theory to fuzzify classical sequentiality, thus introducing sequential fuzzy topological spaces. We discuss the relationships between fuzzy sequentiality and various fuzzy countability properties and between these notions themselves. We test these properties for Lowen goodness and also for initial goodness (which we introduce).

Studying convergence in some special fuzzy topological spaces strengthens our preference for Chang's definition of fuzzy topology, rathen than Lowen's. This is also supported by an observation in part II about the so called crisp fuzzy topology. We establish two interesting facts. The first is that Pu and Liu's quasi-first countable fuzzy topological space (rather than fuzzy first countable) plays in fuzzy topology the role played by first countable space in classical topology. Secondly, the fuzzy extension of the notion of constant sequence is not simply the obvious definition of constant fuzzy sequence, but instead a more general type of fuzzy sequence which we call semiconstant.

Part IV is about fuzzy topological properties of functions and contains all assertions about functions used in the next two parts. Some of these assertions can be found scattered in several papers with a few of them containing minor errors. Alterations have been made to these and erroneous claims corrected. We also introduce the concept of fuzzy sequential continuity and show that it is a good extension of classical sequential continuity. Investigating the goodness of the other fuzzy properties of functions, we give the first published proof of the goodness of fuzzy closedness of a function.

In part V we study the problem of extending to fuzzy topology the classical induced and coinduced topological spaces. These are more often known as the initial and final spaces. Since we use the term initial topology for the particular space defined in part II, we will stick to the names "induced" and "coinduced". Here, our approach is different from Pu and Liu's and some other authors', but similar to Lowen's in that we start by fuzzifying induction and coinduction and then deduce the fuzzy extensions of the special cases of subspace and product on one hand and quotient and disjoint sum on the other. Such an approach was helpful in arriving at what we think are the right extensions to fuzzy topology of these constructions, and in avoiding some awkward definitions of them that have appeared in the literature.

Our proofs of the goodness of fuzzy induction and coinduction, however, differ from those of Lowen in that they are the outcome of results obtained in parts II and IV and in that they are based on Pu and Liu's Q-theory.

These proofs have the advantage of being simpler and fuzzy topological in nature, thus throwing more light on the kind of relationships involved. Our proof of goodness of fuzzy induction provides a simpler fuzzy topological proof of the extension theorem for lower semicontinuous functions.

In the last part of the thesis we concentrate on the behaviour of fuzzy sequentiality and fuzzy countability properties under the basic constructions studied in part V. We introduce the notion of excellence as the criterion by which we judge the reasonableness of fuzzy extensions of classical concepts. We establish the excellence of fuzzy sequentiality, quasi-first countability and fuzzy second countability. As in parts II and III, our approach of combining the goodness criterion with Q-theory proves its advantage. We replace long and detailed proofs of three important results proposed by Pu and Liu by a single simple proof encompassing all three cases.

Throughout the thesis, definitions and theorems adopted or adapted from other authors are attributed as they appear. All unattributed ideas and results should be understood to be our own contributions.

PART I

FUZZY SETS AND FUZZY POINTS

In this part of the thesis, we present all the definitions and properties of fuzzy sets and fuzzy points that are used elsewhere in our study. For the sake of clarity, we divide this part into three sections.

The first section contains the standard definitions related to fuzzy sets. The notions and properties given in this section are independent of the type of approach adopted, "pointless" or otherwise.

The second section is devoted to the ideas of fuzzy point and fuzzy membership. We start by discussing the alternative definitions of these concepts which appeared in the literature. We explain our preference for Pu and Liu's choice and state its consequences.

In the third section, we study quasi-coincidence. Deducing the relevant relationships involving this notion, we explain its role in fuzzy set theory.

1. Fuzzy Sets

In the following, let X be a set and $I = [0,1]$.

Definition 1.1.1 (Zadeh)

A fuzzy set in X is a function from X into I.

Definition 1.1.2

Let $x \in X$ and A be a fuzzy set in X. The value of A at x is called the degree of membership of x in A and denoted by $A(x)$.

Definition 1.1.3 (Weiss [53])

Let A be a fuzzy set in X and $\lambda \in I - \{1\}$. The set $A^{-1}(\lambda,1] = \{x : A(x) > \lambda\}$ is called the strong λ-cut for A. In particular the strong 0-cut $A^{-1}(0,1]$ for A is called the support of A and denoted by supt A . Thus supt $A = \{x : A(x) > 0\}$. If supt A is a singleton, say $\{x\}$, then we adopt the (abused) notation supt $A = x$.

Definition 1.1.4

Let A be a fuzzy set in X. The set $A^{-1}[\{1\}] = \{x : A(x) = 1\}$ is called the crisp support of A.

Definition 1.1.5 (Weiss)

A fuzzy set A in X satisfying the relation supt $A = A^{-1}[\{1\}]$ is called a crisp set. In other words, a crisp set in X is a characteristic function from X into I. The crisp set of support Z, where $Z \subseteq X$, is denoted by χ_Z.

Definition 1.1.6

The crisp set in X of support ϕ is called the empty fuzzy set in X and denoted by ϕ_X (or by χ_ϕ if no confusion arises as to which original set X we are referring).

The crisp set in X of support X is called the full set in X and denoted by F_X (and by χ_X if no confusion arises).

Definition 1.1.7

A fuzzy set A in X satisfying, for some $\lambda \in I$, the equation $A^{-1}[\{\lambda\}] = X$ is called a constant set in X, and denoted by η_λ. The number λ is referred to as the (constant) value of A in X. Obviously, the constant set of value 0 is the empty fuzzy set while that of value 1 is the full set (i.e. $\eta_0 = \chi_\phi = \phi_X$ and $\eta_1 = \chi_X = F_X$).

Definition 1.1.8

A fuzzy set A in X is said to be a lower (upper) semicrisp set iff these is a $\delta \in I$ and a subset Z of X such that,

$$A(x) = \begin{cases} \delta & \text{if} \quad x \in Z \\ 0 \ \text{(respectively 1)} & \text{if} \quad x \in Z^c \end{cases}$$

For the lower semicrisp set A, the number δ is referred to as the value of A while A itself is denoted by $Z^{(\delta)}$.

Obviously, every crisp set and every constant set is lower semicrisp (that is because $\chi_Z = Z^{(1)}$ and $\eta_\lambda = X^{(\lambda)}$, for all $Z \subseteq X$ and $\lambda \in I$).

In the following, let A and B be fuzzy sets in X.

Definition 1.1.9 (Zadeh)

The fuzzy set A is said to be contained in B (or equivalently to be a subset of B) iff $A(x) \leq B(x)$ for all $x \in$ supt A. This inclusion relation is denoted by $A \subseteq B$. Equivalently we write $B \supseteq A$ and say that B contains A.

If A is not a subset of B, then we write $A \nsubseteq B$ or equivalently, $B \nsupseteq A$.

Definition 1.1.10

The fuzzy set A is said to be equal to B (we denote that by $A = B$) iff $A \subseteq B$ and $B \subseteq A$. If the statement $A = B$ is false, then we write $A \neq B$.

We say that A is a proper subset of B and write $A \subset B$ iff $A \subseteq B$ and $A \neq B$.

Definition 1.1.11 (Zadeh)

A fuzzy set is said to be the complement of the fuzzy set A and denoted by A^c iff $A^c(x) = 1 - A(x)$, for all $x \in X$. It is obvious that, for all $Z \subseteq X$, we have $(\chi_Z)^c = \chi_{Z^c}$ and that for all $\delta \in I$, we have $(\eta_\delta)^c = \eta_{1-\delta}$. It is also obvious that the complement of a lower semicrisp set is an upper semicrisp set and visa versa.

In the following, let Ω be a nonempty indexing set and C a collection of fuzzy sets in X defined by $C = \{A_\alpha : \alpha \in \Omega\}$.

Definition 1.1.12 (Zadeh)

A fuzzy set A is said to be the union of the fuzzy sets in the collection C and denoted by $\bigcup_{\alpha\in\Omega} A_\alpha$ iff $A(x) = \sup \{A_\alpha(x): \alpha \in \Omega\}$, for all $x \in X$.

Definition 1.1.13 (Zadeh)

A fuzzy set A is said to be the intersection of the fuzzy sets in the collection C and denoted by $\bigcap_{\alpha\in\Omega} A_\alpha$ iff $A(x) = \inf \{A_\alpha(x) : \alpha \in \Omega\}$, for all $x \in X$.

The following are consequences of the preceding definitions.

Consequence 1.1.14

Let C be a nonempty collection of fuzzy sets in X. Then, for every $\lambda \in I - \{1\}$, we have;

(a) $(\bigcup_{A\in C} A)^{-1} (\lambda,1] = \bigcup_{A\in C} A^{-1}(\lambda,1]$

(b) $(\bigcap_{A\in C} A)^{-1} (\lambda,1] \subseteq \bigcap_{A\in C} A^{-1}(\lambda,1]$. The equality holds for finite C.

Proof

(a) $x \in (\bigcup_{A\in C} A)^{-1} (\lambda,1]$ iff $(\bigcup_{A\in C} A)(x) > \lambda$

iff $A_0(x) > \lambda$, for some $A_0 \in C$, by property of l.u.b.

$$\text{iff} \quad x \in A_0^{-1}(\lambda,1]$$

$$\text{iff} \quad x \in \bigcup_{A\in C} A^{-1}(\lambda,1].$$

Thus $\bigcup_{A\in C} A^{-1}(\lambda,1] \;=\; (\bigcup_{A\in C} A)^{-1}\,(\lambda,1].$

(b) First,

$$x \in (\bigcap_{A\in C} A)^{-1}\,(\lambda\,,1] \quad \text{iff} \quad (\bigcap_{A\in C} A)(x) > \lambda$$

$$\text{implies} \quad A(x) > \lambda, \text{ for all } A \in C, \text{ by property of g.l.b.}$$

$$\text{iff} \quad x \in A^{-1}(\lambda,1], \text{ for all } A\in C$$

$$\text{iff} \quad x \in \bigcap_{A\in C} A^{-1}(\lambda,1].$$

Thus $(\bigcap_{A\in C} A)^{-1}\,(\lambda,1] \subseteq \bigcap_{A\in C} A^{-1}(\lambda,1].$

If C is finite then we can similarly show that $\bigcap_{A\in C} A^{-1}(\lambda,1] \subseteq (\bigcap_{A\in C} A)^{-1}\,(\lambda,1]$, and hence the equality holds.

To illustrate that the equality does not necessarily hold for infinite C, let $C = \{\eta_{\frac{1}{n}} \subseteq F_X \;:\; n \in \mathbf{N}\}$. Then $(\bigcap_{A\in C} A)^{-1}(0,1] = \phi_X^{-1}(0,1] = \phi$. On the other hand $A^{-1}(0,1] = X$, for all $A \in C$, and so $\bigcap_{A\in C} A^{-1}(0,1] = X$. Hence $(\bigcap_{A\in C} A)^{-1}\,(0,1] \neq \bigcap_{A\in C} A^{-1}(0,1]$. □

Corollary 1.1.15 (Malghan and Benchalli [38])

(a) $\text{supt}\,(\bigcup_{A\in C} A) = \bigcup_{A\in C} \text{supt}\, A$

(b) $\text{supt}\,(\bigcap_{A\in C} A) \subseteq \bigcap_{A\in C} \text{supt}\, A$. The equality holds for finite C.

Proof

Put $\lambda = 0$ in (a) and (b) of 1.1.14. □

Consequence 1.1.16

A fuzzy set A in X is the empty fuzzy set (the full set) in X iff for every fuzzy set B in X, we have $A \subseteq B$ (respectively $B \subseteq A$).

Proof

First, for any $x \in X$ and for each fuzzy set B in X, we have $\phi(x) = 0 \leq B(x) \leq 1 = F_X(x)$. Hence $\phi \subseteq B \subseteq F_X$.

Now let A be a fuzzy set in X, such that for every fuzzy set B in X, we have $A \subseteq B$ (respectively $B \subseteq A$). Let $B = \phi$ (respectively, $B = F_X$) then $A \subseteq \phi$ (respectively $F_X \subseteq A$). Hence for every $x \in X$, we have $A(x) \leq 0$ (respectively $1 \leq A(x)$). But then $A(x) = 0$ (respectively $A(x) = 1$), for every $x \in X$. Thus $A = \phi_X$ (respectively $A = F_X$). □

Implied in consequence 1.1.16 is a useful fact; that is every fuzzy set is a subset of the full set. This provides us with a way of symbolizing statements indicating that a certain mathematical object is a fuzzy set in some given set. Instead of saying "A is a fuzzy set in X", we can simply write "$A \subseteq F_X$". Such a symbol agrees with that used to indicate the corresponding statement for classical sets.

In the following, by the "lattice I" we mean the set of all numbers in I together with the partial ordering "$\leq$". The join and meet for this lattice are the usual "sup" and "inf". Such a lattice is complete, with the numbers '0' and '1' being respectively its universal lower and upper bounds.

Consequence 1.1.17

Fuzzy inclusion is reflexive, transitive and antisymmetric. Symbolically if A, B and C are fuzzy sets in X, then.

a. $A \subseteq A$

b. $A \subseteq C$, whenever $A \subseteq B$ and $B \subseteq C$

c. $A = B$, whenever $A \subseteq B$ and $B \subseteq A$

Proof

These properties follow from Definition 1.1.9 and the fact that I (together with $\leq$) is a poset. □

In further discussions, let $\mathbb{H}(I)$ be the family I^X of all fuzzy sets in X. Consequence 1.1.17 says that $\mathbb{H}(I)$ with the binary relation "$\subseteq$" is a poset. The join "$\cup$" and the meet "$\cap$" for this lattice are given by definitions 1.1.12 and 1.1.13. Consequence 1.1.16 supplies $\mathbb{H}(I)$ with universal lower and upper bounds Φ_X and F_X; thus making it a complete lattice.

The following properties of union and intersection of fuzzy sets can be proved directly, but it is more appropriate to think of them as properties of lattices guaranteed by lemma 1.5.1 in Birkhoff [3].

Consequence 1.1.18

The laws of idempotence, commutativity, associativity absorption and consistency hold for fuzzy sets. Symbolically, if A, B and C are fuzzy sets in X, then

a. $A \cup A = A, \quad A \cap A = A$

b. $A \cup B = B \cup A, \quad A \cap B = B \cap A$

c. $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C$

d. $A \cup (A \cap B) = A = A \cap (A \cup B)$

e. $A \cup B = B$ iff $A \subseteq B$ iff $A \cap B = A$

Consequence 1.1.19

For every fuzzy set A in X, we have;

a. $A \cup F_X = F_X, \quad A \cap F_X = A$

b. $A \cup \phi_X = A\ , \quad A \cap \phi_X = \phi_X$

Proof

This follows from consequences 1.1.16 and 1.1.18(e). □

Consequence 1.1.20

Let C be a nonempty collection of fuzzy sets. The union (intersection) of the fuzzy sets of C is the "smallest"("largest") fuzzy set containing (contained in) every member of C

Symbolically,

(a) for every $A \in \mathbf{C}$, we have $A \subseteq \bigcup_{A \in \mathbf{C}} A$ and $\bigcap_{A \in \mathbf{C}} A \subseteq A$.

(b) If B is a fuzzy set in X such that $A \subseteq B$ (respectively $B \subseteq A$) for every $A \in \mathbf{C}$, then $\bigcup_{A \in \mathbf{C}} A \subseteq B$ (respectively $B \subseteq \bigcap_{A \in \mathbf{C}} A$).

Proof

This follows from the properties of the least upper bounds and the greatest lower bounds in the lattice I. □

Consequence 1.1.21

Let Ω be an indexing set and for every $\alpha \in \Omega$, let $A_\alpha \subseteq B_\alpha$. Then,

(a) $\bigcup_{\alpha \in \Omega} A_\alpha \subseteq \bigcup_{\alpha \in \Omega} B_\alpha$

(b) $\bigcap_{\alpha \in \Omega} A_\alpha \subseteq \bigcap_{\alpha \in \Omega} B_\alpha$

Proof

This follows from consequence 1.1.20. □

Goguen [23], observed that properties of the lattice $H(L)$ of functions from X into a lattice L (called L-sets) reflect the properties of the lattice L itself.

Let L be a complete lattice with a partial order '$\leq$' and with symbols '$\vee$' and '$\wedge$' indicating 'join' and 'meet' respectively. Then $\mathbf{L}$ is said to be Brouwerian (respectively dually Brouwerian) iff for every couple of elements λ_1 and λ_2 in $\mathbf{L}$, the set $\{\delta : \lambda_1 \wedge \delta \leq \lambda_2\}$ (respectively the set $\{\delta : \lambda_1 \vee \delta \geq \lambda_2\}$) contains its least upper bound (respectively its greatest lower bound).

De Luca and Termini [11] showed that if $\mathbf{L}$ is Brouwerian, then so is $\mathbf{H}(\mathbf{L})$. Using theorem 5.10.24 in Birkhoff [3], which says that a complete Brouwerian lattice has complete distributivity of meet over joins, they concluded that if $\mathbf{L}$ is Brouwerian then $\mathbf{H}(\mathbf{L})$ has complete distributivity of meet over joins. However, as indicated by De Luca and Termini, who fault Goguen [23] who assumed the equivalence of the two distributivity laws in lattices, this is not enough to conclude that $\mathbf{H}(\mathbf{L})$ has complete distributivity of join over meets. This is only possible if $\mathbf{L}$ is dually Brouwerian.

Since the lattice I is both Brouwerian and dually Brouwerian, then so is $\mathbf{H}(I)$. Thus $\mathbf{H}(I)$ is a complete Brouwerian and dually Brouwerian lattice. We have the following consequence.

<u>Consequence 1.1.22</u>

The laws of complete distributivity hold for fuzzy sets. Symbolically, if A and C are a fuzzy set and a family of fuzzy sets in X respectively. Then, we have;

(a) $A \cap (\bigcup_{B\in C} B) = \bigcup_{B\in C} (A \cap B)$

(b) $A \cup (\bigcap_{B\in C} B) = \bigcap_{B\in C} (A \cup B)$

Let us define a complement λ^c of an element of the lattice I by $\lambda^c = 1-\lambda$. It is easy to check that the following five properties hold in I.

(a) $(\lambda^c)^c = \lambda$, for all $\lambda \in I$

(b) $\lambda \leqslant \delta$ iff $\delta^c \leqslant \lambda^c$, for all λ, $\delta \in I$

(c) $\min\{\lambda,\lambda^c\} = 0$ (respectively, $\max\{\lambda,\lambda^c\} = 1$) iff $\lambda \in \{0,1\}$. A consequence of this is that I is not "complemented".

(d) $\min\{\lambda,\delta\} = 0$ implies $\lambda \leqslant \delta^c$ (and equivalently $\delta \leqslant \lambda^c$)

(e) If $\mathbf{C}$ I, then,

$$1 - \sup_{\mathbf{C}} \lambda = \inf_{\mathbf{C}} \lambda^c \quad \text{and} \quad 1 - \inf_{\mathbf{C}} \lambda = \sup_{\mathbf{C}} \lambda^c.$$

A consequence of this is the validity in I of De Morgan's laws, that is

$$(\sup_{\mathbf{C}} \lambda)^c = \inf_{\mathbf{C}} \lambda^c \quad \text{and} \quad (\inf_{\mathbf{C}} \lambda)^c = \sup_{\mathbf{C}} \lambda^c.$$

Now, the complement of an element A in $\mathbf{H}(I)$ is defined so that the degree of membership of a point x in X is equal to the 'complement' in I of the element $\lambda = A(x)$.

Observing this "point by point" correlation, it follows that properties of I involving complements are carried on to $\mathbf{H}(I)$. This is expressed formally in the following consequence

<u>Consequence 1.1.23</u>

Let A and B be fuzzy sets in X and $\mathbf{C}$ a nonempty family of fuzzy sets in X. We have,

(a) $(A^c)^c = A$

(b) $A \subseteq B$ iff $B^c \subseteq A^c$

(c) $A \cap A^c = \Phi$ (respectively $A \cup A^c = F_X$) iff A is crisp.

(d) $A \cap B = \Phi$ implies $A \subseteq B^c$ (and equivalently $B \subseteq A^c$)

(e) De Morgan's laws hold for fuzzy sets. Symbolically,

$$(\bigcup_{A \in C} A)^c = \bigcap_{A \in C} A^c \quad \text{and} \quad (\bigcap_{A \in C} A)^c = \bigcup_{A \in C} A^c$$

Proof

All five properties follow from the corresponding five properties in I. A detailed proof for (e) is provided in Deng [12]. □

Consequence 1.1.24

(a) A fuzzy set is lower semicrisp iff it is the intersection of a crisp set and a constant set. More specifically, if $Z \subseteq X$ and $\delta \in I$: then $\eta_\delta \cap \chi_Z = Z^{(\delta)}$

(b) Let C be a family of subsets of X and $\lambda \in I$. Then,

$$(\bigcup_{Z \in C} Z)^{(\lambda)} = \bigcup_{Z \in C} Z^{(\lambda)} \quad \text{and} \quad (\bigcap_{Z \in C} Z)^{(\lambda)} = \bigcap_{Z \in C} Z^{(\lambda)}$$

(c) Let $\mathbf{b} \subseteq I$, $\lambda = \sup \mathbf{b}$ and $\lambda' = \inf \mathbf{b}$. Then,

$$Z^{(\lambda)} = \bigcup_{\delta \in \mathbf{b}} Z^{(\delta)} \quad \text{and} \quad Z^{(\lambda')} = \bigcap_{\delta \in \mathbf{b}} Z^{(\delta)}$$

Proof

Let $x \in X$

(a) $(\eta_\delta \cap \chi_Z)(x) = \min\{\eta_\delta(x), \chi_Z(x)\} = \min\{\delta, \chi_Z(x)\} =$

$$\begin{cases} \delta & \text{if } x \in Z \\ 0 & \text{if } x \in Z^c \end{cases} = Z^{(\delta)}(x).$$

Therefore, $\eta_\delta \cap \chi_Z = Z^{(\delta)}$.

(b) 1. $(\bigcup_{Z \in \mathbf{C}} Z^{(\lambda)})(x) = \sup\{Z^{(\lambda)}(x) : Z \in \mathbf{C}\} = \begin{cases} \lambda & \text{if } x \in \bigcup_{Z \in C} Z \\ 0 & \text{otherwise} \end{cases} =$

$(\bigcup_{Z \in \mathbf{C}} Z)^{(\lambda)}(x)$. Hence $(\bigcup_{Z \in \mathbf{C}} Z)^{(\lambda)} = \bigcup_{Z \in \mathbf{C}} Z^{(\lambda)}$.

2. The case for the intersection is proved similarly.

(c) 1. $\bigcup_{\delta \in \mathbf{b}} Z^{(\delta)}(x) = \sup\{Z^{(\delta)}(x) : \delta \in \mathbf{b}\} = \begin{cases} \sup\{\delta : \delta \in \mathbf{b}\} = \lambda & \text{if } x \in Z \\ 0 & \text{if } x \notin Z \end{cases}$

$= Z^{(\lambda)}(x)$.

Therefore, $\bigcup_{\delta \in \mathbf{b}} Z^{(\delta)} = Z^{(\lambda)}$.

2. The case for the intersection is proved similarly. □

2. Fuzzy Points

The concept of fuzzy membership had its difficulties from the start. Zadeh [58] commented that "the notion of "belonging" which plays a fundamental role in the case of ordinary sets, does not have the same role in the case of fuzzy sets". And as mentioned before, the first three papers on fuzzy topology written by Chang [8] and Wong [54, 55] did not even consider the concept.

In his third paper [56], Wong gave the first definitions for fuzzy point and fuzzy membership which we will refer to respectively as "the excluding definition" and the "strict inequality definition".

Definition 1.2.*1 (The excluding definition)

A fuzzy point in X is a fuzzy set e given, for some $x_0 \in X$ and $\lambda \in I - \{0,1\}$ by,

$$e(x) = \begin{cases} \lambda & x = x_0 \\ 0 & x \in X - \{x_0\}. \end{cases}$$

According to this definition, crisp sets of singleton support are excluded from being fuzzy points.

Definition 1.2.*2 (The strict inequality definition)

Let A and e be a fuzzy set and a fuzzy point in X respectively. We say that e belongs to A and write $e \Leftarrow A$ iff $e(x) < A(x)$, for all $x \in X$. Otherwise we write $e \not\Leftarrow A$.

Assuming definitions 1.2.*1 and 1.2.*2, Wong claimed that a fuzzy point e "belongs" to the union of a nonempty family $\mathbf{C}$ of fuzzy sets iff there is some fuzzy set $A \in \mathbf{C}$, such that $e < A$. (Theorem 3.1 in [56]). Other assertions (e.g. theorem 3.2 in [56]) were deduced from that claim. Gottwald [25] constructed an example to show the falsehood of these theorems. In the following, we present a much simpler counterexample that contradicts theorem 3.1 in [56].

Counterexample 1.2.*3

Let $X = \{x, y, z\}$ and define the fuzzy sets A and B in X by:

$$A(t) = \begin{cases} \frac{1}{2} & t = x,y \\ 0 & t = z \end{cases} \quad \text{and} \quad B(t) = \begin{cases} \frac{1}{2} & t = x,z \\ 0 & t = y \end{cases}.$$

Then $A \bigcup B = \eta_{\frac{1}{2}}$. Now, consider the fuzzy point e in X with support z and non-zero value $\frac{1}{3}$. Then, $e(t) = \begin{cases} \frac{1}{3} & t = z \\ 0 & t \neq z \end{cases}$. Thus $e(t) < \frac{1}{2} = \eta_{\frac{1}{2}}(t) = (A \bigcup B)(t)$, for all $t \in X$.

Hence $e \prec A \bigcup B$. On the other hand, we have $e(z) = \frac{1}{3} \nless 0 = A(z)$ and $e(y) = 0 \nless 0 = B(y)$. Therefore, e "belongs" to neither A nor B. $\square$

Wong's definitions also lead to the following undesirable consequence:

Consequence 1.2.*4

If A is a fuzzy set in X with support other than X, then no fuzzy point belongs to A.

Proof

Let supt $A \neq X$, then $(\text{supt } A)^c \neq \phi$. Let $x \in (\text{supt } A)^c$. Hence $A(x) = 0$. If e is a fuzzy point in X, then $e(x) \nless 0 = A(x)$, and so, $e \not\in A$. □

Implied in consequence 1.2.*4 is that the empty fuzzy set ϕ_X is not the unique solution for the assertion " $e \not\in A$, for all fuzzy points e in X". This deprives ϕ_X of the uniqueness associated with the empty set in the classical case.

Sarkar [45, 46], Srivastava et al [47, 48], Deng [13], Bülbül [6,7] and others chose to adopt "the excluding definition" of fuzzy point while rejecting "the strict inequality definition" of fuzzy membership and replacing it by the following definition to which we shall refer by the name "the proper membership definition". However, they retained the notion of fuzzy inclusion as given in definition 1.1.9.

Definition 1.2.*5 (Proper membership)

Let A be a fuzzy set and e a fuzzy point (in the sense of definition 1.1.*1) of support x. Then e is said to belong to (be in) A iff $e(x) < A(x)$. We write $e \rhd A$.

Starting from definitions 1.2.*1 and 1.2.*5, Srivastava et al [47] arrived at a correct version of theorem 3.1 in [56]. However, these definitions have two undesirable features. First, 1.2.*5 does not agree with the definition of fuzzy inclusion since we can have a fuzzy set e with a singleton support which is contained in some other fuzzy set A but which, as a fuzzy point, does not belong to A. That is, if $x \in$ supt A and $\lambda = A(x)$, then for the fuzzy point e with supt $e = x$ and $e(x) = \lambda$, we have e not in A although, as a fuzzy set, e is contained in A.

The other undesirable result is that, as Pu and Liu [43] pointed out, a classical point would not be a special case of fuzzy point. This view is also shared by Kerre [30].

Now, we introduce the definition of fuzzy points and fuzzy membership which we propose to adopt.

Definition 1.2.1 (Pu & Liu [43])

A fuzzy set in X is said to be a fuzzy point in X iff it has a singleton support.

If e is a fuzzy point in X, then according to this definition, there is a point x in X and a number λ in $I - \{0\}$ such that $e(x) = \lambda$ and $e(y) = 0$, for all $y \in X - \{x\}$. The number λ is called the value of e. A fuzzy point of support x and value λ is denoted by $(x)_\lambda$.

Before moving to the next definition, it may be worthwhile to point out that our notation, $(x)_\lambda$, differs from that of Pu & Liu and others which is simply x_λ. It was motivated by the fact that our work contains a great amount of discussion involving fuzzy sequences and hence a lot of indexing. The clumsiness resulting from adding subscripts to an already indexed symbol is, as pointed out by Tong [50], very inconvenient and confusing.

Our convention allows the recognition of the indices by the separation provided by the parentheses. Thus $(x_n)_\lambda$ is a fuzzy point of support x_n and value λ while $(x)_{\lambda_n}$ is a fuzzy point of support x and value λ_n.

Definition 1.2.2 (Pu & Liu [43])

A fuzzy point with value 1 is called a crisp point.

Definition 1.2.3 (Pu & Liu [43])

Let $e = (x)_\lambda$ be a fuzzy point in X. The fuzzy point e is said to belong to (be in) a fuzzy set A iff $\lambda \leq A(x)$. We denote this by $e \in A$.

If e does not belong to A, then we write $e \notin A$.

It follows that e belongs to A iff, as a fuzzy set, e is contained in A. This is actually the way Wang [51] defined fuzzy membership.

Definition 1.2.4

A fuzzy point is said to properly belong to a fuzzy set if it "belongs" to that fuzzy set in the sense of the proper membership definition 1.2.*5. Thus $(x)_\lambda$ properly belongs to A iff $(x)_\lambda \rhd A$ (or equivalently $\lambda < A(x)$). A fuzzy point that properly belongs to A is called a proper point of A.

We have the following consequences of definitions 1.2.1 and 1.2.3.

Consequence 1.2.5

For any fuzzy point e in X, we have $e \in F_X$ and $e \notin \Phi_X$.

Proof

This follows from consequence 1.1.16. $\square$

The previous result provides a convenient notation for indicating that a given mathematical object is a fuzzy point in a certain set. In place of the sentence "e is a fuzzy point in X", we can substitute the symbolic expression "$e \in F_X$". We recall that consequence 1.1.16 legitimized the use of the notation "$A \subseteq F_X$" as an equivalent way of saying that A is a fuzzy set in X.

Consequence 1.2.6

Let $A, B \subseteq F_X$.

$A \subseteq B$ iff for every $e \in A$, we have $e \in B$.

Proof

The trivial case is a corollary of consequence 1.1.17(b). To show the converse, let $x \in \text{supt } A$ and $\lambda = A(x)$. Then $(x)_\lambda \in A$ and hence, by hypothesis, $(x)_\lambda \in B$. But then $\lambda \leq B(x)$. Thus $A(x) \leq B(x)$. Therefore, by definition 1.1.9, $A \subseteq B$. $\square$

Consequence 1.2.7

Let C be a nonempty collection of fuzzy sets in X. For every $e \in F_X$, we have,

(a) If $e \in A_0$, for some $A_0 \in C$, then $e \in \bigcup_{A \in C} A$

(b) $e \in \bigcap_{A \in C} A$ iff $e \in A$, for all $A \in C$

Proof

(a) Since $A_0 \subseteq \bigcup_{A \in C} A$ by consequence 1.1.20(a), then given $e \in A_0$, we have $e \in \bigcup_{A \in C} A$ by consequence 1.2.6.

To show that the converse need not hold, let $\lambda \in (0,1)$, and consider the family $C = \{\eta_\delta : \delta \in (0,\lambda)\}$ of constant sets in X. Let $x \in X$ and $e = (x)_\lambda$. Then $e \in \bigcup_C \eta_\delta$ but $e \notin \eta_\delta$, for all $\eta_\delta \in C$.

(b) If $e \in \bigcap_{A \in C} A$, then by combining consequence 1.1.20(a) and 1.2.6, we get $e \in A$, for all $A \in C$.

On the other hand if for every $A \in C$, we have $e \in A$, then $e \in \bigcap_{A \in C} A$, by consequence 1.1.20(b). $\square$

Before concluding this section, we mention three interesting remarks concerning fuzzy points.

The first has to do with the union of fuzzy points belonging to a fuzzy set. Srivastava, et al [47] observed that a fuzzy set is the union of all fuzzy points 'belonging' to it in the sense of the "proper membership" definition 1.2.*5. Thus, in our terminology, a fuzzy set is the union of all of its proper points. It is easy to show that a fuzzy set is the union of all of its fuzzy points (in the sense of our definitions). Moreover if for a fuzzy set A, we call the fuzzy point of support x and value A(x), the <u>dominating</u> point of A at x, then a fuzzy set is obviously also equal to the union of all of its dominating points.

The second is about Wong's motivation for his strict inequality definition of fuzzy membership. Wong was interested in preserving in fuzzy theory the classical fact that a point never belongs to a point that is distinct from it. If we refer to any couple of fuzzy points sharing the same support as comparable, then, according to Pu and Liu's definition of fuzzy membership, no two fuzzy points which are not comparable belong to each other. Since distinct classical points are surely not comparable, then we still have a form of generalization of the classical fact that so much concerned Wong.

The third is that a fuzzy point can belong to both a fuzzy set and its complement. For example let $A \subseteq F_X$, $x \in X$ and $A(x) = \frac{1}{4}$, then $A^c(x) = \frac{3}{4}$. But then for the fuzzy point $e = (x)_{\frac{1}{5}}$, we have $e \in A$ and $e \in A^c$. Thus in fuzzy thoery, the relation '$\notin$' does not mimic its classical counterpart in assuming the role of the antithesis of '$\in$' with respect to complementation. The ingenious device of quasi-coincidence provides us with a fuzzy version of that duality as we will see later. It also leads to a fuzzy extension of the classical assertion behind theorem 3.1 in Wong [56] (namely the converse of 1.2.7(a)) which is still not reached in the context of our definition of fuzzy membership.

3. Quasi-Coincidence

Definition 1.3.1 (Pu and Liu)

Let $e = (x)_\lambda \in F_X$ and $A \subseteq F_X$. e is said to be quasi-coincident with A iff $A(x) + \lambda > 1$. We denote that by eqA. If e is not quasi-coincident with A, we write $e\not{q}A$. The relation $(x)_\lambda qA$ will often be expressed in the equivalent form "$A(x) > 1 - \lambda$".

In the following, let $A, B \subseteq F_X$.

Definition 1.3.2

The set $\{x \in X : A(x) + B(x) > 1\}$ is called the set of quasi coincidence for A and B and denoted by $SQ(A,B)$.

We say that A is quasi-coincident with B at x and write $Aq(x)B$ iff $x \in SQ(A,B)$. We say that A is quasi-coincident with B and write AqB iff $SQ(A,B) \neq \phi$. Otherwise, we say that A is nowhere quasi-coincident with B and write $A\not{q}B$. We say that A is everywhere quasi-coincident with B and write $Aq(X)B$ iff $SQ(A,B) = X$.

Since the quasi-coincidence relation is, obviously, symmetric, the statements "A is quasi-coincident with B" and "B is quasi-coincident with A" are equivalent and will sometimes be replaced by the statement "A and B are quasi-coincident (with each other)".

It is also clear that $SQ(A,B) \subseteq$ supt $(A \cap B)$. Note that the equality holds for crisp sets.

Consequence 1.3.3

(a) Let $A, B \subseteq F_X$.

$A = B$ iff for all $e \in F_X$, we have eqA iff eqB.

(b) Let $e, d \in F_X$.

$e = d$ iff for all $A \subseteq F_X$, we have eqA iff dqA.

Proof

This is obvious. □

Consequence 1.3.4

(a) Every fuzzy point is quasi-coincident to the full set and no fuzzy point is quasi-coincident to the empty fuzzy set.

(b) If $A \subseteq F_X$ and for every fuzzy point e in X, we have $e\not{q}A$ (respectively eqA), then $A = \phi_X$ (respectively $A = F_X$).

(c) For every fuzzy set (respectively every nonempty fuzzy set) A in X, we have $A\not{q}\phi_X$ (respectively AqF_X).

Proof

(a) Let $e = (x)_\lambda \in F_X$, then $0 < \lambda \leq 1$ and hence $F_X(x) + \lambda = 1 + \lambda > 1$ and $\phi_X(x) + \lambda = 0 + \lambda \leq 1$. Thus eqF_X and $e\not{q}\phi_X$.

(b) Considering part (a), it is enough to show that if $A \neq \phi_X$ (respectively $A \neq F_X$), then there is a fuzzy point e in

X such that eqA (respectively $e\not{q}A$).

First, let $A \neq \phi_X$, then there is an $x \in X$ such that $A(x) > 0$. Let $e = (x)_1$, then $A(x) + 1 > 1$ and so eqA.

Now, let $A \neq F_X$, then there is a $y \in X$ such that $A(y) < 1$. Let $\lambda \in (A(y),1)$ and consider the fuzzy point $d = (y)_{1-\lambda}$. We have, $A(y) + (1-\lambda) < \lambda + (1-\lambda) = 1$. Hence $d\not{q}A$.

(c) First, let $A \subseteq F_X$. Then $SQ(A,\phi_X) = \{x : A(x) + 0 > 1\} = \phi$, since $A(x) \leq 1$. Hence $A\not{q}\phi_X$.

Now, let $\phi \subset A \subseteq F_X$. Then there is an $x \in X$ such that $A(x) > 0$. But then $A(x) + F_X(x) = A(x) + 1 > 1$. Hence AqF_X . □

Consequence 1.3.5

Let $A, B, C \subseteq F_X$, $B \subseteq C$, $y \in X$ and $e = (x)_\lambda \in F_X$.

Then,

(a) eqB implies eqC

(b) $Aq(y)B$ implies $Aq(y)C$.

Proof

(a) Let eqB. We have; $B(x) + \lambda > 1$ and $C(x) \geq B(x)$. But then $C(x) + \lambda > 1$, and hence eqC.

(b) Let $Aq(y)B$. Then $A(y) + B(y) > 1$. But $C(y) \geq B(y)$. Therefore, $A(y) + C(y) > 1$, and so, $Aq(y)C$. □

It is worth mentioning that the q-relation is not transitive. For instance $\eta_{\frac{1}{2}} q\, \eta_{\frac{3}{4}}$ and $\eta_{\frac{3}{4}} q\, \eta_{\frac{1}{3}}$ but $\eta_{\frac{1}{2}} \not{q}\, \eta_{\frac{1}{3}}$. Nor is it reflexive ($\eta_{\frac{1}{3}} \not{q}\, \eta_{\frac{1}{3}}$).

The next consequence of definitions 1.3.1 and 1.3.2 provides a substitute for theorem 3.1 in [56]. (See discussion following definition 1.2.*2).

In the following let $\mathbf{C}$ be a nonempty collection of fuzzy sets, $B \subseteq F_X$, $e \in F_X$ and $x \in X$.

<u>Consequence 1.3.6</u>

(a) $Bq(x)(\bigcup_{A \in \mathbf{C}} A)$ iff $Bq(x)A_0$, for some $A_0 \in \mathbf{C}$.

(b) $eq(\bigcup_{A \in \mathbf{C}} A)$ iff eqA_0, for some $A_0 \in \mathbf{C}$ (Pu and Liu [43]).

<u>Proof</u>

It is enough to prove (a), since (b) is a corollary of (a).

First, let $Bq(x)A_0$, for some $A_0 \in \mathbf{C}$. Since, $A_0 \subseteq \bigcup_{A \in \mathbf{C}} A$ by consequence 1.1.20(a), then $Bq(x)(\bigcup_{A \in \mathbf{C}} A)$, by consequence 1.3.5(b)

Now, let $Bq(x)(\bigcup_{A \in \mathbf{C}} A)$. Let $\mathbf{b} = \{A(x) : A \in \mathbf{C}\}$ & $\lambda = \sup \mathbf{b}$. Then $\lambda > 1 - B(x)$. Let $\delta \in (1-B(x), \lambda)$. But then δ is not an upper bound for $\mathbf{b}$ and so, there is an $A_0 \in \mathbf{C}$, such that $A_0(x) \geq \delta$. Hence,

$$B(x) + A_0(x) \geq B(x) + \delta > B(x) + 1 - B(x) = 1.$$

Therefore, $Bq(x)A_0$. □

Consequence 1.3.7

(a) $Bq(x)(\bigcap_{A\in C} A)$ implies $Bq(x)A$, for all $A \in C$. The converse holds for finite C.

(b) $eq(\bigcap_{A\in C} A)$ implies eqA, for all $A \in C$. The converse holds for finite C.

Proof

Since (b) follows from (a), it is sufficient to show only (a).

Let $Bq(x)(\bigcap_{A\in C} A)$. For all $A \in C$, by consequence 1.1.20(a), we have, $\bigcap_{A\in C} A \subseteq A$, and hence, by consequence 1.3.5(b), $Bq(x)A$.

If C is finite, then $(\bigcap_{A\in C} A)(x) = \min\{A(x) : A \in C\} = A_0(x)$, for some $A_0 \in C$. Thus if $A(x) + B(x) > 1$, for all $A \in C$ then $B(x) + A_0(x) > 1$, and so, $Bq(x)(\bigcap_{A\in C} A)$.

To show that the converse is not true for nonfinite C. Let $C = \{\eta_{\frac{1}{2}+\frac{1}{n}} \subseteq F_X : n \in N\}$. Consider the fuzzy set $\eta_{\frac{1}{2}}$ (respectively the fuzzy point $(x)_{\frac{1}{2}}$). Then $\eta_{\frac{1}{2}}q(X)(\eta_{\frac{1}{2}+\frac{1}{n}})$ (repectively $(x)_{\frac{1}{2}}q(\eta_{\frac{1}{2}+\frac{1}{n}})$), for all $n \in N$. But $\eta_{\frac{1}{2}}\not{q}(\bigcap_{n=1}^{\infty}\eta_{\frac{1}{2}+\frac{1}{n}})$ (respectively $(x)_{\frac{1}{2}}\not{q}(\bigcap_{n=1}^{\infty}\eta_{\frac{1}{2}+\frac{1}{n}})$), since $\bigcap_{n=1}^{\infty}\eta_{\frac{1}{2}+\frac{1}{n}} = \eta_{\frac{1}{2}}$. □

As has been indicated before, the intersection of a fuzzy set A with its complement A^c is not necessarily the empty fuzzy set ϕ_X, and hence the set supt $(A \bigcap A^c)$ need not be equal to ϕ. Consequence 1.3.8 provides us with a fuzzy version of the excluded middle principle. It also explains our assertion in the introduction that the q-relation together with the ϵ-relation serves as a fuzzy extension of classical membership. The former two relations are related in the same manner as classical membership and its antithesis.

<u>Consequence 1.3.8</u> (Pu and Liu [43])

Let $A, B \subseteq F_X$ and $e \in F_X$. Then

(a) $SQ(A,A^c) = \phi$ (or equivalently $A \not q A^c$).

(b) $A \subseteq B$ iff $A \not q B^c$. Equivalently, AqB iff $A \not\subseteq B^c$.

(c) $e \in A$ iff $e \not q A^c$. Equivalently, eqA iff $e \notin A^c$.

<u>Proof</u>

(a) $SQ(A,A^c) = \{x : A(x) + A^c(x) > 1\} = \{x : 1 > 1\} = \phi$, since $A(x) + A^c(x) = A(x) + (1 - A(x)) = 1$.

(b) AqB^c iff $A(x) > 1 - (B^c)(x) = (B^c)^c(x)$, for some $x \in X$

iff $A(x) > B(x)$

iff $A(x) \leqslant B(x)$

iff $A \not\subseteq B$.

To get the equivalent form replace B^c by B.

(c) This follows from (b). $\square$

<u>Definition 1.3.9</u> (Gerla [20,21])

Let L be a complete lattice. Let $H(L)$ be the lattice L^X of L-sets with the symbols '$\vee$' and '$\wedge$' indicating join and meet in $H(L)$. Moreover, let $P(L)$ be a subfamily of $H(L)$ and $\bar{\epsilon}$ a relation from $P(L)$ to $H(L)$. Then the pair

$(\mathbf{P}(\mathbf{L}), \bar{\epsilon})$ is called an <u>L-point space</u> for $\mathbf{H}(\mathbf{L})$ if and only if the following conditions are satisfied:

(a) If $\mathbf{C}$ is a finite subfamily of $\mathbf{H}(\mathbf{L})$ and $e \in \mathbf{P}(\mathbf{L})$, then

(1) $e \mathrel{\bar{\epsilon}} \bigvee_{A \in \mathbf{C}} A$ iff $e \mathrel{\bar{\epsilon}} A_0$, for some $A_0 \in \mathbf{C}$

(2) $e \mathrel{\bar{\epsilon}} \bigwedge_{A \in \mathbf{C}} A$ iff $e \mathrel{\bar{\epsilon}} A$, for all $A \in \mathbf{C}$

(b) If 0 and 1 are the universal lower and upper bounds in $\mathbf{H}(\mathbf{L})$, then for every $e \in \mathbf{P}(\mathbf{L})$, $e \mathrel{\bar{\notin}} 0$ and $e \mathrel{\bar{\epsilon}} 1$

(c) If $A, B \in \mathbf{H}(\mathbf{L})$, then

$A = B$ iff $\{e \in \mathbf{P}(\mathbf{L}) : e \mathrel{\bar{\epsilon}} A\} = \{e \in \mathbf{P}(\mathbf{L}) : e \mathrel{\bar{\epsilon}} B\}$

(d) If $e, d \in \mathbf{P}(\mathbf{L})$, then

$e = d$ iff $\{A \in \mathbf{H}(\mathbf{L}) : e \mathrel{\bar{\epsilon}} A\} = \{A \in \mathbf{H}(\mathbf{L}) : d \mathrel{\bar{\epsilon}} A\}$.

In particular an I-point space for $\mathbf{H}(I)$ is called a fuzzy point space.

Gerla refers to the L-point space $(\mathbf{P}(\mathbf{L}), \bar{\epsilon})$ as <u>good</u> if and only if condition (a) holds for arbitrary subfamilies $\mathbf{C}$ of $\mathbf{H}(\mathbf{L})$. This is clearly the "obvious" generalization of the properties of the classical "point space" $(\mathbf{P}(\{0,1\}), \epsilon) = (X, \epsilon))$, where "$\epsilon$" is the classical membership relation.

However this "goodness" in the sense of Gerla is stronger than we need for the task of fuzzifying classical topological assertions involving points. Gerla [20] also refers to $(\mathbf{P}(\mathbf{L}), \bar{\epsilon})$ as a disjunction L-point space iff the condition (a)1 holds for arbitrary subfamilies $\mathbf{C}$ of $\mathbf{H}(\mathbf{L})$. The following is an adaptation of this idea.

<u>Definition 1.3.10</u>

Let $(\mathbf{P}(I)), \bar{\epsilon})$ be a fuzzy point space. Then $(\mathbf{P}(I), \bar{\epsilon})$ is said to be <u>suitable</u> if and only if for every family C of fuzzy sets, the following condition is satisfied:

(a) (1)' $e \bar{\in} \bigcup_{A \in C} A$ iff $e \bar{\in} A_0$, for some $A_0 \in C$

Corollary 1.3.11

Let M be the family of all fuzzy points in X. Then the pair (M, q) is a suitable fuzzy point space.

Proof

Obviously $M \subseteq H(I)$ and q is a relation from M to $H(I)$. The conditions (a) and a(1)' are satisfied by consequences 1.3.6(b) and 1.3.7(b). Conditions (c) and (d) are satisfied by consequence 1.3.3. Lastly consequence 1.3.4(a) satisfies condition (b). Therefore, (M, q) is a suitable I-point (or fuzzy point) space for $H(I)$. □

However, (M, q) is not the only suitable fuzzy point space for $H(I)$. To show this, Let M^* be the family of all proper points of F_X (i.e.the noncrisp fuzzy points in X) and '$\rhd$' the proper membership symbol of definition 1.2.4. Then $\rhd$ is a relation from M^* to $H(I)$. It turns out that the pair $(M^*, \rhd)$ is also a suitable fuzzy point space.

The question to be asked is: can $(M^*, \rhd)$ be a 'reasonable' alternative for $(M. q)$? The answer, in our opinion, is an "<u>almost</u> yes". No other author has illustrated this point for us more than Hu [26].

Hu introduced the idea of 'dual point'. If $(x)_\lambda \in M^*$ (i.e. $\lambda \in I - \{0,1\}$), then the fuzzy point $(x)_{1-\lambda}$ is called the <u>dual</u>

point of $(x)_\lambda$ (it was also called the complementary point by Deng [12]). If we denote the dual point of e by $\hat{e}$, then given a fuzzy set A in X, we have eqA iff $\hat{e} \rhd A$.

Thus the fuzzy point space $(M^*, \rhd)$ can also be a cornerstone for a non-pointless fuzzy topology. This is exactly what Hu and other authors chose to adopt.

However, $(M^*, \rhd)$ has the disadvantages (explained in section (2)) that do not allow it to reduce to the classical case. In contrast, classical points and classical membership are respectively a special case of fuzzy points and the q-relation as expressed in the fuzzy point space (M, q).

The different choices of fuzzy point spaces led to alternative ways of fuzzifying some topological concepts. Unfortunately, different fuzzy extensions of a given concept were sometimes given the same name. In the paragraph following proposition 3.1.26 in part III, we discuss a particular case where such a problem occurs.

PART II

FUZZY TOPOLOGICAL SPACES

Different definitions of fuzzy topology have appeared since Chang [8] introduced the concept. A radical alternative to Chang's notion is the one proposed by Lowen [33]. Lowen requires that a fuzzy topology should have one more axiom, namely that it includes the constant sets. Here we adopt Chang's definition of fuzzy topology and consider that of Lowen as a special case (definition 2.2.1). Such preference is motivated by two reasons. The first is connected with the so called 'crisp fuzzy topology' (example 2.2.2). The other reason has to do with the theory of fuzzy convergence which we explain in part III.

In section (1), we introduce the basic notions of fuzzy topology. Of interest are the fuzzy extension of a classical assertion about a base for a topology (proposition 2.1.10) and an examination of the idea of a fuzzy subbase (proposition 2.1.11). A characterization of open fuzzy sets is also given (proposition 2.1.9).

Section (2) contains special examples of fuzzy topological spaces of which the fuzzy topological spaces in the sense of Lowen are one. We concentrate in particular on the natural fuzzy topology and produce a base for it. Other related concepts such as the initial and the Martin topologies are discussed. We also introduce the "λ-cut fuzzy topology" which we use in the following parts of the thesis to provide important counterexamples.

1. Basic Definitions and Assertions

In the following, let X be a set.

Definition 2.1.1 (Chang)

A nonempty collection J of fuzzy sets in X is called a fuzzy topology on F_X if the following conditions are met

(a) Φ_X , $F_X \in J$

(b) $A \cap B \in J$, whenever $A, B \in J$

(c) $\bigcup_{A \in C} A \in J$, for every nonempty subfamily C of J.

In the following, let J be a fuzzy topology on F_X.

Definition 2.1.2

(a) The pair (F_X, J) is called a fuzzy topological space (or in short, fts)

(b) A subset O of F_X is said to be J-open or open in the fts (F_X, J) iff $O \in J$

(c) A subset E of F_X is said to be J-closed or closed in the fts (F_X, J) iff $E^c \in J$

(d) A fuzzy topology J_1 on F_X is said to be stronger (or larger) than J iff $J \subseteq J_1$. If J_1 is stronger than J, the fuzzy topology J is said to be weaker (or smaller) than J_1.

Definition 2.1.3 (Wong [56])

A subfamily $\mathbf{B}$ of J is said to be a base for J iff for each $0 \in J$, there is a subfamily $\mathbf{C}$ of $\mathbf{B}$ such that $0 = \bigcup_{B \in \mathbf{C}} B$.

Definition 2.1.4 (Wong [56])

A subfamily $\mathbf{D}$ of J is said to be a subbase for J iff the family of all finite intersection of members of $\mathbf{D}$ forms a base for J.

Definition 2.1.5 (Pu & Liu)

Let (F_X, J) be a fts, $e \in F_X$ and $A \subseteq F_X$. The fuzzy set A is said to be a neighbourhood (Q-neighbourhood) of e iff there is an $0 \in J$ such that $e \in 0 \subseteq A$ (respectively $eq0 \subseteq A$).

The family of all neighbourhoods (Q-neighbourhoods) of e is called the system of neighbourhoods (respectively the system of Q-neighbourhoods) of e and denoted by $\mathbf{V}_e$ (respectively $\mathbf{Q}_e$).

It follows from the definition that while a neighbourhood of a fuzzy point e necessarily contains it, a Q-neighbourhood of e need not contain e. To illustrate that, let J be the fuzzy topology on F_X given by $J = \{F_X, \Phi_X, Z^{(\frac{2}{5})}\}$, where $\Phi \neq Z \subset X$. Let $x \in Z$; $e = (x)_{\frac{4}{5}}$ and $A = Z^{(\frac{3}{5})}$. Then $eqZ^{(\frac{2}{5})} \subseteq A$, and so A is a Q-neighbourhood of e. But $e = (x)_{\frac{4}{5}} \notin A$.

In keeping with our stated intentions to improve on assertions by other authors that contain incorrect claims (so as to avoid misunderstanding when applying them to our study), we discuss the following proposition given without proof by Pu and Liu (proposition 2.2 in [43]).

Proposition 2.1.*1

For every $e \in F_X$, let $\mathfrak{U}_e$ be a family of subsets of F_X such that the following conditions are satisfied

(1) eqU, for all $U \in \mathfrak{U}_e$ (respectively $e \in U$, for all $U \in \mathfrak{U}_e$)

(2) $U \cap V \in \mathfrak{U}_e$, whenever $U, V \in \mathfrak{U}_e$

(3) $V \in \mathfrak{U}_e$, whenever $U \subseteq V$ and $U \in \mathfrak{U}_e$.

Then the family J of all subsets A of F_X, such that $A \in \mathfrak{U}_d$ whenever $d \in F_X$ and dqA (respectively $d \in A$) is a fuzzy topology on F_X.

The "q" version of proposition 2.1.*1 is easily proved, but the "ϵ" version is untrue as the following counterexamples shows.

Counterexample 2.1.6

For a set X of at least two points, let $e = (x)_{\frac{1}{3}} \in F_X$, for some $x \in X$. Let $\mathfrak{U}_{(y)_\lambda} = \{ U : \lambda \leq U(y)\}$, for every fuzzy point $(y)_\lambda$ in X that is distinct from e and let $\mathfrak{U}_e = \{U : \frac{1}{2} \leq U(x)\}$. Then for every $d \in F_X$ the family $\mathfrak{U}_d$ satisfies conditions (1), (2) and (3) of proposition 2.1.*1. Since for every $y \in X$ and $\lambda \in I - \{0\}$, such that $(y)_\lambda \neq e$, we have $\lambda \leq U(y)$, and hence $U \in \mathfrak{U}_{(y)_\lambda}$ whenever $(y)_\lambda \in U \subseteq F_X$, then the family J as described in proposition 2.1.*1 can be expressed explicitly by:

$$J = \{U : (x)_{\frac{1}{3}} \notin U \text{ or } \tfrac{1}{2} \leq u(x)\} = \{U : U(x) \notin [\tfrac{1}{3} , \tfrac{1}{2})\}.$$

Consider the family $\mathbf{C} = \{\eta_{\delta_n} : \delta_n = \frac{n}{3(n+1)}, \quad n \in \mathbf{N}\}$.

Since $\eta_{\delta_n}(x) = \frac{n}{3(n+1)} < \frac{1}{3}$, for all $n \in \mathbf{N}$, then $\mathbf{C} \subset J$.

But $\bigcup_{\mathbf{C}} \eta_{\delta_n} = \eta_{\frac{1}{3}} \notin J$, since $\eta_{\frac{1}{3}}(x) = \frac{1}{3} \in [\frac{1}{3}, \frac{1}{2})$.

But then J is not a fuzzy topology on F_X. □

Thus, we state a modified version of proposition 2.1.*1.

<u>Proposition 2.1.7</u> (Pu and Liu)

For every $e \in F_X$, let $\mathcal{U}_e$ be a nonempty family of subsets of F_X such that eqU, for all $U \in \mathcal{U}_e$ and that conditions (2) and (3) of proposition 2.1.*1 hold. Then the family J defined by $J = \{U \subseteq F_X$: "$d \in F_X$ and dqU" implies $U \in \mathcal{U}_d\}$ is a fuzzy topology on F_X. Moreover we have $\mathcal{U}_e = \mathcal{Q}_e$ (with respect to this topology) iff the following condition holds:

If $U \in \mathcal{U}_e$, then there is a $V \in \mathcal{U}_e$, such that $V \subseteq U$ and $V \in \mathcal{U}_d$, whenever dqV..............(4)

<u>Proof</u> (cf Kotze [31])

J is closed under arbitrary union (finite intersection) thanks to the combination of consequences 1.3.6(b) and 1.1.20(a) and condition (3) of proposition 2.1.*1 (respectively consequence 1.3.7(b) and condition (2)). ϕ_X is in J since $e \not{q} \phi_X$, for all $e \in F_X$, by consequence 1.3.4(a). F_X is guaranteed to be in $\mathcal{U}_e$ for all $e \in F_X$ by condition (3) and the fact that $\mathcal{U}_e$ is nonempty, and so $F_X \in J$.

If $\mathcal{U}_e$ is the Q-neighbourhood system $\mathcal{Q}_e$ of e, for $e \in F_X$, then conditions (4) follows from definition 2.1.5.

Conversely, assume that the condition holds, $e \in F_X$, and $U \in \mathcal{Q}_e$. Then there is an $O \in J$ such that $eqO \subseteq U$. But then $O \in \mathcal{U}_e$, by definition of J, and hence $U \in \mathcal{U}_e$, by condition (3). Therefore, $\mathcal{Q}_e \subseteq \mathcal{U}_e$. Now, let $U \in \mathcal{U}_e$. Then by condition (4), there is a $V \in \mathcal{U}_e$ such that $V \subseteq U$ and $V \in \mathcal{U}_d$ whenever dqV. Hence $V \in J$, by definition of J and $eqV \subseteq U$. Therefore, $U \in \mathcal{Q}_e$, and hence $\mathcal{U}_e \subseteq \mathcal{Q}_e$. Thus $\mathcal{U}_e = \mathcal{Q}_e$. □

Proposition 2.1.8 (Pu & Liu)

Let (F_X, J) be a fts.

A nonempty subfamily $\mathcal{B}$ of J is a base for J iff for all $O \in J$ and $e \in F_X$ such that eqO there is a $B \in \mathcal{B}$ such that $eqB \subseteq O$.

Proof

See proposition 2.4 in Pu & Liu [43]. □

The following characterization of open fuzzy set was formulated independently by us and Ma and Yu [37].

Proposition 2.1.9 (A characterization of open fuzzy sets)

$A \in J$ iff for all $e \in F_X$ such that eqA, there is an $O \in J$ such that $eqO \subseteq A$.

Proof (cf Pu and Liu [43])

(An alternative proof can be found in Ma and Yu [37])

First if $A \in J$, take $O = A$.

Conversely, assume $A \notin J$ and let $C = \{O : O \in J, O \subset A\}$. Since $\bigcup_{O \in C} O \in J$, then $A \neq \bigcup_{O \in C} O$, and combining that with consequence 1.1.20, we have, $0 \subseteq \bigcup_{0 \in C} 0 \subset A$, for all $0 \in C$. But then there is an $x \in X$, such that $0(x) \leq (\bigcup_{0 \in C} 0)(x) < A(x)$, for all $0 \in C$. Let $\lambda = (\bigcup_{0 \in C} 0)(x)$,then $\lambda < 1$, and so $1 - \lambda > 0$. Let $e = (x)_{1-\lambda} \in F_X$. Then $A(x) + e(x) > \lambda + 1 - \lambda = 1$, but, $0(x) + e(x) \leq \lambda + 1 - \lambda = 1$, for all $0 \in C$. Hence eqA, but $e\not{q}0$, for all $0 \in C$. □

The following proposition is an extension of a useful classical assertion.

<u>Proposition 2.1.10</u>

Let **B** be a nonempty family of subsets of F_X . Then **B** is a base for some fuzzy topology on F_X iff the following two conditions are satisfied.

(1) For all $e \in F_X$,there is a $B \in \mathbf{B}$ such that eqB

(2) For every $B_1, B_2 \in \mathbf{B}$ and $e \in F_X$ such that $eq(B_1 \cap B_2)$, there is a $B \in \mathbf{B}$ such that $eqB \subseteq B_1 \cap B_2$

<u>Proof</u>

First, let **B** be a base for some fuzzy topology J on F_X . Let e be a fuzzy point in X, then eqF_X and hence by proposition 2.1.8, there is a $B \in \mathbf{B}$ such that $eqB \subseteq F_X$. Thus condition (1) is satisfied. To check for condition (2), let B_1, $B_2 \in \mathbf{B}$, $e \in F_X$ and $eq(B_1 \cap B_2)$. Since $\mathbf{B} \subseteq J$, then $B_1 \cap B_2 \in J$, and hence by proposition 2.1.8 there is a $B \in \mathbf{B}$ such that $eqB \subseteq B_1 \cap B_2$. Thus condition (2) is satisfied.

Conversely, let $J = \{A$: "eqA and $e \in F_X$" implies $eqB \subseteq A$, for some $B \in \mathbf{B}\}$. Considering proposition 2.1.8, it is enough to show that J is a fuzzy topology on F_X which contains $\mathbf{B}$.

First $\mathbf{B} \subseteq J$, since $B \subseteq B$, for all $B \in \mathbf{B}$. The full set is in J, by condition (1) and the fact that $B \subseteq F_X$, for all $B \in \mathbf{B}$ (consequence 1.1.16). The empty fuzzy set is in J, since $e\not{q}\Phi_X$, for all $e \in F_X$.

Now, let $A_1, A_2 \in J$ and $eq(A_1 \cap A_2)$. Then there are B_1, $B_2 \in \mathbf{B}$ such that $eqB_1 \subseteq A_1$ and $eqB_2 \subseteq A_2$. Hence, by consequences 1.3.7(b) and 1.1.21(b), $eq(B_1 \cap B_2) \subseteq A_1 \cap A_2$. But then by condition (2), there is a $B \in \mathbf{B}$ such that $eqB \subseteq B_1 \cap B_2$. Hence $eqB \subseteq A_1 \cap A_2$ by consequence 1.1.17(b). Therefore, $A_1 \cap A_2 \in J$.

And last, let $\mathbf{C} \subseteq J$ and $eq(\bigcup_{A \in \mathbf{C}} A)$. Then, by consequence 1.3.6(b), there is an $A_0 \in \mathbf{C}$ such that eqA_0. Hence $A_0 \in J$, and so there exists a $B \in \mathbf{B}$ such that $eqB \subseteq A_0$. But then by consequence 1.1.20(a), $eqB \subseteq \bigcup_{A \in \mathbf{C}} A$. Hence $\bigcup_{A \in \mathbf{C}} A \in J$.

Thus J is a fuzzy topology on F_X. □

In classical topology if $\mathbf{C}$ is a family of subsets of X, then it is a subbase for some topology on the set $\bigcup_{Z \in \mathbf{C}} Z$ which is not necessarily equal to X. Thus <u>any</u> nonempty collection of subsets of X is a subbase for some topology. However, in fuzzy topology, the situation is different. For if $\mathbf{C}$ is a collection of subsets of F_X, then so would be the collection $\mathbf{B}$ of finite intersections of members in $\mathbf{C}$ and, thus $\mathbf{B}$ can be only a base for a fuzzy topology <u>on F_X</u>. But then F_X has to be the union of members of $\mathbf{B}$. Thus unlike in the classical case, an axiom is needed for a subbase for fuzzy topology.

Proposition 2.1.11 (Axiom for a subbase)

A collection $\mathbf{W}$ of subsets of F_X is a subbase for some fuzzy topology on F_X iff $\bigcup_{D\in\mathbf{W}} D = F_X$.

Proof

First, if $\mathbf{W}$ is a subbase for some fuzzy topology J on F_X , then it follows from definitions 2.1.4 and 2.1.3 that $\bigcup_{D\in\mathbf{W}} D = F_X$.

Conversely, let $\mathbf{W}$ be a nonempty family of subsets of F_X such that $\bigcup_{D\in\mathbf{W}} D = F_X$. We need to show that the family $\mathbf{B}$ of all finite intersections of members in $\mathbf{W}$ is a base for some topology on F_X . But this will be the case if $\mathbf{B}$ satisfies the two conditions of proposition 2.1.10.

Thus, first, let $e \in F_X$,then $eq(\bigcup_{D\in\mathbf{W}} D)$ and hence by consequence 1.3.6(a), there is a $D_0 \in \mathbf{W}$ such that eqD_0. Since $\mathbf{W} \subseteq \mathbf{B}$, then condition (1) of 2.1.10 is satisfied.

Now, let $B_1, B_2 \in \mathbf{B}$ and $eq(B_1 \cap B_2)$. Let us express B_1 and B_2 as $\bigcap_{n=1}^{N} D_n$ and $\bigcap_{n=1}^{K} D'_k$ respectively, for some $N, K \in \mathbf{N}$ and $\{D_1, D_2, \ldots, D_N\}$, $\{D'_1, D'_2, \ldots, D'_k\} \subseteq \mathbf{W}$. Rewrite $D'_n = D_{N+n}$. Then, by the associativity of fuzzy intersection (consequence 1.1.18(c)), we have $B_1 \cap B_2 = \bigcap_{n=1}^{N+k} D_n$. But then $B_1 \cap B_2 \in \mathbf{B}$. Let $B = B_1 \cap B_2$, then $eqB \subseteq B_1 \cap B_2$, and hence condition (2) of 2.1.10 is satisfied.

Therefore, **B** is a base for some fuzzy topology on F_X ,and so **W** is a subbase for that fuzzy topology. $\square$

2. Special Fuzzy Topological Spaces

Definition 2.2.1 (Pu and Liu [44])

A fts (F_X, L) is said to be fully stratified iff $\eta_\lambda \in L$ for all $\lambda \in (0,1)$; i.e. all constant fuzzy sets are in L.

It is clear from definitions 2.1.1 and 2.1.2 that a fuzzy topological space in the sense of Lowen [33] coincides with the definition of a fully stratified fts. In order to show the advantages and disadvantages of Lowen's definition of a fts as we progress a fully stratified fts will be referred to as a Lowen fts. The latter is therefore the only name we will use for this type of fuzzy topological space.

In the following, we give examples of some special fuzzy topological spaces. The notation introduced here will then be used throughout.

Example 2.2.2

Let τ be a topology on X. The crisp fuzzy topology $cr(\tau)$ of the topological space (X, τ) is defined by

$cr(\tau) = \{\chi_Z : Z \in \tau\}$.

Notice that the crisp fts is not Lowen. This is a disadvantage of Lowen's definition of fuzzy topology. In the words of Ganther et al [19] Lowen "has lost the concept that a fuzzy topology generalizes topology".

Example 2.2.3

The discrete fuzzy topology J_Δ on F_X is defined by $J_\Delta = \{A : A \subseteq F_X\}$.

Example 2.2.4

Let $\mathbf{b}$ be a closed from above subset of I that contains both 0 and 1. The collection $J(\mathbf{b}) = \{\eta_\lambda \subseteq F_X : \lambda \in \mathbf{b}\}$ is a fuzzy topology on F_X which we call trivial. (Notice that the from-above closedness of $\mathbf{b}$ guarantees that every subset of $\mathbf{b}$ has its least upper bound in $\mathbf{b}$ and hence that $J(\mathbf{b})$ is closed under arbitrary unions).

The trivial fuzzy topology on F_X for which $\mathbf{b} = \{0,1\}$ is called the weakest trivial or the indiscrete fuzzy topology on F_X and denoted by J_0. Thus $J_0 = \{F_X, \phi_X\}$.

The trivial fuzzy topology on F_X for which $\mathbf{b} = I$ is called the strongest trivial or the indiscrete Lowen fuzzy topology on F_X and denoted by L_0. Thus $L_0 = \{\eta_\lambda : \lambda \in I\}$.

Pu and Liu [44] used the names "purely stratified" for trivial fuzzy topological spaces and "simply stratified" for the indiscrete fts. We shall use only the names introduced in example 2.2.4 since they better describe the respective spaces.

Example 2.2.5

Let X be a one point set, say $X = \{x\}$. Then the discrete fuzzy topology on F_X coincides with the indiscrete Lowen fuzzy topology on F_X. That is;

$$J_\Delta = \{(x)_\lambda : \lambda \in (0,1]\} \bigcup \{\phi_X\} = L_0.$$

Here the indiscrete fuzzy topology $J_0 = \{F_X, \phi_X\} =$

$\{(x)_1, \phi_X\}$ is, of course, distinct from J_Δ.

Since in classical topology the discrete and the indiscrete topologies coincide in the case of one point spaces, this is a point at which Lowen's definition seems to have an advantage over Chang's.

Let b be a closed from above subset of I, such that $\{0,1\} \subseteq b$, then for any $x \in X$, the family $J(b) = \{(x)_\lambda : \lambda \in b - \{0\}\} \bigcup \{\phi_X\}$ is a trivial fuzzy topology on F_X, for the one-point set $X = \{x\}$.

This shows that unlike the case in classical topology where only one topology "can be defined" on a singleton, in fuzzy theory, there are as many fuzzy topologies on a full set with a singleton support as there are subsets b of I which contain 0 and 1 and are closed from above.

The natural fuzzy topological space

It is a known classical result (e.g. Bourbaki [4] p.360) that the family of all lower semicontinuous functions from a given topological space to the unit interval is closed under finite infima and arbitrary suprema and contains all the constant functions. In fuzzy language, such a family is actually a Lowen fuzzy topology. This special type of fuzzy topology was first mentioned by Wong [54] who called it "the semi-continuous fuzzy topology". Since then other names appeared in the literature, Weiss [53] referred to it as "the induced fuzzy topology" which was also used by Pu and Liu [43], Martin [39] and Hu[26] who also called it "the product-induced topology". Lowen

[33] gave it the name "topologically generated" which was adopted by Srivastava, et al [48], Artico and Moresco [2], Zhou [60] and others. Mohannadi and Warner [40] referred to it as "the corresponding fuzzy topology". Conrad [10] introduced the label "the natural topology" which was adapted by Ghanim et al [22] and Kotze [31] as the "natural fuzzy topology". This last name is the one most appropriate for us, since its wording does not overlap with that of other concepts contained in this thesis. For instance, we reserve the term "induced fuzzy topology" to describe the fuzzy extension of classical induction (see part V).

Throughout this thesis, we will denote the usual topology on I by the symbol 1^* while using the symbol 1^*_r to indicate the topology $\{(\lambda,1] : \lambda \in I - \{1\}\} \bigcup \{\phi,I\}$.

In the following, let (X, I) be a topological space. Recall (e.g. Bourbaki [4] p.360) that a function A is lower semi-continuous from (X, τ) to $(I, 1^*)$ iff for every $\lambda \in I - \{1\}$ and $x \in X$, such that $A(x) > \lambda$, there is a basic τ-open neighbourhood Z of x satisfying the inequality $A(y) > \lambda$, for all $y \in Z$. Equivalently, A is lower semi-continuous from (X, τ) into $(I, 1^*)$ iff $A^{-1}(\lambda,1] \in \tau$, for all $\lambda \in I - \{1\}$, that is iff A is continuous from (X, τ) into $(I, 1^*_r)$.

Definition 2.2.6

The collection of all lower semi-continuous functions from (X, τ) into $(I, 1^*)$ is called the natural fuzzy topology of (X, τ) and denoted by $\omega(\tau)$. The latter notation was introduced by Lowen [33] and has become by now standard. The pair

$(F_X, \omega(\tau))$ is called the natural fts of (X, τ).

It follows that a subset A of F_X is open in $(F_X, \omega(\tau))$ iff for every $\lambda \in I - \{1\}$, the strong λ-cut $A^{-1}(\lambda,1]$ is in τ. On the other hand a subset D of F_X is closed in $(F_X, \omega(\tau))$ iff D is an upper semi-continuous function from (X, τ) into $(I, 1^*)$, or equivalently for every $\lambda \in I - \{0\}$, $D^{-1}[0,\lambda) \in \tau$.

Since constant and crisp open sets in X are lower semi-continuous from (X, τ) into $(I, 1^*)$, we have,

$$L_0 \bigcup cr(\tau) \subseteq \omega(\tau).$$

In general, a fts (F_X, J) will be called natural if J is the natural fuzzy topology of some topology on X.

In the following, we introduce a base for the natural fts of (X, τ).

Proposition 2.2.7

(a) Let $B(\tau) = \{Z^{(\delta)} : Z \in \tau$ and $\delta \in I\}$, where $Z^{(\delta)}$ is defined as in 1.1.8 . Then $B(\tau)$ is a base for $\omega(\tau)$ and $L_0 \bigcup cr(\tau) \subseteq B(\tau)$.

(b) Let C be a base for τ, and $B'(\tau) = \{Z^{(\delta)} : Z \in C$ and $\delta \in I\}$. Then $B'(\tau)$ is a base for $\omega(\tau)$.

Proof

(a) Let $e = (x)_\lambda \in F_X$ and $O \in \omega(\tau)$ such that eqO. Then $O(x) > 1 - \lambda$ and so $O(x) > 1 - \lambda + \epsilon$, for some $\epsilon \in (0,\lambda)$. Denote $\delta = 1 - \lambda + \epsilon$, then $\delta \in I$ and $O(x) > \delta$. Let $Z = O^{-1}(\delta,1]$. Hence $x \in Z \in \tau$. Let $B = Z^{(\delta)}$, then $B \in B(\tau)$, $B(x) + e(x) = (1 - \lambda + \epsilon) + \lambda = 1 + \epsilon > 1$ and $O(y) > \delta = B(y)$, for all $y \in Z$. Therefore, $eqB \subseteq O$. But then, by proposition 2.1.8, $\mathbf{B}(\tau)$ is a base for $\omega(\tau)$.

Since every constant set and every crisp set is lower semicrisp, then they are in $\mathbf{B}(\tau)$ and so $L_0 \bigcup cr(\tau) \subseteq \mathbf{B}(\tau)$.

(b) Considering (a), it is enough to show that every member of $\mathbf{B}(\tau)$ is a union of some members of $\mathbf{B}'(\tau)$. If $B \in \mathbf{B}(\tau)$, then $B = Z^{(\delta)}$, for some $Z \in \tau$ and $\delta \in I$. But since $\mathbf{C}$ is a base for τ, there is a subfamily $\mathbf{C}'$ of $\mathbf{C}$ such that, $Z = \bigcup_{Y \in \mathbf{C}'} Y$. Hence, $B = Z^{(\delta)} = (\bigcup_{Y \in \mathbf{C}'} Y)^{(\delta)} = \bigcup_{Y \in \mathbf{C}'} Y^{(\delta)}$, by consequence 1.1.24(b). □

Corollary 2.2.8

Let $\Gamma = \{r_n : n \in \mathbf{N}\}$ be the set of all rationals in I. Then, the family $\mathbf{B}''(\tau) = \{Z^{(r_n)} : Z \in \tau,\ n \in \mathbf{N}\}$ is a base for $\omega(\tau)$. Moreover, if $\mathbf{C}$ is a base for τ, then the family $\{Z^{(r_n)} : Z \in \mathbf{C},\ n \in \mathbf{N}\}$ is a base for $\omega(\tau)$.

Proof

Considering proposition 2.2.7, it is enough to show that every member of $\mathbf{B}'(\tau)$ is the union of some members of $\mathbf{B}''(\tau)$. Let $Z^{(\delta)} \in \mathbf{B}'(\tau)$. Since $\delta \in I$, then there is a set $\mathbf{b}$ of rationals in I, such that $\delta = \sup \mathbf{b}$. But then, by consequence 1.1.24(c), $Z^{(\delta)} = \bigcup_{r \in \mathbf{b}} Z^{(r)}$. □

Corollary 2.2.9

Let τ be a topology (on X) and J a Lowen fuzzy topology that contains $cr(\tau)$. Then $\omega(\tau) \subseteq J$.

Proof

Let $\mathbf{B} = \{Z^{(\delta)} : Z \in \tau, \delta \in I\}$. Considering proposition 2.2.7, it is enough to show that $\mathbf{B} \subseteq J$. Thus, let $B \in \mathbf{B}$. Then $B = Z^{(\delta)}$, for some $Z \in \tau$ and $\delta \in I$. Since J is Lowen, $\eta_\delta \in J$, and by the hypothesis $(cr(\tau) \subseteq J)$, we have $\chi_Z \in J$. But then $\eta_\delta \cap \chi_Z \in J$. By consequence 1.1.24(a), we have, $B = Z^{(\delta)} = \eta_\delta \cap \chi_Z \in J$. Therefore, $\mathbf{B} \subseteq J$. □

Definition 2.2.10 (Lowen [33])

Let (F_X, J) be a fts.

A topology on X having the subbase $\{A^{-1}(\lambda,1] : A \in J, \lambda \in I - \{1\}\}$ is called the initial topology of J and denoted by $\iota(J)$. (In a subsequent paper [36], Lowen called it the topological modification of J).

It follows that $\iota(J)$ is the smallest topology on X that makes every member of J continuous from (X, τ) into $(I, 1_r^*)$.

It is a classical result that the family L_0 of all lower semi-continuous functions from the indiscrete topological space (X, τ_0) is actually the family of all constant functions from X. In our terminology, we write;

$$L_0 = \omega(\tau_0).$$

By definition 2.2.10, we have; $\iota(L_0) = \tau_0$.

The following proposition is a generalization of this result.

Proposition 2.2.11

Let J be a fuzzy topology on F_X. Then (F_X, J) is trivial iff $\iota(J) = \tau_0$.

Proof

First, let J be trivial and $A \in J$. Then $A = \eta_\delta$, for some $\delta \in I$. Hence for every $\lambda \in I - \{1\}$, we have,

$$A^{-1}(\lambda,1] = \begin{cases} X & \text{if } \lambda < \delta \\ \phi & \text{if } \delta \leq \lambda \end{cases}.$$ Therefore, by definition 2.2.10, $\{X, \phi\}$ is a subbase for $\iota(J)$ and so $\iota(J) = \{X, \phi\} = \tau_0$.

Now, let $\iota(J) = \tau_0$ and assume that (F_X, J) is not trivial. Then there is an $A \in J$ and $x, y \in X$, such that $A(y) > A(x)$. Let $A(x) = \delta$, then $\delta \in I - \{1\}$. Hence $A(y) > \delta$, and so $y \in A^{-1}(\delta,1]$ and thus, $A^{-1}(\delta,1] \neq \phi$. But $A^{-1}(\delta,1] \in \iota(J) = \tau_0$, and hence $A^{-1}(\delta,1] = X$. But then $x \in A^{-1}(\delta,1]$, and so $A(x) > \delta$. Thus, we arrive at a contradiction. Therefore every member of J is a constant set, and hence (F_X, J) is trivial. $\square$

The following results are direct consequences of definitions 2.2.10 and 2.2.6.

Consequence 2.2.12

Let (X, τ) be a topological space and (F_X, J) a fts. Then,

(a) $\iota(\omega(\tau)) = \tau$ and $J \subseteq \omega(\iota(J))$ (Lowen).

(b) If J_1 is a fuzzy topology on F_X, such that $J \subseteq J_1$, then $\iota(J) \subseteq \iota(J_1)$.

(c) If τ_1 is a topology on X, then

$\omega(\tau) \subseteq \omega(\tau_1)$ iff $\tau \subseteq \tau_1$ ($\omega(\tau) = \omega(\tau_1)$ iff $\tau = \tau_1$).

(d) $\omega(\iota(J))$ is the smallest natural fuzzy topology on F_X containing J (Lowen).

<u>Proof</u>

(a)1. If $Z \in \tau$, then $\chi_Z \in \omega(\tau)$ and so $Z \in \iota(\omega(\tau))$. Thus, $\tau \subseteq \iota(\omega(\tau))$. On the other hand for every $Y \in \{A^{-1}(\lambda,1] : A \in \omega(\tau),\ \lambda \in I - \{1\}\}$, we have $Y \in \tau$, by the definition of the natural fuzzy topology. Hence τ contains a subbase of $\iota(\omega(\tau))$, and so $\iota(\omega(\tau)) \subseteq \tau$. Therefore, $\iota(\omega(\tau)) = \tau$.

2. If $A \in J$, then by definition of $\iota(J)$, $A^{-1}(\lambda,1] \in \iota(J)$, for all $\lambda \in I - \{1\}$, and so $A \in \omega(\iota(J))$, by definition of the natural fuzzy topology. Therefore, $J \subseteq \omega(\iota(J))$.

(b) If $J \subseteq J_1$, then $\{A^{-1}(\lambda,1] : A \in J,\ \lambda \in I - \{1\}\} \subseteq \{A^{-1}(\lambda,1] : A \in J_1,\ \lambda \in I - \{1\}\}$ and so, by definition 2.2.10, $\iota(J) \subseteq \iota(J_1)$.

(c) First if $\omega(\tau) \subseteq \omega(\tau_1)$, then by (b), $\iota(\omega(\tau)) \subseteq \iota(\omega(\tau_1))$. By (a), $\tau = \iota(\omega(\tau))$ and $\tau_1 = \iota(\omega(\tau_1))$, and hence $\tau \subseteq \tau_1$. Similarly, if $\omega(\tau) = \omega(\tau_1)$, then $\tau = \tau_1$.

Now, let $\tau \subseteq \tau_1$ and $O \in \omega(\tau)$. Then, for every $\lambda \in I - \{1\}$; $O^{-1}(\lambda,1] \in \tau$, by definition of $\omega(\tau)$. But then, $O^{-1}(\lambda,1] \in \tau_1$, for every $\lambda \in I - \{1\}$, and so, $O \in \omega(\tau_1)$. Hence $\omega(\tau) \subseteq \omega(\tau_1)$.

Similarly if $\tau = \tau_1$, then $\omega(\tau) = \omega(\tau_1)$.

(d) Let J_1 be a natural fuzzy topology on F_X that contains J. Then $J_1 = \omega(\tau_1)$, for some topology τ_1 on X and $J \subseteq \omega(\tau_1)$. But then by (b) and (a), $\iota(J) \subseteq \iota(\omega(\tau_1)) = \tau_1$. Hence, $\omega(\iota(J)) \subseteq \omega(\tau_1) = J_1$, by (c). Thus every natural fuzzy topology on F_X that contains J also contains $\omega(\iota(J))$, and hence $\omega(\iota(J))$ is the smallest fuzzy topology containing J. □

Definition 2.2.13

Let (F_X, J) be a fts.

Then the family $\mu(J) = \{Z : \chi_Z \in J\}$ is clearly a topology on X. It is said to be the Martin topology of J.

The idea of defining such a topology is due to Martin [39] who did not give it a name.

It follows from the definition that $cr(\mu(J)) \subseteq J$.

Consequence 2.2.14

Let (X, τ) be a topological space and (F_X, J) a fts. Then,

(a) $\tau = \mu(\omega(\tau))$.

(b) $\mu(J) \subseteq \iota(J)$.

(c) If J_1 is a fuzzy topology on F_X such that $J \subseteq J_1$, then $\mu(J) \subseteq \mu(J_1)$.

Proof

(a) $Z \in \tau$ iff $\chi_Z \in \omega(\tau)$ iff $Z \in \mu(\omega(\tau))$, by definition 2.2.13. Therefore, $\tau = \mu(\omega(\tau))$.

(b) Let $Z \in \mu(J)$, then $\chi_Z \in J$, by definition of Martin's topology. Hence $Z \in \iota(J)$, by definition 2.2.10, and so $\mu(J) \subseteq \iota(J)$.

To show that the converse is false, let $J = \{F_X, \Phi_X, Z^{(\frac{1}{2})}\}$, where Z is a nonempty proper subset of X. Then $\iota(J) = \{X, \Phi, Z\} \nsubseteq \{X, \Phi\} = \mu(J)$.

(c) Let $J \subseteq J_1$ and $Z \in \mu(J)$. Then $\chi_Z \in J$, by definition 2.2.13, and so $\chi_Z \in J_1$. Hence $Z \in \mu(J_1)$. Thus $\mu(J) \subseteq \mu(J_1)$.

To show that the converse need not hold, let $J = \{\Phi_X, F_X, Z^{(\frac{1}{3})}\}$ and $J_1 = \{\Phi_X, F_X, Z^{(\frac{1}{4})}\}$, where Z is a nonempty proper subset of X. Then $\mu(J) = \{\Phi, X\} = \mu(J_1)$, but $J \neq J_1$. □

<u>Consequence 2.2.15</u>

Let (F_X, J) be a Lowen fts. Then,

(a) $\omega(\mu(J)) \subseteq J$ (Martin).

(b) $\omega(\mu(J))$ is the largest natural fuzzy topology on F_X contained in J.

(c) $\mu(J) = \iota(J)$ iff (F_X, J) is natural.

<u>Proof</u>

(a) The inequality is obtained by substituting $\mu(J)$ for τ in corollary 2.2.9. This is possible since J is Lowen and by the remark following definition 2.2.13, $cr(\mu(J)) \subseteq J$. Compare this proof with the one given in a different context by Martin (theorem 2.4 in Martin [39]) and which contains a minor error, which, however, does not affect the outcome.

(b) Let T be a natural fuzzy topology on F_X that is contained in J. Then $T = \omega(\tau)$, for some topology τ on X and $\omega(\tau) \subseteq J$. But then by consequence 2.2.13(c), $\mu(\omega(\tau)) \subseteq \mu(J)$. But, by consequence 2.2.14(a), $\mu(\omega(\tau)) = \tau$, and hence, $\tau \subseteq \mu(J)$. Therefore, by consequence 2.2.12(c) $T = \omega(\tau) \subseteq \omega(\mu(J))$. Thus every natural fuzzy topology on F_X that is contained in J is also contained in $\omega(\mu(J))$, and hence $\omega(\mu(J))$ is the largest natural fuzzy topology contained in J.

(c) Since J is Lowen, then $\omega(\mu(J)) \subseteq J \subseteq \omega(\iota(J))$, by (a) and consequence 2.2.12(a). Thus,

$\mu(J) = \iota(J)$ iff $\omega(\mu(J)) = \omega(\iota(J))$, by consequence 2.2.12(c)

iff $\omega(\mu(J)) = J = \omega(\iota(J))$, by part (b) and consequence 2.2.12(d)

iff J is natural. □

Now, we introduce a topology and a fuzzy topology which owe their existence to consequence 1.1.14. The former was mentioned by Lowen [35], Hu [26] and Wuyts [57]. The latter we construct because of its usefulness, especially in providing counterexamples.

Definition 2.2.16

Let (F_X, J) be a fts. Then, as observed by Lowen [35], for every $\lambda \in I - \{1\}$, the family $\{A^{-1}(\lambda,1] : A \in J\}$ is a topology on X. Lowen denoted such a topology by $\iota_\lambda(J)$ and later Wuyts [57] called it the <u>λ-level topology</u> of J. It is obvious, that for every $\lambda \in I - \{1\}$, $\iota_\lambda(J) \subseteq \iota(J)$, and that, as indicated by Lowen, $\iota(J)$ is the union of the family $\{\iota_\lambda(J) : \lambda \in I - \{1\}\}$.

The 0-level topology was called by Kotze [31] <u>the support topology</u>.

<u>Definition 2.2.17</u>

Let (X, τ) be a topological space. For each $\lambda \in I - \{1\}$, define the family $\omega_\lambda(\tau)$ of subsets of F_X by $\omega_\lambda(\tau) = \{A \subseteq F_X : A^{-1}(\lambda,1] \in \tau\}$. This family, as we will show, is a fuzzy topology on F_X, which we call the <u>λ-cut fuzzy topology</u> of τ.

It is obvious that $\omega(\tau) = \bigcap\limits_{\lambda \in I-\{1\}} \omega_\lambda(\tau)$.

The 0-cut fuzzy topology of τ appears in Zheng [59] in its explicit form $\{A : \text{supt } A \in \tau\}$ and under the name "the fuzzy topology introduced by τ ".

<u>Proposition 2.2.18</u>

Let (X, τ) be a topological space. Then for each $\lambda \in I - \{1\}$, $(F_X, \omega_\lambda(\tau))$ is a fts.

<u>Proof</u>

Let $\lambda \in I - \{1\}$.

First $\phi_X, F_X \in \omega_\lambda(\tau)$, since $\phi_X^{-1}(\lambda,1] = \phi \in \tau$ and $F_X^{-1}(\lambda,1] = X \in \tau$.

Now, let $A_1, A_2 \in \omega_\lambda(\tau)$. Then $A_1^{-1}(\lambda,1], A_2^{-1}(\lambda,1] \in \tau$, and hence $A_1^{-1}(\lambda,1] \bigcap A_2^{-1}(\lambda,1] \in \tau$. By consequence 1.1.14(b), we have $(A_1 \bigcap A_2)^{-1}(\lambda,1] = A_1^{-1}(\lambda,1] \bigcap A_2^{-1}(\lambda,1] \in \tau$. Therefore, $A_1 \bigcap A_2 \in \omega_\lambda(\tau)$.

Lastly, let $C \subseteq \omega_\lambda(\tau)$. Then $A^{-1}(\lambda,1] \in \tau$, for all $A \in C$, and hence, $\bigcup\limits_{A \in C} A^{-1}(\lambda,1] \in \tau$. By consequence 1.1.14(a), we have, $\left(\bigcup_{A\in C} A\right)^{-1}(\lambda,1] = \bigcup_{A\in C} A^{-1}(\lambda,1] \in \tau$. Therefore, $\bigcup_{A\in C} A \in \omega_\lambda(\tau)$.

Thus $(F_X, \omega_\lambda(\tau))$ is a fts. □

Proposition 2.2.19

Denote the discrete topology on X by Δ.

Let τ be a topology on X and $\lambda \in I - \{1\}$. Then,

(a) $\iota(\omega_\lambda(\tau)) = \Delta$

(b) $\omega_\lambda(\Delta) = \omega(\Delta) = J_\Delta$

(c) $\tau = \Delta$ iff there is a $\delta \in I - \{\lambda, 1\}$ such that either $\omega_\lambda(\tau) \subseteq \omega_\delta(\tau)$ or $\omega_\delta(\tau) \subseteq \omega_\lambda(\tau)$.

Proof

(a) Let Z be a subset of X. Choose $\delta \in (\lambda, 1)$ and define the upper semicrisp set A in X by,

$$A(x) = \begin{cases} 1 & x \in Z \\ \delta & \text{otherwise} \end{cases}.$$

Then $A^{-1}(\lambda, 1] = X \in \tau$. But then $A \in \omega_\lambda(\tau)$, and so by definition of the initial topology, $A^{-1}(\delta,1] \in \iota(\omega_\lambda(\tau))$. But $A^{-1}(\delta, 1] = Z$. Therefore, $Z \in \iota(\omega_\lambda(\tau))$. Thus $\iota(\omega_\lambda(\tau)) = \Delta$.

(b) It is true, since for every $A \subseteq F_X$ and every $\lambda \in I - \{1\}$, the strong λ-cut $A^{-1}(\lambda,1]$ is open in the discrete topological space (X, Δ).

(c) Considering (b), it is enough to show that if $\tau \neq \Delta$, then for every $\delta \in I - \{\lambda, 1\}$, we have $\omega_\lambda(\tau) \not\subseteq \omega_\delta(\tau)$ and $\omega_\delta(\tau) \not\subseteq \omega_\lambda(\tau)$.

Thus, let $\tau \neq \Delta$, and $\delta \in I - \{\lambda, 1\}$. We can assume that $\lambda > \delta$. Since $\tau \neq \Delta$, then there is a subset Z of X such that $Z \notin \tau$.

First, consider the lower semicrisp set $A = Z^{(\lambda)}$. We have, $A^{-1}(\lambda,1] = \phi \in \tau$, and hence $A \in \omega_\lambda(\tau)$. On the other hand, $A^{-1}(\delta,1] = Z \notin \tau$, and so $A \notin \omega_\delta(\tau)$. Therefore, $\omega_\lambda(\tau) \nsubseteq \omega_\delta(\tau)$.

Now, define the upper semicrisp set B by,

$$B(x) = \begin{cases} 1 & x \in Z \\ \lambda & \text{otherwise} \end{cases}.$$

We have, $B^{-1}(\delta,1] = X \in \tau$, and hence $B \in \omega_\delta(\tau)$. On the other hand, $B^{-1}(\lambda,1] = Z \notin \tau$, and so $B \notin \omega_\lambda(\tau)$. Therefore, $\omega_\delta(\tau) \nsubseteq \omega_\lambda(\tau)$. □

We will return to the λ-cut fuzzy topological space in other parts of this thesis. It will be used to produce counterexamples in part III (e.g. proposition 3.1.21) IV (proposition 4.3.9) and V (counterexample 5.3.13).

PART III

FUZZY COUNTABILITY AND FUZZY SEQUENTIALITY

This part of the thesis focuses on four extensions of classical properties, namely fuzzy second countability as defined by Wong [54], fuzzy first countability and quasi-first countability as given by Pu and Liu [43] and fuzzy sequentiality which we introduce. (The latter was also discussed in Mohannadi and Warner [41]).

In section (1), we investigate the goodness (in the sense of Lowen) and the initial goodness (definition 3.1.13) of the three countability properties and establish their mutual relationship. Such investigation is made simpler by our approach of combining Q-theory (viz the theory of quasi-coincidence) and Lowen's goodness (e.g. propositions 3.1.25 and 3.1.27).

In section (2), we use Pu and Liu's fuzzy convergence theory to determine the types of fuzzy sequences converging in some special fuzzy topological spaces so as to 'discover' the fuzzy versions of familiar objects such as constant sequences. We find that certain fuzzy sequences, which we call semiconstant sequences behave in a manner similar to that of constant sequences in classical topology (propositions 3.2.11 and 3.2.17(a)). We also discover that only one type of fuzzy sequence, which we call predictable, can converge in a Lowen fts (Corollary 3.2.13(a)) and that it is in the nondiscrete fts in the sense of Chang [8] and not in the sense of Lowen [33] that every fuzzy sequence converges. The result demonstrates the advantage of Chang's definition of fuzzy topology.

In section (3), we fuzzify the concept of sequentiality and define the ensuing sequential fuzzy topology. We investigate the goodness and the initial goodness of this new concept and establish its relationship to the fuzzy countability properties.

An important conclusion arrived at in this part of the thesis is that Pu and Liu's quasi-first countable fts plays in fuzzy theory the role of first countable space in classical topology.

1. Fuzzy Countability Properties

In the following, let (F_X, J) be a fts.

Definition 3.1.1 (Wong [54])

The fts (F_X, J) is said to be fuzzy second countable (or C_2^f) iff there is a countable base for J.

In the following three definitions, let $e \in F_X$.

Definition 3.1.2 (Pu and Liu [43])

A subfamily $\mathbf{B}_{\mathbf{Q}_e}$ of the Q-neighbourhood system $\mathbf{Q}_e(J)$ of e is said to be a Q-neighbourhood base for J at e iff every member of $\mathbf{Q}_e(J)$ contains a member of $\mathbf{B}_{\mathbf{Q}_e}$.

In other words, $\mathbf{B}_{\mathbf{Q}_e}$ is a Q-neighbourhood base for J at e iff $\mathbf{B}_{\mathbf{Q}_e} \subseteq \mathbf{Q}_e(J)$ and $\mathbf{Q}_e(J) = \{A \subseteq F_X : D \subseteq A,\ D \in \mathbf{B}_{\mathbf{Q}_e}\}$.

Definition 3.1.3 (Pu and Liu [43])

A subfamily $\mathbf{B}_{\mathbf{V}_e}$ of the neighbourhood system $\mathbf{V}_e(J)$ of e is said to be a neighbourhood base for J at e iff every member of $\mathbf{V}_e(J)$ contains a member of $\mathbf{B}_{\mathbf{V}_e}$.

Equivalently, $\mathbf{B}_{\mathbf{V}_e}$ is a neighbourhood base for J at e iff $\mathbf{B}_{\mathbf{V}_e} \subseteq \mathbf{V}_e(J)$ and $\mathbf{V}_e(J) = \{A \subseteq F_X : D \subseteq A,\ D \in \mathbf{B}_{\mathbf{V}_e}\}$.

Definition 3.1.4

Let B be a family of open Q-neighbourhoods (respectively neighbourhoods) of e (i.e., $B \subseteq J \cap Q_e$ (respectively $B \subseteq J \cap V_e$)). Then B is said to be an open Q-neighbourhood base (respectively open neighbourhood base) at e iff for every $U \in J \cap Q_e$, (respectively, $U \in J \cap V_e$), there is a $B \in B$, such that $B \subseteq U$.

We can easily see that an open Q-neighbourhood base (respectively open neighbourhood base) is a Q-neighbourhood base (respectively neighbourhood base).

Definition 3.1.5 (Pu and Liu [43])

The fts (F_X, J) is said to be quasi-first countable (or $Q\text{-}C_1$) iff there is a countable Q-neighbourhood base for J at every fuzzy point in X.

Definition 3.1.6 (Wong /Pu and Liu)

The fts (F_X, J) is said to be fuzzy first countable (or C_1^f) iff there is a countable neighbourhood base for J at every fuzzy point in X.

Consequence 3.1.7

The fts (F_X, J) is $Q\text{-}C_1$ (respectively C_1^f) iff it has a countable open Q-neighbourhood base (respectively neighbourhood base) for J at every fuzzy point in X.

Proof

This follows almost vebatim the classical case. □

Definition 3.1.8

A family C of fuzzy sets in X is said to be nested iff for every couple A and B in C, either $A \subseteq B$ or $B \subseteq A$.

Consequence 3.1.9

The fts (F_X, J) is Q-C_1 (respectively C_1^f) iff there is a nested countable (open) Q-neighbourhood base (respectively neighbourhood base) for J at every fuzzy point in X.

Proof

We will prove the assertion for the Q-C_1 case. The proof for the C_1^f case is obtained similarly by replacing "q" by "ϵ" and the reference to consequence 1.3.7(b) by the reference to consequence 1.2.7(b). We will only show the nontrivial part.

Thus, let (F_X, J) be Q-C_1. Then given $e \in F_X$, there is a countable open neighbourhood base $\mathbf{B}_e = \{B_n : n \in \mathbf{N}\}$ at e. For every $n \in \mathbf{N}$, let $D_n = \bigcap_{k=1}^{n} B_k$. Then for every $n \in \mathbf{N}$, we have $D_n \in J$ and, by consequence 1.3.7(b), eqD_n. By consequence 1.1.20(a), we have $D_n \subseteq B_n$, for all $n \in \mathbf{N}$. Then the family $\mathbf{B}'_e = \{D_n : n \in \mathbf{N}\}$ is also a countable Q-neighbourhood base for J at e. This family is nested, since for every $n \in \mathbf{N}$, we have;

$D_{n+1} = \bigcap_{k=1}^{n+1} B_k = B_{n+1} \cap (\bigcap_{n=1}^{n} B_k) = B_{n+1} \cap D_n \subseteq D_n$, by consequences 1.1.18(c) and 1.1.20(a). □

Consequence 3.1.10

A base for J contains an open Q-neighbourhood base for J at every fuzzy point in X.

Proof

Let $\mathbf{B}$ be a base for J and $e \in F_X$. Consider $\mathbf{B}_e = \{B \in \mathbf{B} : eqB\}$. Obviously, $\mathbf{B}_e \subseteq \mathbf{Q}_e \cap J$. To show that $\mathbf{B}_e$ is an open Q-neighbourhood base for J at e, let $A \in \mathbf{Q}_e \cap J$. Then, by proposition 2.1.8, there is a $B \in \mathbf{B}$ such that $eqB \subseteq A$. But $B \in \mathbf{B}_e$, by definition of $\mathbf{B}_e$, and so $\mathbf{B}_e$ is an open Q-neighbourhood base for J at e. □

Corollary 3.1.11 (Pu and Liu [43])

Every fuzzy second countable fts is quasi-first countable.

Proof

This follows from consequence 3.1.10 and the fact that a subset of a countable set is countable. □

Proposition 3.1.12 (Pu and Liu)

Every fuzzy first countable fts is quasi-first countable.

Proof

See proposition 3.1 in Pu and Liu [43]. □

Now we introduce the concept of initial goodness and Lowen's criterion for goodness of fuzzy extensions of classical topological properties. In the following, let P be a classical topological property and P^f some claimed extension of that property to fuzzy topology.

Definition 3.1.13 (Initial goodness)

The fuzzy property P^f is said to be an initially good extension of P iff whenever a fts (F_X, J) has the property P^f, its initial topological space $(X, \iota(J))$ has the property P.

Definition 3.1.14 (Lowen's goodness criterion)

The fuzzy property P^f is said to be a good extension of P iff for every topological space (X, τ),

(X, τ) has the property P iff the natural fts

$(F_X, \omega(\tau))$ has the property P^f.

It follows that an initially good fuzzy extension is good if, $(F_X, \omega(\tau))$ has P^f, whenever (X, τ) has P.

Initial goodness

It turns out that all three fuzzy countability properties are initially good. In the following, let (F_X, J) be a fts and let Γ^* be the set $\Gamma - \{1\}$, (recall that Γ is the set of all rationals in I).

Proposition 3.1.15

Let **B** be a base for J. Then the family $\mathbf{C}' = \{B^{-1}(r, 1] : B \in \mathbf{B} : r \in \Gamma^*\}$ is a subbase for $\iota(J)$.

Proof

We will show that the family **C** of all finite intersections of members of **C'** is a base for $\iota(J)$.

First, by definition of $\iota(J)$, we have $\mathbf{C} \subseteq \iota(J)$.

Now, let $x \in Z \in \iota(J)$. Then by definition of $\iota(J)$, we have $x \in \bigcap_{n=1}^{N} A_n^{-1}(\lambda_n, 1] \subseteq Z$, $A_n \in J$, $\lambda_n \in I - \{1\}$, $1 \leq n \leq N$, for some $N \in \mathbf{N}$. Hence $x \in A_n^{-1}(\lambda_n, 1]$, and so $(x)_{1-\lambda_n} q A_n$, $1 \leq n \leq N$. But then, there is a $B_n \in \mathbf{B}$ such that $(x)_{1-\lambda_n} q B_n \subseteq A_n$, and hence $x \in B_n^{-1}(\lambda_n, 1] \subseteq A_n^{-1}(\lambda_n, 1]$, $1 \leq n \leq N$.

Since, $B_n(x) > \lambda_n$, $1 \leq n \leq N$, then for each $n \in \{1, 2, \ldots, N\}$, there is an $r_n \in \Gamma^*$, such that $B_n(x) > r_n > \lambda_n$, and hence $B_n^{-1}(r_n, 1] \subseteq B_n^{-1}(\lambda_n, 1]$. Thus, we have $x \in B_n^{-1}(r_n, 1] \subseteq A_n^{-1}(\lambda_n, 1]$ $1 \leq n \leq N$. But then $x \in \bigcap_{n=1}^{N} B_n^{-1}(r_n, 1] \subseteq \bigcap_{n=1}^{N} A_n^{-1}(\lambda_n, 1] \subseteq Z$. Let $Y = \bigcap_{n=1}^{N} B_n^{-1}(r_n, 1]$. Then $Y \in \mathbf{C}$.

Thus given an $x \in X$ and $Z \in \iota(J)$, there is a $Y \in \mathbf{C} \subseteq \iota(J)$ such that $x \in Y \subseteq Z$. Therefore, $\mathbf{C}$ is a base for $\iota(J)$. □

Proposition 3.1.16

Fuzzy second countability is an initially good extension of classical second countability. That is, if (F_X, J) is C_2^f, then $(X, \iota(J))$ is C_2.

Proof

If (F_X, J) is C_2^f, then there is a countable base $\mathbf{B}$ for J. Let $\mathbf{C}' = \{B_n^{-1}(r_k, 1] : r_k \in \Gamma^*, B_n \in \mathbf{B}, k, n \in \mathbf{N}\}$ and consider the family $\mathbf{C}$ of all finite intersections of members of $\mathbf{C}'$. Then $\mathbf{C}$ is countable (see theorem 18 in Kelley [29]), and, by proposition 3.1.15, $\mathbf{C}$ is a base for $\iota(J)$. Therefore, $(X, \iota(J))$ is C_2. □

Proposition 3.1.17

Let x be a point in X, $\Gamma^{**} = \Gamma - \{0\}$. For each $r \in \Gamma^{**}$ let $\mathbf{B}_r$ be an open Q-neighbourhood base for J at $(x)_r$. Define $\mathbf{C}_r$ by $\mathbf{C}_r = \{B^{-1}(1-r,1] \ : \ B \in \mathbf{B}_r\}$. Then the family $\mathbf{C}_x$ of all finite intersections of members in $\bigcup_{r\in\Gamma^{**}} \mathbf{C}_r$ is an open local base for $\iota(J)$ at x.

Proof

First $\mathbf{C}_x$ is a subfamily of open neighbourhoods of x (relative to $\iota(J)$). Let Z be an open neighbourhood of x. Continuing in a similar way to that used in the proof of proposition 3.1.15, we can show that there is a $Y \in \mathbf{C}_x$, such that $Y \subseteq Z$. $\square$

Proposition 3.1.18

Quasi-first countability is an initially good extension of classical first countability. That is if (F_X, J) is $Q\text{-}C_1$, then $(X, \iota(J))$ is C_1.

Proof

Let $x \in X$.

Since (F_X, J) is $Q\text{-}C_1$, then there is a countable open Q-neighbourhood base $\mathbf{B}_{r_k}$ for J at each fuzzy point $(x)_{r_k}$, where, $r_k \in \Gamma^{**}$. Let $\mathbf{C}_{r_k} = \{B_n^{-1}(1-r_k,1] \ : \ B_n \in \mathbf{B}_{r_k}\}$ and consider the family $\mathbf{C}_x$ of all finite intersections of members of $\bigcup_{k=1}^{\infty} \mathbf{C}_{r_k}$. Since $\bigcup_{k=1}^{\infty} \mathbf{C}_{r_k}$ is countable, then $\mathbf{C}_x$ is countable (see theorem 18 in Kelley [29]). By proposition 3.1.17, $\mathbf{C}_x$ is an open local base for $\iota(J)$ at x. Therefore, $(X, \iota(J))$ is C_1. $\square$

Proposition 3.1.19

Fuzzy first countability is an initially good extension of classical first countability. That is, if (F_X, J) is C_1^f, then $(X, \iota(J))$ is C_1.

Proof

Let (F_X, J) be C_1^f. Then, by proposition 3.1.12 (F_X, J) is Q-C_1. But then, by proposition 3.1.18, $(X, \iota(J))$ is C_1. □

However, if the initial topological space of some fts is first countable (second countable) this does not imply that the fuzzy topological space is fuzzy first countable or even quasi-first countable (respectively second countable). To show this, we use the λ-cut fts we introduced in part II (see Definition 2.2.17). In the following, let $\lambda \in I - \{1\}$, (X, τ) be a topological space and $(F_X, \omega_\lambda(\tau))$ its λ-cut fts.

Proposition 3.1.20

If $(F_X, \omega_\lambda(\tau))$ is C_2^f, then (X, τ) is C_2.

Proof

Let $(F_X, \omega_\lambda(\tau))$ be C_2^f, then there is a countable base $B = \{B_n : n \in \mathbb{N}\}$ for $\omega_\lambda(\tau)$. Consider the family $C = \{B_n^{-1}(\lambda, 1] : n \in \mathbb{N}\}$. Obviously C is countable and, by the definition of $\omega_\lambda(\tau)$, it is a subfamily of τ. To show that it is a base for τ, let $x \in Z \in \tau$. Let $A = \chi_Z$. Then $A \in \omega(\tau)$, and so $A \in \omega_\lambda(\tau)$. Since $\lambda \in I - \{1\}$, then $1-\lambda > 0$. Consider the fuzzy point $e = (x)_{1-\lambda}$. Then eqA, and so there is a $B_n \in B$ such that $eqB_n \subseteq A$. But then $B_n^{-1}(\lambda,1] \in \mathbf{C}$, $B_n^{-1}(\lambda,1] \subseteq A^{-1}(\lambda,1] = Z$, and $B_n(x) > 1-(1-\lambda) = \lambda$. Therefore, $x \in B_n^{-1}(\lambda,1] \subseteq Z$. Thus the countable family $\mathbf{C}$ is a base for τ, and so (X, τ) is C_2. □

Proposition 3.1.21

If $(F_X, \omega_\lambda(\tau))$ is Q-C_1, then (X, τ) is C_1.

Proof

Let $(F_X, \omega_\lambda(\tau))$ be Q-C_1 and $x \in X$.

Consider the fuzzy point $e = (x)_{1-\lambda}$. By the quasi-first countability of $\omega_\lambda(\tau)$, there is a countable open Q-neighbourhood base $\mathbf{B}_e = \{B_n : n \in \mathbf{N}\}$ for $\omega_\lambda(\tau)$ at e. Let $\mathbf{C} = \{B_n^{-1}(\lambda,1] : n \in \mathbf{N}\}$. Following the procedure used in the previous proof, it can be shown that the countable subfamily $\mathbf{C}$ of τ is an (open) local base for τ at x. Therefore, (X, τ) is C_1. □

In the following, let (F_X, J) be a fts.

Proposition 3.1.22

The second countability of $(X, \iota(J))$ does not imply the fuzzy second countability of (F_X, J).

Proof

Let $X = \{(x, y) : x, y \in \mathbf{N} \cup \{0\}\}$. Then X is countable. Define the topology τ on X, by the following.

(1) For every $(x, y) \in X - \{(0,0)\}$, $\{(x, y)\}$ is an open neighbourhood of (x,y) in the topological space (X, τ).

(2) A subset Z of X is an open neighbourhood of the point $(0,0)$ in (X, τ) iff for all , except, may be, a finite number of elements x in X, the set $\{y : (x, y) \in Z\}$ is infinite.

The topological space (X, τ) which is called the Arens-Fort topological space is not C_2 (see II (26) in Steen and Seebach [49]).

Now, let $\lambda \in I - \{1\}$ and consider the fts $J = \omega_\lambda(\tau)$. (In particular we can consider $J = \omega_0(\tau) = \{A \subseteq F_X : \text{supt } A \in \tau\}$). Then, by proposition 2.2.19(a), $\iota(J) = \Delta$. Since X is countable, then the discrete topology Δ on X is C_2, and so the initial topology $(X, \iota(J))$ is C_2. But since τ is not C_2, then, by the counterpositive of proposition 3.1.20, J is not C_2^f. $\square$

<u>Proposition 3.1.23</u>

The first countability of $(X, \iota(J))$ does not imply the quasi-first countability of (F_X, J).

<u>Proof</u>

Let (X, τ) be any non-first countable topological space, (for example the countable complement topology on $\mathbf{R}$). Let $\lambda \in I - \{1\}$ and consider the fts, $J = \omega_\lambda(\tau)$. By proposition 2.2.19(a), $\iota(J) = \Delta$. Since the discrete topological space is first countable, then $(X, \iota(J))$ is C_1. But since τ is not C_1, then, by the counterpositive of 3.1.21, J is not $Q\text{-}C_1$. $\square$

Corollary 3.1.24

The first countability of $(X, \iota(J))$ does not imply the fuzzy first countability of (F_X, J).

Goodness

In the following, we investigate the goodness of our fuzzy countability properties.

Proposition 3.1.25

Fuzzy second countability is a good fuzzy extension of classical second countability. That is, a topological space (X, τ) is C_2 iff its natural fts $(F_X, \omega(\tau))$ is C_2^f.

Proof

Considering proposition 3.1.16, it is enough to show that the second countability of (X, τ) implies the fuzzy second countability of $(F_X, \omega(\tau))$.

Thus, let (X, τ) be C_2. Then there is a countable base $\mathbf{C} = \{Z_n : n \in \mathbf{N}\}$ for τ.

Consider the family $\mathbf{B} = \{Z_n^{(r_k)} : n, k \in \mathbf{N}\}$ of all lower semicrisp sets of values r_k in Γ and supports in $\mathbf{C}$. Then, by corollary 2.2.8 the countable family $\mathbf{B}$ is a base for $\omega(\tau)$. Therefore, $\omega(\tau)$ is C_2^f. □

Proposition 3.1.26

Quasi-first countability is a good fuzzy extension of classical first countability. That is, a topological space (X, τ) is C_1 iff its natural fts $(F_X, \omega(\tau))$ is $Q\text{-}C_1$.

Proof

Considering proposition 3.1.18, we need only show that if (X, τ) is C_1, then $(F_X, \omega(\tau))$ is Q-C_1.

Thus, let (X, τ) be C_1 and $e = (x)_\lambda \in F_X$. By the first countability of (X, τ), there is an open countable base $\mathbf{C} = \{Z_n : n \in \mathbf{N}\}$ for J at x. Let $\mathbf{b} = \Gamma \bigcap (1-\lambda, 1]$, i.e. $\mathbf{b}$ is the set of all rationals in $(1-\lambda,1]$. Notice that $1 - \lambda \neq 1$.

Consider the family $\mathbf{B} = \{Z_n^{(\delta_k)} : \delta_k \in \mathbf{b}, n,k \in \mathbf{N}\}$ of all lower semicrisp sets with values in $\mathbf{b}$ and supports in $\mathbf{C}$. Then $\mathbf{B}$ is a countable family of open Q-neighbourhoods of e.

To show that $\mathbf{B}$ is an open Q-neighbourhood base for J at e, let A be an open Q-neighbourhood of e. Then $x \in A^{-1}(1-\lambda,1] \in \tau$. Hence $A(x) > 1-\lambda$, and so we can choose some rational $\delta_k \in (1-\lambda, A(x))$. But then $\delta_k \in \mathbf{b}$, $x \in A^{-1}(\delta_k, 1] \in \tau$ and $A^{-1}(\delta_k,1] \subseteq A^{-1}(1-\lambda,1]$.

Since $\mathbf{C}$ is a local base for τ at x, then there is a $Z_n \in \mathbf{C}$ such that $Z_n \subseteq A^{-1}(\delta_k, 1]$, (of course $x \in Z_n$). Therefore, the lower semicrisp set $B = Z_n^{(\delta_k)}$ is in $\mathbf{B}$. Now $B(y) = 0 < A(y)$, for all $y \in Z^c$ and $B(y) = \delta_k < A(y)$, for all $y \in Z_n$. Thus $B \subseteq A$, and therefore $\mathbf{B}$ is an open Q-neighbourhood base for $\omega(\tau)$ at e. Hence $(F_X, \omega(\tau))$ is Q-C_1. □

Our proof of goodness for fuzzy second countability given in proposition 3.1.25 is based on proposition 2.2.8 which connects the concept of a base with that of a Q-neighbourhood. An alternative proof that does not involve the notion of quasi-coincidence can be found in Bülbül [7]. Bülbül also showed that "fuzzy first countability" is good. But his definition of fuzzy first countability is different. It follows from his definitions of fuzzy point and fuzzy membership which are that of the "excluding definition" and the "proper membership definition" given in 1.2.*1 and 1.2.*5 respectively.

To show the relationship between Bülbül's first countability and our fuzzy countability properties, recall the fuzzy point space $(M^*, \rhd)$, where M^* is the family of all proper points of F_X (i.e. the noncrisp fuzzy points in X) and $\rhd$ is the proper membership relation defined in 1.2.4. For an $e \in M^*$, define a proper neighbourhood of e in (F_X, J) as a fuzzy set V in X such that $e \rhd A \subseteq V$, for some $A \in J$. Define a proper neighbourhood base B^* for J at e as a family of proper neighbourhoods of e such that every proper neighbourhood of e contains a member of B^*. Lastly call a fts that has a countable proper neighbourhood base at every $e \rhd F_X$ by the name proper first countable (or simply $P\text{-}C_1$). These are Bülbül's definitions for a neighbourhood, neighbourhood base and fuzzy first countability respectively.

Hu [26] observed that if $e = (x)_\lambda \in M^*$, then a proper neighbourhood of e is actually a Q-neighbourhood of the dual point $\hat{e} = (x)_{1-\lambda}$ of e. It follows that a proper neighbourhood base at e is a Q-neighbourhood base at $\hat{e}$. Thus a quasi-first countable fts is $P\text{-}C_1$ or "first countable" in the sense of Bülbül and Hu.

Proposition 3.1.27

Fuzzy first countability is not a good extension of classical first countability.

Proof

Consider the topological space $(I, 1^*)$ which, as we know is first countable. But, as has been shown in Pu and Liu [43], the natural fts $(F_I, \omega(1^*))$ is not C_1^f (it does not have a countable base at any crisp point in I). □

The fts $(F_I, \omega(1^*))$ used in proposition 3.1.27 was actually used in Pu and Liu [43] as an example of a fts which is C_2^f but not C_1^f. To show that $(F_I, \omega(1^*))$ is C_2^f, Pu and Liu gave a long and detailed proof (see lemmas 3.1 and 3.2 in [43]). It turns out now that their result is a particular case of the more general and simpler assertion; proposition 3.1.25. This is shown in the following proposition.

Proposition 3.1.28 (Pu and Liu)

Fuzzy second countability does not imply fuzzy first countability.

Proof

Consider again the fts $(F_I, \omega(1^*))$. Since $(I, 1^*)$ is C_2, then $(F_I, \omega(1^*))$ is C_2^f by proposition 3.1.25. On the other hand $(F_I, \omega(1^*))$ is not C_1^f (see proof of proposition 3.1.27). □

Proposition 3.1.29

Fuzzy first countability does not imply fuzzy second countability.

Prooof

Consider the fts $(F_{\mathbf{R}}, \omega(\Delta))$, where Δ is the discrete topology on $\mathbf{R}$. By proposition 2.2.19(b), $\omega(\Delta) = J_{\Delta}$. And since at every $e \in F_X$, $\{e\}$ is a countable neighbourhood base for J_{Δ} at e, then $(F_{\mathbf{R}}, \omega(\Delta))$ is C_1^f. On the other hand $(\mathbf{R}, \Delta)$ is not C_2, and hence, by proposition 3.1.25, $(F_{\mathbf{R}}, \omega(\Delta))$ is not C_2^f. □

Proposition 3.1.30

Quasi-first countability implies neither fuzzy first countability nor fuzzy second countability. Moreover there is a fts which is Q-C_1 but neither C_1^f nor C_2^f.

Proof

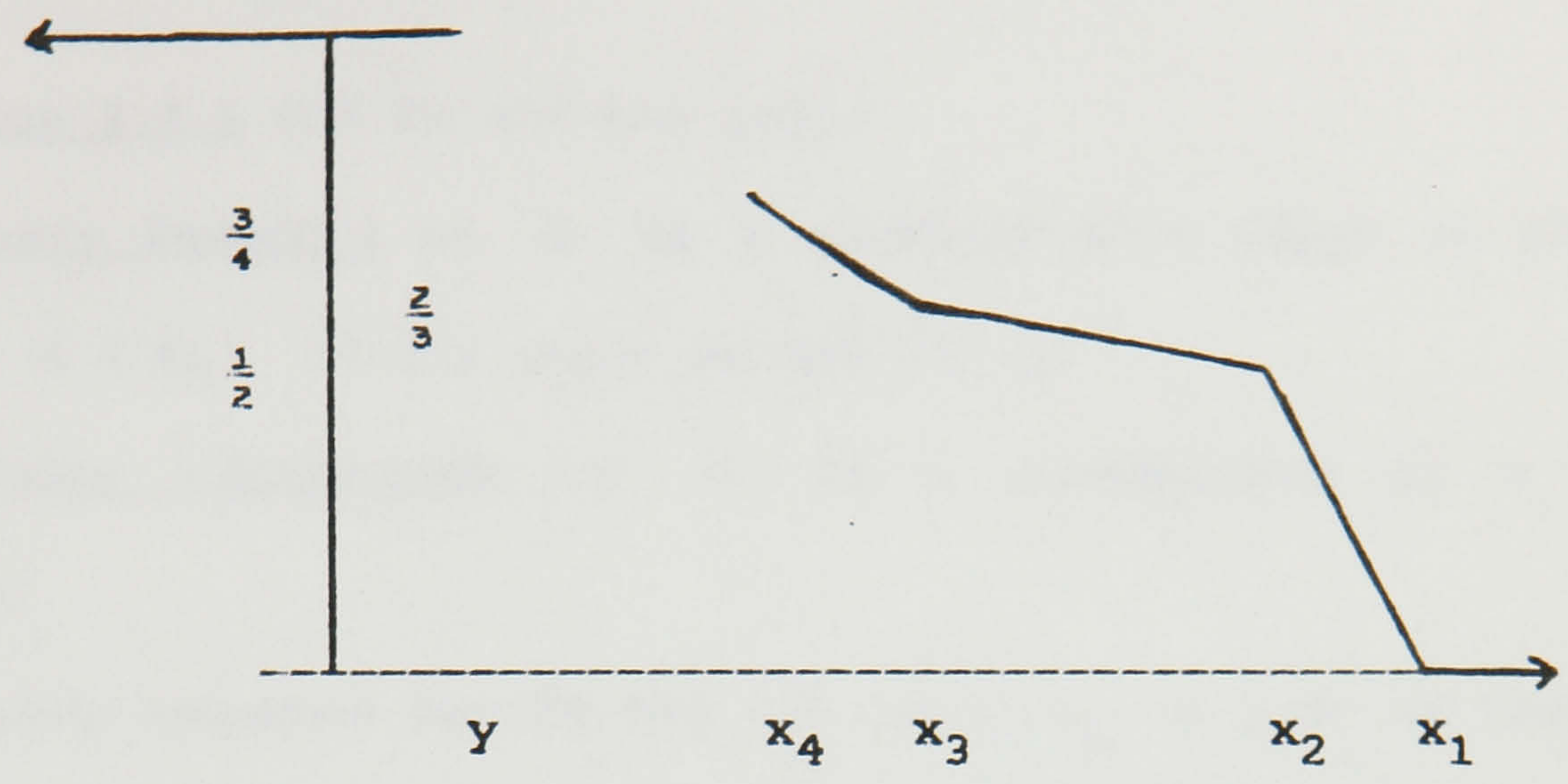
We have given an example of a C_2^f fts which is not C_1^f (proposition 3.1.28) and an example of C_1^f fts which is not C_2^f (proposition 3.1.29). But, by corollary 3.1.11 and proposition 3.1.12, every C_1^f and every C_2^f fts is Q-C_1. Hence we have an example of a Q-C_1 fts which is not C_1^f, and an example of a Q-C_1 fts which is not C_2^f. In the following we give an example of a Q-C_1 fts which is neither C_1^f nor C_2^f.

Consider the natural fts $(F_{\mathbf{R}}, \omega(\tau))$ of the right half-open interval topological space $(\mathbf{R}, \tau)$. Recall that this topological space is C_1 and not C_2 (see Steen and Seebach [49]), and hence $(F_{\mathbf{R}}, \omega(\tau))$ is Q-C_1 but not C_2^f. Recall also that for every $x \in \mathbf{R}$, $(-\infty, x) \in \tau$ and $C_x = \{[x, x+\zeta) : \zeta > 0\}$ is an open neighbourhood base for τ at x. In the following we borrow Pu and Liu's method of theorem 3.1 in [43] to show that $(F_{\mathbf{R}}, \omega(\tau))$ is not C_1^f.

Assume that $(F_{\mathbf{R}}, \omega(\tau))$ is C_1^f. Consider any crisp point e in $\mathbf{R}$ and let $y = \text{supt } e$ (for example take $y = 5$, and hence $e = (5)_1$). Then there is a nested countable open neighbourhood base $\mathbf{B}_e = \{B_n : n \in \mathbf{N}\}$ for $\omega(\tau)$ at e with $B_{n+1} \subseteq B_n$, $n \in \mathbf{N}$. Then for every $n \in \mathbf{N}$, $e \in B_n$, and hence $B_n(y) = 1$. But then for every $n \in \mathbf{N}$, $B_n(y) > 1 - \frac{1}{n}$, and so $y \in B_n^{-1}(1 - \frac{1}{n}, 1]$. Now, for all $n \in \mathbf{N}$, we have $B_n \in \omega(\tau)$ and therefore, $B_n^{-1}(1 - \frac{1}{n}, 1] \in \tau$. By the nature of members of the neighbourhood base $C_y = \{[y, y+\zeta) : \zeta > 0\}$ for τ at y and the nestedness of $\mathbf{B}_e$, for every $n \in \mathbf{N}$, there is an $\zeta_n > 0$ such that $\zeta_n > \zeta_{n+1}$ and $y \in [y, y + \zeta_n) \subseteq B_n^{-1}(1 - \frac{1}{n}, 1]$.
For each $n \in \mathbf{N}$ choose $x_n \in (y + \zeta_{n+1}, y + \zeta_n)$. Then $x_n > x_{n+1} > y$ and $x_n \in B_n^{-1}(1 - \frac{1}{n}, 1]$, and hence $B_n(x_n) > 1 - \frac{1}{n}$, $n \in \mathbf{N}$.

Define the fuzzy set A in $\mathbf{R}$ by,

$$A(x) = \begin{cases} 1 & x \leq y \\ (1 - \frac{1}{n}) + (\frac{x_n - x}{x_n - x_{n+1}})(\frac{1}{n(n+1)}) & x \in [x_n, x_{n+1}] \\ 0 & x > x_1 \end{cases}$$



Since $A(y) = 1$, then $e \in A$. Now, for every $\lambda \in I - \{1\}$, $A^{-1}(\lambda, 1] = (-\infty, A^{-1}(\lambda)) \in \tau$, and hence $A \in \omega(\tau)$. But on the other hand $A(x_n) = 1 - \frac{1}{n} < B_n(x_n)$, for all $n \in \mathbf{N}$. Thus we have an open neighbourhood A of e that does not contain any member of $\mathbf{B}_e$. Therefore, $\mathbf{B}_e$ cannot be a neighbourhood base for $\omega(\tau)$ at e. $\square$

2. Fuzzy Sequences and Fuzzy Convergence

Definition 3.2.1 (cf Pu and Liu [43])

A fuzzy sequence in X is a sequence with range in the set $E = \{e : e \in F_X\}$ of all fuzzy points in X.

A fuzzy subsequence in X is a subsequence of a fuzzy sequence.

A fuzzy sequence having the nth term e_n, $n \in \mathbf{N}$ is denoted by $\langle e_n \rangle$. A subsequence of $\langle e_n \rangle$ having the kth term, e_{n_k}, $k \in \mathbf{N}$ is denoted by $\langle e_{n_k} \rangle$.

In the following, let $\langle e_n \rangle$ be a fuzzy sequence in X.

Definition 3.2.2

Let $e_n = (x_n)_{\lambda_n}$, $n \in \mathbf{N}$.

The sequence $\langle x_n \rangle$ is said to be the support sequence of $\langle e_n \rangle$ and denoted by supt $\langle e_n \rangle$.

The sequence $\langle \lambda_n \rangle$ is said to be the value sequence of $\langle e_n \rangle$ and denoted by Val $\langle e_n \rangle$.

Definition 3.2.3

The fuzzy sequence $\langle e_n \rangle$ is said to be of (eventual) constant support iff supt $\langle e_n \rangle$ is an (eventually) constant sequence.

Similarly, $\langle e_n \rangle$ is said to be of (eventual) constant value iff Val $\langle e_n \rangle$ is an (eventually) constant sequence.

It follows that $\langle e_n \rangle$ is <u>(eventually) constant</u> iff it is of both (eventual) constant support and (eventual) constant value.

In the following, let $A \subseteq F_X$.

<u>Definition 3.2.4</u> (Pu and Liu [43])

The fuzzy sequence $\langle e_n \rangle$ is said to be <u>eventually quasi-coincident to A</u> iff there is an $N \in \mathbf{N}$ such that $e_n q A$, for all $n \geq N$. To indicate this, we write $\langle e_n \rangle q(N)A$. When $N = 1$, we simply write $\langle e_n \rangle qA$ and say that <u>$\langle e_n \rangle$ is quasi-coincident to A.</u>

If for every $n \in \mathbf{N}$, there is an $m > n$ such that $e_m qA$, then we say that $\langle e_n \rangle$ is <u>frequently quasi-coincident to A</u>.

We write $\langle e_n \rangle \not{q} A$ to indicate that $e_n \not{q} A$, for all $n \in \mathbf{N}$.

<u>Definition 3.2.5</u> (Pu and Liu)

The fuzzy sequence $\langle e_n \rangle$ is said to be <u>eventually</u> in A iff there is an $N \in \mathbf{N}$ such that $e_n \in A$, for all $n \geq N$. We say that $\langle e_n \rangle$ is in A iff $e_n \in A$, for all $n \in \mathbf{N}$.

We mention two definitions of convergence proposed by Wong [56] and Bülbül [6].

Avoiding the notion of fuzzy points, Chang [8] introduced the term "sequences of fuzzy sets" for which he defined clustering and other related concepts. Wong [56] observed that, only by introducing the idea of fuzzy points, was a meaningful discussion of convergence in fuzzy topological spaces possible.

The following is Wong's definition expressed in our notation.

Definition 3.2.*1

Let $\langle e_n \rangle$ be a sequence of fuzzy points in a fts (F_X, J) and $x_n = \text{supt } e_n$, $n \in \mathbf{N}$. Let e be a fuzzy point with support $\underline{x \neq x_n,\ n \geq n_0}$, for some $n_0 \in \mathbf{N}$. Then $\langle e_n \rangle$ is said to converge to e in (F_X, J) iff for every $A \in J$ such that $e \leftarrow A$, there exists an $N \in \mathbf{N}$ such that $e_n \leftarrow A$, for all $n \geq N$. Here "$\leftarrow$" is Wong's membership relation (see definition 1.2.*2).

Bülbül [6] rightly rejected the underlined restriction which he described as somehow "unnatural" and observed that it prevents constant fuzzy sequences from converging in any fuzzy topological space. He modifies Wong's definition by both omitting the restriction and replacing the relation "$\leftarrow$" by the proper membership relation "$\rhd$" (see definition 1.2.*5). If we substitute for the latter the q-relation, we obtain Pu and Liu's concept of sequential convergence which we adopt here.

In the following, let $\langle e_n \rangle$ be a fuzzy sequence in X.

Definition 3.2.6 (Pu and Liu [43])

The fuzzy sequence $\langle e_n \rangle$ is said to converge in a fts

(F_X, J) to a fuzzy point e in X (called a limit of $\langle e_n \rangle$) iff, for every $A \in J$ such that eqA there is an $N \in \mathbf{N}$ such that $\langle e_n \rangle q(N)A$. We denote this by $e_n \xrightarrow{J} e$. If this is not the case, we write $e_n \overset{J}{\not\longrightarrow} e$.

It follows from the definition that if a fuzzy point e in X is not quasi-coincident to any member of $J - \{F_X\}$, then every fuzzy sequence in X converges to e.

<u>Proposition 3.2.7</u>

(a) $e_n \xrightarrow{J} e$ iff every subsequence of $\langle e_n \rangle$ converges to e in (F_X, J)

(b) $e_n \xrightarrow{J} e$ iff every subsequence of $\langle e_n \rangle$ has a subsequence that converges to e

(c) If J_1 is a fuzzy topology on F_X which is stronger than J, then $e_n \xrightarrow{J} e$, whenever $e_n \xrightarrow{J_1} e$ and $e_n \overset{J_1}{\not\longrightarrow} e$, whenever $e_n \overset{J}{\not\longrightarrow} e$.

<u>Proof</u>

The proofs follow almost vebatim those of the corresponding classical assertions. □

The following three definitions will be useful when we investigate later sequential convergence in some special fts's.

Let $\langle \lambda_n \rangle = \mathrm{Val}\, \langle e_n \rangle$ and $\lambda \in I - \{0\}$.

Definition 3.2.8

The fuzzy sequence $\langle e_n \rangle$ is said to be predictable by λ iff for every $\zeta \in (0,\lambda)$, there is an $N \in \mathbf{N}$, such that $\lambda_n \in (\lambda-\zeta,1]$, for all $n \geq N$.

If $\langle e_n \rangle$ is predictable by λ, then a fuzzy point e in X of value λ is said to be a predictable point of $\langle e_n \rangle$.

Definition 3.2.9

Let τ be a topology on X.

The fuzzy sequence $\langle e_n \rangle$ is said to be τ-convergent in support iff supt $\langle e_n \rangle$ converges in (X, τ).

A corresponding definition for Val $\langle e_n \rangle$ is not needed. A fuzzy sequence $\langle e_n \rangle$ for which Val $\langle e_n \rangle$ converges in $(I, 1^*)$ is simply a special case of a predictable sequence. Another special case is the situation when Val $\langle e_n \rangle$ is eventually equal to or greater than some number λ in $I - \{0\}$.

Definition 3.2.10

The fuzzy sequence $\langle e_n \rangle$ is said to be semiconstant iff $\langle e_n \rangle$ is (eventually) constant in support and there is a $\lambda \in I - \{0\}$ such that $\langle e_n \rangle$ is predictable by λ. If this is the case and x is the eventual support of $\langle e_n \rangle$, then a predictable point of support x (i.e. $(x)_\lambda$) is called a trivial limit of $\langle e_n \rangle$.

Semiconstant sequences play in fuzzy the role played in topology by constant sequences as the following proposition shows.

Proposition 3.2.11

A semiconstant sequence converges to each of its trivial limits in every fuzzy topological space.

Proof

Let (F_X, J) be a fts and $\langle x_n \rangle = \text{supt } \langle e_n \rangle$.

Let $\langle e_n \rangle$ be predictable by λ and supt $\langle e_n \rangle$ eventually x. Let $A \in J$ and $(x)_\lambda qA$. Then $A(x) > 1-\lambda$, and hence $A(x) > 1-\lambda+\zeta$, for some $\zeta \in (0,\lambda)$. Since $\langle x_n \rangle$ is eventually x, there is an $N_1 \in \mathbf{N}$ such that $A(x_n) = A(x) > 1-\lambda+\zeta$, $n \geq N_1$. On the other hand $\langle e_n \rangle$ is predictable by λ, and hence for that same ζ, there is an $N_2 \in \mathbf{N}$, such that $e_n(x_n) \in (\lambda-\zeta, 1]$, $n \geq N_2$. Let $N = \max(N_1, N_2)$. Then, $A(x_n) + e_n(x_n) > (1-\lambda+\zeta) + (\lambda-\zeta) = 1$, $n \geq N$. Therefore, $\langle e_n \rangle q(N)A$. Thus $e_n \xrightarrow{J} (x)_\lambda$. □

Proposition 3.2.12

(a) In the indiscrete fts (F_X, J_0) every fuzzy sequence converges to every fuzzy point.

(b) A fuzzy sequence converges in the Lowen indiscrete fts (F_X, L_0) if and only if it is predictable. It converges to and only to its predictable points.

Proof

(a) It follows from the fact that the full set is the only open fuzzy set that it is quasi-coincident to any fuzzy point.

(b) Let $\langle x_n \rangle$ = supt $\langle e_n \rangle$.

First, let $\langle e_n \rangle$ be predictable by λ and $x \in X$. Let $\eta_\delta \in L_0$ and $(x)_\lambda q \eta_\delta$. Then $\delta > 1-\lambda$, and hence $\delta > 1-\lambda+\zeta$, for some $\zeta \in (0,\lambda)$. Since $\langle e_n \rangle$ is predictable by λ, then for that same ζ there is an $N \in \mathbf{N}$ such that $e_n(x_n) \in (\lambda-\zeta,1]$, $n \geq N$. Hence, $\eta_\delta(x_n) + e_n(x_n) = \delta + e_n(x_n) > (1-\lambda+\zeta) + (\lambda-\zeta) = 1$, $n \geq N$. Thus, $\langle e_n \rangle q(N)\eta_\delta$. Therefore, $e_n \xrightarrow{L_0} (x)_\lambda$.

Now, let $\lambda \in I - \{0\}$ and assume that $\langle e_n \rangle$ is not predictable by λ. Let $x \in X$ and $e = (x)_\lambda \in F_X$. We will show that $e_n \not\xrightarrow{L_0} e$. Since $\langle e_n \rangle$ is not predictable by λ, there is a subsequence $\langle e_{n_k} \rangle$ of $\langle e_n \rangle$ and a $\zeta \in (0,\lambda)$, such that $e_{n_k}(x_{n_k}) \in (0,\lambda-\zeta]$, for all $k \in \mathbf{N}$. Let $\delta = 1-\lambda+\zeta$, then $0 < \delta < 1$. Consider the constant set η_δ. Since $\delta+\lambda = (1-\lambda+\zeta) + \lambda = 1+\zeta > 1$, then $e q \eta_\delta$. But $\eta_\delta(x_{n_k}) + e_{n_k}(x_{n_k}) \leq \delta + (\lambda-\zeta) = (1-\lambda+\zeta) + (\lambda-\zeta) = 1$, and hence $\langle e_{n_k} \rangle \not{q} \eta_\delta$. Thus $e_{n_k} \not\xrightarrow{L_0} e$. But then, by proposition 3.2.7(a), $e_n \not\xrightarrow{L_0} e$. □

<u>Corollary 3.2.13</u>

(a) Only a predictable sequence converges in a Lowen fts. It converges to and only to its predictable points.

(b) Only a predictable sequence converges in a natural fts. It converges to and only to its predictable points.

<u>Proof</u>

(a) Combine proposition 3.2.12 and 3.2.7(c) with the fact that a Lowen topology contains L_0.

(b) It follows from (a), since a natural fts is Lowen. □

Notice that in a non-Lowen fts a fuzzy sequence can converge to some fuzzy point e without having to be predictable by the value of e. This is trivially true in the indiscrete fts where every fuzzy sequence converges to every fuzzy point. The following generalizes this observation.

Example 3.2.14

Let $\mathbf{b}$ be any closed from above subset of I that contains 0 and 1 and $J(\mathbf{b})$ a trivial fts (see example 2.2.4). Then each fuzzy sequence in X converges in $(F_X, J(\mathbf{b}))$ to every fuzzy point of value $1-\lambda$, for each $\lambda \in [\sup(\mathbf{b}-\{1\}), 1)$.

Proof

Let $\mathbf{a} = \mathbf{b} - \{1\}$ and $\lambda \in [\sup \mathbf{a}, 1)$. Then $1-\lambda \leq 1 - \sup \mathbf{a}$. For every $\eta_\delta \in J(\mathbf{b}) - \{F_X\}$, we have $\delta \leq \sup \mathbf{a}$, and hence $\delta + (1-\lambda) \leq \sup \mathbf{a} + (1-\sup \mathbf{a}) = 1$. Therefore, $e \not{q} \eta_\delta$, for every $\eta_\delta \in J(\mathbf{b}) - \{F_X\}$ and every fuzzy point e with value $1-\lambda$. But then every fuzzy sequence in X converges to e, by remark following definition 3.2.6.

As an illustration, let $\mathbf{b} = [0,\frac{1}{2}] \bigcup \{1\}$. Then $J(\mathbf{b}) = \{\eta_\delta : \delta \in [0,\frac{1}{2}] \bigcup \{1\}\}$, $\mathbf{a} = [0,\frac{1}{2}]$ and $\sup \mathbf{a} = \frac{1}{2}$. If we take $\lambda = \frac{7}{8}$, then $\frac{7}{8} \in [\sup \mathbf{a}, 1) = [\frac{1}{2}, 1)$, and so every fuzzy sequence in X converges to any fuzzy point of value $\frac{1}{8}$. That includes, for instance, any constant fuzzy sequence of value $\frac{1}{16}$. But this latter fuzzy sequence is not predictable by $\frac{1}{8}$, since $\frac{1}{16} \notin (\frac{1}{8}-\zeta, 1]$, for $\zeta = \frac{1}{16}$. □

In the following, let (X, τ) be a topological space, $\langle e_n \rangle$ a fuzzy sequence in X, supt $\langle e_n \rangle = \langle x_n \rangle$, and $e = (x)_\lambda \in F_X$.

Proposition 3.2.15

$$e_n \xrightarrow{cr(\tau)} e \quad \text{iff} \quad x_n \xrightarrow{\tau} x.$$

Proof

First, let $x_n \xrightarrow{\tau} x$ and $e q \chi_Z$, $\chi_Z \in cr(\tau)$. Then $x \in Z$ and $Z \in \tau$. Hence, by the τ-convergence of $\langle x_n \rangle$, there is an $N \in \mathbf{N}$, such that, $x_n \in Z$, $n \geq N$. But then, $\langle e_n \rangle q(N)\chi_Z$, and so $e_n \xrightarrow{cr(\tau)} e$.

Since all implications are reversible, the converse follows. □

Proposition 3.2.16

$$e_n \xrightarrow{\omega(\tau)} (x)_\lambda \quad \text{iff} \quad \langle e_n \rangle \text{ is predictable by } \lambda \text{ and } x_n \xrightarrow{\tau} x.$$

Proof

First, let $e_n \xrightarrow{\omega(\tau)} (x)_\lambda$.

By proposition 3.2.13(b), $\langle e_n \rangle$ is predictable by λ. Since $cr(\tau) \subseteq \omega(\tau)$, then by propositions 3.2.15 and 3.2.7(c), $x_n \xrightarrow{\tau} x$.

Now, let $\langle e_n \rangle$ be predictable by λ and $x_n \xrightarrow{\tau} x$. Let $(x)_\lambda qA$ and $A \in \omega(\tau)$. Then $A(x) > 1-\lambda$, and so $A(x) > 1-\lambda+\zeta$, for some $\zeta \in (0,\lambda)$. Let $\delta = 1-\lambda+\zeta$. Then $x \in A^{-1}(\delta,1] \in \tau$. But then, there is an $N_1 \in \mathbf{N}$, such that $x_n \in A^{-1}(\delta,1]$, $n \geq N_1$.

Hence $A(x_n) > \delta = 1-\lambda+\zeta$, for all $n \geq N_1$.

On the other hand for the same ζ, by the predictability of $\langle e_n \rangle$, there is an $N_2 \in \mathbf{N}$, such that $e_n(x_n) \in (\lambda-\zeta,1]$, $n \geq N_2$. Let $N = \max(N_1, N_2)$. Then, $A(x_n) + e_n(x_n) > (1-\lambda+\zeta) + (\lambda-\zeta) = 1$, $n \geq N$. Therefore $\langle e_n \rangle q(N)A$, and so $e_n \xrightarrow{\omega(\tau)} e$. □

Corollary 3.2.17

In the following fuzzy topological spaces only a semiconstant sequence converges. It converges to and only to its trivial points.

(a) The discrete fts.

(b) The natural fts of the countable complement topological space (X, τ_{CC}) for an uncountable set X.

(c) The λ-cut fts of the countable complement topological space (X, τ_{CC}) for an uncountable set X, for all $\lambda \in I - \{1\}$.

Proof

(a) It follows from proposition 3.2.16, since $J_\Delta = \omega(\Delta)$ and since in the topological space (X, Δ), the sequence $\langle x_n \rangle$ converges iff it is eventually constant.

(b) Recalling that if X is uncountable, a sequence converges in (X, τ_{CC}) iff it is eventually constant, the proof follows from applying proposition 3.2.16 to the convergence in $(F_X, \omega(\tau_{CC}))$.

(c) Combine (b) with proposition 3.2.7(c) and the fact that $\omega(\tau_{CC}) \subseteq \omega_\lambda(\tau_{CC})$, for all $\lambda \in I - \{1\}$. $\square$

The combination of corollary 3.2.17(a) and proposition 3.2.11 establishes the semiconstant sequences as the inheritors, in fuzzy topology, of the role of classical constant sequences.

3. Sequential Fuzzy Topological Spaces

Before we introduce fuzzy sequentiality, it is worthwhile discussing briefly the classical notion of sequentiality and the ensuing sequential spaces. These spaces have been studied extensively by Franklin [16, 17] who referred to them as "spaces in which sequences suffice". The following summary is from Franklin (loc.cit.) and Antosik et al [1].

If (X, τ) is a topological space, then a subset Z of X is said to be sequentially open in (X, τ) or sequentially τ-open iff every sequence in X that converges to a point of Z is eventually in Z. The subset Z is said to be sequentially closed or sequentially τ-closed iff every sequence in Z has its limits in Z. It follows that every open set is sequentially open and that a set is sequentially open iff its complement is sequentially closed.

The family of all sequentially open sets in (X, τ) defines a topology on X. Such a topology has been called the Franklin topology of (X, τ) and denoted by τ_s. It follows that the Franklin topology of (X, τ) contains τ. If the converse of this also holds, that is if the Franklin topology of (X, τ) is equal to τ, then (X, τ) is said to be sequential. Thus a topological space is sequential iff every sequentially open set in it is open. An equivalent statement is that a topological space is sequential iff every sequentially closed set in it is closed.

It turns out that all first countable spaces, and hence the discrete and the indiscrete spaces, are sequential. However, the converse of that statement is false. A counterexample based on the invariance of sequentiality under coinduction can be found in Franklin [16]. Sequentiality is not, however, invariant under induction and even a subspace of a sequential space or a product of a couple of sequential spaces need not be sequential. This can be seen from counterexamples provided in Franklin [16] and Antosik et al [1].

The information on sequential spaces provided in this summary is all that is needed to understand the content of this section.

In the following, let (F_X, J) be a fts and A a fuzzy set in X.

Definition 3.3.1

The fuzzy set A is said to be sequentially open in (F_X, J) or sequentially J-open iff every fuzzy sequence converging to a fuzzy point which is quasi-coincident with A is itself eventually quasi-coincident to A. Symbolically,

A is sequentially J-open iff "$e_n \xrightarrow{J} e q A$" implies "$\langle e_n \rangle q(N)A$, for some $N \in \mathbb{N}$".

Definition 3.3.2

The fuzzy set A is said to be sequentially closed in (F_X, J) or sequentially J-closed iff every fuzzy sequence in A has its limits in A.

Proposition 3.3.3

A fuzzy set is sequentially J-open iff its complement is sequentially J-closed.

Proof

First, let A be sequentially J-open. Let $e_n \xrightarrow{J} e$ and $e_n \in A^c$, $n \in \mathbb{N}$. Hence $\langle e_n \rangle \not{q} A$, and so, by the sequential J-openness of A, $e \not{q} A$. But then $e \in A^c$. Therefore, A^c is sequentially J-closed.

Now, let A be sequentially J-closed. Let $e_n \xrightarrow{J} eqA^c$. Then $e \notin A$. By the sequential J-closedness of A and proposition 3.2.7(a), every subsequence of $\langle e_n \rangle$ is not in A. Hence there is an $N \in \mathbb{N}$, such that $e_n \notin A$, $n \geq N$. But then $\langle e_n \rangle q(N)A$. Therefore, A is sequentially J-open. □

Proposition 3.3.4

(a) Every J-open (J-closed) fuzzy set is sequentially J-open (sequentially J-closed). In particular the full and the empty fuzzy sets are both sequentially J-open and sequentially J-closed.

(b) The family J_s of all sequentially J-open fuzzy sets in X is a fuzzy topology on F_X which is stronger than J.

Proof

(a) is trivial. Thus we only need to show that J_s is closed under finite intersection and arbitrary union.

First, let $A_1, A_2 \in J_s$ and $e_n \xrightarrow{J} eq(A_1 \cap A_2)$. Then, by consequence 1.3.7(b), eqA_1 and eqA_2, and hence there are N_1, $N_2 \in \mathbb{N}$, such that $\langle e_n \rangle q(N_1)A_1$ and $\langle e_n \rangle q(N_2)A_2$. Let $N = \max(N_1, N_2)$. Then $\langle e_n \rangle q(N)(A_1 \cap A_2)$, by consequence

1.3.7(b). Thus, $A_1 \cap A_2 \in J_s$.

Now, let $C \subseteq J_s$ and $e_n \xrightarrow{J} eq(\bigcup_{A \in C} A)$. Then, by consequence 1.3.6(b), there is an $A_0 \in C$, such that eqA_0, and hence there is an $N \in \mathbb{N}$, such that $\langle e_n \rangle q(N) A_0$. But then $\langle e_n \rangle q(N)(\bigcup_{A \in C} A)$, by consequence 1.3.6(b). Thus, $\bigcup_{A \in C} A \in J_s$. □

<u>Definition 3.3.5</u>

The family J_s of all sequentially open sets in (F_X, J) is called the <u>Franklin fuzzy topology of</u> (F_X, J). By proposition 3.3.4(a), $J \subseteq J_s$. The converse of this inclusion need not be true, as is shown by the following proposition.

<u>Proposition 3.3.6</u>

Let X be uncountable and for every $\lambda \in I - \{1\}$, let J_λ be the λ-cut fuzzy topology $\omega_\lambda(\tau_{cc})$ of the countable complement topological space (X, τ_{cc}). Then for every $\lambda \in I - \{1\}$, $(J_\lambda)_s \not\subseteq J_\lambda$.

<u>Proof</u>

Choose $\delta_0 \in (\lambda, 1)$ and $y \in X$, and consider the fuzzy set $E = (y)_{\delta_0}$. We have $E^{-1}(\lambda,1] = \{y\} \neq \phi$, and $(E^{-1}(\lambda,1])^c = X - \{y\}$, and hence $E^{-1}(\lambda,1]$ is a nonempty subset of X the complement of which is uncountable. Hence $E^{-1}(\lambda,1] \notin \tau_{cc}$, and so $E \notin \omega_\lambda(\tau_{cc}) = J_\lambda$.

On the other hand if $\langle e_n \rangle$ is a fuzzy sequence in X, $e = (x)_\delta \in F_X$ and $e_n \xrightarrow{J_\lambda} eqE$, then $x = y$ and $\delta > 1-\delta_0$. Hence $\delta = 1-\delta_0+\zeta$, for some $\zeta \in (0,\delta_0]$. By the convergence in

$\omega_\lambda(\tau_{CC})$ (proposition 3.2.17(c)) supt $\langle e_n\rangle$ is eventually x and $\langle e_n\rangle$ is predictable by δ. Thus for the same ζ, there is an $N \in \mathbb{N}$, such that $e_n(x_n) = e_n(x) \in (\delta-\zeta,1]$, $E(x_n) = \delta_0$, $n \geq N$. Hence $E(x_n) + e_n(x_n) = \delta_0 + e_n(x) > \delta_0 + (\delta-\zeta) = \delta_0 + (1-\delta_0+\zeta-\zeta) = 1$, $n \geq N$. Thus, $\langle e_n\rangle q(N)E$. It follows that E is sequentially J_λ-open.

Therefore, $E \in (J_\lambda)_s - J_\lambda$,and so $(J_\lambda)_s \not\subseteq J_\lambda$. □

Definition 3.3.7

A fts (F_X ,J) is said to be fuzzy sequential or sequential^f iff $J_s \subseteq J$.

It follows that (F_X ,J) is sequential^f iff $J_s = J$ or equivalently every sequentially J-open fuzzy set is J-open.

Corollary 3.3.8

(F_X ,J) is sequential^f iff every sequentially J-closed fuzzy set is J-closed.

Proof

It follows from proposition 3.3.3 and definition 3.3.7. □

Proposition 3.3.9

(a) If T is a fuzzy topology on F_X which is stronger than J, then T_s is stronger than J_s

(b) If $A \in J_s$,then $A^{-1}(\lambda,1] \in (\iota(J))_s$,for each $\lambda \in I - \{1\}$

(c) If τ is a topology on X, then Z is sequentially τ-open iff χ_Z is sequentially $\omega(\tau)$-open.

Proof

(a) This follows from the fact that convergence in (F_X, T) implies convergence in (F_X, J) (proposition 3.2.7(c)).

(b) Let $A \in J_s$. Since $J \subseteq \omega(\iota(J))$, then, by (a), $J_s \subseteq (\omega\iota(J))_s$. Let $x_n \xrightarrow{\iota(J)} x \in A^{-1}(\lambda,1]$. Since $\lambda \in I - \{1\}$, then $1 - \lambda \in I - \{0\}$. Consider the fuzzy point $(x)_{1-\lambda}$. Since $A(x) > \lambda$, then $(x)_{1-\lambda}qA$.
Consider the fuzzy sequence $\langle e_n \rangle$ of constant value $1-\lambda$ and with supt $\langle e_n \rangle = \langle x_n \rangle$. Given any $\zeta \in (0,1-\lambda)$, we have $1 - \lambda > 1 - \lambda - \zeta$, hence $e_n(x_n) \in (1-\lambda-\zeta, 1]$, for all $n \in \mathbf{N}$. Therefore, the fuzzy sequence $\langle e_n \rangle$ is predictable by $1-\lambda$, and hence by the convergence of $\langle x_n \rangle$ to x in $(X, \iota(J))$ and proposition 3.2.16, $e_n \xrightarrow{\omega(\iota(J))} (x)_{1-\lambda}$. Therefore, there is an $N \in \mathbf{N}$ such that $\langle e_n \rangle q(N)A$. Hence $A(x_n) + e_n(x_n) > 1$, $n \geq N$. But then $A(x_n) > 1 - e_n(x_n) = 1 - (1-\lambda) = \lambda$, $n \geq N$. So, $x_n \in A^{-1}(\lambda,1]$, $n \geq N$. Thus $A^{-1}(\lambda,1]$ is sequentially $\iota(J)$-open.

(c) Considering (b), we need only show that if Z is sequentially τ-open then χ_Z is sequentially $\omega(\tau)$-open. Thus, let $e_n \xrightarrow{\omega(\tau)} eq\chi_Z$ and Z be sequentially τ-open. Then $x_n \xrightarrow{\tau} x$, by proposition 3.2.16, and $x \in Z$, where $\langle x_n \rangle = $ supt $\langle e_n \rangle$ and $x =$ supt e. But then, by the sequential τ-openness of Z, $x_n \in Z$, $n \geq N$, for some $N \in \mathbf{N}$. But then $\langle e_n \rangle q(N)\chi_Z$. Therefore, χ_Z is sequentially J-open.

Proposition 3.3.10

Fuzzy sequentiality is a good fuzzy extension of classical sequentiality. That is, a topological space (X, τ) is sequential iff its natural fts $(F_X, \omega(\tau))$ is sequential^f.

Proof

First, let $(F_X, \omega(\tau))$ be sequential^f. Let $Z \in \tau_s$. Then $\chi_Z \in (\omega(\tau))_s$, by proposition 3.3.9(c). But then, by the sequentiality of $(F_X, \omega(\tau))$, $\chi_Z \in \omega(\tau)$. Hence $Z \in \tau$. Thus (X, τ) is sequential.

Now, let (X, τ) be sequential. Let $A \in (\omega(\tau))_s$. Then by proposition 3.3.9(b), $A^{-1}(\lambda,1] \in (\tau)_s$, for every $\lambda \in I - \{1\}$. But then, by the sequentiality of (X, τ), $A^{-1}(\lambda,1] \in \tau$, for every $\lambda \in I - \{1\}$. Hence $A \in \omega(\tau)$. Thus, $(F_X, \omega(\tau))$ is sequential^f. □

Proposition 3.3.11

Quasi-first countability implies fuzzy sequentiality.

Proof

Let (F_X, J) be Q-C_1. We will show that if $A \notin J$, then $A \notin J_s$.

Thus, let $A \notin J$. Then, by consequence 2.1.9, there is an $e \in F_X$ such that eqA, and no open Q-neighbourhood of e is contained in A. By consequence 3.1.9, there is an open nested Q-neighbourhood base $\mathbf{B}_e = \{B_n : n \in \mathbf{N}\}$ for J at e with $B_{n+1} \subseteq B_n$, $n \in \mathbf{N}$. Then for each $n \in \mathbf{N}$, we have $B_n \not\subseteq A$, and hence $B_n(x_n) > A(x_n)$, for some $x_n \in X$. Then, $1 - B(x_n) < 1 - A(x_n)$ $n \in \mathbf{N}$. For each $n \in \mathbf{N}$, let $\lambda_n = 1 - A(x_n)$, then $\lambda_n \in (0,1]$, and hence we can consider the fuzzy points $e_n = (x_n)_{\lambda_n}$, and subsequently the fuzzy sequence $\langle e_n \rangle$. Now $B_n(x_n) + e_n(x_n) = B_n(x_n) + \lambda_n > B_n(x_n) + 1 - B_n(x_n) = 1$, hence $e_n q B_n$; $n \in \mathbf{N}$. But $B_{n+1} \subseteq B_n$, $n \in \mathbf{N}$, and so $e_n q B_k$, $n \geq k$. Thus for every $B_k \in \mathbf{B}_e$ there is an $N_k \in \mathbf{N}$, (namely $N_k = k$), such that $e_n q B_k$, $n \geq N_k$, and so $\langle e_n \rangle q(N_k) B_k$. Thus $\langle e_n \rangle$ is eventually quasi-coincident to every member of $\mathbf{B}_e$, and hence to every open Q-neighbourhood of e. Therefore, $e_n \xrightarrow{J} e$.

On the other hand, $A(x_n) + e_n(x_n) = A(x_n) + \lambda_n = A(x_n) + (1-A(x_n)) = 1$, and so $\langle e_n \rangle \not q A$. Therefore, A is not sequentially J-open. □

Corollary 3.3.12

Both fuzzy first countability and fuzzy second countability imply fuzzy sequentiality.

Proof

This follows from propositions 3.3.11 and 3.1.12 (respectively 3.1.11). □

Proposition 3.3.13

Fuzzy sequentiality does not imply quasi-first countability.

Proof

Let $\mathbf{R}^* = \mathbf{R} - \bigcup_{n=1}^{\infty} \{n,-n\}$ and define the function h^* by

$h^* : \mathbf{R} \longrightarrow \mathbf{R}^*$ and $h^*(x) = \begin{cases} x & \text{if } x \in \mathbf{R}^* \\ 0 & \text{otherwise} \end{cases}$. Consider the topology

τ_{h^*} on $\mathbf{R}^*$ defined by, $\tau_{h^*} = \{Z \subseteq \mathbf{R}^* : {h^*}^{-1}[Z]$ is open in $\mathbf{R}$ (equipped with its usual topology)$\}$.

The topological space (X, τ_{h*}) was shown to be sequential and not first countable (Corollary 1.3 in Franklin [16]). Hence by propositions 3.3.10 and 3.1.26, $(F_{R*}, \omega(\tau_{h*}))$ is sequential^f but not $Q\text{-}C_1$. □

Proposition 3.3.14

Let (F_X, J) be a fts.

The sequentiality of $(X, \iota(J))$ does not imply the fuzzy sequentiality of (F_X, J).

Proof

Let X be an uncountable set, $\lambda \in I - \{1\}$ and $J = \omega_\lambda(\tau_{CC})$. Then, by proposition 2.2.19(a), $\iota(J) = \Delta$, and hence $(X, \iota(J))$ is sequential. On the other hand, by proposition 3.3.6, (F_X, J) is not sequential^f. □

We have been unable to prove the converse of the previous proposition. However, we make the following conjecture.

Conjecture 3.3.15

Fuzzy sequentiality is not initially good.

The following is a summary combining results from sections (1) and (2) about relationships between fuzzy sequential spaces, quasi-first countable spaces and fuzzy first and second countable spaces.

Conclusion 3.3.16

We have the following diagram for fuzzy topological spaces. None of the arrows in the diagram is reversible and the missing ones indicate the lack of any implications.

$$\begin{tikzcd} C_1^f \arrow[dr, Rightarrow] & & C_2^f \arrow[dl, Rightarrow] \\ & Q\text{-}C_1 \arrow[d, Rightarrow] & \\ & \text{Sequential}^f & \end{tikzcd}$$

PART IV

FUNCTIONS IN FUZZY TOPOLOGICAL SPACES

The idea of extending to fuzzy topology classical results involving functions by fuzzifying classical properties of functions rather than introducing fuzzy functions was implicit in Chang's work [8]. As was observed by Warner [52] and others, this is quite sufficient for the fuzzification of topology. Extending the notion of function to fuzzy theory, as was done by Erceg [14] is a generalization which we do not find necessary for the purpose of this thesis. We follow Chang's point of view which is now standard.

In section (1), we give Chang's definitions of fuzzy image and fuzzy inverse image and present their consequences. In the process, we correct some faulty claims which have appeared in the literature.

In section (2), we study basic fuzzy properties of functions. Introducing fuzzy sequential continuity, we show that it provides another way of defining sequential fuzzy topological spaces.

In section (3), we investigate the goodness of the fuzzy extensions introduced in section (2). In particular it has been implied as obvious that fuzzy closedness of a function is good [26], and no proof of this assertion appears in the literature. We present a proof of the goodness of fuzzy closedness which in itself demonstrates the non-triviality of the claim.

1. Images and Inverse Images of Fuzzy Sets

In the following, let X and Y be sets and let $f : X \longrightarrow Y$.

Definition 4.1.1. (Chang)

Let $E \subseteq F_X$.

A subset of F_Y is said to be the image (in Y) of E under f and denoted by $f[E]$ iff, for every y in Y, we have

$$f[E]\,(y) = \begin{cases} \sup \{E(x) : f(x) = y\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}.$$

It follows that for every $x \in X$, we have $f[E]\,(f(x)) \geq E(x)$ and that $f[E]\,(f(x)) = E(x)$ if the function f is injective.

Definition 4.1.2

The image of a fuzzy point e in X is defined to be the image of the fuzzy set e of singleton support.

Definition 4.1.3 (Chang)

Let $D \subseteq F_Y$.

A subset of F_X is said to be the inverse image (in X) of D under f and denoted by $f^{-1}[D]$ iff, for every x in X, we have;

$$f^{-1}[D]\,(x) = D\,(f(x)).$$

It follows that $f^{-1}[D]$ is the composition function $D_\circ f$: $X \longrightarrow I$, as the following diagram shows:

$$\begin{tikzcd}
X \arrow[rr, dashed, "f"] \arrow[dr, dashed, "f^{-1}[D] = D_{\circ}f"'] & & Y \arrow[dl, dashed, "D"] \\
& I &
\end{tikzcd}$$

It also follows that if g is a function from Y to some set Z, then for a fuzzy set $A \subseteq F_Z$, we have;

$$(g_{\circ}f)^{-1}[A] = A_{\circ}(g_{\circ}f) = (A_{\circ}g)_{\circ}f = g^{-1}[A]_{\circ}f = f^{-1}[g^{-1}[A]].$$

This is illustrated by the following diagram.

$$\begin{tikzcd}
X \arrow[r, dashed, "f"] \arrow[dr, dashed, "{f^{-1}[g^{-1}[A]]}"'] & Y \arrow[r, dashed, "g"] \arrow[d, "{g^{-1}[A]}"] & Z \arrow[dl, dashed, "A"] \\
& I &
\end{tikzcd}$$

<u>Consequence 4.1.4</u>

Let $x \in X$, $y \in Y$ and $\lambda \in I - \{0\}$.

(a) $f((x)_\lambda) \quad = \quad (f(x))_\lambda$

(b) $f^{-1}((y)_\lambda) \quad = \quad Z^{(\lambda)}$, where $Z = f^{-1}(y)$. If f is injective then $f^{-1}((y)_\lambda) \quad = \quad (f^{-1}(y))_\lambda$.

Note: we have not used square brackets for fuzzy points.

<u>Proof</u>

(a) The formula in (a) was given in Pu and Liu [44] as the definition of the image of fuzzy point. However it can be proved as a result of definitions 4.1.1 and 4.1.2 as we now show.

Let $t \in Y$

$$f((x)_\lambda)(t) = \begin{cases} \sup\{(x)_\lambda(z) : z \in f^{-1}(t)\} & \text{if } f^{-1}(t) \neq \phi \\ 0 & \text{if } f^{-1}(t) = \phi \end{cases}$$

Now, by definition of $(x)_\lambda$, $(x)_\lambda(z) = \begin{cases} \lambda & \text{if } z = x \\ 0 & \text{if } z \in X - \{x\} \end{cases}$

and since $x \in f^{-1}(t)$ iff $t = f(x)$, then,

$$f((x)_\lambda)(t) = \begin{cases} \lambda & \text{if } t = f(x) \\ 0 & \text{if } t \neq f(x) \end{cases}. \text{ Therefore,}$$

$$f((x)_\lambda) = (f(x))_\lambda$$

(b) Let $x \in X$.

$$f^{-1}((y)_\lambda)(x) = (y)_\lambda(f(x)) = \begin{cases} \lambda & \text{if } x \in f^{-1}(y) = Z \\ 0 & \text{if } x \notin f^{-1}(y) = Z \end{cases}.$$

$$= Z^{(\lambda)}(x)$$

Therefore, $f^{-1}((y)_\lambda) = Z^{(\lambda)}$.

If f is injective, then Z is a singleton, and hence $f^{-1}((y)_\lambda) = (f^{-1}(y))_\lambda$. □

Notation 4.1.5

To distinguish constant sets in the domain X of f (i.e. constant subsets of F_X) from those in the codomain Y (i.e. constant subsets of F_Y), we adopt the following convention. For a $\lambda \in I$, the already established symbol η_λ represents a constant set in X with value λ, while the primed symbol η'_λ denotes a constant set in Y of the same value λ. In the same manner we will notationally distinguish between crisp sets in X and those in Y.

Consequence 4.1.6

Let $Z \subseteq X$, $Z_1 \subseteq Y$ and $\lambda \in I$.

(a) 1. $f[Z^{(\lambda)}] = (f[Z])^{(\lambda)}$

2. $f^{-1}[Z_1^{(\lambda)}] = (f^{-1}[Z_1])^{(\lambda)}$

(b) 1. $f[\chi_Z] = \chi'_{f[Z]}$

2. $f^{-1}[\chi'_{Z_1}] = \chi_{f^{-1}[Z_1]}$

(c) 1. $f[\eta_\lambda] = (f[X])^{(\lambda)}$. If f is surjective, then $f[\eta_\lambda] = \eta'_\lambda$

2. $f^{-1}[\eta'_\lambda] = \eta_\lambda$

(d) 1. $f[\phi_X] = \phi_Y$ and $f[F_X] = \chi'_{f[X]}$. If f is surjective, then $f[F_X] = F_Y$

2. $f^{-1}[\phi_Y] = \phi_X$ and $f^{-1}[F_Y] = F_X$.

Proof

(a) 1. Let $t \in Y$.

$$f[Z^{(\lambda)}](t) = \begin{cases} \sup\{Z^{(\lambda)}(z) : z \in f^{-1}(t)\} & \text{if } f^{-1}(t) \neq \phi \\ 0 & \text{if } f^{-1}(t) = \phi \end{cases}$$

$$= \begin{cases} \lambda & \text{if } f^{-1}(t) \cap Z \neq \phi \\ 0 & \text{if } f^{-1}(t) \cap Z = \phi \end{cases}.$$

But $f^{-1}(t) \cap Z \neq \phi$ iff $t \in f[Z]$. Thus

$$f[Z^{(\lambda)}](t) = \begin{cases} \lambda & \text{if } t \in f[Z] \\ 0 & \text{if } t \notin f[Z] \end{cases}$$

$$= (f[Z])^{(\lambda)}(t).$$

Therefore, $f[Z^{(\lambda)}] = (f[Z])^{(\lambda)}$.

2. Let $z \in X$.

$$f^{-1}[Z_1^{(\lambda)}](z) = Z_1^{(\lambda)}(f(z)) = \begin{cases} \lambda & \text{if } f(z) \in Z_1 \\ 0 & \text{if } f(z) \notin Z_1 \end{cases}$$

$$= \begin{cases} \lambda & \text{if } z \in f^{-1}[Z_1] \\ 0 & \text{if } z \notin f^{-1}[Z_1] \end{cases}$$

$$= (f^{-1}[Z_1])^{(\lambda)}(z).$$

Therefore, $f^{-1}[Z_1{}^{(\lambda)}] = (f^{-1}[Z_1])^{(\lambda)}$.

(b) $\chi_Z = Z^{(1)}$ and $\chi'_{Z_1} = Z_1{}^{(1)}$. By part (a), we have,

1. $f[\chi_Z] = f[Z^{(1)}] = (f[Z])^{(1)} = \chi'_{f[Z]}$

2. $f^{-1}[\chi'_{Z_1}] = f^{-1}[Z_1{}^{(1)}] = (f^{-1}[Z_1])^{(1)} = \chi_{f^{-1}[Z_1]}$

(c) $\eta_\lambda = X^{(\lambda)}$ and $\eta'_\lambda = Y^{(\lambda)}$. By part (a), we have,

1. $f[\eta_\lambda] = f[X^{(\lambda)}] = (f[X])^{(\lambda)}$. If f is surjective then $Y = f[X]$ and so $f[\eta_\lambda] = Y^{(\lambda)} = \eta'_\lambda$

2. $f^{-1}[\eta'_\lambda] = f^{-1}[Y^{(\lambda)}] = (f^{-1}[Y])^{(\lambda)} = X^{(\lambda)} = \eta_\lambda$.

(d) 1. $\Phi_X = \chi_\Phi$ and $F_X = \chi_X$. Hence, by part (b), we have, $f[\Phi_X] = f[\chi_\Phi] = \chi'_{f[\Phi]} = \chi'_\Phi = \Phi_Y$, and $f[F_X] = f[\chi_X] = \chi'_{f[X]}$. If f is surjective then, $Y = f[X]$, and so $f[F_X] = \chi'_Y = F_Y$.

2. $\Phi_Y = \eta'_0$ and $F_Y = \eta'_1$. Hence, by part (c), we have, $f^{-1}[\Phi_Y] = f^{-1}[\eta'_0] = \eta_0 = \Phi_X$, and $f^{-1}[F_Y] = f^{-1}[\eta'_1] = \eta_1 = F_X$. □

<u>Consequence 4.1.7</u>

Let $E, E_1 \subseteq F_X$,$D, D_1 \subseteq F_Y$, $e \in F_X$,$d \in F_Y$, and $\langle e_n \rangle$ a fuzzy sequence in X.

(a) 1. If $E \subseteq E_1$, then $f[E] \subseteq f[E_1]$

2. If $D \subseteq D_1$, then $f^{-1}[D] \subseteq f^{-1}[D_1]$ (Chang)

(b) 1. If $e \in E$, then $f(e) \in f[E]$

If $e q E$, then $f(e)\, q\, f[E]$

2. $e \in f^{-1}[D]$ iff $f(e) \in D$ (cf Bülbül [6])

$e\ q\ f^{-1}[D]$ iff $f(e)\ q\ D$.

(c) If $\lambda \in I - \{0\}$ and $\langle e_n \rangle$ is predictable by λ, then so is $\langle f(e_n) \rangle$. In particular if $\langle e_n \rangle$ is semiconstant with e being a trivial limit of $\langle e_n \rangle$, then $\langle f(e_n) \rangle$ is also semiconstant and $f(e)$ is a trivial limit of $\langle f(e_n) \rangle$.

Proof

(a) See theorem 4.1, c and d in Chang [8].

(b) 1. The first assertion follows from part a.1. To prove the second assertion, let $x = \text{supt } e$, then $f(x) = \text{supt } f(e)$, by consequence 4.1.4(a), and recalling that $f[E](f(x)) \geq E(x)$, we have,

$$f[E](f(x)) + f(e)(f(x)) \geq E(x) + e(x)$$
$$> 1 \quad , \text{ if } eqE.$$

Hence, $f(e)\ q\ f[E]$ if eqE.

2. Let $x = \text{supt } e$, then $f(x) = \text{supt } f(e)$, $f(e)(f(x)) = e(x)$, and $f^{-1}[D](x) = D(f(x))$. We have,

$f(e)qD$ iff $1 < f(e)(f(x)) + D(f(x)) = e(x) + f^{-1}[D](x)$

iff $eqf^{-1}[D]$. And

$f(e) \in D$ iff $f(e)(f(x)) \leq D(f(x))$ iff $e(x) \leq f^{-1}[D](x)$

iff $e \in f^{-1}[D]$.

(c) 1. Let $\langle x_n \rangle = \text{supt } \langle e_n \rangle$, then $\langle f(x_n) \rangle = \text{supt } \langle f(e_n) \rangle$ and $f(e_n)(f(x_n)) = e_n(x_n)$, $n \in \mathbf{N}$, by consequence 4.1.4(a). Let $\zeta \in (0,\lambda)$ and $\langle e_n \rangle$ be predictable by λ. Then, there is an $N \in \mathbf{N}$, such that, $e_n(x_n) \in (\lambda-\zeta,1]$, $n \geq N$. But then, $f(e_n)\ (f(x_n)) \in (\lambda-\zeta,1]$, $n \geq N$. Therefore, $\langle f(e_n) \rangle$ is predictable by λ.

2. Now let $\langle e_n \rangle$ be semiconstant and e a trivial point of $\langle e_n \rangle$. Let $e = (x)_\delta$, for some $x \in X$ and $\delta \in I - \{0\}$. Then $\langle e_n \rangle$ is predictable by δ and supt $\langle e_n \rangle$ is eventually x. But then, $\langle f(e_n) \rangle$ is predictable by δ and supt $\langle f(e_n) \rangle$ is eventually $f(x)$. Therefore, $\langle f(e_n) \rangle$ is semiconstant and $(f(x))_\delta = f((x)_\delta) = f(e)$ is a trivial point of $\langle f(e_n) \rangle$. $\square$

<u>Consequence 4.1.8</u>

Let **C** and **C'** be families of subsets of F_X and F_Y respectively. Then

(a) 1. $f[\bigcup_{E \in \mathbf{C}} E] = \bigcup_{E \in \mathbf{C}} f[E]$

2. $f^{-1}[\bigcup_{D \in \mathbf{C}'} D] = \bigcup_{D \in \mathbf{C}'} f^{-1}[D]$

(b) 1. $f[\bigcap_{E \in \mathbf{C}} E] \subseteq \bigcap_{E \in \mathbf{C}} f[E]$. The equality holds if f is injective.

2. $f^{-1}[\bigcap_{D \in \mathbf{C}'} D] = \bigcap_{D \in \mathbf{C}'} f^{-1}[D]$.

<u>Proof</u>

(a) 1. Let $y \in Y$. If $f^{-1}(y) = \phi$, then $f[E](y) = 0$, for all $E \in \mathbf{C}$, and hence $(\bigcup_{E \in \mathbf{C}} f[E])(y) = \sup\{f[E](y) : E \in \mathbf{C}\}$ $= 0 = f[\bigcup_{E \in \mathbf{C}} E](y)$.

If $f^{-1}(y) \neq \phi$, then

$$f[\bigcup_{E \in \mathbf{C}} E](y) = \sup\{(\bigcup_{E \in \mathbf{C}} E)(x) : x \in f^{-1}(y)\}$$
$$= \sup\{\sup\{E(x) : E \in \mathbf{C}\} : x \in f^{-1}(y)\}$$
$$= \sup\{\sup\{E(x) : x \in f^{-1}(y)\} : E \in \mathbf{C}\}$$

$= \sup\{f[E](y) : E \in C\} = (\bigcup_{E\in C} f[E])(y).$

Therefore, $f[\bigcup_{E\in C} E] = \bigcup_{E\in C} f[E]$.

2. Let $x \in X$.

$$f^{-1}[\bigcup_{D\in C'} D](x) = \bigcup_{D\in C'} D(f(x)) = \sup\{D(f(x)) : D \in C'\}$$
$$= \sup\{f^{-1}[D](x) : D \in C'\}$$
$$= \bigcup_{D\in C'} f^{-1}[D](x).$$

Therefore, $f^{-1}[\bigcup_{D\in C'} D] = \bigcup_{D\in C'} f^{-1}[D]$.

(b) 1. See Foster [15]. The equalilty holds if f is injective, because then, for all $y \in f[X]$, $f^{-1}(y)$ is a singleton, say $\{x\}$, and hence $f[\bigcap_{E\in C} E](y) = (\bigcap_{E\in C} E)(x) = \inf\{E(x) : E \in C\} = \inf\{f[E](y) : E \in C\} = \bigcap_{E\in C} f[E](y)$.

The converse of (b)1 need not be true, since it does not necessarily hold in the classical case. For instance, let $X = \{x_1, x_2, z\}$, $Y = \{y,t\}$ and define $f: X \longrightarrow Y$, by $f(x_1) = y = f(x_2)$ and $f(z) = t$. Then choosing $Z_1 = \{x_1\}$ and $Z_2 = \{x_2, z\}$, we have $f[Z_1 \bigcap Z_2] = f[\phi] = \phi$, but $f[Z_1] \bigcap f[Z_2] = \{y\} \bigcap \{y,t\} = \{y\}$. Thus, for the crisp fuzzy sets, $X_{Z_1}, X_{Z_2} \subseteq F_X$,we have $f[X_{Z_1} \bigcap X_{Z_2}] = X'_{f[Z_1 \bigcap Z_2]} = X'_\phi = \phi_Y$, while $f[X_{Z_1}] \bigcap f[X_{Z_2}] = X'_{f[Z_1]} \bigcap X'_{f[Z_2]} = X'_{f[Z_1] \bigcap f[Z_2]} = X'_{\{y\}} \not\subseteq \phi_Y$.

Thus $f[X_{Z_1}] \bigcap f[X_{Z_2}] \not\subseteq f[X_{Z_1} \bigcap X_{Z_2}]$.

2. It is proved in a similar fashion as (a)2. $\square$

Ganguly & Saha [18], in result 2.7, gave and proved the following statement;

If $A, B \subseteq F_X$, then $f[A \bigcup B] \subseteq f[A] \bigcup f[B]$.

This result, although not false, gives the impression that the converse need not be true. This has made it necessary to supply the straightforward proof of (a)1.

Another inaccuracy appeared in Chang [8]. In theorem 4.1(b) in [8], Chang claimed that, given an $A \subseteq F_X$, we have $(f([A])^c \subseteq f[A^c]$. This faulty statement was borrowed later by Malghan and Benchalli [38]. Since this claim is not even true in classical set theory, it is certainly false for fuzzy sets. In the following, we give the correct version of that property.

Consequence 4.1.9

Let $E \subseteq F_X$ and $D \subseteq F_Y$, then

(a) If f is surjective, then $(f[E])^c \subseteq f[E^c]$.

If f is injective, then $f[E^c] \subseteq (f[E])^c$.

If f is bijective, then $(f[E])^c = f[E^c]$.

(b) $(f^{-1}[D])^c = f^{-1}[D^c]$ (Chang).

Proof

(a) First, let f be surjective. Then, for every $y \in Y$, we have $f^{-1}(y) \neq \phi$, and hence,

$$\begin{aligned}(f[E])^c(y) = 1 - f[E](y) &= 1 - \sup\{E(x) : y = f(x)\}\\ &= \inf\{1 - E(x) : y = f(x)\}\\ &\leq \sup\{1 - E(x) : y = f(x)\}\\ &= \sup\{E^c(x) : y = f(x)\}\\ &= f[E^c](y).\end{aligned}$$

Thus, $(f[E])^c \subseteq f[E^c]$.

Now, let f be injective, then for every $y \in f[X]$,

$f^{-1}(y)$ is a singleton, say $\{x\}$, and hence $f[E](y) = E(x)$ and $f[E^c](y) = E^c(x)$, (see definition 4.1.1). Thus, $(f[E])^c(y) = 1 - f[E](y) = 1 - E(x) = E^c(x) = f[E^c](y)$. If $y \in Y - f[X]$, then $f[E^c](y) = 0 \leq (f[E])^c(y)$.

Therefore, $f[E^c] \subseteq (f[E])^c$.

Lastly, if f is bijective, then combining the two former results, we have $(f[E])^c = f[E^c]$.

To show that neither of the converses of the first two results needs hold for an arbitrary function, let $X = \{x_1, x_2\}$, $Y = \{y, t\}$ and define f by, $f(x_1) = y = f(x_2)$. Let $E = \chi_{\{x_1\}}$. Then $f[E] = \chi_{\{y\}}$ and $(f[E])^c = \chi_{\{t\}}$. On the other hand $E^c = \chi_{\{x_2\}}$ and $f[E^c] = \chi_{\{y\}}$. Thus, $f[E^c] \not\subseteq (f[E])^c$ and $(f[E])^c \not\subseteq f[E^c]$.

(b) See theorem 4.1(a) in Chang [8]. □

Consequence 4.1.10

Let $E \subseteq F_X$, $D \subseteq F_Y$, $e \in F_X$ and $d \in F_Y$.

(a) 1. $E \subseteq f^{-1}[f[E]]$ and $f[f^{-1}[D]] \subseteq D$ (Chang).

2. If for every $y \in f[X]$, there is a $\lambda_y \in I$, such that $E(x) = \lambda_y$ for all $x \in f^{-1}(y)$, then $E = f^{-1}[f[E]]$. In particular, if f is injective, then $E = f^{-1}[f[E]]$ (Pu & Liu).

3. If $D \subseteq f[X]$, then $D = f[f^{-1}[D]]$. In particular if f is surjective, then $D = f[f^{-1}[D]]$ (Pu & Liu).

(b) 1. $e \in f^{-1}(f(e))$ and $f(f^{-1}(d) \subseteq d$.

2. If f is injective, then $e = f^{-1}(f(e))$.

3. If f is surjective, then $d = f(f^{-1}(d))$.

Proof

(a) 1. See theorem 4.1(f,g) in Chang [8].

2. Let $x \in X$,

$f^{-1}[f[E]](x) = f[E](f(x)) = \sup\{E(x) : x \in f^{-1}(y)\}$

$= E(x)$, since $E(x)$ is the same for all $x \in f^{-1}(y)$,

by hypothesis.

Thus, $E = f^{-1}[f[E]]$.

3. Let $y \in \text{supt } D$. Then $y \in f[X]$, by hypothesis. Thus,

$f[f^{-1}[D]](y) = \sup\{f^{-1}[D](x) : x \in f^{-1}(y)\} =$

$\sup\{D(f(x)) : y = f(x)\} = D(y)$.

Therefore, $D = f[f^{-1}[D]]$.

(b) This follows from (a). Notice that if $\text{supt } d \notin f[X]$, then, $f^{-1}(d) = \phi_X$ and so $f(f^{-1}(d)) = \phi_Y$. □

The following three consequences will be useful later. The first consequence will be frequently used.

In the following, let $\lambda \in I - \{1\}$.

Consequence 4.1.11

Let $E \subseteq F_X$, $D \subseteq F_Y$.

(a) 1. $f(E^{-1}(\lambda,1]) = (f[E])^{-1}(\lambda,1]$.

2. $f^{-1}(D^{-1}(\lambda,1] = (f^{-1}[D])^{-1}(\lambda,1]$.

(b) In particular,

1. $f(\text{supt } E) = \text{supt}(f[E])$.

2. $f^{-1}(\text{supt } D) = \text{supt}(f^{-1}[D])$.

(Malghan and Benchalli [38])

Proof

(a) 1. Let $y \in Y$.

$y \in f(E^{-1}(\lambda,1])$ iff there is an $x \in f^{-1}(y)$ such that $E(x) > \lambda$

iff $\sup\{E(x) : y = f(x)\} > \lambda$

iff $f[E](y) > \lambda$

iff $y \in (f[E])^{-1}(\lambda,1]$.

Therefore, $f(E^{-1}(\lambda,1]) = (f[E])^{-1}(\lambda,1]$.

2. $(f^{-1}[D])^{-1}(\lambda,1] = (D \circ f)^{-1}(\lambda,1]$, by remark following definition 4.1.3

$= f^{-1}(D^{-1}(\lambda,1])$.

(b) Substitute $\lambda = 0$ in (a). □

Consequence 4.1.12

Let Ω be an indexing set and for every $\alpha \in \Omega$, let Y_α be a set and f_α a function from X into Y_α. Moreover, for every $\alpha \in \Omega$, let $D_\alpha \subseteq F_{Y_\alpha}$. Then,

(a) $(\bigcup_{\alpha\in\Omega} f_\alpha^{-1}[D_\alpha])^{-1}(\lambda,1] = \bigcup_{\alpha\in\Omega} f_\alpha^{-1}[D_\alpha^{-1}(\lambda,1]]$.

(b) $(\bigcap_{\alpha\in\Omega} f_\alpha^{-1}[D_\alpha])^{-1}(\lambda,1] \subseteq \bigcap_{\alpha\in\Omega} f_\alpha^{-1}[D_\alpha^{-1}(\lambda,1]]$. The equality holds for finite Ω.

(c) In particular,

supt $\bigcup_{\alpha\in\Omega} f_\alpha^{-1}[D_\alpha] = \bigcup_{\alpha\in\Omega} f_\alpha^{-1}(\text{supt } D_\alpha)$ and

supt $\bigcap_{\alpha\in\Omega} f_\alpha^{-1}[D_\alpha] \subseteq \bigcap_{\alpha\in\Omega} f_\alpha^{-1}(\text{supt } D_\alpha)$. (The equality holds for finite Ω).

Proof

(a) $(\bigcup_{\alpha\in\Omega} f_\alpha^{-1}[D_\alpha])^{-1}(\lambda,1] = \bigcup_{\alpha\in\Omega}(f_\alpha^{-1}[D_\alpha])^{-1}(\lambda,1]$, by consequence 1.1.14(a).

$$= \bigcup_{\alpha\in\Omega} f_\alpha^{-1}(D_\alpha^{-1}(\lambda,1]),$$ by consequence 4.1.11(a)2.

(b) $(\bigcap_{\alpha\in\Omega} f_\alpha^{-1}[D_\alpha])^{-1}(\lambda,1] \subseteq \bigcap_{\alpha\in\Omega}(f_\alpha^{-1}[D_\alpha])^{-1}(\lambda,1]$, (with the equality holding for finite Ω) by consequence 1.1.14(b).

$$= \bigcap_{\alpha\in\Omega} f_\alpha^{-1}[D_\alpha^{-1}(\lambda,1],$$ by consequence 4.1.11(b)2.

(c) Put $\lambda = 0$ in (a) and (b). □

Consequence 4.1.13

Let Y be a set and Ω an indexing set and for every $\alpha \in \Omega$, let X_α be a set and f_α a function from X_α into Y. Moreover, for every $\alpha \in \Omega$, let $E_\alpha \subseteq F_{X_\alpha}$. Then,

(a) $(\bigcup_{\alpha\in\Omega} f_\alpha[E_\alpha])^{-1}(\lambda,1] = \bigcup_{\alpha\in\Omega} f_\alpha(E^{-1}(\lambda,1]$.

(b) $(\bigcap_{\alpha\in\Omega} f_\alpha[E_\alpha])^{-1}(\lambda,1] \subseteq \bigcap_{\alpha\in\Omega} f_\alpha(E^{-1}(\lambda,1]$. The equality holds for finite Ω.

(c) In particular

supt $\bigcup_{\alpha\in\Omega} f_\alpha[E_\alpha] = \bigcup_{\alpha\in\Omega} f_\alpha(\text{supt } E_\alpha)$ and

supt $\bigcap_{\alpha\in\Omega} f_\alpha[E_\alpha] \subseteq \bigcap_{\alpha\in\Omega} f_\alpha(\text{supt } E_\alpha)$. (with the equality holding for finite Ω).

Proof

Combine consequence 4.1.11(a)1 with consequence 1.1.14 in a similar manner as in the proof of the previous consequence and put $\lambda = 0$, to obtain the particular cases. □

2. Basic Fuzzy Topological Properties of Functions

In the following, let $f: X \longrightarrow Y$, and let J and T be fuzzy topologies on F_X and F_Y respectively.

Definition 4.2.1 (Chang)

The function f is said to be fuzzy continuous (fuz-continuous) from (F_X, J) into (F_Y, T) iff

$U \in T$ implies $f^{-1}[U] \in J$.

It follows from definition 4.2.1 that constant functions are fuzzy continuous between Lowen fts's. This was actually the main reason which Lowen [33] gave in justifying his definition of fuzzy topology.

Extending the idea of 'sequential continuity' to fuzzy topology, we introduce the following concept.

Definition 4.2.2

The function f is said to be sequentially fuzzy continuous (sequentially fuz-continuous) from (F_X, J) into (F_Y, T) iff for every fuzzy sequence $\langle e_n \rangle$ and every fuzzy point e in X,

$$e_n \xrightarrow{J} e \quad \text{implies} \quad f(e_n) \xrightarrow{T} f(e).$$

Consequence 4.2.3 (Chang/Pu & Liu)

The following statements are equivalent

(a) f is fuz-continuous from (F_X, J) into (F_Y, T).

(b) $f^{-1}[D]$ is J-closed, whenever D is T-closed, where $D \subseteq F_Y$

(c) For every fuzzy point e in X and every neighbourhood D of $f(e)$ (i.e. $D \in V_{f(e)}(T)$), there is an $O \in J$, such that $e \in O$ and $f[O] \subseteq D$.

(d) For every fuzzy point e in X and every Q-neighbourhood D of $f(e)$ (i.e. $D \in Q_{f(e)}(T)$), there is an $O \in J$, such that eqO and $f[O] \subseteq D$.

Proof

See thoerem 4.2 in Chang [8] for the equivalence of the statements (a) and (b). For the implication (a) $\to$ (d), see Pu and Liu [44]. The implication (a) $\to$ (c) is proved similarly. Notice that consequences 4.1.7(b)2, 4.1.7(a)2 and 4.1.10(a) come in handy.

To show that (d) $\to$ (a), assume that f is not fuz-continuous. Then there is a $U \in T$, such that $f^{-1}[U] \notin J$. But, then by proposition 2.1.9, there is an $e \in F_X$ such that $eqf^{-1}[U]$. But, for every $O \in J \cap Q_e(J)$, we have $O \nsubseteq f^{-1}[U]$. Hence $f(e)qU$, by consequence 4.1.7(b)2, and $f^{-1}[U](x_0) < O(x_0)$ for some $x_0 \in X$. Thus, $U(f(x_0)) = f^{-1}[U](x_0) < O(x_0) \leq f[O](f(x_0))$, by definitions 4.1.3 and 4.1.1. Hence $f[O] \nsubseteq U$. Therefore, by the counterpositive argument, (d) implies (a).

The implication (c) $\to$ (a) is proved similarly. Combining all the results, we have (a) $\leftrightarrow$ (b) $\leftrightarrow$ (c) $\leftrightarrow$ (d). □

Consequence 4.2.4

(a) The composition of fuzzy continuous functions is fuzzy continuous (Chang).

(b) The composition of fuzzy sequentially continuous functions is fuzzy sequentially continuous.

Proof

(a) See comment following definition 4.2 in Chang [8].

(b) This is obvious. □

Proposition 4.2.5

Every fuzzy continuous function is sequentially fuzzy continuous.

Proof

Let f be fuz-continuous from (F_X, J) into (F_Y, T) and $e_n \xrightarrow{J} e$. Let $f(e)qU$ and $U \in T$. Then $f^{-1}[U] \in J$ and, by consequence 4.1.7(b)2, $eqf^{-1}[U]$. But then there is an $N \in \mathbb{N}$, such that $\langle e_n \rangle q(N) f^{-1}[U]$, and hence by the same consequence, $\langle f(e_n) \rangle q(N) U$. Therefore, $f(e_n) \xrightarrow{T} f(e)$, and so f is sequentially fuz-continuous. □

The converse of consequence 4.2.5, however, is not true. The following counterexample proves this point.

Proposition 4.2.6

A sequentially fuzzy continuous function need not be fuzzy continuous.

Proof

Consider the countable complement topology τ_{cc} on an uncountable set X. Choose a set Y, a function $f: X \longrightarrow Y$ and a fuzzy topology T on F_Y, such that f is not fuz-continuous from $(F_X, \omega(\tau_{cc}))$ to (F_Y, T). (e.g. let Y = X, f be the identity map

i_d: $X \longrightarrow Y = X$, and T the discrete fuzzy topology J_Δ on F_Y .)

Let $\langle e_n \rangle$ be a fuzzy sequence that converges to e in $(F_X, \omega(\tau_{cc}))$. By corollary 3.2.17(b), $\langle e_n \rangle$ is semiconstant and e is a trivial limit of $\langle e_n \rangle$. But then, by consequence 4.1.7(c), $\langle f\langle e_n \rangle)$ is semiconstant and $f(e)$ is a trivial limit of $\langle f(e_n)\rangle$. Therefore, by proposition 3.2.11, $f(e_n) \xrightarrow{T} f(e)$. Thus, f is sequentially fuzzy continuous. □

<u>Proposition 4.2.7</u>

Let J_s be the Franklin fuzzy topology of the fts (F_X, J). Then, the identity map on X is sequentially fuzzy continuous from (F_X, J) into (F_X, J_s).

<u>Proof</u>

Let $e_n \xrightarrow{J} e$, $0 \in J_s$ and $eq0$. Then 0 is sequentially J-open, and hence $\langle e_n \rangle$ is eventually quasi-coincident to 0. Thus $i_d(e_n) = e_n \xrightarrow{J_s} e = i_d(e)$. Hence i_d is sequentially fuz-continuous from (F_X, J) into (F_X, J_s). □

<u>Proposition 4.2.8</u> (Characterization of sequential fts's)

A fts is sequential^f iff every sequentially fuzzy continuous function in it is fuzzy continuous.

<u>Proof</u>

First, let (F_X, J) be sequential^f and let $f: X \longrightarrow Y$ be a sequentially fuz-continuous function from (F_X, J) into some fts (F_Y, T). Let $0 \in T$, and let $e_n \xrightarrow{J} eqf^{-1}[0]$. By the sequential fuz-continuity of f and consequence 4.1.7(b)2, $f(e_n) \xrightarrow{T} f(e)q0$.

Hence $\langle f(e_n)\rangle q(N) 0$, for some $N \in \mathbb{N}$, and so, by the consequence mentioned earlier, $\langle e_n\rangle q(N) f^{-1}[0]$. Hence, $f^{-1}[0]$ is sequentially J-open. But then, by the fuzzy sequentiality of (F_X, J), $f^{-1}[0] \in J$. Therefore, f is fuz-continuous.

Conversely, let (F_X, J) be a fts having the property that every sequentially fuz-continuous function in it is fuz-continuous. Consider the identity map $i_d: X \longrightarrow X$. By proposition 4.2.7, i_d is sequentially fuz-continuous from (F_X, J) into (F_X, J_s). But then, by hypothesis, i_d is fuz-continuous from (F_X, J) into (F_X, J_s). Hence $J_s \subseteq J$, and so (F_X, J) is sequential^f. □

<u>Proposition 4.2.9</u>

(a) A fuzzy continuous image of a sequential^f fts need not be sequential^f.

(b) A fuzzy continuous image of a Q-C_1 fts need not be Q-C_1.

(c) A fuzzy continuous image of a C_1^f (respectively C_2^f) fts need not be C_1^f (respectively C_2^f).

<u>Proof</u>

Let X be an uncountable set, J_Δ the discrete fuzzy topology, and $T = \omega(\tau_{cc})$. The identity function $i_d : X \longrightarrow X$ is fuzzy continuous from (F_X, J_Δ) to (F_X, T).

Now (F_X, J_Δ) is C_1^f, Q-C_1 and sequential^f. But (F_X, T) is not sequential^f (by proposition 3.3.6), and hence not Q-C_1, by proposition 3.3.11, nor C_1^f by corollary 3.3.12.

For the case C_2^f, let (X, τ) be the Arens-Fort space of proposition 3.1.22. Then $(F_X, \omega(\tau))$ is not C_2^f but is the image of the fuzzy second countable fts (F_X, J_Δ) under the identity function $id : X \longrightarrow X$. □

In the following, let $f: X \longrightarrow Y$, and let J and T be fuzzy topologies on F_X and F_Y respectively.

<u>Definition 4.2.9</u> (Wong [56])

The function f is said to be fuzzy open (fuz-open) from (F_X, J) into (F_Y, T) iff for every $O \subseteq F_X$

$O \in J$ implies $f[O] \in T$.

<u>Definition 4.2.10</u> (Christoph [9])

The function f is said to be fuzzy closed (fuz-closed) from (F_X, J) into (F_Y, T) iff for every $E \subseteq F_X$

E is J-closed implies $f[E]$ is T-closed.

3. Goodness of Fuzzy Extensions

Now, we expand Lowen's idea of goodness, which was originally defined for properties of fuzzy topological spaces, to include fuzzy properties of functions.

In the following, let P be a classical property of functions and P^f a fuzzy extension of P.

Definition 4.3.1

The fuzzy property P^f is said to be a good fuzzy extension of P iff for every couple of topological spaces (X, τ) and (Y, τ_1) and every function $g: X \longrightarrow Y$, we have,

g has the property P from (X, τ) into (Y, τ_1) iff it has the fuzzy property P^f from $(F_X, \omega(\tau))$ into $(F_Y, \omega(\tau_1))$.

Similarly, we expand the concept of initial goodness.

Definition 4.3.2

The fuzzy property P^f is said to be an initially good fuzzy extension of P iff for every couple of fuzzy topological spaces (F_X, J) and (F_Y, T) and every function $g: X \longrightarrow Y$, we have,

g has the fuzzy property P^f from (F_X, J) into (F_Y, T) implies it has the property P from $(X, \iota(J))$ into $(Y, \iota(T))$.

Proposition 4.3.3

Fuzzy continuity is an initially good fuzzy extension of classical continuity. That is; if a function f is fuzzy continuous from a fts (F_X, J) into a fts (F_Y, T), then f is continuous from $(X, \iota(J))$ into $(Y, \iota(T))$.

Proof

Let f be fuz-continuous from (F_X, J) into (F_Y, T). It is enough to show that the inverse image under f of every member of the subbase $\{U^{-1}(\lambda,1] : U \in T, \lambda \in J - \{1\}\}$ of $\iota(T)$ is in $\iota(J)$.

Thus, let $U \in T$ and $\lambda \in I - \{1\}$. Then $f^{-1}[U] \in J$, by the fuzzy continuity of f, and hence, $(f^{-1}[U])^{-1}(\lambda,1] \in \iota(J)$. But $(f^{-1}[U])^{-1}(\lambda,1] = f^{-1}(U^{-1}(\lambda,1])$, by consequence 4.1.11(a)2. Therefore, $f^{-1}(U^{-1}(\lambda,1]) \in \iota(J)$. □

Proposition 4.3.4 (Weiss)

Fuzzy continuity is a good fuzzy extension of classical continuity. That is, a function f is continuous from a topological space (X, τ) into a topological space (Y, τ_1) iff it is fuzzy continuous from $(F_X, \omega(\tau))$ into $(F_Y, \omega(\tau_1))$.

Proof

First, we have $\iota(\omega(\tau)) = \tau$ and $\iota(\omega(\tau_1)) = \tau_1$, and so, by proposition 4.3.3, the fuzzy continuity of f from $(F_X, \omega(\tau))$ into $(F_Y, \omega(\tau_1))$ implies the continuity of f from (X, τ) into (Y, τ_1).

The converse is the result of consequence 4.1.11(a)2 (see Conrad [10]). In Weiss [53], the proof is a little bit confused, because of the sudden departure from the definition of fuzzy set as a function into I to that into $\mathbf{R}$.

The following diagram illustrates the idea of the simpler proof for this part of the result.

$$\begin{tikzcd}
(X,\ \tau) \arrow[rr, dashed, "f"] \arrow[dr, dashed, "{f^{-1}[A] = A_\circ f \atop \in \omega(\tau)}"'] & & (Y,\ \tau_1) \arrow[dl, dashed, "A \in \omega(\tau_1)"] \\
& (I, 1_r^*) &
\end{tikzcd} \quad . \ \square$$

Proposition 4.3.5

Sequential fuzzy continuity is a good fuzzy extension of classical sequential continuity. That is; a function f is sequentially fuzzy continuous from a topological space $(X,\ \tau)$ into the topological space $(Y,\ \tau_1)$ iff it is fuzzy continuous from $(F_X\ ,\omega(\tau))$ into $(F_Y\ ,\omega(\tau_1))$.

Proof

First, let f be sequentially continuous: $(X,\ \tau) \longrightarrow (Y,\ \tau_1)$. Let $\langle e_n \rangle$ be a fuzzy sequence in X, $\langle x_n \rangle$ = supt $\langle e_n \rangle$ and $e_n \xrightarrow{\omega(\tau)} (x)_\lambda$,for some $x \in X$ and $\lambda \in I - \{0\}$. Then, by proposition 3.2.16, $\langle e_n \rangle$ is predictable by λ and $x_n \xrightarrow{\tau} x$. But then, $\langle f(e_n) \rangle$ is predictable by λ, by consequence 4.1.7(c) and $f(x_n) \xrightarrow{\tau_1} f(x)$, by the sequential continuity of f. Hence, by proposition 3.2.16 $f(e_n) \xrightarrow{\omega(\tau_1)} (f(x))_\lambda$. Therefore, f is sequentially fuzzy continuous from $(F_X\ ,\omega(\tau))$ into $(F_Y\ ,\omega(\tau_1))$.

Now, let f be fuzzy sequentially continuous: $(F_X\ ,\omega(\tau)) \longrightarrow (F_Y\ ,\omega(\tau_1))$. Let $x_n \xrightarrow{\tau} x$. Consider the fuzzy sequence $\langle e_n \rangle$ in X with all its term being crisp points and with $\langle x_n \rangle$ = supt $\langle e_n \rangle$. Then $\langle e_n \rangle$ is predictable by 1 and hence $e_n \xrightarrow{\omega(\tau)} (x)_1$, by proposition 3.2.16. But then, by the sequential fuzzy continuity of f, $f(e_n) \xrightarrow{\omega(\tau_1)} (f(x))_1$. Hence, by proposition 3.2.16, and since $f(e_n) = (f(x_n))_1$, we have, $f(x_n) \xrightarrow{\tau_1} f(x)$. Therefore, f is sequentially continuous from (X, τ) into (Y, τ_1). □

Proposition 4.3.6 (Conrad)

Fuzzy openness of a function is a good fuzzy extension of classical openness. That is, a function f is open from a topological space (X, τ) into a topological space (Y, τ_1) iff it is fuzzy open from $(F_X, \omega(\tau))$ into $(F_Y, \omega(\tau_1))$.

Proof

The proof is obtained in one way by using consequence 4.1.11(a)1 and conversely by using the facts that a subset Z of X is in τ iff $\chi_Z \in \omega(\tau)$ and that $f[\chi_Z] = \chi'_{f[Z]}$. (See Conrad [10]). Hu [26], also proved this proposition using different terminology; namely that of "shapes" of fuzzy sets. □

Hu [26], also implicitly claims that a proof of a similar proposition for closedness of functions is straightforward. We consider this not to be the case. So here we give a proof of the proposition which is in tune with our approach. But first, we prove the following lemma.

Lemma 4.3.7

Let $f : X \longrightarrow Y$, $A \subseteq F_X$ and $\lambda \in I - \{0\}$. Then,

$$(f[A])^{-1}\,[0,\lambda) \;=\; \bigcup_{\delta\in(0,\lambda)} (f(A^{-1}[\lambda-\delta,1]))^{c}.$$

Proof

Let $y \in Y$.

$y \in (f[A])^{-1}\,[\lambda,1]$ iff $f[A](y) \geq \lambda$

iff for every $\delta \in (0,\lambda)$, there is an $x \in f^{-1}(y)$, such that $A(x) \geq \lambda - \delta$

iff for every $\delta \in (0,\lambda)$, there is an $x \in f^{-1}(y)$, such that $x \in A^{-1}[\lambda-\delta,1]$

iff for all $\delta \in (0,\lambda)$, $y \in f(A^{-1}[\lambda-\delta,1])$

iff $y \in \bigcap_{\delta\in(0,\lambda)} f(A^{-1}[\lambda-\delta,1])$.

Thus,

$$(f[A])^{-1}\,[\lambda,1] = \bigcap_{\delta\in(0,\lambda)} f(A^{-1}[\lambda-\delta,1]).$$

Taking the complements, we have;

$$(f[A])^{-1}\,[0,\lambda) = \bigcup_{\delta\in(0,\lambda)} (f(A^{-1}[\lambda-\delta,1]))^{c}. \quad \square$$

Proposition 4.3.8

Fuzzy closedness of a function is a good fuzzy extension of classical closedness. That is, a function f is closed from a topological space (X, τ) into a topological space (Y, τ_1) iff it is fuzzy closed from $(F_X, \omega(\tau))$ into $(F_Y, \omega(\tau_1))$.

Proof

First, let f be closed from (X, τ) into (Y, τ_1) and let A be $\omega(\tau)$-closed. Then $A^{-1}[0,\delta) \in \tau$, for all, $\delta \in I - \{0\}$, by the remark following definition 2.2.6. But then for every $\delta \in I - \{0\}$, we have $A^{-1}[\delta,1]$ is τ-closed, and hence, by the closedness of f, $f(A^{-1}[\delta,1])$ is τ_1-closed. Thus, $(f(A^{-1}[\delta,1]))^c \in \tau_1$, for every $\delta \in I - \{0\}$...... (*)

Now, let $\lambda \in I - \{0\}$. By lemma 4.3.7, we have;

$$(f[A])^{-1}[0,\lambda) = \bigcup_{\delta\in(0,\lambda)} (f(A^{-1}[\lambda-\delta,1]))^c$$

$$\in \tau_1 \quad , \text{ by } (*).$$

Therefore, $f[A]$ is an upper semicontinuous function in (X, τ_1) and hence $\omega(\tau_1)$-closed. Thus f is fuz-closed from $(F_X, \omega(\tau))$ into $(F_Y, \omega(\tau_1))$.

Now, let f be fuz-closed from $(F_X, \omega(\tau))$ into $(F_Y, \omega(\tau_1))$ and let Z be τ-closed. Then χ_Z is $\omega(\tau)$-closed, and hence, by the fuzzy closedness of f, $f[\chi_Z]$ is $\omega(\tau_1)$-closed. But, by consequence 4.1.6(b)1, $f[\chi_Z] = \chi'_{f[Z]}$. Hence $f[Z]$ is τ_1-closed. Thus f is closed from (X, τ) into (Y, τ_1). □

Proposition 4.3.9

Let (F_X, J) and (F_Y, T) be fts's and $f: X \longrightarrow Y$.

The openness (closedness) of the function f from $(X, \iota(J))$ into $(Y, \iota(T))$ does not imply its fuzzy openness (respectively closedness) from (F_X, J) into (F_Y, T).

Proof

Let $X = \{x_1, x_2, x_3\}$, $Z = \{x_1, x_2\}$ and consider the topology $\tau = \{X, \Phi, Z\}$ on X. Let $J = \omega_{\frac{1}{8}}(\tau)$ and $T = \omega_{\frac{1}{4}}(\tau)$ be respectively the $\frac{1}{8}$-cut and $\frac{1}{4}$-cut fuzzy topologies on F_X.

By proposition 2.2.19(a), $\iota(J) = \Delta = \iota(T)$, and hence the identity map $i_d : X \longrightarrow X$ is open from $(X, \iota(J))$ into $(X, \iota(T))$.

On the other hand, let $A \subseteq F_X$ be defined by, $A(x_3) = \frac{1}{2}$ and $A(x_1) = \frac{1}{4} = A(x_2)$. Then $A^{-1}(\frac{1}{8}, 1] = X \in \tau$, and hence $A \in \omega_{\frac{1}{8}}(\tau) = J$, but $A^{-1}(\frac{1}{4}, 1] = \{x_3\} \notin \tau$, and so $A \notin \omega_{\frac{1}{4}}(\tau) = T$. Therefore, $i_d[A] \notin T$, and hence i_d is not fuz-open from (F_X, J) into (F_Y, T). The map i_d is not fuz-closed either, since the fuzzy set A^c is J-closed but not T-closed. □

We have been unable to prove the converse of the preceding proposition. However, we make the following conjecture.

<u>Conjecture 4.3.10</u>

Fuzzy openness (closedness) of a function is not initially good.

PART V

BASIC FUZZY TOPOLOGICAL CONSTRUCTIONS

This part of the thesis deals with the fuzzification of the classical concepts of induction and coinduction. Many authors attempted to extend to fuzzy topology some or other special instances of these general constructions. Great attention has been given to the particular cases of subspace and product. Goguen [24] defined a product of L-topological spaces. He was followed by Wong [55] who gave the first definitions of product and quotient fuzzy topologies. Lowen [34] defined fuzzy induction and coinduction and applied the former to formulate a definition for the product fuzzy topology. He also studied the goodness of these fuzzy constructions. The notion of fuzzy subspace was treated in many papers including Foster [15] and Pu and Liu [43] (who also studied the fuzzy product [44]), Sarkar [45], Ghanim, et al [22], and Zhou [60]. Special cases of fuzzy coinduction were considered separately by Ghanim et al [22] who were the first to define the fuzzy topological sum and Christoph [9] who adapted Wong's definition of quotient fuzzy topology.

We adopt Lowen's concepts of fuzzy induction and fuzzy coinduction. We apply the former to arrive at fuzzy subspace and the latter to formulate the definitions of the quotient and the disjoint sum fuzzy topologies. We follow each of these definitions by its consequence; an explicit description of the respective fuzzy topology. This is succeeded by a discussion of the relevant alternative descriptions appearing in the literature and their relationships to ours. This approach is observed in both sections (1) and (2).

In section (1), we make a correction to a useful result given by Pu and Liu [44] and investigate a fuzzy version of the box topology showing its relationship to the fuzzy product.

In section (2), we introduce the disjoint sum sets and use them to describe the disjoint sum fuzzy topology. Proving a "disjoint sum" version of consequence 4.1.8(a), we arrive at a fuzzy version of the classical assertion that the quotient of a sum is the sum of the quotients.

In section (3), we investigate the goodness and initial goodness of the concepts introduced in the preceding two sections. All the proofs of goodness and initial goodness are based on our previous results derived from Pu & Liu's Q-theory. They are thus fuzzy topological in nature and considerably simpler than those of Lowen.

1. Induced Fuzzy Topological Spaces

In the following let X be a set and Ω an indexing set. For every $\alpha \in \Omega$, let Y_α be a set, f_α a function from X into Y_α and J_α a fuzzy topology on F_{Y_α}.

For any $\alpha \in \Omega$, we have $f_\alpha^{-1}[F_{Y_\alpha}] = F_X$. Thus the union of the family $\{f_\alpha^{-1}[U_\alpha] : U_\alpha \in J_\alpha, \alpha \in \Omega\}$ is equal to F_X. It follows that this family satisfies the axiom for a fuzzy subbase (proposition 2.1.11) and hence is a subbase for some fuzzy topology on F_X.

Definition 5.1.1. (Lowen [34])

A fuzzy topology on F_X is said to be the induced fuzzy topology for the family of functions $\{f_\alpha : \alpha \in \Omega\}$ and the family of fuzzy topological spaces $\{(F_{Y_\alpha}, J_\alpha) : \alpha \in \Omega\}$, and denoted by J_i iff it has as a subbase the family $\mathbf{B}_i = \{f_\alpha^{-1}[U_\alpha] : U_\alpha \in J_\alpha, \alpha \in \Omega\}$.

It follows that J_i is the smallest fuzzy topology on F_X making every f_α $(\alpha \in \Omega)$ fuzzy continuous. It is also called the weak fuzzy topology on F_X.

Using this notation, we have the following two propositions.

Proposition 5.1.2 (Adaptation of theorem 3.1(iii) in Wong [55])

If (F_Z, T) is a fts and g is a surjective function from Z to X. Then,

g is fuzzy continuous from (F_Z, T) into (F_X, J_i) iff $f_\alpha \circ g$ is fuzzy continuous from (F_Z, T) into (F_{Y_α}, J_α) for every $\alpha \in \Omega$.

Proof

Considering consequences 4.1.8(a)2 and 4.1.8(b)2 about the inverse images of union and intersection and the remark following definition 4.1.3 about the inverse image under a composition function, the proof follows the classical version. □

Proposition 5.1.3 (Adaptation of theorem 2.4 in Pu and Liu [44] to induction and fuzzy sequences)

Let $\langle e_n \rangle$ be a fuzzy sequence in X and $e \in F_X$.

Then,

$$e_n \xrightarrow{J_i} e \quad \text{iff} \quad f_\alpha(e_n) \xrightarrow{J_\alpha} f_\alpha(e), \text{ for every } \alpha \in \Omega.$$

Proof

First, let $e_n \xrightarrow{J_i} e$. By definition of J_i, for every $\alpha \in \Omega$, f_α is fuz-continuous, and hence, by consequence 4.2.5, fuzzy sequentially continuous from (F_X, J_i) into (F_{Y_α}, J_α). Therefore, $f_\alpha(e_n) \xrightarrow{J_\alpha} f_\alpha(e)$.

Conversely, let $f_\alpha(e_n) \xrightarrow{J_\alpha} f_\alpha(e)$, for every $\alpha \in \Omega$. Let $O \in J_i$ and eqO. Then, by proposition 2.1.8 and definition of J_i, there is a $K \in \mathbb{N}$ and a family $\{O_{\alpha_k} : 1 \leq k \leq K\}$, $O_{\alpha_k} \in J_{\alpha_k}$, such that $eq(\bigcap_{k=1}^{K} f_{\alpha_k}^{-1}[O_{\alpha_k}]) \subseteq O$. But then, for every $k \in \{1, 2, \ldots, K\}$, $eq(f_{\alpha_k}^{-1}[O_{\alpha_k}])$, and hence, $f_{\alpha_k}(e)qO_{\alpha_k}$, by consequence 4.1.7(b)2. Therefore, by the convergence of $\langle f_{\alpha_k}(e_n) \rangle$ in $(F_{Y_{\alpha_k}}, J_{\alpha_k})$, for every $k \in \{1, 2, \ldots, K\}$, there is an $N_k \in \mathbb{N}$, such that, $\langle f_{\alpha_k}(e_n) \rangle q(N_k) O_{\alpha_k}$. Hence, by consequence 4.1.7(b)2, for every $k \in \{1, 2, \ldots, K\}$, $\langle e_n \rangle q(N_k) f_{\alpha_k}^{-1}[O_{\alpha_k}]$. Let $N = \max\{N_1, N_2, \ldots N_K\}$. Then $\langle e_n \rangle q(N) f_{\alpha_k}^{-1}[O_{\alpha_k}]$, $1 \leq k \leq K$. By consequence 1.3.7(b) $\langle e_n \rangle q(N) (\bigcap_{k=1}^{K} f_{\alpha_k}^{-1}[O_{\alpha_k}])$. Hence, $\langle e_n \rangle q(N) O$, by consequence 1.3.5(a).

Thus $e_n \xrightarrow{J_i} e$. □

In the following, let $X \subseteq Y$ and $D \subseteq F_Y$.

Definition 5.1.4

A subset of F_X is said to be the restriction of D to X and denoted by $\underset{X}{\text{res}}\, D$ (or simply res D if no confusion arises) iff for every $x \in X$, res D (x) = D(x).

The classical notation for restriction will be used for arbitrary functions while our notation is reserved for fuzzy sets.

In the following, let i_n be the inclusion function from X into Y.

Consequence 5.1.5

(a) $i_n^{-1}[D]$ = res D.

(b) $(\text{res } D)^c = \text{res } D^c$.

(c) If $D \subseteq D_1$, then res $D \subseteq$ res D_1.

(d) The restriction of a fuzzy point in Y with support in X is a fuzzy point in X. Moreover if $e \in F_Y$, such that supt $e \in X$, then, for the fuzzy point d = res e, we have, dq(res D) iff eqD (and $d \in$ res D iff $e \in D$).

Proof

(a) By definition 4.1.3, $i_n^{-1}[D] \subseteq F_X$ and for every $x \in X$, $i_n^{-1}[D](x) = D(i_n(x)) = D(x)$. Therefore, $i_n^{-1}[D]$ = res D.

(b) $(\text{res } D)^c = (i_n^{-1}[D])^c$, by (a)

$= i_n^{-1}[D^c]$, by consequence 4.1.9(b)

$= \text{res } D^c$.

(c) res $D = i_n^{-1}[D] \subseteq i_n^{-1}[D_1]$, by consequence 4.1.7(a)2

$= \text{res } D_1$.

(d) This is true, since e and d share the same value and support and on the other hand D and res D have identical values at all points of X. □

Definition 5.1.6

Let J be a fuzzy topology on F_Y. A fuzzy topology on F_X is said to be the X-relative fuzzy topology of (F_Y, J) and denoted by J' iff it is the induced fuzzy topology for the singleton family of functions $\{i_n : X \longrightarrow Y\}$ and the singleton family of fts's $\{(F_Y, J)\}$.

The fts (F_X, J') is called a (fuzzy) subspace of (F_Y, J).

Consequence 5.1.7

$J' = \{\text{res } U : U \in J\}$.

Proof

By definitions 5.1.1. and 5.1.6, we have, $J' = \{U : i_n^{-1}[U] \in J\}$. But $i_n^{-1}(U) = \text{res } U$, by consequence 5.1.5(a). Hence the result follows. □

The description of the fuzzy subspace given by the previous consequence coincides with the definition proposed by Pu and Liu [43]. The fact that we have arrived at this definition, starting from the general concept of fuzzy induction reinforces its reasonableness as the proper extension of the classical notion of a subspace. In section (3) of this part the validity of this definition will be even more strengthened by a goodness proposition.

Other definitions of a fuzzy subspace have appeared in the literature. Zhou [60] defines a relative topology J_A of a fts (F_Y, J) on a fuzzy set A in Y by $J_A = \{A \cap U : U \in J\}$. He then claims that J_A is a fuzzy topology. This is, of course, not true since J_A does not contain the full set in Y. Such a claim also was made by Sarkar [45]. The same construction was called by Foster [15], "the induced fuzzy topology on A", where J here is assumed to be Lowen. He comments that "in general" J_A does not contain the constant sets. Actually J_A <u>never</u> contains the constant sets or even the full set F_X, except in the trivial case when $J_A = J$ (i.e. when $A = F_X$).

Another extension of the concept of subspace appears in Ghanim, et al [22]. Their definition was made in such a way as to help in fuzzifying the classical topological sum. We will discuss both definitions in section (2).

Another special case of fuzzy induction is the product fuzzy topology.

<u>Definition 5.1.8</u> (Lowen [34])

Let Ω be an indexing set, $C = \{(F_{Y_\alpha}, J_\alpha) : \alpha \in \Omega\}$ be a family of fuzzy topological spaces, and $X = \prod_{\alpha\in\Omega} Y_\alpha$.

A fuzzy topology on F_X is said to be the product of the fuzzy topological spaces in the family C and denoted by $\prod_{\alpha\in\Omega} J_\alpha$ (or simply $\prod_\Omega$) iff it is the induced fts for the family C and the family of projections $\{P_\alpha : X \longrightarrow Y_\alpha : \alpha \in \Omega\}$.

It follows that $\prod_{\Omega}$ is the smallest fuzzy topology on F_X making every projection P_α fuzzy continuous. The pair $(F_X, \prod_{\Omega})$ is called the product fts of the fuzzy topological spaces in **C**.

It also follows from this definition that the family $B_\pi = \{\bigcap_{\alpha\in\Lambda} P_\alpha^{-1}[U_\alpha] : U_\alpha \in J_\alpha, \Lambda$ is a finite subset of $\Omega\}$ is a base for $\prod_{\Omega}$. This is precisely the definition of the product fuzzy topology given by Wong [56].

Until the end of this section we use the notation of definition 5.1.8.

Foster [15] introduced the idea of a product of fuzzy sets. The following is an adaptation of Foster's definition.

Definition 5.1.9

Let $D_\alpha \subseteq F_{Y_\alpha}$, $\alpha \in \Omega$.

A fuzzy set in X is said to be the product set of the fuzzy sets D_α ($\alpha \in \Omega$) and denoted by $\prod_{\Omega} D_\alpha$ iff for every $x \in X$, we have

$$(\prod_{\Omega} D_\alpha)(x) = \inf\{D_\alpha(P_\alpha(x)) : \alpha \in \Omega\}.$$

It follows that $\prod_{\Omega} D_\alpha = \bigcap_{\alpha\in\Omega} P_\alpha^{-1}[D_\alpha]$.

Using definition 5.1.9, we arrive at an alternative description of the product fts.

Consequence 5.1.10

Let $B'_\pi = \{\prod_{\alpha\in\Omega} U_\alpha : U_\alpha \in J_\alpha$ and $U_\alpha = F_{Y_\alpha}$ for all but a finite number of $\alpha\}$. Then, $B'_\pi = B_\pi$.

Proof

First, let $B \in B'_\pi$. Then $B = \prod_{\alpha\in\Omega} U_\alpha$, for some family $\{U_\alpha \in J_\alpha : \alpha \in \Omega\}$ with $U_\alpha = F_{Y_\alpha}$, for all but a finite number α. Let $\Lambda \subseteq \Omega$, such that $U_\alpha \neq F_{Y_\alpha}$, $\alpha \in \Lambda$. Then Λ is finite and for every $x \in X$ and $\alpha \in \Omega - \Lambda$, we have,

$U_\alpha(P_\alpha(x)) = F_{Y_\alpha}(P_\alpha(x)) = 1$. Therefore

$$\begin{aligned}(\prod_{\alpha\in\Omega} U_\alpha)(x) &= (\bigcap_{\alpha\in\Omega} P_\alpha^{-1}[U_\alpha])(x) \\ &= \inf\{U_\alpha(P_\alpha(x)) : \alpha \in \Omega\} \\ &= \min\{U_\alpha(P_\alpha(x) : \alpha \in \Lambda\} = (\bigcap_{\alpha\in\Lambda} P_\alpha^{-1}[U_\alpha])(x).\end{aligned}$$

Therefore $\prod_{\alpha\in\Omega} U_\alpha = \bigcap_{\alpha\in\Lambda} P_\alpha^{-1}[D_\alpha] \in B_\pi$.

The same argument can be inverted to show that every member of B_π is actually in B'_π. Therefore, $B_\pi = B'_\pi$. □

The following results for the product fuzzy topology are due to Pu and Liu [44]. The first result contained a minor inaccuracy, and since also no proof was provided, we thought it useful to prove it. We use the following lemma.

Lemma 5.1.11

Let Z be a set, ψ a function from Z into I and $b \in I$.

Then

$$\sup_{z\in Z} \min\{\psi(z), b\} = \min\{\sup_{z\in Z} \psi(z), b\}.$$

Proof

Denote $f_1 = \sup_{z\in Z} \min\{\psi(z), b\}$ and $f_2 = \min\{\sup_{z\in Z} \psi(z), b\}$.

Either $\psi(z) \leq b$ for all $z \in Z$ or there is a subset Z' of Z, such that $\psi(z') > b$, for all $z' \in Z'$.

First, if $\psi(z) \leq b$ for all $z \in Z$, then $\sup_{z \in Z} \psi(z) \leq b$. Hence $f_1 = \sup_{z \in Z} \psi(z) = f_2$.

On the other hand if there is a $Z' \subseteq Z$, such that $\psi(z') > b$ for all $z' \in Z'$, then $\sup_{z \in Z} \psi(z') > b$, and hence $f_2 = b$.

Now,

$$\min \{\psi(z), b\} = \begin{cases} b & \text{if } z \in Z' \\ \psi(z) \leq b & \text{if } z \notin Z' \end{cases}.$$ But then,

$f_1 = \sup_{z \in Z} \min \{\psi(z), b\} = b = f_2$.

Therefore, $f_1 = f_2$. □

Consequence 5.1.12 (Pu and Liu)

For every $B \in \mathbf{B}_\pi$ (defined on p.147) and $\alpha \in \Omega$, $P_\alpha[B]$ is either a constant set or an intersection of a constant set and a member of J_α.

Symbolically, if $B = \bigcap_{n=1}^{N} P_{\alpha_n}^{-1}[0_{\alpha_n}]$. Then,

$$P_\alpha[B] = \begin{cases} \eta_\lambda & \text{, for some } \lambda \in I, \text{ if } \alpha \notin \{\alpha_1, \alpha_2, .., \alpha_N\} \\ \eta_\delta \cap 0_\alpha & \text{, for some } \delta \in I, \text{ if } \alpha \in \{\alpha_1, \alpha_2, .., \alpha_N\}. \end{cases}$$

Proof

First, let $\alpha \notin \{\alpha_1, \alpha_2, .., \alpha_N\}$. Let $y \in Y_\alpha$.

$$P_\alpha[B](y) = P_\alpha[\bigcap_{n=1}^{N} P_{\alpha_n}^{-1}[0_{\alpha_n}]](y) = \sup_{x \in P_\alpha^{-1}(y)} \min_{1 \leq n \leq N} 0_{\alpha_n}(P_{\alpha_n}(x)).$$

Now, for every family $\{y_{\alpha_n} \in Y_{\alpha_n} : 1 \leq n \leq N\}$, there is an $x \in P_\alpha^{-1}(y)$, such that $P_{\alpha_n}(x) = y_{\alpha_n}$, $1 \leq n \leq N$, (e.g. any element of X having y as its α-th coordinate and y_{α_n} as its α_n-th coordinate, $1 \leq n \leq N$).

Therefore, $P_\alpha[B](y) = \sup_{\prod_{n=1}^{N} y_{\alpha_n} \in \prod_{n=1}^{N} Y_{\alpha_n}} \min_{1 \le n \le N} O_{\alpha_n}(y_{\alpha_n})$. Hence the value of $P_\alpha[B](y)$ does not depend on y, and so it is equal to some constant say λ, for all $y \in Y$. Thus $P_\alpha[B] = \eta_\lambda$.

On the other hand, if $\alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$, then $\alpha = \alpha_k$ for some $k \in \{1, 2, \ldots, N\}$. Let $\Lambda = \{1, 2, \ldots, N\} - \{k\}$ and $y \in Y_\alpha$.

Now, for every family $\{y_{\alpha_n} \in Y_{\alpha_n} : n \in \Lambda\}$, there is an $x \in P_\alpha^{-1}(y)$ such that $P_{\alpha_n}(x) = y_{\alpha_n}$, $n \in \Lambda$ (any element of X having y as its α_k-th coordinate and y_{α_n} as its α_n-th coordinate, $n \in \Lambda$).

Therefore, $P_\alpha[B](y) = \sup_{x \in P_\alpha^{-1}(y)} \min_{1 \le n \le N} O_{\alpha_n}(P_{\alpha_n}(x))$

$$= \sup_{x \in P_\alpha^{-1}(y)} \min \{\min_{n \in \Lambda} \{O_{\alpha_n}(P_{\alpha_n}(x)), O_\alpha(y)\}\}$$

$$= \min \{ \sup_{\prod_{n\in\Lambda} y_{\alpha_n} \in \prod_{n\in\Lambda} Y_{\alpha_n}} \min_{n \in \Lambda}(O_{\alpha_n}(y_{\alpha_n}), O_\alpha(y)\}$$

by lemma 5.1.11.

Since the value $\sup_{\prod_{n\in\Lambda} y_{\alpha_n} \in \prod_{n\in\Lambda} Y_{\alpha_n}} \min_{n\in\Lambda} O_{\alpha_n}(y_{\alpha_n})$ does not depend on y, then it is equal to a constant, say δ, for some $\delta \in I$. Hence $P_\alpha[B](y) = \min \{\delta, O_\alpha(y)\} = (\eta_\delta \cap O_\alpha)(y)$. Therefore, $P_\alpha[B] = \eta_\delta \cap O_\alpha$. □

<u>Proposition 5.1.13</u> (Pu & Liu)

For every $\alpha \in \Omega$, the projection P_α is fuzzy open from $(F_X, \prod_\Omega)$ into (F_{Y_α}, J_α), whenever (F_{Y_α}, J_α) is Lowen.

Proof

This follows from consequence 5.1.12. □

Now we introduce the box fuzzy topology which, although related to the product fuzzy topology, is not a special case of fuzzy induction.

Remark 5.1.14

Let $\mathbf{C} = \{(F_{Y_\alpha}, J_\alpha) : \alpha \in \Omega\}$ and $X = \prod_{\alpha\in\Omega} Y_\alpha$. Define the family $\mathbf{B}_b$ by,

$\mathbf{B}_b = \{ \prod_{\alpha\in\Omega} U_\alpha : U_\alpha \in J_\alpha\}$. Then $\mathbf{B}_b$ is a base for some fuzzy topology on F_X.

Proof

Considering proposition 2.1.10, it is enough to show that $\mathbf{B}_b$ satisfies the two conditions of this proposition.

Condition 1. First, $F_X = \bigcap_{\alpha\in\Omega} P_\alpha^{-1}[F_{Y_\alpha}] = \prod_{\alpha\in\Omega} F_{Y_\alpha} \in \mathbf{B}_\pi$, and hence every fuzzy point in X is quasi-coincident to a member of $\mathbf{B}_\pi$.

Condition 2. Now, let $B_1, B_2 \in \mathbf{B}_b$. Then $B_1 = \prod_{\alpha\in\Omega} D_\alpha$ and $B_2 = \prod_{\alpha\in\Omega} G_\alpha$, where $D_\alpha, G_\alpha \in J_\alpha, \alpha \in \Omega$. Let $U_\alpha = D_\alpha \bigcap G_\alpha$, then $U_\alpha \in J_\alpha$, $\alpha \in \Omega$. Condition (2) will be surely satisfied if the intersection of B_1 and B_2 is in $\mathbf{B}_b$, which we now show.

$$B_1 \bigcap B_2 = (\prod_{\alpha\in\Omega} D_\alpha) \bigcap (\prod_{\alpha\in\Omega} G_\alpha) = (\bigcap_{\alpha\in\Omega} P_\alpha^{-1}[D_\alpha]) \bigcap (\bigcap_{\alpha\in\Omega} P_\alpha^{-1}[G_\alpha])$$

$$= \bigcap_{\alpha\in\Omega} (P_\alpha^{-1}[G_\alpha] \bigcap P_\alpha^{-1}[D_\alpha]),$$ by the associativity property (consequence 1.1.18(c))

$$= \bigcap_{\alpha\in\Omega} P_\alpha^{-1}[G_\alpha \cap D_\alpha], \text{ by consequence 4.1.8(b)2}$$

$$= \bigcap_{\alpha\in\Omega} P_\alpha^{-1}[U_\alpha] = \prod_{\alpha\in\Omega} U_\alpha \in \mathbf{B}_b. \ \square$$

Definition 5.1.15

The fuzzy topology on F_X having $\mathbf{B}_b$ as its base is called the box fuzzy topology and denoted by J_b.

Unlike the situation for the product fuzzy topology, where the fuzzy continuity of the projections is assumed in the definition, the definition of the box fuzzy topology does not contain an explicit reference to fuzzy continuity. The following consequence shows that the projections are fuzzy continuous also in this case.

Consequence 5.1.16

For every $\alpha \in \Omega$, P_α is fuzzy continuous from (F_X, J_b) into (F_{Y_α}, J_α).

Proof

Let $\alpha \in \Omega$ and $U \in J_\alpha$. For every $\beta \in \Omega$, define D_β, by

$D_\beta = \begin{cases} U & \beta = \alpha \\ F_{Y_\beta} & \beta \neq \alpha \end{cases}$. Then for every $\beta \in \Omega$, $D_\beta \in J_\beta$ and for every $\beta \in \Omega - \{\alpha\}$, $P_\beta^{-1}[D_\beta] = P_\beta^{-1}[F_{Y_\beta}] = F_X$ (consequence 4.1.6(d)2).

Hence, $P_\alpha^{-1}[U] = \bigcap_{\beta\in\Omega} P_\beta^{-1}[D_\beta] = \prod_{\beta\in\Omega} D_\beta \in \mathbf{B}_b \subseteq J_b$.

Therefore, P_α is fuz-continuous from (F_X, J_b) into (F_{Y_α}, J_α). $\square$

Implied in consequence 5.1.16 is the fact that $\prod_{\Omega} \subseteq J_b$, since J_b is a fuzzy topology on F_X that makes the projections fuzzy continuous while $\prod_{\Omega}$ was defined to be the induced fuzzy topology for this family of projections, that is the smallest fuzzy topology on F_X making all of them fuzzy continuous.

The idea of the box fuzzy topology was a consequence of the notion of a fuzzy product set. The latter concept is trouble-free when we consider the fuzzy product of a finite family of fuzzy sets or nonfinite family that has only a finite number of nonfull fuzzy sets, where the situation resembles that for classical product sets. The radical differences from the classical case appear when considering nonfinite fuzzy products of non-full fuzzy sets.

First for a family $\{D_\alpha \subseteq F_{Y_\alpha} : \alpha \in \Omega\}$, the support of $\prod_{\alpha\in\Omega} D_\alpha$ (or equivalently supt $(\bigcap_{\alpha\in\Omega} P_\alpha^{-1}[D_\alpha])$) need not equal $\bigcap_{\alpha\in\Omega}$ supt $P_\alpha^{-1}[D_\alpha]$ (see consequence 1.1.15(b)). Thus, it is not surprising, as indicated by Wong [56], that the product of a collection $\{e_\alpha \in F_\alpha : \alpha \in \Omega\}$ of fuzzy points need not be a fuzzy point. For instance, let $x \in \prod_{n=1}^{\infty} Y_n = X$ and define $e_n \in F_{Y_n}$,by $e_n = (P_n(x))_{\frac{1}{n}}$, $n \in \mathbf{N}$. Then $(\prod_{n=1}^{\infty} e_n)(x) = \inf \{e_n(P_n(x)) : n \in \mathbf{N}\} = \inf \{\frac{1}{n} : n \in \mathbf{N}\} = 0$. Since for any $z \in X - \{x\}$, there is a $k \in \mathbf{N}$, such that $P_k(z) \neq P_k(x)$, then $e_k(P_k(z)) = 0$ and hence $(\prod_{n=1}^{\infty} e_n)(z) = 0$. Therefore, $\prod_{n=1}^{\infty} e_n = \phi_X$.

Another important difference is that we can have two distinct families $C_1 = \{D_\alpha \subseteq F_{Y_\alpha} : \alpha \in \Omega\}$ and $C_2 = \{B_\alpha \subseteq F_{Y_\alpha} : \alpha \in \Omega\}$ such that $\prod_{\alpha\in\Omega} D_\alpha = \prod_{\alpha\in\Omega} B_\alpha$; something that does not happen in the classical case. As an illustration, let $\Omega = \mathbf{N}$, and $Y_n = Y$, $n \in \mathbf{N}$

and define $C_1 = \{D_n : n \in \mathbb{N}\}$ and $C_2 = \{B_n : n \in \mathbb{N}\}$ by, $D_n = \eta'_{\frac{1}{4}} \subseteq F_Y$ and $B_n = \eta'_{\frac{1}{4}+\frac{1}{n}} \subseteq F_Y$. Then, for every $x \in X$, we have;

$$(\prod_{n=1}^{\infty} B_n)(x) = \tfrac{1}{4} = (\prod_{n=1}^{\infty} D_n)(x), \text{ and so, } \prod_{n=1}^{\infty} B_n = \prod_{n=1}^{\infty} D_n.$$

2. Coinduced Fuzzy Topological Spaces

In the following, let Y be a set and Ω an indexing set and for every $\alpha \in \Omega$, let (F_{X_α}, J_α) be a fts and f_α a function from X_α into Y.

Remark 5.2.1

Let $J_C = \{D \subseteq F_Y : f_\alpha^{-1}[D] \in J_\alpha$, for every $\alpha \in \Omega\}$. Then J_C is a fuzzy topology on F_Y.

Proof (cf classical case)

First, for every $\alpha \in \Omega$, $f_\alpha^{-1}[\Phi_Y] = \Phi_{X_\alpha} \in J_\alpha$ and $f_\alpha^{-1}[F_Y] = F_{X_\alpha} \in J_\alpha$, by consequence 4.1.6(d)2. Hence, $\Phi_Y, F_Y \in J_C$.

Now, let D_1, $D_2 \in J_C$. Then, for every $\alpha \in \Omega$, $f_\alpha^{-1}[D_1]$, $f_\alpha^{-1}[D_2] \in J_\alpha$, and hence $f_\alpha^{-1}[D_1] \bigcap f_\alpha^{-1}[D_2] \in J_\alpha$. By consequence 4.1.8(b)2, $f_\alpha^{-1}[D_1 \bigcap D_2] = f_\alpha^{-1}[D_1] \bigcap f_\alpha^{-1}[D_2] \in J_\alpha$, $\alpha \in \Omega$. Therefore, $D_1 \bigcap D_2 \in J_C$.

Lastly, let $C \subseteq J_C$. Then, for every $D \in C$ and every $\alpha \in \Omega$, $f_\alpha^{-1}[D] \in J_\alpha$, and hence $\bigcup_{D \in C} f_\alpha^{-1}[D] \in J_\alpha$. By consequence 4.1.8(a)2, $f_\alpha^{-1}[\bigcup_{D \in C} D] = \bigcup_{D \in C} f_\alpha^{-1}[D] \in J_\alpha$, $\alpha \in \Omega$. Therefore, $\bigcup_{D \in C} D \in J_C$.

Thus, J_C is a fuzzy topology on F_Y. $\square$

Definition 5.2.2 (Lowen [34])

The fuzzy topology of remark 5.2.1 is called the coinduced fuzzy topology on F_Y for the family of functions $\{f_\alpha : X_\alpha \to Y, \alpha \in \Omega\}$ and the family of fuzzy topological spaces $\{(F_{X_\alpha}, J_\alpha) : \alpha \in \Omega\}$. It is the largest fuzzy topology on F_Y making every f_α ,($\alpha \in \Omega$) fuzzy continuous. It is also called the strong fuzzy topology on F_Y.

An important special case of coinduced fuzzy topological spaces is the quotient fts.

Definition 5.2.3

Let $f : X \longrightarrow Y$ be a surjection and J a fuzzy topology on F_X . A fuzzy topology is said to be the quotient fuzzy topology of (F_X, J) under f and denoted by J_f iff it is the coinduced fuzzy topology on F_Y for the singleton family of functions $\{f\}$ and the singleton family of fuzzy topological spaces $\{(F_X, J)\}$.

It follows that if J' is a fuzzy topology on F_Y that makes f fuzzy continuous (from (F_X, J)), then $J' \subseteq J_f$.

The fts (F_Y, J_f) is called the quotient under f of the fts (F_X, J) and the function f is referred to as the fuzzy quotient map.

Consequence 5.2.4

Let J_f be the quotient fuzzy topology on F_Y under the function f of the fts (F_X, J). Then J_f is the largest fuzzy topology on F_Y making f fuzzy continuous. More precisely,

$$J_f = \{U \subseteq F_Y : f^{-1}[U] \in J\}.$$

Proof

It follows from definition 5.2.2 and 5.2.3. $\square$

The description of the quotient fts given by the previous consequence coincides with the definition introduced by Christoph [9].

The following four propositions are due to Christoph. The detailed proofs which can be seen in Christoph [9] follow the corresponding classical versions. Here, we only mention which consequences of definitions 4.1.1 and 4.1.3 are needed for these proofs. In the next four propositions, let $f : X \longrightarrow Y$ be a surjection, J a fuzzy topology on F_X and J_f the quotient fuzzy topology of (F_X, J) under f.

Proposition 5.2.5

If $g : Y \longrightarrow Z$ and T is a fuzzy topology on F_Z. Then,

g is fuzzy continuous from (F_Y, J_f) into (F_Z, T) iff $g_\circ f$ is fuzzy continuous from (F_X, J) into (F_Z, T).

Proof

It follows from consequence 4.2.4(a) and the remark following definition 4.1.3 concerning a composition function. □

In the following two propositions, let J' be a fuzzy topology on F_Y.

Proposition 5.2.6

Let f be fuzzy continuous from (F_X, J) into (F_Y, J'). Then, $J' = J_f$ iff for every fts (F_Z, T) and every function,

$g : Y \longrightarrow Z$, the fuzzy continuity of $g_\circ f$ from (F_X, J) into (F_Z, T) implies the fuzzy continuity of g from (F_Y, J') into (F_Z, T).

<u>Proof</u>

Considering proposition 5.2.5, it is sufficient to prove the converse. Since $J' \subseteq J_f$ (by remark following definition 5.2.3), we only need to show that $J_f \subseteq J'$ or equivalently that the identity map $i_d : Y \longrightarrow Y$ is fuz-continuous from (F_Y, J') into (F_Y, J_f). This follows from the fact that $i_d{}_\circ f = f$ is fuz-continuous from (F_X, J) onto (F_Y, J_f). □

<u>Proposition 5.2.7</u>

Let f be a surjective function from X onto Y. If f is fuzzy continuous and either fuzzy open or fuzzy closed from (F_X, J) into (F_Y, J'), then $J' = J_f$.

<u>Proof</u>

We only need to show that $J_f \subseteq J'$. That is proved using consequence 4.1.10(a)3 (in addition to consequence 4.1.9(b) for the case of fuz-closedness of f). □

<u>Proposition 5.2.8</u>

Let (F_Z, T) be a fts and g a function from Y into Z. Then,

(F_Z, T) is the quotient of (F_Y, J_f) under g iff it is the quotient of (F_X, J) under $g_\circ f$.

Proof

It is proved using the remark following definition 4.1.3 concerning the inverse image of a fuzzy set under a composition function. □

Another important special case of the coinduced fts is the fuzzy disjoint sum. We first introduce some useful tools.

Definition 5.2.9

Let $X \subseteq Y$ and $A \subseteq F_X$.

A subset of F_Y is said to be the (proper) extension set of A to Y and denoted by $\underset{Y}{\text{ext}}\, A$ (or simply ext A, if no confusion arises) iff

$$\text{ext } A\,(x) = \begin{cases} A(x), & x \in X \\ 0, & x \in Y-X \end{cases}.$$

It follows that an extension of a fuzzy point d in X is a fuzzy point in Y and that if e = ext d, then dqA iff eq(ext A) (and $d \in A$ iff $e \in \text{ext } A$). It also follows that if $B \subseteq A$, then $\text{ext } B \subseteq \text{ext } A$.

Definition 5.2.10

Let $C = \{X_\alpha : \alpha \in \Omega\}$ be a family of mutually disjoint sets, $Y = \bigcup_{\alpha\in\Omega} X_\alpha$ and $E_\alpha \subseteq F_{X_\alpha}$, $\alpha \in \Omega$.

A subset of F_Y is said to be the <u>(disjoint) sum</u> of the fuzzy sets E_α and denoted by $\sum_{\alpha\in\Omega} E_\alpha$ iff

$$\sum_{\alpha\in\Omega} E_\alpha = \bigcup_{\alpha\in\Omega} \underset{Y}{\text{ext}}\, E_\alpha .$$

Consequence 5.2.11

Let C and Y be as in definition 5.2.10.

(a) For every $D \subseteq F_Y$, we have

$$D = \sum_{\alpha\in\Omega} \operatorname{res}_{X_\alpha} D .$$

(b) If $E_\alpha \subseteq F_{X_\alpha}$, $\alpha \in \Omega$. Then,

$$D = \sum_{\alpha\in\Omega} E_\alpha \quad \text{iff} \quad \operatorname{res}_{X_\alpha} D = E_\alpha \; , \alpha \in \Omega.$$

Proof

(a) is obvious from definition 5.2.9 and (b) follows from (a). □

Through definition 5.2.10 and the succeeding consequence, we notice an important difference between working with classical sets and working with fuzzy sets. Had the E_α been classical sets, the union '$\bigcup$' would have replaced the sign 'Σ', regardless of the fact that the E_α are subsets of different (and even disjoint) sets. In fuzzy theory, however, the situation is not quite the same. The fuzzy union is defined for fuzzy subsets of the same full set. (i.e. functions from the same set). Thus, for the E_α in definition 5.2.10, the expression " $\bigcup_{\alpha\in\Omega} E_\alpha$" is meaningless. In the discussion of the (disjoint) sum fuzzy topological space, something that replaces this expression is needed, and will turn out to be provided by definition 5.2.10.

Definition 5.2.12

Let Ω be an indexing set and $C = \{(F_{X_\alpha}, J_\alpha\} : \alpha \in \Omega\}$ a family of fuzzy topological spaces with all the X_α being mutually disjoint, and $Y = \bigcup_{\alpha\in\Omega} X_\alpha$.

A fuzzy topology on F_Y is said to be <u>the (disjoint) sum</u> of the fuzzy topologies in C and denoted by $\sum_{\alpha\in\Omega} J_\alpha$ (or simply $\sum_\Omega$) iff it is the coinduced fuzzy topology for the family of inclusion functions $\{(i_n)_\alpha : X_\alpha \dashrightarrow Y : \alpha \in \Omega\}$ and the family C of fts's.

It follows that $\sum_\Omega$ is the largest fuzzy topology making each of the $(i_n)_\alpha$ fuzzy continuous from (F_{X_α}, J_α).

The fts $(F_Y, \sum_\Omega)$ is called the <u>disjoint (sum)</u> fts of the topological spaces in C.

<u>Consequence 5.2.13</u>

Let C, Y and $\sum_\Omega$ be as in the previous definition. Then,

$$\sum_\Omega = \{O \subseteq F_Y : \operatorname{res}_{X_\alpha} O \in J_\alpha, \alpha \in \Omega\} = \{\sum_{\alpha\in\Omega} O_\alpha : O_\alpha \in J_\alpha, \alpha \in \Omega\}.$$

<u>Proof</u>

Let $O \subseteq F_Y$ and $O_\alpha = \operatorname{res}_{X_\alpha} O$. By consequence 5.2.11, we have $O = \sum_{\alpha\in\Omega} O_\alpha$. Let $(i_n)_\alpha : X_\alpha \dashrightarrow Y$. Then,

$O \in \sum_\Omega$ iff $(i_n)_\alpha^{-1}[O] \in J_\alpha, \alpha \in \Omega$, by definition of 5.2.12

iff $O_\alpha = \operatorname{res}_{X_\alpha} O \in J_\alpha, \alpha \in \Omega$, by consequence 5.1.5(a)

iff $O = \sum_{\alpha\in\Omega} O_\alpha, O_\alpha \in J_\alpha, \alpha \in \Omega$, by consequence 5.2.11. □

In Ghanim et al [22], we come across a description of the (disjoint) sum fuzzy topology which although appearing to be different, is equivalent to that given in the previous consequence. However, it is not presented rigorously. The authors define the (disjoint) sum fuzzy topology $\sum_{\Omega}$ of the fts's in $\mathbf{C} = \{(F_{X_\alpha}, J_\alpha) : \alpha \in \Omega\}$, by $\sum_{\Omega} = \{0 : 0 \cap X_\alpha \in J_\alpha$, for all $\alpha \in \Omega\}$. The symbol "X_α" is what they and other authors term the full set in X_α (i.e. F_{X_α}), but here it actually stands for the crisp set χ_{X_α} in Y; a flexibility which they allow. Now the intersection $0 \cap X_\alpha$ (or $0 \cap \chi_{X_\alpha}$ in our terminology) is a fuzzy set in Y (i.e. $0 \quad \chi_{X_\alpha} \subseteq F_Y$) and not in X_α, and so it can not be in J_α. But again thanks to an "agreement" stated in the start of the paper, the zero values of $0 \cap \chi_{X_\alpha}$ at $Y - X_\alpha$ are disregarded, so that it is thought of as a fuzzy set in X_α. This awkwardness can be overcome by substituting restrictions for intersections as we have done in 5.2.13. This adjustment has also the conceptual advantage of allowing the sum fts to be a special case of coinduction. Their definition of a fuzzy subspace which is evident from their description of fuzzy sum given above invites similar comments. They also defined openness for a fuzzy subspace, so that (F_X, J') would be an open subspace of a fts (F_Y, J) iff $\chi_X \in J$. This notion is obviously not very useful, since a fuzzy topology need not contain any crisp set other than the empty fuzzy set and the full set. It is clear that they had the fuzzy sum in mind when they made this definition.

The following two consequences are extensions of some results in classical topology. Compare them with the corresponding ideas in Ghanim et al [22].

Let $(F_Y, \underset{\Omega}{\Sigma})$ be the (disjoint) sum fts of the family $\{(F_{X_\alpha}, J_\alpha) : \alpha \in \Omega\}$ of fts's.

Consequence 5.2.14

For every $\alpha \in \Omega$, (F_{X_α}, J_α) is a subspace of $(F_Y, \underset{\Omega}{\Sigma})$.

Proof

Let $\alpha \in \Omega$. We will show that J_α is the X_α-relative fuzzy topology $J'_\alpha = \{\underset{X_\alpha}{\text{res}}\ O : O \in \underset{\Omega}{\Sigma}\}$ of $\underset{\Omega}{\Sigma}$.

First, let $U \in J'_\alpha$. Then there is an $O \in \underset{\Omega}{\Sigma}$, such that $U = \underset{X_\alpha}{\text{res}}\ O$. But then by consequence 5.2.13, $U \in J_\alpha$. Thus, $J'_\alpha \subseteq J_\alpha$.

Now, let $V \in J_\alpha$. Define $O \subseteq F_Y$, by $O = \sum_{\beta\in\Omega} O_\beta$, where

$$O_\beta = \begin{cases} V & \text{if } \beta = \alpha \\ F_{X_\beta} & \text{if } \beta \in \Omega - \{\alpha\} \end{cases}.$$

Since $O_\beta \in J_\beta$, $\beta \in \Omega$, then $O \in \underset{\Omega}{\Sigma}$, by consequence 5.2.13. But then by the definition of J'_α, we have $\underset{X_\alpha}{\text{res}}\ O \in J'_\alpha$. But $\underset{X_\alpha}{\text{res}}\ O = V$, and so $V \in J'_\alpha$. Thus $J_\alpha \subseteq J'_\alpha$.

Therefore, $J_\alpha = J'_\alpha$. □

Consequence 5.2.15

(a) For every $\alpha \in \Omega$ and $U \in J_\alpha$, we have $\text{ext}\ U \in \underset{\Omega}{\Sigma}$.

(b) For every $\alpha \in \Omega$, the fuzzy set χ_{X_α} is both open and closed in (F_{X_α}, J_α).

Proof

(a) Let $\alpha \in \Omega$, $U \in J_\alpha$ and $A = \text{ext } U$. Then for every $\beta \in \Omega$,

$$\underset{X_\beta}{\text{res}}\, A = \begin{cases} U & \text{if } \beta = \alpha \\ \Phi_{X_\beta} & \text{if } \beta \neq \alpha \end{cases}.$$ Therefore, $\underset{X_\beta}{\text{res}}\, A \in J_\beta$, $\beta \in \Omega$,

and hence $A \in \underset{\Omega}{\Sigma}$.

(b) First, since $\chi_{X_\alpha} = \text{ext } F_{X_\alpha}$, then, by (a), $\chi_{X_\alpha} \in \underset{\Omega}{\Sigma}$.

Now, let $A = \chi_{X_\alpha}$. Then,

$$\begin{aligned}
\underset{X_\beta}{\text{res}}\, A^c &= (\underset{X_\beta}{\text{res}}\, A)^c \text{, by consequence 5.1.5(b)} \\
&= (\underset{X_\beta}{\text{res}}\, \chi_{X_\alpha})^c \\
&= \begin{cases} (F_{X_\alpha})^c & \text{if } \beta = \alpha \\ (\Phi_{X_\alpha})^c & \text{if } \beta \neq \alpha \end{cases} \\
&= \begin{cases} \Phi_{X_\alpha} & \beta = \alpha \\ F_{X_\beta} & \beta \neq \alpha \end{cases} \\
&\in J_\beta \ , \quad \beta \in \Omega .
\end{aligned}$$

Thus, by consequence 5.2.13, $A^c \in \underset{\Omega}{\Sigma}$. But then A is $\underset{\Omega}{\Sigma}$-closed. □

A useful result in classical topology connecting the concepts of quotient topology and (disjoint) sum is the following.

Let Ω be an indexing set and let $\{X_\alpha : \alpha \in \Omega\}$ and $\{Y_\alpha : \alpha \in \Omega\}$ be two families of mutually disjoint sets. For every $\alpha \in \Omega$, let

$(Y_\alpha, \tau_{f_\alpha})$ be the quotient of the topological space (X_α, τ_α) under the surjective map $f_\alpha : X_\alpha \longrightarrow Y_\alpha$. Let $X = \bigcup_{\alpha\in\Omega} X_\alpha$ and $Y = \bigcup_{\alpha\in\Omega} Y_\alpha$ and define the function $f : X \longrightarrow Y$, by, $f|_{X_\alpha} = f_\alpha$; $\alpha \in \Omega$. Then the topological sum of the family $\{(Y_\alpha, \tau_{f_\alpha}) : \alpha \in \Omega\}$ is the quotient under f of the topological sum of the family $\{(X_\alpha, \tau_\alpha) : \alpha \in \Omega\}$.

To arrive at a fuzzy version of this classical theorem, we need the following consequence of definition 5.2.11 of (disjoint) sum set and definitions 4.1.1 and 4.1.3 of image and inverse image of fuzzy sets.

Using the foregoing notations (with f_α not necessarily surjective), we introduce the 'disjoint sum' version of consequence 4.1.8(a).

Consequence 5.2.16

(a) Let $D = \sum_{\alpha\in\Omega} D_\alpha$, $D_\alpha \subseteq F_{Y_\alpha}$, $\alpha \in \Omega$. Then

$$f^{-1}[D] = \sum_{\alpha\in\Omega} f_\alpha^{-1}[D_\alpha].$$

(b) Let $E = \sum_{\alpha\in\Omega} E_\alpha$, $E_\alpha \subseteq F_{X_\alpha}$, $\alpha \in \Omega$. Then

$$f[E] = \sum_{\alpha\in\Omega} f_\alpha[E_\alpha].$$

Proof

(a) Let $\beta \in \Omega$. Since $D = \sum_{\alpha\in\Omega} D_\alpha$, then $\operatorname{res}_{Y_\beta} D = D_\beta$, and so,

for every $y \in Y_\beta$, $D(y) = D_\beta(y)$.

Let $x \in X_\beta$, then $f(x) = f_\beta(x) \in f[X_\beta] \subseteq Y_\beta$. Therefore,

$f^{-1}[D](x) = D(f(x)) = D(f_\beta(x)) = D_\beta(f_\beta(x)) = f_\beta^{-1}[D_\beta](x)$.

Thus, $\operatorname{res}_{X_\beta} f^{-1}[D] = f_\beta^{-1}[D_\beta]$, and hence, $f^{-1}[D] = \sum_{\beta\in\Omega} f_\beta^{-1}[D_\beta]$.

(b) Let $\beta \in \Omega$. Since $E = \sum_{\alpha\in\Omega} E_\alpha$, then $\operatorname{res}_{X_\beta} E = E_\beta$, and so,

for every $x \in X_\beta$, $E(x) = E_\beta(x)$.

Let $y \in Y_\beta$, then, since the X_α (and the Y_α) are mutually disjoint, we have $f^{-1}(y) = f_\beta^{-1}(y)$.

Either $f^{-1}(y) = \phi$, and hence $f[E](y) = 0 = f_\beta[E_\beta](y)$, or $f^{-1}(y) \neq \phi$, and hence,

$f[E](y) = \sup \{E(x) : x \in f^{-1}(y) = f_\beta^{-1}(y)\} = \sup \{E_\beta(x) : x \in f_\beta^{-1}(y)\} = f_\beta[E_\beta](y)$.

Thus, $\operatorname{res}_{Y_\beta} f[E] = f_\beta[E_\beta]$, and hence $f[E] = \sum_{\beta\in\Omega} f_\beta[E_\beta]$. □

.

Proposition 5.2.17

Using the foregoing notations, for every $\alpha \in \Omega$, let $(F_{Y_\alpha}, J_{f_\alpha})$ be the quotient of the fts (F_{X_α}, J_α) under the surjective function f_α. Then,

the fts $(F_Y, \sum_{\alpha\in\Omega} J_{f_\alpha})$ is the quotient under f of $(F_X, \sum_{\alpha\in\Omega} J_\alpha)$.

Proof

$$0 \in \sum_{\alpha\in\Omega} J_{f_\alpha} \quad \text{iff} \quad 0 = \sum_{\alpha\in\Omega} 0_\alpha, \quad 0_\alpha \in J_{f_\alpha}$$

$$\text{iff} \quad f^{-1}[0] = \sum_{\alpha\in\Omega} f_\alpha^{-1}[0_\alpha], \; f_\alpha^{-1}[0_\alpha] \in J_\alpha, \text{ by}$$

consequences 4.1.10(a)3 and 5.2.16

$$\text{iff} \quad f^{-1}[0] \in \sum_{\alpha\in\Omega} J_\alpha .$$

Therefore, $\sum_{\alpha\in\Omega} J_{f_\alpha} = \{0 : f^{-1}[0] \in \sum_{\alpha\in\Omega} J_\alpha\}$, and hence

$(F_Y, \sum_{\alpha\in\Omega} J_{f_\alpha})$ is the quotient of $(F_X, \sum_{\alpha\in\Omega} J_\alpha)$ under f. □

3. Goodness and Initial Goodness of Extensions

For the purpose of brevity, we introduce the following notations.

Notation 5.3.1

Let **C** be a family of topological spaces. The family of the natural fuzzy topological spaces of all members of **C** is denoted by $\omega[\mathbf{C}]$. Thus

$$\omega[\mathbf{C}] = \{(F_X, \omega(\tau)) : (X,\tau) \in \mathbf{C}\}.$$

Notation 5.3.2

Let **C'** be a family of fuzzy topological spaces. The family of the initial topological spaces of all members of **C'** is denoted by $\iota[\mathbf{C}']$. Thus,

$$\iota[\mathbf{C}'] = \{(X, \iota(J)) : (F_X, J) \in \mathbf{C}'\}.$$

Goodness of fuzzy induction

In looking for a criterion for goodness of a definition of fuzzy induction, we consider the case of one function $f:X \longrightarrow Y$. Let τ be a topology on Y and τ_i the induced topology on X for f and (Y,τ).

$$\begin{array}{lcl} (X, \tau_i) & \longrightarrow & (Y, \tau) \\ \begin{array}{l} (F_X , \omega(\tau_i)) \\ (F_X , (\omega(\tau))_i) \end{array} & \longrightarrow & (F_Y , \omega(\tau)) \end{array}$$

The above diagram illustrates two possible ways of constructing a fuzzy topology on F_X ,namely induction followed by taking the natural fuzzy topology and taking the natural fuzzy topology followed by fuzzy induction. It seems reasonable that goodness should require that those two fuzzy topologies coincide, that is; $\omega(\tau_i) = (\omega(\tau))_i$. This motivates the definition of goodness for fuzzy induction, which, as we will show, is satisfied by definition 5.1.1.

Proposition 5.3.3

Fuzzy induction is a good extension to fuzzy topology of classical induction. More precisely for every set X and every family of topological spaces $C = \{(Y_\alpha , \tau_\alpha) : \alpha \in \Omega\}$ and family of functions $\Psi = \{f_\alpha : X \longrightarrow Y_\alpha : \alpha \in \Omega\}$, we have;

$$J_i = \omega(\tau_i),$$

where τ_i is the induced topology on X for Ψ and C and J_i the induced fuzzy topology on F_X for Ψ and $\omega[C]$.

Proof

Let B be the family of all finite intersections of the family $D = \{f_\alpha^{-1}[O_\alpha] : O_\alpha \in \omega(\tau_\alpha), \alpha \in \Omega\}$. By definition 5.1.1, B is a base for J_i. We will show that B is a base for $\omega(\tau_i)$.

First, we show that $B \subseteq \omega(\tau_i)$. It is enough to show that each member of $\mathbf{D}$ is in $\omega(\tau_i)$. Let $D \in \mathbf{D}$, then $D = f_\alpha^{-1}[O_\alpha]$ for some $O_\alpha \in \omega(\tau_\alpha)$ and $\alpha \in \Omega$. Since, by the definition of induction, f_α is continuous form (X,τ_i) into (Y_α, τ_α), then, by proposition 4.3.4, f_α is fuz-continuous from $(F_X, \omega(\tau_i))$ into $(F_{Y_\alpha}, \omega(\tau_\alpha))$. But then, $f_\alpha^{-1}[O_\alpha] \in \omega(\tau_i)$ and hence, $D \in \omega(\tau_i)$.

Now, let $O \in \omega(\tau_i)$ and $e = (x)_\lambda \in F_X$, such that eqO. Then $O(x) > 1 - \lambda + \zeta$, for some $\zeta \in (0,\lambda)$. Let $\delta = 1 - \lambda + \zeta$, and $Z = O^{-1}(\delta,1]$. Then $x \in Z \in \tau_i$. By the definition of τ_i, $x \in \bigcap_{n=1}^{N} f_{\alpha_n}^{-1}[Z_{\alpha_n}] = Z' \subseteq Z$, for some $N \in \mathbf{N}$, and $Z_{\alpha_n} \in \tau_{\alpha_n}$, $1 \leqslant n \leqslant N$. For every $n \in \{1,2,\ldots,N\}$, let $O_{\alpha_n} = Z_{\alpha_n}^{(\delta)}$, (i.e. the semicrisp set in Y_{α_n} of support Z_{α_n} and value δ). Then,

$O_{\alpha_n} \in \omega(\tau_{\alpha_n})$, $1 \leqslant n \leqslant N$. Consider $B = \bigcap_{n=1}^{N} f_{\alpha_n}^{-1}[O_{\alpha_n}]$. Hence, we

have $B \in \mathbf{B}$ and, by consequence 4.1.12(c), $\operatorname{supt} B = \bigcap_{n=1}^{N} f_\alpha^{-1}(\operatorname{supt} O_{\alpha_n})$ $= \bigcap_{n=1}^{N} f_{\alpha_n}^{-1}(Z_{\alpha_n}) = Z'$. For every $z \in X$, we have;

$$B(z) = \min\{O_{\alpha_n}(f_{\alpha_n}(z)) : 1 \leqslant n \leqslant N\} = \begin{cases} \delta < O(z) & \text{if } z \in Z' \\ 0 \leqslant O(z) & \text{if } z \in X - Z' \end{cases}.$$

Therefore $B \subseteq O$, and since $x \in Z'$, then $B(x) = \delta = 1 - \lambda + \zeta$, and so eqB. Thus $eqB \subseteq O$, and so $\mathbf{B}$ is a base for $\omega(\tau_i)$. □

<u>Remark 5.3.4</u>

Taking the initial topology of both sides of the equation $J_i = \omega(\tau_i)$ in the previous proposition, we get, $\iota(J_i) = \tau_i$.

Later, we will show that the latter equality holds even if the family $\omega[C]$ is replaced by a family of arbitrary fuzzy topological spaces.

Corollary 5.3.5

In the sense of proposition 5.3.3,

(a) the fuzzy subspace is a good extension of the classical subspace.

(b) the product fts is a good extension of the classical topological space.

Corollary 5.3.6

Let (X,τ') be a subspace of the topological space (Y,τ). Then a lower semicontinuous function from this subspace into $(I, 1^*)$ can be extended to the space (Y,τ).

Proof

Let $U \in \omega(\tau')$. Let (F_X , J') be a fuzzy subspace of the fts $(F_Y , \omega(\tau))$. By the goodness of the fuzzy subspace, we have $J' = \omega(\tau')$. Hence $U \in J'$, and so, by definition of J', there is an $A \in J$ such that, $U = \text{res } A = i_n^{-1}[A] = A_\circ i_n$. This is illustrated by the following diagram.

$$
\begin{tikzcd}
(X, \tau') \arrow[rr, dashed, "i_n"] \arrow[dr, dashed, "{U = A_\circ i_n \in J'}"'] & & (Y, \tau) \arrow[dl, dashed, "A \in J"] \\
& (I, 1^*) &
\end{tikzcd}
$$

. □

Goodness of fuzzy coinduction

To motivate a criterion of goodness for a definition of fuzzy coinduction, we consider again the case of one function $f: X \longrightarrow Y$. But this time, let τ be a topology on X and τ_C the induced topology on Y for f and (X,τ)

$$\begin{array}{ccc} (X,\ \tau) & \longrightarrow & (Y,\ \tau_C) \\ & & \\ (F_X\ ,\ \omega(\tau)) & \dashrightarrow & \boxed{\begin{array}{c} (F_Y\ ,\ \omega(\tau_C)) \\ (F_Y\ ,\ (\omega(\tau))_C) \end{array}} \end{array}$$

The above illustration shows two possible ways of constructing a fuzzy topology on F_Y , namely coinduction followed by taking the natural fuzzy topology and taking the natural topology followed by fuzzy coinduction. Again it is reasonable that goodness should demand that the two fuzzy topologies coincide, that is; $\omega(\tau_C) = (\omega(\tau))_C$. The following proposition shows that our definition of fuzzy coinduction is good in this sense.

Proposition 5.3.7

Fuzzy coinduction is a good extension to fuzzy topology of classical coinduction. More precisely, for every set Y and every family of topological spaces $C = \{(X_\alpha\ ,\ \tau_\alpha) : \alpha \in \Omega\}$ and family of functions $\Phi = \{f_\alpha : X_\alpha \longrightarrow Y,\ \alpha \in \Omega\}$, we have;

$$J_C = \omega(\tau_C),$$

where τ_C is the coinduced topology on Y for Φ and C, and J_C is the coinduced fuzzy topology on F_Y for Φ and $\omega[C]$.

Proof

By definition of τ_C, for every $\alpha \in \Omega$, f_α is continuous from (X_α, τ_α) into (Y, τ_C), and hence, by proposition 4.3.4, fuz-continuous from $(F_{X_\alpha}, \omega(\tau_\alpha))$ into $(F_Y, \omega(\tau_C))$. Thus, for every $O \in \omega(\tau_C)$, we have; $f_\alpha^{-1}[O] \in \omega(\tau_\alpha)$; $\alpha \in \Omega$, and hence, $O \in J_C$. Therefore, $\omega(\tau_C) \subseteq J_C$.

On the other hand, if $A \in J_C$, then, by definition of J_C, $f_\alpha^{-1}[A] \in \omega(\tau_\alpha)$, $\alpha \in \Omega$. So, for every $\alpha \in \Omega$ and $\lambda \in I - \{1\}$, we have; $(f_\alpha^{-1}[A])^{-1}(\lambda,1] \in \tau_\alpha$. But $(f_\alpha^{-1}[A])^{-1}(\lambda,1] = f_\alpha^{-1}(A^{-1}(\lambda,1])$, by consequence 4.1.11(a)2. Therefore, by definition of τ_C, $A^{-1}(\lambda,1] \in \tau_C$, $\lambda \in I - \{1\}$. But then, $A \in \omega(\tau_C)$. Therefore, $J_C \subseteq \omega(\tau_C)$.

Thus, $J_C = \omega(\tau_C)$. $\square$

Alternative proofs of propositions 5.3.7 and 5.3.3 can be found in Lowen [34]. The proofs we gave are more in tune with our approach. They are also in our opinion simpler and <u>fuzzy</u> topological in nature.

Remark 5.3.8

Taking the initial topology of both sides of the equation, $J_C = \omega(\tau_C)$ in the previous proposition, we get, $\iota(J_C) = \tau_C$.

The latter equality, however, does not necessarily hold if the family $\omega[C]$ is replaced by a family of fts's which are not all natural. This we discuss later.

Corollary 5.3.9

In the sense of proposition 5.3.7,

(a) the quotient fts is a good extension of the classical topological space.

(b) the (disjoint) sum fts is a good extension of the classical (disjoint) sum topological space.

Initial goodness

In the following, let J_i be the induced fuzzy topology on F_X, for the family of functions $\Psi = \{f_\alpha : X \longrightarrow Y_\alpha : \alpha \in \Omega\}$ and the family of fuzzy topological spaces, $C' = \{(F_{Y_\alpha}, J_\alpha) : \alpha \in \Omega\}$.

Lemma 5.3.10

Let $\mathbf{C} = \{(U_\alpha \circ f_\alpha)^{-1}(\lambda,1] : U_\alpha \in J_\alpha, \alpha \in \Omega, \lambda \in I - \{1\}\}$. Then $\mathbf{C}$ is a subbase for $\iota(J_i)$.

Proof

First, if $Z \in \mathbf{C}$, then, $Z = (U_\alpha \circ f_\alpha)^{-1}(\lambda,1] = f_\alpha^{-1}(U_\alpha^{-1}(\lambda,1])$ for some $\lambda \in I - \{1\}$, $\alpha \in \Omega$, and $U_\alpha \in J_\alpha$. By consequence 4.1.11(a)2, $Z = (f_\alpha^{-1}[U_\alpha])^{-1}(\lambda,1]$, and since, by definition of J_i, $f_\alpha^{-1}[U_\alpha] \in J_i$, we have $Z \in \iota(J_i)$. Therefore, $\mathbf{C} \subseteq \iota(J_i)$.

Now, let $Z \in \iota(J_i)$ and $x \in Z$. By definition 2.2.10,

$$x \in \bigcap_{n=1}^{N} A_n^{-1}(\lambda_n,1] \subseteq Z,$$ for some $N \in \mathbb{N}$, $A_n \in J_i$, $\lambda_n \in I - \{1\}$, $1 \leq n \leq N$. But then $x \in A_n^{-1}(\lambda_n,1]$, $1 \leq n \leq N$. For each $n \in \{1,2,\ldots,N\}$, define $e_n \in F_X$, by $e_n = (x)_{1-\lambda_n}$. Hence, for every $n \in \{1,2,\ldots,N\}$, $e_n q A_n$, and so by definition of J_i,

$e_n q(\bigcap_{k=1}^{K} f_{\alpha_k}^{-1}[\overset{n}{U}_{\alpha_k}]) \subseteq A_n$, for some $K \in \mathbb{N}$, $\alpha_k \in \Omega$, $\overset{n}{U}_{\alpha_k} \in J_{\alpha_k}$, $1 \leq k \leq K$. But then, by consequence 1.3.7(b), for every $n \in \{1,2,..,N\}$, $e_n q(f_{\alpha_k}^{-1}[\overset{n}{U}_{\alpha_k}])$, and hence $x \in (f_{\alpha_k}^{-1}[\overset{n}{U}_{\alpha_k}])^{-1}(\lambda_n ,1]$ $= (\overset{n}{U}_{\alpha_k} \circ f_k)^{-1}(\lambda_n,1]$, by definition 4.1.3, for every $k \in \{1,2,...,K\}$.

Therefore, $x \in \bigcap_{k=1}^{K} (U_{\alpha_k} \circ f_{\alpha_k})^{-1}(\lambda_n , 1] \subseteq A_n^{-1}(\lambda_n ,1]$, $1 \leq n \leq N$

and so $x \in \bigcap_{n=1}^{N} (\bigcap_{k=1}^{K} (\overset{n}{U}_{\alpha_k} \circ f_{\alpha_k})^{-1}(\lambda_n ,1] \subseteq \bigcap_{n=1}^{N} A_n^{-1}(\lambda_n ,1] \subseteq Z$.

Therefore, the family of all intersections of members of $\mathbf{C}$ is a base for $\iota(J)$, and hence, $\mathbf{C}$ is a subbase for it. □

Now, we are ready to prove the following proposition.

<u>Proposition 5.3.11</u> (Lowen [34])

Fuzzy induction is an initially good extension to fuzzy topology of classical induction. More precisely, for every set X and every family. $\mathbf{C}' = \{(F_{Y_\alpha} , J_\alpha) : \alpha \in \Omega\}$ of fuzzy topological spaces and family of functions $\Psi = \{f_\alpha : X \longrightarrow Y_\alpha , \alpha \in \Omega\}$, we have;

$$\iota(J_i) = \tau_i ,$$

where J_i is the induced fuzzy topology on F_X for Ψ and $\mathbf{C}'$ and τ_i the induced topology on X for Ψ and $\iota[\mathbf{C}']$.

<u>Proof</u>

For every $\alpha \in \Omega$, let Λ_α be the cardinal number of J_α and $\mathbf{C}_\alpha$ the family of identical topological spaces $(I, 1_r^*)$ indexed by

Λ_α.

By definition 2.2.10 of initial topology, for every $\alpha \in \Omega$, $\iota(J_\alpha)$ is the induced topology on Y_α for the family of topological spaces C_α and the family of functions $\{U : Y_\alpha \longrightarrow I : U \in J_\alpha\}$.

Now, for each $\alpha \in \Omega$, we have; $f_\alpha : X \longrightarrow (Y_\alpha , \iota(J_\alpha))$ and $U : (Y_\alpha , \iota(J_\alpha)) \longrightarrow (I, 1_r^*)$, for every $U \in J_\alpha$. Hence by the transitivity of classical induction (e.g. James [28]), τ_i is the induced topology on X, for the family $\bigcup_{\alpha\in\Omega} C_\alpha$ of identical topological spaces $(I, 1_r^*)$ and the family of functions $\{U_\alpha \circ f_\alpha : U_\alpha \in J_\alpha , \alpha \in \Omega\}$. Thus, by the definition of induction, the family $\{(U_\alpha \circ f_\alpha)^{-1} (\lambda, 1] : U_\alpha \in J_\alpha , \alpha \in \Omega, \lambda \in I - \{1\}\}$ is a subbase for τ_i. But by lemma 5.3.10, this family is a subbase for $\iota(J_i)$. Thus $\tau_i = \iota(J_i)$. □

Fuzzy coinduction, however, is not initially good. The following proposition and counterexample complete the picture of this basic fuzzy topological construction.

<u>Proposition 5.3.12</u>

Let Y be a set, $C' = \{(F_{X_\alpha} , J_\alpha) : \alpha \in \Omega\}$ a family of fts's and $\Psi = \{f_\alpha : X_\alpha \longrightarrow Y : \alpha \in \Omega\}$ a family of functions. Then,

$$\iota(J_C) \subseteq \tau_C ,$$

where J_C is the coinduced fuzzy topology on F_Y for Ψ and C', and τ_C the coinduced topology on Y for Ψ and $\iota[C']$.

Proof (cf Lowen [34])

It is enough to show that every member of the subbase of $\iota(J_C)$ is in τ_C. Thus, let D be a member of this subbase, then $D = A^{-1}(\lambda,1]$, for some $A \in J_C$ and $\lambda \in I - \{1\}$. Hence $f_\alpha^{-1}[A] \in J_\alpha$, $\alpha \in \Omega$, by definition of J_C. Therefore, for every, $\alpha \in \Omega$, by definition of $\iota(J_\alpha)$, $(f_\alpha^{-1}[A])^{-1}(\lambda,1] \in \iota(J_\alpha)$. But, $(f_\alpha^{-1}[A])^{-1}(\lambda,1] = f_\alpha^{-1}(A^{-1}(\lambda,1])$, by consequence 4.1.11(a)2. Hence, $f_\alpha^{-1}(A^{-1}(\lambda,1]) \in \iota(J_\alpha)$, $\alpha \in \Omega$, and so, by definition of τ_C, $A^{-1}(\lambda,1] \in \tau_C$. □

The converse of proposition 5.3.12 need not hold, as the following counterexample shows.

Counterexample 5.3.13

Let τ be any topology on X such that $\tau \neq \Delta$. Let $\Omega = I - \{1\}$, $X_\alpha = X$ and $J_\alpha = \omega_\alpha(\tau)$, $\alpha \in \Omega$, and let Ψ be the family of identical identity functions $\{i_d : i_d : X \longrightarrow Y = X\}$. Then $J_C = \{A : i_d^{-1}[A] \in \omega_\alpha(\tau),\ \alpha \in \Omega\} = \bigcap_{\alpha\in\Omega} \omega_\alpha(\tau) = \omega(\tau)$, by the remark following definition 2.2.17, and so $\iota(J_C) = \iota(\omega(\tau)) = \tau$, by consequence 2.2.12(a) and (b).

On the other hand,

$$\tau_C = \{Z : i_d^{-1}[Z] \in \iota(\omega_\alpha(\tau)),\ \alpha \in \Omega\}$$

$$= \{Z : Z \in \Delta\} = \Delta, \text{ by proposition 2.2.19(a).}$$

Therefore, $\tau_C = \Delta \neq \tau = \iota(J_C)$. □

Since in the preceding counterexample each of the J_α is Lowen, then the converse of proposition 5.3.12 need not hold even if C' is a family of Lowen fuzzy topological spaces. Lowen [34] provides another counterexample to illustrates the same point.

Now we give a simpler counterexample but for which members of C' are not Lowen.

Counterexample 5.3.14

Let X be a set of more than one point, Z_1 a nonempty proper subset of X and J_1 and J_2 the fuzzy topologies on F_X given by, $J_1 = \{F_X , \Phi_X , D\}$ and $J_2 = \{F_X , \Phi_X , Z_1^{(\frac{1}{3})}\}$, where D is the upper semicrisp set given by, $D(x) = \{\begin{matrix} 1 & x \in Z_1 \\ \frac{2}{3} & x \notin Z_1 \end{matrix}$. Let $C' = \{(F_X , J_1), (F_X , J_2)\}$ and $\Psi = \{f_n = i_d : X \longrightarrow Y = X, 1 \leq n \leq 2\}$. Then,

$J_c = \{A \subseteq F_X : f_n^{-1}[A] \in J_n , 1 \leq n \leq 2\}$

$= \{A : A \in J_1 \bigcap J_2\} = \{F_X , \Phi_X\}$.

Therefore, $\iota(J_c) = \{X, \Phi\}$.

On the other hand,

$\tau_c = \{Z \subseteq X : f_n^{-1}[Z] \in \iota(J_n), 1 \leq n \leq 2\} = \{Z : Z \in \iota(J_1) \bigcap \iota(J_2)\}$

$= \{X, \Phi, Z_1\}$

Therfore, $\tau_c \not\subseteq \iota(J_c)$. □

PART VI

THE BEHAVIOUR OF FUZZY PROPERTIES UNDER FUZZY TOPOLOGICAL CONSTRUCTIONS

In part three we introduced fuzzy sequentiality and showed it, together with fuzzy second countability and quasi-first countability, to be a good extension of the corresponding classical property. Here, we submit these fuzzy properties to another test which further establishes their suitability as the right fuzzy extensions of these respective classical notions. This new criterion will be called <u>constructive goodness</u>. It demands that, under the four basic topological constructions of part V, the behaviour of a classical property agrees with that of its fuzzy extension.

The advantage of our Q-goodness approach is demonstrated very clearly in this part. Long and complicated proofs given by Pu and Liu [44] of three important results are replaced by a single simple proof encompassing all three cases.

1. Constructive Goodness and Excellence

Notations and definitions 6.1.1

We will refer to the four basic (fuzzy) topological constructions of part V as *operations*. An operation of this type is denoted by $\circledast$ (respectively $\circledast^f$).

A (fuzzy) property is said to be *inductive* (respectively *coinductive*) iff it is invariant under (fuzzy) induction (respectively (fuzzy) coinduction).

We have the following four special cases of (fuzzy) induction and coinduction.

(1) *(Fuzzy) subtraction* : The construction of a subspace from a given (fuzzy) topological space.

(2) *(Fuzzy) division* : The construction of a quotient (fuzzy) topological space from a given (fuzzy) topological space.

(3) *(Fuzzy) addition* : The construction of a (fuzzy) topological sum of a family of (fuzzy) topological spaces.

(4) *(Fuzzy) multiplication* : The construction of a (fuzzy) topological product of a family of (fuzzy) topological spaces.

A (fuzzy) property is said to be *subtractive* iff it is preserved under subtraction. In the same manner we define, *divisibility*, *(countable) additivity* and *(finite, countable) multiplicativity*.

In the following, let P be a topological property and P^f a fuzzy extension of P.

Definition 6.1.2

The fuzzy property P^f is said to be a *constructively good* extension of P iff for every basic topological operation $\circledast$ we have;

P^f is invariant under $\circledast^f$ iff P is invariant under $\circledast$.

Definition 6.1.3

The fuzzy property P^f is said to be an *excellent* extension of P iff it is both a good and constructively good fuzzy extension of P.

Notation 6.1.4

Let C (respectively C') be a family of (fuzzy) topological spaces. A (fuzzy) topological space resulting from C (respectively C') by means of a (fuzzy) topological operation $\circledast$ (respectively $\circledast^f$) is denoted by $\circledast C$ (respectively $\circledast^f C'$).

Remark 6.1.5

Let P^f be a good fuzzy extension of P. For every basic topological operation $\circledast$, we have;

If P is not invariant under $\circledast$, then P^f is not invariant under $\circledast^f$.

Proof

Since P is not invariant under $\circledast$, then there is a family C of topological spaces, such that every member of C has the property P, but $\circledast C$ does not have it. Let $C' = \omega[C]$. By the goodness of both P^f and $\circledast^f$ (see section 3 of part V), every member of $\mathbf{C}'$ has the fuzzy property P^f, but $\circledast^f \mathbf{C}'$ does not have it. Thus, P^f is not invariant under $\circledast^f$. □

Consequence 6.1.6

The fuzzy property P^f is an excellent extension of P iff it is good and for every topological operation $\circledast$, the invariance of P under $\circledast$ implies the invariance of P^f under $\circledast^f$.

Proof

This follows from remark 6.1.5. □

Considering the goodness of the basic fuzzy topological operation shown in part V, one might wonder if goodness or initial goodness of a fuzzy property implies its contructive goodness and hence its excellence, thus the equivalence of the two concepts. The answer to this is provided in the following counterexamples.

Counterexample 6.1.7

Goodness does not imply constructive goodness.

Proof

Consider the fuzzy extension of classical compactness given by Lowen (definition 4.3 in Lowen [33] and definition VI in Lowen [35]) and called "weak fuzzy compactness". (A fts (F_X, J) is said to be weakly compact iff for every subfamily $\mathbf{B}$ of J such that $\bigcup_{O \in \mathbf{B}} O = F_X$ and for every $\zeta \in (0,1]$ there is a finite subfamily $\mathbf{B}_0$ of $\mathbf{B}$ such that $\bigcup_{U \in \mathbf{B}_0} U \supseteq \eta_{1-\zeta}$).

By theorem 2.1 in Lowen [35], weak compactness is a good extension of classical compactness, but, by the counterexample in page 17 in Lowen [34], it is not multiplicative. Since classical compactness is multiplicative by Tychonoff's theorem (e.g. theorem 5.13 in Kelley [29]), then weak compactness is not constructively good. □

<u>Counterexample 6.1.8</u>

Initial goodness does not imply constructive goodness.

<u>Proof</u>

Consider fuzzy first countability. By proposition 3.1.19 it is initially good, but by theorem 4.3 in Pu and Liu [44] it is not finitely multiplicative. Since classical first countability is countably multiplicative, then fuzzy first countability is not constructively good. □

<u>Proposition 6.1.9</u>

Let both P^f and $\otimes^f$ be initially good. Assume that the following statement is true;

"For every family C of topological spaces, if $\otimes C$ has the property P, then all but a countable number of members of C are indiscrete".

Then the following statement is true;

"For every family C' of fuzzy topological spaces, if $\otimes^f C'$ has the property P^f, then all but a countable number of members of C' are trivial".

Proof

Let $\mathbf{C}'$ be a family of fts's such that $\circledast^f \mathbf{C}'$ is P^f. By the initial goodness of both P^f and $\circledast^f$, $\circledast(\iota[\mathbf{C}'])$ has the property P. But then, by hypothesis, all but a countable number of members of $\iota[\mathbf{C}']$ are indiscrete. Therefore, by proposition 2.2.11, all but a countable number of $\mathbf{C}'$ are trivial. $\square$

2. The Excellence of Quasi-first Countability and Fuzzy Second Countability

Proposition 6.2.1

Neither quasi-first countability nor fuzzy second countability is divisive.

Proof

This follows from remark 6.1.5 and the fact that neither classical first countability nor classical second countability is divisive. As a classical counterexample consider the quotient topological space $(\mathbf{R}^*, \tau_h{}^*)$ described in the proof of proposition 3.3.13. As indicated there, $(\mathbf{R}^*, \tau_h{}^*)$ is not C_1 (and hence not C_2) although it is a quotient of a second countable (and hence first countable) topological space (i.e. $\mathbf{R}$ with its usual topology). □

Although, as demonstrated by the previous proposition, quasi-first countability and fuzzy second countability are not preserved under fuzzy division in general, they are preserved under a special type of fuzzy division, namely that of proposition 5.2.7. This we now prove.

Propositon 6.2.2

A fuzzy continuous fuzzy-open image of a quasi-first countable fts is quasi-first countable.

Proof

Let f be a function from X onto Y.

First, let f be fuz-continuous fuzzy-open from (F_X , J) into (F_Y , T) and let (F_X , J) be Q-C_1. Let $d \in F_Y$. We will show that there is a countable Q-neighbourhood base for T at d.

By the surjectivity of f, there is a fuzzy point e in X such that d = f(e) (take any fuzzy point in X with the same value as d and with its support lying in the inverse image of support d). By the quasi-first countability of (F_X , J), there is a countable open Q-neighbourhood base **B** at e. Let **B'** = {f[B] : B $\in$ **B**}. Then **B'** is countable, and by the fuz-openess of f, **B'** $\subseteq$ T. Also, since for every B $\in$ **B** we have eqB, then, f(e)qf[B], by consequence 4.1.7(b)1. Thus every member of **B'** is an open Q-neighbourhood of e. To show that **B'** is a Q-neighbourhood base at d, let U $\in$ T and dqU. Then $f^{-1}[U] \in J$ and, by consequence 4.1.7(b)2, $eqf^{-1}[U]$. But then there is a B $\in$ **B**, such that $B \subseteq f^{-1}[U]$. Therefore, f[B] $\in$ **B'**, and $f[B] \subseteq f[f^{-1}[U]] = U$, by consequence 4.1.7(a)1 and 4.1.10(a)3.

Therefore, **B'** is a countable Q-neighbourhood base at e, and so (F_Y , T) is Q-C_1. ☐

Proposition 6.2.3

A fuzzy continuous fuzzy open image of a fuzzy second countable fts is fuzzy second countable.

Proof

This is proved in a similar manner to proposition 6.2.2. □

In the following, we show that while quasi-first countability is additive, second countability is only countably additive.

Proposition 6.2.4

Quasi-first countability is additive.

Proof

Let (F_Y , Σ) be the (disjoint) sum fts of the family $\mathbf{C} = \{(F_{X_\alpha} , J_\alpha): \alpha \in \Omega\}$ of quasi-first countable fuzzy topological spaces. Let $e \in F_Y$. We will show that there is a countable Q-neighbourhood base at e.

There exists an $\alpha \in \Omega$ and $d \in X_\alpha$, such that $\underset{X_\alpha}{\text{res}}\ e = d$ (and, hence $\underset{Y}{\text{ext}}\ d = e$). By the quasi-first countability of $(F_{X_\alpha} , J_\alpha)$, there is a countable open Q-neighbourhood base $\mathbf{B}$ for J_α at d. Let $\mathbf{B}' = \{\underset{Y}{\text{ext}}\ B : B \in \mathbf{B}\}$. Then $\mathbf{B}'$ is countable, and by consequence 5.2.15(a), $\mathbf{B}' \subseteq \Sigma$. Also, since for every $B \in \mathbf{B}'$, we have dqB, then $eq(\underset{Y}{\text{ext}} B)$, by remark following definition 5.2.9. Thus every member of $\mathbf{B}'$ is an open Q-neighbourhood of e. To show that $\mathbf{B}'$ is a Q-neighbourhood base at d, let $O \in \Sigma$ and eqO. Let $U = \underset{X_\alpha}{\text{res}}\ O$, then $U \in J_\alpha$, by consequence 5.2.13, and dqU. But then, there is a $B \in \mathbf{B}$, such that $B \subseteq U$. Hence $\underset{Y}{\text{ext}}\ B \subseteq \underset{Y}{\text{ext}}\ U \subseteq O$ and $\underset{Y}{\text{ext}}\ B \in \mathbf{B}'$.

Therefore, $\mathbf{B}'$ is a countable Q-neighbourhood base for Σ at e, and hence (F_Y , Σ) is $Q\text{-}C_1$. □

Proposition 6.2.5

Fuzzy second countability is countably additive.

Proof (Cf Ghanim et al [22] and classical case)

Let (F_Y, Σ) be the (disjoint) sum fts of the family $C = \{(F_{X_n}, J_n) : n \in \mathbb{N}\}$ of fuzzy second countable fts. For each $n \in \mathbb{N}$, let B_n be an open countable base for J_n and define B'_n by $B'_n = \{\underset{Y}{\text{ext}}\ B : B \in B_n\}$. Then, we can show that the family $B = \bigcup_{n=1}^{\infty} B'_n$ is α base for (F_Y, Σ) following similar steps to those used in the proof of proposition 6.2.4. □

Proposition 6.2.6

Fuzzy second countability is not additive.

Proof

This follows from remark 6.1.5 and the fact that classical second countability is not additive.

As an illustration consider the uncountable family $\{X_\alpha = \{x_\alpha\} : \alpha \in \Omega\}$ of mutually disjoint singletons, and for every $\alpha \in \Omega$, let $J_\alpha = \{(x_\alpha)_\lambda : \lambda \in I - \{0,1\}\} \bigcup \{F_{X_\alpha}, \phi_{X_\alpha}\}$. Then, for every $\alpha \in \Omega$, J_α is the natural fuzzy topology of the one point topological space $(X_\alpha, \{X_\alpha, \phi\})$ and hence is C_2^f. Let $Y = \bigcup_{\alpha\in\Omega} X_\alpha$ and (F_Y, Σ) be the (disjoint) sum fts of the family $\{(F_{X_\alpha}, J_\alpha) : \alpha \in \Omega\}$. Then for every $A \subseteq F_Y$ and $\alpha \in \Omega$, we have;

$$\underset{X_\alpha}{\text{res}}\ A = \begin{cases} (x_\alpha)_\lambda, & \text{for some } \lambda \in I - \{0\}, \text{ if } x_\alpha \in \text{supt } A \\ \phi_{X_\alpha}, & \text{if } x_\alpha \notin \text{supt } A \end{cases}$$

$$\in J_\alpha .$$

Therefore, $A \in \Sigma$, and so Σ is the discrete fuzzy topology on

F_Y . Since Y is uncountable, then (F_Y , Σ) is not C_2^f. □

Lastly, we investigate multiplication.

Proposition 6.2.7 (Pu and Liu)

Both quasi-first countability and fuzzy second countability are countably multiplicative.

Proof

See theorem 4.2 in Pu and Liu [44] for the case of quasi-first countability. The case of fuzzy second countability is proved similarly. □

Proposition 6.2.8 (Pu and Liu)

If a product fuzzy topology of a family C of fuzzy topological spaces is quasi-first countable or fuzzy second countable, then all but a countable number of members of C are trivial.

Proof

Considering the initial goodness of quasi-first countability and the product fuzzy topology (propositions 3.1.18 and corollary 5.3.11), the result follows from proposition 6.1.9 and the corresponding classical result (e.g. theorem 3.5 in Kelley [29]). Note that fuzzy second countability implies quasi-first countability (corollary 3.1.11). □

Pu and Liu [44] gave a long and complicated proof of the previous result. This proof consisted of two lemmas (lemma 3.1 and 3.2), a theorem (4.1) and the bulk of the proof of another theorem (3.1). Our simpler proof is a consequence of the overall approach adopted in this thesis.

Proposition 6.2.9

Both quasi-first countability and fuzzy second countability are not multiplicative.

Proof

Let $\mathbf{C}$ be an uncountable family of fts's such that each member of $\mathbf{C}$ is both $\mathbf{Q}$-C_1 (respectively C_2^f) and not trivial (for instance we can take each member of $\mathbf{C}$ to be the natural fts of $\mathbf{R}$ equipped with its usual topology). Then, by proposition 6.2.8, the product fts of the fts's in $\mathbf{C}$ can not be $\mathbf{Q}$-C_1 (respectively C_2^f). $\square$

The following proposition completes the picture for products.

Proposition 6.2.10

The Lowen factors of a quasi-first countable (fuzzy second countable) product fts are quasi-first countable (respectively fuzzy second countable).

Proof

This follows from propositions 5.1.13, 6.2.2 and 6.2.3. $\square$

Proposition 6.2.11

Both quasi-first countability and fuzzy second countability are subtractive.

Proof

Let (F_X, J') be a subspace of the quasi-first countable fts (F_Y, J). Let $d \in F_X$, then there is an $e \in F_Y$, such that $d = \underset{X}{\text{res}}\, e$. Let $\mathbf{B}_e$ be a countable open Q-neighbourhood base for J at e. Consider the family $\mathbf{B}'_d = \{\underset{X}{\text{res}}\, B : B \in \mathbf{B}_e\}$. We have $\mathbf{B}'_d \subseteq J'$ and, by consequence 5.1.5(d), every member of $\mathbf{B}'_d$ is quasi-coincident with d. Now, let $U \in J'$ such that dqU. Then, there is an $O \in J$, such that $U = \underset{X}{\text{res}}\, O$, and hence eqO. But then there is a $B \in \mathbf{B}_e$, such that $B \subseteq O$. Therefore, $\underset{X}{\text{res}}\, B \in \mathbf{B}'_d$ and $\underset{X}{\text{res}}\, B \subseteq \underset{X}{\text{res}}\, O = U$, by consequence 5.1.5(c). Thus the countable family $\mathbf{B}'_d$ is an open Q-neighbourhood base for J' at d, and hence (F_X, J') is $Q\text{-}C_1$.

Similarly if $\mathbf{B}$ is a countable base for J, then the family $\mathbf{B}' = \{\underset{X}{\text{res}}\, B : B \in \mathbf{B}\}$ is a base for J'. Thus, if (F_Y, J) is C_2^f, then so is (F_X, J'). $\square$

We now state a result that sums up previous propositions.

Propositions 6.2.12

Both quasi-first countability and fuzzy second countability are excellent.

Proof

Constructive goodness follows from the propositions of this section and goodness from propositions 3.1.26 and 3.1.25 respectively. □

Notice 6.2.13

We have shown that fuzzy first countability is neither good (proposition 3.1.27) nor constructively good (counterexample 6.1.8). However proofs similar to those given for quasi-first countability can be easily provided to show that fuzzy first countability is subtractive and additive and that a fuzzy continuous fuzzy open image of a C_1^f fts is C_1^f. Virtually the same proofs as for propositions 6.2.10 and 6.2.8 show respectively that the Lowen factors of a C_1^f fts are C_1^f and that if the product fts of a family $\mathbf{C}$ of fts's is C_1^f , then all but a countable number of members of $\mathbf{C}$ are trivial.

3. The Excellence of Fuzzy Sequentiality

Proposition 6.3.1

Fuzzy sequentiality is neither subtractive nor finitely multiplicative.

Proof

This follows from proposition 6.1.5 and the fact that classical sequentiality is neither subtractive (see example 1.8 in Franklin [16]) nor multiplicative (see example of section (2) in Antosik et al [1]). □

To investigate the behaviour of fuzzy sequentiality under addition and division, we need the following lemma.

Lemma 6.3.2

Let J_C be the coinduced fuzzy topology on F_Y for the family of functions $\Psi = \{f_\alpha : X_\alpha \longrightarrow Y, \alpha \in \Omega\}$ and the family of fts's $C = \{(F_{X_\alpha}, J_\alpha) : \alpha \in \Omega\}$ and let $D \subseteq F_Y$. Then, the J_C-sequential openness of D implies the J_α-sequential openness of $f_\alpha^{-1}[D]$, $\alpha \in \Omega$.

Proof

Let D be J_C-sequentially open. Let $\alpha \in \Omega$, and let $\langle e_n \rangle$ be a fuzzy sequence in X_α, $e_n \xrightarrow{J_\alpha} e$ and $e q f_\alpha^{-1}[D]$. Then by the fuz-continuity, and hence the sequential fuz-continuity, of

f_α from (F_{X_α}, J_α) into (F_Y, J_c), and by consequence 4.1.7(b)2, $f_\alpha(e_n) \xrightarrow{J_c} f_\alpha(e)qD$. So, there is an $N \in \mathbf{N}$, such that $\langle f_\alpha(e_n)\rangle q(N)D$, and hence again, by consequence 4.1.7(b)2, $\langle e_n\rangle q(N)f_\alpha^{-1}[D]$. Therefore, $f_\alpha^{-1}[D]$ is J_α-sequentially open. □

Proposition 6.3.3

Fuzzy sequentiality is coinductive.

Proof

Using the foregoing notations, assume that for every $\alpha \in \Omega$, (F_{X_α}, J_α) is sequential.

Let D be J_c-sequentially open. Then, by lemma 6.3.2, $f_\alpha^{-1}[D]$ is J_α-sequentially open, $\alpha \in \Omega$. But then for every $\alpha \in \Omega$, $f_\alpha^{-1}[D] \in J_\alpha$, and hence $D \in J_c$. Therefore (F_Y, J_c) is sequential^f. □

Corollary 6.3.4

Fuzzy sequentiality is both divisive and additive.

Proof

This follows from proposition 6.3.3. □

Corollary 6.3.5

A fuzzy continuous fuzzy open (closed) image of a sequential fts is sequential.

Proof

This follows from corollary 6.3.4 (the fact that fuzzy sequentiality is divisive) and proposition 5.2.7. □

Corollary 6.3.6

Lowen factors of a sequential product fts are sequential.

Proof

This follows from corollary 6.3.5 and proposition 5.1.13. □

Proposition 6.3.7

Fuzzy sequentiality is excellent.

Proof

Combine the goodness of fuzzy sequentiality (proposition 3.3.10) with its constructive goodness which follows from corollary 6.3.4 and proposition 6.3.1. □

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