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On Algebraic Singularities, Finite Graphs and D-Brane Gauge Theories: A String Theoretic Perspective

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Abstract

In this writing we shall address certain beautiful inter-relations between the construction of 4-dimensional supersymmetric gauge theories and resolution of algebraic singularities, from the perspective of String Theory. We review in some detail the requisite background in both the mathematics, such as orbifolds, symplectic quotients and quiver representations, as well as the physics, such as gauged linear sigma models, geometrical engineering, Hanany-Witten setups and D-brane probes.

We investigate aspects of world-volume gauge dynamics using D-brane resolutions of various Calabi-Yau singularities, notably Gorenstein quotients and toric singularities. Attention will be paid to the general methodology of constructing gauge theories for these singular backgrounds, with and without the presence of the NS-NS B-field, as well as the T-duals to brane setups and branes wrapping cycles in the mirror geometry. Applications of such diverse and elegant mathematics as crepant resolution of algebraic singularities, representation of finite groups and finite graphs, modular invariants of affine Lie algebras, etc. will naturally arise. Various viewpoints and generalisations of McKay's Correspondence will also be considered.

The present work is a transcription of excerpts from the first three volumes of the author's PhD thesis which was written under the direction of Prof. A. Hanany - to whom he is much indebted - at the Centre for Theoretical Physics of MIT, and which, at the suggestion of friends, he posts to the ArXiv pro hac vice; it is his sincerest wish that the ensuing pages might be of some small use to the beginning student.

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Præfatio et Agnitio

Forsan et haec olim meminisse iuvabit. *Vir. Aen. I.1.203*

N*ot that I merely owe this title to the font, my education, or the clime wherein I was born, as being bred up either to confirm those principles my parents instilled into my understanding, or by a general consent proceed in the religion of my country; but having, in my riper years and confirmed judgment, seen and examined all, I find myself obliged, by the principles of grace, and the law of mine own reason, to embrace no other name but this.*

So wrote Thomas Browne in *Religio Medici* of his conviction to his Faith. Thus too let me, with regard to that title of "Physicist," of which alas I am most unworthy, with far less wit but with equal devotion, confess my allegiance to the noble Cause of Natural Philosophy, which I pray that in my own riper years I shall embrace none other. Therefore prithee gentle reader, bear with this fond fool as he here leaves his rampaging testimony to your clemency.

Some nine years have past and gone, since when the good Professor H. Verlinde, of Princeton, first re-embraced me from my straying path, as Saul was upon the road to Damascus - for, Heaven forbid, that in the even greater folly of my youth I had once blindly fathomed to be my destiny the more pragmatic career of an Engineer (pray mistake me not, as I hold great esteem for this Profession, though had I pursued her my own heart and soul would have been greatly misplaced indeed) - to the Straight and Narrow path leading to Theoretical Physics, that Holy Grail of Science.

I have suffered, wept and bled sweat of labour. Yet the divine Bach reminds us in the Passion of Our Lord according to Matthew, "*Ja! Freilich will in uns das Fleisch und Blut zum Kreuz gezwungen sein; Je mehr es unsrer Seele gut, Je herber geht es ein.*" Ergo, I too have rejoiced, laughed and shed tears of jubilation. Such is the nature of Scientific Research, and indeed the grand *Principia Vitæ*. These past half of a decade has been constituted of thousands of nightly lucubrations, each a battle, each *une petite mort*, each with its *te Deum* and *Non Nobis Domine*. I carouse to these five years past, short enough to be one day deemed a mere passing period, long enough to have earned some silvery strands upon my idle rank.

And thus commingled, the *fructus labori* of these years past, is the humble work I shall present in the ensuing pages. I beseech you o gentle reader, to indulge its length, I regret to confess that what I lack in content I can only supplant with volume, what I lack in wit I can only distract with loquacity. To that great Gaussian principle of *Pauca sed Matura* let me forever bow in silent shame.

Yet the poorest offering does still beseech painstaking preparation and the lowliest work, a helping hand. How blessed I am, to have a flight souls aiding me in bearing the great weight!

For what is a son, without the wings of his parent? How blessed I am, to have my dear mother and father, my aunt DaYi and grandmother, embrace me with four-times compounded love! Every fault, a tear, every wrong, a guiding hand and every triumph, an exaltation.

For what is Dante, without his Virgil? How blessed I am, to have the perspicacious guidance of the good Professor Hanany, who in these 4 years has taught me so much! His ever-lit lamp and his ever-open door has been a beacon for home amidst the nightly storms of life and physics. In addition thereto, I am indebted to Professors Zwiebach, Freedman and Jaffe, together with all my honoured Professors and teachers, as well as the ever-supportive staff: J. Berggren, R. Cohen, S. Morley and E. Sullivan at the Centre for Theoretical Physics, to have brought me to my intellectual manhood.

For what is Damon, without his Pythias? How blessed I am, to have such multitudes of friends! I drink to their health! To the Ludwigs: my brother, mentor and colleague in philosophy and mathematics, J. S. Song and his JJFS; my brother and companion in wine and Existentialism, N. Moeller and his Marina. To my collaborators: my colleagues and brethren, B. Feng, I. Ellwood, A. Karch, N. Prezas and A. Uranga. To my brothers in Physics and remembrances past: I. Savonije and M. Spradlin, may that noble Nassau-Orange thread bind the colourless skeins of our lives. To my Spiritual counsellors: M. Serna and his ever undying passion for Physics, D. Matheu and his Franciscan soul, L. Pantelidis and his worldly wisdom, as well as the Schmidts and the Domesticity which they symbolise. To the fond memories of one beauteous adventuress Ms. M. R. Warden, who once wept with me at the times of sorrow and danced with me at the moments of delight. And to you all my many dear beloved friends whose names, though I could not record here, I shall each and all engrave upon my heart.

And so composed is a fledgling, through these many years of hearty battle, and amidst blood, sweat and tears was formed another grain of sand ashore the Vast Ocean of Unknown. Therefore at this eve of my reception of the title *Doctor Philosophiae*, though I myself could never dream to deserve to be called either "learned" or a "philosopher," I shall fast and pray, for henceforth I shall bear, as Atlas the weight of Earth upon his shoulders, the name "Physicist" upon my soul. And so I shall prepare for this my initiation into a Brotherhood of Dreamers, as an incipient neophyte intruding into a Fraternity of Knights, accoladed by the sword of *Regina Mathematica*, who dare to uphold that Noblest calling of *"Sapere Aude"*.

Let me then embrace, not with merit but with homage, not with arms eager but with knees bent, and indeed not with a mind deserving but with a heart devout, naught else but this dear cherished Title of "Physicist."

I call upon ye all, gentle readers, my brothers and sisters, all the Angels and Saints, and Mary, ever Virgin, to pray for me, *Dei Sub Numine*, as I dedicate this humble work and my worthless self,

Ad Catharinae Sanctae Alexandriae et Ad Majorem Dei Gloriam...

De Singularitatis Algebraicæ, Graphicæ Finitatis, & Theorica Mensuræ Branæ Dirichletiensis: Aspectus Theoricæ Chordæ, cum digressi super theorica campi chordae. Libellus in Quattuor Partibus, sub Auspicio CTP et LNS, MIT, atque DOE et NSF, sed potissimum, Sub Numine Dei.

Invocatio et Apologia

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We live in an Age of Dualism. The Absolutism which has so long permeated through Western Thought has been challenged in every conceivable fashion: from philosophy to politics, from religion to science, from sociology to aesthetics. The ideological conflicts, so often ending in tragedy and so much a theme of the twentieth century, had been intimately tied with the recession of an archetypal norm of undisputed Principles. As we enter the third millennium, the Zeitgeist is already suggestive that we shall perhaps no longer be victims but beneficiaries, that the uncertainties which haunted and devastated the proceeding century shall perhaps serve to guide us instead.

Speaking within the realms of Natural Philosophy, beyond the wave-particle duality or the Principle of Equivalence, is a product which originated in the 60's and 70's, a product which by now so well exemplifies a dualistic philosophy to its very core.

What I speak of, is the field known as String Theory, initially invented to explain the dual-resonance behaviour of hadron scattering. The dualism which I emphasise is more than the fact that the major revolutions of the field, string duality and D-branes, AdS/CFT Correspondence, etc., all involve dualities in a strict sense, but more so the fact that the essence of the field still remains to be defined. A chief theme of this writing shall be the dualistic nature of String theory as a scientific endeavour: it has thus far no experimental verification to be rendered physics and it has thus far no rigorous formulations to be considered mathematics. Yet String theory has by now inspired so much activity in both physics and mathematics that, to quote C. N. Yang in the early days of Yang-Mills theory, its beauty alone certainly merits our attention.

I shall indeed present you with breath-taking beauty; in Books I and II, I shall carefully guide the readers, be them physicists or mathematicians, to a preparatory journey to the requisite mathematics in Liber I and to physics in Liber II. These two books will attempt to review a tiny fraction of the many subjects developed in the last few decades in both fields in relation to string theory. I quote here a saying of E. Zaslow of which I am particularly fond, though it applies to me far more appropriately: in the Book on mathematics I shall be the physicist and the Book on physics, I the mathematician, so as to beg the reader to forgive my inexpertise in both.

Books III and IV shall then consist of some of my work during my very enjoyable stay at the Centre for Theoretical Physics at MIT as a graduate student. I regret that I shall tempt the readers with so much elegance in the first two books and yet lead them to so humble a work, that the journey through such a beautiful garden would end in such a witless swamp. And I take the opportunity to apologise again to the reader for the excruciating length, full of sound and fury and signifying nothing. Indeed as Saramago points out that the shortness of life is so incompatible with the verbosity of the world.

Let me speak no more and let our journey begin. Come then, ye Muses nine, and with strains divine call upon mighty Diane, that she, from her golden quiver may draw the arrow, to pierce my trembling heart so that it could bleed the ink with which I shall hereafter compose this my humble work...

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De Singularitatis Algebraicæ, Graphicæ Finitatis, & Theorica Mensuræ Branæ Dirichletiensis: Aspectus Theoricæ Chordæ

Chapter 1

INTROIT

The two pillars of twentieth century physics, General Relativity and Quantum Field Theory, have brought about tremendous progress in Physics. The former has described the macroscopic, and the latter, the microscopic, to beautiful precision. However, the pair, in and of themselves, stand incompatible. Standard techniques of establishing a quantum theory of gravity have met uncancellable divergences and unrenormalisable quantities.

As we enter the twenty-first century, a new theory, born in the mid-1970's, has promised to be a candidate for a Unified Theory of Everything. The theory is known as **String Theory**, whose basic tenet is that all particles are vibrational modes of strings of Plankian length. Such elegant structure as the natural emergence of the graviton and embedding of electromagnetic and large N dualities, has made the theory more and more attractive to the theoretical physics community. Moreover, concurrent with its development in physics, string theory has prompted enormous excitement among mathematicians. Hitherto unimagined mathematical phenomena such as Mirror Symmetry and orbifold cohomology have brought about many new directions in algebraic geometry and representation theory.

Promising to be a Unified Theory, string theory must incorporate the Standard Model of interactions, or minimally supersymmetric extensions thereof. The purpose of this work is to study various aspects of a wide class of gauge theories arising from string theory in the background of singularities, their dynamics, moduli spaces, duality transformations etc. as well as certain branches of associated mathematics. We will investigate how these gauge theories, of various supersymmetry and in various dimensions, arise as low-energy effective theories associated with hypersurfaces in String Theory known as D-branes.

It is well-known that the initial approach of constructing the real world from String Theory had been the compactification of the 10 dimensional superstring or the 10(26) dimensional heterotic string on Calabi-Yau manifolds of complex dimension three. These are complex manifolds described as algebraic varieties with Ricci-flat curvature so as to preserve supersymmetry. The resulting theories are $\mathcal{N} = 1$ supersymmetric gauge theories in 4 dimensions that would be certain minimal extensions of the Standard Model.

This paradigm has been widely pursued since the 1980's. However, we have a host of Calabi-Yau threefolds to choose from. The inherent length-scale of the superstring and deformations of the world-sheet conformal field theory, made such violent behaviour as topology changes in space-time natural. These changes connected vast classes of manifolds related by, notably, mirror symmetry. For the physics, these mirror manifolds which are markedly different mathematical objects, give rise to the same conformal field theory.

Physics thus became equivalent with respect to various different compactifications. Even up to this equivalence, the plethora of Calabi-Yau threefolds (of which there is still yet no classification) renders the precise choice of the compactification difficult to select. A standing problem then has been this issue of "vacuum degeneracy."

Ever since Polchinski's introduction of D-branes into the arena in the Second String Revolution of the mid-90's, numerous novel techniques appeared in the construction of gauge theories of various supersymmetries, as low-energy effective theories of the ten dimensional superstring and eleven dimensional M-theory (as well as twelve dimensional F-theory).

The natural existence of such higher dimensional surfaces from a theory of strings proved to be crucial. The Dp-branes as well as Neveu-Schwarz (NS) 5-branes are carriers of Ramond-Ramond and NS-NS charges, with electromagnetic duality (in

10-dimensions) between these charges (forms). Such a duality is well-known in supersymmetric field theory, as exemplified by the four dimensional Montonen-Olive Duality for $\mathcal{N} = 4$, Seiberg-Witten for $\mathcal{N} = 2$ and Seiberg's Duality for $\mathcal{N} = 1$. These dualities are closely associated with the underlying S-duality in the full string theory, which maps small string coupling to the large.

Furthermore, the inherent winding modes of the string includes another duality contributing to the dualities in the field theory, the so-called T-duality where small compactification radii are mapped to large radii. By chains of applications of S and T dualities, the Second Revolution brought about a unification of the then five disparate models of consistent String Theories: types I, IIA/B, Heterotic $E_8 \times E_8$ and Heterotic Spin$(32)/\mathbb{Z}_2$.

Still more is the fact that these branes are actually solutions in 11-dimensional supergravity and its dimensional reduction to 10. Subsequently proposals for the enhancement for the S and T dualities to a full so-called U-Duality were conjectured. This would be a symmetry of a mysterious underlying M-theory of which the unified string theories are but perturbative limits. Recently Vafa and collaborators have proposed even more intriguing dualities where such U-duality structure is intimately tied with the geometric structure of blow-ups of the complex projective 2-space, viz., the del Pezzo surfaces.

With such rich properties, branes will occupy a central theme in this writing. We will exploit such facts as their being BPS states which break supersymmetry, their dualisation to various pure geometrical backgrounds and their ability to probe sub-stringy distances. We will investigate how to construct gauge theories empowered with them, how to realise dynamical processes in field theory such as Seiberg duality in terms of toric duality and brane motions, how to study their associated open string states in bosonic string field theory as well as many interesting mathematics that emerge.

We will follow the thread of thought of the trichotomy of methods of fabricating low-energy effective super-Yang-Mills theories which soon appeared in quick succession in 1996, after the D-brane revolution.

One method was very much in the geometrical vein of compactification: the so-named **geometrical engineering** of Katz-Klemm-Lerche-Vafa. With branes of various dimensions at their disposal, the authors wrapped (homological) cycles in the Calabi-Yau with branes of the corresponding dimension. The supersymmetric cycles (i.e., cycles which preserve supersymmetry), especially the middle dimensional 3-cycles known as Special Lagrangian submanifolds, play a crucial rôle in Mirror Symmetry.

In the context of constructing gauge theories, the world-volume theory of the wrapped branes are described by dimensionally reduced gauge theories inherited from the original D-brane and supersymmetry is preserved by the special properties of the cycles. Indeed, at the vanishing volume limit gauge enhancement occurs and a myriad of supersymmetric Yang-Mills theories emerge. In this spirit, certain global issues in compactification could be addressed in the analyses of the local behaviour of the singularity arising from the vanishing cycles, whereby making much of the geometry tractable.

The geometry of the homological cycles, together with the wrapped branes, determine the precise gauge group and matter content. In the language of sheafs, we are studying the intersection theory of coherent sheafs associated with the cycles. We will make usage of these techniques in the study of such interesting behaviour as "toric duality."

The second method of engineering four dimensional gauge theories from branes was to study the world-volume theories of configurations of branes in 10 dimensions. Heavy use were made especially of the D4 brane of type IIA, placed in a specific position with respect to various D-branes and the solitonic NS5-branes. In the limit of low energy, the world-volume theory becomes a supersymmetric gauge theory in 4-dimensions.

Such configurations, known as **Hanany-Witten setups**, provided intuitive realisations of the gauge theories. Quantities such as coupling constants and beta functions were easily visualisable as distances and bending of the branes in the setup. Moreover, the configurations lived directly in the flat type II background and the intricacies involved in the curved compactification spaces could be avoided altogether.

The open strings stretching between the branes realise as the bi-fundamental and adjoint matter of the resulting theory while the configurations are chosen judiciously to break down to appropriate supersymmetry. Motions of the branes relative to each other correspond in the field theory to moving along various Coulomb and Higgs branches of the Moduli space. Such dynamical processes as the Hanany-Witten Effect of brane creation lead to important string theoretic realisations of Seiberg's duality.

We shall too take advantage of the insights offered by this technique of brane setups which make quantities of the product gauge theory easily visualisable.

The third method of engineering gauge theories was an admixture of the above two, in the sense of utilising both brane dynamics and singular geometry. This became known as the **brane probe** technique, initiated by Douglas and Moore. Stacks of parallel D-branes were placed near certain local Calabi-Yau manifolds; the world-volume theory, which would otherwise be the uninteresting parent $U(n)$ theory in flat space, was projected into one with product gauge groups, by the geometry of the singularity on the open-string sector.

Depending on chosen action of the singularity, notably orbifolds, with respect to the $SU(4)$ R-symmetry of the parent theory, various supersymmetries can be achieved. When we choose the singularity to be $SU(3)$ holonomy, a myriad of gauge theories of $\mathcal{N} = 1$ supersymmetry in 4-dimensions could be thus fabricated given local structures of the algebraic singularities. The moduli space, as solved by the vacuum conditions of D-flatness and F-flatness in the field theory, is then by construction, the Calabi-Yau singularity. In this sense space-time itself becomes a derived concept, as realised by the moduli space of a D-brane probe theory.

As Maldacena brought about the Third String Revolution with the AdS/CFT conjecture in 1997, new light shone upon these probe theories. Indeed the $SU(4)$ R-symmetry elegantly manifests as the $SO(6)$ isometry of the 5-sphere in the $AdS_5 \times S^5$ background of the bulk string theory. It was soon realised by Kachru, Morrison, Silverstein et al. that these probe theories could be harnessed as numerous checks for the correspondence between gauge theory and near horizon geometry.

Into various aspects of these probes theories we shall delve throughout the writing and attention will be paid to two classes of algebraic singularities, namely orbifolds and toric singularities,

With the wealth of dualities in String Theory it is perhaps of no surprise that the three methods introduced above are equivalent by a sequence of T-duality (mirror) transformations. Though we shall make extensive usage of the techniques of all three throughout this writing, focus will be on the latter two, especially the last. We shall elucidate these three main ideas: geometrical engineering, Hanany-Witten brane configurations and D-branes transversely probing algebraic singularities, respectively in Chapters 6, 7 and 8 of Book II.

The abovementioned, of tremendous interest to the physicist, is only half the story. In the course of this study of compactification on Ricci-flat manifolds, beautiful and unexpected mathematics were born. Indeed, our very understanding of classical geometry underwent modifications and the notions of "stringy" or "quantum" geometry emerged. Properties of algebro-differential geometry of the target space-time manifested as the supersymmetric conformal field theory on the world-sheet. Such delicate calculations as counting of holomorphic curves and intersection of homological cycles mapped elegantly to computations of world-sheet instantons and Yukawa couplings.

The mirror principle, initiated by Candelas et al. in the early 90's, greatly simplified the aforementioned computations. Such unforeseen behaviour as pairs of Calabi-Yau manifolds whose Hodge diamonds were mirror reflections of each other naturally arose as spectral flow in the associated world-sheet conformal field theory. Though we shall too make usage of versions of mirror symmetry, viz., the **local mirror**, this writing will not venture too much into the elegant inter-relation between the mathematics and physics of string theory through mirror geometry.

What we shall delve into, is the local model of Calabi-Yau manifolds. These are the *algebraic singularities* of which we speak. In particular we concentrate on **canonical Gorenstein** singularities that admit crepant resolutions to smooth Calabi-Yau varieties. In particular, attention will be paid to orbifolds, i.e., quotients of flat space by finite groups, as well as toric singularities, i.e., local behaviour of toric varieties near the singular point.

As early as the mid 80's, the string partition function of Dixon-Harvey-Vafa-Witten (DHVW) proposed a resolution of orbifolds then unknown to the mathematician and made elegant predictions on the Euler characteristic of orbifolds. These gave new directions to such remarkable observations as the **McKay Correspondence** and its generalisations to beyond dimension 2 and beyond du Val-Klein singularities. Recent work by Bridgeland, King, and Reid on the generalised McKay from the derived category of coherent sheafs also tied deeply with similar structures arising in D-brane technologies as advocated by Aspinwall, Douglas et al. Stringy orbifolds thus became a topic of pursuit by such noted mathematicians as Batyrev, Kontsevich and Reid.

Intimately tied thereto, were applications of the construction of certain hyper-Kähler quotients, which are themselves moduli spaces of certain gauge theories, as gravitational instantons. The works by Kronheimer-Nakajima placed the McKay Correspondence under the light of representation theory of *quivers*. Douglas-Moore's construction mentioned above for the orbifold gauge theories thus brought these quivers into a string theoretic arena.

With the technology of D-branes to probe sub-stringy distance scales, Aspinwall-Greene-Douglas-Morrison-Plesser made space-time a derived concept as moduli space of world-volume theories. Consequently, novel perspectives arose, in the understanding of the field known as Geometric Invariant Theory (GIT), in the light of gauge invariant operators in the gauge theories on the D-brane. Of great significance, was the realisation that the Landau-Ginzberg/Calabi-Yau correspondence in the linear sigma model of Witten, could be used to translate between the gauge theory as a world-volume theory and the moduli space as a **GIT quotient**.

In the case of toric varieties, the sigma-model fields corresponded nicely to generators of the homogeneous coördinate ring in the language of Cox. This provided us with a alternative and computationally feasible view from the traditional approaches to toric varieties. We shall take advantage of this fact when we deal with toric duality later on.

This work will focus on how the above construction of gauge theories leads to various intricacies in algebraic geometry, representation theory and finite graphs, and vice versa, how we could borrow techniques from the latter to address the physics of the former. In order to refresh the reader's mind on the requisite mathematics, Book I is devoted to a review on the relevant topics. Chapter 2 will be an overview of the geometry, especially algebraic singularities and Picard-Lefschetz theory. Also included will be a discussion on symplectic quotients as well as the special case of toric varieties. Chapter 3 then prepares the reader for the orbifolds, by reviewing the pertinent concepts from representation theory of finite groups. Finally in Chapter 4, a unified outlook is taken by studying quivers as well as the constructions of Kronheimer and Nakajima.

Thus prepared with the review of the mathematics in Book I and the physics in II, we shall then take the reader to Books III and IV, consisting of some of the author's work in the last four years at the Centre for Theoretical Physics at MIT.

We begin with the D-brane probe picture. In Chapters 9 and 11 we classify and study the singularities of the orbifold type by discrete subgroups of $SU(3)$ and $SU(4)$ [292, 294]. The resulting physics consists of catalogues of finite four dimensional Yang-Mills theories with 1 or 0 supersymmetry. These theories are nicely encoded by certain finite graphs known as **quiver diagrams**. This generalises the work of Douglas and Moore for abelian ALE spaces and subsequent work by Johnson-Meyers for all ALE spaces as orbifolds of $SU(2)$. Indeed McKay's Correspondence facilitates the ALE case; moreover the ubiquitous ADE meta-pattern, emerging in so many seemingly unrelated fields of mathematics and physics greatly aids our understanding.

In our work, as we move from two-dimensional quotients to three and four dimensions, interesting observations were made in relation to generalised McKay's Correspondences. Connections to Wess-Zumino-Witten models that are conformal field theories on the world-sheet, especially the remarkable resemblance of the McKay graphs from the former and fusion graphs from the latter were conjectured in [292]. Subsequently, a series of activities were initiated in [293, 297, 300] to attempt to address why weaker versions of the complex of dualities which exists in dimension two may persist in higher dimensions. Diverse subject matters such as symmetries of the modular invariant partition functions, graph algebras of the conformal field theory, matter content of the probe gauge theory and crepant resolution of quotient singularities all contribute to an intricate web of inter-relations. Axiomatic approaches such as the quiver and ribbon categories were also attempted. We will discuss these issues in Chapters 10, 12 and 13.

Next we proceed to address the T-dual versions of these D-brane probe theories in terms of Hanany-Witten configurations. As mentioned earlier, understanding these would greatly enlighten the understanding of how these gauge theories embed into string theory. With the help of orientifold planes, we construct the first examples of non-Abelian configurations for $\mathbb{C}^3$ orbifolds [295, 296]. These are direct generalisations of the well-known elliptic models and brane box models, which are a widely studied class of conformal theories. These constructions will be the theme for Chapters 14 and 15.

Furthermore, we discuss the steps towards a general method [302], which we dubbed as "stepwise projection," of finding Hanany-Witten setups for arbitrary orbifolds in Chapter 16. With the help of Frøbenius' induced representation theory, the stepwise procedure of systematically obtaining non-Abelian gauge theories from the Abelian theories, stands as a non-trivial step towards solving the general problem of T-dualising pure geometry into Hanany-Witten setups.

Ever since Seiberg and Witten's realisation that the NS-NS B-field of string theory, turned on along world-volumes of D-branes, leads to non-commutative field theories, a host of activity ensued. In our context, Vafa generalised the DHVW closed sector orbifold partition function to include phases associated with the B-field. Subsequently, Douglas and Fiol found that the open sector analogue lead to projective representation of the orbifold group.

This inclusion of the background B-field has come to be known as turning on **discrete torsion**. Indeed a corollary of a theorem due to Schur tells us that orbifolds of dimension two, i.e., the ALE spaces do not admit such turning on. This is in perfect congruence with the rigidity of the $\mathcal{N} = 2$ superpotential. For $\mathcal{N} = 0, 1$ theories however, we can deform the superpotential consistently and arrive at yet another wide class of field theories.

With the aid of such elegant mathematics as the Schur multiplier, covering groups and the Cartan-Leray spectral sequence, we systematically study how and when it is possible to arrive at these theories with discrete torsion by studying the projective representations of orbifold groups [301, 303] in Chapters 17 and 18.

Of course orbifolds, the next best objects to flat (complex-dimensional) space, are but one class of local Calabi-Yau singularities. Another intensively studied class of algebraic varieties are the so-called toric varieties. As finite group representation theory is key to the former, combinatorial geometry of convex bodies is key to the latter. It is pleasing to have such powerful interplay between such esoteric mathematics and our gauge theories.

We address the problem of constructing gauge theories of a D-brane probe on toric singularities [298] in Chapter 19. Using the technique of partial resolutions pioneered by Douglas, Greene and Morrison, we formalise a so-called "Inverse Algorithm" to Witten's gauged linear sigma model approach and carefully investigate the type of theories which arise given the type of toric singularity.

Harnessing the degree of freedom in the toric data in the above method, we will encounter a surprising phenomenon which we call **Toric Duality**. [306]. This in fact gives us an algorithmic technique to engineer gauge theories which flow to the same fixed point in the infra-red moduli space. The manifestation of this duality as Seiberg Duality for $\mathcal{N} = 1$ [308] came as an additional bonus. Using a combination of field theory calculations, Hanany-Witten-type of brane configurations and the intersection theory of the mirror geometry [312], we check that all the cases produced by our algorithm do indeed give Seiberg duals and conjecture the validity in general [313]. These topics will constitute Chapters 20 and 21.

All these intricately tied and inter-dependent themes of D-brane dynamics, construction of four-dimensional gauge theories, algebraic singularities and quiver graphs, will be the subject of this present writing.

I

LIBER PRIMUS: Invocatio Mathematicæ

Chapter 2

Algebraic and Differential Geometry

Nomenclature

Unless otherwise stated, we shall adhere to the following notations throughout the writing:

X	Complex analytic variety
T_pX, T_p^*X	Tangent and cotangent bundles (sheafs) of X at point p
$\mathcal{O}(X)$	Sheaf of analytic functions on X
$\mathcal{O}^*(X)$	Sheaf of non-zero analytic functions on X
$\Gamma(X,\mathcal{O})$	Sections of the sheaf (bundle) $\mathcal{O}$ over X
$\Omega^{p,q}(X)$	Dolbeault (p,q)-forms on X
ω_X	The canonical sheaf of X
$f:\tilde{X}\to X$	Resolution of the singularity X
$\mathfrak{g}=Lie(G)$	The Lie Algebra of the Lie group G
$\widetilde{\mathfrak{g}}$	The Affine extension of $\mathfrak{g}$
$\mu: M\to Lie(G)^*$	Moment map associated with the group G
$\mu^{-1}(c)//G$	Symplectic quotient associated with the moment map μ
$\|G\|$	The order of the finite group G
$\chi_\gamma^{(i)}(G)$	Character for the i-th irrep in the γ-th conjugacy class of G

As the subject matter of this work is on algebraic singularities and their applications to string theory, what better place to commence our mathematical invocations indeed, than a brief review on some rudiments of the vast field of singularities in algebraic varieties. The material contained herein shall be a collage from such canonical texts as [1, 2, 3, 4], to which the reader is highly recommended to refer.

2.1 Singularities on Algebraic Varieties

Let M be an m-dimensional complex algebraic variety; we shall usually deal with projective varieties and shall take M to be $\mathbb{P}^m$, the complex projective m-space, with projective coördinates $(z_1, \ldots, z_m) = [Z_0 : Z_2 : \ldots : Z_m] \in \mathbb{C}^{m+1}$. In general, by Chow's Theorem, any analytic subvariety X of M can be locally given as the zeores of a finite collection of holomorphic functions $g_i(z_1, \ldots, z_m)$. Our protagonist shall then be the variety $X := \{z | g_i(z_1, \ldots, z_m) = 0 \;\; \forall \, i = 1, \ldots, k\}$, especially the singular points thereof. The following definition shall distinguish such points for us:

DEFINITION 2.1.1 *A point $p \in X$ is called a smooth point of X if X is a submanifold of M near p, i.e., the Jacobian $\mathcal{J}(X) := \left(\frac{\partial g_i}{\partial z_j}\right)_p$ has maximal rank, namely k.*

Denoting the locus of smooth points as X^*, then if $X = X^*$, X is called a smooth variety. Otherwise, a point $s \in V \setminus V^*$ is called a **singular point**.

Given such a singularity s on a X, the first exercise one could perform is of course its **resolution**, defined to be a birational morphism $f : \tilde{X} \to X$ from a nonsingular variety $\tilde{X}$. The preimage $f^{-1}(s) \subset \tilde{X}$ of the singular point is called the **exceptional divisor** in $\tilde{X}$. Indeed if X is a projective variety, then if we require the resolution f to be projective (i.e., it can be composed as $\tilde{X} \to X \times \mathbb{P}^N \to X$), then $\tilde{X}$ is a projective variety.

The singular variety X, of (complex) dimension n, is called *normal* if the structure sheafs obey $\mathcal{O}_X = f^* \mathcal{O}_{\tilde{X}}$. We henceforth restrict our attention to normal varieties. The point is that as a topological space the normal variety X is simply the quotient

$$X = \tilde{X} / \sim,$$

where $\sim$ is the equivalence which collapses the exceptional divisor to a point[1], the so-called process of blowing down. Indeed the reverse, where we replace the singularity s by a set of directions (i.e., a projective space), is called blowing up . As we shall mostly concern ourselves with Calabi-Yau manifolds (CY) of dimensions 2 and 3, of the uttermost importance will be exceptional divisors of dimension 1, to these we usually refer as **$\mathbb{P}^1$-blowups**.

Now consider the **canonical divisors** of $\tilde{X}$ and X. We recall that the canonical divisor K_X of X is any divisor in the linear equivalence (differing by principal divisors) class as the canonical sheaf ω_X, the n-th (hence maximal) exterior power of the sheaf of differentials. Indeed for X Calabi-Yau, K_X is trivial. In general the canonical sheaf of the singular variety and that of its resolution $\tilde{X}$ are not so naïvely related but differ by a term depending on the exceptional divisors E_i:

$$K_{\tilde{X}} = f^*(K_X) + \sum_i a_i E_i.$$

The term $\sum\limits_i a_i E_i$ is a formal sum over the exceptional divisors and is called the *discrepancy* of the resolution and the values of the numbers a_i categorise some commonly encountered subtypes of singularities characterising X, which we tabulate below:

$a_i \geq 0$	canonical	$a_i > 0$	terminal
$a_i \geq -1$	log canonical	$a_i > -1$	log terminal

The type which shall be pervasive throughout this work will be the canonical singularities. In the particular case when all $a_i = 0$, and the discrepancy term vanishes, we have what is known as a **crepant resolution**. In this case the canonical sheaf of the resolution is simply the pullback of that of the singularity, when the latter is trivial, as in the cases of orbifolds which we shall soon see, the former remains trivial and hence Calabi-Yau. Indeed crepant resolutions always exists for dimensions 2 and 3, the situations of our interest, and are related by flops. Although in dimension 3, the

[1]And so X has the structure sheaf $f^*\mathcal{O}_{\tilde{X}}$, the set of regular functions on $\tilde{X}$ which are constant on $f^{-1}(s)$.

resolution may not be unique (q.v. e.g. [5]). On the other hand, for terminal singularities, any resolution will change the canonical sheaf and such singular Calabi-Yau's will no longer have resolutions to Calabi-Yau manifolds.

In this vein of discussion on Calabi-Yau's, of the greatest relevance to us are the so-called[2] **Gorenstein** singularities , which admit a nowhere vanishing global holomorphic n-form on $X \setminus s$; these are then precisely those singularities whose resolutions have the canonical sheaf as a trivial line bundle, or in other words, these are the local Calabi-Yau singularities.

Gorenstein canonical singularities which admit crepant resolutions to smooth Calabi-Yau varieties are therefore the subject matter of this work.

2.1.1 Picard-Lefschetz Theory

We have discussed blowups of singularities in the above, in particular $\mathbb{P}^1$-blowups. A most useful study is when we consider the vanishing behaviour of these S^2-cycles. Upon this we now focus. Much of the following is based on [6]; The reader is also encouraged to consult e.g. [7, 56] for aspects of Picard-Lefschetz monodromy in string theory.

Let X be an n-fold, and $f : X \to U \subset \mathbb{C}$ a holomorphic function thereupon. For our purposes, we take f to be the embedding equation of X as a complex algebraic variety (for simplicity we here study a hypersurface rather than complete intersections). The singularities of the variety are then, in accordance with Definition 2.1.1, $\{\vec{x}|f'(\vec{x}) = 0\}$ with $\vec{x} = (x_1, ..., x_n) \in M$. f evaluated at these critical points $\vec{x}$ is called a critical value of f.

We have level sets $F_z := f^{-1}(z)$ for complex numbers z; these are $n-1$ dimensional varieties. For any non-critical value z_0 one can construct a loop γ beginning and ending at z_0 and encircling no critical value. The map $h_\gamma : F_{z_0} \to F_{z_0}$, which generates

[2]The definition more familiar to algebraists is that a singularity is Gorenstein if the local ring is a Gorenstein ring, i.e., a local Artinian ring with maximal ideal m such that the annihilator of m has dimension 1 over A/m. Another commonly encountered terminology is the $\mathbb{Q}$-Gorenstein singularity; these have $\Gamma(X \setminus p, K_X^{\otimes n})$ a free $\mathcal{O}(X)$-module for some finite n and are cyclic quotients of Gorenstein singularities.

the monodromy as one cycles the loop, the main theme of Picard-Lefschetz Theory. In particular, we are concerned with the induced action $h_{\gamma *}$ on the homology cycles of F_{z_0}.

When f is Morse[3], in the neighbourhood of each critical point p_i , f affords the Taylor series $f(x_1,\ldots,x_n) = z_i + \sum_{j=1}^{n}(x_j - p_j)^2$ in some coordinate system. Now adjoin a critical value $z_i = f(p_i)$ with a non-critical value z_0 by a path $u(t) : t \in [0,1]$ which does not pass through any other critical value. Then in the level set $F_{u(t)}$ we fix sphere $S(t) = \sqrt{u(t) - z_i}S^{n-1}$ (with S^{n-1} the standard $(n-1)$-sphere $\{(x_1,\ldots,x_n) : |x|^2 = 1, \mathrm{Im} x_i = 0\}$. In particular $S(0)$ is precisely the critical point p_i. Under these premises, we call the homology class $\Delta \in H_{n-1}(F_{z_0})$ in the non-singular level set F_{z_0} represented by the sphere $S(1)$ the **Picard-Lefschetz vanishing cycle**.

Fixing z_0, we have a set of such cycles, one from each of the critical values z_i. Let us consider what are known as *simple loops*. These are elements of $\pi_1(U\backslash\{z_i\}, z_0)$, the fundamental group of loops based at z_0 and going around the critical values. For these simple loops τ_i we have the corresponding *Picard-Lefschetz monodromy operator*

$$h_i = h_{\tau_i *} : H_\bullet(F_{z_0}) \to H_\bullet(F_{z_0}).$$

On the other hand if $\pi_1(U\backslash\{z_i\}, z_0)$ is a free group then the cycles $\{\Delta_i\}$ are *weakly distinguished.*

The *point d'appui* is the Picard-Lefschetz Theorem which determines the monodromy of f under the above setup:

THEOREM 2.1.1 *The monodromy group of the singularity is generated by the Picard-Lefschetz operators h_i, corresponding to a weakly distinguished basis $\{\Delta_i\} \subset H_{n-1}$ of the non-singular level set of f near a critical point. In particular for any cycle $a \in H_{n-1}$ (no summation in i)*

$$h_i(a) = a + (-1)^{\frac{n(n+1)}{2}}(a \circ \Delta_i)\Delta_i.$$

[3]That is to say, at all critical points x_i, the Hessian $\frac{\partial f}{\partial x_i \partial x_j}$ has non-zero determinant and all critical values $z_i = f(x_i)$ are distinct.

2.2 Symplectic Quotients and Moment Maps

We have thus far introduced canonical algebraic singularities and monodromy actions on exceptional $\mathbb{P}^1$-cycles. The spaces we shall be concerned are Kähler (Calabi-Yau) manifolds and therefore naturally we have more structure. Of uttermost importance, especially when we encounter moduli spaces of certain gauge theories, is the symplectic structure.

DEFINITION 2.2.2 *Let M be a complex algebraic variety, a* symplectic form ω *on M is a holomorphic 2-form, i.e.* $\omega \in \Omega^2(M) = \Gamma(M, \bigwedge^2 T^*M)$*, such that*

- ω *is closed:* $d\omega = 0$;
- ω *is non-degenerate:* $\omega(X,Y) = 0$ *for any* $Y \in T_pM \Rightarrow X = 0$.

Therefore on the *symplectic manifold* (M, ω) (which by the above definition is locally a complex symplectic vector space, implying that $\dim_{\mathbb{C}} M$ is even) ω induces an isomorphism between the tangent and cotangent bundles by taking $X \in TM$ to $i_X(\omega) := \omega(X, \cdot) \in \Omega^1(M)$. Indeed for any global analytic function $f \in \mathcal{O}(M)$ we can obtain its differential $df \in \Omega^1(M)$. However by the (inverse map of the) above isomorphism, we can define a vector field X_f, which we shall call the *Hamiltonian* vector field associated to f (a scalar called the Hamiltonian). In the language of classical mechanics, this vector field is the generator of infinitesimal canonical transformations[4]. In fact, $[X_f, X_g]$, the commutator between two Hamiltonian vector fields is simply $X_{\{f,g\}}$, where $\{f, g\}$ is the familiar Poisson bracket.

The vector field X_f is actually symplectic in the sense that

$$L_{X_f}\omega = 0,$$

where L_X is the Lie derivative with respect to the vector field X. This is so since $L_{X_f}\omega = (d \circ i_{X_f} + i_{X_f} \circ d)\omega = d^2 f + i_{X_f} d\omega = 0$. Let $H(M)$ be the Lie subalgebra

[4]If we were to write local coördinates (p_i, q_i) for M, then $\omega = \sum_i dq_i \wedge dq_i$ and the Hamiltonian vector field is $X_f = \sum_i \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - (p_i \leftrightarrow q_i)$ and our familiar Hamilton's Equations of motion are $i_{X_f}(\omega) = \omega(X_f, \cdot) = df$.

of Hamiltonian vector fields (of the tangent space at the identity), then we have an obvious exact sequence of Lie algebras (essentially since energy is defined up to a constant),

$$0 \to \mathbb{C} \to \mathcal{O}(M) \to H(M) \to 0,$$

where the Lie bracket in $\mathcal{O}(M)$ is the Poisson bracket.

Having presented some basic properties of symplectic manifolds, we proceed to consider quotients of such spaces by certain equivariant actions. We let G be some algebraic group which acts symplectically on M. In other words, for the action g^* on $\Omega^2(M)$, induced from the action $m \to gm$ on the manifold for $g \in G$, we have $g^*\omega = \omega$ and so the symplectic structure is preserved. The infinitesimal action of G is prescribed by its Lie algebra, acting as symplectic vector fields; this gives homomorphisms $k : Lie(G) \to H(M)$ and $\tilde{k} : Lie(G) \to \mathcal{O}(M)$. The action of G on M is called Hamiltonian if the following modification to the above exact sequence commutes

$$\begin{array}{ccccccc} 0 \to \mathbb{C} \to & \mathcal{O}(M) & \to H(M) & \to & 0 \\ & \tilde{k} \nwarrow & \uparrow k & & \\ & & Lie(G) & & \end{array}$$

DEFINITION 2.2.3 *Any such Hamiltonian G-action on M gives rise to a G-equivariant* **Moment Map** $\mu : M \to Lie(G)^*$ *which corresponds*[5] *to the map $\tilde{k}$ and satisfies*

$$k(A) = X_{A\circ\mu} \quad \textit{for any} \quad A \in Lie(G),$$

i.e., $d(A \circ \mu) = i_{k(A)}\omega$.

Such a definition is clearly inspired by the Hamilton equations of motion as presented in Footnote 4. We shall not delve into many of the beautiful properties of the moment map, such as when G is translation in Euclidean space, it is nothing more than momentum, or when G is rotation, it is simply angular momentum; for what we shall interest ourselves in the forthcoming, we are concerned with a crucial property of the moment map, namely the ability to form certain smooth quotients.

[5] *Because* $\hom(Lie(G), \hom(M, \mathbb{C})) = \hom(M, Lie(G)^*)$.

Let $\mu : M \to Lie(G)^*$ be a moment map and $c \in [Lie(G)^*]^G$ be the G-invariant subalgebra of $Lie(G)^*$ (in other words the co-centre), then the equivariance of μ says that G acts on the fibre $\mu^{-1}(c)$ and we can form the quotient of the fibre by the group action. This procedure is called the *symplectic quotient* and the subsequent space is denoted $\mu^{-1}(c)//G$. The following theorem guarantees that the result still lies in the category of algebraic varieties.

THEOREM 2.2.2 *Assume that G acts freely on $\mu^{-1}(c)$, then the* **symplectic quotient** *$\mu^{-1}(c)//G$ is a symplectic manifold, with a unique symplectic form $\bar{\omega}$, which is the pullback of the restriction of the symplectic form on M $\omega|_{\mu^{-1}(c)}$; i.e., $\omega|_{\mu^{-1}(c)} = q^*\bar{\omega}$ if $q : \mu^{-1}(c) \to \mu^{-1}(c)//G$ is the quotient map.*

A most important class of symplectic quotient varieties are the so-called toric varieties. These shall be the subject matter of the next section.

2.3 Toric Varieties

The types of algebraic singularities with which we are most concerned in the ensuing chapters in Physics are quotient and toric singularities. The former are the next best thing to flat spaces and will constitute the topic of the Chapter on finite groups. For now, having prepared ourselves with symplectic quotients from the above section, we give a lightening review on the vast subject matter of toric varieties, which are the next best thing to tori. The reader is encouraged to consult [10, 11, 12, 13, 14] as canonical mathematical texts as well as [17, 18, 19] for nice discussions in the context of string theory.

As a holomorphic quotient, a toric variety is simply a generalisation of the complex projective space $\mathbb{P}^d := (\mathbb{C}^{d+1}\ \{0\})/\mathbb{C}^*$ with the $\mathbb{C}^*$-action being the identification $x \sim \lambda x$. A toric variety of complex dimension d is then the quotient

$$(\mathbb{C}^n \setminus F)/\mathbb{C}^{*(n-d)}.$$

Here the $\mathbb{C}^{*(n-d)}$-action is given by $x_i \sim \lambda_a^{Q_i^a} x_i$ ($i = 1, \ldots, n; a = 1, \ldots, n-d$) for some integer matrix (of charges) Q_i^a. Moreover, $F \in \mathbb{C}^n \setminus \mathbb{C}^{*n}$ is a closed set of points one must remove to make the quotient well-defined (Hausdorff).

In the language of symplectic quotients, we can reduce the geometry of such varieties to the combinatorics of certain convex sets.

2.3.1 The Classical Construction

Before discussing the quotient, let us first outline the standard construction of a toric variety. What we shall describe is the classical construction of a toric variety from its defining fan, due originally to MacPherson. Let $N \simeq \mathbb{Z}^n$ be an integer lattice and let $M = \hom_{\mathbb{Z}}(N, \mathbb{Z}) \simeq \mathbb{Z}^n$ be its dual. Moreover let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ (and similarly for $M_{\mathbb{R}}$). Then

DEFINITION 2.3.4 *A (strongly convex) polyhedral cone σ is the positive hull of a finitely many vectors $v_1, \ldots, v_k$ in N, namely*

$$\sigma = \mathrm{pos}\{v_{i=1,\ldots,k}\} := \sum_{i=1}^{k} \mathbb{R}_{\geq 0} v_i.$$

From σ we can compute its **dual cone** $\sigma^\vee$ as

$$\sigma^\vee := \{u \in M_{\mathbb{R}} | u \cdot v \geq 0 \forall v \in \sigma\} .$$

Subsequently we have a finitely generated monoid

$$S_\sigma := \sigma^\vee \cap M = \{u \in M | u \cdot \sigma \geq 0\} .$$

We can finally associate maximal ideals of the monoid algebra of the polynomial ring adjoint S_σ to points in an algebraic (variety) scheme. This is the affine toric variety X_σ associated with the cone σ:

$$X_\sigma := \mathrm{Spec}(\mathbb{C}[S_\sigma]).$$

To go beyond affine toric varieties, we simply paste together, as coördinate patches, various X_{σ_i} for a collection of cones σ_i; such a collection is called a **fan** $\Sigma = \bigsqcup_i \sigma_i$ and we finally arrive at the general toric variety X_Σ.

As we are concerned with the singular behaviour of our varieties, the following definition and theorem shall serve us greatly.

DEFINITION 2.3.5 *A cone $\sigma = \text{pos}\{v_i\}$ is* **simplicial** *is all the vectors v_i are linearly independent; it is* **regular** *if $\{v_i\}$ is a $\mathbb{Z}$-basis for N. The fan Σ is* **complete** *if its cones span the entirety of $\mathbb{R}^n$ and it is regular if all its cones are regular and simplicial.*

Subsequently, we have

THEOREM 2.3.3 *X_Σ is compact iff Σ is complete; it is* **non-singular** *iff Σ is regular.*

Finally we are concerned with Calabi-Yau toric varieties, these are associated with what is know (recalling Section 1.1 regarding Gorenstein resolutions) as **Gorenstein cones**. It turns out that an n-dimensional toric variety satisfies the Ricci-flatness condition if all the endpoints of the vectors of its cones lie on a single $n-1$-dimensional hypersurface, in other words,

THEOREM 2.3.4 *The cone σ is called* Gorenstein *if there exists a vector $w \in N$ such that $\langle v_i, w \rangle = 1$ for all the generators v_i of σ. Such cones give rise to toric Calabi-Yau varieties.*

We refer the reader to [20] for conditions when Gorenstein cones admit crepant resolutions.

The name *toric* may not be clear from the above construction but we shall see now that it is crucial. Consider each point t the algebraic torus $T^n := (\mathbb{C}^*)^n \simeq N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq \hom(M, \mathbb{C}^*) \simeq \text{spec}(\mathbb{C}[M])$ as a group homomorphism $t : M \to \mathbb{C}^*$ and each point $x \in X_\sigma$ as a monoid homomorphism $x : S_\sigma \to \mathbb{C}$. Then we see that there is a *natural torus action* on the toric variety by the algebraic torus T^n as $x \to t \cdot x$ such that $(t \cdot x)(u) := t(u)x(u)$ for $u \in S_\sigma$. For $\sigma = \{0\}$, this action is nothing other than the group multiplication in $T^n = X_{\sigma=\{0\}}$.

2.3.2 The Delzant Polytope and Moment Map

How does the above tie in together with what we have discussed on symplectic quotients? We shall elucidate here. It turns out such a construction is canonically done for compact toric varieties embedded into projective spaces, so we shall deal more with *polytopes* rather than *polyhedral cones*. The former is simply a compact version of the latter and is a bounded set of points instead of extending as a cone. The argument below can be easily extended for fans and non-compact (affine) toric varieties. For now our toric variety X_Δ is encoded in a polytope Δ.

Let (X, ω) be a symplectic manifold of real dimension $2n$. Let $\tau : T^n \to \mathrm{Diff}(X, \omega)$ be a Hamiltonian action from the n-torus to vector fields on X. This immediately gives us a moment map $\mu : X \to \mathbb{R}^n$, where $\mathbb{R}^n$ is the dual of the Lie algebra for T^n considered as the Lie group $U(1)^n$. The image of μ is a polytope Δ, called a moment or **Delzant Polytope**. The inverse image, up to equivalence of the T^n-action, is then nothing but our toric variety X_Δ. But this is precisely the statement that

$$X_\Delta := \mu^{-1}(\Delta)//T^n$$

and the toric variety is thus naturally a symplectic quotient.

In general, given a convex polytope, Delzant's theorem guarantees that if the following conditions are satisfied, then the polytope is Delzant and can be used to construct a toric variety:

THEOREM 2.3.5 (Delzant) *A convex polytope $\Delta \subset \mathbb{R}^n$ is Delzant if:*

1. *There are n edges meeting at each vertex p_i;*

2. *Each edge is of the form $p_i + \mathbb{R}_{\geq 0} v_i$ with $v_{i=1,\dots,n}$ a basis of $\mathbb{Z}^n$.*

We shall see in Liber II and III, that the moduli space of certain gauge theories arise as toric singularities. In Chapter 5, we shall in fact see a third, physically motivated construction for the toric variety. For now, let us introduce another class of Gorenstein singularities.

Chapter 3

Representation Theory of Finite Groups

A wide class of Gorenstein canonical singularities are of course quotients of flat spaces by appropriate discrete groups. When the groups are chosen to be discrete subgroups of special unitary groups, i.e., the holonomy groups of Calabi-Yau's, and when crepant resolutions are admissible, these quotients are singular limits of CY's and provide excellent local models thereof. Such quotients of flat spaces by discrete finite subgroups of certain Lie actions, are called **orbifolds** (or V-manifolds, in their original guise in [21]). It is therefore a natural *point de départ* for us to go from algebraic geometry to a brief discussion on finite group representations (q.v. e.g. [22] for more details of which much of the following is a condensation).

3.1 Preliminaries

We recall that a **representation** of a finite group G on a finite dimensional (complex) vector space V is a homomorphism $\rho : G \to GL(V)$ to the group of automorphisms $GL(V)$ of V. Of great importance to us is the **regular** representation, where V is the vector space with basis $\{e_g | g \in G\}$ and G acts on V as $h \cdot \sum a_g e_g = \sum a_g e_{hg}$ for $h \in G$.

Certainly the corner-stone of representation theory is Schur's Lemma:

THEOREM 3.1.6 (Schur's Lemma) *If V and W are irreducible representations of G and $\phi : V \to W$ is a G-module homomorphism, then (a) either ϕ is an isomorphism or $\phi = 0$. If $V = W$, then ϕ is a homothety (i.e., a multiple of the identity).*

The lemma allows us to uniquely decompose any representation R into irreducibles $\{R_i\}$ as $R = R_1^{\oplus a_1} \oplus \ldots \oplus R_n^{\oplus a_n}$. The three concepts of regular representations, Schur's lemma and unique decomposition we shall extensively use later in Liber III. Another crucial technique is that of character theory into which we now delve.

3.2 Characters

If V is a representation of G, we define its character χ_V to be the $\mathbb{C}$-function on $g \in G$:

$$\chi_V(g) = \mathrm{Tr}(V(g)).$$

Indeed the character is a **class function**, constant on each conjugacy class of G; this is due to the cyclicity of the trace: $\chi_V(hgh^{-1}) = \chi_V(g)$. Moreover χ is a homomorphism from vector spaces to $\mathbb{C}$ as

$$\chi_{V \oplus W} = \chi_V + \chi_W \qquad \chi_{V \otimes W} = \chi_V \chi_W.$$

From the following theorem

THEOREM 3.2.7 *There are precisely the same number of conjugacy classes are there are irreducible representations of a finite group G,*

and the above fact that χ is a class function, we can construct a square matrix, the so-called **character table**, whose entries are the characters $\chi_\gamma^{(i)} := \mathrm{Tr}(R_i(\gamma))$, as i goes through the irreducibles R_i and γ, through the conjugacy classes. This table will be of tremendous computational use for us in Liber III.

The most important important properties of the character table are its two *orthogonality conditions*, the first of which is for the rows, where we sum over conjugacy classes:

$$\sum_{g\in G}\chi_g^{(i)*}\chi_g^{(j)*} = \sum_{\gamma=1}^{n} r_\gamma \chi_\gamma^{(i)*}\chi_\gamma^{(j)*} = |G|\delta_{ij},$$

where n is the number of conjugacy classes (and hence irreps) and r_γ the size of the γ-th conjugacy class. The other orthogonality is for the columns, where we sum over irreps:

$$\sum_{i=1}^{n}\chi_k^{(i)*}\chi_l^{(i)*} = \frac{|G|}{r_k}\delta_{kl}.$$

We summarise these relations as

THEOREM 3.2.8 *With respect to the inner product* $(\alpha,\beta) := \frac{1}{|G|}\sum\limits_{g\in G}\alpha^*(g)\beta(g) = \frac{1}{|G|}\sum\limits_{\gamma=1}^{n} r_\gamma\alpha^*(\gamma)\beta(\gamma)$*, the characters of the irreducible representations (i.e. the character table) are orthonormal.*

Many interesting corollaries follow. Of the most useful are the following. Any representation R is irreducible iff $(\chi_R,\chi_R) = 1$ and if not, then (χ_R,χ_{R_i}) gives the multiplicity of the decomposition of R into the i-th irrep.

For the *regular representation* R_r, the character is simply $\chi(g) = 0$ if $g \neq \mathbb{I}$ and it is $|G|$ when $g = \mathbb{I}$ (this is simply because any group element h other than the identity will permute $g \in G$ and in the vector basis e_g correspond to a non-diagonal element and hence do not contribute to the trace). Therefore if we were to decompose the R_r in to irreducibles, the i-th would receive a multiplicity of $(R_r, R_i) = \frac{1}{|G|}\chi_{R_i}(\mathbb{I})|G| = \dim R_i$. Therefore any irrep R_i appears in the regular representation precisely $\dim R_i$ times.

3.2.1 Computation of the Character Table

There are some standard techniques for computing the character table given a finite group G; the reader is referred to [23, 24, 25] for details.

For the j-th conjugacy class c_j, define a *class operator* $C_j := \sum\limits_{g\in c_j} g$, as a formal sum of group elements in the conjugacy class. This gives us a **class multiplication**:

$$C_jC_k = \sum_{g\in c_j;h\in c_k} gh = \sum_k c_{jkl}C_l,$$

where c_{jkl} are "fusion coefficients" for the class multiplication and can be determined from the multiplication table of the group G. Subsequently one has, by taking characters,

$$r_j r_k \chi_j^{(i)} \chi_k^{(i)} = \dim R_i \sum_{l=1}^{n} c_{jkl} r_l \chi_k^{(l)}.$$

These are n^2 equations in $n^2 + n$ variables $\{\chi_j^{(i)}; \dim R_i\}$. We have another n equations from the orthonormality $\frac{1}{|G|} \sum_{j=1}^{n} r_j |\chi_j^{(i)}|^2 = 1$; these then suffice to determine the characters and the dimensions of the irreps.

3.3 Classification of Lie Algebras

In Book the Third we shall encounter other aspects of representation theory such as induced and projective representation; we shall deal therewith accordingly. For now let us turn to the representation of Lie Algebras. It may indeed seem to the reader rather discontinuous to include a discussion on the the classification of Lie Algebras in a chapter touching upon finite groups. However the reader's patience shall soon be rewarded in Chapter 4 as well as Liber III when we learn that certain classifications of finite groups are intimately related, by what has become known as McKay's Correspondence, to that of Lie Algebras. Without further ado then let us simply present, for the sake of refreshing the reader's memory, the classification of complex Lie algebras.

Given a complex Lie algebra $\mathfrak{g}$, it has the *Levi Decomposition*

$$\mathfrak{g} = \mathrm{Rad}(\mathfrak{g}) \oplus \tilde{\mathfrak{g}} = \mathrm{Rad}(\mathfrak{g}) \oplus \bigoplus_i \mathfrak{g}_i,$$

where $\mathrm{Rad}(\mathfrak{g})$ is the radical, or the maximal solvable ideal, of $\mathfrak{g}$. The representation of such solvable algebras is trivial and can always be brought to $n \times n$ upper-triangular matrices by a basis change. On the other hand $\tilde{\mathfrak{g}}$ is semisimple and contains no nonzero solvable ideals. We can decompose $\tilde{\mathfrak{g}}$ further into a direct sum of **simple** Lie algebras $\mathfrak{g}_i$ which contain no nontrivial ideals. The $\mathfrak{g}_i$'s are then the nontrivial pieces of $\mathfrak{g}$.

The great theorem is then the complete classification of the complex simple Lie algebras due to Cartan, Dynkin and Weyl. These are the

- **Classical Algebras:** $A_n := \mathfrak{sl}_{n+1}(\mathbb{C})$, $B_n := \mathfrak{so}_{2n+1}(\mathbb{C})$, $C_n := \mathfrak{sp}_{2n}(\mathbb{C})$ and $D_n := \mathfrak{so}_{2n}(\mathbb{C})$ for $n = 1, 2, 3\ldots$;
- **Exceptional Algebras:** $E_{6,7,8}$, F_4 and G_2.

The Dynkin diagrams for these are given in Figure 3-1. The nodes are marked with the so-called comarks $a_i^\vee$ which we recall to be the expansion coefficients of the highest root θ into the simple coroots $\alpha_i^\vee := 2\alpha_i/|\alpha_i|^2$ (α_i are the simple roots)

$$\theta = \sum_i^r a_i^\vee \alpha_i^\vee,$$

where r is the rank of the algebra (or the number of nodes).

The dual Coxeter numbers are defined to be

$$c := \sum_i^r a_i^\vee + 1$$

and the **Cartan Matrix** is

$$C_{ij} := (\alpha_i, \alpha_j^\vee).$$

We are actually concerned more with **Affine** counterparts of the above simple algebras. These are central extensions of the above in the sense that if the commutation relation in the simple $\mathfrak{g}$ is $[T^a, T^b] = f_c^{ab} T^c$, then that in the affine $\widehat{\mathfrak{g}}$ is $[T_m^a, T_m^b] = f_c^{ab} T_{m+n}^c + kn\delta_{ab}\delta_{m,-n}$. The generators T^a of $\mathfrak{g}$ are seen to be generalised to $T_m^a := T^a \otimes t^m$ of $\widehat{\mathfrak{g}}$ by Laurent polynomials in t. The above concepts of roots etc. are directly generalised with the inclusion of the affine root. The Dynkin diagrams are as in Figure 3-1 but augmented with an extra affine node.

We shall see in Liber III that the comarks and the dual Coxeter numbers will actually show up in the dimensions of the irreducible representations of certain fi-

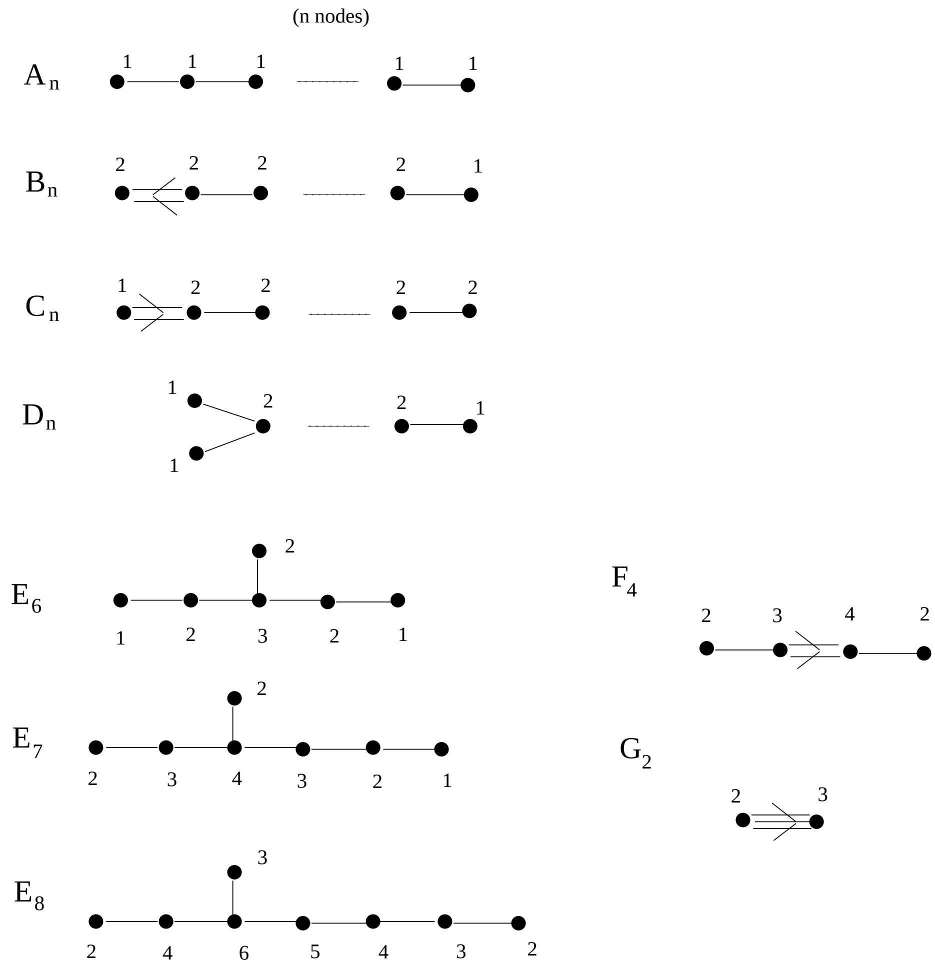

Figure 3-1: The Dynkin diagrams of the simple complex Lie Algebras; the nodes are labelled with the comarks.

nite groups. Moreover, the Cartan matrices will correspond to certain graphs constructable from the latter.

Chapter 4

Finite Graphs, Quivers, and Resolution of Singularities

We have addressed algebraic singularities, symplectic quotients and orbifolds in relation to finite group representations. It is now time to embark on a journey which would ultimately give a unified outlook. To do so we must involve ourselves with yet another field of mathematics, namely the theory of graphs.

4.1 Some Rudiments on Graphs and Quivers

As we shall be dealing extensively with algorithms on finite graphs in our later work on toric singularities, let us first begin with the fundamental concepts in graph theory. The reader is encouraged to consult such classic texts as [26, 27].

DEFINITION 4.1.6 *A finite graph is a triple (V, E, I) such that V, E are disjoint finite sets (respectively the set of vertices and edges) with members of E joining those of V according to the incidence relations I.*

The graph is *undirected* if for each edge e joining vertex i to j there is another edge e' joining j to i; it is directed otherwise. The graph is *simple* if there exists no *loops* (i.e., edges joining a vertex to itself). The graph is connected if any two vertices can be linked a series of edges, a so-called *walk*. Two more commonly encountered concepts are the *Euler* and *Hamilton* cycles, the first of which is walk returning to the beginning vertex which traverses each edge only once and each vertex at least once, while the latter, the vertices only once. Finally we call two graphs isomorphic if they are topologically homeomorphic; we emphasise the unfortunate fact that the graph isomorphism problem (of determining whether two graphs are isomorphic) is thus far unsolved; it is believed to be neither P nor NP-complete. This will place certain restrictions on our computations later.

We can represent a graph with n vertices and m edges by an $n \times n$ matrix, the so-named **adjacency matrix** a_{ij} whose ij-th entry is the number of edges from i to j. If the graph is simple, then we can also represent the graph by an **incidence matrix**, an $n \times m$ matrix d_{ia} in whose a-th columns there is a -1 (resp. 1) in row i (resp. row j) if there an a-th edge going from i to j. We emphasise that the graph must be *simple* for the incidence matrix to fully encapture its information. Later on in Liber III we will see this is a shortcoming when we are concerned with gauged linear sigma models.

4.1.1 Quivers

Now let us move onto a specific type of directed graphs, which we shall call a **quiver**. To any such a quiver (V, E, I) is associated the abelian category $\text{Rep}(V, E, I)$, of its representations (over say, $\mathbb{C}$). A (complex) *representation* of a quiver associates to every vertex $i \in V$ a vector space V_i and to any edge $i \xrightarrow{a} j$ a linear map $f_a : V_i \to V_j$. The vector $\vec{d} = (d_i := \dim_{\mathbb{C}} V_i)$ is called the dimension of the representation.

Together with its representation dimension, we can identify a quiver as a *labelled graph* (i.e., a graph with its nodes associate to integers) $(V, E, I; \vec{d})$. Finally, as we shall encounter in the case of gauge theories, one could attribute certain algebraic meaning to the arrows by letting them be formal variables which satisfy certain sets algebraic relations R; now we have to identify the quiver as a quintuple $(V, E, I; \vec{d}, R)$. These labelled directed finite quivers with relations are what concern string theorist the most.

In Liber III we shall delve further into the representation theory of quivers in relation to gauge theories, for now let us introduce two more preliminary concepts. We say a representation with dimension $\vec{d'}$ is a sub-representation of that with $\vec{d}$ if $(V, E, I; \vec{d'}) \hookrightarrow (V, E, I; \vec{d})$ is an injective morphism. In this case given a vector θ such that $\theta \cdot d = 0$, we call a representation with dimension d **θ-semistable** if for any subrepresentation with dimension d', $\theta \cdot d' \geq 0$; we call it θ-stable for the strict inequality. King's beautiful work [28] has shown that θ-stability essentially implies existence of solutions to certain BPS equations in supersymmetric gauge theories, the so-called F-D flatness conditions. But pray be patient as this discussion would have to wait until Liber II.

4.2 du Val-Kleinian Singularities

Having digressed some elements of graph and quiver theories, let us return to algebraic geometry. We shall see below a beautiful link between the theory of quivers and that of orbifold of $\mathbb{C}^2$.

First let us remind the reader of the classification of the quotient singularities of $\mathbb{C}^2$, these date as far back as F. Klein [30]. The affine equations of these so-called **ALE** (Asymptotically Locally Euclidean) singularities can be written in $\mathbb{C}[x, y, z]$ as

$$\boxed{\begin{array}{c} A_n : xy + z^n = 0 \\ D_n : x^2 + y^2 z + z^{n-1} = 0 \\ E_6 : x^2 + y^3 + z^4 = 0 \\ E_7 : x^2 + y^3 + yz^3 = 0 \\ E_8 : x^2 + y^3 + z^5 = 0. \end{array}}$$

We have not named these ADE by coincidence. The resolutions of such singularities were studied extensively by [31] and one sees in fact that the $\mathbb{P}^1$-blowups intersect precisely in the fashion of the Dynkin diagrams of the simply-laced Lie algebras ADE. For a illustrative review upon this elegant subject, the reader is referred to [9].

4.2.1 McKay's Correspondence

Perhaps it is a good point here to introduce the famous McKay correspondence, which will be a major part of Liber III. We shall be brief now, promising to expound upon the matter later.

Due to the remarkable observation of McKay in [32], there is yet another justification of naming the classification of the discrete finite subgroups Γ of $SU(2)$ as ADE. Take the defining representation R of Γ, and consider its tensor product with all the irreducible representations R_i:

$$R \otimes R_i = \bigoplus_j a_{ij} R_j.$$

Now consider a_{ij} as an adjacency matrix of a finite quiver with labelling the dimensions of the irreps. Then McKay's Theorem states that a_{ij} of the ADE finite group is precisely the Dynkin diagram of the *affine* ADE Lie algebra and the dimensions correspond to the comarks of the algebra. Of course for any finite group we can perform such a procedure, and we shall call the quiver so-obtained the **McKay Quiver**.

4.3 ALE Instantons, hyper-Kähler Quotients and McKay Quivers

It is the unique perspective of Kronheimer's work [33] which uses the methods of certain symplectic quotients in conjunction with quivers to study the resolution of the $\mathbb{C}^2$ orbifolds. We must digress one last time, to introduce instanton constructions.

4.3.1 The ADHM Construction for the E^4 Instanton

For the Yang-Mills equation $D^a F_{ab} := \nabla^a F_{ab} + [A^a, F_{ab}] = 0$ obtained from the action $L_{\rm YM} = -\frac{1}{4} F_{ab} F^{ab}$ with connexion A_a and field strength $F_{ab} := \nabla_{[a} A_{b]} + [A_a, A_b]$, we seek *finite action* solutions. These are known as **instantons**. A theorem due to Uhlenbeck [34] ensures that finding such an instanton solution in Euclidean space E^4

amounts to investigating G-bundles over S^4 since finite action requires the gauge field to be well-behaved at infinity and hence the one-point compactification of E^4 to S^4.

Such G-bundles, at least for simple G, are classified by integers, viz., the second Chern number of the bundle E, $c_2(E) := \frac{1}{8\pi}\int_{S^4} \mathrm{Tr}(F \wedge F)$; this is known as the *instanton number* of the gauge field. In finding the saddle points, so as to enable the evaluation of the Feynman path integral for L_{YM}, one can easily show that only the self-dual and self-anti-dual solutions $F_{ab} = \pm F_{ab}$ give rise to absolute minima in each topological class (i.e., for fixed instanton number). Therefore we shall focus in particular on the self-dual instantons. We note that self-duality implies solution to the Yang-Mills equation due to the Bianchi identity. Hence we turn our attention to self-dual gauge fields. There is a convenient theorem (see e.g. [35]) which translates the duality condition into the language of holomorphic bundles:

THEOREM 4.3.9 (Atiyah et al.) *There is a natural 1-1 correspondence between*

- *Self-dual SU(n) gauge fields*[1] *on U, an open set in S^4, and*
- *Holomorphic rank n vector bundles E over $\hat{U}$, an open set*[2] *in $\mathbb{P}^3$, such that (a) $E|_{\hat{x}}$ is trivial $\forall x \in U$; (b) $\det E$ is trivial; (c) E admits a positive real form.*

Therefore the problem of constructing self-dual instantons amounts to constructing a holomorphic vector bundle over $\mathbb{P}^3$. The key technique is due to the monad concept of Horrocks [36] where a sequence of vector bundles $F \stackrel{A}{\rightarrow} G \stackrel{B}{\rightarrow} H$ is used to produce the bundle E as a quotient $E = \ker B / \mathrm{Im} A$. Atiyah, Hitchin, Drinfeld and Manin then utilised this idea in their celebrated paper [37] to reduce the self-dual Yang-Mills instanton problem from partial differential equations to matrix equations; this is now known as the **ADHM construction**. Let V and W be complex vector spaces of dimensions $2k+n$ and k respectively and $A(Z)$ a linear map

$$A(Z) : W \rightarrow V$$

[1] *Other classical groups have also been done, but here we shall exemplify with the unitary groups.*

[2] *There is a canonical mapping from $x \in U$ to $\hat{x} \in \hat{U}$ into which we shall not delve.*

depending linearly on coordinates $\{Z^{a=0,1,2,3}\}$ of $\mathbb{P}^3$ as $A(Z) := A_a Z^a$ with A_a constant linear maps from W to V. For any subspace $U \subset V$, we define

$$U^0 := \{v \in V | (u, v) = 0 \ \ \forall u \in U\}$$

with respect to the symplectic (nondegenerate skew bilinear) form (,). Moreover we introduce antilinear maps $\sigma : W \to W$ with $\sigma^2 = 1$ and $\sigma : V \to V$ with $\sigma^2 = -1$ and impose the conditions

$$\begin{array}{ll} (1) & \forall Z^a \neq 0, U_Z := A(Z)W \text{ has dimension } k \text{ and is isotropic } (U_Z \subset U_Z^0); \\ (2) & \forall w \in W, \sigma A(Z)w = A(\sigma Z)\sigma w. \end{array} \tag{4.3.1}$$

Then the quotient space $E_Z := U_Z^0/U_Z$ of dimension $(2k+n-k)-k = n$ is precisely the rank n $SU(n)$-bundle E over $\mathbb{P}^3$ which we seek. One can further check that E satisfies the 3 conditions in theorem 4.3.9, whereby giving us the required self-dual instanton. Therefore we see that the complicated task of solving the non-linear partial differential equations for the self-dual instantons has been reduced to finding $(2k+n) \times k$ matrices $A(Z)$ satisfying condition (4.3.1), the second of which is usually known - though perhaps here not presented in the standard way - as the ADHM equation.

4.3.2 Moment Maps and Hyper-Kähler Quotients

The other ingredient we need is a generalisation of the symplectic quotient discussed in Section 1.2, the so-called Hyper-Kähler Quotients of Kronheimer [33] (see also the elucidation in [38]). A Riemannian manifold X with three covariantly constant complex structures $i := I, J, K$ satisfying the quaternionic algebra is called **Hyper-Kähler**[3]. From these structures we can define closed (hyper-)Kähler 2-forms:

$$\omega_i(V, W) := g(V, iW) \qquad \text{for} \quad i = I, J, K$$

[3] In dimension 4, simply-connectedness and self-duality of the Ricci tensor suffice to guarantee hyper-Kählerity.

mapping tangent vectors $V, W \in T(X)$ to $\mathbb{R}$ with g the metric tensor.

On a hyper-Kähler manifold with Killing vectors V (i.e., $\mathcal{L}_V g = 0$) we can impose **triholomorphicity**: $\mathcal{L}_V \omega_i = V^\nu (d\omega_i)_\nu + d(V^\nu(\omega_i)_\nu) = 0$ which together with closedness $d\omega_i = 0$ of the hyper-Kähler forms imply the existence of potentials μ_i, such that $d\mu_i = V^\nu(\omega_i)_\nu$. Since the dual of the Lie algebra $\mathfrak{g}$ of the group of symmetries G generated by the Killing vectors V is canonically identifiable with left-invariant forms, we have an induced map of such potentials:

$$\mu_i : X \to \mu_i^a \in \mathbb{R}^3 \otimes \mathfrak{g}^* \qquad i = 1,2,3; \quad a = 1, ..., \dim(G)$$

These maps are the (hyper-Kähler) **moment maps** and usually grouped as $\mu_{\mathbb{R}} = \mu_3$ and $\mu_{\mathbb{C}} = \mu_1 + i\mu_2$

Thus equipped, for any hyper-Kähler manifold Ξ of dimension $4n$ admitting k freely acting triholomorphic symmetries, we can construct another, X_ζ, of dimension $4n - 4k$ by the following two steps:

1. We have $3k$ moment maps and can thus define a level set of dimension $4n-3k$:

$$P_\zeta := \{\xi \in \Xi | \mu_i^a(\xi) = \zeta_i^a\};$$

2. When $\zeta \in \mathbb{R}^3 \otimes \mathrm{Centre}(\mathfrak{g}^*)$, P_ζ turns out to be a principal G-bundle over a new hyper-Kähler manifold

$$X_\zeta := P_\zeta / G \cong \{\xi \in \Xi | \mu_{\mathbb{C}}^a(\xi) = \zeta_{\mathbb{C}}^a\}/G^{\mathbb{C}}.$$

This above construction , where in fact the natural connection on the bundle $P_\zeta \to X_\zeta$ is self-dual, is the celebrated **hyper-Kähler quotient** construction [33].

Now we present a remarkable fact which connects these moment maps to the previous section. If we write (4.3.1) for $SU(n)$ groups into a (perhaps more standard) component form, we have the ADHM data

$$M := \{A, B; s, t^\dagger | A, B \in \mathrm{End}(V); s, t^\dagger \in \mathrm{Hom}(V, W)\},$$

with the ADHM equations

$$\begin{aligned}
[A,B]+ts &= 0;\\
([A,A^\dagger]+[B,B^\dagger]) - ss^\dagger + tt^\dagger &= 0.
\end{aligned}$$

Comparing with the hyper-Kähler forms $\omega_{\mathbb{C}} = \mathrm{Tr}(dA \wedge dB) + \mathrm{Tr}(dt \wedge ds)$ and $\omega_{\mathbb{R}} = \mathrm{Tr}(dA \wedge dA^\dagger + dB \wedge dB^\dagger) - \mathrm{Tr}(ds^\dagger \wedge ds - dt \wedge dt^\dagger)$ which are invariant under the action by $A, B, s, t^\dagger$, we immediately arrive at the following fact:

PROPOSITION 4.3.1 *The moment maps for the triholomorphic $SU(n)$ isometries precisely encode the ADHM equation for the $SU(n)$ self-dual instanton construction.*

4.3.3 ALE as a Hyper-Kähler Quotient

Kronheimer subsequently used the above construction for the case of X being the ALE space, i.e. the orbifolds $\mathbb{C}^2/(\Gamma \in SU(2))$. Let us first clarify some notations: $\Gamma \subset SU(2)$:= Finite discrete subgroup of $SU(2)$, i.e., A_n, D_n, or $E_{6,7,8}$; Q := The defining $\mathbb{C}^2$-representation; R := The regular $|\Gamma|$-dimensional complex representation; $R_{i=0,..,r}$:= irreps(Γ) of dimension n_i with 0 corresponding to the affine node (the trivial irrep); $(\ )_\Gamma$:= The Γ-invariant part; a_{ij} := The McKay quiver matrix for Γ, i.e., $Q \otimes R_i = \bigoplus\limits_j a_{ij} R_j$; T := A one dimensional quaternion vector space $= \{x_0 + x_1 i + x_2 j + x_3 k | x_i \in \mathbb{R}\}$; $\Lambda^+ T^*$:= The self-dual part of the second exterior power of the dual space = span{hyper-Kähler forms $\omega_{i=I,J,K}$}; $[y \wedge y] := (T^* \wedge T^*) \otimes [\mathrm{End}(V), \mathrm{End}(V)]$, for $y \in T^* \otimes \mathrm{End}(V)$; Endskew($R$) := The anti-Hermitian endomorphisms of R; Z := Trace free part of Centre(Endskew$_\Gamma(R)$); $G := \prod\limits_{i=1}^{r} U(n_i)$ = The group of unitary automorphisms of R commuting with the action of Γ, modded out by $U(1)$ scaling[4] $X_\zeta := \{y \in (T^* \otimes_{\mathbb{R}} \mathrm{Endskew}(R))_\Gamma | [y \wedge y]^+ = \zeta\}/G$ for generic$\zeta \in \Lambda^+ T^* \otimes Z$; $\mathcal{R}$:= The natural bundle over X_ζ, viz., $Y_\zeta \times_G R$, with $Y_\zeta := \{y | [y \wedge y]^+ = \zeta\}$; and finally ξ := A tautological vector-bundle endormorphism as an element in $T^* \otimes_{\mathbb{R}}$ Endskew($\mathcal{R}$).

[4]This is in the sense that the group $U(|\Gamma|)$ is broken down, by Γ-invariance, to $\prod\limits_{i=0}^{r} U(n_i)$, and then further reduced to G by the modding out.

We now apply the hyper-Kähler construction in the previous subsection to the ALE manifold

$$\boxed{\begin{array}{rl} \Xi := & (Q \otimes \mathrm{End}(R))_\Gamma = \{\xi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\} \\ = & \bigoplus\limits_{ij} a_{ij} \,\mathrm{hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) \\ \cong & (T^* \otimes_{\mathbb{R}} \mathrm{Endskew}(R))_\Gamma = \{\xi = \begin{pmatrix} \alpha & -\beta^\dagger \\ \beta & \alpha^\dagger \end{pmatrix}\} \end{array}}$$

where α and β are $|\Gamma| \times |\Gamma|$ matrices satisfying $\begin{pmatrix} R_\gamma \alpha R_{\gamma^{-1}} \\ R_\gamma \beta R_{\gamma^{-1}} \end{pmatrix} = Q_\gamma \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ for $\gamma \in \Gamma$. Of course this is simply the Γ-invariance condition; or in a physical context, the projection of the matter content on orbifolds. In the second line we have directly used the definition of the McKay matrices[5] a_{ij} and in the third, the canonical isomorphism between $\mathbb{C}^4$ and the quaternions.

The hyper-Kähler forms are $\omega_{\mathbb{R}} = \mathrm{Tr}(d\alpha \wedge d\alpha^\dagger) + \mathrm{Tr}(d\beta \wedge d\beta^\dagger)$ and $\omega_{\mathbb{C}} = \mathrm{Tr}(d\alpha \wedge d\beta)$, the moment maps, $\mu_{\mathbb{R}} = [\alpha, \alpha^\dagger] + [\beta, \beta^\dagger]$ and $\mu_{\mathbb{C}} = [\alpha, \beta]$. Moreover, the group of triholomorphic isometries is $G = \prod\limits_{i=1}^{r} U(n_i)$ with a trivial $U(n_0) = U(1)$ modded out. It is then the celebrated theorem of Kronheimer [33] that

THEOREM 4.3.10 (Kronheimer) *The space*

$$X_\zeta := \{\xi \in \Xi | \mu_i^a(\xi) = \zeta_i^a\}/G$$

is a smooth hyper-Kähler manifold of dimension[6] *four diffeomorphic to the resolution of the ALE orbifold* $\mathbb{C}^2/\Gamma$. *And conversely all ALE hyper-Kähler four-folds are obtained by such a resolution.*

We remark that in the metric, $\zeta_{\mathbb{C}}$ corresponds to the complex deformation while

[5] The steps are as follows: $(Q \otimes \mathrm{End}(R))_\Gamma = (Q \otimes \mathrm{Hom}(\bigoplus\limits_i R_i \otimes \mathbb{C}^{n_i}, \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})))_\Gamma = (\bigoplus\limits_{ijk} a_{ik} \mathrm{Hom}(R_k, R_j))_\Gamma \otimes \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j}) = \bigoplus\limits_{ij} a_{ij} \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_j})$ by Schur's Lemma.

[6] *Since* $\dim(X_\zeta) = \dim(\Xi) - 4\dim(G) = 2\sum\limits_{ij} a_{ij} n_i n_j - 4(|\Gamma| - 1) = 4|\Gamma| - 4|\Gamma| + 1 = 4$.

$\zeta_{\mathbb{R}} = 0$ corresponds to the singular limit $\mathbb{C}^2/\Gamma$.

4.3.4 Self-Dual Instantons on the ALE

Kronheimer and Nakajima [39] subsequently applied the ADHM construction on the ALE quotient constructed in the previous section. In analogy to the usual ADHM construction, we begin with the data $(V, W, \mathcal{A}, \Psi)$ such that

$$\begin{array}{rl} V, W := & \text{A pair of unitary } \Gamma\text{-modules of complex dimensions} \\ & k \text{ and } n \text{ respectively;} \\ A, B := & \Gamma\text{-equivariant endomorphisms of } V; \\ \mathcal{A} := & \begin{pmatrix} A & -B^\dagger \\ B & A^\dagger \end{pmatrix} \in (T^* \otimes_{\mathbb{R}} \text{Endskew}(R))_\Gamma = \bigoplus\limits_{ij} a_{ij} \text{Hom}(V_i, V_j); \\ s, t^\dagger := & \text{homomorphisms from } V \text{ to } W; \\ \Psi := & (s, t^\dagger) \in \text{Hom}(S \otimes V, W)_\Gamma. \end{array}$$

Let us explain the terminology above. By Γ-module we simply mean that V and W admit decompositions into the irreps of Γ in the canonical way: $V = \bigoplus\limits_i V_i \otimes R_i$ with $V_i \cong \mathbb{C}^{v_i}$ such that $k = \dim(V) = \sum\limits_i v_i n_i$ and similarly for W. By Γ-equivariance we mean the operators as matrices can be block-decomposed (into $n_i \times n_j$) according to the decomposition of the modules V and W. In the definition of $\mathcal{A}$ we have used the McKay matrices in the reduction of $(T^* \otimes_{\mathbb{R}} \text{Endskew}(R))_\Gamma$ in precisely the same fashion as was in the definition of Ξ. For Φ, we use something analogous to the standard spin-bundle decomposition of tangent bundles $T^* \otimes \mathbb{C} = S \otimes \bar{S}$, to positive and (dual) negative spinors S and $\bar{S}$. We here should thus identify S as the right-handed spinors and Q, the left-handed.

Finally we have an additional structure on X_ζ. Now since X_ζ is constructed as a quotient, with P_ζ as a principal G-bundle, we have an induced natural bundle $\mathcal{R} := P_\zeta \times_G R$ with trivial R fibre. From this we have a **tautological bundle** $\mathcal{T}$ whose endomorphisms are furnished by $\xi \in T^* \otimes_{\mathbb{R}} \text{Endskew}(\mathcal{R})$. This is tautological in the sense that $\xi \in \Xi$ and the points of the base X_ζ are precisely the endomorphisms of the fibre R.

On X_ζ we define operators $\mathcal{A}\otimes \mathrm{Id}_{\mathcal{T}}, \mathrm{Id}_V\otimes\xi$ and $\Psi\otimes\mathrm{Id}_{\mathcal{T}} : S\otimes V\otimes\mathcal{T}\to W\otimes\mathcal{T}$. Finally we define the operator (which is a $(2k+n)|\Gamma|\times 2k|\Gamma|$ matrix because S and Q are of complex dimension 2, V, of dimension k and $\mathcal{R}$ and $\mathcal{T}$, of dimension $|\Gamma|$)

$$\mathcal{D} := (\mathcal{A}\otimes\mathrm{Id} - \mathrm{Id}\otimes\xi)\oplus\Psi\otimes\mathrm{Id}$$

mapping $S\otimes V\otimes\mathcal{R}\to Q\otimes V\otimes\mathcal{T}\oplus W\otimes\mathcal{R}$. We can restrict this operator to the Γ-invariant part, viz., $\mathcal{D}_\Gamma$, which is now a $(2k+n)\times 2k$ matrix. The adjoint is given by

$$\mathcal{D}_\Gamma^\dagger : \left(\bar{Q}\otimes\bar{V}\otimes\mathcal{T}\right)_\Gamma\oplus\left(\bar{W}\otimes\mathcal{T}\right)_\Gamma\to S\otimes\left(\bar{V}\otimes\mathcal{T}\right)_\Gamma,$$

where $\bar{V},\bar{W}$ and $\bar{Q}$ denote the trivial (Cartesian product) bundle over X_ζ with fibres V,W and Q.

Now as with the $\mathbb{R}^4$ case, the moment maps encode the ADHM equations, except that instead of the right hand side being zero, we now have the deformation parametres ζ. In other words, we have $[\mathcal{A}\wedge\mathcal{A}]^+ + \{\Psi^\dagger,\Psi\} = -\zeta_V$, where $\{\Psi^\dagger,\Psi\}\in\Lambda^+T^*\otimes\mathrm{Endskew}(V)$ is the symmetrisation in the S indices and contracting in the W indices of $\Psi^\dagger\otimes\Psi$, and ζ_V is such that $\zeta_V\otimes\mathrm{Id}\in\Lambda^+T^*\otimes\mathrm{End}((V\otimes R)^\Gamma)$. In component form this reads

$$\begin{aligned}[A,B]+ts &= -\zeta_{\mathbb{C}};\\ ([A,A^\dagger]+[B,B^\dagger])-ss^\dagger+tt^\dagger &= \zeta_{\mathbb{R}},\end{aligned}\qquad(4.3.2)$$

where as before $\zeta = \bigoplus_{i=1}^{r}\zeta_i\mathrm{Id}_{v_i}\in\mathbb{R}^3\otimes Z$.

Thus equipped, the anti-self-dual[7] instantons can be constructed by the following theorem:

THEOREM 4.3.11 (Kronheimer-Nakajima) *For $\mathcal{A}$ and Ψ satisfying injectivity of $\mathcal{D}_\Gamma$ and (4.3.2), all anti-self-dual $U(n)$ connections of instanton number k, on ALE can be obtained as the induced connection on the bundle $E=\mathrm{Coker}(\mathcal{D}_\Gamma)$.*

[7]The self-dual ones are obtained by reversing the orientation of the bundle.

More explicitly, we take an orthonormal frame U of sections of $\mathrm{Ker}(\mathcal{D}_\Gamma^\dagger)$, i.e., a $(2k+n)\times n$ complex matrix such that $\mathcal{D}_\Gamma^\dagger U = 0$ and $U^\dagger U = \mathrm{Id}$. Then the required connection (gauge field) is given by

$$A_\mu = U^\dagger \nabla_\mu U.$$

4.3.5 Quiver Varieties

We can finally take a unified perspective, combining what we have explained concerning the construction of ALE-instantons as Hyper-Kähler quotients and the quivers for th orbifolds of $\mathbb{C}^2$. Given an $SU(2)$ quiver (i.e., a McKay quiver constructed out of Γ, a finite discrete subgroup of $SU(2)$) Q with edges $H = \{h\}$, vertices $\{1, 2, ..., r\}$, and beginning (resp. ends) of h as $\alpha(h)$ (resp. $\beta(h)$), we study the representation by associating vector spaces as follows: to each vertex q, we associate a pair of hermitian vector spaces V_q and W_q. We then define the complex vector space:

$$\begin{aligned} M(v,w) &:= \left(\bigoplus_{h\in H} \mathrm{Hom}(V_{\alpha(h)}, V_{\beta(h)})\right) \oplus \left(\bigoplus_{q=1}^{r} \mathrm{Hom}(W_q, V_q) \oplus \mathrm{Hom}(V_q, W_q)\right) \\ &:= \bigoplus_{h,q} \{B_h, i_q, j_q\} \end{aligned}$$

with $v := (\dim_{\mathbb{C}} V_1,\ ...\ , \dim_{\mathbb{C}} V_n)$ and $w := (\dim_{\mathbb{C}} W_1,\ ...\ , \dim_{\mathbb{C}} W_n)$ being vectors of dimensions of the spaces associated with the nodes.

Upon $M(v,w)$ we can introduce the action by a group

$$G := \prod_q U(V_q) : \{B_h, i_q, j_q\} \to \left\{ g_{\alpha(h)} B_h g_{\beta(h)}^{-1}, g_q i_q, j_q g_q^{-1} \right\}$$

with each factor acting as the unitary group $U(V_q)$. We shall be more concerned with $G' := G/U(1)$ where the trivial scalar action by an overall factor of $U(1)$ has been modded out.

In Q we can choose an orientation Ω and hence a signature for each (directed) edge h, viz., $\epsilon(h) = 1$ if $h \in \Omega$ and $\epsilon(h) = -1$ if $h \in \bar{\Omega}$. Hyper-Kähler moment maps are subsequently given by:

$$\begin{aligned}
\mu_{\mathbb{R}}(B,i,j) &:= \frac{i}{2}\left(\sum_{h\in H, q=\alpha(h)} B_h B_h^\dagger - B_{\bar{h}}^\dagger B_{\bar{h}} + i_q i_q^\dagger - j_q^\dagger j_q\right) \in \bigoplus_q \mathfrak{u}(V_q) := \mathfrak{g}, \\
\mu_{\mathbb{C}}(B,i,j) &:= \left(\sum_{h\in H, q=\alpha(h)} \epsilon(h) B_h B_{\bar{h}} + i_q j_q\right) \in \bigoplus_q \mathfrak{gl}(V_q) := \mathfrak{g}\otimes\mathbb{C}.
\end{aligned} \tag{4.3.3}$$

These maps (4.3.3) we recognise as precisely the ADHM equations in a different guise. Moreover, the center Z of $\mathfrak{g}$, being a set of scalar $r\times r$ matrices, can be identified with $\mathbb{R}^n$. For Dynkin graphs[8] we can then define R_+, the set of positive roots, $R_+(v)$, the positive roots bounded by v and D_θ, the wall defined by the root θ.

We rephrase Kronheimer's theorem as [39]:

THEOREM 4.3.12 *For the discrete subgroup $\Gamma \in SU(2)$, let $v = (n_0, n_1, ..., n_n)$, the vector of Dynkin labels of the Affine Dynkin graph associated with Γ and let $w = 0$, then for*[9] $\zeta := (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in \left\{\mathbb{R}^3\otimes Z\right\} \setminus \bigcup_{\theta\in\mathbb{R}_+\setminus\{n\}} \mathbb{R}^3\otimes D_\theta$*, the manifold*

$$X_\zeta := \{B \in M(v,0) | \mu(B) = \zeta\}//G'$$

is the smooth resolution of $\mathbb{C}^2/\Gamma$ with corresponding ALE metric.

For our purposes this construction induces a natural bundle which will give us the required instanton. In fact, we can identify $G' = \prod_{q\neq 0} U(V_q)$ as the gauge group over the non-Affine nodes and consider the bundle

$$\mathcal{R}_l = \mu^{-1}(\zeta)\times_{G'}\mathbb{C}^{n_l}$$

for $l = 1, ..., r$ indexing the non-Affine nodes where $\mathbb{C}^{n_l}$ is the space acted upon by the irreps of Γ (whose dimensions, by the McKay Correspondence, are precisely the Dynkin labels) such that $U(V_q)$ acts trivially (by Schur's Lemma) unless $q = l$. For the

[8] In general they are defined as $R_+ := \{\theta \in \mathbb{Z}^n_{\geq 0} | \theta^t\cdot C\cdot\theta \leq 2\}\backslash\{0\}$ for generalised Cartan matrix $C := 2I - A$ with A the adjacency matrix of the graph; $R_+(v) := \{\theta \in R_+ | \theta_q \leq v_q = \dim_{\mathbb{C}} V_q\ \forall q\}$ and $D_\theta := \{x \in \mathbb{R}^n | x\cdot\theta = 0\}$.

[9] *Z is the trace-free part of the centre and $\mu(B) = \zeta$ means, component-wise $\mu_{\mathbb{R}} = \zeta_{\mathbb{R}}$ and $\mu_{\mathbb{C}} = \zeta_{\mathbb{C}}$.*

affine node, we define $\mathcal{R}_0$ to be the trivial bundle (inspired by the fact that this node corresponds to the trivial principal 1-dimensional irrep of Γ). There is an obvious tautological bundle endomorphism:

$$\xi := (\xi_h) \in \bigoplus_{h \in H} \mathrm{Hom}(\mathcal{R}_{\alpha(h)}, \mathcal{R}_{\beta(h)}).$$

We now re-phrase the Kronheimer-Nakajima theorem above as

THEOREM 4.3.13 *The following sequence of bundle endomorphisms*

$$\bigoplus_q V_q \otimes \mathcal{R}_q \xrightarrow{\sigma} \left(\bigoplus_{h \in H} V_{\alpha(h)} \otimes \mathcal{R}_{\beta(h)} \right) \oplus \left(\bigoplus_q W_q \otimes \mathcal{R}_q \right) \xrightarrow{\tau} \bigoplus_q V_q \otimes \mathcal{R}_q,$$

where

$$\begin{array}{rcl} \sigma & := & \left(B_{\bar{h}} \otimes \mathrm{Id}_{\mathcal{R}_{\beta(h)}} + \epsilon(h) \mathrm{Id}_{V_{\alpha(h)}} \otimes \xi_h \right) \oplus \left(j_q \otimes \mathrm{Id}_{\mathcal{R}_q} \right) \\ \tau & := & \left(\epsilon(h) B_{\bar{h}} \otimes \mathrm{Id}_{\mathcal{R}_{\beta(h)}} - \mathrm{Id}_{V_{\alpha(h)}} \otimes \xi_{\bar{h}}, i_q \otimes \mathrm{Id}_{V_q} \right) \end{array}$$

is a complex (since the ADHM equation $\mu_{\mathbb{C}}(B,i,j) = -\zeta_{\mathbb{C}}$ implies $\tau\sigma = 0$) and the induced connection A on the bundle

$$E := \mathrm{Coker}(\sigma, \tau^\dagger) \subset \left(\bigoplus_{h \in H} V_{\alpha(h)} \otimes \mathcal{R}_{\beta(h)} \right) \oplus \left(\bigoplus_q W_q \otimes \mathcal{R}_q \right)$$

is anti-self-dual. And conversely all such connections are thus obtained.

We here illustrate the discussions above via explicit quiver diagrams; though we shall use the $\widehat{A_2}$ as our diagrammatic example, the generic structure should be captured. The quiver is represented in Figure 4-1 and the concepts introduced in the previous sections are elucidated therein. In the figure, the vector space V of dimension k is decomposed into $V_0 \oplus V_1 \oplus ... \oplus V_r$, each of dimension v_i and associated with the i-th node of Dynkin label $n_i = \dim(R_i)$ in the affine Dynkin diagram of rank r. This is simply the usual McKay quiver for $\Gamma \subset SU(2)$. Therefore we have $k = \sum\limits_i n_i v_i$.

To this we add the vector space W of dimension n decomposing similarly as $W = W_0 \oplus W_1 \oplus ... \oplus W_r$, each of dimension w_i and $n = \sum\limits_i n_i w_i$. Now we have the McKay quiver with extra legs. Between each pair of nodes V_{q_1} and V_{q_2} we have the

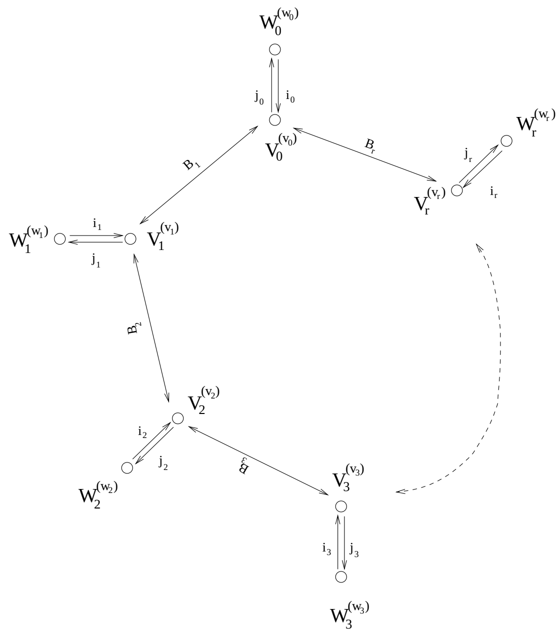

Figure 4-1: The Kronheimer-Nakajima quiver for $\mathbb{C}^2/A_n$, extending the McKay quiver to also encapture the information for the construction of the ALE instanton.

map B_h with h the edge between these two nodes. We note of course that due to McKay h is undirected and single-valence for $SU(2)$ thus making specifying merely one map between two nodes sufficient. Between each pair V_q and W_q we have the maps $i_q : W_q \to V_q$ and j_q, in the other direction. The group $U(k)$ is broken down to $(\prod_{q=0}^{r} U(v_q))/U(1)$. This is the group of Γ-compatible symplectic diffeomorphisms. This latter gauge group is our required rank $n = \dim(W)$ unitary bundle with anti-self-dual connection, i.e., an $U(n)$ instanton with instanton number $k = \dim(V)$.

Epilogue

Thus we conclude Liber I, our preparatory journey into the requisite mathematics. We have introduced canonical Gorenstein singularities and monodromies thereon. Thereafter we have studied symplectic structures one could impose, especially in the context of symplectic quotients and moment maps. As a powerful example of such quotients we have reviewed toric varieties.

We then digressed to the representation of finite groups, in preparation of studying a wide class of Gorenstein singularities: the orbifolds. We shall see in Liber III how all of the Abelian orbifolds actually afford toric descriptions. Subsequently we digressed again to the theory of finite graphs and quiver, another key constituent of this writing.

A unified outlook was finally performed in the last sections of Chapter 4 where symplectic quotients in conjunction with quivers were used to address orbifolds of $\mathbb{C}^2$, the so-called ALE spaces. With all these tools in hand, let us now proceed to string theory.

II

LIBER SECUNDUS: Invocatio Philosophiæ Naturalis

Chapter 5

Calabi-Yau Sigma Models and $\mathcal{N} = 2$ Superconformal Theories

Nomenclature

We have by now prepared the reader, in the spirit of the Landau School, with the requisite mathematics. Now let us move onto the theme of this writing: string theory. The following 4 chapters will serve as an introduction of the requisite background in physics. First, to parallel Liber I, ket us clarify some notations:

α'	String tension
l_s, g_s	String Length and Coupling
CY3	Calabi-Yau threefold
Dp	Dirichlet p-brane
$NS5$	Neveu-Schwarz 5-brane
g_{YM}	Yang-Mills coupling
GLSM	Gauged linear σ-model
LG	Landau-Ginsberg Theories
$\mathcal{N}$	Number of supersymmetries
VEV	Vacuum Expectation Value
ζ	Fayet-Illiopoulos Parametre

A key feature of the type II superstring is the 2-dimensional world-sheet $\mathcal{N} = 2$ superconformal field theory with central charge $c = 15$. In compactification down to $\mathbb{R}^4 \times CY3$, the difficult part to study is the $c = 9$ $\mathcal{N} = 2$ theory internal to the Calabi-Yau, the properties of which determine the $c = 6$ theory on the $\mathbb{R}^4$ that is ultimately to give our real world.

A main theme therefore, is the construction of the various $c = 9$ so-called "internal" $\mathcal{N} = 2$ superconformal theories. Three major subtypes have been widely studied (q.v. [18] for an excellent pedagogical review). These are

1. The non-linear sigma model, embedding the worldsheet, into the $CY3$ endowed with a metric $g_{\mu\nu}$ and anti-symmetric 2-form $B_{\mu\nu}$, with action

$$\frac{1}{\alpha'} \int_{w.s} (g_{\mu\nu} + B_{\mu\nu}) \partial^\mu X \partial^\nu X + \text{fermion};$$

2. The Landau-Ginsberg (LG) theory, constructed from chiral superfields $\Psi_i, \bar{\Psi}_i$, and with a holomorphic polynomial superpotential $W(\Psi_i)$ giving a unique vacuum. The action is an integral over the $\mathcal{N} = 2$ superspace

$$\int dz^2 d\theta^4 K(\Psi_i, \bar{\Psi}_i) + (W(\Psi_i) + h.c.).$$

   We usually start with a non-conformal case and let it flow to a superconformal fixed point into IR;

3. The minimal models, being rational conformal field theories with a finite number of primary fields (and $c < 1$ in the bosonic case or $c < 3/2$ in $\mathcal{N} = 1$), furnishing unitary highest-weight representations of the (super)-Virasoro algebra. These can then be tensored together to achieve $c = 9$.

Now the LG theories can be seen as explicit Langrangian realisation of tensor products of the minimal models [41]. On the other hand, the Gepner construction [42] relates the chiral primaries in the minimal models with coördinates in certain Calabi-Yau hypersurfaces, thereby relating 1 and 3. Hence we shall focus on the inter-relation between 1 and 2.

Indeed this inter-relation between LG theories and Calabi-Yau sigma models is what interests us most. The theme of this writing is to study the behaviour of string theory on Calabi-Yau varieties, modeled as algebraic singularities. The physics with which we are concerned are supersymmetric gauge theories of $\mathcal{N} = 0, 1, 2$ in 4 dimensions. These, with their matter content and superpotential, can be written precisely in LG form. In establishing the proposed correspondence, quantities in the gauge theory can then be mapped to geometrical properties in the Calabi-Yau. This correspondence was first provided by Witten in [17]. With a brief review thereupon let us begin our invocations in physics.

5.1 The Gauged Linear Sigma Model

According to [17], let us begin with neither the Calabi-Yau sigma model nor the LG theory with superpotenetial, let us begin instead with a **linear sigma model** with gauge group $U(1)$. The action is

$$S = S_{kinetic} + S_D + \int d^2 z d^2 \theta \; W,$$

where W is our superpotential in terms of the chiral super-fields $X = \{P, s_{1,\ldots,5}\}$, with $U(1)$ charges $Q := (-5, 1, ..., 1)$. We choose W to be of the form $W = P \cdot G(s_i)$ where G is a homogeneous polynomial of degree 5. On the other hand, S_D is the D-term of Fayet-Illiopoulos, of the form

$$D = -e^2 \left(\sum_i Q_i |X_i|^2 - r \right) = -e^2 \left(\sum_i |s_i|^2 - 5|p|^2 - r \right).$$

The bosonic part of our potential then becomes

$$U = |G(s_i)|^2 + |p|^2 \sum_i |\frac{\partial G}{\partial s_i}|^2 + \frac{1}{2e^2} + 2|\sigma|^2 \sum_i Q_i^2 |X_i|^2,$$

with σ a scalar field in the (twisted) chiral multiplet. The vacuum of the theory, i.e., the moduli space, is then determined by the minimum of U, which being a sum of squares, attains its minimum when each of the terms does so.

What is crucial is the FI-parametre r which we shall see as an interpolator between phases.

The Phase $r > 0$

When $r > 0$, minimising the D^2 term in U implies that at least one s_i is non-zero. This forces the second term in U to attain its minimum at $p = 0$, so too the argument applies to the last term to force $\sigma = 0$ and the first, to imply $G = 0$.

Therefore our vacuum is parametrised by $\sum_i |s_i|^2 = r$, together with the identification due to gauge symmetry, viz., $s_i \sim e^{i\theta} s_i$. In other words, the superfields live in $\mathbb{CP}^4$ (a toric variety).

The one more condition we obtained, namely $G = 0$, implies that for $r > 0$ the fields actually live in a hypersurface in $\mathbb{CP}^4$. Of course such hypersurface, the homogenenous quintic, is a Calabi-Yau manifold.

We note therefore, in the limit of $r > 0$, certain fields whose masses in the original Lagrangian are determined by r, play no rôle in recovering the Calabi-Yau and are effectively integrated out. We have therefore obtained, in the IR, a conformal non-linear sigma model on the CY as a hypersurface in a toric variety.

The Phase $r < 0$

In the case of $r < 0$, reasoning as above, we conclude that all s_i vanish and $p = \sqrt{-r/5}$ which gives an unbroken $\mathbb{Z}_5$ gauge symmetry because p is of charge 5. We actually arrive at a single point for the vacuum and the s_i act as fluctuations around it. The configuration is thus $\mathbb{C}^5/\mathbb{Z}_5$ and is an orbifold of a LG theory.

We conclude therefore that the gauged linear sigma model has 2 limits, a Calabi-Yau non-linear sigma model ($r > 0$) and an (orbifolded) Landau-Ginsberg theory ($r < 0$). In fact the complexified form of r, namely $e^{2\pi i(b+ir)}$ serves as the Kähler parametre of the moduli space.

5.2 Generalisations to Toric Varieties

The above approach of relating LG theories and Calabi-Yau sigma models not only gave a physically enlightening way to intimately tie together two methods of constructing $\mathcal{N} = 2$ superconformal theories, but also presented mathematicians with a novel perspective on toric varieties. The construction was soon generalised to other toric varieties as well as hypersurfaces therein [43, 44, 45] (cf. also [19] and [14]).

As we shall later describe the method in painstaking detail in Liber III, where we shall construct gauge theories for D-brane probes on arbitrary toric singularities, we shall be brief for the moment. The idea is to generalise the charge vector Q discussed above to a product of $n - d$ $U(1)$ groups for n superfields, whereupon the charges become encoded by an $n \times (n - d)$ integer matrix $Q_{i=1,\dots,n}^{a=1,\dots,n-d}$ such that $\sum_i Q_i^a = 0$ so that the D-term equations are written as

$$\sum_i Q_i^a v_i = 0 \ \forall \ a.$$

It is with foresight in the above that we have written $v_i := |X_i|^2$ for the modulus-squared of the superfields. We identify v_i as generators of a polyhedral cone (cf. Liber I, Section 1.3) and define the toric variety accordingly, the toroidal $\mathbb{C}^{*(n-d)}$ action is prescribed exactly as

$$\lambda_a : x_i \to \lambda_a^{Q_i^a} x_i$$

for $x_i \in \mathbb{C}^n$.

In this description therefore, the moment map defining the toric variety is simply the D-term and the charge matrix of the linear sigma model gives the relations among all the generators of the cone. In the case of the toric variety being singular, the desingularisation thereof simply corresponds to the acquisition of non-zero values of the FI-parametre r.

In this way we can describe any toric variety as a gauged linear sigma model with charge matrix Q_i^a whose integer kernel has $\mathbb{Z}$-span v_i, which are the generators of the cone. The *homogeneous coördinate ring* is given as the subring of $\mathbb{C}[x_1, \dots, x_n]$, invariant under the above $\mathbb{C}^*$ action by Q_i^a, namely

$$\mathbb{C}[x_1, \ldots, x_n]^Q = \{z_a = \prod_i x_i^{v_i^a}\}.$$

Our above construction of the moduli space in the IR, will turn out to be a crucial ingredient in the construction of gauge theories from string theory. Indeed if we use D-branes to probe background (Calabi-Yau) geometry, the IR moduli space of the world-volume theory will precisely be the background.

This construction of gauge theories brings us to the motivation behind all of our discussions. Indeed if string theory promises to be Grand Unified Theory, one must be able to construct the Standard Model gauge theory therefrom. In the following 3 chapters we shall present 3 alternative methods towards this noble goal.

Chapter 6

Geometrical Engineering of Gauge Theories

A natural approach to the construction of four dimensional (supersymmetric) gauge theories is of course to consider the low energy limit of String/M/F-theory in the context of compactifications on Calabi-Yau spaces. Such an endeavour, of using the geometrical properties of the underlying Calabi-Yau space to explain the perturbative and non-perturbative effects of the field theory, was pioneered in the beautiful papers [46, 47, 48].

Historical trends have shown that the more supersymmetry one has, the easier the techniques become. The above papers initiated the study of $\mathcal{N} = 2$ theories; those with $\mathcal{N} = 1$ came later (q.v. e.g [291]). The construction was based on the fabrication of $\mathcal{N} = 2$ theories by compactifying the heterotic $E_8 \times E_8$ or $Spin(32)/\mathbb{Z}_2$ string theory on $K3 \times T^2$, which by string duality [51], is equivalent to type IIA/B on a Calabi-Yau threefold.

6.1 Type II Compactifications

Let us first briefly remind ourselves of some key facts in type II compactifications (q. v. [54] for an excellent review). The spaces with which we are concerned are

Ricci-flat Kähler manifolds of $SU(3)$ holonomy with Hodge diamond

$$
h^{p,q} = \begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

There are hence two parametres serving to characterise such a (restricted) Calabi-Yau 3-fold, namely $h^{2,1}$, the space of complex structure and $h^{1,1}$, the space of Kähler structure. Indeed string theory on such curved backgrounds gives rise to a $(2,2)$ super-conformal sigma model, the spectrum of which is therefore in one-to-one correspondence with the above cohomologies of the Calabi-Yau. From the point of view of the resulting $\mathcal{N}=2$ theory in four dimensions, the aforementioned deformations of complex and Kähler structures realise as the moduli space of vector (M_V) and hyper-multiplets (M_H) of the supersymmetry algebra. Indeed for type IIA compactifications, M_V corresponds to the complexified[1] Kähler deformations and is of complex dimension $h^{2,1}$ while M_H corresponds to complex deformations together with RR fields and the dilation-axion, and has quaternionic dimension $h^{2,1}+1$. In other words, the abelian gauge symmetry including the graviphoton corresponding to the vector multiplets is $U(1)^{h^{1,1}+1}$. In addition, there are $h^{2,1}+1$ massless hyper-multiplets. One important fact to note is that since the dilaton lives in the hypermultiplet, the vector couplings (gauge coupling and moduli space metric) are purely classical.

The situation for type IIB is reversed, and the complex dimension $\dim_{\mathbb{C}}(M_V) = h^{2,1}$ while the quaternionic dimension $\dim_Q(M_H) = h^{1,1}+1$. Thus the vector (gauge) couplings here are not affected by Kähler deformations which correspond to world-sheet instantons and can be calculated purely geometrically. This is of course a

[1] The Kähler form J is complexified by the type II NS-NS B-field as $J + iB$.

manifestation of mirror symmetry upon which we shall touch lightly later in this chapter.

6.2 Non-Abelian Gauge Symmetry and Geometrical Engineering

In the above, we have addressed the massless spectrum of type II compactifications on Calabi-Yau 3-folds (CY3) where one could see the emergence of an Abelian gauge symmetry. The construction of non-Abelian gauge theories with adjoint matter fields was initiated in [46, 47, 48, 49]. As with all studies in compactification, the method of attack was to start with the Calabi-Yau 2-fold, namely the K3 surface and consider complex fibrations of K3 to obtain the 3-fold. The crucial realisation was that, due to the duality between heterotic on $K3 \times T^2$ and type IIA on the CY3, itself as a $K3$-fibre bundle [51], the relevant QFT moduli space comes from the $K3$ singularities so that the gauge fields are obtained from wrapping type IIA D2 branes on the vanishing 2-cycles thereof and that the matter comes from the extra singularities of the base of the CY3.

Let us digress a moment to remind the reader of the key features of K3 surfaces needed in the construction. We recall that a local singularity of K3 can be modeled as an **Asymptotically Locally Euclidean** or ALE space. These are quotient spaces of $\mathbb{C}^2$ by discrete subgroups of the monodromy group $SU(2)$. We have learnt in Liber I, Chapter 3 that such quotients are the 2-dimensional orbifolds, or the du-Val-Klein singularities, with an ADE-classification. The steps of geometrical engineering are therefore as follows: (i) specify the type of ADE singularity of the K3 fibre; (ii) the gauge coupling is related to the volume of the base as

$$\frac{1}{g_{YM}} = \sqrt{V(B)};$$

take the large $V(B)$ limit so that gravity decouples and so that only the gauge dynamics becomes relevant; and (iii) consider the behaviour of the string theory as

D2-branes wrap the vanishing cycles corresponding to the singularities of the fibre. In so doing, our study of the vanishing cycles in the context of Picard-Lefschetz theory in 2.1.1 will be of significance.

Let us illustrate with the canonical example of the A_1 singularity corresponding to a $\mathbb{Z}_2$ quotient of $\mathbb{C}^2$, fibred over $\mathbb{P}^1$. The singularity is described by $xy = z^2$. We can set $x = \phi_1^2\phi_2, y = \phi_3^2\phi_2$ and $z = \phi_1\phi_2\phi_3$ with ϕ_i the complex fields of a two-dimensional SUSY gauged linear sigma model (GLSM); the D-term is given by $U(\phi_1, \phi_2, \phi_3) = (\phi_1\phi_1 + \phi_3\phi_3 - 2\phi_2\phi_2 - \zeta)^2$, with Fayet-Illiopoulos parametre ζ serving as a Kähler resolution of the singularity as a $\mathbb{P}^1$-blowup.

Now let D2-branes wrap the $\mathbb{P}^1$-blowups, which are the vanishing cycles of the fibre. We obtain two vector particles $W_\mu^\pm$ depending on the orientation of wrapping, with masses proportional to the volume of the blowup. These are charged under the $U(1)$ field Z_0^μ obtained from decomposing the RR 3-form of IIA onto the harmonic form of the $\mathbb{P}^1$. As we shrink the size of the blow-up, the W and Z become massless and form an adjoint of $SU(2)$ and we obtain a 6D $SU(2)$ gauge theory. Further compactification upon the base over which our type A_1 $K3$ is fibred to give the CY3 finally gives us a 4D $\mathcal{N} = 2$ pure $SU(2)$ Yang-Mills. The analysis extends to all other ADE groups and it is easy to remember that a singularity of type A (respectively D, E) gives a gauge group which is the compact Lie groups under Dynkin classification type A (respectively D, E)[2].

To obtain matter, we consider *collisions of fibres*. For example, letting an A_{m-1} singularity of the K3 fibre meet with an A_{n-1} one would give a gauge group $SU(m) \times SU(n)$. The base geometry would consist of two intersecting $\mathbb{P}^1$'s whose volumes determine the gauge couplings of each factor. Wrapping a linear combination of the 2 vanishing cycles will give rise to **bi-fundamental** fields transforming as $(m, \bar{n})$ of the gauge group. Moreover, taking the limit of one of the base volumes would make the gauge factor a flavour symmetry and henceforth give rise to fundamental matter.

We can thus geometrically engineer 4 dimensional $\mathcal{N} = 2$ Yang-Mills theories with

[2]The non-simply laced cases of $BCFG$ can be obtained as well after some modifications (q. v. e.g. [50])

product gauge groups with (bi-)fundamental matter by the pure classical geometry of CY3 modeled as $K3$-fibrations over $\mathbb{P}^1$.

6.2.1 Quantum Effects and Local Mirror Symmetry

The above construction gave us classical aspects of the gauge theory as one had to take the $\alpha' \to 0$ limit to decouple gravity and consider only the low energy physics. Therefore we consider only local geometry, or the non-compact singularities which model the Calabi-Yau. This is why we discussed at length the singularity behaviour of complex varieties in Liber I and why we shall later make extensive usage of these local, singular varieties. The large volume limits are suppressed by powers of α'.

However it is well-known that the classical moduli space of $\mathcal{N} = 2$ Super-Yang-Mills receives quantum corrections. The prepotential of the pure $SU(2)$ case for example is of the form $\mathcal{F}(A) = \frac{1}{2}\tau_0 A^2 + \frac{i}{\pi} A^2 \log\left(\frac{A}{\Lambda}\right)^2 + F_{inst}$ in terms of the scalar in the $\mathcal{N} = 2$ vector multiplet. The log-term describes the 1-loop effects while F_{inst} is the instanton corrections as determined by the Seiberg-Witten curve [68]. The corresponding prepotential in type IIA has the structure [55]

$$\mathcal{F} = -\frac{1}{6} C_{ABC} t_A t_B t_C - \frac{\chi\zeta(3)}{4\pi^3} + \frac{1}{8\pi^3} \sum\_ d_1, \ldots, d_h n_{d_1,\ldots,d_h} Li_3(\exp(i \sum_A d_A t_A)),$$

in terms of the Kähler moduli $t_{A=1,\ldots,h^{1,1}}$, where $n_{d_1,\ldots,d_h}$ are the rational curves in the Calabi-Yau corresponding to the instantons.

To compute these instanton effects one evokes the mirror principle and map the discussion to type IIB compactified on the mirror Calabi-Yau. Now we need to consider $D3$ branes wrapping vanishing 3-cycles (conifold-type singularities). In the double-scaling limit as we try to decouple gravity ($\alpha' \to 0$) and study low energy dynamics (volume of cycles $\to 0$), we are finding mirrors of non-compact Calabi-Yau's. Such a procedure, with the prototypical example being the ALE-fibrations, is referred to as **local mirror** transformation as opposed to that for the compact manifolds studied in the original context of mirror symmetry.

We shall not delve to much into the matter, a rich and beautiful field in itself.

Suffice to say that mathematicians and physicists alike have made much progress in the local mirror phenomenon, especially in the context of (our interested) toric varieties (cf e.g. [58, 59, 60, 61, 56, 57, 62]). The original conjecture was the statement in [59], that "every pair of d dimensional dual reflexive Gorenstein σ and $\sigma^\vee$ of index r gives rise to an $\mathcal{N}=2$ superconformal theory with central charge $c = 3(d-2(r-1))$. Moreover, the superpotentials of the corresponding LG theories define two families of generalised toric Calabi-Yau manifolds related by mirror symmetry." We recall from Section 1.1 of Liber I the definitions of dual and Gorenstein cones. Here we elucidate two more. By reflexive we mean that the Gorenstein cone σ has a dual cone $\sigma^\vee$ which is also Gorenstein. The index r is the inner product of w and $w^\vee$, the two vectors guaranteeing the Calabi-Yau conditions ($\langle w, \sigma\rangle = \langle w^\vee, \sigma^\vee\rangle = 0$).

In terms of the complex equations. If M is the variety corresponding to σ generated by v_i satisfying the charge relation (cf. Section 4.2)

$$\sum_{i=1}^{n+d} Q_i^a v_i = 0 \qquad a = 1, \ldots, n,$$

then the mirror W is defined by the equation $\sum_i a_i m_i = 0$, where a_i are coefficients and m_i, monomials which satisfy

$$\prod_{i=1}^{n+d} m_i^{Q_i^a} = 1.$$

Having addressed the method of geometrical engineering, we now move onto a more physical realisation of gauge theories, involving certain configurations of branes in the 10-dimensions of the superstring.

Chapter 7

Hanany-Witten Configurations of Branes

7.1 Type II Branes

It is well-known that type IIA (restively type IIB) superstring theory has Dirichlet p-branes of world volume dimension $p+1$ for $p = 0, 2, 4, 6, 8$ (resp. $-1, 1, 3, 5, 7, 9$) which are coupled to the Ramond-Ramond $p+1$-form electro-magnetically. They are of tension and hence RR charge, in units of the fundamental string scale l_s,

$$T_p = \frac{1}{g_s l_s^{p+1}}.$$

The Dp-branes are BPS saturated objects preserving half of the 32 supercharges of type II, namely those of the form

$$\epsilon_L Q_L + \epsilon_R Q_R \qquad \text{s.t.} \quad \epsilon_L = \Gamma^0 \ldots \Gamma^p \epsilon_R,$$

where $Q_{L,R}$ are the spacetime supercharges generated by left and right moving world-sheet degree of freedom of opposite chirality.

7.1.1 Low Energy Effective Theories

The low-energy world-volume theory on an infinite Dp-brane is a $p+1$-dimensional field theory with 16 SUSY's describing the dynamics of the ground state of the open string which end on the brane (for a pedantic review upon this subject, q.v. [63], upon which much of the ensuing in this section is based). The theory is obtained by dimensional reduction of the $9+1$-D $\mathcal{N}=1$ $U(1)$ super-Yang-Mills (SYM) with gauge coupling $g_{YM}^2 = g_s l_s^{p-3}$ by dimensional analysis. Gravity can thus be decoupled by holding g_{YM} fixed while sending $l_s \to 0$. The massless spectrum includes a $p+1$-D $U(1)$ gauge field A_μ whose world-volume degrees of freedom carry the *Chan-Paton* factors of the open strings, as well as $9-p$ scalars X^I described by the transverse directions to the world-volume.

As BPS objects, parallel Dp-branes are shown in a celebrated calculation of [64] to exert zero force upon each other. This subsequently inspired the famous result of [65], stating that the low-energy dynamics of N_c parallel coincident Dp-branes gives a $U(N_c)$ SYM in $p+1$ dimensions with 16 supercharges. With the addition of **orientifold p-planes**, which are fixed planes of a $\mathbb{Z}_2$ action on the 10-D spacetime and are of charge $\pm 2^{p-4}$ times the corresponding Dp-brane charge, we can similarly fabricate SYM with Sp and SO gauge groups.

A last player upon our stage is of course the solitonic NS-NS 5-brane, of tension $T_{NS} = \frac{1}{g_s^2 l_s^6}$, which couples magnetically to the NS-NS B-field. It too is BPS object preserving 16 supercharges. The low-energy theory of a stack of k IIB NS-branes is a 6-D $(1,1)$ $U(k)$ SYM while that of the IIA NS-brane is more exotic, being a non-Abelian generalisation of a non-trivial $(2,0)$ tensor-multiplet theory in 6 dimensions. The most crucial fact with which we shall concern ourselves is the above tension formula. Indeed in the low-energy limit as $g_S \to 0$, the NS-brane is heavier than any of the D-branes and can be considered as relatively non-dynamical.

7.1.2 Webs of Branes and Chains of Dualities

Having addressed stacks of Dp-branes, now consider N_c Dp-branes occupying $x^{0,\dots,p}$ directions with N_f $D(p+4)$-branes in the $x^{0,\dots,p+4}$ directions. The SUSY preservation conditions become more constrained: $\epsilon_L = \Gamma^0 \dots \Gamma^p \epsilon_R = \Gamma^0 \dots \Gamma^{p+4} \epsilon_R$, subsequently another 1/2 SUSY is broken. This is the famous $Dp - D(p+4)$ system where the Dp probes the geometry of the latter and the relative positions of the branes give various moduli of the gauge theory.

More precisely, the locations of the $D(p+4)$ give the masses of the N_f fundamentals, those of Dp, the VEV's in the adjoint of $U(N_c)$ and parametrise the Coulomb branch, and finally the Dp directions in the $D(p+4)$ are the VEV's of adjoint hypermultiplets. The Higgs branch, parametrised by the VEV's of the fundamentals, is then the moduli space of N_c instantons with gauge group $U(N_f)$.

From the above setup, in conjunction with the usage of a chain of dualities which we summarise below, we may arrive at a sequence of other useful setups. Here then are the effects of S and T dualities on various configurations (R_i is the compactification radius):

T-duality along the i-th direction:	R_i	$\leftrightarrow$	$\frac{l_s^2}{R_i}$
	g_s	$\leftrightarrow$	$\frac{g_s l_s}{R_i}$
	Dp wrapped on x^i	$\leftrightarrow$	$D(p-1)$ at a point on x^i
	$NS5_{IIA}$ wrapped on x^i	$\leftrightarrow$	$NS5_{IIB}$ wrapped on x^i
	$NS5$ at a point on x^i	$\leftrightarrow$	KK monopole

Type IIB S-duality:	g_s	$\leftrightarrow$	$\frac{1}{g_s}$
	l_s^2	$\leftrightarrow$	$l_s^2 g_s$
	Fundamental String	$\leftrightarrow$	$D1$
	$D3$	$\leftrightarrow$	$D3$
	$NS5$	$\leftrightarrow$	$D5$
	$(p,q)7$ brane	$\leftrightarrow$	$(p',q')7$ brane

7.2 Hanany-Witten Setups

Equipped with the above chain of dualities, from the $Dp - D(p+4)$ system we can arrive at $Dp - D(p+2)$ by compactifying 2 directions as well its T-dual version $D(p+1) - D(p+3)$. Notably one has the $D3 - D5$ system. Subsequent S-duality leads to $D3 - NS5$ configuration as well as all $Dp - NS5$ for other values of p by repeated T-dualities.

Of particular interest is the type IIA setup, directly liftable to M-theory, of a stack of N_c $D4$-branes, stretched between 2 parallel infinite NS5 branes (cf. Figure 7-1). The D4-branes occupy directions $x^{0,\ldots 3,6}$ and the two NS5 occupy $x^{0,\ldots 5}$, but at a distant L_6 apart. As discussed earlier the SUSY condition become more restricted and the theory with 32 supercharges had the NS-brane been absent now becomes one with a quarter as much, or 8. More precisely, the Lorentz group breaks as $SO(1,9) \to SO(1,3) \times SO(2) \times SO(3)$ respectively on $x^{0,1,2,3}$, $x^{4,5}$ and $x^{7,8,9}$. The $SO(3)$ becomes a global $SU(2)$ R-symmetry of an $\mathcal{N} = 2$ SYM while the $SO(2)$, a $U(1)$ R-symmetry. At low energies, to an 4-dimensional observer in $x^{0,1,2,3}$, bulk 10-D spacetime modes as well as those on the NS5 branes are higher dimensional excitations and for length scales larger than L_6 the excitations on the D4-branes essentially describe a 4-dimensional (instead of 4+1) physics. What results is an $\mathcal{N} = 2$ pure SYM theory in 4 dimensions with gauge group $U(N_c)$. The gauge coupling is $\frac{1}{g^2} = \frac{L_6}{g_s l_s}$ and in order to decouple gravity effects and go to the low energy limit we need once again take the double limit $g_s \to 0, L_6/l_s \to 0$.

What we have described above, is a prototypical example of the celebrated **Hanany-Witten** brane configuration where one fabricates 4 dimensional supersymmetric gauge theories by suspending D4-brane between NS-branes.

7.2.1 Quantum Effects and M-Theory Solutions

Of course the above discussion had been classical. Just as in the geometrical engineering picture one has to use local mirror symmetry to consider quantum effects, here too must one be careful. Indeed, in type IIA the endpoints of the D4-branes

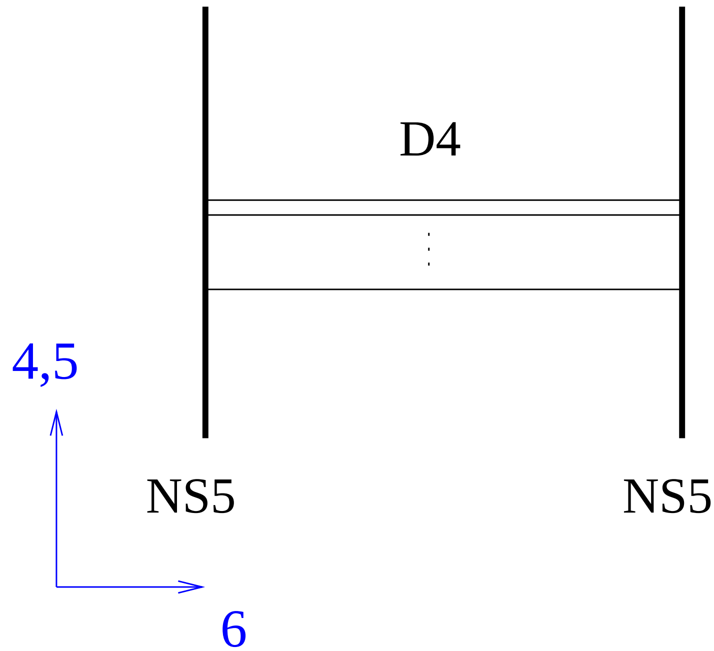

Figure 7-1: The canonical example of the Hanany-Witten setup where a stack of D4-branes is stretched between 2 parallel NS5-branes.

on the NS5-branes are singular and are governed by a Laplace-type of equation. An approximate solution is to let the D4 exert a force and cause the two NS5-branes to bend so that they are no longer strictly parallel with respect to the 6th direction. In fact, the NS5-branes bend logarithmically and the separation (which as we saw governs the gauge coupling) varies and determines the logarithmic running of the coupling.

The shapes of the branes thus incorporate the 1-loop effects. Now since our theory is $\mathcal{N} = 2$, there are no higher-loop contributions due to non-renormalisation. Therefore what remains to be considered are the non-perturbative instanton effects which we saw above as the $Dp - D(p+4)$ system, or here, $D0$-branes in the $D4$.

The solution is the elegant "lift to M-Theory" [67]. Of course both the $D4$ and $NS5$ are different manifestation of the same object in M-Theory, namely the $M5$-brane; the former is the $M5$ wrapped around the compact 11-th dimensional S^1 in going from M-theory to type IIA while the latter is the $M5$ situated at a point on the S^1.

The lift of the Hanany-Witten setup is then a Riemann surface Σ in 11-dimensions. The bending condition from the 1-loop effects determine the embedding equation of Σ while the instantons are automatically included since the $D0$-branes in M-theory are simply Kaluza-Klein modes of the compactification. A most beautiful result of [67] is that Σ is precisely the Seiberg-Witten curve [68] describing the 4-dimensional field theory.

From geometrical engineering we have moved to configurations of branes. Our next method will be D-branes at singular points in the geometry.

Chapter 8

Brane Probes and World Volume Theories

The third method of constructing gauge theories from string theory which we shall now review in detail is the method of D-branes probing background geometries. This is in some sense a mixture of the two methods described above: it utilises both the geometry of local Calabi-Yau as well as world-volume gauge theories living on D-branes.

The pioneering work in this direction was initiated by Douglas and Moore in [69]. Their technique is a physical realisation of the mathematics which we described in Liber I, Chapter 3 and gives a unifying application of such concepts as Hyper-Kähler quotients, McKay quivers, Finite group representations and instanton moduli spaces.

8.1 The Closed Sector

Before we introduce D-branes and hence the open sector to our story let us first briefly remind ourselves of the closed sector, in the vein of the geometric engineering and compactifications presented earlier in this Liber II as well as the mathematics of ALE spaces introduced in Section 3.3 of Liber I. We recall that the ALE space M_Γ is the local model for K3 surfaces, being (resolutions of) the orbifolds $\mathbb{C}^2/(\Gamma \in SU(2))$. It is also known as a **gravitational instanton** in the sense that it is endowed with a anti-self-dual hyper-Kähler metric (with $SU(2)$ holonomy).

For $\Gamma = A_{n-1}$, the metric is explicitly given as the multi-centre Eguchi-Hanson metric [70]:

$$ds^2 = \left(\sum_{i=1}^{n} \frac{1}{|\vec{x} - \vec{x_i}|}\right)^{-1} (dt\vec{A} \cdot d\vec{x})^2 + V dx^2,$$

where $-\vec{\nabla} V = \vec{\nabla} \times \vec{A}$, t is the angular coördinate, and x_i the n singular points. Choosing a basis Σ_i of $H^2(M_\Gamma; \mathbb{Z})$, the quantity $\vec{\zeta} := \vec{x_{i+1}} - \vec{x_i}$ is then equal to $\int_{\Sigma_i} \vec{\omega}$, where $\vec{\omega} = \omega_{I,J,K}$ are the three hyper-Kähler symplectic forms introduced in Section 1.2 of Liber I. The ζ_i's govern the size of the $\mathbb{P}^1$-blowups and are hence the Kähler parametres of the ALE space. The moduli space is of dimension $3n - 6$, our familiar result for moduli space of instantons.

When considering the ALE as the (two complex dimensional) target-space for non-linear sigma models, we are left with $\mathcal{N} = (4,4)$ supersymmetry. On the other hand in the context of considering superstrings propagating in the background $\mathbb{R}^6 \times M_\Gamma$, we have $\mathcal{N} = (0,1), (1,1)$ and $(0,2)$ respectively for types I, IIA and IIB. The $SU(2)$ R-symmetry of the 6-dimensional gauge theory sits as an unbroken subgroup of the $SO(4)$ isometry of the space.

8.2 The Open Sector

Now let us add D5 branes to the picture. We do so for the obvious reason that we shall consider D5 with its world-volume extending the $\mathbb{R}^6$ and transverse to the 4-dimensional M_Γ (which together constitute the 10-dimensions of type II superstring theory). Also historically, Witten in [71] considered the 5-brane built as an instanton in the gauge theory of [72]. The 6-dimensional $\mathcal{N} = 1$ theory on the world-volume leads to a hyper-Kähler quotient description of the vacuum moduli space.

Consider a stack of N D5-branes each filling the $\mathbb{R}^6$ and at a point in $\mathbb{C}^2$. This gives us, as discussed in the previous chapter, an $U(N)$ gauge theory in 6-dimensions with $\mathcal{N} = 1$. Open strings ending on the i,j-th D-brane carry Chan-Paton factors corresponding to the gauge fields A_μ^{ij} as $N \times N$ Hermitian matrices; we can write the states as

$$|A\rangle = A_\mu^{ij}\psi^\mu|ij\rangle,$$

where ψ^μ are fermions; similarly we have scalars X^i as $N\times N$ matrices by dimensional reduction.

Thus prepared, let us move on to the configuration in question, viz., the stack of D5-branes situated at a point in the ALE orbifold of $\mathbb{C}^2$. The group Γ has an induced action on the vectors as well as scalars (and hence by supersymmetry the fermions), namely for $g \in \Gamma$, $g : A_\mu(x) \to \gamma(g)A_\mu(x')\gamma(g)^{-1}$ and $g : X^i(x) \to R(g)^i_j\gamma(g)X^j(x')\gamma(g)^{-1}$ where γ is a representation acting on the Chan-Paton indices and R is a representation that act additionally on space-time.

Due to this projection by the orbifold group, only a subsector of the theory survives, namely

$$A_\mu(x) = \gamma(g)A_\mu(x)\gamma(g)^{-1} \qquad X^i(x) = R(g)^i_j\gamma(g)X^j(x')\gamma(g)^{-1}. \tag{8.2.1}$$

In Liber III, we shall present a detailed method of explicitly solving these equations. For now we shall point out to the reader that such a configuration of a stack of D-branes, placed transversely to a singular point of the geometry, is called a **brane probe**.

8.2.1 Quiver Diagrams

We shall certainly delve into this matter further in Liber III, a chief theme of which shall in fact be the encoding of solutions to equations (8.2.1), namely those which describe the matter content (and interaction) of the world-volume probe theory. Now, let us here entice the reader with a few advertisements.

We shall learn that the world-volume super-Yang-Mills (SYM) theory can be represented by a **quiver diagram**, which we recall from Section 3.1 to be a labelled directed finite graph together with a (complex) representation.

To each vertex we associate the vector multiplet and to each edge, the hypermul-

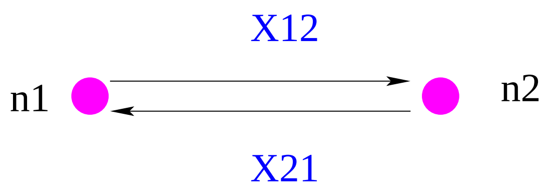

Figure 8-1: An example of a quiver diagram encoding the matter content. Here theory has gauge group $U(n_1) \times U(n_2)$ with hypermultiplet (X_{12}, X_{21}).

tiplet. Generically we have product $U(n_i)$'s for the gauge groups of the theory (with the inclusion of orientifolds we can also obtain other groups). Therefore we attribute a vector space V_i as well as the semisimple component (i.e., the $U(n_i)$ factor) for the gauge group which acts on V_i, to each vertex v_i.

In other words, the vector multiplets are seen as (Hermitian) matrices, representing adjoint gauge fields, acting on the space V. On the other hand, an edge from vertex v_i to v_j is a complex scalar transforming in the representation $\bar{V}_i \otimes V_j = \mathrm{Hom}(V_i, V_j)$ and hence constitutes a mapping between the two vector spaces. An undirected edge consisting of two oppositely directed edges composes a single hypermultiplet. And so with this we can encode the matter content of a SYM theory on the D-brane probe as a quiver. The example in Figure 8-1 shall serve to clarify.

8.2.2 The Lagrangian

Having addressed the matter content for the theory discussed, namely $\mathcal{N} = 1$ in 6-dimensions (or $\mathcal{N} = 2$ in 4), enough supercharges exist to allow us to actually write down the Lagrangian, rather conveniently in terms of hyper-Kähler geometry. The action is of the form

$$L = L_{BI} + L_{HM} + L_{CS} + \text{fermions},$$

where L_{BI} is the familiar Dirac-Born-Infeld action, L_{HM}, the kinetic energy of the hypermultiplets, L_{CS}, a Chern-Simons coupling term and the fermions form the SUSY completion. We concentrate on the first terms as they are purely in terms of the scalars and shall provide the moduli space of the vacuum.

We recall from the previous subsection that the hypermultiplets take values in $\mathrm{Hom}(V,V)$ for the vector space $V := \{z^{a=1,\ldots,n}\}$ attributed to a vertex (and hence a semisimple factor G of the gauge group). Then letting the dual space V^* have coördinates $\{w_a\}$, the hypermultiplets can then be written as Hermitian matrices $X^a := \begin{pmatrix} z^a & \bar{w}^a \\ -w_a & \bar{z}_a \end{pmatrix}$ which form a quaternionic vector space with the Pauli matrices $\vec{\sigma}$ serving as the 3 complex structures. More generally, the X^a's form a *hyper-Kähler manifold* with a triplet of symplectic forms: $\omega^{\mathbb{R}} = \frac{i}{2} dz^a d\bar{z}_a + dw_a d\bar{w}^a$ and $\omega^{\mathbb{C}} = dz^a \wedge dw_a$.

Finally $\mathfrak{g} := Lie(G)$ has a natural action on X as $\delta_{j=1,\ldots\dim(\mathfrak{g})} X^a = (t_j)^a_b X^b$. This action is symplectic with respect to the above triplet of ω's and we can write down a triplet of hyper-Kähler moment maps

$$\vec{\mu}_j := \frac{1}{2} \mathrm{tr} \vec{\sigma} X^\dagger_a (t_j)^a_b X^b,$$

being Noether charges of the symplectic action.

The scalar part of the action then reads

$$L_{HM} + L_{BI} = \int_{D6} \sum_{j \in \mathfrak{g}} \vec{D}_j \cdot (\vec{D}_j + \vec{\mu}_j),$$

where $\vec{D}_j$ is the triple of auxiliary fields from the D-fields in the vector-multiplet. For D3 branes (and hence $\mathcal{N} = 2$ in 4-dimensions), $D^{\mathbb{R}}$ is the FI D auxiliary field while $D^{\mathbb{C}}$ gives the F auxiliary field.

8.2.3 The Vacuum Moduli Space

Integrating out the D-fields from the Lagrangian above (we have to include the L_{CS} as well which we shall not discuss here), we obtain an effective potential energy for the hypermultiplet: $\sum\limits_j (\vec{\mu_j} - \vec{\phi_j})^2$. Here ϕ_j's are scalars in the hypermultiplet corresponding to the centre of $\mathfrak{g}$ acting as $\mathrm{Hom}(V,V)$. Letting the VEV of ϕ_j be

$\langle\vec{\phi_j}\rangle = \vec{\zeta_j}$, the vacuum manifold is then

$$\vec{\mu_j} = \vec{\zeta_j},$$

modulo gauge transformations. We remark that in fact for D5 and D4 branes, this classical moduli space is the same the quantum one and for D3 probes, the hyper-Kähler metric does not obtain quantum corrections.

We have of course seen this already in Section 3.3 of Liber I. This moduli space is a hyper-Kähler quotient, with respect to the moment maps. Such a space $M_\zeta = \mu^{-1}(\zeta)/G$ is precisely the Kronheimer's **ALE-instanton** [33] as a resolution of the orbifold $\mathbb{C}^2/\Gamma$ in the case of the type IIB D5-brane probing the ALE space as a local K3. More generically, when we include pairs of vector spaces (V, W) to each node in the spirit of Figure 4-1, we can actually obtain the **quiver manifold** for the ALE space [39]. This manifold is actually the moduli space of instantons on M_ζ and we shall refer the reader to Section 3.3 for the details.

Epilogue

We have addressed three methods of constructing gauge theories in 4-dimensions, from which hopefully one day we can uniquely identify our real world. It should be no surprise to us of course, that these three prescriptions: geometrical engineering, Hanany-Witten setups and D-brane probes, are all different guises of a single concept.

The key of course is T-duality, or for the mathematician, Mirror Symmetry [60]. Using fractional branes and 3 consecutive T-dualities, [53] showed the equivalence between Hanany-Witten and geometrical engineering. Furthermore, [128] showed that T-dual of NS-branes is precisely the ALE instanton, whereby effectively establishing the equivalence between NS-D-brane setups and D-brane probes.

Thus concludes our invocations. Prepared with some rudiments in the mathematics and physics of a beautiful subfield of string theory, let us trudge on...

III

LIBER TERTIUS: Sanguis, Sudor, et Larcrimæ Mei

Prologue

Having hopefully by now conjured up the spirits of our gentle readers, by these our invocations in mathematics and in physics, let us proceed to the heart of this writing. I shall regret, to have enticed so much, and yet shall soon provide so little. Though the ensuing pages will be voluminous, my sheer want of wit shall render them uninspiring.

Yet I have laboured upon them and for some four years shed my blood, sweat and tears upon these pages. I shall thus beg ye readers to open your magnanimous hearts, to peruse and not to scoff, to criticise and not to scorn.

Without further ado then allow me to summarise the contents of the following chapters. This Book the Third itself divides into three parts. The first, consists of chapters 9 till 13. They deal with gauge theory living on D-brane probes transverse to quotient singularities of dimensions two, three (Chap. 9) and four (Chap. 11). Certain unified perspectives, from such diverse points of view as modular invariants of WZW models, quiver categories and generalised McKay's correspondences are discussed in Chapters 10, 12 and 13. Extensive use will be made of the techniques of Chapters 2, 3 and 4 of Liber I.

The next part consists of Chapters 14 till 16 where we address the more physical question of realising the above probe theories as brane configurations of the Hanany-Witten type. Thereafter, the two chapters 17 and 18 consider the additional complication when there is a background of the NS-NS B-field, which subsequently leads to the study of projective representations of the orbifold group.

Finally the remaining chapters of the present Liber III are dedicated to a detailed study of the IR moduli space of certain gauge theories, in particular we venture beyond the orbifolds and study toric singularities. Chapter 2 of Liber I and chapter 5 of Liber II will therefore be of great use.

Chapter 9

Orbifolds I: $SU(2)$ and $SU(3)$

Synopsis

This is the first chapter on D-brane probes on orbifold singularities where we study the world-volume $\mathcal{N}=4$ $U(n)$ super-Yang-Mills theory orbifolded by discrete subgroups of $SU(2)$ and $SU(3)$. We have reached many interesting observations that have graph-theoretic interpretations.

For the subgroups of $SU(2)$, we have McKay's correspondence to our aid. In the case of $SU(3)$ we have constructed a catalogue of candidates for finite (chiral) $\mathcal{N}=1$ theories, giving the gauge group and matter content.

To generalise the case of $SU(2)$, we conjecture a McKay-type correspondence for Gorenstein singularities in dimension 3 with modular invariants of WZW conformal models. This implies a connection between a class of finite $\mathcal{N}=1$ supersymmetric gauge theories in four dimensions and the classification of affine $SU(3)$ modular invariant partition functions in two dimensions [292].

9.1 Introduction

Recent advances on finite four dimensional gauge theories from string theory constructions have been dichotomous: either from the geometrical perspective of studying algebro-geometric singularities such as orbifolds [75] [76] [77], or from the intuitive perspective of studying various configurations of branes such as the so-called brane-box models [78]. (See [79] and references therein for a detailed description of these models. A recent paper discusses the bending of non-finite models in this context [80].) The two approaches lead to the realisation of finite, possibly chiral, $\mathcal{N}=1$ supersymmetric gauge theories, such as those discussed in [81]. Our ultimate dream is of course to have the flexibility of the equivalence and completion of these approaches, allowing us to compute say, the duality group acting on the moduli space of marginal gauge couplings [82]. (The duality groups for the $\mathcal{N}=2$ supersymmetric theories were discussed in the context of these two approaches in [48] and [67].) The brane-box method has met great success in providing the intuitive picture for orbifolds by Abelian groups: the elliptic model consisting of $k \times k'$ branes conveniently reproduces the theories on orbifolds by $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ [79]. Orbifolds by $\mathbb{Z}_k$ subgroups of $SU(3)$ are given by Brane Box Models with non-trivial identification on the torus [82] [79]. Since by the structure theorem that all finite Abelian groups are direct sums of cyclic ones, this procedure can be presumably extended to all Abelian quotient singularities. The non-Abelian groups however, present difficulties. By adding orientifold planes, the dihedral groups have also been successfully attacked for theories with $\mathcal{N}=2$ supersymmetry [83]. The question still remains as to what could be done for the myriad of finite groups, and thus to general Gorenstein singularities.

In this chapter we shall present a catalogue of these Gorenstein singularities in dimensions 2 and 3, i.e., orbifolds constructed from discrete subgroups of $SU(2)$ and $SU(3)$ whose classification are complete. In particular we shall concentrate on the gauge group, the fermionic and bosonic matter content resulting from the orbifolding of an $\mathcal{N}=4$ $U(n)$ super-Yang-Mills theory. In Section 2, we present the general arguments that dictate the matter content for arbitrary finite group Γ. Then in Section

3, we study the case of $\Gamma \subset SU(2)$ where we notice interesting graph-theoretic descriptions of the matter matrices. We analogously analyse case by case, the discrete subgroups of $SU(3)$ in Section 4, followed by a brief digression of possible mathematical interest in Section 5. This leads to a Mckay-type connection between the classification of two dimensional $SU(3)_k$ modular invariant partition functions and the class of finite $\mathcal{N} = 1$ supersymmetric gauge theories calculated in this chapter. Finally we tabulate possible chiral theories obtainable by such orbifolding techniques for these $SU(3)$ subgroups.

9.2 The Orbifolding Technique

Prompted by works by Douglas, Greene, Moore and Morrison on gauge theories which arise by placing D3 branes on orbifold singularities [69] [73], [74], Kachru and Silverstein [75] and subsequently Lawrence, Nekrasov and Vafa [76] noted that an orbifold theory involving the projection of a supersymmetric $\mathcal{N} = 4$ gauge theory on some discrete subgroup $\Gamma \subset SU(4)$ leads to a conformal field theory with $\mathcal{N} \leq 4$ supersymmetry. We shall first briefly summarise their results here.

We begin with a $U(n)$ $\mathcal{N} = 4$ super-Yang-Mills theory which has an R-symmetry of $Spin(6) \simeq SU(4)$. There are gauge bosons A_{IJ} ($I, J = 1, ..., n$) being singlets of $Spin(6)$, along with adjoint Weyl fermions $\Psi^{\mathbf{4}}_{IJ}$ in the fundamental $\mathbf{4}$ of $SU(4)$ and adjoint scalars $\Phi^{\mathbf{6}}_{IJ}$ in the antisymmetric $\mathbf{6}$ of $SU(4)$. Then we choose a discrete (finite) subgroup $\Gamma \subset SU(4)$ with the set of irreducible representations $\{\mathbf{r}_i\}$ acting on the gauge group by breaking the I-indices up according to $\{\mathbf{r}_i\}$, i.e., by $\bigoplus\limits_i \mathbf{r}_i = \bigoplus\limits_i \mathbb{C}^{N_i}\mathbf{r}_i$ such that $\mathbb{C}^{N_i}$ accounts for the multiplicity of each $\mathbf{r}_i$ and $n = \sum\limits_{i=1} N_i \dim(\mathbf{r}_i)$. In the string theory picture, this decomposition of the gauge group corresponds to permuting n D3-branes and hence their Chan-Paton factors which contain the IJ indices, on orbifolds of $\mathbb{R}^6$. Subsequently by the Maldecena large N conjecture [84], we have an orbifold theory on $AdS_5 \times S^5$, with the R-symmetry manifesting as the $SO(6)$ symmetry group of S^5 in which the branes now live [75]. The string perturbative calculation in this context, especially with respect to vanishing theorems for β-functions, has been performed [77].

Having decomposed the gauge group, we must likewise do so for the matter fields: since an orbifold is invariant under the Γ-action, we perform the so-called projection on the fields by keeping only the Γ-invariant fields in the theory. Subsequently we arrive at a (superconformal) field theory with gauge group $G = \bigotimes_i SU(N_i)$ and Yukawa and quartic interaction respectively as (in the notation of [76]):

$$Y = \sum_{ijk} \gamma_{ijk}^{f_{ij},f_{jk},f_{ki}} \operatorname{Tr} \Psi_{f_{ij}}^{ij} \Phi_{f_{jk}}^{jk} \Psi_{f_{ki}}^{ki}$$

$$V = \sum_{ijkl} \eta_{f_{ij},f_{jk},f_{kl},f_{li}}^{ijkl} \operatorname{Tr} \Phi_{f_{ij}}^{ij} \Phi_{f_{jk}}^{jk} \Phi_{f_{kl}}^{kl} \Phi_{f_{li}}^{li},$$

where

$$\gamma_{ijk}^{f_{ij},f_{jk},f_{ki}} = \Gamma_{\alpha\beta,m} \left(Y_{f_{ij}}\right)_{v_i\bar{v}_j}^{\alpha} \left(Y_{f_{jk}}\right)_{v_j\bar{v}_k}^{m} \left(Y_{f_{ki}}\right)_{v_k\bar{v}_i}^{\beta}$$

$$\eta_{f_{ij},f_{jk},f_{kl},f_{li}}^{ijkl} = \left(Y_{f_{ij}}\right)_{v_i\bar{v}_j}^{[m} \left(Y_{f_{jk}}\right)_{v_j\bar{v}_k}^{n]} \left(Y_{f_{kl}}\right)_{v_k\bar{v}_l}^{[m} \left(Y_{f_{li}}\right)_{v_l\bar{v}_i}^{n]},$$

such that $\left(Y_{f_{ij}}\right)_{v_i\bar{v}_j}^{\alpha}$, $\left(Y_{f_{ij}}\right)_{v_i\bar{v}_j}^{m}$ are the f_{ij}'th Clebsch-Gordan coefficients corresponding to the projection of $4 \otimes \mathbf{r}_i$ and $6 \otimes \mathbf{r}_i$ onto $\mathbf{r}_j$, and $\Gamma_{\alpha\beta,m}$ is the invariant in $\mathbf{4} \otimes \mathbf{4} \otimes \mathbf{6}$.

Furthermore, the matter content is as follows:

1. Gauge bosons transforming as

$$\hom\left(\mathbb{C}^n, \mathbb{C}^n\right)^{\Gamma} = \bigoplus_i \mathbb{C}^{N_i} \otimes \left(\mathbb{C}^{N_i}\right)^*,$$

   which simply means that the original (R-singlet) adjoint $U(n)$ fields now break up according to the action of Γ to become the adjoints of the various $SU(N_i)$;

2. $a_{ij}^{\mathbf{4}}$ Weyl fermions $\Psi_{f_{ij}}^{ij}$ ($f_{ij} = 1, ..., a_{ij}^{\mathbf{4}}$)

$$\left(\mathbf{4} \otimes \hom\left(\mathbb{C}^n, \mathbb{C}^n\right)\right)^{\Gamma} = \bigoplus_{ij} a_{ij}^{\mathbf{4}} \mathbb{C}^{N_i} \otimes \left(\mathbb{C}^{N_j}\right)^*,$$

   which means that these fermions in the fundamental $\mathbf{4}$ of the original R-symmetry now become $\left(N_i, \overline{N}_j\right)$ bi-fundamentals of G and there are $a_{ij}^{\mathbf{4}}$ copies of them;

3. $a^{\mathbf{6}}_{ij}$ scalars $\Phi^{ij}_{f_{ij}}$ $(f_{ij} = 1, ..., a^{\mathbf{6}}_{ij}\ )$ as

$$(\mathbf{6} \otimes \hom(\mathbb{C}^n, \mathbb{C}^n))^\Gamma = \bigoplus_{ij} a^{\mathbf{6}}_{ij} \mathbb{C}^{N_i} \otimes \left(\mathbb{C}^{N_j}\right)^*,$$

similarly, these are G bi-fundamental bosons, inherited from the $\mathbf{6}$ of the original R-symmetry.

For the above, we define $a^{\mathcal{R}}_{ij}$ ($\mathcal{R} = \mathbf{4}$ or $\mathbf{6}$ for fermions and bosons respectively) as the composition coefficients

$$\mathcal{R} \otimes \mathbf{r}_i = \bigoplus_j a^{\mathcal{R}}_{ij} \mathbf{r}_j \tag{9.2.1}$$

Moreover, the supersymmetry of the projected theory must have its R-symmetry in the commutant of $\Gamma \subset SU(4)$, which is $U(2)$ for $SU(2)$, $U(1)$ for $SU(3)$ and trivial for $SU(4)$, which means: if $\Gamma \subset SU(2)$, we have an $\mathcal{N} = 2$ theory, if $\Gamma \subset SU(3)$, we have $\mathcal{N} = 1$, and finally for $\Gamma \subset$ the full $SU(4)$, we have a non-supersymmetric theory.

Taking the character χ for element $\gamma \in \Gamma$ on both sides of (9.2.1) and recalling that χ is a $(\otimes, \oplus)$-ring homomorphism, we have

$$\chi^{\mathcal{R}}_\gamma \chi^{(i)}_\gamma = \sum_{j=1}^{r} a^{\mathcal{R}}_{ij} \chi^{(j)}_\gamma \tag{9.2.2}$$

where $r = |\{\mathbf{r}_i\}|$, the number of irreducible representations, which by an elementary theorem on finite characters, is equal to the number of inequivalent conjugacy classes of Γ. We further recall the orthogonality theorem of finite characters,

$$\sum_{\gamma=1}^{r} r_\gamma \chi^{(i)*}_\gamma \chi^{(j)}_\gamma = g\delta^{ij}, \tag{9.2.3}$$

where $g = |\Gamma|$ is the order of the group and r_γ is the order of the conjugacy class containing γ. Indeed, χ is a class function and is hence constant for each conjugacy class; moreover, $\sum\limits_{\gamma=1}^{r} r_\gamma = g$ is the class equation for Γ. This orthogonality allows us to invert (9.2.2) to finally give the matrix a_{ij} for the matter content

$$a_{ij}^{\mathcal{R}} = \frac{1}{g} \sum_{\gamma=1}^{r} r_\gamma \chi_\gamma^{\mathcal{R}} \chi_\gamma^{(i)} \chi_\gamma^{(j)*} \tag{9.2.4}$$

where $\mathcal{R} = \mathbf{4}$ for Weyl fermions and $\mathbf{6}$ for adjoint scalars and the sum is effectively that over the columns of the Character Table of Γ. Thus equipped, let us specialise to Γ being finite discrete subgroups of $SU(2)$ and $SU((3)$.

9.3 Checks for $SU(2)$

The subgroups of $SU(2)$ have long been classified [30]; discussions and applications thereof can be found in [32] [85] [33] [88]. To algebraic geometers they give rise to the so-called Klein singularities and are labeled by the first affine extension of the simply-laced simple Lie groups $\widehat{A}\widehat{D}\widehat{E}$ (whose associated Dynkin diagrams are those of ADE adjointed by an extra node), i.e., there are two infinite series and 3 exceptional cases:

1. $\widehat{A}_n = \mathbb{Z}_{n+1}$, the cyclic group of order $n+1$;
2. $\widehat{D}_n$, the binary lift of the ordinary dihedral group d_n;
3. the three exceptional cases, $\widehat{E}_6$, $\widehat{E}_7$ and $\widehat{E}_8$, the so-called binary or double[1] tetrahedral, octahedral and icosahedral groups $\mathcal{T}, \mathcal{O}, \mathcal{I}$.

The character tables for these groups are known [23] [91] [93] and are included in Appendix 22.1 for reference. Therefore to obtain (9.2.4) the only difficulty remains in the choice of $\mathcal{R}$. We know that whatever $\mathcal{R}$ is, it must be 4 dimensional for the

[1] For $SO(3) \cong SU(2)/\mathbb{Z}_2$ these would be the familiar symmetry groups of the respective regular solids in $\mathbb{R}^3$: the dihedron, tetrahedron, octahedron/cube and icosahedron/dodecahedron. However since we are in the double cover $SU(2)$, there is a non-trivial $\mathbb{Z}_2$- lifting,

$$\begin{array}{ccccccccc} 0 & \rightarrow & \mathbb{Z}_2 & \rightarrow & SU(2) & \rightarrow & SO(3) & \rightarrow & 0, \\ & & & & \bigcup & & \bigcup & & \\ & & & & \widehat{D}, \mathcal{T}, \mathcal{O}, \mathcal{I} & \rightarrow & d, T, O, I & & \end{array}$$

hence the modifier "binary". Of course, the A-series, being abelian, receives no lifting. Later on we shall briefly touch upon the ordinary d, T, I, O groups as well.

fermions and 6 dimensional for the bosons inherited from the fundamental **4** and antisymmetric **6** of $SU(4)$. Such an $\mathcal{R}$ must therefore be a 4 (or 6) dimensional irrep of Γ, or be the tensor sum of lower dimensional irreps (and hence be reducible); for the character table, this means that the row of characters for $\mathcal{R}$ (extending over the conjugacy classes of Γ) must be an existing row or the sum of existing rows. Now since the first column of the character table of any finite group precisely gives the dimension of the corresponding representation, it must therefore be that $\dim(\mathcal{R}) = 4, 6$ should be partitioned into these numbers. Out of these possibilities we must select the one(s) consistent with the decomposition of the **4** and **6** of $SU(4)$ into the $SU(2)$ subgroup[2], namely:

$$\begin{array}{ccc} SU(4) & \to & SU(2)\times SU(2)\times U(1) \\ \mathbf{4} & \to & (\mathbf{2},\mathbf{1})_{+1}\bigoplus(\mathbf{1},\mathbf{2})_{-1} \\ \mathbf{6} & \to & (\mathbf{1},\mathbf{1})_{+2}\bigoplus(\mathbf{1},\mathbf{1})_{-2}\bigoplus(\mathbf{2},\mathbf{2})_{0} \end{array} \tag{9.3.5}$$

where the subscripts correspond to the $U(1)$ factors (i.e., the trace) and in particular the $\pm$ forces the overall traceless condition. From (9.3.5) we know that $\Gamma \subset SU(2)$ inherits a **2** while the complement is trivial. This means that the 4 dimensional represention of Γ must be decomposable into a nontrivial 2 dimensional one with a trivial 2 dimensional one. In the character language, this means that $\mathcal{R} = \mathbf{4} = \mathbf{2_{trivial}} \oplus \mathbf{2}$ where $\mathbf{2_{trivial}} = \mathbf{1_{trivial}} \oplus \mathbf{1_{trivial}}$, the tensor sum of two copies of the (trivial) principal representation where all group elements are mapped to the identity, i.e., corresponding to the first row in the character table. Whereas for the bosonic case we have $\mathcal{R} = \mathbf{6} = \mathbf{2_{trivial}} \oplus \mathbf{2} \oplus \mathbf{2}'$. We have denoted $\mathbf{2}'$ to signify that the two **2**'s may not be the same, and correspond to inequivalent representations of Γ with the same dimension. However we can restrict this further by recalling that the antisymmetrised tensor product $[\mathbf{4}\otimes\mathbf{4}]_A \to 1\oplus 2\oplus 2\oplus [2\otimes 2]_A$ must in fact contain the **6**. Whence we conclude that $\mathbf{2} = \mathbf{2}'$. Now let us again exploit the additive property of the group character, i.e., a homomorphism from a $\oplus$-ring to a $+$-subring of a number

[2]We note that even though this decomposition is that into irreducibles for the full continuous Lie groups, such irreducibility may not be inherited by the discrete subgroup, i.e., the **2**'s *may* not be irreducible representations of the finite Γ.

field (and indeed much work has been done for the subgroups in the case of number fields of various characteristics); this means that we can simplify $\chi^{\mathcal{R}=x\oplus y}$ as $\chi^x + \chi^y$. Consequently, our matter matrices become:

$$
\begin{array}{l}
a^{\mathbf{4}}_{ij} = \frac{1}{g}\sum\limits_{\gamma=1}^{r} r_\gamma \left(2\chi^{\mathbf{1}}_\gamma + \chi^{\mathbf{2}}_\gamma\right)\chi^{(i)}_\gamma \chi^{(j)*}_\gamma = 2\delta_{ij} + \frac{1}{g}\sum\limits_{\gamma=1}^{r} r_\gamma \chi^{\mathbf{2}}_\gamma \chi^{(i)}_\gamma \chi^{(j)*}_\gamma \\
a^{\mathbf{6}}_{ij} = \frac{1}{g}\sum\limits_{\gamma=1}^{r} r_\gamma \left(2\chi^{\mathbf{1}}_\gamma + \chi^{\mathbf{2}\oplus\mathbf{2}}_\gamma\right)\chi^{(i)}_\gamma \chi^{(j)*}_\gamma = 2\delta_{ij} + \frac{2}{g}\sum\limits_{\gamma=1}^{r} r_\gamma \chi^{\mathbf{2}}_\gamma \chi^{(i)}_\gamma \chi^{(j)*}_\gamma
\end{array}
$$

where we have used the fact that χ of the trivial representation are all equal to 1, thus giving by (9.2.3), the δ_{ij}'s. This simplification thus limits our attention to only 2 dimensional representations of Γ; however there still may remain many possibilities since the **2** may be decomposed into nontrivial **1**'s or there may exist many inequivalent irreducible **2**'s.

We now appeal to physics for further restriction. We know that the $\mathcal{N} = 2$ theory (which we recall is the resulting case when $\Gamma \subset SU(2)$) is a non-chiral supersymmetric theory; this means our bifundamental fields should not distinguish the left and right indices, i.e., the matter matrix a_{ij} must be *symmetric*. Also we know that in the $\mathcal{N} = 2$ vector multiplet there are 2 Weyl fermions and 2 real scalars, thus the fermionic and bosonic matter matrices have the same entries on the diagonal. Furthermore the hypermultiplet has 2 scalars and 1 Weyl fermion in $(N_i, \bar{N}_j)$ and another 2 scalars and 1 Weyl fermion in the complex conjugate representation, whence we can restrict the off-diagonals as well, viz., $2a^{\mathbf{4}}_{ij} - a^{\mathbf{6}}_{ij}$ must be some multiple of the identity. This supersymmetry matching is of course consistent with (19).

Enough said on generalities. Let us analyse the groups case by case. For the cyclic group, the **2** must come from the tensor sum of two **1**'s. Of all the possibilities, only the pairing of dual representations gives symmetric a_{ij}. By dual we mean the two **1**'s which are complex conjugates of each other (this of course includes when $\mathbf{2} = \mathbf{1}^{\mathbf{2}}_{\mathbf{trivial}}$, which exist for all groups and gives us merely δ_{ij}'s and can henceforth be eliminated as uninteresting). We denote the nontrivial pairs as $\mathbf{1}'$ and $\mathbf{1}''$. In this case we can easily perform yet another consistency check. From (9.3.5), we have a traceless condition seen as the cancelation of the $U(1)$ factors. That was on the Lie algebra level; as groups, this is our familiar determinant unity condition. Since in the block decomposition (9.3.5) the $\mathbf{2_{trivial}} \subset$ the complement $SU(4)\backslash\Gamma$ clearly has determinant 1, this forces our $\mathbf{2}$ matrix to have determinant 1 as well. However in this cyclic case, Γ is abelian, whence the characters are simply presentations of the group, making the $\mathbf{2}$ to be in fact diagonal. Thus the determinant is simply the product of the entries of the two rows in the character table. And indeed we see for dual representations, being complex conjugate roots of unity, the two rows do multiply to 1 for all members. Furthermore we note that different dual pairs give a_{ij}'s that are mere permutations of each other. We conclude that the fermion matrix arises from $\mathbf{1^2} \oplus \mathbf{1}^{'} \oplus \mathbf{1}^{''}$. For the bosonic matrix, by (19), we have $\mathbf{6} = (\mathbf{1} \oplus \mathbf{1}^{'} \oplus \mathbf{1}^{''})^{\mathbf{2}}$. These and ensuing a_{ij}'s are included in Appendix 22.2.

For the dihedral case, the $\mathbf{1}$'s are all dual to the principal, corresponding to some $\mathbb{Z}_2$ inner automorphism among the conjugacy classes and the characters consist no more than ± 1's, giving us a_{ij}'s which are block diagonal in $((1,0),(0,1))$ or $((0,1),(1,0))$ and are not terribly interesting. Let us rigorise this statement. Whenever we have the character table consisting of a row that is composed of cycles of roots of unity, which is a persistent theme for $\mathbf{1}$ irreps, this corresponds in general to some $\mathbb{Z}_k$ action on the conjugacy classes. This implies that our a_{ij} for this choice of $\mathbf{1}$ will be the Kronecker product of matrices obtained from the cyclic groups which offer us nothing new. We shall refer to these cases as "blocks"; they offer us another condition of elimination whose virtues we shall exploit much. In light of this, for the dihedral the choice of the $\mathbf{2}$ comes from the irreducible $\mathbf{2}$'s which again give symmetric a_{ij}'s that are permutations among themselves. Hence $\mathcal{R} = \mathbf{4} = \mathbf{1^2} \oplus \mathbf{2}$ and $\mathcal{R} = \mathbf{6} = \mathbf{1^2} \oplus \mathbf{2^2}$. For reference we have done likewise for the dihedral series not in the full $SU(2)$, the choice for $\mathcal{R}$ is the same for them.

Finally for the exceptionals $\mathcal{T}, \mathcal{O}, \mathcal{I}$, the $\mathbf{1}$'s again give uninteresting block diagonals and out choice of $\mathbf{2}$ is again unique up to permutation. Whence still $\mathcal{R} = \mathbf{4} = \mathbf{1^2} \oplus \mathbf{2}$ and $\mathcal{R} = \mathbf{6} = \mathbf{1^2} \oplus \mathbf{2^2}$. For reference we have computed the ordinary exceptionals T, O, I which live in $SU(2)$ with its center removed, i.e., in $SU(2)/\mathbb{Z}_2 \cong SO(3)$. For them the $\mathbf{2}$ comes from the $\mathbf{1}^{'} \oplus \mathbf{1}^{''}$, the $\mathbf{2}$, and the trivial $\mathbf{1^2}$ respectively.

Of course we can perform an *a posteriori* check. In this case of $SU(2)$ we already know the matter content due to the works on quiver diagrams [69] [85] [83]. The theory dictates that the matter content a_{ij} can be obtained by looking at the Dynkin diagram of the $\widehat{A}\widehat{D}\widehat{E}$ group associated to Γ whereby one assigns 2 for a_{ij} on the diagonal as well as 1 for every pair of connected nodes $i \to j$ and 0 otherwise, i.e., a_{ij} is essentially the adjacency matrix for the Dynkin diagrams treated as unoriented graphs. Of course adjacency matrices for unoriented graphs are symmetric; this is consistent with our nonchiral supersymmetry argument. Furthermore, the dimension of $a_{ij}^{\mathbf{4}}$ is required to be equal to the number of nodes in the associated affine Dynkin diagram (i.e., the rank). This property is immediately seen to be satisfied by examining the character tables in Appendix 22.1 where we note that the number of conjugacy classes of the respective finite groups (which we recall is equal to the number of irreducible representations) and hence the dimension of a_{ij} is indeed that for the ranks of the associated affine algebras, namely $n+1$ for $\widehat{A_n}$ and $\widehat{D_n}$ and 7,8,9 for $\widehat{E_{6,7,8}}$ respectively. We note in passing that the conformality condition $N_f = 2N_c$ for this $\mathcal{N} = 2$ [75] [76] nicely translates to the graph language: it demands that for the one loop β-function to vanish the label of each node (the gauge fields) must be $\frac{1}{2}$ that of those connected thereto (the bi-fundamentals).

Our results for a_{ij} computed using (9.2.4), Appendix 22.1, and the aforementioned decomposition of $\mathcal{R}$ are tabulated in Appendix 22.2. They are precisely in accordance with the quiver theory and present themselves as the relevant adjacency matrices. One interesting point to note is that for the dihedral series, the ordinary d_n (which are in $SO(3)$ and not $SU(2)$) for even n also gave the binary $\widehat{D_{n'=\frac{n+6}{2}}}$ Dynkin diagram while the odd n case always gave the ordinary $D_{n'=\frac{n+3}{2}}$ diagram.

These results should be of no surprise to us, since a similar calculation was in fact done by J. Mckay when he first noted his famous correspondence [32]. In the paper he computed the composition coefficients m_{ij} in $R \bigotimes R_j = \bigoplus\limits_k m_{jk} R_k$ for $\Gamma \subset SU(2)$ with R being a faithful representation thereof. He further noted that for all these Γ's there exists (unique up to automorphism) such R, which is precisely the 2 dimensional irreducible representation for $\widehat{D}$ and $\widehat{E}$ whereas for $\widehat{A}$ it is the direct sum of a pair of dual 1 dimensional representations. Indeed this is exactly the decomposition of $\mathcal{R}$ which we have argued above from supersymmetry. His *Theorema Egregium* was then

Theorem: The matrix m_{ij} is $2I$ minus the cartan matrix, and is thus the adjacency matrix for the associated affine Dynkin diagram treated as undirected C_2-graphs (i.e., maximal eigenvalue is 2).

Whence m_{ij} has 0 on the diagonal and 1 for connected nodes. Now we note from our discussions above and results in Appendix 22.2, that our $\mathcal{R}$ is precisely Mckay's R (which we henceforth denote as R_M) plus two copies of the trivial representation for the **4** and R_M plus the two dimensional irreps in addition to the two copies of the trivial for the **6**. Therefore we conclude from (9.2.4):

$$\begin{array}{c}
a_{ij}^{\mathbf{4}} = \frac{1}{g}\sum\limits_{\gamma=1}^{r} r_\gamma \chi_\gamma^{R_M \oplus \mathbf{1}^2} \chi_\gamma^{(i)} \chi_\gamma^{(j)*} \\
a_{ij}^{\mathbf{6}} = \frac{1}{g}\sum\limits_{\gamma=1}^{r} r_\gamma \chi_\gamma^{R_M \oplus R_M \oplus \mathbf{1}^2} \chi_\gamma^{(i)} \chi_\gamma^{(j)*}
\end{array}$$

which implies of course, that our matter matrices should be

$$\begin{array}{c}
a_{ij}^{\mathbf{4}} = 2\delta_{ij} + m_{ij} \\
a_{ij}^{\mathbf{6}} = 2\delta_{ij} + 2m_{ij}
\end{array}$$

with Mckay's m_{ij} matrices. This is exactly the results we have in Appendix 22.2. Having obtained such an elegant graph-theoretic interpretation to our results, we remark that from this point of view, oriented graphs means chiral gauge theory and connected means interacting gauge theory. Hence we have the foresight that the $\mathcal{N} = 1$ case which we shall explore next will involve oriented graphs.

Now Mckay's theorem explains why the discrete subgroups of $SU(2)$ and hence Klein singularities of algebraic surfaces (which our orbifolds essentially are) as well as subsequent gauge theories thereupon afford this correspondence with the affine simply-laced Lie groups. However they were originally proven on a case by case basis, and we would like to know a deeper connection, especially in light of quiver theories. We can partially answer this question by noting a beautiful theorem due to Gabriel

[86] [87] which forces the quiver considerations by Douglas et al. [69] to have the ADE results of Mckay.

It turns out to be convenient to formulate the theory axiomatically. We define $\mathcal{L}(\gamma, \Lambda)$, for a finite connected graph γ with orientation Λ, vertices γ_0 and edges γ_1, to be the category of quivers whose objects are any collection (V, f) of spaces $V_{\alpha\in\gamma_0}$ and mappings $f_{l\in\gamma_1}$ and whose morphisms are $\phi : (V, f) \to (V', f')$ a collection of linear mappings $\phi_{\alpha\in\Gamma_0} : V_\alpha \to V'_\alpha$ compatible with f by $\phi_{e(l)} f_l = f'_l \phi_{b(l)}$ where $b(l)$ and $e(l)$ are the beginning and end of the directed edge l. Then we have

Theorem: If in the quiver category $\mathcal{L}(\gamma, \Lambda)$ there are only finitely many non-isomorphic indecomposable objects, then γ coincides with one of the graphs $A_n, D_n, E_{6,7,8}$.

This theorem essentially compels any finite quiver theory to be constructible only on graphs which are of the type of the Dynkin diagrams of ADE. And indeed, the theories of Douglas, Moore et al. [69] [83] have explicitly made the physical realisations of these constructions. We therefore see how McKay's calculations, quiver theory and our present calculations nicely fit together for the case of $\Gamma \subset SU(2)$.

9.4 The case for $SU(3)$

We repeat the above analysis for $\Gamma = SU(3)$, though now we have no quiver-type theories to aid us. The discrete subgroups of $SU(3)$ have also been long classified [88]. They include (the order of these groups are given by the subscript), other than all those of $SU(2)$ since $SU(2) \subset SU(3)$, the following new cases. We point out that in addition to the cyclic group in $SU(2)$, there is now in fact another Abelian case $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ for $SU(3)$ generated by the matrix $((e^{\frac{2\pi i}{k}}, 0, 0), (0, e^{\frac{2\pi i}{k'}}, 0), (0, 0, e^{-\frac{2\pi i}{k} - \frac{2\pi i}{k'}}))$ much in the spirit that $((e^{\frac{2\pi i}{n}}, 0), (0, e^{-\frac{2\pi i}{n}}))$ generates the $\mathbb{Z}_n$ for $SU(2)$. Much work has been done for this $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ case, q. v. [79] and references therein.

1. Two infinite series Δ_{3n^2} and Δ_{6n^2} for $n \in \mathbb{Z}$, which are analogues of the dihedral series in $SU(2)$:

(a) $\Delta \subset$ only the full $SU(3)$: when $n = 0 \bmod 3$ where the number of classes for $\Delta(3n^2)$ is $(8 + \frac{1}{3}n^2)$ and for $\Delta(6n^2)$, $\frac{1}{6}(24 + 9n + n^2)$;

(b) $\Delta \subset$ both the full $SU(3)$ and $SU(3)/\mathbb{Z}_3$: when $n \neq 0 \bmod 3$ where the number of classes for $\Delta(3n^2)$ is $\frac{1}{3}(8 + n^2)$ and for $\Delta(6n^2)$, $\frac{1}{6}(8 + 9n + n^2)$;

2. Analogues of the exceptional subgroups of $SU(2)$, and indeed like the later, there are two series depending on whether the $\mathbb{Z}_3$-center of $SU(3)$ has been modded out (we recall that the binary $\mathcal{T}, \mathcal{O}, \mathcal{I}$ are subgroups of $SU(2)$, while the ordinary T, O, I are subgroups of the center-removed $SU(2)$, i.e., $SO(3)$, and not the full $SU(2)$):

   (a) For $SU(3)/\mathbb{Z}_3$:
   $\Sigma_{36}, \Sigma_{60} \cong A_5$, the alternating symmetric-5 group, which incidentally is precisely the ordinary icosahedral group $I, \Sigma_{72}, \Sigma_{168} \subset S_7$, the symmetric-7 group, $\Sigma_{216} \supset \Sigma_{72} \supset \Sigma_{36}$, and $\Sigma_{360} \cong A_6$, the alternating symmetric-6 group;

   (b) For the full[3] $SU(3)$:
   $\Sigma_{36\times3}, \Sigma_{60\times3} \cong \Sigma_{60} \times \mathbb{Z}_3, \Sigma_{168\times3} \cong \Sigma_{168} \times \mathbb{Z}_3, \Sigma_{216\times3}$, and $\Sigma_{360\times3}$.

Up-to-date presentations of these groups and some character tables may be found in [89] [90]. The rest have been computed with [92]. These are included in Appendix 22.3 for reference. As before we must narrow down our choices for $\mathcal{R}$. First we note that it must be consistent with the decomposition:

[3]In his work on Gorenstein singularities [89], Yau points out that since the cases of $\Sigma_{60\times3}$ and $\Sigma_{168\times3}$ are simply direct products of the respective cases in $SU(3)/\mathbb{Z}_3$ with $\mathbb{Z}_3$, they are usually left out by most authors. The direct product simply extends the class equation of these groups by 3 copies and acts as an inner automorphism on each conjugacy class. Therefore the character table is that of the respective center-removed cases, but with the entries each multiplied by the matrix $((1,1,1),(1,w,w^2),(1,w^2,w))$ where $w = \exp(2\pi i/3)$, i.e., the full character table is the Kronecker product of that of the corresponding center-removed group with that of $\mathbb{Z}_3$. Subsequently, the matter matrices a_{ij} become the Kronecker product of a_{ij} for the center-removed groups with that for $\Gamma = \mathbb{Z}_3$ and gives no interesting new results. In light of this, we shall adhere to convention and call Σ_{60} and Σ_{168} subgroups of both $SU(3)/\mathbb{Z}_3$ and the full $SU(3)$ and ignore $\Sigma_{60\times3}$ and $\Sigma_{168\times3}$.

$$\begin{array}{ccc} SU(4) & \to & SU(3) \times U(1) \\ \mathbf{4} & \to & \mathbf{3}_{-1} \bigoplus \mathbf{1}_3 \\ \mathbf{6} & \to & \mathbf{3}_2 \bigoplus \bar{\mathbf{3}}_{-2} \end{array} \tag{9.4.6}$$

This decomposition (9.4.6), as in the comments for (9.3.5), forces us to consider only 3 dimensionals (possibly reducible) and for the fermion case the remaining $\mathbf{1}$ must in fact be the trivial, giving us a δ_{ij} in $a_{ij}^{\mathbf{4}}$.

Now as far as the symmetry of a_{ij} is concerned, since $SU(3)$ gives rise to an $\mathcal{N}=1$ chiral theory, the matter matrices are no longer necessarily symmetric and we can no longer rely upon this property to guide us. However we still have a matching condition between the bosons and the fermions. In this $\mathcal{N}=1$ chiral theory we have 2 scalars and a Weyl fermion in the chiral multiplet as well as a gauge field and a Weyl fermion in the vector multiplet. If we denote the chiral and vector matrices as C_{ij} and V_{ij}, and recalling that there is only one adjoint field in the vector multiplet, then we should have:

$$\begin{aligned} a_{ij}^{\mathbf{4}} &= V_{ij} + C_{ij} = \delta_{ij} + C_{ij} \\ a_{ij}^{\mathbf{6}} &= C_{ij} + C_{ji}. \end{aligned} \tag{9.4.7}$$

This decomposition is indeed consistent with (9.4.6); where the δ_{ij} comes from the principal $\mathbf{1}$ and the C_{ij} and C_{ji}, from dual pairs of $\mathbf{3}$; incidentally it also implies that the bosonic matrix should be symmetric and that dual $\mathbf{3}$'s should give matrices that are mutual transposes. Finally as we have discussed in the A_n case of $SU(2)$, if one is to compose only from 1 dimensional representations, then the rows of characters for these $\mathbf{1}$'s must multiply identically to 1 over all conjugacy classes. Our choices for $\mathcal{R}$ should thus be restricted by these general properties.

Once again, let us analyse the groups case by case. First the Σ series. For the members which belong to the center-removed $SU(2)$, as with the ordinary T, O, I of $SU(2)/\mathbb{Z}_2$, we expect nothing particularly interesting (since these do not have non-trivial 3 dimensional representations which in analogy to the non-trivial 2 dimensional irreps of $\widehat{D_n}$ and $\widehat{E_{6,7,8}}$ should be the ones to give interesting results). However, for completeness, we shall touch upon these groups, namely, $\Sigma_{36,72,216,360}$. Now the $\mathbf{3}$ in (9.4.6) must be composed of $\mathbf{1}$ and $\mathbf{2}$. The obvious choice is of course again the trivial one where we compose everything from only the principal $\mathbf{1}$ giving $4\delta_{ij}$ and $6\delta_{ij}$ for the fermionic and bosonic a_{ij} respectively. We at once note that this is the only possibility for Σ_{360}, since its first non-trivial representation is 5 dimensional. Hence this group is trivial for our purposes. For Σ_{36}, the $\mathbf{3}$ can come only from $\mathbf{1}$'s for which case our condition that the rows must multiply to 1 implies that $\mathbf{3} = \Gamma_1 \oplus \Gamma_3 \oplus \Gamma_4$, or $\Gamma_1 \oplus \Gamma_2^2$, both of which give uninteresting blocks, in the sense of what we have discussed in Section 2. For Σ_{72}, we similarly must have $\mathbf{3} = \Gamma_2 \oplus \Gamma_3 \oplus \Gamma_4$ or $\mathbf{1}\oplus$ the self-dual $\mathbf{2}$, both of which again give trivial blocks. Finally for Σ_{216}, whose conjugacy classes consist essentially of $\mathbb{Z}_3$-cycles in the 1 and 2 dimensional representations, the $\mathbf{3}$ comes from $\mathbf{1} \oplus \mathbf{2}$ and the dual $\mathbf{3}$, from $\mathbf{1} \oplus \mathbf{2}'$.

For the groups belonging to the full $SU(3)$, namely $\Sigma_{168,60,36\times3,216\times3,360\times3}$, the situation is clear: as to be expected in analogy to the $SU(2)$ case, there always exist dual pairs of $\mathbf{3}$ representations. The fermionic matrix is thus obtained by tensoring the trivial representations with one member from a pair selected in turn out of the various pairs, i.e., $\mathbf{1} \oplus \mathbf{3}$; and indeed we have explicitly checked that the others (i.e., $\mathbf{1} \oplus \mathbf{3}'$) are permutations thereof. On the other hand, the bosonic matrix is obtained from tensoring any choice of a dual pair $\mathbf{3} \oplus \mathbf{3}'$ and again we have explicitly checked that other dual pairs give rise to permutations. We may be tempted to construct the $\mathbf{3}$ out of the $\mathbf{1}$'s and $\mathbf{2}$'s which do exist for $\Sigma_{36\times3,216\times3}$, however we note that in these cases the $\mathbf{1}$ and $\mathbf{2}$ characters are all cycles of $\mathbb{Z}_3$'s which would again give uninteresting blocks. Thus we conclude still that for all these groups, $\mathbf{4} = \mathbf{1} \oplus \mathbf{3}$ while $\mathbf{6} = \mathbf{3}\oplus$ dual $\bar{\mathbf{3}}$. These choices are of course obviously in accordance with the decomposition (9.4.6) above. Furthermore, for the Σ groups that belong solely to the full $SU(3)$, the dual pair of $\mathbf{3}$'s always gives matrices that are mutual transposes, consistent with the requirement in (9.4.7) that the bosonic matrix be symmetric.

Moving on to the two Δ series. We note[4], that for $n = 1$, $\Delta_3 \cong \mathbb{Z}_3$ and $\Delta_6 \cong d_6$ while for $n = 2$, $\Delta_{12} \cong T := E_6$ and $\Delta_{24} \cong O := E_7$. Again we note that for all $n > 1$ (we have already analysed the $n = 1$ case[5] for $\Gamma \subset SU(2)$), there exist the dual $\mathbf{3}$ and $\mathbf{3}'$ representations as in the $\Sigma \subset$ full $SU(3)$ above; this is expected of course since as noted before, all the Δ groups at least belong to the full $SU(3)$. Whence we again form the fermionic a_{ij} from $\mathbf{1} \oplus \mathbf{3}'$, giving a generically nonsymmetric matrix (and hence a good chiral theory), and the bosonic, from $\mathbf{3} \oplus \mathbf{3}'$, giving us always a symmetric matrix as required. We note in passing that when $n = 0$ mod 3, i.e,. when the group belongs to both the full and the center-removed $SU(3)$, the Δ_{3n^2} matrices consist of a trivial diagonal block and an L-shaped block. Moreover, all the Δ_{6n^2} matrices are block decomposable. We shall discuss the significances of this observation in the next section. Our analysis of the discrete subgroups of $SU(3)$ is now complete; the results are tabulated in Appendix 22.4.

9.5 Quiver Theory? Chiral Gauge Theories?

Let us digress briefly to make some mathematical observations. We recall that in the $SU(2)$ case the matter matrices a_{ij}, due to McKay's theorem and Moore-Douglas quiver theories, are encoded as adjacency matrices of affine Dynkin diagrams considered as unoriented graphs as given by Figures 9-1 and 9-2.

We are of course led to wonder, whether in analogy, the a_{ij} for $SU(3)$ present themselves as adjacency matrices for quiver diagrams associated to some *oriented* graph theory because the theory is chiral. This is very much in the spirit of recent works on extensions of Mckay correspondences by algebraic geometers [98] [99]. We here present these quiver graphs in figures 9-3 9-4 and 9-5, hoping that it may be of academic interest.

[4] Though congruence in this case really means group isomorphisms, for our purposes since only the group characters concern us, in what follows we might use the term loosely to mean identical character tables.

[5] Of course for $\mathbb{Z}_3$, we must have a different choice for $\mathcal{R}$, in particular to get a good chiral model, we take the $\mathbf{3} = \mathbf{1}' \oplus \mathbf{1}'' \oplus \mathbf{1}'''$

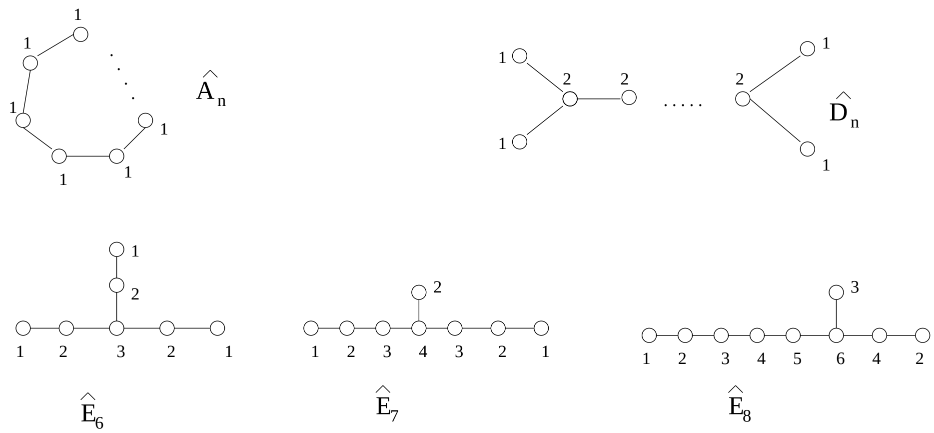

Figure 9-1: $\Gamma \subset$ full $SU(2)$ correspond to affine Dynkin diagrams with the Dynkin labels N_i on the nodes corresponding to the dimensions of the irreps. In the quiver theory the nodes correspond to gauge groups and the lines (or arrows for chiral theories), matter fields. For finite theories each N_i must be $\frac{1}{2}$ of the sum of neighbouring labels and the gauge group is $\bigoplus_i U(N_i)$.

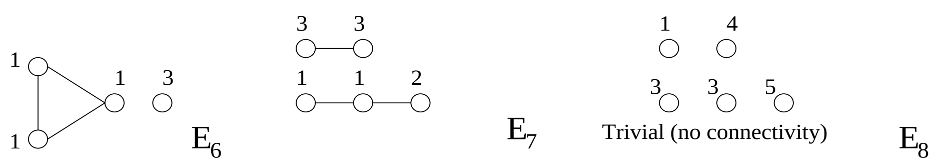

Figure 9-2: $\Gamma \subset SU(2)/\mathbb{Z}_2$ give disconnected graphs

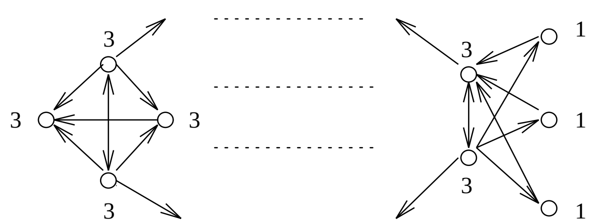

Figure 9-3: $\Delta_{3n^2} \subset SU(3)$ for $n \neq 0$ mod 3. These belong to both the full and center-removed $SU(3)$.

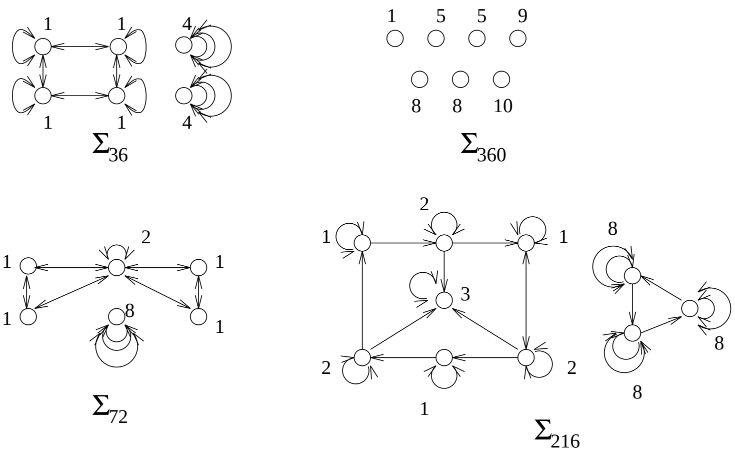

Figure 9-4: $\Sigma \subset SU(3)/\mathbb{Z}_3$ gives unconnected graphs.

Indeed we note that for the center-removed case, as with $SU(2)$, we get disconnected (or trivial) graphs; this of course is the manifestation of the fact that there are no non-trivial **3** representations for these groups (just as there are no non-trivial **2**'s of $\Gamma \subset SU(2)/\mathbb{Z}_2$). On the other hand for $\Gamma \subset$ full $SU(3)$, we do get interesting connected and oriented graphs, composed of various directed triangular cycles.

Do we recognise these graphs? The answer is sort of yes and the right place to look for turns out to be in conformal field theory. In the work on general modular invariants in the WZW model for $\widehat{su(n)}_k$ (which is equivalent to the study of the modular properties of the characters for affine Lie algebras), an ADE classification was noted for $n = 2$ [93] [94] [95]; this should somewhat be expected due to our earlier discussion on Gabriel's Theorem. For $n = 3$, work has been done to extract coefficients in the fusion rules and to treat them as entries of adjacency matrices; this fundamentally is analogous to what we have done since fusion rules are an affine version of finite group composition coefficients. So-called generalised Dynkin diagrams have been constructed for $\widehat{su(3)}$ in analogy to the 5 simply-laced types corresponding to $SU(2)$, they are: $\mathcal{A}_n, \mathcal{D}_{3n}, \mathcal{E}_5, \mathcal{E}_9$ and $\mathcal{E}_{21}$ where the subscripts denote the level in the representation of the affine algebra [93] [104] [96]. We note a striking resemblance between these graphs (they are some form of a dual and we hope to rigorise this similarity in future work) with our quiver graphs: the $\mathcal{E}_5, \mathcal{E}_9$ and $\mathcal{E}_{21}$ correspond to $\Sigma_{216\times 3}$, $\Sigma_{360\times 3}$, and $\Sigma_{36\times 3}$ respectively. Incidentally these Σ groups are the *only* ones

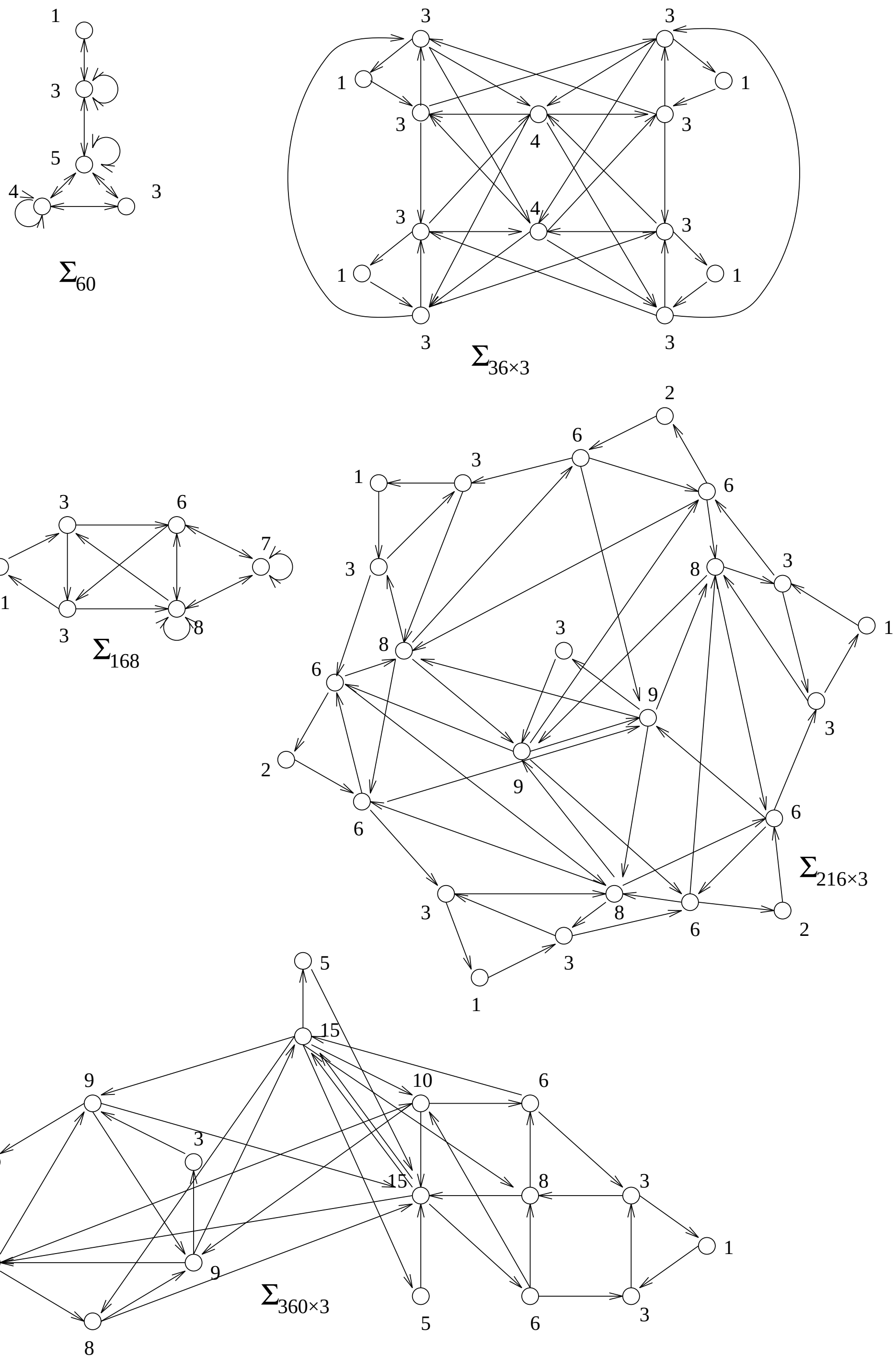

Figure 9-5: $\Sigma \subset$ full $SU(3)$. Only $\Sigma_{36\times3,216\times3,360\times3}$ belong only to the full $SU(3)$, for these we have the one loop β-function vanishing condition manifesting as the label of each node equaling to $\frac{1}{3}$ of that of the incoming and outgoing neighbours respectively. The matrix representation for these graphs are given in Appendix 22.4.

that belong solely to the full and not the center-removed $SU(3)$. The $\mathcal{D}_{3n}$ corresponds to Δ_{3n^2} for $n \neq 0$ mod 3, which are the non-trivial ones as observed in the previous section and which again are those that belong solely to the full $SU(3)$. The Δ_{6n^2} series, as noted above, gave non-connected graphs, and hence do not have a correspondent. Finally the $\mathcal{A}_n$, whose graph has complete $\mathbb{Z}_3$ symmetry must come from the Abelian subgroup of $SU(3)$, i.e., the $\widehat{A_n}$ case of $SU(2)$ but with $\mathcal{R} = \mathbf{3}$ and not $\mathbf{2}$. This beautiful relationship prompts us to make the following conjecture upon which we may labour in the near future:

Conjecture: There exists a McKay-type correspondence between Gorenstein singularities and the characters of integrable representations of affine algebras $\widehat{su(n)}$ (and hence the modular invariants of the WZW model).

A physical connection between $\widehat{SU(2)}$ modular invariants and quiver theories with 8 supercharges has been pointed out [106]. We remark that our conjecture is in the same spirit and a hint may come from string theory. If we consider a D1 string on our orbifold, then this is just our configuration of D3 branes after two T-dualities. In the strong coupling limit, this is just an F1 string in such a background which amounts to a non-linear sigma model and therefore some (super) conformal field theory whose partition function gives rise to the modular invariants. Moreover, connections between such modular forms and Fermat varieties have also been pointed out [97], this opens yet another door for us and many elegant intricacies arise.

Enough digression on mathematics; let us return to physics. We would like to conclude by giving a reference catalogue of chiral theories obtainable from $SU(3)$ orbifolds. Indeed, though some of the matrices may not be terribly interesting graph-theoretically, the non-symmetry of $a_{ij}^{\mathbf{4}}$ is still an indication of a good chiral theory.

For the original $U(n)$ theory it is conventional to take a canonical decomposition [76] as $n = N|\Gamma|$ [76], whence the (orbifolded) gauge group must be $\bigotimes\limits_i SU(N_i)$ as discussed in Section 3, such that $N|\Gamma| = n = \sum\limits_i N_i |\mathbf{r}_i|$. By an elementary theorem on finite characters: $|\Gamma| = \sum\limits_i |\mathbf{r}_i|^2$, we see that the solution is $N_i = N|\mathbf{r}_i|$. This thus immediately gives the form of the gauge group. Incidentally for $SU(2)$, the McKay correspondence gives more information, it dictates that the dimensions of the irreps of Γ are actually the Dynkin labels for the diagrams. This is why we have labeled the nodes in the graphs above. Similarly for $SU(3)$, we have done so as well; these should be some form of generalised Dynkin labels.

Now for the promised catalogue, we shall list below all the chiral theories obtainable from orbifolds of $\Gamma \subset SU(3)$ ($\mathbb{Z}_3$ center-removed or not). This is done so by observing the graphs, connected or not, that contain unidirectional arrows. For completeness, we also include the subgroups of $SU(2)$, which are of course also in $SU(3)$, and which do give non-symmetric matter matrices (which we eliminated in the $\mathcal{N} = 2$ case) if we judiciously choose the $\mathbf{3}$ from their representations. We use the short hand $(n_1^{k_1}, n_2^{k_2}, ..., n_i^{k_i})$ to denote the gauge group $\bigoplus^{k_1} SU(n_1) ... \bigoplus^{k_i} SU(n_i)$. Analogous to the discussion in Section 3, the conformality condition to one loop order in this $\mathcal{N} = 1$ case, viz., $N_f = 3N_c$ translates to the requirement that the label of each node must be $\frac{1}{3}$ of the sum of incoming and the sum of outgoing neighbours individually. (Incidentally, the gauge anomaly cancelation condition has been pointed out as well [80]. In our language it demands the restriction that $N_j a_{ij} = \bar{N}_j a_{ji}$.) In the following table, the * shall denote those groups for which this node condition is satisfied. We see that many of these models contain the group $SU(3) \times SU(2) \times U(1)$

and hope that some choice of orbifolds may thereby contain the Standard Model.

$\Gamma \subset SU(3)$	Gauge Group
$\widehat{A_n} \cong \mathbb{Z}_{n+1}$	(1^{n+1})
$\mathbb{Z}_k \times \mathbb{Z}_{k'}$	$(1^{kk'})*$
$\widehat{D_n}$	$(1^4, 2^{n-3})$
$\widehat{E_6} \cong \mathcal{T}$	$(1^3, 2^3, 3)$
$\widehat{E_7} \cong \mathcal{O}$	$(1^2, 2^2, 3^2, 4)$
$\widehat{E_8} \cong \mathcal{I}$	$(1, 2^2, 3^2, 4^2, 5, 6)$
$E_6 \cong T$	$(1^3, 3)$
$E_7 \cong O$	$(1^2, 2, 3^2)$
$E_8 \cong I$	$(1, 3^2, 4, 5)$
$\Delta_{3n^2} (n = 0 \bmod 3)$	$(1^9, 3^{\frac{n^2}{3}-1})*$
$\Delta_{3n^2} (n \neq 0 \bmod 3)$	$(1^3, 3^{\frac{n^2-1}{3}})*$
$\Delta_{6n^2} (n \neq 0 \bmod 3)$	$(1^2, 2, 3^{2(n-1)}, 6^{\frac{n^2-3n+2}{6}})*$
Σ_{168}	$(1, 3^2, 6, 7, 8)*$
Σ_{216}	$(1^3, 2^3, 3, 8^3)$
$\Sigma_{36\times3}$	$(1^4, 3^8, 4^2)*$
$\Sigma_{216\times3}$	$(1^3, 2^3, 3^7, 6^6, 8^3, 9^2)*$
$\Sigma_{360\times3}$	$(1, 3^4, 5^2, 6^2, 8^2, 9^3, 10, 15^2)*$

9.6 Concluding Remarks

By studying gauge theories constructed from orbifolding of an $\mathcal{N} = 4$ $U(n)$ super-Yang-Mills theory in 4 dimensions, we have touched upon many issues. We have presented the explicit matter content and gauge group that result from such a procedure, for the cases of $SU(2)$ and $SU(3)$. In the first we have shown how our calculations agree with current quiver constructions and in the second we have constructed possible candidates for chiral theories. Furthermore we have noted beautiful graph-theoretic interpretations of these results: in the $SU(2)$ we have used Gabriel's theorem to partially explain the ADE outcome and in the $SU(3)$ we have noted connections with generalised Dynkin diagrams and have conjectured the existence of a McKay-type correspondence between these orbifold theories and modular invariants of WZW conformal models.

Much work of course remains. In addition to proving this conjecture, we also have numerous questions in physics. What about $SU(4)$, the full group? These would give interesting non-supersymmetric theories. How do we construct the brane box version of these theories? Roan has shown how the Euler character of these orbifolds correspond to the class numbers [99]; we know the blow-up of these singularities correspond to marginal operators. Can we extract the marginal couplings and thus the duality group this way? We shall hope to address these problems in forth-coming work. Perhaps after all, string orbifolds, gauge theories, modular invariants of conformal field theories as well as Gorenstein singularities and representations of affine Lie algebras, are all manifestations of a fundamental truism.

Chapter 10

Orbifolds II: Avatars of McKay Correspondence

Synopsis

Continuing with the conjecture from the previous chapter, we attempt to view the ubiquitous ADE classification, manifesting as often mysterious correspondences both in mathematics and physics, from a string theoretic perspective.

On the mathematics side we delve into such matters as quiver theory, ribbon categories, and the McKay Correspondence which relates finite group representation theory to Lie algebras as well as crepant resolutions of Gorenstein singularities. On the physics side, we investigate D-brane orbifold theories, the graph-theoretic classification of the WZW modular invariants, as well as the relation between the string theory nonlinear σ-models and Landau-Ginzburg orbifolds.

We here propose a unification scheme which naturally incorporates all these correspondences of the ADE type in two complex dimensions. An intricate web of interrelations is constructed, providing a possible guideline to establish new directions of research or alternate pathways to the standing problems in higher dimensions [293].

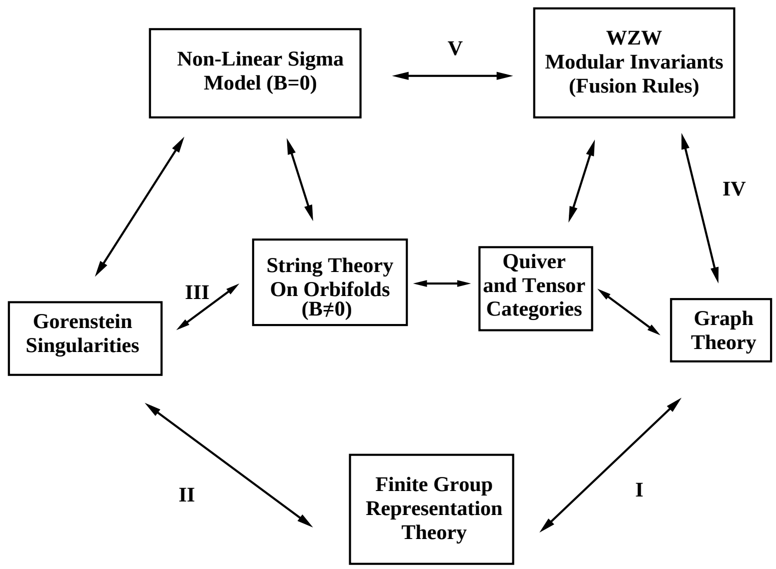

Figure 10-1: The Myriad of Correspondences: it is the purpose of this chapter to elucidate these inter-relations in 2-dimensions, so as to motivate a similar coherent picture in higher dimensions. Most of the subsectors in this picture have been studied separately by mathematicians and physicists, but they are in fact not as disparate as they are guised.

10.1 Introduction

This chapter reviews the known facts about the various ADE classifications that arise in mathematics and string theory and organizes them into a unified picture. This picture serves as a guide for our on-going work in higher dimensions and naturally incorporates diverse concepts in mathematics.

In the course of their research on supersymmetric Yang-Mills theories resulting from the type IIB D-branes on orbifold singularities (Chap. 9), as prompted by collective works in constructing (conformal) gauge theories in the physics literature (cf. previous chapter), it was conjectured that there may exist a McKay-type correspondence between the bifundamental matter content and the modular invariant partition functions of the Wess-Zumino-Witten (WZW) conformal field theory. Phrased in another way, the correspondence, if true, would relate the Clebsch-Gordan coefficients for tensor products of the irreducible representations of finite subgroups of $SU(n)$ with the integrable characters for the affine algebras $\widehat{SU(n)}_k$ of some integral weight k.

Such a relation has been well-studied in the case of $n = 2$ and it falls into an ADE classification scheme [93, 94, 101, 102]. Evidences for what might occur in the case of $n = 3$ were presented in Chap. 9 by computing the Clebsch-Gordan coefficients extensively for the subgroups of $SU(3)$. Indications from the lattice integrable model perspective were given in [104].

The natural question to pose is why there should be such correspondences. Indeed, why should there be such an intricate chain of connections among string theory on orbifolds, finite representation theory, graph theory, affine characters and WZW modular invariants? In this chapter, we hope to propose a unified quest to answer this question from the point of view of the conformal field theory description of Gorenstein singularities. We also observe that category theory seems to prove a common basis for all these theories.

We begin in two dimensions, where there have been numerous independent works in the past few decades in both mathematics and physics to establish various correspondences. In this case, the all-permeating theme is the ADE classification. In particular, there is the original McKay's correspondence between finite subgroups of $SU(2)$ and the ADE Dynkin diagrams [32] to which we henceforth refer as the *Algebraic McKay Correspondence*. On the geometry side, the representation rings of these groups were related to the Groethendieck (cohomology) rings of the resolved manifolds constructed from the Gorenstein singularity of the respective groups [130, 99]; we shall refer to this as the *Geometric McKay Correspondence*. Now from physics, studies in conformal field theory (CFT) have prompted many beautiful connections among graph theory, fusion algebra, and modular invariants [93, 94, 101, 102, 111, 112]. The classification of the modular invariant partition function associated with $\widehat{SU(2)}$ Wess-Zumino-Witten (WZW) models also mysteriously falls into an ADE type [100]. There have been some recent attempts to explain this seeming accident from the supersymmetric field theory and brane configurations perspective [106, 108]. In this chapter we push from the direction of the Geometric McKay Correspondence and see how Calabi-Yau (CY) non-linear sigma models constructed on the Gorenstein singularities associated with the finite groups may be related to Kazama-Suzuki coset models [111, 112, 113, 114, 118, 17], which in turn can be related to the WZW models. This link would provide a natural setting for the emergence of the ADE classification of the modular invariants. In due course, we will review and establish a catalog of inter-relations, whereby forming a complex web and unifying many independently noted correspondences. Moreover, we find a common theme of categorical axioms that all of these theories seem to satisfy and suggest why the ADE classification and its extensions arise so naturally. This web, presented in Figure 10-1, is the central idea of our chapter. Most of the correspondences in Figure 10-1 actually have been discussed in the string theory literature although not all at once in a unified manner with a mathematical tint.

Our purpose is two-fold. Firstly, we shall show that tracing through the arrows in Figure 10-1 may help to enlighten the links that may seem accidental. Moreover, and perhaps more importantly, we propose that this program may be extended beyond two dimensions and hence beyond A-D-E. Indeed, algebraic geometers have done extensive research in generalizing McKay's original correspondence to Gorenstein singularities of dimension greater than 2 ([129] to [136]); many standing conjectures exist in this respect. On the other hand, there is the conjecture mentioned above regarding the $\widehat{SU(n)}_k$ WZW and the subgroups of $SU(n)$ in general. It is our hope that Figure 10-1 remains valid for $n > 2$ so that these conjectures may be attacked by the new pathways we propose. We require and sincerely hope for the collaborative effort of many experts in mathematics and physics who may take interest in this attempt to unify these various connections.

The outline of the chapter follows the arrows drawn in Figure 10-1. We begin in §10.2 by summarizing the ubiquitous ADE classifications, and §10.3 will be devoted to clarifying these ADE links, while bearing in mind how such ubiquity may permeate to higher dimensions. It will be divided in to the following subsections:

- I. The link between representation theory for finite groups and quiver graph theories (Algebraic McKay);
- II. The link between finite groups and crepant resolutions of Gorenstein singu-

larities (Geometric McKay);

- III. The link between resolved Gorenstein singularities, Calabi-Yau manifolds and chiral rings for the associated non-linear sigma model (Stringy Gorenstein resolution);

- IV. The link between quiver graph theory for finite groups and WZW modular invariants in CFT, as discovered in the study of of string orbifold theory (Conjecture in Chap. 9);

and finally, to complete the cycle of correspondences,

- V. The link between the singular geometry and its conformal field theory description provided by an orbifoldized coset construction which contains the WZW theory.

In §10.4 we discuss arrow V which fills the gap between mathematics and physics, explaining why WZW models have the magical properties that are so closely related to the discrete subgroups of the unitary groups as well as to geometry. From all these links arises §10.6 which consists of our conjecture that there exists a conformal field theory description of the Gorenstein singularities for higher dimensions, encoding the relevant information about the discrete groups and the cohomology ring. In §10.5, we hint at how these vastly different fields may have similar structures by appealing to the so-called ribbon and quiver categories.

Finally in the concluding remarks of §10.7, we discuss the projection for future labors.

We here transcribe our observations with the hope they would spark a renewed interest in the study of McKay correspondence under a possibly new light of CFT, and vice versa. We hope that Figure 10-1 will open up many interesting and exciting pathways of research and that not only some existing conjectures may be solved by new methods of attack, but also further beautiful observations could be made.

Notations and Nomenclatures

We put a $\tilde{}$ over a singular variety to denote its resolved geometry. By dimension we mean complex dimension unless stated otherwise. Also by "representation ring of Γ," we mean the ring formed by the tensor product decompositions of the irreducible representations of Γ. The capital Roman numerals, I–V, in front of the section headings correspond to the arrows in Figure 10-1.

10.2 Ubiquity of ADE Classifications

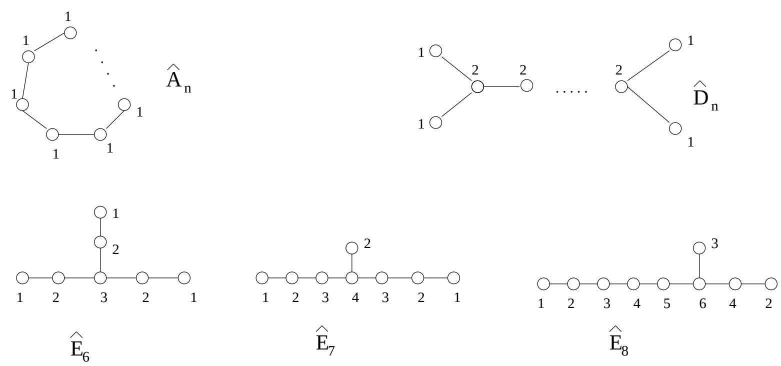

Figure 10-2: The Affine Dynkin Diagrams and Labels.

In this section, we summarize the appearance of the ADE classifications in physics and mathematics and their commonalities.

It is now well-known that the complexity of particular algebraic and geometric structures can often be organized into classification schemes of the ADE type. The first hint of this structure began in the 1884 work of F. Klein in which he classified the discrete subgroups Γ of $SU(2)$ [30]. These were noted to be in 1-1 correspondence with the Platonic solids in $\mathbb{R}^3$, and with some foresight, we write them as:

- type A: the cyclic groups (the regular polygons);
- type D: the binary dihedral groups (the regular dihedrons) and
- type E: the binary tetrahedral (the tetrahedron), octahedral (the cube and the octahedron) and icosahedral (the dodecahedron and the icosahedron) groups,

	Theory	Nodes	Matrices
(a)	Finite Subgroup Γ of $SU(2)$	Irreducible Representations	Clebsch-Gordan Coefficients
(b)	Simple Lie algebra of type ADE	Simple Roots	Extended Cartan matrix
(c)	Quiver Dynkin Diagrams	Dynkin Labels	Adjacency Matrix
(d)	Minimal Resolution $X \to \mathbb{C}^2/\Gamma$	Irreducible Components of the Exceptional Divisor (Basis of $H_2(X, \mathbb{Z})$)	Intersection Matrix
(e)	$\widehat{SU(2)}_k$ WZW Model	Modular Invariants / WZW Primary Operators	Fusion Coefficients
(f)	Landau-Ginzburg	Chiral Primary Operators	Chiral Ring Coefficients
(g)	CY Nonlinear Sigma Model	Twisted Fields	Correlation Functions

Table 10.1: ADE Correspondences in 2-dimensions. The same graphs and their affine extensions appear in different theories.

where we have placed in parenthesis next to each group the geometrical shape for which it is the double cover of the symmetry group.

The ubiquity of Klein's original hint has persisted till the present day. The ADE scheme has manifested itself in such diverse fields as representation theory of finite groups, quiver graph theory, Lie algebra theory, algebraic geometry, conformal field theory, and string theory. It will be the intent of the next section to explain the details of the correspondences appearing in Table 10.1, and we will subsequently propose their extensions in the remainder of the chapter.

10.3 The Arrows of Figure 1.

In this section, we explain the arrows appearing in Figure 1. We verify that there are compelling evidences in favor of the picture for the case of $2_{\mathbb{C}}$-dimensions, and we will propose its generalization to higher dimensions in the subsequent sections, hoping that it will lead to new insights on the McKay correspondence as well as conformal field theory.

10.3.1 (I) The Algebraic McKay Correspondence

In the full spirit of the omnipresent ADE classification, it has been noticed in 1980 by J. McKay that there exists a remarkable correspondence between the discrete subgroups of $SU(2)$ and the affine Dynkin graphs [32]. Indeed, this is why we have labeled the subgroups in the manner we have done.

DEFINITION 10.3.7 *For a finite group Γ, let $\{r_i\}$ be its set of irreducible representations (irreps), then we define the coefficients m_{ij}^k appearing in*

$$r_k \otimes r_i = \bigoplus_j m_{ij}^k r_j \tag{10.3.1}$$

to be the **Clebsch-Gordan coefficients** *of Γ.*

For $\Gamma \subset SU(2)$ McKay chose a fixed (not necessarily irreducible) representation R in lieu of general k in 10.3.1 and defined matrices m_{ij}^R. He has noted that up to automorphism, there always exists a unique 2-dimensional representation, which for type A is the tensor sum of 2 dual 1-dimensional irreps and for all others the self-conjugate 2-dimensional irrep. It is this $R = 2$ which we choose and simply write the matrix as m_{ij}. The remarkable observation of McKay can be summarized in the following theorem; the original proof was on a case-to-case basis and Steinberg gave a unified proof in 1981 [32].

THEOREM 10.3.14 (McKay-Steinberg) *For $\Gamma = A, D, E$, the matrix m_{ij} is $2I$ minus the Cartan matrix of the affine extensions of the respective simply-laced simple Dynkin diagrams $\widehat{A}, \widehat{D}$ and $\widehat{E}$, treated as undirected C_2-graphs (i.e., maximal eigenvalue of the adjacency matrix is 2).*

Moreover, the Dynkin labels of the nodes of the affine Dynkin diagrams are precisely the dimensions of the irreps. Given a discrete subgroup $\Gamma \subset SU(2)$, there thus exists a Dynkin diagram that encodes the essential information about the representation ring of Γ. Indeed the number of nodes should equal to the number of irreps and thus by a rudimentary fact in finite representation theory, subsequently equals the number of conjugacy classes of Γ. Furthermore, if we remove the node corresponding to the trivial 1-dimensional (principal) representation, we obtain the regular ADE Dynkin diagrams. We present these facts in the following diagram:

Clebsch-Gordan Coefficients for $\Gamma = A, D, E$	$\longleftrightarrow$	Dynkin Diagram of $\widehat{A}, \widehat{D}, \widehat{E}$	$\longleftrightarrow$	Cartan matrix and dual Coxeter labels of $\widehat{A}, \widehat{D}, \widehat{E}$

This is Arrow I of Figure 10-1.

Proofs and extension of McKay's results from geometric perspectives of this originally combinatorial/graph-theoretic theorem soon followed; they caused fervent activities in both algebraic geometry and string theory (see e.g., [122, 129, 130, 99]). Let us first turn to the former.

10.3.2 (II) The Geometric McKay Correspondence

In this section, we are interested in crepant resolutions of Gorenstein quotient singularities.

DEFINITION 10.3.8 *The singularities of* $\mathbb{C}^n/\Gamma$ *for* $\Gamma \subset GL(n, \mathbb{C})$ *are called* **Gorenstein** *if there exists a nowhere-vanishing holomorphic n-form*[1] *on regular points.*

Restricting Γ to $SU(n)$ would guarantee that the quotient singularities are Gorenstein.

DEFINITION 10.3.9 *We say that a smooth variety* $\widetilde{M}$ *is a* **crepant** *resolution of a singular variety* M *if there exists a birational morphism* $\pi : \widetilde{M} \to M$ *such that the canonical sheaves* $\mathcal{K}_M$ *and* $\mathcal{K}_{\widetilde{M}}$ *are the same, or more precisely, if* $\pi^*(\mathcal{K}_M) = \mathcal{K}_{\widetilde{M}}$.

[1] *Gorenstein singularities thus provide local models of singularities on Calabi-Yau manifolds.*

For $n \leq 3$, Gorenstein singularities always admit crepant resolutions [130, 99]. On the other hand, in dimensions greater than 3, there are known examples of terminal Gorenstein singularities which do not admit crepant resolutions. It is believed, however, that when the order of Γ is sufficiently larger than n, there exist crepant resolutions for most of the groups.

The traditional ADE classification is relevant in studying the discrete subgroups of $SU(2)$ and resolutions of Gorenstein singularities in two complex-dimensions. Since we can choose an invariant Hermitian metric on $\mathbb{C}^2$, finite subgroups of $GL(2,\mathbb{C})$ and $SL(2,\mathbb{C})$ are conjugate to finite subgroups of $U(2)$ and $SU(2)$, respectively. Here, motivated by the string compactification on manifolds of trivial canonical bundle, we consider the linear actions of non-trivial discrete subgroups Γ of $SU(2)$ on $\mathbb{C}^2$. Such quotient spaces $M = \mathbb{C}^2/\Gamma$, called *orbifolds*, have fixed points which are isolated Gorenstein singularities of the ADE type studied by Felix Klein.

As discussed in the previous sub-section, McKay[32] has observed a 1-1 correspondence between the non-identity conjugacy classes of discrete subgroups of $SU(2)$ and the Dynkin diagrams of A-D-E simply-laced Lie algebras, and this relation in turn provides an indirect correspondence between the orbifold singularities of M and the Dynkin diagrams. In fact, there exists a direct geometric correspondence between the crepant resolutions of M and the Dynkin diagrams. Classical theorems in algebraic geometry tell us that there exists a unique crepant resolution $(\widetilde{M}, \pi)$ of the Gorenstein singularity of M for all $\Gamma \subset SU(2)$. Furthermore, the exceptional divisor $E = \pi^{-1}(0)$ is a compact, connected union of irreducible $1_{\mathbb{C}}$-dimensional curves of genus zero[2] such that their intersection matrix is represented by the simply-laced Dynkin diagram associated to Γ. More precisely, each node of the diagram corresponds to an irreducible $\mathbb{P}^1$, and the intersection matrix is negative of the Cartan matrix of the Dynkin diagram such that two $\mathbb{P}^1$'s intersect transversely at one point if and only if the two nodes are connected by a line in the diagram. In particular, we see that the curves have self-intersection numbers -2 which exhibits the singular nature of the orbifold upon blowing them down. Simple consideration shows that these curves

[2] We will refer to them as $\mathbb{P}^1$ blow-ups.

$f(x,y,z)$	Subgroup Γ	Order of Γ
$x^2+y^2+z^{k+1}$	A_k Cyclic	$k+1$
$x^2+y^2z+z^{k-1}$	D_k Binary Dihedral	$4(k-2)$
$x^2+y^4+z^3$	E_6 Binary Tetrahedral	24
$x^2+y^3z+z^3$	E_7 Binary Octahedral	48
$x^2+y^5+z^3$	E_8 Binary Icosahedral	120

Table 10.2: Algebraic Surfaces with Quotient Singularities

form a basis of the homology group $H_2(\widetilde{M},\mathbb{Z})$ which is seen to coincide with the root lattice of the associated Dynkin diagram by the above identification. Now, combined with the algebraic McKay correspondence, this crepant resolution picture yields a 1-1 correspondence between the basis of $H_2(\widetilde{M},\mathbb{Z})$ and the non-identity conjugacy classes of Γ. We recapitulate the above discussion in the following diagram:

$H_2(\widetilde{M},\mathbb{Z})$ of the blow-up	$\longleftrightarrow$	Dynkin Diagram of Γ	$\longleftrightarrow$	Non-identity Conjugacy Classes of Γ

This is Arrow II in Figure 10-1. Note incidentally that one can think of irreducible representations as being dual to conjugacy classes and hence as basis of $H^2(\widetilde{M},\mathbb{Z})$. This poses a subtle question of which correspondence is more natural, but we will ignore such issues in our discussions.

It turns out that M is not only analytic but also algebraic; that is, M is isomorphic to $f^{-1}(0)$, where $f:\mathbb{C}^3\to\mathbb{C}$ is one of the polynomials in Table 10.2 depending on Γ. The orbifolds defined by the zero-loci of the polynomials are commonly referred to as the singular ALE spaces.

10.3.3 (II, III) McKay Correspondence and SCFT

One of the first relevance of ADE series in conformal field theory appeared in attempts to classify $N=2$ superconformal field theories (SCFT) with central charge $c<3$ [111]. Furthermore, the exact forms of the ADE polynomials in Table 10.2 appeared in a similar attempt to classify certain classes of $N=2$ SCFT in terms of Landau-Ginzburg (LG) models. The LG super-potentials were precisely classified by the polynomials, and the chiral ring and quantum numbers were computed with applications of singularity theory [114]. The LG theories which realize coset models would appear again in this chapter to link the WZW to geometry.

In this subsection, we review how string theory, when the B-field is non-vanishing, resolves the orbifold singularity and how it encodes the information about the cohomology of the resolved manifold. Subsequently, we will consider the singular limit of the conformal field theory on orbifolds by turning off the B-field, and we will argue that, in this singular limit, the $\widehat{SU(2)}_k$ WZW fusion ring inherits the information about the cohomology ring from the smooth theory.

Orbifold Resolutions and Cohomology Classes

Our discussion here will be general and not restricted to $n = 2$. Many remarkable features of string theory stem from the fact that we can "pull-back" much of the physics on the target space to the world-sheet, and as a result, the resulting world-sheet conformal field theory somehow encodes the geometry of the target space. One example is that CFT is often[3] insensitive to Gorenstein singularities and quantum effects revolve the singularity so that the CFT is smooth. More precisely, Aspinwall [125] has shown that non-vanishing of the NS-NS B-field makes the CFT smooth. In fact, string theory predicts the Euler characteristic of the *resolved* orbifold [122]; the local form of the statement is

CONJECTURE 10.3.1 (Stringy Euler characteristic) *Let $M = \mathbb{C}^n/\Gamma$ for $\Gamma \subset SU(n)$ a finite subgroup. Then, there exists a crepant resolution $\pi : \widetilde{M} \to M$ such that*

$$\chi(\widetilde{M}) = |\{\textit{Conjugacy Classes of}\ \Gamma\}| \ . \tag{10.3.2}$$

[3]Not all CFT on singular geometry are smooth. For example, there are examples of singular CFT's defined on singular backgrounds, such as in the case of gauge symmetry enhancement of the Type IIA string theory compactified on singular $K3$ where the B-field vanishes [126]. Later, we will discuss a tensored coset model [128] describing this singular non-linear sigma model and relate it to the algebraic McKay Correspondence.

Furthermore, the Hodge numbers of resolved orbifolds were also predicted by Vafa for CY manifolds realized as hypersurfaces in weighted projective spaces and by Zaslow for Kähler manifolds [119]. In dimension three, it has been proved [130, 99, 132] that every Gorenstein singularity admits a crepant resolution[4] and that every crepant resolution satisfies the Conjecture 10.3.1 and the Vafa-Zaslow Hodge number formulae. For higher dimensions, there are compelling evidences that the formulae are satisfied by all crepant resolutions, when they exist.

As the Euler Characteristic in mathematics is naturally defined by the Hodge numbers of cohomology classes, motivated by the works of string theorists and the fact that $\widetilde{M}$ has no odd-dimensional cohomology[5], mathematicians have generalized the classical McKay Correspondence [130, 99, 132, 133] to geometry.

The geometric McKay Correspondence in 2-dimensions actually identifies the cohomology ring of $\widetilde{M}$ and the representation ring of Γ not only as vector spaces but as rings. Given a finite subgroup $\Gamma \subset SU(2)$, the intersection matrix of the irreducible components of the exceptional divisor of the resolved manifold is given by the negative of the Cartan matrix of the associated Dynkin diagram which is specified by the algebraic McKay Correspondence. Hence, there exists an equivalence between the tensor product decompositions of conjugacy classes and intersection pairings of homology classes. Indeed in [134], Ito and Nakajima prove that for all $\Gamma \subset SU(2)$ and for abelian $\Gamma \subset SU(3)$, the Groethendieck (cohomology) ring of $\widetilde{M}$ is isomorphic as a $\mathbb{Z}$-module to the representation ring of Γ and that the intersection pairing on its dual, the Groethendieck group of coherent sheaves on $\pi^{-1}(0)$, can be expressed as the Clebsch-Gordan coefficients. Furthermore, string theory analysis also predicts a similar relation between the two ring structures [123].

The geometric McKay Correspondence can thus be stated as

CONJECTURE 10.3.2 (Geometric McKay Correspondence) *Let* Γ, M, *and* $\widetilde{M}$

[4]In fact, a given Gorenstein singularity generally admits many crepant resolutions [139]. String theory so far has yielded two distinguished desingularizations: the traditional CFT resolution without discrete torsion and deformation with discrete torsion [124]. In this chapter, we are concerned only with Kähler resolutions without discrete torsion.

[5]See [132] for a discussion on this point.

be as in Conjecture 10.3.1. Then, there exist bijections

$$\boxed{\textit{Basis of } H^*(\widetilde{M}, \mathbb{Z})} \longleftrightarrow \boxed{\{\textit{Irreducible Representations of } \Gamma\}}$$
$$\boxed{\textit{Basis of } H_*(\widetilde{M}, \mathbb{Z})} \longleftrightarrow \boxed{\{\textit{Conjugacy Classes of } \Gamma\}},$$

and there is an identification between the two ring structures.

Question of Ito and Reid and Chiral Ring

In [132], Ito and Reid raised the question whether the cohomology ring[6] $H^*(\widetilde{M})$ is generated by $H^2(\widetilde{M})$. In this subsection, we rephrase the question in terms of $N = 2$ SCFT on $M = \mathbb{C}^n/\Gamma$, $\Gamma \subset SU(n)$. String theory provides a way[7] of computing the cohomology of the resolved manifold $\widetilde{M}$. Let us briefly review the method for the present case [122]:

The cohomology of $\widetilde{M}$ consists of those elements of $H^*(\mathbb{C}^n)$ that survive the projection under Γ and new classes arising from the blow-ups. In this case, $H^0(\mathbb{C}^n)$ is a set of all constant functions on $\mathbb{C}^n$ and survives[8] the projection, while all other cohomology classes vanish. Hence, all other non-trivial elements of $H^*(\widetilde{M})$ arise from the blow-up process; in string theory language, they correspond to the twisted chiral primary operators, which are not necessarily all marginal. In the $N = 2$ SCFT of non-linear sigma-model on a compact CY manifold, the $U(1)$ spectral flow identifies the chiral ring of the SCFT with the cohomology ring of the manifold, modulo quantum corrections. For non-compact cases, by considering a topological non-linear σ-model, the A-model chiral ring matches the cohomology ring and the blow-ups still correspond to the twisted sectors.

An $N = 2$ non-linear sigma model on a CY n-fold X has two topological twists called the A and B-models, of which the "BRST" non-trivial observables [143] encode

[6]Henceforth, $\dim M = n$ is not restricted to 2.
[7]It is believed that string theory somehow picks out a distinguished resolution of the orbifold, and the following discussion pertains to such a resolution when it exists.
[8]This cohomology class should correspond to the trivial representation in the McKay correspondence.

the information about the Kähler and complex structures of X, respectively. The correlation functions of the A-model receive instanton corrections whereas the classical computations of the B-model give exact quantum answers. The most efficient way of computing the A-model correlation functions is to map the theory to a B-model on another manifold Y which is a mirror[9] of X [140]. Then, the classical computation of the B-model on Y yields the full quantum answer for the A-model on X.

In this chapter, we are interested in Kähler resolutions of the Gorenstein singularities and, hence, in the A-model whose chiral ring is a quantum deformation of the classical cohomology ring. Since all non-trivial elements of the cohomology ring, except for H^0, arise from the twisted sector or blow-up contributions, we have the following reformulation of the Geometric McKay Correspondence which is well-established in string theory:

PROPOSITION 10.3.2 (String Theory McKay Correspondence) *Let Γ be a discrete subgroup of $SU(n)$ such that the Gorenstein singularities of $M = \mathbb{C}^n/\Gamma$ has a crepant resolution $\pi : \widetilde{M} \to M$. Then, there exists a following bijection between the cohomology and A-model data:*

$$\boxed{\textit{Basis of } \bigoplus_{i>0} H^i(\widetilde{M})} \quad \longleftrightarrow \quad \boxed{\{\textit{Twisted Chiral Primary Operators}\}}\ , \qquad (10.3.3)$$

or equivalently, by the Geometric McKay Correspondence,

$$\boxed{\{\textit{Conjugacy classes of } \Gamma\}} \quad \longleftrightarrow \quad \boxed{\{\textit{Twisted Elements of the Chiral Ring}\}}\ . \qquad (10.3.4)$$

Thus, since all $H^i, i > 0$ arise from the twisted chiral primary but not necessarily marginal fields and since the marginal operators correspond to H^2, we can now reformulate the question of whether H^2 generates H^* as follows:

[9]Mirror symmetry has been intensely studied by both mathematicians and physicists for the past decade, leading to many powerful tools in enumerative geometry. A detailed discussion of mirror symmetry is beyond the scope of this chapter, and we refer the reader to [140] for introductions to the subject and for references.

Do the marginal twisted chiral primary fields generate the entire twisted chiral ring?

This kind of string theory resolution of orbifold singularities is Arrow III in Figure 10-1. In §10.4, we will see how a conformal field theory description of the singular limit of these string theories naturally allows us to link geometry to representation theory. In this way, we hint why McKay correspondence and the discoveries of [100] are not mere happy flukes of nature, as it will become clearer as we proceed.

10.3.4 (I, IV) McKay Correspondence and WZW

When we calculate the partition function for the WZW model with its energy momentum tensor associated to an algebra $\widehat{g}_k$ of level k, it will be of the form[10]:

$$Z(\tau) = \sum_{\widehat{\lambda}, \widehat{\xi} \in P_+^{(k)}} \chi_{\widehat{\lambda}}(\tau) \mathcal{M}_{\widehat{\lambda}, \widehat{\xi}} \overline{\chi_{\widehat{\xi}}}(\overline{\tau})$$

where $P_+^{(k)}$ is the set of dominant weights and $\chi_{\widehat{\lambda}}$ is the affine character of $\widehat{g}_k$. The matrix $\mathcal{M}$ gives the multiplicity of the highest weight modules in the decomposition of the Hilbert space and is usually referred to as the *mass matrix*. Therefore the problem of classifying the modular invariant partition functions of WZW models is essentially that of the integrable characters χ of affine Lie algebras.

In the case of $\widehat{g}_k = \widehat{SU(2)}_k$, all the modular invariant partition functions are classified, and they fall into an ADE scheme ([93] to [102]). In particular, they are of the form of sums over modulus-squared of combinations of the weight k Weyl-Kac character χ_λ^k for $\widehat{SU(2)}$ (which is in turn expressible in term of Jacobi theta functions), where the level k is correlated with the rank of the ADE Dynkin diagrams as shown in Table 10.3 and λ are the eigenvalues for the adjacency matrices of the ADE Dynkin diagrams. Not only are the modular invariants classified by these graphs, but some of the fusion ring algebra can be reconstructed from the graphs.

[10] we henceforth use the notation in [93]

Dynkin Diagram of Modular Invariants	Level of WZW
A_n	$n-1$
D_n	$2n-4$
E_6	10
E_7	16
E_8	28

Table 10.3: The ADE-Dynkin diagram representations of the modular invariants of the $\widehat{SU(2)}$ WZW.

Though still largely a mystery, the reason for this classification can be somewhat traced to the so-called *fusion rules*. In a rational conformal field theory, the fusion coefficient $N_{\phi_i\phi_j}^{\phi_k^*}$ is defined by

$$\phi_i \times \phi_j = \sum_{\phi_k^*} \mathcal{N}_{\phi_i\phi_j}^{\phi_k^*} \phi_k^* \tag{10.3.5}$$

where $\phi_{i,j,k}$ are chiral[11] primary fields. This fusion rule provides such vital information as the number of independent coupling between the fields and the multiplicity of the conjugate field ϕ_k^* appearing in the operator product expansion (OPE) of ϕ_i and ϕ_j. In the case of the WZW model with the energy-momentum tensor taking values in the algebra $\widehat{g}_k$ of level k, we can recall that the primary fields have integrable representations $\widehat{\lambda}$ in the dominant weights of $\widehat{g}_k$, and subsequently, (10.3.5) reduces to

$$\widehat{\lambda} \otimes \widehat{\mu} = \bigoplus_{\widehat{\nu} \in P_+^k} \mathcal{N}_{\widehat{\lambda}\widehat{\mu}}^{\widehat{\nu}}\ \widehat{\nu}.$$

Indeed now we see the resemblance of (10.3.5) coming from conformal field theory to (10.3.1) coming from finite representation theory, hinting that there should be some underlying relation. We can of course invert (10.3.1) using the properties of finite characters, just as we can extract $\mathcal{N}$ by using the Weyl-Kac character formula (or by the Verlinde equations).

Conformal field theorists, inspired by the ADE classification of the minimal mod-

[11]Chirality here means left- or right-handedness not chirality in the sense of $N = 2$ superfields.

els, have devised similar methods to treat the fusion coefficients. It turns out that in the simplest cases the fusion rules can be generated entirely from one special case of $\widehat{\lambda} = f$, the so-called fundamental representation. This is of course in analogy to the unique (fundamental) 2-dimensional representation R in McKay's paper. In this case, all the information about the fusion rule is encoded in a matrix $[N]_{ij} = \mathcal{N}_{fi}^{j}$, to be treated as the adjacency matrix of a finite graph. Conversely we can define a commutative algebra to any finite graph, whose adjacency matrix is defined to reproduce the fusion rules for the so-called *graph algebra*. It turns out that in the cases of A_n, D_{2n}, E_6 and E_8 Dynkin diagrams, the resulting graph algebra has an subalgebra which reproduces the (extended) fusion algebra of the respective ADE $\widehat{SU(2)}$ WZW models.

From another point of view, we can study the WZW model by quotienting it by discrete subgroups of $SU(2)$; this is analogous to the twisted sectors in string theory where for the partition function we sum over all states invariant under the action of the discrete subgroup. Of course in this case we also have an A-D-E-type classification for the finite groups due to the McKay Correspondence, therefore speculations have risen as to why both the discrete subgroups and the partition functions are classified by the same graphs [93, 104], which also reproduce the associated ring structures. The reader may have noticed that this connection is somewhat weaker than the others hitherto considered, in the sense that the adjacency matrices do not correspond 1-1 to the fusion rules. This subtlety will be addressed in §10.4 and §10.5.

Indeed, the graph algebra construction has been extended to $\widehat{SU(3)}$ and a similar classification of the modular invariants have in fact been done and are shown to correspond to the so-called *generalized Dynkin Diagrams* [93, 96, 104]. On the other hand, the Clebsch-Gordan coefficients of the McKay type for the discrete subgroups of $SU(3)$ have been recently computed in the context of studying D3-branes on orbifold singularities (Chap 9). It was noted that the adjacency graphs drawn in the two different cases are in some form of correspondence and was conjectured that this relationship might extend to $\widehat{SU(n)_k}$ model for n other than 2 and 3 as well. It is hoped that this problem may be attacked by going through the other arrows.

We have now elucidated arrows I and IV in Figure 10-1.

10.4 The Arrow V: σ-model/LG/WZW Duality

We here summarize the link V in Figure 10-1 for ALE spaces, as has been established in [128].

It is well-known that application of catastrophe theory leads to the ADE classification of Landau-Ginzburg models [114]. It has been subsequently shown that the renormalization group fixed points of these theories actually provide the Lagrangian formulations of $N = 2$ discrete minimal models [118]. What is even more surprising and beautiful is Gepner's another proposal [112] that certain classes of $N = 2$ non-linear sigma-models on CY 3-folds are equivalent to tensor products of $N = 2$ minimal models with the correct central charges and $U(1)$ projections. Witten has successfully verified the claim in [17] using a gauged linear-sigma model which interpolates between Calabi-Yau compactifications and Landau-Ginzburg orbifolds.

In a similar spirit, Ooguri and Vafa have considered LG orbifolds[12] of the tensor product of $SL(2,\mathbb{R})/U(1)$ and $SU(2)/U(1)$ Kazama-Suzuki models[13] [151] and have shown that the resulting theory describes the singular conformal field theory of the non-linear sigma-model with the B-field turned off. In particular, they have shown that the singularity on A_{n-1} ALE space is described by the

$$\frac{\frac{SL(2)_{n+2}}{U(1)} \times \frac{SU(2)_{n-2}}{U(1)}}{\mathbb{Z}_n} \tag{10.4.6}$$

[12] The universality classes of the LG models are completely specified by their superpotentials W, and such a simple characterization leads to very powerful methods of detailed computations [113, 116]. Generalizations of these models have many important applications in string theory, and the OPE coefficients of topological LG theories with judiciously chosen non-conformal deformations yield the fusion algebra of rational conformal field theories. In [149], Gepner has shown that the topological LG models with deformed Grassmannian superpotentials yield the fusion algebra of the $\widehat{SU(n)}_k$ WZW, illustrating that much information about non-supersymmetric RCFT can be extracted from their $N = 2$ supersymmetric counterparts. Gepner's superpotential could be viewed as a particular non-conformal deformation of the superpotential appearing in Ooguri and Vafa's model.

[13] The $SL(2,\mathbb{R})/U(1)$ coset model describes the two-dimensional black hole geometry [152], while the $SU(2)/U(1)$ Kazama-Suzuki model is just the $N = 2$ minimal model.

ALE Type	Level of WZW
A_n	$n-1$
D_n	$2n-4$
E_6	10
E_7	16
E_8	28

Table 10.4: The WZW subsector of the Ooguri-Vafa conformal field theory description of the singular non-linear sigma-model on ALE.

orbifold model which contains the $\widehat{SU(2)}_{n-2}$ WZW theory at level $k = n-2$. The coset descriptions of the non-linear σ-models on D and E-type ALE spaces also contain the corresponding WZW theories whose modular invariants are characterized by the D and E-type resolution graphs of the ALE spaces. The full orbifoldized Kazama-Suzuki model has fermions as well as an extra Feigin-Fuchs scalar, but we will be interested only in the WZW sector of the theory, for this particular sector contains the relevant information about the discrete group Γ and the cohomology of $\widetilde{\mathbb{C}^2/\Gamma}$. We summarize the results in Table 10.4.

We now assert that many amazing ADE-related properties of the $\widehat{SU(2)}$ WZW conformal field theory and the McKay correspondence can be interpreted as consequences of the fact that the conformal field theory description of the singularities of ALE spaces contains the $\widehat{SU(2)}$ WZW. That is, we argue that the WZW theory inherits most of the geometric information about the ALE spaces.

10.4.1 Fusion Algebra, Cohomology and Representation Rings

Comparing the Table 10.4 with the Table 10.3, we immediately see that the graphical representations of the homology intersections of $H_2(\widetilde{\mathbb{C}^2/\Gamma}, \mathbb{Z})$ and the modular invariants of the associated $\widehat{SU(2)}$ WZW subsector are identical.

Let us recall how $\widehat{SU(2)}_k$ WZW model has been historically related to the finite subgroups of $SU(2)$. Meanwhile we shall recapitulate some of the key points in §10.3.4. The finite subgroups Γ of $SU(2)$ have two infinite and one finite series. The Algebraic McKay Correspondence showed that the representation ring of each finite group admits a graphical representation such that the two infinite series have the precise A and D Dynkin diagrams while the finite series has the $E_{6,7,8}$ Dynkin diagrams. Then, it was noticed that the same Dynkin diagrams classify the modular invariants of the $\widehat{SU(2)}_k$ WZW model, and this observation was interesting but there was no *a priori* connection to the representation theory of finite subgroups. It was later discovered that the Dynkin diagrams also encode the $\widehat{SU(2)}_k$ WZW fusion rules or their extended versions[14]. Independently of the WZW models, the Dynkin diagrams are also well-known to represent the homological intersection numbers on $\widetilde{\mathbb{C}^2/\Gamma}$, which are encoded the chiral ring structure of the sigma-model when $B \neq 0$. What Ooguri and Vafa have shown us is that when the B-field is set to zero, the information about the chiral ring and the discrete subgroup Γ do not get destroyed but get transmitted to the orbifoldized Kazama-Suzuki model which contains the $\widehat{SU(2)}_k$ WZW.

Let us demonstrate the fusion/cohomology correspondence for the A-series. Let C_i be the basis of $H^2(\widetilde{\mathbb{C}^2/\mathbb{Z}_n}, \mathbb{Z})$ and Q_{ij} their intersection matrix inside the A_{n-1} ALE space. The $\widehat{SU(2)}_k$ WZW at level $k = n-2$ has $k+1$ primary fields $\phi_a, a = 0, 1, \ldots n-2$. Then, the fusion of the fundamental field ϕ_1 with other primary fields

$$\phi_1 \times \phi_a = \mathcal{N}_{1a}{}^b \; \phi_b \tag{10.4.7}$$

is precisely given by the intersection matrix, i.e. $\mathcal{N}_{1a}{}^b = Q_{ab}$. Now, let N_1 be the matrix whose components are the fusion coefficients $(N_1)_{ab} = \mathcal{N}_{1a}{}^b$, and define $k-1$ matrices $N_i, i = 2, \ldots, k$ recursively by the following equations

$$\begin{aligned}
N_1 N_1 &= N_0 + N_2 \\
N_1 N_2 &= N_1 + N_3 \\
N_1 N_3 &= N_2 + N_4 \; \cdots \\
N_1 N_{k-1} &= N_{k-2} + N_k \\
N_1 N_k &= N_{k-1}
\end{aligned}$$

[14] See [93] for a more complete discussion of this point.

where $N_0 = \mathrm{Id}_{(k+1)\times(k+1)}$. That is, multiplication by N_1 with N_j just lists the neighboring nodes in the A_{k+1} Dynkin diagram with a sequential labeling. Identifying the primary fields ϕ_i with the matrices N_i, it is easy to see that the algebra of N_i generated by the defining equations (10.4.8) precisely reproduces the fusion algebra of the ϕ_i for the $\widehat{SU(2)}_k$ WZW at level $k = n - 2$. This algebra is the aforementioned graph algebra in conformal field theory. The graph algebra has been known for many years, but what we are proposing in this chapter is that the graph algebra is a consequence of the fact that the WZW contains the information about the cohomology of the corresponding ALE space.

Furthermore, recall from §10.3.1 that the intersection matrix is identical to the Clebsch-Gordan coefficients m_{ij}, ignoring the affine node. This fact is in accordance with the proof of Ito and Nakajima [134] that the cohomology ring of $\widetilde{\mathbb{C}^2/\Gamma}$ is isomorphic to the representation ring $\mathcal{R}(\Gamma)$ of Γ. At first sight, it appears that we have managed to reproduce only a subset of Clebsch-Gordan coefficients of $\mathcal{R}(\Gamma)$ from the cohomology or equivalently the fusion ring. For the A-series, however, we can easily find all the Clebsch-Gordan coefficients of the irreps of $\mathbb{Z}_n$ from the fusion algebra by simply relabeling the irreps and choosing a different self-dual 2-dimensional representation. This is because the algebraic McKay correspondence produces an A_{n-1} Dynkin diagram for any self-dual 2-dimensional representation R and choosing a different R amounts to relabeling the nodes with different irreps. The graph algebras of the $\widehat{SU(2)}_k$ WZW theory for the D and E-series actually lead not to the fusion algebra of the original theory but to that of the extended theories, and these cases require further investigations.

String theory is thus telling us that the cohomology ring of $\widetilde{\mathbb{C}^2/\Gamma}$, fusion ring of $\widehat{SU(2)}$ WZW and the representation ring of Γ are all equivalent. We summarize the noted correspondences and our observations in Figure 10-3.

10.4.2 Quiver Varieties and WZW

In this subsection, we suggest how affine Lie algebras may be arising so naturally in the study of two-dimensional quotient spaces.

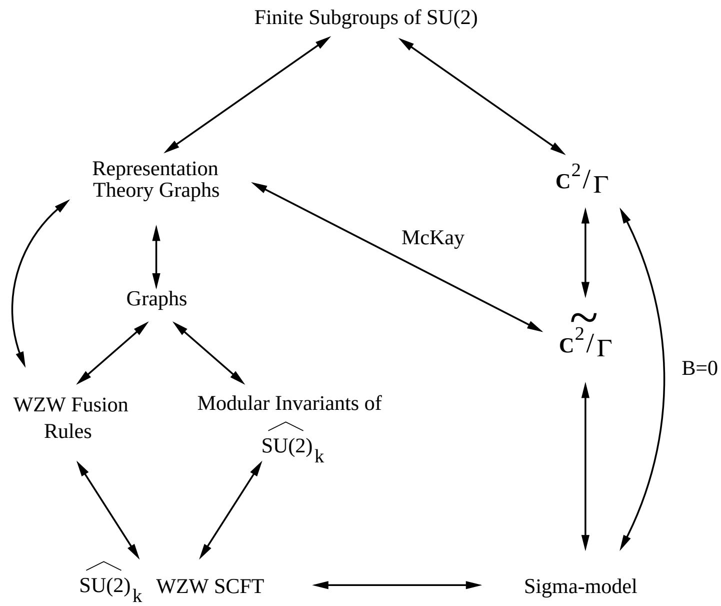

Figure 10-3: Web of Correspondences: Each finite group $\Gamma \subset SU(2)$ gives rise to an isolated Gorenstein singularity as well as to its representation ring $\mathcal{R}$. The cohomology ring of the resolved manifold is isomorphic to $\mathcal{R}$. The $\widehat{SU(2)}_k$ WZW theory at level $k =$ # Conjugacy classes of $\Gamma - 2$ has a graphical representation of its modular invariants and its fusion ring. The resulting graph is precisely the non-affine version of McKay's graph for Γ. The WZW model arises as a subsector of the conformal field theory description of the quotient singularity when the B-field has been set to zero. We further note that the three rings in the picture are equivalent.

Based on the previous studies of Yang-Mills instantons on ALE spaces as in [33, 39], Nakajima has introduced in [137] the notion of a quiver variety which is roughly a hyper-Kähler moduli space of representations of a quiver associated to a finite graph (We shall turn to quivers in the next section). There, he presents a beautiful geometric construction of representations of affine Lie algebras. In particular, he shows that when the graph is of the ADE type, the middle cohomology of the quiver variety is isomorphic to the weight space of integrable highest-weight representations. A famous example of a quiver variety with this kind of affine Lie algebra symmetry is the moduli space of instantons over ALE spaces.

In a separate paper [134], Nakajima also shows that the quotient space $\mathbb{C}^2/\Gamma$ admits a Hilbert scheme resolution X which itself can be identified with a quiver variety associated with the affine Dynkin diagram of Γ. The analysis of [137] thus seems to suggest that the second cohomology of the resolved space X is isomorphic to the weight space of some affine Lie algebra. We interpret Nakajima's work as telling physicists that the $\widehat{SU(2)}_k$ WZW has every right to be present and carries the geometric information about the second cohomology. Let us demonstrate our thoughts when $\Gamma = \mathbb{Z}_n$. In this case, we have $\dim H^2 = n - 1$, consisting of $n - 1$ $\mathbb{P}^1$ blow-ups in a linear chain. We interpret the H^2 basis as furnishing a representation of the $\widehat{SU(2)}_k$ WZW at level $k = n - 2$, as the basis matches the primary fields of the WZW. This interpretation agrees with the analysis of Ooguri and Vafa, but we are not certain how to reproduce the result directly from Nakajima's work.

10.4.3 T-duality and Branes

In [105, 106, 107, 108], the $\widehat{SU(2)}_k$ WZW theory arose in a different but equivalent context of brane dynamics. As shown in [128], the type IIA (IIB) string theory on an A_{n-1} ALE space is T-dual to the type IIB (IIA) theory in the background of n $NS5$-branes. The world-sheet description of the near-horizon geometry of the colliding $NS5$-branes is in terms of the $\widehat{SU(2)}_k$ WZW, a Feigin-Fuchs boson, and their superpartners. More precisely, the near-horizon geometry of n $NS5$-branes is given by the WZW at level $n-2$, which is consistent with the analysis of Ooguri and Vafa.

It was conjectured in [106], and further generalized in [107], that the string theory on the near horizon geometry of the $NS5$-branes is dual to the decoupled theory on the world-volume of the $NS5$-branes. In this chapter, our main concern has been the singularity structure of the ALE spaces, and we have thus restricted ourselves only to the transverse directions of the $NS5$-branes in the T-dual picture.

10.5 Ribbons and Quivers at the Crux of Correspondences

There is a common theme in all the fields relevant to our observations so far. In general we construct a theory and attempt to encode its rules into some matrix, whether it be fusion matrices, Clebsch-Gordan coefficients, or intersection numbers. Then we associate this matrix with some graph by treating the former as the adjacency matrix of the latter and study the properties of the original theory by analyzing the graphs[15].

Therefore there appears to be two steps in our program: firstly, we need to study the commonalities in the minimal set of axioms in these different fields, and secondly, we need to encode information afforded by these axioms by certain graphical representations. It turns out that there has been some work done in both of these steps, the first exemplified by the so-called ribbon categories and the second, quiver categories.

10.5.1 Ribbon Categories as Modular Tensor Categories

Prominent work in the first step has been done by A. Kirillov [144] and we shall adhere to his notations. We are interested in monoidal additive categories, in particular, we need the following:

DEFINITION 10.5.10 *A* **ribbon category** *is an additive category* $\mathcal{C}$ *with the following additional structures:*

- *BRAIDING: A bifunctor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *along with functorial associativity and commutativity isomorphisms for objects* V *and* W*:*

$$a_{V_1,V_2,V_3} : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3),$$
$$\check{R}_{V,W} : V \otimes W \to W \otimes V;$$

- *MONOIDALITY: A unit object* $\mathbf{1} \in Obj\,\mathcal{C}$ *along with isomorphisms* $\mathbf{1} \otimes V \to V, V \otimes \mathbf{1} \to V$*;*

- *RIGIDITY of duals: for every object* V *we have a (left) dual* V^* *and homomorphisms*

$$e_V : V^* \otimes V \to \mathbf{1},$$
$$i_V : \mathbf{1} \to V \otimes V^*;$$

[15]There is interesting work done to formalize to sub-factors and to investigate the graphs generated [148].

- *BALANCING: functorial isomorphisms* $\theta_V : V \to V$*, satisfying the compatibility condition*

$$\theta_{V\otimes W} = \check{R}_{W,V}\check{R}_{V,W}(\theta_V \otimes \theta_W).$$

Of course we see that all the relevant rings in Figure 10-1 fall under such a category. Namely, we see that the representation rings of finite groups, chiral rings of non-linear σ-models, Groethendieck rings of exceptional divisors or fusion rings of WZW, together with their associated tensor products, are all different realizations of a ribbon category [16]. This fact is perhaps obvious from the point of view of orbifold string theory, in which the fusion ring naturally satisfies the representation algebra of the finite group and the WZW arises as a singular limit of the vanishing B-field. The ingredients of each of these rings, respectively the irreps, chiral operators and cohomology elements, thus manifest as the objects in $\mathcal{C}$. Moreover, the arrows of Figure 10-1, loosely speaking, become functors among these various representations of $\mathcal{C}$ whereby making our central diagram a (meta)graph associated to $\mathcal{C}$. What this means is that as far as the ribbon category is concerned, all of these theories discussed so far are axiomatically identical. Hence indeed any underlying correspondences will be natural.

What if we impose further constraints on $\mathcal{C}$?

DEFINITION 10.5.11 *We define* $\mathcal{C}$ *to be* **semisimple** *if*

- *It is defined over some field* $\mathbb{K}$ *and all the spaces of homomorphisms are finite-dimensional vector spaces over mathbbK;*

- *Isomorphism classes of simple objects* X_i *in* $\mathcal{C}$ *are indexed by elements* i *of some set* I*. This implies involution* $^* : I \to I$ *such that* $X_i^* \simeq X_{i^*}$ *(in particular,* $0^* = 0$*);*

- *"Schur's Lemma":* $\hom(X_i, X_j) = \mathbb{K}\delta_{ij}$*;*

[16] Of course they may possess additional structures, e.g., these rings are all finite. We shall later see how finiteness becomes an important constraint when going to step two.

- *Complete Finite Reducibility:* $\forall\ V \in Obj\,\mathcal{C}$, $V = \bigoplus_{i\in I} N_i X_i$, *such that the sum is finite, i.e., almost all* $N_i \in \mathbb{Z}_+$ *are zero.*

Clearly we see that in fact our objects, whether they be WZW fields or finite group irreps, actually live in a semisimple ribbon category. It turns out that semisimplicity is enough to allow us to define composition coefficients of the "Clebsch-Gordan" type:

$$X_i \oplus X_j = \bigoplus N_{ij}^k X_k,$$

which are central to our discussion.

Let us introduce one more concept, namely the matrix s_{ij} mapping $X_i \to X_j$ represented graphically by the simple ribbon tangle, i.e., a link of 2 closed directed cycles of maps from X_i and X_j respectively into themselves. The remarkable fact is that imposing that

- s_{ij} be invertible and that
- $\mathcal{C}$ have only a finite number of simple objects (i.e., the set I introduced above is finite)

naturally gives rise to modular properties. We define such semisimple ribbon category equipped with these two more axioms as a **Modular Tensor Category**. If we define the matrix $t_{ij} = \delta_{ij}\theta_i$ with θ_i being the functorial isomorphism introduced in the balancing axiom for $\mathcal{C}$, the a key result is the following [144]:

THEOREM 10.5.15 *In the modular tensor category* $\mathcal{C}$*, the matrices* s *and* t *generate precisely the modular group* $SL(2,\mathbb{Z})$*.*

Kirillov remarks in [144] that it might seem mysterious that modular properties automatically arise in the study of tensor categories and argues in two ways why this may be so. Firstly, a projective action of $SL(2,\mathbb{Z})$ may be defined for certain objects in $\mathcal{C}$. This is essentially the construction of Moore and Seiberg [145] when they have found new modular invariants for WZW, showing how WZW primary operators are objects in $\mathcal{C}$. Secondly, he points out that geometrically one can associate a topological quantum field theory (TQFT) to each tensor category, whereby the mapping class group of the Riemann surface associated to the TQFT gives rise to the modular group. If the theories in Figure 10-1 are indeed providing different but equivalent realizations of $\mathcal{C}$, we may be able to trace the origin of the $SL(2, \mathbb{Z})$ action on the category to the WZW modular invariant partition functions. That is, it seems that in two dimensions the ADE scheme, which also arises in other representations of $\mathcal{C}$, naturally classifies some kind of modular invariants. In a generic realization of the modular tensor category, it may be difficult to identify such modular invariants, but they are easily identified as the invariant partition functions in the WZW theories.

10.5.2 Quiver Categories

Quivers!quiver category We now move onto the second step. Axiomatic studies of the encoding procedure (at least a version thereof) have been done even before McKay's result. In fact, in 1972, Gabriel has noticed that categorical studies of quivers lead to A-D-E-type classifications [86].

DEFINITION 10.5.12 *We define the* **quiver category** $\mathcal{L}(\Gamma, \Lambda)$*, for a finite connected graph* Γ *with orientation* Λ*, vertices* Γ_0 *and edges* Γ_1 *as follows: The objects in this category are any collection* (V, f) *of spaces* $V_\alpha, \alpha \in \Gamma_0$ *and mappings* $f_l, l \in \Gamma_1$*. The morphisms are* $\phi : (V, f) \to (V', f')$ *a collection of linear mappings* $\phi_\alpha : V_\alpha \to V'_\alpha$ *compatible with* f *by* $\phi_{e(l)} f_l = f'_l \phi_{b(l)}$ *where* $b(l)$ *and* $e(l)$ *are the beginning and the ending nodes of the directed edge* l*.*

Finally we define decomposability in the usual sense that

DEFINITION 10.5.13 *The object* (V, f) *is* **indecomposable** *iff there do not exist objects* $(V_1, f_1), (V_2, f_2) \in \mathcal{L}(\Gamma, \Lambda)$ *such that* $V = V_1 \oplus V_2$ *and* $f = f_1 \oplus f_2$*.*

Under these premises we have the remarkable result:

THEOREM 10.5.16 (GABRIEL-TITS) *The graph* Γ *in* $\mathcal{L}(\Gamma, \Lambda)$ *coincides with one of the graphs* $A_n, D_n, E_{6,7,8}$*, if and only if there are only finitely many non-isomorphic indecomposable objects in the quiver category.*

By this result, we can argue that the theories, which we have seen to be different representations of the ribbon category $\mathcal{C}$ and which all have ADE classifications in two dimensions, each must in fact be realizable as a finite quiver category $\mathcal{L}$ in dimension two. Conversely, the finite quiver category has representations as these theories in 2-dimensions. To formalize, we state

PROPOSITION 10.5.3 *In two dimensions, finite group representation ring, WZW fusion ring, Gorenstein cohomology ring, and non-linear σ-model chiral ring, as representations of a ribbon category $\mathcal{C}$, can be mapped to a finite quiver category $\mathcal{C}$. In particular the "Clebsch-Gordan" coefficients $\mathcal{N}_{ij}^k$ of $\mathcal{C}$ realize as adjacency matrices of graphs in $\mathcal{L}$* [17].

Now $\mathcal{L}$ has recently been given a concrete realization by the work of Douglas and Moore [69], in the context of investigating string theory on orbifolds. The objects in the quiver category have found representations in the resulting $\mathcal{N} = 2$ Super Yang-Mills theory. The modules V (nodes) manifest themselves as gauge groups arising from the vector multiplet and the mappings f (edges which in this case are really bidirectional arrows), as bifundamental matter. This is the arrow from graph theory to string orbifold theory in the center of Figure 10-1. Therefore it is not surprising that an ADE type of result in encoding the physical content of the theory has been obtained. Furthermore, attempts at brane configurations to construct these theories are well under way (e.g. [83]).

Now, what makes ADE and two dimensions special? A proof of the theorem due to Tits [86] rests on the fact that the problem can essentially be reduced to a Diophantine inequality in the number of nodes and edges of Γ, of the general type:

$$\sum_i \frac{1}{p_i} \geq c$$

where c is some constant and $\{p_i\}$ is a set of integers characterizing the problem at hand. This inequality has a long history in mathematics [147]. In our context, we

[17] *Here the graphs are ADE Dynkin diagrams. For higher dimension we propose that there still is a mapping, though perhaps not to a finite quiver category.*

recall that the uniqueness of the five perfect solids in $\mathbb{R}^3$ (and hence the discrete subgroups of $SU(2)$) relies essentially on the equation $1/p + 1/q \geq 1/2$ having only 5 pairs of integer solutions. Moreover we recall that Dynkin's classification theorem of the simple Lie algebras depended on integer solutions of $1/p + 1/q + 1/r \geq 1$.

Since Gabriel's theorem is so restrictive, extensions thereto have been done to relax certain assumptions (e.g., see [146]). This will hopefully give us give more graphs and in particular those appearing in finite group, WZW, orbifold theories or non-linear σ-models at higher dimensions. A vital step in the proof is that a certain quadratic form over the $\mathbb{Q}$-module of indices on the nodes (effectively the Dynkin labels) must be positive-definite. It was noted that if this condition is relaxed to positive semi-definity, then Γ would include the affine cases $\widehat{A}, \widehat{D}, \widehat{E}$ as well. Indeed we hope that further relaxations of the condition would admit more graphs, in particular those drawn for the $SU(3)$ subgroups. This inclusion on the one hand would relate quiver graphs to Gorenstein singularities in dimension three due to the link to string orbifolds[18] and on the other hand to the WZW graph algebras by the conjecture in Chap. 9. Works in this direction are under way. It has been recently suggested that since the discrete subgroups of SU(4,5,6,7) have also been classified [102], graphs for these could be constructed and possibly be matched to the modular invariants corresponding to $\widehat{SU(n)}$ for $n = 4, .., 7$ respectively. Moreover, proposals for unified schemes for the modular invariants by considering orbifolds by abelian Γ in SU(2,3,..,6) have been made in [103].

Let us summarize what we have found. We see that the representation ring of finite groups with its associated $(\otimes, \oplus)$, the chiral ring of nonlinear σ-model with its $(\otimes, \oplus)$, the fusion ring of the WZW model with its $(\times, \oplus)$ and the Groethendieck ring of resolved Gorenstein singularities with it $(\otimes, \oplus)$ manifest themselves as different realizations of a semisimple ribbon category $\mathcal{C}$. Furthermore, the requirement of finiteness and an invertible s-matrix makes $\mathcal{C}$ into a modular tensor category. The ADE schemes in two dimensions, if they arise in one representation of $\mathcal{C}$, might naturally appear in another. Furthermore, the quiver category $\mathcal{L}$ has a physical

[18]In this case we get $\mathcal{N} = 1$ Super-Yang-Mills theory in 4 dimension.

realization as bifundamentals and gauge groups of SUSY Yang-Mills theories. The mapping of the Clebsch-Gordan coefficients in $\mathcal{C}$ to the quivers in $\mathcal{L}$ is therefore a natural origin for the graphical representations of the diverse theories that are objects in $\mathcal{C}$.

10.6 Conjectures

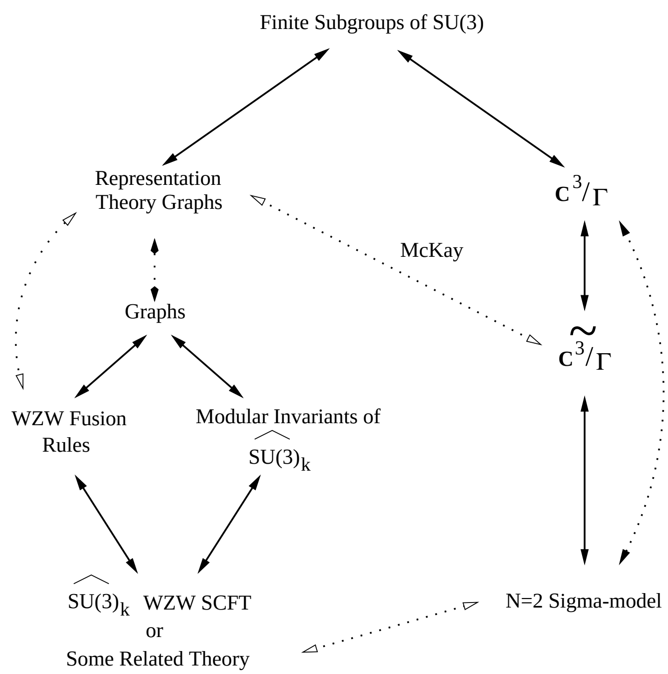

Figure 10-4: Web of Conjectures: Recently, the graphs from the representation theory side were constructed and were noted to resemble those on WZW $\widehat{SU(3)}_k$ side (Chap. 9). The solid lines have been sufficiently well-established while the dotted lines are either conjectural or ill-defined.

We have seen that there exists a remarkably coherent picture of inter-relations in two dimensions among many different branches of mathematics and physics. The organizing principle appears to be the mathematical theory of quivers and ribbon category, while the crucial bridge between mathematics and physics is the conformal field theory description of the Gorenstein singularities provided by the orbifoldized coset construction.

Surprisingly, similar features have been noted in three dimensions. The Clebsch-

Gordan coefficients for the tensor product of irreducible representations for all discrete subgroups of $SU(3)$ were computed in [141, 142] and Chap. 9, and a possible correspondence was noted, and conjectured for $n \geq 3$, between the resulting Dynkin-like diagrams and the graphic representations of the fusion rules and modular invariants of $\widehat{SU(3)}_k$ WZW models. Furthermore, as discussed previously, the Geometric McKay Correspondence between the representation ring of the abelian discrete subgroups $\Gamma \subset SU(3)$ and the cohomology ring of $\widetilde{\mathbb{C}^3/\Gamma}$ has been proved in [134]. Hence, the situation in 3-dimensions as seen in Figure 10-4 closely resembles that in 2-dimensions.

Now, one naturally inquires:

> *Are there graphical representations of the fusion rules and modular invariants of the $\widehat{SU(n)}_k$ WZW model or some related theory that contain the Clebsch-Gordan coefficients for the representations of $\Gamma \subset SU(n)$? And, in turn, are the Clebsch-Gordan coefficients related to the (co)-homological intersections on the resolved geometry $\widetilde{\mathbb{C}^n/\Gamma}$ that are contained in the chiral ring of the $N = 2$ σ-model on $\mathbb{C}^n/\Gamma$ with a non-vanishing B-field?* **Most importantly, what do these correspondences tell us about the two conformal field theories and their singular limits?**

As physicists, we believe that the McKay correspondence and the classification of certain modular invariants in terms of finite subgroups are consequences of orbifolding and of some underlying quantum equivalence of the associated conformal field theories.

We thus believe that a picture similar to that seen in this chapter for 2-dimensions persists in higher dimensions and conjecture that there exists a conformal field theory description of the Gorenstein singularities in higher dimensions. If such a theory can be found, then it would explain the observation made in Chap. 9 of the resemblance of the graphical representations of the representation ring of the finite subgroups of $SU(3)$ and the modular invariants of the $\widehat{SU(3)}_k$ WZW. We have checked that the correspondence, if any, between the finite subgroups of $SU(3)$ and the $\widehat{SU(3)}_k$ WZW theory is not one-to-one. For example, the number of primary fields generically does not match the number of conjugacy classes of the discrete subgroups. It has been observed in Chap. 9, however, that some of the representation graphs appear to be subgraphs of the graphs encoding the modular invariants. We hope that the present chapter serves as a motivation for finding the correct conformal field theory description in three dimensions which would tell us how to "project" the modular invariant graphs to retrieve the representation graphs of the finite graphs.

Based on the above discussions, we summarize our speculations, relating geometry, generalizations of the ADE classifications, representation theory, and string theory in Figure 10-4.

10.6.1 Relevance of Toric Geometry

It is interesting to note that the toric resolution of certain Gorenstein singularities also naturally admits graphical representations of fans. In fact, the exceptional divisors in the Geometric McKay Correspondence for $\Gamma = \mathbb{Z}_n \subset SU(2)$ in 2-dimensions can be easily seen as the vertices of new cones in the toric resolution, and these vertices precisely form the A_{n-1} Dynkin digram. Thus, at least for the abelian case in 2-dimensions, the McKay correspondence and the classification of $\widehat{SU(2)}$ modular invariants seem to be most naturally connected to geometry as toric diagrams of the resolved manifolds $\widetilde{C^2/\Gamma}$.

Surprisingly—perhaps not so much so in retrospect—we have noticed a similar pattern in 3-dimensions. That is, the toric resolution diagrams of $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ singularities reproduce the graphs that classify the $\mathcal{A}$-type modular invariants of the $\widehat{SU(3)}_k$ WZW models. For which k? It has been previously observed in [153] that there seems to be a correspondence, up to some truncation, between the subgroups $\mathbb{Z}_n \times \mathbb{Z}_n \subset SU(3)$ and the $\mathcal{A}$-type $\widehat{SU(3)}_{n-1}$ modular invariants, which do appear as subgraphs of the $\widetilde{\frac{\mathbb{C}^3}{\mathbb{Z}_n \times \mathbb{Z}_n}}$ toric diagrams. On the other hand, a precise formulation of the correspondence with geometry and the conformal field theory description of Gorenstein singularities still remains as an unsolved problem and will be presented elsewhere [154].

10.7 Conclusion

Inspired by the ubiquity of ADE classification and prompted by an observation of a mysterious relation between finite groups and WZW models, we have proposed a possible unifying scheme. Complex and intricate webs of connections have been presented, the particulars of which have either been hinted at by collective works in the past few decades in mathematics and physics or are conjectured to exist by arguments in this chapter. These webs include the McKay correspondences of various types as special cases and relate such seemingly disparate subjects as finite group representation theory, graph theory, string orbifold theory and sigma models, as well as conformal field theory descriptions of Gorenstein singularities. We note that the integrability of the theories that we are considering may play a role in understanding the deeper connections.

This chapter catalogs many observations which have been put forth in the mathematics and physics literature and presents them from a unified perspective. Many existing results and conjectures have been phrased under a new light. We can summarize the contents of this chapter as follows:

1. In two dimensions, all of the correspondences mysteriously fall into an ADE type. We have provided, via Figure 10-1, a possible setting how these mysteries might arise naturally. Moreover, we have pointed out how axiomatic works done by category theorists may demystify some of these links. Namely, we have noted that the relevant rings of the theories can be mapped to the quiver category.

2. We have also discussed the possible role played by the modular tensor category in our picture, in which the modular invariants arise very naturally. Together with the study of the quiver category and quiver variety, the ribbon category seems to provide the reasons for the emergence of affine Lie algebra symmetry and the ADE classification of the modular invariants.

3. We propose the validity of our program to higher dimensions, where the picture is far less clear since there are no ADE schemes, though some hints of generalized

graphs have appeared.

4. There are three standing conjectures:

   - We propose that there exists a conformal field theory description of the Gorenstein singularities in dimensions greater than two.
   - As noted in Chap. 9, we conjecture that the modular invariants and the fusion rings of the $\widehat{SU(n)}, n > 2$ WZW, or their generalizations, may be related to the discrete subgroups of the $SU(n)$.
   - Then, there is the mathematicians' conjecture that there exits a McKay correspondence between the cohomology ring $H^*(\widetilde{C^n/\Gamma}, \mathbb{Z})$ and the representation ring of Γ, for finite subgroup $\Gamma \subset SU(n)$.

   We have combined these conjectures into a web so that proving one of them would help proving the others.

We hope that Figure 10-1 essentially commutes and that the standing conjectures represented by certain arrows therewithin may be solved by investigating the other arrows. In this way, physics may provide us with a possible method of attack and explanation for McKay's correspondence and many other related issues, and likewise mathematical structures may help to clarify and rigorize some observations made from string theory.

It is the purpose of this writing to inform the physics and mathematics community of a possibly new direction of research which could harmonize ostensibly different and diverse branches of mathematics and physics into a unified picture.

Chapter 11

Orbifolds III: $SU(4)$

Synopsis

Whereas chapter 9 studied the $SU(2)$ and $SU(3)$ orbifolds as local Calabi-Yau surfaces and threefolds, we here present, in modern notation, the classification of the discrete finite subgroups of $SU(4)$ as well as the character tables for the exceptional cases thereof (Cf. http://pierre.mit.edu/∼yhe/su4.ct).

We hope this catalogue will be useful to works on string orbifold theories on Calabi-Yau fourfolds, quiver theories, WZW modular invariants, Gorenstein resolutions, nonlinear sigma-models as well as the inter-connections among them proposed in Chapter 10 [294].

11.1 Introduction

It is well known that the discrete finite subgroups of $SL(n = 2, 3; \mathbb{C})$ have been completely classified; works related to string orbifold theories and quiver theories have of late used these results (see for example [293, 69, 76, 75, 141, 142] and Chap. 9 as well as references therein). Conjectures regarding higher n have been raised and works toward finite subgroups of $SU(4)$ are under way. Recent works by physicists and mathematicians alike further beckon for a classification of the groups, conveniently presented, in the case of $SU(4)$ [160, 96]. Compounded thereupon is the disparity of language under which the groups are discussed: the classification problem in the past decades has chiefly been of interest to either theoretical chemists or to pure mathematicians, the former of whom disguise them in Bravais crystallographic notation (e.g. [161]) while the latter abstract them in fields of finite characteristic (e.g. [162]). Subsequently, there is a need within the string theory community for a list of the finite subgroups of $SU(4)$ tabulated in our standard nomenclature, complete with the generators and some brief but not overly-indulgent digression on their properties.

The motivations for this need are manifold. There has recently been a host of four dimensional finite gauge theories constructed by placing D3 branes on orbifold singularities [69, 76, 75]; brane setups have also been achieved for some of the groups [79, 78]. In particular, a theory with $\mathcal{N} = 2, 1, 0$ supercharges respectively is obtained from a $\mathbb{C}^N/\{\Gamma \subset SU(n = 2, 3, 4)\}$ singularity with $N = 2, 3$ (see Chap. 9, [69, 76, 75] and references therein). Now as mentioned above $n = 2, 3$ have been discussed, and $n = 4$ has yet to be fully attacked. This last case is of particular interest because it gives rise to an $\mathcal{N} = 0$, non-supersymmetric theory. On the one hand these orbifold theories provide interesting string backgrounds for checks on the AdS/CFT Correspondence [77, 156]. On the other hand, toric descriptions for the Abelian cases of the canonical Gorenstein singularities have been treated while the non-Abelian still remain elusive [158, 157]. Moreover, the quiver theories arising from these string orbifold theories (or equivalently, representation rings of finite subgroups of $SU(n)$) have been hinted to be related to modular invariants of $\widehat{su(n)}$-WZW models (or equivalently, affine characters of $\widehat{su(n)}$) for arbitrary n [104], and a generalised McKay Correspondence, which would also relate non-linear sigma models, has been suggested to provide a reason [293]. Therefore a need for the discrete subgroups of $SU(4)$ arises in all these areas.

Indeed the work has been done by Blichfeldt [88] in 1917, or at least all the exceptional cases, though in an obviously outdated parlance and moreover with many infinite series being "left to the reader as an exercise." It is therefore the intent of the ensuing monograph to present the discrete subgroups Γ of $SL(4,\mathbb{C})$ in a concise fashion, hoping it to be of use to impending work, particularly non-supersymmetric conformal gauge theories from branes on orbifolds, resolution of Gorenstein singularities in higher dimension, as well as $\widehat{su(4)}$-WZW models.

Nomenclature

Unless otherwise stated we shall adhere to the convention that Γ refers to a discrete subgroup of $SU(n)$ (i.e., a finite collineation group), that $< x_1, .., x_n >$ is a finite group generated by $\{x_1, .., x_n\}$, that $H \triangleleft G$ means H is a normal subgroup of G, that S_n and A_n are respectively the symmetric and alternating permutation groups on n elements, and that placing $*$ next to a group signifies that it belongs to $SU(4) \subset SL(4;\mathbb{C})$.

11.2 Preliminary Definitions

Let Γ be a finite discrete subgroup of the general linear group, i.e., $\Gamma \subset GL(n,\mathbb{C})$. From a mathematical perspective, quotient varieties of the form $\mathbb{C}^n/\Gamma$ may be constructed and by the theorem of Khinich and Watanabe [155, 89], the quotient is Gorenstein[1] if and only if Γ is in fact in $SL(n,\mathbb{C})$. Therefore we would like to focus on the discrete subgroups of linear transformations *up to linear equivalence*, which are what has been dubbed in the old literature as **finite collineation groups** [88]. From a physics perspective, discrete subgroups of $SU(n) \subset SL(n;\mathbb{C})$ have been subject to investigation in the early days of particle phenomenology [90] and have lately been of renewed interest in string theory, especially in the context of orbifolds (see for example Chap. 9 and [293, 69, 76, 75, 160, 96]).

There are some standard categorisations of finite collineation groups [88, 89]. They first fall under the division of transitivity and intransitivity as follows:

DEFINITION 11.2.14 *If the n variables upon which Γ acts as a linear transformation*

[1] That is, if there exists a nowhere-vanishing holomorphic n-form. These varieties thus provide local models of Calabi-Yau manifolds and are recently of great interest.

can be separated into 2 or more sets either directly or after a change of variables, such that the variables of each set are transformed into linear functions only of themselves, then Γ *is called* **Intransitive***; it is called Transitive otherwise.*

The transitive Γ can be further divided into the primitive and imprimitive cases:

DEFINITION 11.2.15 *If for the transitive* Γ *the variables may be separated*[2] *into 2 or more sets such that the variables of each are transformed into linear functions of only those in any set according to the separation (either the same or different), then* Γ *is called* **Imprimitive***; it is called Primitive otherwise.*

Therefore in the matrix representation of the groups, we may naïvely construe intransitivity as being block-diagonalisable and imprimitivity as being block off-diagonalisable, whereby making primitive groups generically having no *a priori* zero entries. We give examples of an intransitive, a (transitive) imprimitive and a (transitive) primitive group, in their matrix forms, as follows:

$$\underset{\text{Intransitive}}{\begin{pmatrix} \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{pmatrix}} \quad \underbrace{\underset{\text{Imprimitive}}{\begin{pmatrix} 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \\ \times & \times & 0 & 0 \\ \times & \times & 0 & 0 \end{pmatrix}} \quad \underset{\text{Primitive}}{\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix}}}_{\text{Transitive}}$$

Let us diagrammatically summarise all these inter-relations as is done in [89]:

$$\Gamma \begin{cases} \text{Intransitive} \\ \text{Transitive} \begin{cases} \text{Imprimitive} \\ \text{Primitive} \begin{cases} \text{Simple} \\ \text{Having Normal Primitive Subgroups} \\ \text{Having Normal Intransitive Subgroups} \\ \text{Having Normal Imprimitive Subgroups} \end{cases} \end{cases} \end{cases}$$

[2] *Again, either directly or after a change of variables.*

In some sense the primitive groups are the fundamental building blocks and pose as the most difficult to be classified. It is those primitive groups that Blichfeldt presented, as linear transformations, in [88]. These groups are what we might call *exceptionals* in the sense that they do not fall into infinite series, in analogy to the $E_{6,7,8}$ groups of $SU(2)$. We present them as well as their sub-classifications first. Thereafter we shall list the imprimitive and intransitives, which give rise to a host of infinite series of groups, in analogy to the A_n and D_n of $SU(2)$.

Let us take a final digression to clarify the so-called **Jordan Notation**, which is the symbol ϕ commonly used in finite group theory. A linear group Γ often has its order denoted as $|\Gamma| = g\phi$ for positive integers g and ϕ; the ϕ signifies the order of the subgroup of homotheties, or those multiples of the identity which together form the center of the $SL(n;\mathbb{C})$. We know that $SU(n) \subset SL(n;\mathbb{C})$, so a subgroup of the latter is not necessarily that of the former. In the case of $SL(n = 2,3;\mathbb{C})$, the situation is simple[3]: the finite subgroups belonged either to (A) $SU(n = 2,3)$, or to (B) the center-modded[4] $SU(n = 2,3)/\mathbb{Z}_{2,3}$, or (C) to both. Of course a group with order g in type (B) would have a natural lifting to type (A) and become a group of order g multiplied by $|\mathbb{Z}_2| = 2$ or $|\mathbb{Z}_3 = 3|$ respectively, which is now a finite subgroup of the full $SU(2)$ or $SU(3)$, implying that the Jordan ϕ is 2 or 3 respectively.

For the case at hand, the situation is slightly more complicated since 4 is not a prime. Therefore ϕ can be either 2 or 4 depending how one lifts with respect to the relation $SU(4)/\mathbb{Z}_2 \times \mathbb{Z}_2 \cong SO(6)$ and we lose a good discriminant of whether or not Γ is in the full $SU(4)$. To this end we have explicitly verified the unitarity condition for the group elements and will place a star $(*)$ next to those following groups which indeed are in the full $SU(4)$. Moreover, from the viewpoint of string orbifold theories which study for example the fermionic and bosonic matter content of the resulting Yang-Mills theory, one naturally takes interest in $Spin(6)$, or the full $\mathbb{Z}_2 \times \mathbb{Z}_2$ cover of $SO(6)$ which admits spinor representations; for these we shall look in particular at the groups that have $\phi = 4$ in the Jordan notation, as will be indicated in the tables

[3]See [90] for a discussion on this point.

[4]For $n = 2$, this our familiar $SU(2)/\mathbb{Z}_2 \cong SO(3)$.

below.

11.3 The Discrete Finite Subgroups of $SL(4;\mathbb{C})$

We shall henceforth let Γ denote a finite subgroup of $SL(4;\mathbb{C})$ unless otherwise stated.

11.3.1 Primitive Subgroups

There are in all 30 types of primitive cases for Γ. First we define the constants $w = e^{\frac{2\pi i}{3}}$, $\beta = e^{\frac{2\pi i}{7}}$, $p = \beta + \beta^2 + \beta^4$, $q = \beta^3 + \beta^5 + \beta^6$, $s = \beta^2 + \beta^5$, $t = \beta^3 + \beta^4$, and $u = \beta + \beta^6$. Furthermore we shall adhere to some standard notation and denote the permutation and the alternating permutation group on n elements respectively as S_n and A_n. Moreover, in what follows we shall use the function $Lift$ to mean the lifting by (perhaps a subgroup) of the Abelian center C according to the exact sequence $0 \rightarrow C \rightarrow SU(4) \rightarrow SU(4)/C \rightarrow 0$.

We present the relevant matrix generators as we proceed:

$$F_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w^2 \end{pmatrix} F_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 0 & -1 \end{pmatrix} F_3 = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$F_2' = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & 2 & -1 \end{pmatrix} F_3' = \frac{1}{4} \begin{pmatrix} -1 & \sqrt{15} & 0 & 0 \\ \sqrt{15} & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \end{pmatrix} F_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta^4 & 0 \\ 0 & 0 & 0 & \beta^2 \end{pmatrix} T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} W = \frac{1}{i\sqrt{7}} \begin{pmatrix} p^2 & 1 & 1 & 1 \\ 1 & -q & -p & -p \\ 1 & -p & -q & -p \\ 1 & -p & -p & -q \end{pmatrix}$$

$$R = \frac{1}{\sqrt{7}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & s & t & u \\ 2 & t & u & s \\ 2 & u & s & t \end{pmatrix} C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & w^2 \end{pmatrix} D = \begin{pmatrix} w & 0 & 0 & 0 \\ 0 & w & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$V = \frac{1}{i\sqrt{3}} \begin{pmatrix} i\sqrt{3} & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & w & w^2 \\ 0 & 1 & w^2 & w \end{pmatrix} F = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We see that all these matrix generators are unitary except R.

Primitive Simple Groups

There are 6 groups of this most fundamental type:

Group	Order	Generators	Remarks
I$*$	60×4	F_1, F_2, F_3	$Lift(A_5)$
II$*$	60	F_1, F_2', F_3'	$\cong A_5$
III$*$	360×4	F_1, F_2, F_3	$Lift(A_6)$
IV$*$	$\frac{1}{2}7! \times 2$	S, T, W	$Lift(A_7)$
V	168×4	S, T, R	
VI$*$	$2^6 3^4 5 \times 2$	T, C, D, E, F	

Groups Having Simple Normal Primitive Subgroups

There are 3 such groups, generated by simple primitives and the following 2 matrices:

$$F' = \frac{1+i}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad F'' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The groups are then:

Group	Order	Generators	Remarks
VII$*$	120×4	(I), F''	$Lift(S_5)$
VIII$*$	120×4	(II), F'	$Lift(S_5)$
IX$*$	720×4	(III), F''	$Lift(S_6)$

Groups Having Normal Intransitive Subgroups

There are seven types of Γ in this case and their fundamental representation matrices turn out to be Kronecker products of those of the exceptionals of $SU(2)$. In other words, for M, the matrix representation of Γ, we have $M = A_1 \otimes_K A_2$ such that A_i are the 2×2 matrices representing $E_{6,7,8}$. Indeed we know that $E_6 = \langle S_{SU(2)}, U^2_{SU(2)} \rangle$, $E_7 =$

$\langle S_{SU(2)}, U_{SU(2)}\rangle, E_8 = \langle S_{SU(2)}, U^2_{SU(2)}, V_{SU(2)}\rangle$, where

$$S_{SU(2)} = \tfrac{1}{2}\begin{pmatrix} -1+i & -1+i \\ 1+i & -1-i \end{pmatrix} U_{SU(2)} = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}.$$

$$V_{SU(2)} = \begin{pmatrix} \frac{i}{2} & \frac{1-\sqrt{5}}{4} - i\frac{1+\sqrt{5}}{4} \\ -\frac{1-\sqrt{5}}{4} - i\frac{1+\sqrt{5}}{4} & -\frac{i}{2} \end{pmatrix}$$

We use, for the generators, the notation $\langle A_i\rangle \otimes \langle B_j\rangle$ to mean that Kronecker products are to be formed between all combinations of A_i with B_j. Moreover the group (XI), a normal subgroup of (XIV), is formed by tensoring the 2-by-2 matrices $x_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$, $x_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$, $x_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$, $x_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$, $x_5 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix}$, and $x_6 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$. The seven groups are:

Group	Order	Generators	Remarks
X∗	144×2	$\langle S_{SU(2)}, U^2_{SU(2)}\rangle \otimes \langle S_{SU(2)}, U^2_{SU(2)}\rangle$	$\cong E_6 \otimes_K E_6$
XI∗	288×2	$x_1 \otimes x_2, x_1 \otimes x_2^T, x_3 \otimes x_4, x_5 \otimes x_6$	$(\mathrm{X}) \triangleleft \Gamma \triangleleft (\mathrm{XIV})$
XII∗	288×2	$\langle S_{SU(2)}, U^2_{SU(2)}\rangle \otimes \langle S_{SU(2)}, U_{SU(2)}\rangle$	$\cong E_6 \otimes_K E_7$
XIII∗	720×2	$\langle S_{SU(2)}, U^2_{SU(2)}\rangle \otimes \langle S_{SU(2)}, V_{SU(2)}, U^2_{SU(2)}\rangle$	$\cong E_6 \otimes_K E_8$
XIV∗	576×2	$\langle S_{SU(2)}, U_{SU(2)}\rangle \otimes \langle S_{SU(2)}, U_{SU(2)}\rangle$	$\cong E_7 \otimes_K E_7$
XV∗	1440×2	$\langle S_{SU(2)}, U_{SU(2)}\rangle \otimes \langle S_{SU(2)}, V_{SU(2)}, U^2_{SU(2)}\rangle$	$\cong E_7 \otimes_K E_8$
XVI∗	3600×2	$\langle S_{SU(2)}, V_{SU(2)}, U^2_{SU(2)}\rangle \otimes \langle S_{SU(2)}, V_{SU(2)}, U^2_{SU(2)}\rangle$	$\cong E_8 \otimes_K E_8$

Groups Having X-XVI as Normal Primitive Subgroups

There are in all 5 of these, generated by the above, together with

$$T_1 = \tfrac{1+i}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$$

The group generated by (XIV) and T_2 is isomorphic to (XXI), generated by (XIV) and T_1 so we need not consider it. The groups are:

Group	Order	Generators
XVII$*$	576×4	$(\mathrm{XI}), T_1$
XVIII$*$	576×4	$(\mathrm{XI}), T_2$
XIX$*$	288×4	$(\mathrm{X}), T_1$
XX$*$	7200×4	$(\mathrm{XVI}), T_1$
XXI$*$	1152×4	$(\mathrm{XIV}), T_1$

Groups Having Normal Imprimitive Subgroups

Finally these following 9 groups of order divisible by 5 complete our list of the primitive Γ, for which we need the following generators:

$$A = \tfrac{1+i}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} B = \tfrac{1+i}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$S' = \tfrac{1+i}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} T' = \tfrac{1+i}{2} \begin{pmatrix} -i & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -i & i & 0 \end{pmatrix} R' = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 & 0 \\ i & 1 & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -1 & -i \end{pmatrix}$$

Moreover these following groups contain the group K of order 16×2, generated by:

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} A_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We tabulate the nine groups:

Group	Order	Generators
XXII$*$	$5 \times 16 \times 4$	$(\mathrm{K}), T'$
XXIII$*$	$10 \times 16 \times 4$	$(\mathrm{K}), T', R'^2$
XXIV$*$	$20 \times 16 \times 4$	$(\mathrm{K}), T, R$
XXV$*$	$60 \times 16 \times 4$	$(\mathrm{K}), T, S'B$
XXVI$*$	$60 \times 16 \times 4$	$(\mathrm{K}), T, BR'$
XXVII$*$	$120 \times 16 \times 4$	$(\mathrm{K}), T, A$
XXVIII$*$	$120 \times 16 \times 4$	$(\mathrm{K}), T, B$
XXIX$*$	$360 \times 16 \times 4$	$(\mathrm{K}), T, AB$
XXX$*$	$720 \times 16 \times 4$	$(\mathrm{K}), T, S$

11.3.2 Intransitive Subgroups

These cases are what could be constructed from the various combinations of the discrete subgroups of $SL(2;\mathbb{C})$ and $SL(3;\mathbb{C})$ according to the various possibilities of diagonal embeddings. Namely, they consist of those of the form $(1,1,1,1)$ which represents the various possible Abelian groups with one-dimensional (cyclotomic) representation[5], $(1,1,2)$, two Abelians and an $SL(2;\mathbb{C})$ subgroup, $(1,3)$, an Abelian and an $SL(3;\mathbb{C})$ subgroup, and $(2,2)$, two $SL(2;\mathbb{C})$ subgroups as well as the various

[5]These includes the $\mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_p$ groups recently of interest in brane cube constructions [163].

permutations thereupon. Since these embedded groups (as collineation groups of lower dimension) have been well discussed in Chap. 9, we shall not delve too far into their account.

11.3.3 Imprimitive Groups

The analogues of the dihedral groups (in both $SL(2;\mathbb{C})$ and $SL(3;\mathbb{C})$), which present themselves as infinite series, are to be found in these last cases of Γ. They are of two subtypes:

- (a) Generated by the canonical Abelian group of order n^3 for $n \in \mathbb{Z}^+$ whose elements are

$$\Delta = \{ \begin{pmatrix} \omega^i & 0 & 0 & 0 \\ 0 & \omega^j & 0 & 0 \\ 0 & 0 & \omega^k & 0 \\ 0 & 0 & 0 & \omega^{-i-j-k} \end{pmatrix} \} \qquad \begin{array}{c} \omega = e^{\frac{2\pi i}{n}} \\ i, j, k = 1, ..., n \end{array}$$

  as well as respectively the four groups A_4, S_4, the Sylow-8 subgroup $Sy \subset S_4$ (or the ordinary dihedral group of 8 elements) and $\mathbb{Z}_2 \times \mathbb{Z}_2$;

- (b) We define H and T'' (where again $i = 1, ..., n$) as:

$$H = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix} T'' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \omega^i & 0 & 0 & 0 \\ 0 & \omega^{-i} & 0 & 0 \end{pmatrix}$$

  where the blocks of H are $SL(2;\mathbb{C})$ subgroups.

We tabulate these last cases of Γ as follows:

Subtype	Group	Order	Generators
(a)	XXXI$*$	$12n^3$	$\langle \Delta, A_4 \rangle$
	XXXII$*$	$24n^3$	$\langle \Delta, S_4 \rangle$
	XXXIII$*$	$8n^3$	$\langle \Delta, Sy \rangle$
	XXXIII$*$	$4n^3$	$\langle \Delta, \mathbb{Z}_2 \times \mathbb{Z}_2 \rangle$
(b)	XXXIV$*$		$\langle H, T'' \rangle$

11.4 Remarks

We have presented, in modern notation, the classification of the discrete subgroups of $SL(4, \mathbb{C})$ and in particular, of $SU(4)$. The matrix generators and orders of these groups have been tabulated, while bearing in mind how the latter fall into subcategories of transitivity and primitivity standard to discussions on collineation groups.

Furthermore, we have computed the character table for the 30 exceptional cases [92]; The interested reader may, at his or her convenience, find the character tables at http://pierre.mit.edu/~yhe/su4.ct. These tables will be crucial to quiver theories. As an example, we present in Figure 11-1 the quiver for the irreducible **4** of the group (I) of order 60×4, which is the lift of the alternating permutation group on 5 elements.

Indeed such quiver diagrams may be constructed for all the groups using the character tables mentioned above. We note in passing that since $\Gamma \subset SU(4)$ gives rise to an $\mathcal{N} = 0$ theory in 4 dimensions, supersymmetry will not come to our aid in relating the fermionic $a_{ij}^{\mathbf{4}}$ and the bosonic $a_{ij}^{\mathbf{6}}$ as was done in Chap. 9. However we can analyse the problem with a slight modification and place a stack of M2 branes on the orbifold, (which in the Maldacena picture corresponds to orbifolds on the S^7 factor in $AdS_4 \times S^7$), and obtain an $\mathcal{N} = 2$ theory in 3 dimensions at least in the IR limit as we lift from type IIA to M Theory [69, 76, 75, 158, 157, 159]. This supersymmetry would help us to impose the constraining relation between the two matter matrices, and hence the two quiver diagrams. This would be an interesting check which we leave to future work.

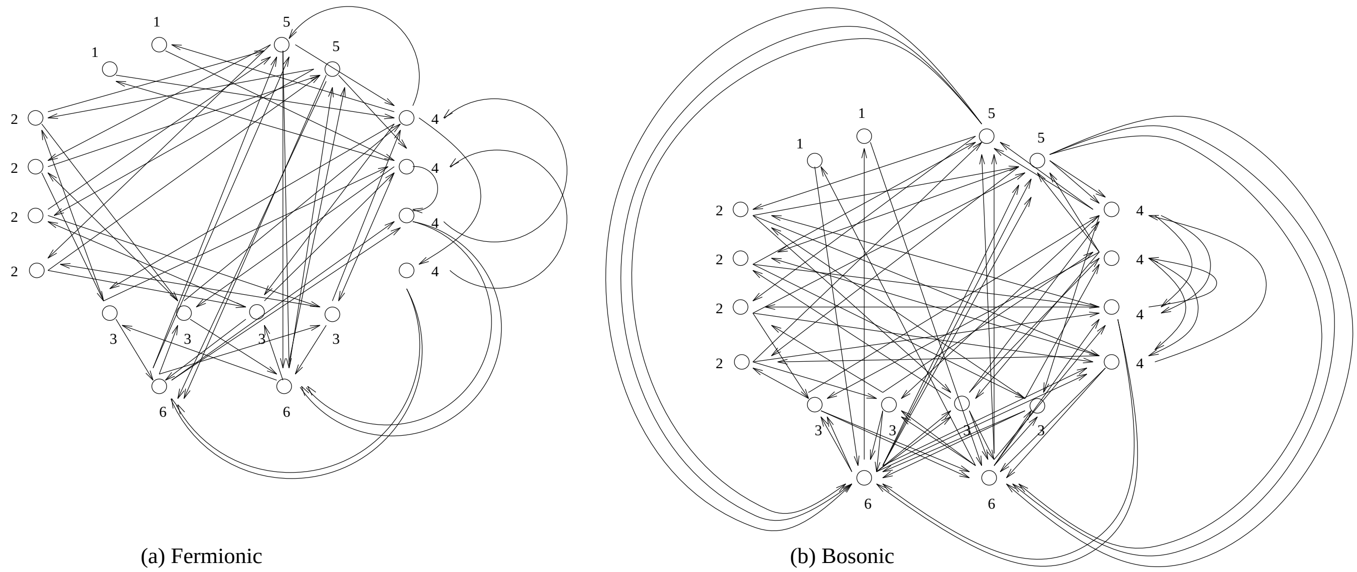

Figure 11-1: The Quiver Diagram for Group (I), constructed for (a) the fermionic a^4_{ij} corresponding to the irreducible $\mathbf{4}_3$ and (b) the bosonic a^6_{ij} corresponding to the irreducible $\mathbf{6}_2$ (in the notation of Chap. 9). We make this choice because we know that $\mathbf{4}_1 \otimes \mathbf{4}_3 = \mathbf{4}_3 \oplus \mathbf{6}_1 \oplus \mathbf{6}_2$ and that the two $\mathbf{6}$'s are conjugates. The indices are the dimensions of the various irreducible representations, a generalisation of Dynkin labels.

We see therefore a host of prospective research in various areas, particularly in the context of string orbifold/gauge theories, WZW modular invariants, and singularity-resolutions in algebraic geometry. It is hoped that this monograph, together with its companion tables on the web, will provide a ready-reference to works in these directions.

Chapter 12

Finitude of Quiver Theories and Finiteness of Gauge Theories

Synopsis

The D-branes probe theories thusfar considered are all finite theories with a conformal fixed point in the IR. Indeed, asymptotic freedom, finitude and IR freedom pose as a trichotomy of the beta-function behaviour in quantum field theories. Parallel thereto is a trichotomy in set theory of finite, tame and wild representation types. At the intersection of the above lies the theory of quivers.

We briefly review some of the terminology standard to the physics and to the mathematics. Then we utilise certain results from graph theory and axiomatic representation theory of path algebras to address physical issues such as the implication of graph additivity to finiteness of gauge theories, the impossibility of constructing completely IR free string orbifold theories and the unclassifiability of $\mathcal{N} < 2$ Yang-Mills theories in four dimensions. This perspective sheds a new light on the speciality of $SU(2)$ ADE orbifolds [297].

12.1 Introduction

In a quantum field theory (QFT), it has been known since the 70's (q.v. e.g. [164]), that the behaviour of physical quantities such as mass and coupling constant are sensitive to the renormalisation and evolve according to momentum scale as dictated by the so-called *renormalisation flows*. In particular, the correlation (Green's) functions, which encode the physical information relevant to Feymann's perturbative analysis of the theory and hence unaffected by such flows, obey the famous Callan-Symanzik Equations. These equations assert the existence two universal functions $\beta(\lambda)$ and $\gamma(\lambda)$ shifting according to the coupling and field renormalisation in such a way so as to compensate for the renormalisation scale.

A class of QFT's has lately received much attention, particularly among the string theorists. These are the so-named **finite** theories, characterised by the vanishing of the β-functions. These theories are extremely well-behaved and no divergences can be associated with the coupling in the ultraviolet; they were thus once embraced as the solution to ultraviolet infinities of QFT's. Four-dimensional finite theories are restricted to supersymmetric gauge theories (or Super-Yang-Mills, SYM's), of which divergence cancelation is a general feature, and have a wealth of interesting structure. $\mathcal{N} = 4$ SYM theories have been shown to be finite to all orders (Cf. e.g. [166, 167]) whereas for $\mathcal{N} = 2$, the Adler-Bardeen Theorem guarantees that no higher than 1-loop corrections exist for the β-function [168]. Finally, for the unextended $\mathcal{N} = 1$ theories, the vanishing at 1-loop implies that for 2-loops [169].

When a **conformal** field theory (CFT) with vanishing β-function also has the anomalous dimensions vanishing, the theory is in fact a finite theory. This class of theories is without divergence and scale – and here we enter the realm of string theory. Recently much attempts have been undertaken in the construction of such theories as low-energy limits of the world-volume theories of D-brane probes on spacetime singularities (Chap. 9, [73, 74, 69, 171, 75, 76]) or of brane setups of the Hanany-Witten type [66, 82, 78, 79]. The construction of these theories not only supplies an excellent check for string theoretic techniques but also, vice versa, facilitate the incorporation of the Standard Model into string unifications. These finite (super-)conformal theories in four dimensions still remain a topic of fervent pursuit.

Almost exactly concurrent with these advances in physics was a host of activities in mathematics. Inspired by problems in linear representations of partially ordered sets over a field [178, 86, 180, 183, 184], elegant and graphical methods have been developed in attacking standing problems in algebra and combinatorics such as the classification of representation types and indecomposables of finite-dimensional algebras.

In 1972, P. Gabriel introduced the concept of a "Köcher" in [86]. This is what is known to our standard parlance today as a "Quiver." What entailed was a plethora of exciting and fruitful research in graph theory, axiomatic set theory, linear algebra and category theory, among many other branches. In particular one result that has spurned interest is the great limitation imposed on the shapes of the quivers once the concept of **finite representation type** has been introduced.

It may at first glance seem to the reader that these two disparate directions of research in contemporary physics and mathematics may never share conjugal harmony. However, following the works of [69, 171, 75, 76] those amusing quiver diagrams have surprisingly - or perhaps not too much so, considering how that illustrious field of String Theory has of late brought such enlightenment upon physics from seemingly most esoteric mathematics - taken a slight excursion from the reveries of the abstract, and manifested themselves in SYM theories emerging from D-branes probing orbifolds. The gauge fields and matter content of the said theories are conveniently encoded into quivers and further elaborations upon relations to beyond orbifold theories have been suggested in Chapters 9 and 10.

It is therefore natural, for one to pause and step back awhile, and regard the string orbifold theory from the perspective of a mathematician, and the quivers, from that of a physicist. However, due to his inexpertise in both, the author could call himself neither. Therefore we are compelled to peep at the two fields as outsiders, and from afar attempt to make some observations on similarities, obtain some vague notions of the beauty, and speculate upon some underlying principles. This is then the purpose of this note: to perceive, with a distant and weak eye; to inform, with a remote and feeble voice.

The organisation of this chapter is as follows. Though the main results are given in §4, we begin with some preliminaries from contemporary techniques in string theory on constructing four dimensional super-Yang-Mills, focusing on what each interprets finitude to mean: §12.2.1 on D-brane probes on orbifold singularities, §12.2.2 on Hanany-Witten setups and §12.2.3 on geometrical engineering. Then we move to the other direction and give preliminaries in the mathematics, introducing quiver graphs and path algebras in §12.3.1, classification of representation types in §12.3.2 and how the latter imposes constraints on the former in §12.3.3. The physicist may thus liberally neglect §2 and the mathematician, §3. Finally in §12.4 we shall see how those beautiful theorems in graph theory and axiomatic set theory may be used to give surprising results in constructing gauge theories from string theory.

Nomenclature

Unless the contrary is stated, we shall throughout this chapter adhere to the convention that k is a field of characteristic zero (and hence infinite), that Q denotes a quiver and kQ, the path algebra over the field k associated thereto, that $\text{rep}(X)$ refers to the representation of the object X, and that $\text{irrep}(\Gamma)$ is the set of irreducible representations of the group Γ. Moreover, $\mathsf{San\ serif}$ type setting will be reserved for categories, calligraphic $\mathcal{N}$ is used to denote the number of supersymmetries and $\widehat{\ }$, to distinguish the Affine Lie Algebras or Dynkin graphs.

12.2 Preliminaries from the Physics

The Callan-Symanzik equation of a QFT dictates the behaviour, under the renormalisation group flow, of the n-point correlator $G^{(n)}(\{\phi(x_i)\}; M, \lambda)$ for the quantum fields $\phi(x)$, according to the renormalisation of the coupling λ and momentum scale

M (see e.g. [164], whose conventions we shall adopt):

$$\left[M\frac{\partial}{\partial M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right] G^{(n)}(\{\phi(x_i)\}; M, \lambda) = 0.$$

The two universal dimensionless functions β and γ are known respectively as the **β-function** and the **anomalous dimension**. They determine how the shifts $\lambda \to \lambda+\delta\lambda$ in the coupling constant and $\phi \to (1+\delta\eta)\phi$ in the wave function compensate for the shift in the renormalisation scale M:

$$\beta(\lambda) := M\frac{\delta\lambda}{\delta M} \qquad \gamma(\lambda) := -M\frac{\delta\eta}{\delta M}.$$

Three behaviours are possible in the region of small λ: (1) $\beta(\lambda) > 0$; (2) $\beta(\lambda) < 0$; and (3) $\beta(\lambda) = 0$. The first has good IR behaviour and admits valid Feynmann perturbation at large-distance, and the second possesses good perturbative behaviour at UV limits and are asymptotically free. The third possibility is where the coupling constants do not flow at all and the renormalised coupling is always equal to the bare coupling. The only possible divergences in these theories are associated with field-rescaling which cancel automatically in physical S-matrix computations. It seems that to arrive at these well-tamed theories, some supersymmetry (SUSY) is needed so as to induce the cancelation of boson-fermion loop effects[1]. These theories are known as the **finite theories** in QFT.

Of particular importance are the finite theories that arise from **conformal** field theories which generically have in addition to the vanishing β-functions, also zero anomalous dimensions. Often this subclass belongs to a continuous manifold of scale invariant theories and is characterised by the existence of exactly marginal operators and whence dimensionless coupling constants, the set of mappings among which constitutes the *duality group* à la Mantonen-Olive of $\mathcal{N} = 4$ SYM, a hotly pursued topic.

A remarkable phenomenon is that if there is a choice of coupling constants such

[1]Proposals for non-supersymmetric finite theories in four dimensions have been recently made in [75, 76, 160, 294]; to their techniques we shall later turn briefly.

that all β-functions as well as the anomalous dimensions (which themselves do vanish at leading order if the manifold of fixed points include the free theory) vanish at first order then the theory is finite to all orders (Cf. references in [82]). A host of finite theories arise as low energy effective theories of String Theory. It will be under this light that our discussions proceed. There are three contemporary methods of constructing (finite, super) gauge theories: (1) geometrical engineering; (2) D-branes probing singularities and (3) Hanany-Witten brane setups. Discussions on the equivalence among and extensive reviews for them have been in wide circulation (q.v. e.g. [175, 53, 172, 295, 296]). Therefore we shall not delve too far into their account; we shall recollect from them what each interprets *finitude* to mean.

12.2.1 D-brane Probes on Orbifolds

When placing n D3-branes on a space-time orbifold singularity $\mathbb{C}^m/\Gamma$, out of the parent $\mathcal{N}=4$ $SU(n)$ SYM one can fabricate a $\prod\limits_i U(N_i)$ gauge theory with irrep$(\Gamma) :=$ $\{\mathbf{r}_i\}$ and $\sum\limits_i N_i \mathrm{dim}\mathbf{r}_i = n$ [76]. The resulting SUSY in the four-dimensional worldvolume is $\mathcal{N}=2$ if the orbifold is $\mathbb{C}^2/\{\Gamma \subset SU(2)\}$ as studied in [69, 171], $\mathcal{N}=1$ if $\mathbb{C}^3/\{\Gamma \subset SU(3)\}$ as in Chap. 9 and non-SUSY if $\mathbb{C}^3/\{\Gamma \subset SU(4)\}$ as in Chap. 11. The subsequent matter fields are $a^{\mathbf{4}}_{ij}$ Weyl fermions $\Psi^{ij}_{f_{ij}=1,\ldots,a^{\mathbf{4}}_{ij}}$ and $a^{\mathbf{6}}_{ij}$ scalars $\Phi^{ij}_{f_{ij}}$ with $i,j=1,\ldots,n$ and $a^{\mathcal{R}}_{ij}$ defined by

$$\mathcal{R} \otimes \mathbf{r}_i = \bigoplus_j a^{\mathcal{R}}_{ij} \mathbf{r}_j \tag{12.2.1}$$

respectively for $\mathcal{R}=4,6$. It is upon these matrices a_{ij}, which we call **bifundamental matter matrices** that we shall dwell. They dictate how many matter fields transform under the $(N_i, \bar{N}_j)$ of the product gauge group. It was originally pointed out in [69, 171] that one can encode this information in **quiver diagrams** where one indexes the vector multiplets (gauge) by nodes and hypermultiplets (matter) by links in a (finite) graph so that the bifundamental matter matrix defines the (possibly oriented) adjacency matrix for this graph. In other words, one draws a_{mn} number of arrows from node m to n. Therefore to each vertex i is associated a vector space V_i and a semisimple component $SU(N_i)$ of the gauge group acting on V_i. Moreover an oriented link from V_1 to V_2 represents a complex field transforming under $\hom(V_1, V_2)$. We shall see in section §12.3.1 what all this means.

When we take the dimension of both sides of (12.2.1), we obtain the matrix equation

$$\dim(\mathcal{R}) r_i = a_{ij}^{\mathcal{R}} r_j \tag{12.2.2}$$

where $r_i := \dim \mathbf{r}_i$. As discussed in Chap. 9, the remaining SUSY must be in the commutant of Γ in the $SU(4)$ R-symmetry of the parent $\mathcal{N}=4$ theory. In the case of $\mathcal{N}=2$ this means that $4 = 1+1+2$ and by SUSY, $6 = 1+1+2+2$ where the 1 is the principal (trivial) irrep and 2, a two-dimensional irrep. Therefore due to the additivity and orthogonality of group characters, it was thus pointed out (*cit. ibid.*) that one only needs to investigate the fermion matrix a_{ij}^4, which is actually reduced to $2\delta_{ij} + a_{ij}^2$. Similarly for $\mathcal{N}=1$, we have $\delta_{ij} + a_{ij}^3$. It was subsequently shown that (12.2.2) necessitates the vanishing of the β-function to one loop. Summarising these points, we state the condition for *finitude* from the orbifold perspective:

$$\begin{array}{c|c} \text{SUSY} & \text{Finitude} \\ \hline \mathcal{N}=2 & 2r_i = a_{ij}^2 r_j \\ \mathcal{N}=1 & 3r_i = a_{ij}^3 r_j \\ \mathcal{N}=0 & 4r_i = a_{ij}^4 r_j \end{array} \tag{12.2.3}$$

In fact it was shown in [76, 48], that the 1-loop β-function is proportional to $dr_i - a_{ij}^d r_j$ for $d = 4-\mathcal{N}$ whereby the vanishing thereof signifies finitude, exceeding zero signifies asymptotical freedom and IR free otherwise[2]. We shall call this expression $d\delta_{ij} - a_{ij}^d$ the **discriminant function** since its relation with respective to zero (once dotted with the vector of labels) discriminates the behaviour of the QFT. This point shall arise once again in §12.4.

[2]As a cautionary note, these conditions are necessary but may not be sufficient. In the cases of $\mathcal{N} < 2$, one needs to check the superpotential. However, throughout the chapter we shall focus on the necessity of these conditions.

12.2.2 Hanany-Witten

In brane configurations of the Hanany-Witten type [66], D-branes are stretched between sets of NS-branes, the presence of which break the SUSY afforded by the 32 supercharges of the type II theory. In particular, parallel sets of NS-branes break one-half SUSY, giving rise to $\mathcal{N} = 2$ in four dimensions [66] whereas rotated NS-branes [165] or grids of NS-branes (the so-called Brane Box Models) [82, 78, 79] break one further half SUSY and gives $\mathcal{N} = 1$ in four dimensions.

The Brane Box Models (BBM) (and possible extensions to brane cubes) provide an intuitive and visual realisation of SYM. They generically give rise to $\mathcal{N} = 1$, with $\mathcal{N} = 2$ as a degenerate case. Effectively, the D-branes placed in the boxes of NS-branes furnish a geometrical way to encode the representation properties of the finite group Γ discussed in §12.2.1. The bi-fundamentals, and hence the quiver diagram, are constructed from oriented open strings connecting the D-branes according to the rule given in [78]:

$$\mathbf{3} \otimes r_i = \bigoplus_{j \in \substack{\text{N, E, SW} \\ \text{Neighbours}}} r_j .$$

This is of course (12.2.1) in a different guise and we clearly see the equivalence between this and the orbifold methods of §12.2.1.

Now in [66], for the classical setup of stretching a D-brane between two NS-branes, the asymptotic bending of the NS-brane controls the evolution of the gauge coupling (since the inverse of which is dictated by the distance between the NS-branes). Whence NS-branes bending towards each other gives an IR free theory (case (1) defined above for the β-functions), while bending away give an UV free (case (2)) theory. No bending thus indicates the non-evolution of the β-function and thus finiteness; this is obviously true for any brane configurations, intervals, boxes or cubes. We quote [82] verbatim on this issue: *Given a brane configuration which has no bending, the corresponding field theory which is read off from the brane configuration by using the rules of [78] is a finite theory.*

Discussions on bending have been treated in [80, 170] while works towards the establishment of the complete correspondence between Hanany-Witten methods and orbifold probes (to beyond the Abelian case) are well under way [172, 295, 296]. Under this light, we would like to lend this opportunity to point out that the anomaly cancelation equations (2-4) of [80] which discusses the implication of tadpole-cancelation to BBM in excellent detail, are precisely in accordance with (12.2.1). In particular, what they referred as the Fourier transform to extract the rank matrix for the $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ BBM is precisely the orthogonality relations for finite group characters (which in the case of the Abelian groups conveniently reduce to roots of unity and hence Fourier series). The generalisation of these equations for non-Abelian groups should be immediate. We see indeed that there is a close intimacy between the techniques of the current subsection with §12.2.1; let us now move to a slightly different setting.

12.2.3 Geometrical Engineering

On compactifying Type IIA string theory on a non-compact Calabi-Yau threefold, we can geometrically engineer [47, 49, 48] an $\mathcal{N} = 2$ SYM. More specifically when we compactify Type IIA on a K3 surface, locally modeled by an ALE singularity, we arrive at an $\mathcal{N} = 2$ SYM in 6 dimensions with gauge group ADE depending on the singularity about which D2-branes wrap in the zero-volume limit. However if we were to further compactify on T^2, we would arrive at an $\mathcal{N} = 4$ SYM in 4 dimensions. In order to kill the extraneous scalars we require a 2-fold without cycles, namely $\mathbf{P}^1$, or the 2-sphere. Therefore we are effectively compactifying our original 10 dimensional theory on a (non-compact) Calabi-Yau threefold which is an ALE (K3) fibration over $\mathbf{P}^1$, obtaining a pure $\mathcal{N} = 2$ SYM in 4 dimensions with coupling $\frac{1}{g^2}$ equaling to the volume of the base $\mathbf{P}^1$.

To incorporate matter [49, 48] we let an A_{n-1} ALE fibre collide with an A_{m-1} one to result in an A_{m+n-1} singularity; this corresponds to a Higgsing of $SU(m+n) \to SU(m) \times SU(n)$, giving rise to a bi-fundamental matter $(n, \bar{m})$. Of course, by colliding the A singular fibres appropriately (i.e., in accordance with Dynkin diagrams) this above idea can easily be generalised to fabricate generic product SU gauge groups. Thus as opposed to §12.2.1 where bi-fundamentals (and hence the quiver diagram) arise from linear maps between irreducible modules of finite group representations, or §12.2.2 where they arise from open strings linking D-branes, in the context of geometrical engineering, they originate from colliding fibres of the Calabi-Yau.

The properties of the β-function from this geometrical perspective were also investigated in [48]. The remarkable fact, using the Perron-Frobenius Theorem, is that the *possible* resulting SYM is highly restricted. The essential classification is that *if the $\mathcal{N} = 2$ β- function vanishes (and hence a* finite *theory), then the quiver diagram encoding the bi-fundamentals must be the affine $\widehat{ADE}$ Dynkin Diagrams* and when it is less than zero (and thus an asymptotically free theory), the quiver must be the ordinary ADE. We shall see later how one may graphically arrive at these results.

Having thus reviewed the contemporary trichotomy of the methods of constructing SYM from string theory fashionable of late, with special emphasis on what the word *finitude* means in each, we are obliged, as prompted by the desire to unify, to ask ourselves whether we could study these techniques axiomatically. After all, the quiver diagram does manifest under all these circumstances. And it is these quivers, as viewed by a graph or representation theorist, that we discuss next.

12.3 Preliminaries from the Mathematics

We now formally study what a quiver is in a mathematical sense. There are various approaches one could take, depending on whether one's interest lies in category theory or in algebra. We shall commence with P. Gabriel's definition, which was the genesis of the excitement which ensued. Then we shall introduce the concept of path algebras and representation types as well as a host of theorems that limit the shapes of quivers depending on those type. As far as convention and nomenclature are concerned, §12.3.1 and §12.3.2 will largely follow [176, 177, 178].

12.3.1 Quivers and Path Algebras

In his two monumental papers [86, 180], Gabriel introduced the following concept:

DEFINITION 12.3.16 *A* **quiver** *is a pair* $Q = (Q_0, Q_1)$*, where* Q_0 *is a set of vertices and* Q_1*, a set of arrows such that each element* $\alpha \in Q_1$ *has a beginning* $s(\alpha)$ *and an end* $e(\alpha)$ *which are vertices, i.e.,* $\{s(\alpha) \in Q_0\} \xrightarrow{\alpha} \{e(\alpha) \in Q_0\}$*.*

In other words a quiver is a (generically) *directed graph*, possibly with multiple arrows and loops. We shall often denote a member γ of Q_1 by the beginning and ending vertices, as in $x \xrightarrow{\gamma} y$.

Given such a graph, we can generalise $Q_{0,1}$ by defining a **path of length** m to be the formal composition $\gamma = \gamma_1\gamma_2 \ldots \gamma_m := (i_0 \xrightarrow{\gamma_1} i_1 \ldots \xrightarrow{\gamma_m} i_m)$ with $\gamma_j \in Q_1$ and $i_j \in Q_0$ such that $i_0 = s(\gamma_1)$ and $i_t = s(\gamma_{t-1}) = e(\gamma_t)$ for $t = 1, ..., m$. This is to say that we follow the arrows and trace through m nodes. Subsequently we let Q_m be the set of all paths of length m and for the identity define, for each node x, a trivial path of length zero, e_x, starting and ending at x. This allows us to associate $Q_0 \sim \{e_x\}_{x \in Q_0}$ and $(i \xrightarrow{\alpha} j) \sim e_i\alpha = \alpha e_j$. Now Q_m is defined for all non-negative m, whereby giving a gradation in Q.

Objects[3] may be assigned to the nodes and edges of the quiver so as to make its conception more concrete. This is done so in two closely-related ways:

1. By the **representation of a quiver**, rep(Q), we mean to associate to each vertex $x \in Q_0$ of Q, a vector space V_x and to each arrow $x \to y$, a linear transformation between the corresponding vector spaces $V_x \to V_y$.

2. Given a field k and a quiver Q, a **path algebra** kQ is an algebra which as a vector space over k has its basis prescribed by the paths in Q.

There is a 1-1 correspondence between kQ-modules and rep(Q). Given rep$(Q) = \{V_{x \in Q_0}, (x \to y) \in Q_1\}$, the associated kQ module is $\bigoplus\limits_x V_x$ whose basis is the set of paths Q_m. Conversely, given a kQ-module V, we define $V_x = e_xV$ and the arrows to be prescribed by the basis element u such that $u \sim e_yu = ue_x$ whereby making u a map from V_x to V_y.

[3]We could take this word literally and indeed we shall later briefly define the objects in a Quiver Category.

Figure 12-1: Two examples of quivers with nodes and edges labeled.

On an algebraic level, due to the gradation of the quiver Q by Q_m, the path algebra is furnished by

$$kQ := \bigoplus_m kQ_m \quad \text{with} \quad kQ_m := \bigoplus_{\gamma \in Q_m} \gamma k \tag{12.3.4}$$

As a k-algebra, the addition and multiplication axioms of kQ are as follows: given $a = \sum\limits_{\alpha \in Q_m;\ a_\alpha \in k} \alpha a_\alpha$ and $b = \sum\limits_{\beta \in Q_n;\ a_\beta \in k} \beta a_\beta$ as two elements in kQ, $a+b = \sum\limits_\alpha \alpha(a_\alpha + b_\alpha)$ and $a \cdot b = \sum\limits_{\alpha,\beta} \alpha\beta a_\alpha b_\beta$ with $\alpha\beta$ being the joining of paths (if the endpoint of one is the beginning of another, otherwise it is defined to be 0).

This correspondence between path algebras and quiver representations gives us the flexibility of freely translating between the two, an advantage we shall later graciously take. As illustrative examples of concepts thus far introduced, we have drawn two quivers in Figure 12-1. In example (I), $Q_0 = \{1, 2\}, Q_1 = \{\alpha, \beta\}$ and $Q_{m>1} = \{\}$. The path algebra is then the so-called Kronecker Algebra:

$$kQ = e_1 k \oplus e_2 k \oplus \alpha k \oplus \beta k = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}.$$

On the other hand, for example (II), $Q_{m \in \{0,1,2,\ldots\}} = \{\beta^m\}$ and the path algebra becomes $\bigoplus\limits_m \beta^m k = k[\beta]$, the infinite dimensional free algebra of polynomials of one variable over k.

In general, kQ is finitely generated if there exists a finite number of vertices and arrows in Q and kQ is finite-dimensional if there does not exist any oriented cycles in Q.

To specify the quiver even further one could introduce labeling schemes for the nodes and edges; to do so we need a slight excursion to clarify some standard terminology from graph theory.

DEFINITION 12.3.17 *The following are common categorisations of graphs:*

- *A* **labeled graph** *is a graph which has, for each of its edge* $(i \xrightarrow{\gamma} j)$*, a pair of positive integers* $(a_{ij}^{\gamma}, a_{ji}^{\gamma})$ *associated thereto;*
- *A* **valued graph** *is a labeled graph for which there exists a positive integer* f_i *for each node* i*, such that* $a_{ij}^{\gamma} f_j = a_{ji}^{\gamma} f_i$ *for each arrow*[4]*.*
- *A* **modulation** *of a valued graph consists of an assignment of a field* k_i *to each node* i*, and a* k_i*-*k_j *bi-module* M_{ij}^{γ} *to each arrow* $(i \xrightarrow{\gamma} j)$ *satisfying*

  (a) $M_{ij}^{\gamma} \cong \hom_{k_i}(M_{ij}^{\gamma}, k_i) \cong \hom_{k_j}(M_{ij}^{\gamma}, k_j)$*;*

  (b) $\dim_{k_i}(M_{ij}^{\gamma}) = a_{ij}^{\gamma}$*.*
- *A* **modulated quiver** *is a valued graph with a modulation (and orientation).*

We shall further adopt the convention that we omit the label to edges if it is $(1, 1)$. We note that of course according to this labeling, the matrices a_{ij} are almost what we call **adjacency matrices**. In the case of unoriented single-valence edges between say nodes i and j, the adjacency matrix has $a_{ij} = a_{ji} = 1$, precisely the label $(1, 1)$. However, directed edges, as in Figure 12-2 and Figure 12-3, are slightly more involved. This is exemplified by $\bullet \Rightarrow \bullet$ which has the label $(2, 1)$ whereas the conventional adjacency matrix would have the entries $a_{ij} = 2$ and $a_{ji} = 0$. Such a labeling scheme is of course so as to be consistent with the entries of the Dynkin-Cartan Matrices of the semi-simple Lie Algebras. To this subtlety we shall later turn.

The canonical examples of labeled (some of them are valued) graphs are what are known as the **Dynkin** and **Euclidean** graphs. The Dynkin graphs are further

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[4] *Thus a labeled graph without any cycles is always a valued graph since we have enough degrees of freedom to solve for a consistent set of* f_i *whereas cycles would introduce extra constraints. (Of course there is no implicit summation assumed in the equation.)*

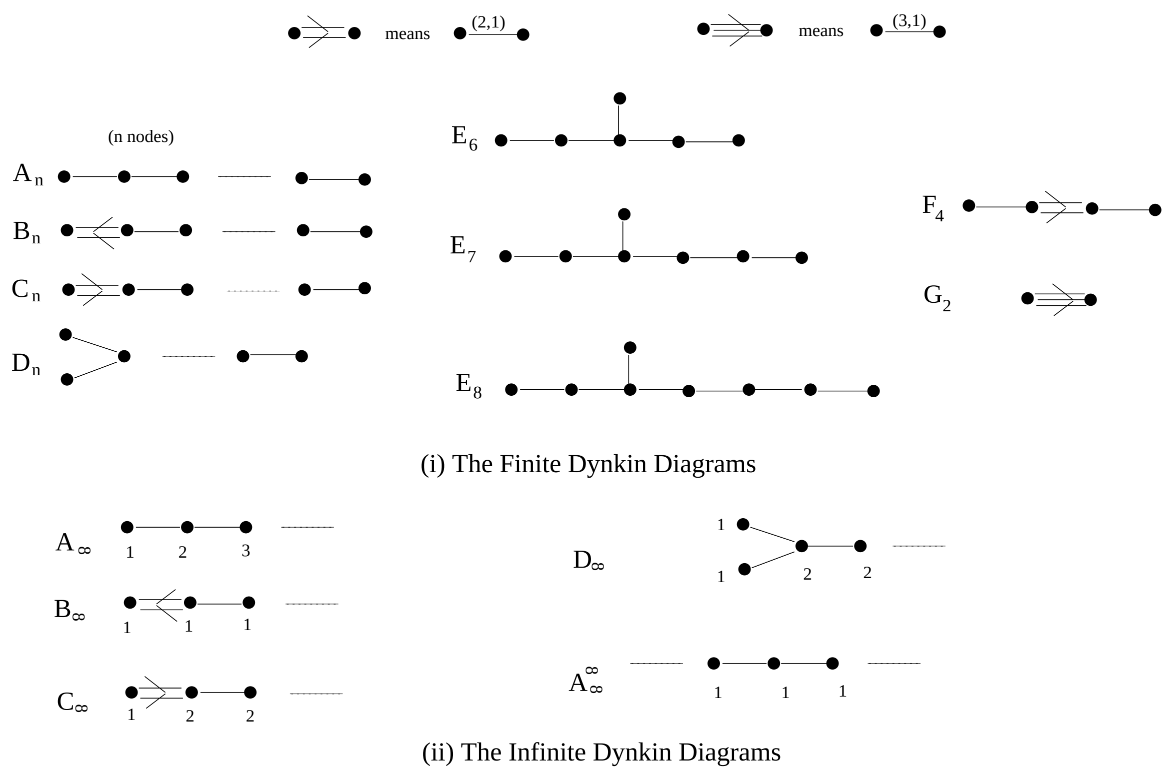


Figure 12-2: The Finite and Infinite Dynkin Diagrams as labeled quivers. The finite cases are the well-known Dynkin-Coxeter graphs in Lie Algebras (from Chapter 4 of [176]).

subdivided into the finite and the infinite; the former are simply the Dynkin-Coxeter Diagrams well-known in Lie Algebras while the latter are analogues thereof but with infinite number of labeled nodes (note that the nodes are not labeled so as to make them valued graphs; we shall shortly see what those numbers signify.) The Euclidean graphs are the so-called Affine Coxeter-Dynkin Diagrams (of the affine extensions of the semi-simple Lie algebras) but with their multiple edges differentiated by oriented labeling schemes. These diagrams are shown in Figure 12-2 and Figure 12-3.

How are these the *canonical* examples? We shall see the reason in §12.3.3 why they are ubiquitous and atomic, constituting, when certain finiteness conditions are imposed, the only elemental quivers. Before doing so however, we need some facts from representation theory of algebras; upon these we dwell next.

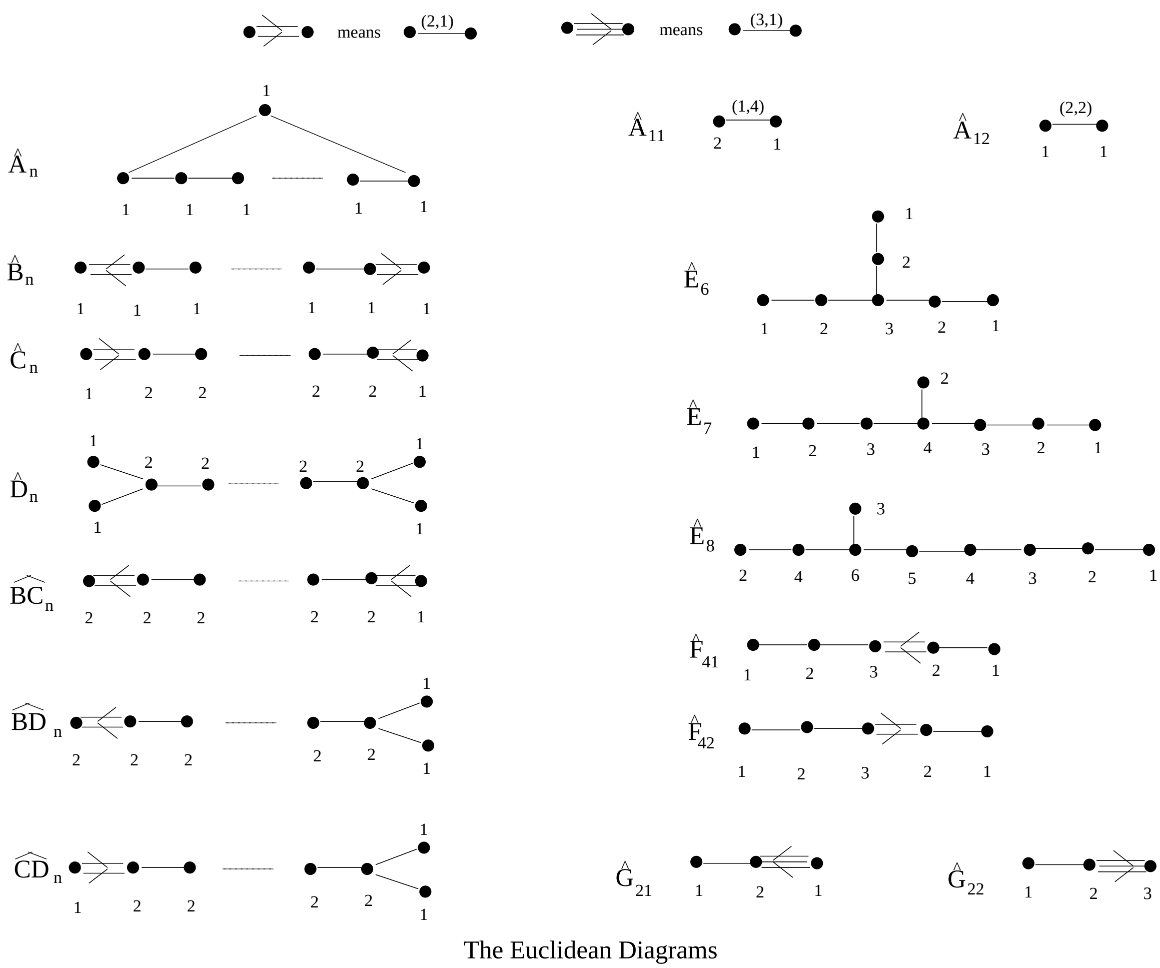


Figure 12-3: The Euclidean Diagrams as labeled quivers; we recognise that this list contains the so-called Affine Dynkin Diagrams (from Chapter 4 of [176]).

12.3.2 Representation Type of Algebras

Henceforth we restrict ourselves to infinite fields, as some of the upcoming definitions make no sense over finite fields. This is of no loss of generality because in physics we are usually concerned with the field $\mathbb{C}$. When given an algebra, we know its quintessential properties once we determine its decomposables (or equivalently the irreducibles of the associated module). Therefore classifying the behaviour of the indecomposables is the main goal of classifying **representation** types of the algebras.

The essential idea is that an algebra is of finite type if there are only finitely many indecomposables; otherwise it is of infinite type. Of the infinite type, there is one well-behaved subcategory, namely the algebras of tame representation type, which has its indecomposables of each dimension coming in finitely many one-parameter families with only finitely many exceptions. Tameness in some sense still suggests classifiability of the infinite indecomposables. On the other hand, an algebra of wild type includes the free algebra on two variables, $k[X, Y]$, (the path algebra of Figure 12-1 (II), but with two self-adjoining arrows), which indicates representations of arbitrary finite dimensional algebras, and hence unclassifiability[5].

We formalise the above discussion into the following definitions:

DEFINITION 12.3.18 *Let k be an infinite field and A, a finite dimensional algebra.*

- *A is of* **finite representation type** *if there are only finitely many isomorphism classes of indecomposable A-modules, otherwise it is of infinite type;*

- *A is of* **tame representation type** *if it is of infinite type and for any dimension n, there is a finite set of A-$k[X]$-bimodules[6] M_i which obey the following:*

    1. *M_i are free as right $k[X]$-modules;*

[5] For precise statements of the unclassifiability of modules of two-variable free algebras as Turing-machine undecidability, cf. e.g. Thm 4.4.3 of [176] and [185].

[6] *Therefore for the polynomial ring $k[X]$, the indeterminate X furnishes the parameter for the one-parameter family mentioned in the first paragraph of this subsection. Indeed the indecomposable $k[X]$-modules are classified by powers of irreducible polynomials over k.*

2. *For some i and some indecomposable $k[X]$-module M, all but a finitely many indecomposable A-modules of dimension n can be written as $M_i \otimes_{k[X]} M$.*

If the M_i may be chosen independently of n, then we say A is of **domestic representation type**.

- *A is of* **wild representation type** *if it is of infinite representation type and there is a finitely generated A-$k[X,Y]$-bimodule M which is free as a right $k[X,Y]$-module such that the functor $M \otimes_{k[X,Y]}$ from finite-dimensional $k[X,Y]$-modules to finite-dimensional A-modules preserves indecomposability and isomorphism classes.*

We are naturally led to question ourselves whether the above list is exhaustive. This is indeed so: what is remarkable is the so-called trichotomy theorem which says that all finite dimensional algebras must fall into one and only one of the above classification of types[7]:

THEOREM 12.3.17 (Trichotomy Theorem) *For k algebraically closed, every finite dimensional algebra A is of finite, tame or wild representation types, which are mutually exclusive.*

To this pigeon-hole we may readily apply our path algebras of §12.3.1. Of course such definitions of representation types can be generalised to additive categories with unique decomposition property. Here by an additive category B we mean one with finite direct sums and an Abelian structure on $\mathsf{B}(X,Y)$, the set of morphisms from object X to Y in B such that the composition map $\mathsf{B}(Y,Z) \times \mathsf{B}(X,Y) \to \mathsf{B}(X,Z)$ is bilinear for X,Y,Z objects in B. Indeed, that (a) each object in B can be finitely decomposed via the direct sum into indecomposable objects and that (b) the ring of endomorphisms between objects has a unique maximal ideal guarantees that B possesses unique decomposability as an additive category [178].

[7]For a discussion on this theorem and how similar structures arises for finite groups, cf. e.g. [176, 178] and references therein.

The category $\mathsf{rep}(Q)$, what [87] calls the **Quiver Category**, has as its objects the pairs (V, α) with linear spaces V associated to the nodes and linear mappings α, to the arrows. The morphisms of the category are mappings $\phi : (V, \alpha) \to (V', \alpha')$ compatible with α by $\phi_{e(l)}\alpha_l = \alpha'_l\phi_{s(l)}$. In the sense of the correspondence between representation of quivers and path algebras as discussed in §12.3.1, the category $\mathsf{rep}(Q)$ of finite dimensional representations of Q, as an additive category, is equivalent to $\mathsf{mod}(kQ)$, the category of finite dimensional (right) modules of the path algebra kQ associated to Q. This equivalence

$$\mathsf{rep}(Q) \cong \mathsf{mod}(kQ)$$

is the axiomatic statement of the correspondence and justifies why we can hereafter translate freely between the concept of representation types of quivers and associated path algebras.

12.3.3 Restrictions on the Shapes of Quivers

Now we return to our quivers and in particular combine §12.3.1 and §12.3.2 to address the problem of how the representation types of the path algebra restricts the shapes of the quivers. Before doing so let us first justify, as advertised in §12.3.1, why Figure 12-2 and Figure 12-3 are canonical. We first need a preparatory definition: we say a labeled graph T_1 is **smaller** than T_2 if there is an injective morphism of graphs $\rho : T_1 \to T_2$ such that for each edge $(i \xrightarrow{\gamma} j)$ in T_1, $a_{ij} \leq a_{\rho(i)\rho(j)}$ (and T_1 is said to be strictly smaller if ρ can not be chosen to be an isomorphism). With this concept, we can see that the Dynkin and Euclidean graphs are indeed our archetypal examples of labeled graphs due to the following theorem:

THEOREM 12.3.18 [176, 177] *Any connected labeled graph T is one and only one of the following:*

1. *T is Dynkin (finite or infinite);*

2. *There exists a Euclidean graph smaller than T.*

This is a truly remarkable fact which dictates that the atomic constituents of all labeled graphs are those arising from semi-simple (ordinary and affine) Lie Algebras. The omni-presence of such meta-patterns is still largely mysterious (see e.g. [293, 96] for discussions on this point).

Let us see another manifestation of the elementarity of the Dynkin and Euclidean Graphs. Again, we need some rudimentary notions.

DEFINITION 12.3.19 *The* **Cartan Matrix** *for a labeled graph T with labels (a_{ij}, a_{ji}) for the edges is the matrix*[8] $c_{ij} := 2\delta_{ij} - \sum\limits_{\gamma} a_{ij}^{\gamma}$

We can symmetrise the Cartan matrix for valued graphs as $\tilde{c}_{ij} = c_{ij} f_j$ with $\{f_j\}$ the valuation of the nodes of the labeled graph. With the Cartan matrix at hand, let us introduce an important function on labeled graphs:

DEFINITION 12.3.20 *A* **subadditive function** $n(x)$ *on a labeled graph T is a function taking nodes $x \in T$ to $n \in \mathbb{Q}^+$ such that $\sum\limits_i n(i) c_{ij} \geq 0 \ \forall \ j$. A subadditive function is* **additive** *if the equality holds.*

It turns out that imposing the existence such a function highly restricts the possible shape of the graph; in fact we are again led back to our canonical constituents. This is dictated by the following

THEOREM 12.3.19 (Happel-Preiser-Ringel [176]) *Let T be a labeled graph and $n(x)$ a subadditive function thereupon, then the following holds:*

1. *T is either (finite or infinite) Dynkin or Euclidean;*

2. *If $n(x)$ is not additive, then T is finite Dynkin or A_∞;*

3. *If $n(x)$ is additive, then T is infinite Dynkin or Euclidean;*

4. *If $n(x)$ is unbounded then $T = A_\infty$*

[8] *This definition is inspired by, but should be confused with, Cartan matrices for semisimple Lie algebras; to the latter we shall refer as Dynkin-Cartan matrices. Also, in the definition we have summed over edges γ adjoining i and j so as to accommodate multiple edges between the two nodes each with non-trivial labels.*

We shall see in the next section what this notion of graph additivity [176, 96] signifies for super-Yang-Mills theories. For now, let us turn to the *Theorema Egregium* of Gabriel that definitively restricts the shape of the quiver diagram once the *finitude* of the representation type of the corresponding path algebra is imposed.

THEOREM 12.3.20 (Gabriel [86, 180, 178]) *A finite quiver Q (and hence its associated path algebra over an infinite field) is of finite representation type if and only if it is a disjoint union of Dynkin graphs of type A_n, D_n and E_{678}, i.e., the ordinary simply-laced ADE Coxeter-Dynkin diagrams.*

In the language of categories [87], where a proof of the theorem may be obtained using Coxeter functors in the Quiver Category, the above proposes that the quiver is (unions of) ADE if and only if there are a finite number of non-isomorphic indecomposable objects in the category $\mathsf{rep}(Q)$.

Once again appears the graphs of Figure 12-2, and in fact only the single-valence ones: that ubiquitous ADE meta-pattern! We recall from discussions in §12.3.1 that only for the simply-laced (and thus simply-valanced quivers) cases, viz. ADE and $\widehat{ADE}$, do the labels a_{ij}^{γ} precisely prescribe the adjacency matrices. To what type of path algebras then, one may ask, do the affine $\widehat{ADE}$ Euclidean graphs correspond? The answer is given by Nazarova as an extension to Gabriel's Theorem.

THEOREM 12.3.21 (Nazarova [182, 178]) *Let Q be a connected quiver without oriented cycles and let k be an algebraically closed field, then kQ is of tame (in fact domestic) representation type if and only if Q is the one of the Euclidean graphs of type $\hat{A}_n$, $\hat{D}_n$ and $\hat{E}_{678}$, i.e., the affine ADE Coxeter-Dynkin diagrams.*

Can we push further? What about the remaining quivers of in our canonical list? Indeed, with the introduction of modulation on the quivers, as introduced in §12.3.1, the results can be further relaxed to include more graphs, in fact all the Dynkin and Euclidean graphs:

THEOREM 12.3.22 (Tits, Bernstein-Gel'fand-Ponomarev, Dlab-Ringel, Nazarova-Ringel [87, 184, 181, 176]) *Let Q be a connected modulated quiver, then*

1. *If Q is of finite representation type then Q is Dynkin;*

2. *If Q is of tame representation type, then Q is Euclidean.*

This is then our dualism, on the one level of having finite graphs encoding a (classifiability) infinite algebra and on another level having the two canonical constituents of all labeled graphs being partitioned by finitude versus infinitude[9].

12.4 Quivers in String Theory and Yang-Mills in Graph Theory

We are now equipped with a small arsenal of facts; it is now our duty to expound upon them. Therefrom we shall witness how axiomatic studies of graphs and representations may shed light on current developments in string theory.

Let us begin then, upon examining condition (12.2.3) and Definition 12.3.20, with the following

OBSERVATION 1 *The condition for finitude of $\mathcal{N} = 2$ orbifold SYM theory is equivalent to the introduction of an additive function on the corresponding quiver as a labeled graph.*

This condition that for the label n_i to each node i and adjacency matrix A_{ij}, $2n_i = \sum\limits_j a_{ij} n_j$ is a very interesting constraint to which we shall return shortly. What we shall use now is Part 3 of Theorem 12.3.19 in conjunction with the above observation to deduce

COROLLARY 12.4.1 *All finite $\mathcal{N} = 2$ super-Yang-Mills Theories with bi-fundamental matter have their quivers as (finite disjoint unions) of the single-valence (i.e., $(1,1)$-labeled edges) cases of the Euclidean (Figure 12-3) or Infinite Dynkin (Figure 12-2) graphs.*

[9]This is much in the spirit of that wise adage, "Cette opposition nouvelle, 'le fini et l'infini', ou mieux 'l'infini dans le fini', remplace le dualisme de l'être et du paraître: ce qui paraît, en effet, c'est seulement un *aspect* de l'objet et l'objet est tout entier *dans* cet aspect et tout entier hors de lui [186]."

A few points to remark. This is slightly a more extended list than that given in [48] which is comprised solely of the $\widehat{ADE}$ quivers. These latter cases are the ones of contemporary interest because they, in addition to being geometrically constructable (Cf. §12.2.3), are also obtainable from the string orbifold technique[10] (Cf. §12.2.1) since after all the finite discrete subgroups of $SU(2)$ fall into an $\widehat{ADE}$ classification due to McKay's Correspondence [32, 73, 74] and Chap. 9. In addition to the above well-behaved cases, we also have the infinite simply-laced Dynkin graphs: A_∞, D_∞ and A_∞^∞. The usage of the Perron-Frobenius Theorem in [48] restricts one's attention to finite matrices. The allowance for infinite graphs of course implies an infinitude of nodes and hence infinite products for the gauge group. One needs not exclude these possibilities as after all in the study of D-brane probes, Maldacena's large N limit has been argued in [75, 76, 82] to be required for conformality and finiteness. In this limit of an infinite stack of D-branes, infinite gauge groups may well arise. In the Hanany-Witten picture, A_∞^∞ for example would correspond to an infinite array of NS5-branes, and A_∞, a semi-infinite array with enough D-branes on the other side to ensure the overall non-bending and parallelism of the NS. Such cases had been considered in [165].

Another comment is on what had been advertised earlier in §12.3.1 regarding the adjacency matrices. Theorem 12.3.19 does not exclude graphs with multiple-valanced oriented labels. This issue does not arise in $\mathcal{N} = 2$ which has only single-valanced and unoriented quivers. However, going beyond to $\mathcal{N} = 1, 0$, requires generically oriented and multiply-valanced quivers (i.e., non-symmetric, non-binary matter matrices) (Cf. Chapters 9 and 11); or, it is conceivable that certain theories not arising from orbifold procedures may also possess these generic traits. Under this light we question ourselves how one may identify the bi-fundamental matter matrices not with strict adjacency matrices of graphs but with the graph-label matrices a_{ij}^γ of §12.3.1 so as to accommodate multiple, chiral bi-fundamentals (i.e. multi-valence, directed graphs). In other words, could Corollary 12.4.1 *actually be relaxed to incorporate all of the Euclidean and infinite Dynkin graphs* as dictated by Theorem 12.3.19? Thoughts

[10] And in the cases of A and D also from Hanany-Witten setups [83, 172, 295, 296].

on this direction, viz., how to realise Hanany-Witten brane configurations for non-simply-laced groups have been engaged but still waits further clarification [173].

Let us now turn to Gabriel's famous Theorem 12.3.20 and see its implications in string theory and vice versa what information the latter provides for graph theory. First we make a companion statement to Observation 1:

OBSERVATION 2 *The condition for asymptotically free ($\beta < 0$) $\mathcal{N} = 2$ SYM theory with bi-fundamentals is equivalent to imposing a subadditive (but not additive) function of the corresponding quiver.*

This may thus promptly be utilised together with Part 2 of Theorem 12.3.19 to conclude that the only such theories are ones with ADE quiver, or, allowing infinite gauge groups, A_∞ as well (and indeed all finite Dynkin quivers once, as mentioned above, non-simply-laced groups have been resolved). This is once again a slightly extended version of the results in [48].

Let us digress, before trudging on, a moment to consider what is means to encode SYM with quivers. Now we recall that for the quiver Q, the assignment of objects and morphisms to the category $\mathsf{rep}(Q)$, or vector spaces and linear maps to nodes Q_0 and edges Q_1 in Q, or bases to the path algebra kQ, are all equivalent procedures. From the physics perspective, these assignments are precisely what we do when we associate vector multiplets to nodes and hypermultiplets to arrows as in the orbifold technique, or NS-branes to nodes and oriented open strings between D-branes to arrows as in the Hanany-Witten configurations, or singularities in Calabi-Yau to nodes and colliding fibres to arrows as in geometrical engineering. In other words the three methods, §12.3.1, §12.2.1 and §12.2.3, of constructing gauge theories in four dimensions currently in vogue are different representations of $\mathsf{rep}(Q)$ and are hence *axiomatically* equivalent as far as quiver theories are concerned.

Bearing this in mind, and in conjunction with Observations 1 and 2, as well as Theorem 12.3.20 together with its generalisations, and in particular Theorem 12.3.22, we make the following

COROLLARY 12.4.2 *To an asymptotically free $\mathcal{N} = 2$ SYM with bi-fundamentals is associated a finite path algebra and to a finite one, a tame path algebra. The association is in the sense that these SYM theories (or some theory categorically equivalent thereto) prescribe representations of the only quivers of such representation types.*

What is even more remarkable perhaps is that due to the Trichotomy Theorem, the path algebra associated to **all other quivers** must be of wild representation type. What this means, as we recall the unclassifiability of algebras of wild representations, is that these quivers are unclassifiable. In particular, if we assume that SYM with $\mathcal{N} = 0, 1$ and arbitrary bi-fundamental matter content can be constructed (either from orbifold techniques, Hanany-Witten, or geometrical engineering), then these theories *can not* be classified, in the strict sense that they are Turing undecidable and there does not exist, in any finite language, a finite scheme by which they could be listed. Since the set of SYM with bi-fundamentals is a proper subset of all SYM, the like applies to general SYM. What this signifies is that however ardently we may continue to provide more examples of say finite $\mathcal{N} = 1, 0$ SYM, the list can never be finished nor be described, unlike the $\mathcal{N} = 2$ case where the above discussions exhaust their classification. We summarise this amusing if not depressing fact as follows:

COROLLARY 12.4.3 *The generic $\mathcal{N} = 1, 0$ SYM in four dimensions are unclassifiable in the sense of being Turing undecidable.*

We emphasise again that by unclassifiable here we mean not *completely* classifiable because we have given a subcategory (the theories with bi-fundamentals) which is unclassifiable. Also, we rest upon the assumption that for any bi-fundamental matter content an SYM could be constructed. Works in the direction of classifying all possible gauge invariant operators in an $\mathcal{N} = 1$ SUSY Lagrangian have been pursued [174]. Our claim is much milder as no further constraints than the possible naïve matter content are imposed; we simply state that the complete generic problem of classifying the $\mathcal{N} < 2$ matter content is untractable. In [174], the problem has been reduced to manipulating a certain cohomological algebra; it would be interesting to see for example, whether such BRST techniques may be utilised in the classification of certain categories of graphs.

Such an infinitude of gauge theories need not worry us as there certainly is no shortage of say, Calabi-Yau threefolds which may be used to geometrically engineer them. This unclassifiability is rather in the spirit of that of, for example, four-manifolds. Indeed, though we may never exhaust the list, we are not precluded from giving large exemplary subclasses which are themselves classifiable, e.g., those prescribed by the orbifold theories. Determining these theories amounts to the classification of the finite discrete subgroups of $SU(n)$.

We recall from Corollary 12.4.1 that $\mathcal{N} = 2$ is given by the affine and infinite Coxeter-Dynkin graphs of which the orbifold theories provide the $\widehat{ADE}$ cases. What remarks could one make for $\mathcal{N} = 0, 1$, i.e., $SU(3, 4)$ McKay quivers (Cf. Chap. 9 and 11)? Let us first see $\mathcal{N} = 2$ from the graph-theoretic perspective, which will induce a relationship between additivity (Theorem 12.3.19) and Gabriel-Nazarova (Theorems 12.3.20 and extensions). The crucial step in Tit's proof of Gabriel's Theorem is the introduction of the quadratic form on a graph [87, 188]:

DEFINITION 12.4.21 *For a labeled quiver $Q = (Q_0, Q_1)$, one defines the (symmetric bilinear)* **quadratic form** *$B(x)$ on the set x of the labels as follows:*

$$B(x) := \sum_{i \in Q_0} x_i^2 - \sum_{\alpha \in Q_1} x_{s(\alpha)} x_{e(\alpha)}.$$

The subsequent work was then to show that finitude of representation is equivalent to the positive-definity of $B(x)$, and in fact, as in Nazarova's extension, that tameness is equivalent to positive-semi-definity. In other words, finite or tame representation type can be translated, in this context, to a Diophantine inequality which dictates the nodes and connectivity of the quiver (incidentally the very same Inequality which dictates the shapes of the Coxeter-Dynkin Diagrams or the vertices and faces of the Platonic solids in $\mathbb{R}^3$):

$$B(x) \geq 0 \Leftrightarrow \widehat{ADE}, ADE \qquad B(x) > 0 \Leftrightarrow ADE.$$

Now we note that $B(x)$ can be written as $\frac{1}{2}x^T \cdot c \cdot x$ where $(c)_{ij}$ is *de facto* the Cartan Matrix for graphs as defined in §12.3.3. The classification problem thus, because $c := 2I - a$, becomes that of classifying graphs whose adjacency matrix a has maximal eigenvalue 2, or what McKay calls C_2-graphs in [32]. This issue was addressed in [189] and indeed the $\widehat{ADE}$ graphs emerge. Furthermore the additivity condition $\sum\limits_j c_{ij}x_j \geq 0 \; \forall \; i$ clearly implies the constraint $\sum\limits_{ij} c_{ij}x_i x_j \geq 0$ (since all labels are positive) and thereby the like on the quadratic form. Hence we see how to arrive at the vital step in Gabriel-Nazarova through graph subadditivity.

The above discussions relied upon the specialty of the number 2. Indeed one could translate between the graph quadratic form $B(x)$ and the graph Cartan matrix precisely because the latter is defined by $2I - a$. From a physical perspective this is precisely the discriminant function for $\mathcal{N} = 2$ orbifold SYM (i.e. $d = 2$) as discussed at the end of §12.2.1. This is why $\widehat{ADE}$ arises in all these contexts. We are naturally led to question ourselves, what about general[11] d? This compels us to consider a **generalised Cartan matrix** for graphs (Cf. Definition in §12.3.3), given by $c_{ij} := d\delta_{ij} - a_{ij}$, our discriminant function of §12.2.1. Indeed such a matrix was considered in [179] for general McKay quivers. As a side remarks, due to such an extension, Theorem 12.3.19 must likewise be adjusted to accommodate more graphs; a recent paper [187] shows an example, the so-dubbed semi-Affine Dynkin Diagrams, where a new class of labeled graphs with additivity with respect to the extended c_{ij} emerge.

Returning to the generalised Cartan matrix, in [179], the McKay matrices a_{ij} were obtained, for an arbitrary finite group G, by tensoring a faithful d-dimensional representation with the set of irreps: $r_d \otimes r_i = \oplus_j a_{ij} r_j$. What was noticed was that the scalar product defined with respect to the matrix $d\delta_{ij} - a_{ij}$ (precisely our generalised Cartan) was positive semi-definite in the vector space $V = \{x_i\}$ of labels. In other words, $\sum\limits_{ij} c_{ij}x_i x_j \geq 0$. We briefly transcribe his proof in Appendix 22.5. What this means for us is that is the following

COROLLARY 12.4.4 *String orbifold theories can not produce a completely IR free*

[11] In the arena of orbifold SYM, $d = 1, 2, 3$, but in a broader settings, as in generalisation of McKay's Correspondence, d could be any natural number.

(i.e., with respect to all semisimple components of the gauge group) QFT (i.e., Type (1), $\beta > 0$).

To see this suppose there existed such a theory. Then $\beta > 0$, implying for our discriminant function that $\sum\limits_j c_{ij}x_j < 0 \; \forall\, i$ for some finite group. This would then imply, since all labels are positive, that $\sum\limits_{ij} c_{ij}x_i x_j < 0$, violating the positive semidefinity condition that it should always be nonnegative for any finite group according to [179]. Therefore by *reductio ad absurdum*, we conclude Corollary 12.4.4.

On a more general setting, if we were to consider using the generalised Cartan matrix $d\delta_{ij} - a_{ij}$ to define a generalised subadditive function (as opposed to merely $d = 2$), could we perhaps have an extended classification scheme? To our knowledge this is so far an unsolved problem for indeed take the subset of these graphs with all labels being 1 and $dn_i = \sum\limits_j a_{ij}n_j$, these are known as d-regular graphs (the only 2-regular one is the $\widehat{A}$-series) and these are already unclassified for $d > 2$. We await input from mathematicians on this point.

12.5 Concluding Remarks and Prospects

The approach of this writing has been bilateral. On the one hand, we have briefly reviewed the three contemporary techniques of obtaining four dimensional gauge theories from string theory, namely Hanany-Witten, D-brane probes and geometrical engineering. In particular, we focus on what finitude signifies for these theories and how interests in quiver diagrams arises. Subsequently, we approach from the mathematical direction and have taken a promenade in the field of axiomatic representation theory of algebras associated to quivers. The common ground rests upon the language of graph theory, some results from which we have used to address certain issues in string theory.

From the expression of the one-loop β-function, we have defined a discriminant function $f := d\delta_{ij} - a^d_{ij}$ for the quiver with adjacency matrix a_{ij} which encodes the bi-fundamental matter content of the gauge theory. The nullity (resp. negativity/positivity) of this function gives a necessary condition for the finitude (resp. IR freedom/asymptotic freedom) of the associated gauge theory. We recognise this function to be precisely the generalised Cartan matrix of a (not necessarily finite) graph and the nullity (resp. negativity) thereof, the additivity (resp. strict subadditivity) of the graph. In the case of $d = 2$, such graphs are completely classified: infinite Dynkin or Euclidean if $f = 0$ and finite Dynkin or A_∞ if $f < 0$. In physical terms, this means that these are the *only* $\mathcal{N} = 2$ theories with bi-fundamental matter (Corollary 12.4.1 and Observation 2). This slightly generalises the results of [48] by the inclusion of infinite graphs, i.e., theories with infinite product gauge groups. From the mathematics alone, also included are the non-simply-laced diagrams, however we still await progress in the physics to clarify how these gauge theories may be fabricated.

For $d > 2$, the mathematical problem of their classification is so far unsolved. A subclass of these, namely the orbifold theories coming from discrete subgroups of $SU(n)$ have been addressed upto $n = 4$ [69, 171, 292, 294]. A general remark we can make about these theories is that, due to a theorem of Steinberg, D-brane probes on orbifolds can never produce a completely IR free QFT (Corollary 12.4.4).

From a more axiomatic stand, we have also investigated possible finite quivers that may arise. In particular we have reviewed the correspondence between a quiver and its associated path algebra. Using the Trichotomy theorem of representation theory, that all finite dimensional algebras over an algebraically closed field are of either finite, tame or wild type, we have seen that all quivers are respectively either ADE, $\widehat{ADE}$ or unclassifiable. In physical terms, this means that asymptotically free and finite $\mathcal{N} = 2$ SYM in four dimensions respectively exhaust the *only* quiver theories of respectively finite and tame type (Corollary 12.4.2). What these particular path algebras mean in a physical context however, is yet to be ascertained. For the last type, we have drawn a melancholy note that all other theories, and in particular, $\mathcal{N} < 2$ in four dimensions, are in general Turing unclassifiable (Corollary 12.4.3).

Much work remains to be accomplished. It is the main purpose of this note, through the eyes of a neophyte, to inform readers in each of two hitherto disparate fields of gauge theories and axiomatic representations, of certain results from the other. It is hoped that future activity may be prompted.

Chapter 13

Orbifolds IV: Finite Groups and WZW Modular Invariants, Case Studies for $SU(2)$ and $SU(3)$

Synopsis

Inspired by Chapters 9 and 10 which contained some attempts to formulate various correspondences between the classification of affine $SU(k)$ WZW modular-invariant partition functions and that of discrete finite subgroups of $SU(k)$, we present a small and perhaps interesting observation in this light.

In particular we show how the groups generated by the permutation of the terms in the exceptional $\widehat{SU(2)}$-WZW invariants encode the corresponding exceptional $SU(2)$ subgroups. We also address a weaker analogue for $SU(3)$ [300].

13.1 Introduction

The ubiquitous ADE meta-pattern of mathematics makes her mysterious emergence in the classification of the modular invariant partition functions in Wess-Zumino-Witten (WZW) models of rational conformal field theory (RCFT). Though this fact is by now common knowledge, little is known about why *a fortiori* these invariants should fall under such classification schemes [93]. Ever since the original work in the completion of the classification for $\widehat{su(2)}$ WZW invariants by Cappelli-Itzykson-Zuber [190, 191] as well as the subsequent case for $\widehat{su(3)}$ by Gannon [192, 193], many efforts have been made to attempt to clarify the reasons behind the said emergence. These include perspectives from lattice integrable systems where the invariants are related to finite groups [104], and from generalised root systems and N-colourability of graphs [195, 196]. Furthermore, there has been a recent revival of interest in the matter as viewed from string theory where sigma models and orbifold constructions are suggested to provide a link [292, 293, 154].

Let us first briefly review the situation at hand (much shall follow the conventions of [93] where a thorough treatment may be found). The $\widehat{g}_k$-WZW model (i.e., associated to an affine Lie algebra g at level k) is a non-linear sigma model on the group manifold G corresponding to the algebra g. Its action is

$$S^{\mathrm{WZW}} = \frac{k}{16\pi}\int_G \frac{d^2x}{X_{\mathrm{rep}}}\mathrm{Tr}(\partial^\mu g^{-1}\partial_\mu g) + k\Gamma$$

where $k \in \mathbb{Z}$ is called the level, $g(x)$, a matrix bosonic field with target space[1] G and X_{rep} the Dynkin index for the representation of g. The first term is our familiar pull back in sigma models while the second

$$\Gamma = \frac{-i}{24\pi}\int_B \frac{d^3y}{X_{\mathrm{rep}}}\epsilon_{\alpha\beta\gamma}\mathrm{Tr}(\tilde{g}^{-1}\partial^\alpha\tilde{g}\tilde{g}^{-1}\partial^\beta\tilde{g}\tilde{g}^{-1}\partial^\gamma\tilde{g})$$

is the WZW term added to ensure conformal symmetry. B is a manifold such that $\partial B = G$ and $\tilde{g}$ is the subsequent embedding of g into B. The conserved cur-

[1]We are really integrating over the pull-back to the world sheet.

rents $J(z) := \sum_a J^a t^a$ and $J^a := \sum_{n \in \mathbb{Z}} J_n^a z^{-n-1}$ (together with an independent anti-holomorphic copy) form a **current algebra** which is precisely the level k affine algebra $\widehat{g}$:

$$\left[J_n^a, J_m^b\right] = i \sum_c f_{abc} J_{n+m}^c + kn\delta_{ab}\delta_{n+m,0}.$$

The energy momentum tensor $T(z) = \frac{1}{d+k} \sum_a J^a J^a$ with d the dual Coxeter number of g furnishes a Virasoro algebra with central charge

$$c(\widehat{g}_k) = \frac{k\mathrm{dim}g}{k+d}.$$

Moreover, the primary fields are in 1-1 correspondence with the heighest weights $\widehat{\lambda} \in P_+^k$ of $\widehat{g}$, which, being of a finite number, constrains the number of primaries to be finite, thereby making WZW a RCFT. The **fusion algebra** of the primaries ϕ for this RCFT is consequently given by $\phi_i \times \phi_j = \sum_{\phi_k^*} \mathcal{N}_{\phi_i \phi_j}^{\phi_k^*} \phi_k^*$, or in the integrable representation language of the affine algebra:

$$\widehat{\lambda} \otimes \widehat{\mu} = \bigoplus_{\widehat{\nu} \in P_+^k} \mathcal{N}_{\widehat{\lambda}\widehat{\mu}}^{\widehat{\nu}} \, \widehat{\nu}.$$

The Hilbert Space of states decomposes into holomorphic and anti-holomorphic parts as $\mathcal{H} = \bigoplus_{\widehat{\lambda},\widehat{\xi} \in P_+^{(k)}} \mathcal{M}_{\widehat{\lambda},\widehat{\xi}} H_{\widehat{\lambda}} \otimes H_{\widehat{\xi}}$ with the **mass matrix** $\mathcal{M}_{\widehat{\lambda},\widehat{\xi}}$ counting the multiplicity of the H-modules in the decomposition. Subsequently, the partition function over the torus, $Z(q) := \mathrm{Tr}_{\mathcal{H}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}$ with $q := e^{2\pi i \tau}$ reduces to

$$Z(\tau) = \sum_{\widehat{\lambda},\widehat{\xi} \in P_+^k} \chi_{\widehat{\lambda}}(\tau) \mathcal{M}_{\widehat{\lambda},\widehat{\xi}} \bar{\chi}_{\widehat{\xi}}(\bar{\tau}) \tag{13.1.1}$$

with χ being the affine characters of $\widehat{g}_k$. Being a partition function on the torus, (13.1.1) must obey the $SL(2;\mathbb{Z})$ symmetry of T^2, i.e., it must be invariant under the **modular group** generated by $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$. Recalling the modular transformation properties of the affine characters, viz.,

$$\begin{aligned} T : \chi_{\widehat{\lambda}} &\rightarrow \sum_{\widehat{\mu} \in P_+^k} \mathcal{T}_{\widehat{\lambda}\widehat{\mu}} \chi_{\widehat{\mu}} \\ S : \chi_{\widehat{\lambda}} &\rightarrow \sum_{\widehat{\mu} \in P_+^k} \mathcal{S}_{\widehat{\lambda}\widehat{\mu}} \chi_{\widehat{\mu}} \end{aligned}$$

with

$$\begin{aligned} \mathcal{T}_{\widehat{\lambda}\widehat{\mu}} &= \delta_{\widehat{\lambda}\widehat{\mu}} e^{\pi i \left(\frac{|\widehat{\lambda}+\widehat{\rho}|^2}{k+d} - \frac{|\widehat{\rho}|^2}{d} \right)} \\ \mathcal{S}_{\widehat{\lambda}\widehat{\mu}} &= K \sum_{w \in W} \epsilon(w) e^{-\frac{2\pi i}{k+d}(w(\lambda+\rho), \mu+\rho)} \end{aligned}$$

where $\widehat{\rho}$ is the sum of the fundamental weights, W, the Weyl group and K, some proportionality constant. Modular invariance of (13.1.1) then implies $[\mathcal{M}, \mathcal{S}] = [\mathcal{M}, \mathcal{T}] = 0$. The problem of classfication of the physical modular invariants of $\widehat{g}_k$-WZW then amounts to solving for *all nonnegative integer matrices* $\mathcal{M}$ *such that* $\mathcal{M}_{00} = 1$ (so as to guarantee uniqueness of vacuum) and *satisfying these commutant relations.*

The fusion coefficients $\mathcal{N}$ can be, as it is with modular tensor categories (q.v. e.g. [293]), related to the matrix $\mathcal{S}$ by the celebrated Verlinde Formula:

$$\mathcal{N}_{rs}^t = \sum_m \frac{\mathcal{S}_{rm}\mathcal{S}_{sm}\mathcal{S}_{mt}^{-1}}{\mathcal{S}_{0m}}. \tag{13.1.2}$$

Furthermore, in light of the famous McKay Correspondence (Cf. e.g. [292, 293] for discussions of the said correspondence in this context), to establish correlations between modular invariants and graph theory, one can chose a fundamental representation f and regard $(N)_{st} := \mathcal{N}_{fs}^t$ as an adjacency matrix of a finite graph. Conversely out of the adjacency matrix $(G)_{st}$ for some finite graph, one can extract a set of matrices $\{(N)_{st}\}_i$ such that $N_0 = \mathbb{1}$ and $N_f = G$. We diagonalise G as $\mathcal{S}\Delta\mathcal{S}^{-1}$ and define, as inspired by (13.1.2), the set of matrices $N_r := \{(N)_{st}\}_r = \sum_m \frac{\mathcal{S}_{rm}\mathcal{S}_{sm}\mathcal{S}_{mt}^{-1}}{\mathcal{S}_{0m}}$, which clearly satisfy the constriants on $N_{0,f}$. This set of matrices $\{N_i\}$, each associated to a vertex in the judiciously chosen graph, give rise to a **graph algebra** and appropriate subalgebras thereof, by virtue of matrix multiplication, constitute a representation for the fusion algebra, i.e., $N_i \cdot N_j = \sum_k \mathcal{N}_{ij}^k N_k$. In a more axiomatic language, the

Verlinde equation (13.1.2) is essentially the inversion of the McKay composition

$$R_r \otimes R_s = \bigoplus_t \mathcal{N}_{rs}^t R_t \tag{13.1.3}$$

of objects $\{R_i\}$ in a (modular) tensor category. The $\mathcal{S}$ matrices are then the characters of these objects and hence the matrix of eigenvectors of $G = \mathcal{N}_{rs}^t$ once fixing some r by definition (13.1.3). The graph algebra is essentially the set of these matrices $\mathcal{N}_{rs}^t$ as we extrapolate r from 0 (giving $\mathbb{1}$) to some fixed value giving the graph adjacency matrix G.

Thus concludes our brief review on the current affair of things. Let us now proceed to present our small observation.

Nomenclature

Throughout the chapter, unless otherwise stated, we shall adhere to the folloing conventions: G_n is group G of order n. $\langle x_i \rangle$ is the group generated by the (matrix) elements $\{x_i\}$. k is the level of the WZW modular invariant partition function Z. χ is the affine character of the algebra $\widehat{g}$. $\mathcal{S}, \mathcal{T}$ are the generators of the modular group $SL(2;\mathbb{Z})$ whereas S, T will be these matrices in a new basis, to be used to generate a finite group. $E_{6,7,8}$ are the ordinary tetrahedral, octahedral and icosahedral groups while $\widehat{E_{6,7,8}}$ are their binary counterparts. Calligraphic font $(\mathcal{A}, \mathcal{D}, \mathcal{E})$ shall be reserved for the names of the modular invariants.

13.2 $\widehat{su(2)}$-WZW

The modular invariants of $\widehat{su(2)}$-WZW were originally classified in the celebrated works of [190, 191]. The only solutions of the abovementioned conditions for $k, \mathcal{S}, \mathcal{T}$ and $\mathcal{M}$ give rise to the following:

$$\mathcal{S}_{ab} = \sqrt{\frac{2}{k+2}} \sin(\pi \frac{(a+1)(b+1)}{k+2}), \qquad \mathcal{T}_{ab} = \exp[\pi i (\frac{(a+1)^2}{2(k+2)} - \frac{1}{4})] \, \delta_{a,b} \qquad a, b = 0, ..., k \tag{13.2.4}$$

with the partition functions

$$
\begin{array}{lll}
k & \mathcal{A}_{k+1} & Z = \sum\limits_{n=0}^{k} |\chi_n|^2 \\
k = 4m & \mathcal{D}_{2m+2} & Z = \sum\limits_{n=0,\mathrm{even}}^{2m-2} |\chi_n + \chi_{k-n}|^2 + 2|\chi_{2m}|^2 \\
k = 4m-2 & \mathcal{D}_{2m+1} & Z = |\chi_{\frac{k}{2}}|^2 + \sum\limits_{n=0,\mathrm{even}}^{4m-2} |\chi_n|^2 + \sum\limits_{n=1,\mathrm{odd}}^{2m-1} (\chi_n \bar{\chi}_{k-n} + c.c.) \\
k = 10 & \mathcal{E}_6 & Z = |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2 \\
k = 16 & \mathcal{E}_7 & Z = |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + (\bar{\chi}_8(\chi_2 + \chi_{14}) + c.c.) \\
k = 28 & \mathcal{E}_8 & Z = |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2
\end{array}
\tag{13.2.5}
$$

We know of course that the simply-laced simple Lie algebras, as well as the discrete subgroups of $SU(2)$ fall precisely under such a classification. The now standard method is to associate the modular invariants to subalgebras of the graph algebras constructed out of the respective ADE-Dynkin Diagram. This is done in the sense that the adjacency matrices of these diagrams[2] are to define N_1 and subsets of N_i determine the fusion rules. The correspondence is rather weak, for in addition to the necessity of the truncation to subalgebras, only A_k, D_{2k} and $E_{6,8}$ have been thus related to the graphs while D_{2k+1} and E_7 give rise to negative entries in $\mathcal{N}_{ij}^k$. However as an encoding process, the above correspondences has been very efficient, especially in generalising to WZW of other algebras.

The first attempt to explain the ADE scheme in the $\widehat{su(2)}$ modular invariants was certainly not in the sophistry of the above context. It was in fact done in the original work of [191], where the authors sought to relate their invariants to the discrete subgroups of $SO(3) \cong SU(2)/\mathbb{Z}_2$. It is under the inspiration of this idea, though initially abandoned (*cit. ibid.*), that the current writing has its birth. We do not promise to find a stronger correspondence, yet we shall raise some observations of interest.

The basic idea is simple. To ourselves we pose the obvious question: what, alge-

[2]These are the well-known symmetric matrices of eigenvalues ≤ 2, or equivalently, the McKay matrices for $SU(2)$; for a discussion on this point q.v. e.g. [297].

braically does it mean for our partition functions (13.2.5) to be modular invariant? It signifies that the action by $\mathcal{S}$ and $\mathcal{T}$ thereupon must permute the terms thereof in such a way so as not to, by virtue of the transformation properties of the characters (typically theta-functions), introduce extraneous terms. In the end of the monumental work [191], the authors, as a diversion, used complicated identities of theta and eta functions to rewrite the $\mathcal{E}_{6,7,8}$ cases of (13.2.5) into sum of terms on whose powers certain combinations of $\mathcal{S}$ and $\mathcal{T}$ act. These combinations were then used to generate finite groups which in the case of $\mathcal{E}_6$, did give the ordinary tetrahedral group E_6 and $\mathcal{E}_8$, the ordinary icosahedral group E_8, which are indeed the finite groups associated to these Lie algebras, a fact which dates back to F. Klein. As a postlude, [191] then speculated upon the reasons for this correspondence between modular invariants and these finite groups, as being attributable to the representation of the modular groups over finite fields, since afterall $E_6 \cong PSL(2;\mathbb{Z}_3)$ and $E_8 \cong PSL(2;\mathbb{Z}_4) \cong PSL(2;\mathbb{Z}_5)$.

We shall not take recourse to the complexity of manipulation of theta functions and shall adhere to a pure group theoretic perspective. We translate the aforementioned concept of the permutation of terms into a vector space language. First we interpret the characters appearing in (13.2.5) as basis upon which $\mathcal{S}$ and $\mathcal{T}$ act. For the k-th level they are defined as the canonical bases for $\mathbb{C}^{k+1}$:

$$\chi_0 := (1,0,...,0); \quad ... \quad \chi_i := (\mathbb{1})_{i+1}; \quad ... \quad \chi_k := (0,0,...,1).$$

Now $\mathcal{T}$ being diagonal clearly maps these vectors to multiples of themselves (which after squaring the modulus remain uneffected); the interesting permutations are performed by $\mathcal{S}$.

13.2.1 The E_6 Invariant

Let us first turn to the illustrative example of $\mathcal{E}_6$. From Z in (13.2.5), we see that we are clearly interested in the vectors $v_1 := \chi_0 + \chi_6 = (1,0,0,0,0,0,1,0,0,0,0)$, $v_2 := \chi_4 + \chi_{10} = (0,0,0,0,1,0,0,0,0,0,1)$ and $v_3 := \chi_3 + \chi_7 = (0,0,0,1,0,0,0,1,0,0,0)$. Hence (13.2.4) gives $\mathcal{T} : v_1 \to e^{\frac{-5\pi i}{24}} v_1$, $\mathcal{T} : v_2 \to e^{\frac{19\pi i}{24}} v_2$ and $\mathcal{T} : v_3 \to e^{\frac{5\pi i}{12}} v_3$.

Or, in other words in the subspace spanned by $v_{1,2,3}$, $\mathcal{T}$ acts as the matrix $T :=$ Diag$(e^{\frac{-5\pi i}{24}}, e^{\frac{19\pi i}{24}}, e^{\frac{5\pi i}{12}})$. Likewise, $\mathcal{S}$ becomes a 3 by 3 matrix; we present them below:

$$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad T = \begin{pmatrix} e^{-\frac{5\pi i}{24}} & 0 & 0 \\ 0 & e^{\frac{19\pi i}{24}} & 0, \\ 0 & 0 & e^{\frac{5\pi i}{12}} \end{pmatrix} \tag{13.2.6}$$

Indeed no extraneous vectors are involved, i.e., of the 11 vectors χ_i and all combinations of sums thereof, only the combinations $v_{1,2,3}$ appear after actions by $\mathcal{S}$ and $\mathcal{T}$. This closure of course is what is needed for modular invariance. What is worth of note, is that we have collapsed an 11-dimensional representation of the modular group acting on $\{\chi_i\}$, to a (non-faithful) 3-dimensional representation which corresponds the subspace of interest (of the initial $\mathbb{C}^{11}$) by virtue of the appearance of the terms in the associated modular invariant. Moreover the new matrices S and T, being of finite order (i.e., $\exists m, n \in \mathbb{Z}_+$ s.t. $S^m = T^n = \mathbb{1}$), actually generate a *finite group*. It is this finite group that we shall compare to the ADE-subgroups of $SU(2)$.

The issue of the finiteness of the initial group generated by $\mathcal{S}$ and $\mathcal{T}$ was addressed in a recent work by Coste and Gannon [197]. Specifically, the group

$$P := \{S, T | T^N = S^2 = (ST)^3 = \mathbb{1}\}, \tag{13.2.7}$$

generically known as the *polyhedral (2,3,N) group*, is infinite for $N > 5$. On the other hand, for $N = 2, 3, 4, 5$, $G \cong \Gamma/\Gamma(N) := SL(2; \mathbb{Z}/N\mathbb{Z})$, which, interestingly enough, for these small values are, the symmetric-3, the tetrahedral, the octahedral and icosahedral groups respectively.

We see of course that our matrices in (13.2.6) satisfy the relations of (13.2.7) with $N = 48$ (along with additional relations of course) and hence generates a subgroup of P. Indeed, P is the modular group in a field of finite characteristic N and since we are dealing with nonfaithful representations of the modular group, the groups generated by S, T, as we shall later see, in the cases of other modular invariants are all finite subgroups of P.

In our present case, $G = \langle S, T \rangle$ is of order 1152. Though G itself may seem unenlightening, upon closer inspection we find that it has 12 normal subgroups $H \triangleleft G$ and *only one* of which is of order 48. In fact this H_{48} is $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3$. The observation is that the quotient group formed between G and H is precisely the binary tetrahedral group $\widehat{E_6}$, i.e.,

$$G_{1152}/H_{48} \cong \widehat{E_6}. \tag{13.2.8}$$

We emphasize again the *uniqueness* of this procedure: as will be with later examples, given $G(\mathcal{E}_6)$, there exists a unique normal subgroup which can be quotiented to give $\widehat{E_6}$, and moreover there does not exist a normal subgroup which could be used to generate the other exceptional groups, viz., $\widehat{E_{7,8}}$. We shall later see that such a 1-1 correpondence between the exceptional modular invariants and the exceptional discrete groups persists.

This is a pleasant surprise; it dictates that the symmetry group generated by the permutation of the terms in the $\mathcal{E}_6$ modular invariant partition function of $\widetilde{SU(2)}$-WZW, upon appropriate identification, is exactly the symmetry group assocaited to the $\widehat{E_6}$ discrete subgroup of $SU(2)$. Such a correspondence may *a priori* seem rather unexpected.

13.2.2 Other Invariants

It is natural to ask whether similar circumstances arise for the remaining invariants. Let us move first to the the case of E_8. By procedures completely analogous to (13.2.6) as applied to the partition function in (13.2.5), we see that the basis is composed of $v_1 = \chi_0+\chi_{10}+\chi_{18}+\chi_{28} = \{1,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,1\}$ and $v_2 = \chi_6+\chi_{12}+\chi_{16}+\chi_{22} = \{0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,0\}$, under which S and T assume the forms as summarised in Table 13.2.11.

This time $G = \langle S, T \rangle$ is of order 720, with one *unique* normal subgroup of order 6 (in fact $\mathbb{Z}_6$). Moreover we find that

$$G_{720}/H_6 \cong \widehat{E_8}, \tag{13.2.9}$$

in complete analogy with (13.2.8). Thus once again, the symmetry due to the permutation of the terms inherently encode the associated discrete $SU(2)$ subgroup.

What about the remaining exceptional invariant, E_7? The basis as well as the matrix forms of S, T thereunder are again presented in Table 13.2.11. The group generated thereby is of order 324, with 2 non-trivial normal subgroups of orders 27 and 108. Unfortunately, no direct quotienting could possibly give the binary octahedral group here. However G/H_{27} gives a group of order 12 which is in fact the *ordinary* octahedral group $E_7 = A_4$, which is in turn isomorphic to $\widehat{E_7}/\mathbb{Z}_2$. Therefore for our present case the situation is a little more involved:

$$G_{324}/H_{27} \cong \widehat{E_7}/\mathbb{Z}_2 \cong E_7. \tag{13.2.10}$$

We recall [93] that a graph algebra (13.1.2) based on the Dynkin diaram of E_7 has actually not been succesully constructed for the E_7 modular invariant. Could we speculate that the slight complication of (13.2.10) in comparison with (13.2.8) and (13.2.9) be related to this failure?

We shall pause here with the exceptional series as for the infinite series the quotient of the polyhedral $(2, 3, N)$ will never give any abelian group other than $\mathbb{Z}_{1,2,3,4,6}$ or any dihedral group other than $D_{1,3}$ [198]. More complicated procedures are called for which are yet to be ascertained [92], though we remark here briefly that for the A_{k+1} series, since Z is what is known as the *diagonal invariant*, i.e., it includes all possible χ_n-bases, we need not perform any basis change and whence S, T are simply the original $\mathcal{S}, \mathcal{T}$ and there is an obvious relationship that $G := \langle T^8 \rangle \cong \mathbb{Z}_{k+2} := A_{k+1}$.

Incidentally, we can ask ourselves whether any such correspondences could possibly hold for the *ordinary* exceptional groups. From (13.2.10) we see that $G(\mathcal{E}_7)/H_{27}$ does indeed correspond to the ordinary octahedral group. Upon further investigation, we find that $G(\mathcal{E}_6)$ could not be quotiented to give the ordinary E_6 while $G(\mathcal{E}_8)$ does have a normal subgroup of order 12 which could be quotiented to give the ordinary

E_8. Without much further ado for now, let us summarise these results:

	$G := \langle S, T \rangle$	Normal Subgroups	Relations	
$\mathcal{E}_6$	G_{1152}	$H_{3,4,12,16,48,64,192,192',384,576}$	$G_{1152}/H_{48} \cong \widehat{E_6}$	–
$\mathcal{E}_7$	G_{324}	$H_{27,108}$	$G_{324}/H_{27} \cong \widehat{E_7}/\mathbb{Z}_2$	$G_{324}/H_{27} \cong E_7$
$\mathcal{E}_8$	G_{720}	$H_{2,3,4,6,12,120,240,360}$	$G_{720}/H_6 \cong \widehat{E_8}$	$G_{720}/H_{12} \cong E_8$

Table of $SU(2)$ Exceptional Invariants

	Matrix Generators	Basis
E_6	$S = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} T = \begin{pmatrix} e^{\frac{-5\pi i}{24}} & 0 & 0 \\ 0 & e^{\frac{19\pi i}{24}} & 0, \\ 0 & 0 & e^{\frac{5\pi i}{12}} \end{pmatrix}$	$v_1 = \chi_0 + \chi_6$ $v_2 = \chi_4 + \chi_{10}$ $v_3 = \chi_3 + \chi_7$
E_7	$S = \frac{1}{3} \begin{pmatrix} \sin(\frac{\pi}{18}) + \sin(\frac{17\,\pi}{18}) & \sin(\frac{5\,\pi}{18}) + \sin(\frac{85\,\pi}{18}) & \sin(\frac{7\,\pi}{18}) + \sin(\frac{119\,\pi}{18}) & 2 & 1 \\ \sin(\frac{5\,\pi}{18}) + \sin(\frac{13\,\pi}{18}) & \sin(\frac{25\,\pi}{18}) + \sin(\frac{65\,\pi}{18}) & \sin(\frac{35\,\pi}{18}) + \sin(\frac{91\,\pi}{18}) & 2 & 1 \\ \sin(\frac{7\,\pi}{18}) + \sin(\frac{11\,\pi}{18}) & \sin(\frac{35\,\pi}{18}) + \sin(\frac{55\,\pi}{18}) & \sin(\frac{49\,\pi}{18}) + \sin(\frac{77\,\pi}{18}) & -2 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -2 & 2 \end{pmatrix}$ $T = \begin{pmatrix} e^{\frac{-2\,i}{9}\,\pi} & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{4\,i}{9}\,\pi} & 0 & 0 & 0 \\ 0 & 0 & e^{\frac{-8\,i}{9}\,\pi} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$v_1 = \chi_0 + \chi_{16}$ $v_2 = \chi_4 + \chi_{12}$ $v_3 = \chi_6 + \chi_{10}$ $v_4 = \chi_8$ $v_5 = \chi_2 + \chi_{14}$
E_8	$S = \frac{1}{\sqrt{15}} \begin{pmatrix} \sin(\frac{\pi}{30}) + \sin(\frac{11\,\pi}{30}) + \sin(\frac{19\,\pi}{30}) + \sin(\frac{29\,\pi}{30}) & \sin(\frac{7\,\pi}{30}) + \sin(\frac{77\,\pi}{30}) + \sin(\frac{133\,\pi}{30}) + \sin(\frac{203\,\pi}{30}) \\ \sin(\frac{7\,\pi}{30}) + \sin(\frac{13\,\pi}{30}) + \sin(\frac{17\,\pi}{30}) + \sin(\frac{23\,\pi}{30}) & \sin(\frac{49\,\pi}{30}) + \sin(\frac{91\,\pi}{30}) + \sin(\frac{119\,\pi}{30}) + \sin(\frac{161\,\pi}{30}) \end{pmatrix}$ $T = \begin{pmatrix} e^{\frac{-7\,i}{30}\,\pi} & 0 \\ 0 & e^{\frac{17\,i}{30}\,\pi} \end{pmatrix}$	$v_1 = \chi_0 + \chi_{10} + \chi_{18} + \chi_{28}$ $v_2 = \chi_6 + \chi_{12} + \chi_{16} + \chi_{22}$

(13.2.11)

13.3 Prospects: $\widehat{su(3)}$-WZW and Beyond?

There has been some recent activity [104, 292, 293, 154] in attempting to explain the patterns emerging in the modular invariants beyond $\widehat{su(2)}$. Whether from the perspective of integrable systems, string orbifolds or non-linear sigma models, proposals of the invariants being related to subgroups of $SU(n)$ have been made. It is natural therefore for us to inquire whether the correspondences from the previous subsection between $\widehat{su(n)}$-WZW and the discrete subgroups of $SU(n)$ for $n=2$ extend to $n=3$.

We recall from [192, 193] that the modular invariant partition functions for $\widehat{su(3)}$-WZW have been classified to be the following:

$$\begin{array}{rcl}
\mathcal{A}_k & := & \sum\limits_{\lambda \in P^k} |\chi_\lambda^k|^2, \qquad \forall k \geq 1; \\
\mathcal{D}_k & := & \sum\limits_{(m,n) \in P^k} \chi_{m,n}^k \chi_{\omega^{k(m-n)}(m,n)}^{k*}, \quad \text{for } k \not\equiv 0 \bmod 3 \text{ and } k \geq 4; \\
\mathcal{D}_k & := & \frac{1}{3} \sum\limits_{\substack{(m,n)\in P^k \\ m \equiv n \bmod 3}} |\chi_{m,n}^k + \chi_{\omega(m,n)}^k + \chi_{\omega^2(m,n)}^k|^2; \\
\mathcal{E}_5 & := & |\chi_{1,1}^5 + \chi_{3,3}^5|^2 + |\chi_{1,3}^5 + \chi_{4,3}^5|^2 + |\chi_{3,1}^5 + \chi_{3,4}^5|^2 + \\
& & |\chi_{3,2}^5 + \chi_{1,6}^5|^2 + |\chi_{4,1}^5 + \chi_{1,4}^5|^2 + |\chi_{2,3}^5 + \chi_{6,1}^5|^2; \\
\mathcal{E}_9^{(1)} & := & |\chi_{1,1}^9 + \chi_{1,10}^9 + \chi_{10,1}^9 + \chi_{5,5}^9 + \chi_{5,2}^9 + \chi_{2,5}^9|^2 + 2|\chi_{3,3}^9 + \chi_{3,6}^9 + \chi_{6,3}^9|^2; \\
\mathcal{E}_9^{(2)} & := & |\chi_{1,1}^9 + \chi_{10,1}^9 + \chi_{1,10}^9|^2 + |\chi_{3,3}^9 + \chi_{3,6}^9 + \chi_{6,3}^9|^2 + 2|\chi_{4,4}^9|^2 \\
& & + |\chi_{1,4}^9 + \chi_{7,1}^9 + \chi_{4,7}^9|^2 + |\chi_{4,1}^9 + \chi_{1,7}^9 + \chi_{7,4}^9|^2 + |\chi_{5,5}^9 + \chi_{5,2}^9 + \chi_{2,5}^9|^2 \\
& & + (\chi_{2,2}^9 + \chi_{2,8}^9 + \chi_{8,2}^9)\chi_{4,4}^{9*} + \chi_{4,4}^9(\chi_{2,2}^{9*} + \chi_{2,8}^{9*} + \chi_{8,2}^{9*}); \\
\mathcal{E}_{21} & := & |\chi_{1,1}^{21} + \chi_{5,5}^{21} + \chi_{7,7}^{21} + \chi_{11,11}^{21} + \chi_{22,1}^{21} + \chi_{1,22}^{21} + \chi_{14,5}^{21} + \chi_{5,14}^{21} + \chi_{11,2}^{21} + \chi_{2,11}^{21} + \chi_{10,7}^{21} + \chi_{7,10}^{21}|^2 \\
& & + |\chi_{16,7}^{21} + \chi_{7,16}^{21} + \chi_{16,1}^{21} + \chi_{1,16}^{21} + \chi_{11,8}^{21} + \chi_{8,11}^{21} + \chi_{11,5}^{21} + \chi_{5,11}^{21} + \chi_{8,5}^{21} + \chi_{5,8}^{21} + \chi_{7,1}^{21} + \chi_{1,7}^{21}|^2;
\end{array} \tag{13.3.12}$$

where we have labeled the level k explicitly as subscripts. Here the highest weights are labeled by two integers $\lambda = (m,n)$ as in the set

$$P^k := \{\lambda = m\beta_1 + n\beta_2 \,|\, m,n \in \mathbb{Z},\ 0 < m,n,m+n < k+3\}$$

and ω is the operator $\omega : (m,n) \to (k+3-m-n,n)$. The modular matrices are simplified to

$$\begin{array}{rcl}
\mathcal{S}_{\lambda\lambda'} & = & \frac{-i}{\sqrt{3}(k+3)}\{e_k(2mm' + mn' + nm' + 2nn') + e_k(-mm' - 2mn' - nn' + nm') \\
& & + e_k(-mm' + mn' - 2nm' - nn') - e_k(-2mn' - mm' - nn' - 2nm') \\
& & - e_k(2mm' + mn' + nm' - nn') - e_k(-mm' + mn' + nm' + 2nn')\} \\
\mathcal{T}_{\lambda\lambda'} & = & e_k(-m^2 - mn - n^2 + k + 3)\, \delta_{m,m'}\, \delta_{n,n'}
\end{array} \tag{13.3.13}$$

with $e_k(x) := \exp[\frac{-2\pi i x}{3(k+3)}]$.

We imitate the above section and attempt to generate various finite groups by S, T under appropriate transformations from (13.3.13) to new bases. We summarise the results below:

	Basis	$G := \langle S, T \rangle$
$\mathcal{E}_5$	$\{\chi_{1,1}+\chi_{3,3}; \chi_{1,3}+\chi_{4,3}; \chi_{3,1}+\chi_{3,4}; \chi_{3,2}+\chi_{1,6}; \chi_{4,1}+\chi_{1,4}; \chi_{2,3}+\chi_{6,1}\}$	G_{1152}
$\mathcal{E}_9^{(1)}$	$\{\chi_{1,1}+\chi_{1,10}+\chi_{10,1}+\chi_{5,5}+\chi_{5,2}+\chi_{2,5}]; \chi_{3,3}+\chi_{3,6}+\chi_{6,3}\}$	G_{48}
$\mathcal{E}_9^{(2)}$	$\{\chi_{1,1}+\chi_{1,10}+\chi_{10,1}; \chi_{5,5}+\chi_{5,2}+\chi_{2,5};$ $\chi_{3,3}+\chi_{3,6}+\chi_{6,3}; \chi_{4,4}; \chi_{4,1}+\chi_{1,7}+\chi_{7,4};$ $\chi_{1,4}+\chi_{7,1}+\chi_{4,7}; \chi_{2,2}+\chi_{2,8}+\chi_{8,2}\}$	G_{1152}
$\mathcal{E}_{21}$	$\{\chi_{1,1}+\chi_{5,5}+\chi_{7,7}+\chi_{11,11}+\chi_{22,1}+\chi_{1,22}+\chi_{14,5}+\chi_{5,14}+$ $\chi_{11,2}+\chi_{2,11}+\chi_{10,7}+\chi_{7,10};$ $\chi_{16,7}+\chi_{7,16}+\chi_{16,1}+\chi_{1,16}+\chi_{11,8}+\chi_{8,11}+$ $\chi_{11,5}+\chi_{5,11}+\chi_{8,5}+\chi_{5,8}+\chi_{1,7}+\chi_{7,1}\}$	G_{144}

We must confess that unfortunately the direct application of our technique in the previous section has yielded no favourable results, i.e., no quotients groups of G gave any of the exceptional $SU(3)$ subgroups $\Sigma_{36\times3,72\times3,216\times3,360\times3}$ or nontrivial quotients thereof (and *vice versa*), even though the fusion graphs for the former and the McKay quiver for the latter have been pointed out to have certain similarities [104, 195, 292]. These similarities are a little less direct than the Mckay Correspondence for $SU(2)$ and involve truncation of the graphs, the above failure of a naïve correspondence by quotients may be related to this complexity.

Therefore much work yet remains for us. Correspondences for the infinite series in the $SU(2)$ case still needs be formulated whereas a method of attack is still pending for $SU(3)$ (and beyond). It is the main purpose of this short note to inform the reader of an intriguing correspondence between WZW modular invariants and finite groups which may hint at some deeper mechanism yet to be uncovered.

Chapter 14

Orbifolds V: The Brane Box Model for $\mathbb{C}^3/Z_k \times D_{k'}$

Synopsis

In the next four chapters we shall study the T-dual aspects of what had been discussed in the previous chapters; these are the so-called Hanany-Witten brane setups.

In this chapter, an example of a non-Abelian Brane Box Model, namely one corresponding to a $Z_k \times D_{k'}$ orbifold singularity of $\mathbb{C}^3$, is constructed. Its self-consistency and hence equivalence to geometrical methods are subsequently shown. It is demonstrated how a group-theoretic twist of the non-Abelian group circumvents the problem of inconsistency that arise from naïve attempts at the construction [295].

14.1 Introduction

Brane setups [66] have been widely attempted to provide an alternative to algebro-geometric methods in the construction of gauge theories (see [63] and references therein). The advantages of the latter include the enlightening of important properties of manifolds such as mirror symmetry, the provision of convenient supergravity descriptions and in instances of pure geometrical engineering, the absence of non-perturbative objects. The former on the other hand, give intuitive and direct treatments of the gauge theory. One can conveniently read out much information concerning the gauge theory from the brane setups, such as the dimension of the Coulomb and Higgs branches [66], the mirror symmetry [66, 199, 83, 200] in 3 dimensions first shown in [85], the Seiberg-duality in 4 dimensions [175], and exact solutions when we lift the setups from Type IIA to M Theory [67].

In particular, when discussing $\mathcal{N} = 2$ supersymmetric gauge theories in 4 dimensions, there are three known methods currently in favour. The first method is *geometrical engineering* exemplified by works in [48]; the second uses D3 branes as *probes* on orbifold singularities of the type $\mathbb{C}^2/\Gamma$ with Γ being a finite discrete subgroup of $SU(2)$ [69], and the third, the usage of *brane setups*. These three approaches are related to each other by proper T or S Dualities [53, 201]. For example, the configuration of stretching Type IIA D4 branes between $n+1$ NS5 branes placed in a circular fashion, the so-called **elliptic model**[1], is precisely T-dual to D3 branes stacked upon ALE[2] singularities of type $\widehat{A_n}$ (see [67, 202, 203, 204, 165] for detailed discussions).

The above constructions can be easily generalised to $\mathcal{N} = 1$ supersymmetric field theories in 4 dimensions. Methods from geometric engineering as well as D3 branes as probes now dictate the usage of orbifold singularities of the type $\mathbb{C}^3/\Gamma$ with Γ being

[1]We call it elliptic even though there is only an S^1 upon which we place the D4 branes; this is because from the M Theory perspective, there is another direction: an S^1 on which we compactify to obtain type Type IIA. The presence of two S^1's makes the theory toroidal, or elliptic. Later we shall see how to make use of $T^2 = S^1 \times S^1$ in Type IIB. For clarity we shall refer to the former as the $\mathcal{N} = 2$ elliptic model and the latter, the $\mathcal{N} = 1$ elliptic model.

[2] Asymptotically Locally Euclidean, i.e., Gorenstein singularities that locally represent Calabi-Yau manifolds.

a finite discrete subgroup of $SU(3)$ [75, 76, 77, 73, 74]. A catalogue of all the discrete subgroups of $SU(3)$ in this context is given in [292, 141]. Now from the brane-setup point of view, there are two ways to arrive at the theory. The first is to rotate certain branes in the configuration to break the supersymmetry from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ [175]. The alternative is to add another type of NS5 branes, viz., a set of NS5$'$ branes placed perpendicularly to the original NS5, whereby constructing the so-called **Brane Box Model** [78, 79]. Each of these two different approaches has its own merits. While the former (rotating branes) facilitates the deduction of Seiberg Duality, for the latter (Brane Box Models), it is easier to construct a class of new, finite, chiral field theories [82]. By finite we mean that in the field theory the divergences may be cancelable. From the perspective of branes on geometrical singularities, this finiteness corresponds to the cancelation of tadpoles in the orbifold background and from that of brane setups, it corresponds to the no-bending requirement of the branes [53, 201, 82, 80]. Indeed, as with the $\mathcal{N} = 2$ case, we can still show the equivalence among these different perspectives by suitable S or T Duality transformations. This equivalence is explicitly shown in [79] for the case of the Abelian finite subgroups of $SU(3)$. More precisely, for the group $Z_k \times Z_{k'}$ or Z_k and a chosen decomposition of $\mathbf{3}$ into appropriate irreducible representations thereof one can construct the corresponding Brane Box Model that gives the same quiver diagram as the one obtained directly from the geometrical methods of attack; this is what we mean by equivalence [75, 76, 77].

Indeed, we are not satisfied with the fact that this abovementioned equivalence so far exists only for Abelian singularities and would like to see how it may be extended to non-Abelian cases. The aim for constructing Brane Box Models of non-Abelian finite groups is twofold: firstly we would generate a new category of finite supersymmetric field theories and secondly we would demonstrate how the equivalence between the Brane Box Model and D3 branes as probes is true beyond the Abelian case and hence give an interesting physical perspective on non-Abelian groups. More specifically, the problem we wish to tackle is that given any finite discrete subgroup Γ of $SU(2)$ or $SU(3)$, what is the brane setup (in the T-dual picture) that corresponds to D3 branes as probes on orbifold singularities afforded by Γ? For the $SU(2)$ case, the answer for the $\widehat{A}$ series was given in [67] and that for the $\widehat{D}$ series, in [83], yet $\widehat{E_{6,7,8}}$ are still unsolved. For the $SU(3)$ case, the situation is even worse. While [78, 79] have given solutions to the Abelian groups Z_k and $Z_k \times Z_{k'}$, the non-Abelian Δ and Σ series have yet to be treated. Though it is not clear how the generalisation can be done for arbitrary non-Abelian singularities, it is the purpose of this writing to take one further step from [78, 79], and address the next simplest series of dimension three orbifold theories, viz., those of $\mathbb{C}^3/Z_k \times D_{k'}$ and construct the corresponding Brane Box Model and show its equivalence to geometrical methods. In addition to equivalence we demonstrate how the two pictures are bijectively related for the group of interest and that given one there exists a unique description in the other. The key input is given by Kutasov, Sen and Kapustin in [83, 205, 206]. Moreover [207] has briefly pointed out how his results may be used, but without showing the consistency and equivalence.

The chapter is organised as follows. In section §14.2 we shall briefly review some techniques of brane setups and orbifold projections in the context of finite quiver theories. Section §14.3 is then devoted to a crucial digression on the mathematical properties of the group of our interest, or what we call $G := Z_k \times D_{k'}$. In section §14.4 we construct the Brane Box Model for G, followed by concluding remarks in section §14.5.

Nomenclature

Unless otherwise stated, we shall, throughout our chapter, adhere to the notation that $\omega_n = e^{\frac{2\pi i}{n}}$, the nth root of unity, that G refers to the group $Z_k \times D_{k'}$, that without ambiguity Z_k denotes $\mathbb{Z}_k$, the cyclic group of k elements, that D_k is the binary dihedral group of order $4k$ and gives the affine Dynkin diagram of $\widehat{D}_{k+2}$, and that d_k denotes the ordinary dihedral group of order $2k$. Moreover δ will be defined as $(k, 2k')$, the greatest common divisor (GCD) of k and $2k'$.

14.2 A Brief Review of D_n Quivers, Brane Boxes, and Brane Probes on Orbifolds

The aim of this chapter is to construct the Brane Box Model of the non-Abelian finite group $Z_k \times D_{k'}$ and to show its consistency as well as equivalence to geometric methods. To do so, we need to know how to read out the gauge groups and matter content from quiver diagrams which describe a particular field theory from the geometry side. The knowledge for such a task is supplied in §14.2.1. Next, as mentioned in the introduction, to construct field theories which could be encoded in the D_k quiver diagram, we need an important result from [83, 205, 206]. A brief review befitting our aim is given in §14.2.2. Finally in §14.2.3 we present the rudiments of the Brane Box Model.

14.2.1 Branes on Orbifolds and Quiver Diagrams

It is well-known that a stack of coincident n D3 branes gives rise to an $\mathcal{N} = 4$ $U(n)$ super-Yang-Mills theory on the four dimensional world volume. The $U(1)$ factor of the $U(n)$ gauge group decouples when we discuss the low energy dynamics of the field theory and can be ignored, therefore giving us an effective $SU(n)$ theory. For $\mathcal{N} = 4$ in 4 dimensions the R-symmetry is $SU(4)$. Under such an R-symmetry, the fermions in the vector multiplet transform in the spinor representation of $SU(4) \simeq Spin(6)$ and the scalars, in the vector representation of $Spin(6)$, the universal cover of $SO(6)$. In the brane picture we can identify the R-symmetry as the $SO(6)$ isometry group which acts on the six transverse directions of the D3-branes. Furthermore, in the AdS/CFT picture, this $SU(4)$ simply manifests as the $SO(6)$ isometry group of the 5-sphere in $AdS_5 \times S^5$ [75, 76, 77].

We shall refer to this gauge theory of the D3 branes as the parent theory and consider the consequences of putting the stack on geometric singularities. A wide class of finite Yang-Mills theories of various gauge groups and supersymmetries is obtained when the parent theory is placed on orbifold singularities of the type $\mathbb{C}^m/\Gamma$ where

$m = 2, 3$. What this means is that we select a discrete finite group $\Gamma \subset SU(4)$ and let its irreducible representations $\{\mathbf{r}_i\}$ act on the Chan-Paton indices $I, J = 1, ..., n$ of the D3 branes by permutation. Only those matter fields of the parent theory that are invariant under the group action of Γ remain, the rest are eliminated by this so-called "orbifold projection". We present the properties of the parent and the orbifolded theory in the following diagram:

	Parent Theory $\xrightarrow{\Gamma,\ \text{irreps } = \{\mathbf{r}_i\}}$	Orbifold Theory
SUSY	$\mathcal{N} = 4$	$\mathcal{N} = 2$, for $\mathbb{C}^2/\{\Gamma \subset SU(2)\}$ $\mathcal{N} = 1$, for $\mathbb{C}^3/\{\Gamma \subset SU(3)\}$ $\mathcal{N} = 0$, for $(\mathbb{C}^3 \simeq \mathbb{R}^6)/\{\Gamma \subset \{SU(4) \simeq SO(6)\}\}$
Gauge Group	$U(n)$	$\prod\limits_i SU(N_i)$, where $\sum\limits_i N_i \dim \mathbf{r}_i = n$
Fermion Boson	$\Psi^{\mathbf{4}}_{IJ}$ $\Phi^{\mathbf{6}}_{IJ}$	$\Psi^{ij}_{f_{ij}}$ $\Phi^{ij}_{f_{ij}}$ where $I, J = 1, ..., n$; $f_{ij} = 1, ..., a^{\mathcal{R}=\mathbf{4},\mathbf{6}}_{ij}$ $\mathcal{R} \otimes \mathbf{r}_i = \bigoplus\limits_j a^{\mathcal{R}}_{ij}$

Let us briefly explain what the above table summarises. In the parent theory, there are, as mentioned above, gauge bosons $A_{IJ=1,...,n}$ as singlets of $Spin(6)$, adjoint Weyl fermions $\Psi^{\mathbf{4}}_{IJ}$ in the fundamental $\mathbf{4}$ of $SU(4)$ and adjoint scalars $\Phi^{\mathbf{6}}_{IJ}$ in the antisymmetric $\mathbf{6}$ of $SU(4)$. The projection is the condition that

$$A = \gamma(\Gamma) \cdot A \cdot \gamma(\Gamma)^{-1}$$

for the gauge bosons and

$$\Psi(\text{ or } \Phi) = R(\Gamma) \cdot \gamma(\Gamma) \cdot \Psi(\text{ or } \Phi) \cdot \gamma(\Gamma)^{-1}$$

for the fermions and bosons respectively (γ and R are appropriate representations of Γ).

Solving these relations by using Schur's Lemma gives the information on the orbifold theory. The equation for A tell us that the original $U(n)$ gauge group is broken to $\prod_i SU(N_i)$ where N_i are positive integers such that $\sum_i N_i \dim\mathbf{r}_i = n$. We point out here that henceforth we shall use the *regular representation* where $n = N|\Gamma|$ for some integer N and $n_i = N\dim\mathbf{r}_i$. Indeed other choices are possible and they give rise to *Fractional Branes*, which not only provide interesting dynamics but are also crucial in showing the equivalence between brane setups and geometrical engineering [208, 53]. The equations for Ψ and Φ dictate that they become bi-fundamentals which transform under various pairs $(N_i, \bar{N}_j)$ within the product gauge group. We have a total of $a_{ij}^{\mathbf{4}}$ Weyl fermions $\Psi^{ij}_{f_{ij}=1,\ldots,a_{ij}^{\mathbf{4}}}$ and $a_{ij}^{\mathbf{6}}$ scalars $\Phi^{ij}_{f_{ij}}$ where $a_{ij}^{\mathcal{R}}$ is defined by

$$\mathcal{R} \otimes \mathbf{r}_i = \bigoplus_j a_{ij}^{\mathcal{R}} \mathbf{r}_j \tag{14.2.1}$$

respectively for $\mathcal{R} = 4, 6$.

The supersymmetry of the orbifold theory is determined by analysing the commutant of Γ as it embeds into the parent $SU(4)$ R-symmetry. For Γ belonging to $SU(2)$, $SU(3)$ or the full $SU(4)$, we respectively obtain $\mathcal{N} = 2, 1, 0$. The corresponding geometric singularities are as presented in the table. Furthermore, the action of Γ clearly differs for $\Gamma \subset SU(2, 3,$ or $4)$ and the $\mathbf{4}$ and $\mathbf{6}$ that give rise to the bi-fundamentals must be decomposed appropriately. Generically, the number of trivial (principal) 1-dimensional irreducible representations corresponds to the co-dimension of the singularity. For the matter matrices a_{ij}, these irreducible representations give a contribution of δ_{ij} and therefore to guaranteed adjoints. For example, in the case of $\mathcal{N} = 2$, there are 2 trivial $\mathbf{1}$'s in the $\mathbf{4}$ and for $\mathcal{N} = 1$, $\mathbf{4} = \mathbf{1}_{\text{trivial}} \oplus \mathbf{3}$. In this chapter, we focus on the latter case since $Z_k \times D_{k'}$ is in $SU(3)$ and gives rise to $\mathcal{N} = 1$. Furthermore we acknowledge the inherent existence of the trivial 1-dimensional irrep and focus on the decomposition of the $\mathbf{3}$.

The matrices $a_{ij}^{\mathcal{R}=\mathbf{4},\mathbf{6}}$ in (14.2.1) and the numbers $\dim\mathbf{r}_i$ contain all the information about the matter fields and gauge groups of the orbifold theory. They can be conveniently encoded into so-called **quiver diagrams**. Each node of such a diagram treated as a finite graph represents a factor in the product gauge group and is labeled by dim$\mathbf{r}_i$. The (possibly oriented) adjacency matrix for the graph is prescribed precisely by a_{ij}. The cases of $\mathcal{N} = 2, 3$ are done [69, 292, 141, 142] and works toward the (non-supersymmetric) $\mathcal{N} = 0$ case are underway [294]. In the $\mathcal{N} = 2$ case, the quivers must coincide with ADE Dynkin diagrams treated as unoriented graphs in order that the orbifold theory be finite [48]. The quiver diagrams in general are suggested to be related to WZW modular invariants [292, 293].

This is a brief review of the construction via geometric methods and it is our intent now to see how brane configurations reproduce examples thereof.

14.2.2 D_k Quivers from Branes

Let us first digress briefly to A_k quivers from branes. In the case of $SU(2) \supset \Gamma = \widehat{A_k} \simeq Z_{k+1}$, the quiver theory should be represented by an affine A_k Dynkin diagram, i.e., a regular polygon with $k+1$ vertices. The gauge group is $\prod\limits_i SU(N_i) \times U(1)$ with N_i being a $k+1$-partition of n since $\mathbf{r}_i$ are all one-dimensional[3]. However, we point out that on a classical level we expect $U(N_i)$'s from the brane perspective rather than $SU(N_i)$. It is only after considering the one-loop quantum corrections in the field theory (or bending in the brane picture) that we realise that the $U(1)$ factors are frozen. This is explained in [67]. On the other hand, from the point of view of D-branes as probes on the orbifold singularity, associated to the anomalous $U(1)$'s are field-dependent Fayet-Illiopoulos terms generating which freezes the $U(1)$ factors. Thess two prespectives are T-dual to each other. Further details can be found in [209].

Now, placing $k+1$ NS5 branes on a circle with N_i stacked D4 branes stretched between the ith and $i+1$st NS5 reproduces precisely this gauge group with the correct bifundamentals provided by open strings ending on the adjacent D4 branes (in the compact direction). This circular model thus furnishes the brane configuration of an A_n-type orbifold theory and is summarised in Figure 14-1. Indeed T-duality in the compact direction transforms the $k+1$ NS5 branes into a nontrivial metric,

[3]The $U(1)$ corresponds to the centre-of-mass motion and decouples from other parts of the theory so that when we discuss the dynamical properties, it does not contribute.

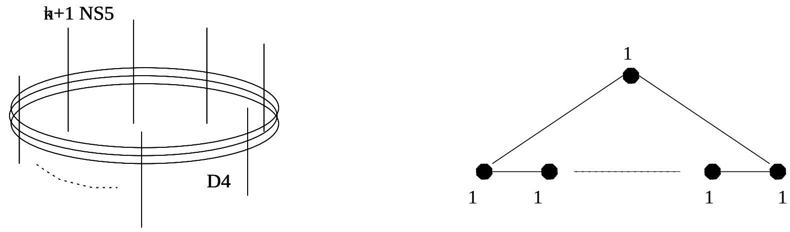


Figure 14-1: The $\mathcal{N} = 2$ elliptic model of D4 branes stretched between NS5 branes to give quiver theories of the $\widehat{A_k}$ type.

viz., the $k+1$-centered Taub-NUT, precisely that expected from the orbifold picture. Since both the NS5 and the D4 are offsprings of the M5 brane, in the M-Theory context, the circular configuration becomes $\mathbb{R}^4 \times \bar{\Sigma}$ in $\mathbb{R}^{10,1}$, where $\bar{\Sigma}$ is a $k+1$-point compactification of a the Riemann surface Σ swept out by the worldvolume of the fivebrane [67]. The duality group, which is the group of automorphisms among the marginal couplings that arise in the resulting field theory, whence becomes the fundamental group of $\mathcal{M}_{k+1}$, the moduli space of an elliptic curve with $k+1$ marked points.

The introduction of ON^0 planes facilitates the next type of $\mathcal{N} = 2, d = 4$ quiver theories, namely those encoded by affine $\widehat{D_k}$ Dynkin diagrams [83]. The gauge group is now $SU(2N)^{k-3} \times SU(N)^4 \times U(1)$ (here $U(1)$ decouples also, as explained before) dictated by the Dynkin indices of the $\widehat{D_k}$ diagrams.

There are two ways to see the $\widehat{D_k}$ quiver in the brane picture: one in Type IIA theory and the other, in Type IIB. Because later on in the construction of the Brane Box Model we will use D5 branes which are in Type IIB, we will focus on Type IIB only (for a complete description and how the two descriptions are related by T-duality, see [83]). In this case, what we need is the ON^0-plane which is the S-dual of a peculiar pair: a D5 brane on top of an $\mathrm{O5}^-$-plane. The one important property of the ON^0-plane is that it has an orbifold description $\mathbb{R}^6 \times \mathbb{R}^4/\mathcal{I}$ where $\mathcal{I}$ is a product of world sheet fermion operator $(-1)^{F_L}$ with the parity inversion of the $\mathbb{R}^4$ [206]. Let us place 2 parallel vertical ON^0 planes and $k-2$ NS5 branes in between and parallel to both as in Figure 14-2. Between the ON^0 and its immediately adjacent NS5, we stretch $2N$

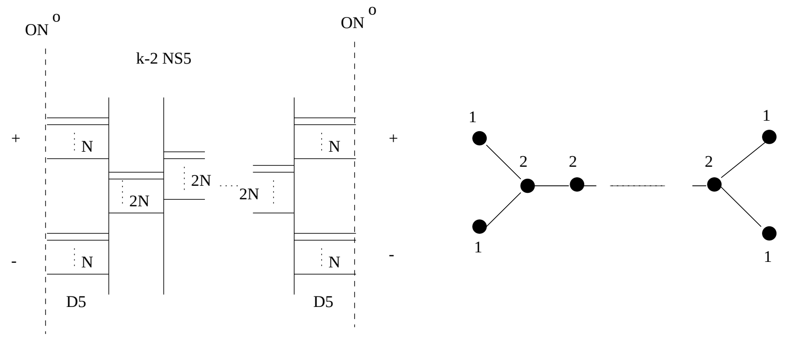


Figure 14-2: D5 branes stretched between ON^0 branes, interrupted by NS5 branes to give quiver theories of the $\widehat{D_k}$ type.

D5 branes; N of positive charge on the top and N of negative charge below. Now due to the projection of the ON^0 plane, N D5 branes of positive charge give one $SU(N)$ gauge group and N D5 branes of negative charge give another. Furthermore, these D5 branes end on NS5 branes and the boundary condition on the NS5 projects out the bi-fundamental hypermultiplets of these two $SU(N)$ gauge groups (for the rules of such projections see [83]). Moreover, between the two adjacent interior NS5's we stretch $2N$ D5 branes, giving $SU(2N)$'s for the gauge group. From this brane setup we immediately see that the gauge theory is encoded in the affine Quiver diagram of $\widehat{D_k}$.

14.2.3 Brane Boxes

We have seen in the last section, that positioning appropriate branes according to Dynkin diagrams - which for $\Gamma \subset SU(2)$ have their adjacency matrices determined by the representation of Γ, due to the McKay Correspondence [292] - branengineers some orbifold theories that can be geometrically engineered. The exceptional groups however, have so far been elusive [83]. For $\Gamma \subset SU(3)$, perhaps related to the fact that there is not yet a general McKay Correspondence[4] above dimension 2, the problem becomes more subtle; brane setups have been achieved for orbifolds of the Abelian

[4]For Gorenstein singularities of dimension 3, only those of the Abelian type such that 1 is not an eigenvalue of g $\forall g \in \Gamma$ are isolated. This restriction perhaps limits naïve brane box constructions to Abelian orbifold groups [79]. For a discussion on the McKay Correspondence as a ubiquitous thread, see [293].

type, a restriction that has been argued to be necessary for consistency [78, 79]. It is thus the purpose of this writing to show how a group-theoretic "twisting" can relax this condition and move beyond Abelian theories; to this we shall turn later.

We here briefly review the so-called $Z_k \times Z_{k'}$ elliptic brane box model. The orbifold theory corresponds to $\mathbb{C}^3/\{\Gamma = Z_k \times Z_{k'} \subset SU(3)\}$ and hence by arguments before we are in the realm of $\mathcal{N} = 1$ super-Yang-Mills. The generators for Γ are given, in its fundamental 3-dimensional representation[5], by diagonal matrices $diag(e^{\frac{2\pi i}{k}}, e^{\frac{-2\pi i}{k}}, 1)$ corresponding to the Z_k which act non-trivially on the first two coordinates of $\mathbb{C}^3$ and $diag(1, e^{\frac{2\pi i}{k'}}, e^{\frac{-2\pi i}{k'}})$ corresponding to the $Z_{k'}$ which act non-trivially on the last two coordinates of $\mathbb{C}^3$.

Since Γ is a direct product of Abelian groups, the representation thereof is simply a Kronecker tensor product of the two cyclic groups. Or, from the branes perspective, we should in a sense take a Cartesian product or sewing between two $\mathcal{N} = 2$ elliptic A_{k-1} and $A_{k'-1}$ models discussed above, resulting in a brane configuration on $S^1 \times S^1 = T^2$. This is the essence of the ($\mathcal{N} = 1$ elliptic) Brane Box Model [78, 79]. Indeed the placement of a perpendicular set of branes breaks the supersymmetry of the $\mathcal{N} = 2$ model by one more half, thereby giving the desired $\mathcal{N} = 1$. More specifically, we place k NS5 branes in the 012345 and k' NS5$'$ branes in the 012367 directions, whereby forming a grid of kk' boxes as in Figure 14-3. We then stretch n_{ij} D5 branes in the 012346 directions within the i,j-th box and compactify the 46 directions (thus making the low-energy theory on the D5 brane to be 4 dimensional). The bi-fundamental fields are then given according to adjacent boxes horizontally, vertically and diagonally and the gauge groups is $(\bigotimes\limits_{i,j} SU(N)) \times U(1) = SU(N)^{kk'} \times U(1)$ (here again the $U(1)$ decouples) as expected from geometric methods. Essentially we construct one box for each irreducible representation of $\Gamma = Z_k \times Z_{k'}$ such that going in the 3 directions as shown in Figure 14-3 corresponds to tensor decomposition of the irreducible representation in that grid and a special **3**-dimension representation

[5] We have chosen the directions in the transverse spacetime upon which each cyclic factor acts; the choice is arbitrary. In the language of finite groups, we have chosen the transitivity of the collineation sets. The group at hand, $Z_k \times Z_{k'}$, is in fact the first example of an intransitive subgroup of $SU(3)$. For a discussion of finite subgroups of unitary groups, see [294] and references therein.

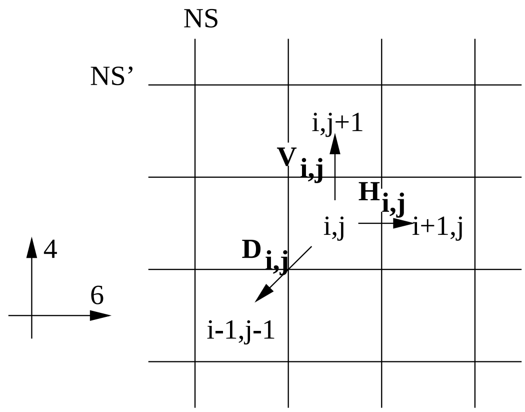


Figure 14-3: Bi-fundamentals arising from D5 branes stretched between grids of NS5 and NS5′ branes in the elliptic brane box model.

which we choose when we construct the Brane Box Model.

We therefore see the realisation of Abelian orbifold theories in dimension 3 as brane box configurations; twisted identifications of the grid can in fact lead to more exotic groups such as $Z_k \times Z_{kk'/l}$. More details can be found in [79].

14.3 The Group $G = Z_k \times D_{k'}$

It is our intent now to investigate the next simplest example of intransitive subgroups of $SU(3)$, i.e., the next infinite series of orbifold theories in dimension 3 (For definitions on the classification of collineation groups, see for example [294]). This will give us a first example of a Brane Box Model that corresponds to non-Abelian singularities.

Motivated by the $Z_k \times Z_{k'}$ treated in section §14.2, we let the second factor be the binary dihedral group of $SU(2)$, or the $D_{k'}$ series (we must point out that in our notation, the $D_{k'}$ group gives the $\widehat{D}_{k'+2}$ Dynkin diagram). Therefore Γ is the group $G = Z_k \times D_{k'}$, generated by

$$\alpha = \begin{pmatrix} \omega_k & 0 & 0 \\ 0 & \omega_k^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{2k'} & 0 \\ 0 & 0 & \omega_{2k'}^{-1} \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

where $w_x := e^{\frac{2\pi i}{x}}$. We observe that indeed α generates the Z_k acting on the first two directions in $\mathbb{C}^3$ while β and γ generate the $D_{k'}$ acting on the second two.

We now present some crucial properties of this group G which shall be used in the next section. First we remark that the $\times$ in G is really an abuse of notation, since G is certainly not a direct product of these two groups. This is the cause why naïve constructions of the Brane Box Model fail and to this point we shall turn later. What we really mean is that the actions on the first two and last two coordinates in the transverse directions by these subgroups are to be construed as separate. Abstractly, we can write the presentation of G as

$$\alpha\beta = \beta\alpha, \quad \beta\gamma = \gamma\beta^{-1}, \quad \alpha^m\gamma\alpha^n\gamma = \gamma\alpha^n\gamma\alpha^m \quad \forall m, n \in \mathbb{Z} \tag{14.3.2}$$

These relations compel all elements in G to be writable in the form $\alpha^m\gamma\alpha^{\tilde{m}}\gamma^n\beta^p$. However, before discussing the whole group, we find it very useful to discuss the subgroup generated by β and γ, i.e the binary dihedral group $D_{k'}$ as a degenerate ($k = 1$) case of G, because the properties of the binary dihedral group turn out to be crucial for the structure of the Brane Box Model and the meaning of "twisting" which we shall clarify later.

14.3.1 The Binary Dihedral $D_{k'} \subset G$

All the elements of $D_{k'}$ can be written as $\beta^p\gamma^n$ with $n = 0, 1$ and $p = 0, 1, ..., 2k' - 1$, giving the order of the group as $4k'$. We now move onto Frobenius characters. It is easy to work out the structure of conjugate classes. We have two conjugate classes $(1), (\beta^{k'})$ which have only one element, $(k'-1)$ conjugate classes $(\beta^p, \beta^{-p}), p = 1, .., k'-1$ which have two elements and two conjugate classes $(\beta^{p \text{ even}}\gamma), (\beta^{p \text{ odd}}\gamma)$ which have k' elements. The class equation is thus as follows:

$$4k' = 1 + 1 + (k' - 1) \cdot 2 + 2 \cdot k'.$$

Moreover there are 4 1-dimensional and $k' - 1$ 2-dimensional irreducible representations such that the characters for the 1-dimensionals depend on the parity of k'.

Now we have enough facts to clarify our notation: the group $D_{k'}$ gives $k'+3$ nodes (irreducible representations) which corresponds to the Dynkin diagram of $\widehat{D_{k'+2}}$.

We summarise the character table as follows:

k'even

	$C_{n=0}^{p=0}$	$C_{n=0}^{p=k'}$	$C_{n=0}^{\pm\text{even } p}$	$C_{n=0}^{\pm\text{odd } p}$	$C_{n=1}^{\text{even } p}$	$C_{n=1}^{\text{odd } p}$
$\lvert C\rvert$	1	1	2	2	k'	k'
$\#C$	1	1	$\frac{k'-1}{2}$	$\frac{k'-1}{2}$	1	1
Γ_1	1	1	1	1	1	1
Γ_2	1	-1	1	-1	1	-1
Γ_3	1	1	1	1	-1	-1
Γ_4	1	-1	1	-1	-1	1
Γ_l	$(\omega_{2k'}^{lp}+\omega_{2k'}^{-lp})$		$l=1,..,k'-1$		0	0

k'odd

	$C_{n=0}^{p=0}$	$C_{n=0}^{p=k'}$	$C_{n=0}^{\pm\text{even } p}$	$C_{n=0}^{\pm\text{odd } p}$	$C_{n=1}^{\text{even } p}$	$C_{n=1}^{\text{odd } p}$
$\lvert C\rvert$	1	1	2	2	k'	k'
$\#C$	1	1	$\frac{k'-2}{2}$	$\frac{k'}{2}$	1	1
Γ_1	1	1	1	1	1	1
Γ_2	1	1	1	-1	ω_4	$-\omega_4$
Γ_3	1	1	1	1	-1	-1
Γ_4	1	1	1	-1	$-\omega_4$	ω_4
Γ_l	$(\omega_{2k'}^{lp}+\omega_{2k'}^{-lp})$		$l=1,..,k'-1$		0	0

In the above tables, $|C|$ denotes the number of group elements in conjugate class C and $\#C$, the number of conjugate classes belonging to this type. Therefore $\sum_C \#C\cdot|C|$ should equal to order of the group. When we try to look for the character of the 1-dimensional irreps, we find it to be the same as the character of the factor group $D_{k'}/N$ where N is the normal subgroup generated by β. This factor group is Abelian of order 4 and is different depending on the parity of k'. When k' = even, it is $Z_2\times Z_2$ and when k' = odd it is Z_4. Furthermore, the conjugate class (β^p,β^{-p}) corresponds to different elements in this factor group depending on the parity of p, and we distinguish the two different cases in the table as $C_{n=0}^{\pm\text{odd }p}$ and $C_{n=0}^{\pm\text{even }p}$.

14.3.2 The whole group $G = Z_k \times D_{k'}$

Now from (14.3.2) we see that all elements of G can be written in the form $\alpha^m\gamma\alpha^{\tilde{m}}\gamma^n\beta^p$ with $m,\tilde{m} = 0,..,k-1$, $n = 0,1$ and $p = 0,..2k'-1$, which we abbreviate as $(m,\tilde{m},n,p)$. In the matrix form of our fundamental representation, they become

$$(m,\tilde{m},n=0,p) = \begin{pmatrix} \omega_k^{m+\tilde{m}} & 0 & 0 \\ 0 & 0 & i\omega_k^{-m}\omega_{2k'}^{-p} \\ 0 & i\omega_k^{-\tilde{m}}\omega_{2k'}^{p} & 0 \end{pmatrix}, \quad (m,\tilde{m},n=1,p) = \begin{pmatrix} \omega_k^{m+\tilde{m}} & 0 & 0 \\ 0 & -\omega_k^{-m}\omega_{2k'}^{p} & 0 \\ 0 & 0 & -\omega_k^{-\tilde{m}}\omega_{2k'}^{-p} \end{pmatrix}.$$

Of course this representation is not faithful and there is a non-trivial orbit; we can easily check the repeats:

$$\begin{aligned} (m,\tilde{m},n=0,p) &= (m+\tfrac{k}{(k,2k')}, \tilde{m}-\tfrac{k}{(k,2k')}, n=0, p-\tfrac{2k'}{(k,2k')}), \\ (m,\tilde{m},n=1,p) &= (m+\tfrac{k}{(k,2k')}, \tilde{m}-\tfrac{k}{(k,2k')}, n=1, p+\tfrac{2k'}{(k,2k')}) \end{aligned} \qquad (14.3.3)$$

where $(k,2k')$ denotes the largest common divisor between them. Dividing by the factor of this repeat immediately gives the order of G to be $\frac{4k'k^2}{(k,2k')}$.

We now move on to the study of the characters of the group. The details of the conjugation automorphism, class equation and irreducible representations we shall leave to the appendix 22.6 and the character tables we shall present below; again we have two cases, depending on the parity of $\frac{2k'}{(k,2k')}$. First however we start with some preliminary definitions. We define η as a function of n, p and $h = 1, 2, 3, 4$.

$$\begin{array}{cccccc}
 & & (n=1, p=\text{even}) & (n=1, p=\text{odd}) & (n=0, p=\text{even}) & (n=0, p=\text{odd}) \\
 & \eta^1 & 1 & 1 & 1 & 1 \\
k'=\text{even} & \eta^2 & 1 & -1 & 1 & -1 \\
 & \eta^3 & 1 & 1 & -1 & -1 \\
 & \eta^4 & 1 & -1 & -1 & 1 \\
 & & (n=1, p=\text{odd}) & (n=1, p=\text{even}) & (n=0, p=\text{even}) & (n=0, p=\text{odd}) \\
 & \eta^1 & 1 & 1 & 1 & 1 \\
k'=\text{odd} & \eta^2 & 1 & -1 & \omega_4 & -\omega_4 \\
 & \eta^3 & 1 & 1 & -1 & -1 \\
 & \eta^4 & 1 & -1 & -\omega_4 & \omega_4
\end{array} \tag{14.3.4}$$

Those two tables simply give the character tables of $Z_2 \times Z_2$ and Z_4 which we saw in the last section.

Henceforth we define $\delta := (k, 2k')$. Furthermore, we shall let Γ_x^n denote an n-dimensional irreducible representation indexed by some (multi-index) x. For $\frac{2k'}{\delta} =$ even, there are $4k$ 1-dimensional irreducible representations indexed by (l, h) with $l = 0, 1, .., k-1$ and $h = 1, 2, 3, 4$ and $k(\frac{k'k}{(k,2k')} - 1)$ 2-dimensionals indexed by (d, l) with $d = 1, .., \frac{k'k}{(k,2k')} - 1; l = 0, .., k-1$. For $\frac{2k'}{\delta} =$ odd, there are $2k$ 1-dimensional irreducible representations indexed by (l, h) with $l = 0, .., k-1; h = 1, 3$ and $k(\frac{k'k}{(k,2k')} - \frac{1}{2})$ 2-dimensionals indexed by (d, l) $d = 1, .., \frac{k'k}{(k,2k')} - 1; l = 0, .., k-1$ and $d = \frac{k'k}{(k,2k')}; l = 0, .., \frac{k}{2} - 1$. Now we present the character tables.

$\frac{2k'}{\delta} = \text{even}$

<table>
<tr><td>$|C|$</td><td>1</td><td>2</td><td>$\frac{k'k}{(k,2k')}$</td></tr>
<tr><td>$\#C$</td><td>$2k$</td><td>$k(\frac{k'k}{(k,2k')} - 1)$</td><td>$2k$</td></tr>
<tr><td></td><td>$m = 0, .., \frac{k}{\delta} - 1;\ i = 0, .., \delta - 1;$
$\tilde{m} = m + \frac{ik}{\delta};\ n = 1;$
$p = k' - \frac{ik'}{(k,2k')}, 2k' - \frac{ik'}{(k,2k')}$</td><td>$m = 0, .., \frac{k}{\delta} - 1;\ i = 0, .., \delta - 1;\ n = 1$
$\begin{cases} s = 0, .., m - 1;\ p = 0, ..2k' - 1; \\ \quad \tilde{m} = s + \frac{ik}{\delta}; \\ s = m; \text{and require further that} \\ \quad p < (-p - \frac{2ik'}{\delta}) \bmod (2k') \end{cases}$</td><td>$m = 0;$
$\tilde{m} = 0, .., k - 1;$
$p = 0, 1;$
$n = 0$</td></tr>
<tr><td>$\Gamma^1_{(l,h)}$</td><td colspan="3">$\omega_k^{(m+\tilde{m})l} \eta^h, \qquad l = 0, 1, .., k - 1; \quad h = 1, .., 4$</td></tr>
<tr><td>$\Gamma^2_{(d,l)}$</td><td colspan="2">$(-1)^d (\omega_k^{-dm} \omega_{2k'}^{dp} + \omega_k^{-d\tilde{m}} \omega_{2k'}^{-dp}) \omega_k^{(m+\tilde{m})l}$
$d \in [1, \frac{k'k}{(k,2k')} - 1]; \quad l \in [0, k)$</td><td>$0$</td></tr>
</table>

$\frac{2k'}{\delta} = \text{odd}$

<table>
<tr><td>$|C|$</td><td>1</td><td>2</td><td>$\frac{k'k}{(k,2k')}$</td></tr>
<tr><td>$\#C$</td><td>k</td><td>$k(\frac{k'k}{(k,2k')} - \frac{1}{2})$</td><td>$k$</td></tr>
<tr><td></td><td>$m = 0, .., \frac{k}{\delta} - 1;$
$i = 0, .., \delta - 1$ and even;
$\tilde{m} = m + \frac{ik}{\delta};\ n = 1;$
$p = k' - \frac{ik'}{(k,2k')},$
$2k' - \frac{ik'}{(k,2k')}$</td><td>$m = 0, .., \frac{k}{\delta} - 1;\ i = 0, .., \delta - 1;\ n = 1$
$\begin{cases} s = 0, .., m - 1;\ p = 0, ..2k' - 1; \\ \quad \tilde{m} = s + \frac{ik}{\delta}; \\ s = m; \text{and require further that} \\ \quad p < (-p - \frac{2ik'}{\delta}) \bmod (2k') \text{ for even } i \\ \quad p \leq (-p - \frac{2ik'}{\delta}) \bmod (2k') \text{ for odd } i \end{cases}$</td><td>$m = 0;$
$\tilde{m} = 0, .., k - 1;$
$p = 0;$
$n = 0$</td></tr>
<tr><td>$\Gamma^1_{(l,h)}$</td><td colspan="3">$\omega_k^{(m+\tilde{m})l} \eta^h, \qquad l = 0, 1, .., k - 1; \quad h = 1, 3$</td></tr>
<tr><td>$\Gamma^2_{(d,l)}$</td><td colspan="2">$(-1)^d (\omega_k^{-dm} \omega_{2k'}^{dp} + \omega_k^{-d\tilde{m}} \omega_{2k'}^{-dp}) \omega_k^{(m+\tilde{m})l}$
$d \in [1, \frac{k'k}{(k,2k')} - 1]; \quad l \in [0, k)$</td><td>$0$</td></tr>
<tr><td>$\Gamma^2_{(d,l)}$</td><td colspan="2">$(-1)^d (\omega_k^{-dm} \omega_{2k'}^{dp} + \omega_k^{-d\tilde{m}} \omega_{2k'}^{-dp}) \omega_k^{(m+\tilde{m})l}$
$d = \frac{k'k}{(k,2k')}; \quad l \in [0, \frac{k}{2})$</td><td>$0$</td></tr>
</table>

Let us explain the above tables in more detail. The third row of each table give the representative elements of the various conjugate classes. The detailed description of the group elements in each conjugacy class is given in appendix 22.6. It is easy to see, by using the above character tables, that given two elements $(m_i, \tilde{m}_i, n_i, p_i)$ $i = 1, 2$, if they share the same characters (as given in the last two rows), they belong to same conjugate class as to be expected since the character is a class function.

We can be more precise and actually write down the 2 dimensional irreducible representation indexed by (d, l) as

$$\begin{array}{l} (m, \tilde{m}, n = 0, p) = \omega_k^{(m+\tilde{m})l} \begin{pmatrix} 0 & i^d \omega_k^{-dm} \omega_{2k'}^{-dp} \\ i^d \omega_k^{-d\tilde{m}} \omega_{2k'}^{dp} & 0 \end{pmatrix} \\ (m, \tilde{m}, n = 1, p) = \omega_k^{(m+\tilde{m})l} \begin{pmatrix} (-1)^d \omega_k^{-dm} \omega_{2k'}^{dp} & 0 \\ 0 & (-1)^d \omega_k^{-d\tilde{m}} \omega_{2k'}^{-dp} \end{pmatrix} \end{array} \quad (14.3.5)$$

14.3.3 The Tensor Product Decomposition in G

A concept crucial to character theory and representations is the decomposition of tensor products into tensor sums among the various irreducible representations, namely the equation

$$\mathbf{r}_k \otimes \mathbf{r}_i = \bigoplus_j a_{ij}^k \mathbf{r}_j.$$

Not only will such an equation enlighten us as to the structure of the group, it will also provide quintessential information to the brane box construction to which we shall turn later. Indeed the $\mathcal{R}$ in (14.2.1) is decomposed into direct sums of irreducible representations $\mathbf{r}_k$, which by the additive property of the characters, makes the fermionic and bosonic matter matrices $a_{ij}^{\mathcal{R}}$ ordinary sums of matrices a_{ij}^k. In particular, knowing the specific decomposition of the $\mathbf{3}$, we can immediately construct the quiver diagram prescribed by $a_{ij}^{\mathbf{3}}$ as discussed in §14.2.1.

We summarise the decomposition laws as follows (using the multi-index notation for the irreducible representations introduced in the previous section), with the case of $\frac{2k'}{\delta}$ = even in (14.3.6) and odd, in (14.3.7).

$\mathbf{1}\otimes\mathbf{1}'$	$(l_1,h_1)_1 \otimes (l_2,h_2)_1 = (l_1+l_2,h_3)_1$ where h_3 is such that $\eta^{h_1}\eta^{h_2}=\eta^{h_3}$ according to (14.3.4).
$\mathbf{2}\otimes\mathbf{1}$	$(d,l_1)_2\otimes(l_2,h_2)_1 = \begin{cases} (d,l_1+l_2)_2 \quad \text{when} \quad h_2=1,3. \\ (\frac{k'k}{(k,2k')}-d, l_1+l_2-d)_2 \quad \text{when} \quad h_2=2,4 \end{cases}$
$\mathbf{2}\otimes\mathbf{2}'$	$(d_1,l_1)_2\otimes(d_2\leq d_1,l_2)_2 =$ $\qquad (d_1+d_2,l_1+l_2)_2\oplus(d_1-d_2,l_1+l_2-d_2)_2$ where $(d_1-d_2,l_1+l_2-d_2)_2 :=$ $\qquad (l_1+l_2-d_2,h=1)_1\oplus(l_1+l_2-d_2,h=3)_1 \quad \text{if} \quad d_1=d_2$ $(d_1+d_2,l_1+l_2)_2 :=$ $\qquad (l_1+l_2,h=2)_1\oplus(l_1+l_2,h=4)_1 \quad \text{if} \quad d_1+d_2=\frac{k'k}{\delta}$ $(d_1+d_2,l_1+l_2)_2 :=$ $\qquad (\frac{2k'k}{(k,2k')}-(d_1+d_2),(l_1+l_2)-(d_1+d_2))_2 \quad \text{if} \quad d_1+d_2>\frac{k'k}{\delta}$

(14.3.6)

$\mathbf{1}\otimes\mathbf{1}'$	$(l_1,h_1)_1 \otimes (l_2,h_2)_1 = \begin{cases} (l_1+l_2,h=1)_1 \quad \text{if} \quad h_1=h_2 \\ (l_1+l_2,h=3)_1 \quad \text{if} \quad h_1\neq h_2 \end{cases}$
$\mathbf{2}\otimes\mathbf{1}$	$(d,l_1)_2\otimes(l_2,h_2)_1 = \begin{cases} (d,l_1+l_2)_2 \\ (d,l_1+l_2-\frac{k}{2})_2 \quad \text{if} \quad d=\frac{k'k}{(k,2k')} \text{ and } l_1+l_2\geq\frac{k}{2} \end{cases}$
$\mathbf{2}\otimes\mathbf{2}'$	$(d_1,l_1)_2\otimes(d_2\leq d_1,l_2)_2 =$ $\qquad (d_1+d_2,l_1+l_2)_2\oplus(d_1-d_2,l_1+l_2-d_2)_2$ where $(d_1-d_2,l_1+l_2-d_2)_2 :=$ $\qquad (l_1+l_2-d_2,h=1)_1\oplus(l_1+l_2-d_2,h=3)_1 \quad \text{if} \quad d_1=d_2$ $(d_1+d_2,l_1+l_2)_2 :=$ $\qquad (d_1+d_2,l_1+l_2-\frac{k}{2})_2 \quad \text{if} \quad d_1+d_2=\frac{k'k}{\delta} \text{ and } l_1+l_2\geq\frac{k}{2}$ $(d_1+d_2,l_1+l_2)_2 :=$ $\qquad (\frac{2k'k}{(k,2k')}-(d_1+d_2),(l_1+l_2)-(d_1+d_2))_2 \quad \text{if} \quad d_1+d_2>\frac{k'k}{\delta}$

(14.3.7)

14.3.4 $D_{\frac{kk'}{\delta}}$, an Important Normal Subgroup

We now investigate a crucial normal subgroup $H \triangleleft G$. The purpose is to write G as a canonical product of H with the factor group formed by quotienting G thereby, i.e., as $G \simeq G/H \times H$. The need for this rewriting of the group will become clear in §14.4 on the brane box construction. The subgroup we desire is the one presented in the following:

LEMMA 14.3.1 *The subgroup*

$$H := \{(m, -m, n, p) | m = 0, .., k-1; n = 0, 1; p = 0, ..., 2k' - 1\}$$

is normal in G and is isomorphic to $D_{\frac{kk'}{\delta}}$.

To prove normality we use the multiplication and conjugation rules in G given in appendix 22.6 as (22.6.1) and (22.6.2). Moreover, let $D_{\frac{kk'}{\delta}}$ be generated by $\tilde{\beta}$ and $\tilde{\gamma}$ using the notation of §14.3.1, then isomorphism can be shown by the following bijection:

$$(m, -m, 1, p) \longleftrightarrow \tilde{\beta}^{\frac{2k'}{\delta}m - \frac{k}{\delta}(p-k')},$$
$$(m, -m, 0, p) \longleftrightarrow \tilde{\beta}^{\frac{2k'}{\delta}m + \frac{k}{\delta}p}\tilde{\gamma}.$$

Another useful fact is the following:

LEMMA 14.3.2 *The factor group G/H is isomorphic to Z_k.*

This is seen by noting that $\alpha^l, l = 0, 1, ...k-1$ can be used as representatives of the cosets. We summarise these results into the following

PROPOSITION 14.3.4 *There exists another representation of G, namely $Z_k \times D_{k'} \simeq Z_k \rtimes D_{\frac{kk'}{\delta}}$, generated by the same α together with*

$$\tilde{\beta}^{\frac{2k'}{\delta}m - \frac{k}{\delta}p} := (m, -m, 1, p+k') = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_k^{-m}\omega_{2k'}^{p} & 0 \\ 0 & 0 & \omega_k^{m}\omega_{2k'}^{-p} \end{pmatrix}, \qquad \tilde{\gamma} := \gamma = (0,0,0,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}.$$

The elements of the group can now be written as $\alpha^a \tilde{\beta}^b \tilde{\gamma}^n$ *with* $a \in [0, k)$, $b \in [0, \frac{2kk'}{\delta})$ *and* $n = 0, 1$, *constrained by the presentation*

$$\{\alpha^k = \tilde{\beta}^{\frac{2kk'}{\delta}} = 1, \tilde{\beta}^{\frac{kk'}{\delta}} = \tilde{\gamma}^2 = -1, \alpha\tilde{\beta} = \tilde{\beta}\alpha, \tilde{\beta}\tilde{\gamma} = \tilde{\gamma}\tilde{\beta}^{-1}, \alpha\tilde{\gamma} = \tilde{\beta}^{\frac{2k'}{\delta}}\tilde{\gamma}\alpha\}$$

In the proposition, by $\rtimes$ we do mean the internal semi-direct product between Z_k and $H := D_{\tilde{k}} := D_{\frac{kk'}{\delta}}$, in the sense [210] that (I) $G = HZ_k$ as cosets, (II) H is normal in G and Z_k is another subgroup, and (III) $H \cap Z_k = 1$. Now we no longer abuse the symbol $\times$ and unambiguously use $\rtimes$ to show the true structure of G. We remark that this representation is in some sense more natural (later we shall see that this naturality is not only mathematical but also physical). The mathematical natuality is seen by the lift from the normal subgroup H. We will see what is the exact meaning of the "twist" we have mentioned before. When we include the generator α and lift the normal subgroup $D_{\frac{kk'}{\delta}}$ to the whole group G, the structure of conjugacy classes will generically change as well. For example, from

$$\alpha(\tilde{\beta}^b\tilde{\gamma})\alpha^{-1} = (\tilde{\beta}^{b+\frac{2k'}{\delta}}\tilde{\gamma}), \tag{14.3.8}$$

we see that the two different conjugacy classes $(\tilde{\beta}^{\text{even } b}\tilde{\gamma})$ and $(\tilde{\beta}^{\text{odd } b}\tilde{\gamma})$ will remain distinct if $\frac{2k'}{\delta} =$ even and collapse into one single conjugacy class if $\frac{2k'}{\delta} =$ odd. We formally call the latter case **twisted**. Further clarifications regarding the structure of the conjugacy classes of G from the new points of view, especially physical, shall be most welcome.

After some algebraic manipulation, we can write down all the conjugacy classes of G in this new description. For fixed a and $\frac{2k'}{\delta} =$ even, we have the following classes: $(\alpha^a\tilde{\beta}^{-\frac{k'}{\delta}a})$, $(\alpha^a\tilde{\beta}^{\frac{kk'}{\delta}-\frac{k'}{\delta}a})$, $(\alpha^a\tilde{\beta}^b, \alpha^a\tilde{\beta}^{-b-\frac{2k'}{\delta}a})$ (with $b \neq -\frac{k'}{\delta}a$ and $\frac{kk'}{\delta} - \frac{k'}{\delta}a$), $(\alpha^b\tilde{\beta}^{p \text{ even}}\tilde{\gamma})$ and $(\alpha^b\tilde{\beta}^{p \text{ odd}}\tilde{\gamma})$. The crucial point here is that, for every value of a, the structure of conjugacy classes is almost the same as that of $D_{\frac{kk'}{\delta}}$. There is a 1-1 correspondence (or the lifting without the "twist") as we go from the conjugacy classes of H to G, making it possible to use the idea of [207] to construct the corresponding Brane Box Model. We will see this point more clearly later. On the other hand, when $\frac{2k'}{\delta} =$ odd, for fixed a, the conjugacy classes are no longer in 1-1 correspondence between H and G. Firstly, the last two classes of H will combine into only one of G. Secondly, the classes which contain only one element (the first two in H) will remain so only for $a =$ even; for $a =$ odd, the they will combine into one single class of G which has two elements.

So far the case of $\frac{2k'}{\delta}$ = odd befuddles us and we do not know how the twist obstructs the construction of the Brane Box Model. This twist seems to suggest quiver theories on non-affine D_k diagrams because the bifurcation on one side collapses into a single node, a phenomenon hinted before in [292, 207]. It is a very interesting problem which we leave to further work.

14.4 The Brane Box for $Z_k \times D_{k'}$

14.4.1 The Puzzle

The astute readers may have by now questioned themselves why such a long digression on the esoterica of G was done; indeed is it not enough to straightforwardly combine the $D_{k'}$ quiver technique with the elliptic model and stack k copies of Kapustin's configuration on a circle to give the $Z_k \times D_{k'}$ brane boxes? Let us investigate where this naïveté fails. According to the discussions in §14.2.3, one must construct one box for each irreducible representation of G. Let us place 2 ON^0 planes with k' parallel NS5 branes in between as in §14.2.2, and then copy this k times in the direction of the ON^0 and compactify that direction. This would give us $k + k$ boxes each containing 2 1-dimensional irreducible representations corresponding to the boxes bounded by one ON^0 and one NS5 on the two ends. And in the middle we would have $k(k' - 1)$ boxes each containing 1 2-dimensional irreducible representation.

Therefrom arises a paradox already! From the discussion of the group $G = Z_k \times D_{k'}$ in §14.3, we recall that there are $4k$ 1-dimensional irreducible representations and $k(\frac{k'k}{(k,2k')} - 1)$ 2-dimensionals if $\frac{2k'}{\delta}$ = even and for $\frac{2k'}{\delta}$ = odd, $2k$ 1-dimensionals and $k(\frac{k'k}{(k,2k')} - \frac{1}{2})$ 2-dimensionals. Our attempt above gives a mismatch of the number the

2-dimensionals by a factor of as large as k; there are far too many 2-dimensionals for G to be placed into the required kk' boxes. This mismatch tells us that such naïve constructions of the Brane Box Model fails. The reason is that in this case what we are dealing with is a non-Abelian group and the noncommutative property thereof twists the naïve structure of the singularity. To correctly account for the property of the singularity after the non-Abelian twisting, we should attack in a new direction. In fact, the discussion of the normal subgroup H in §14.3.4 is precisely the way to see the structure of singularity more properly. Indeed we have hinted, at least for $\frac{2k'}{\delta}$ = even, that the naïve structure of the Brane Box Model can be applied again with a little modification, i.e., with the replacement of $D_{k'}$ by $D_{\frac{kk'}{\delta}}$. Here again we have the generator of Z_k acting on the first two coordinates of $\mathbb{C}^3$ and the generators of $D_{\frac{kk'}{\delta}}$ acting on the last two. This is the subject of the next sub section where we will give a consistent Brane Box Model for $G = Z_k \times D_{k'}$.

14.4.2 The Construction of Brane Box Model

Let us first discuss the decomposition of the fermionic **4** for which we shall construct the brane box (indeed the model will dictate the fermion bi-fundamentals, bosonic matter fields will be given therefrom by supersymmetry). As discussed in [292] and §14.2.1, since we are in an $\mathcal{N} = 1$ (i.e., a co-dimension one theory in the orbifold picture), the **4** must decompose into $\mathbf{1} \oplus \mathbf{3}$ with the **1** being trivial. More precisely, since G has only 1-dimensional or 2-dimensional irreducible representations, for giving the correct quiver diagram which corresponds to the Brane Box Model the **4** should go into one trivial 1-dimensional, one non-trivial 1-dimensional and one 2-dimensional according to

$$\mathbf{4} \longrightarrow (0,1)_1 \oplus (l',h')_1 \oplus (d,l)_2.$$

Of course we need a constraint so as to ensure that such a decomposition is consistent with the unity-determinant condition of the matrix representation of the groups. Since from (14.3.5) we can compute the determinant of the $(d,l)_2$ to be $(-1)^{(n+1)(d+1)}\omega_k^{(m+\tilde{m})(2l-d)}$, the constraining condition is $l' + 2l - d \equiv 0(\mathrm{mod} k)$. In particular we choose

$$\mathbf{3} \longrightarrow (l' = 1, h' = 1)_1 + (d = 1, l = 0)_2; \tag{14.4.9}$$

indeed this choice is precisely in accordance with the defining matrices of G in §14.3 and we will give the Brane Box Model corresponding to this decomposition and check consistency.

Now we construct the brane box using the basic idea in [207] . Let us focus on the case of $\delta := (k, 2k')$ being even where we have $4k$ 1-dimensional irreducible representations and $k(\frac{k'k}{(k,2k')} - 1)$ 2-dimensionals. We place 2 ON^0 planes vertically at two sides. Between them we place $\frac{kk'}{\delta}$ vertically parallel NS5 branes (which give the structure of $D_{\frac{kk'}{\delta}}$). Next we place k NS5$'$ branes horizontally (which give the structure of Z_k) and identify the kth with the zeroth. This gives us a grid of $k(\frac{kk'}{\delta} + 1)$ boxes. Next we put N D5 branes with positive charge and N with negative charge in those grids. Under the decomposition (14.4.9), we can connect the structure of singularity to the structure of Brane Box Model by placing the irreducible representations into the grid of boxes à la [78, 79] as follows (the setup is shown in Figure 14-4).

First we place the $4k$ 1-dimensionals at the two sides such that those boxes each contains two: at the left we have $(l' = 0, h' = 1)_1$ and $(l' = 0, h' = 3)_1$ at the lowest box and with the upper boxes containing subsequent increments on l'. Therefore we have the list, in going up the boxes, $\{(0,1)_1$ & $(0,3)_1; (1,1)_1$ & $(1,3)_1; (2,1)_1$ & $(2,3)_1; ...(k-1,1)_1$ & $(k-1,3)_1\}$. The right side has a similar list: $\{(0,2)_1$ & $(0,4)_1; (1,2)_1$ & $(1,4)_1$; $(2,2)_1$ & $(2,4)_1; ...(k-1,2)_1$ & $(k-1,4)_1\}$. Into the middle grids we place the 2-dimensionals, one to a box, such that the bottom row consists of $\{(d = 1, l = 0)_2, (2,0)_2, (3,0)_2, ...(\frac{kk'}{\delta} - 1, 0)_2\}$ from left to right. And as we go up we increment l until $l = k - 1$ ($l = k$ is identified with $l = 0$ due to our compactification). Now we must check the consistency condition. We choose the bi-fundamental directions according to the conventions in [78, 79], i.e., East, North and Southwest. The consistency condition is that for the irreducible representation in box i, forming the tensor product with the $\mathbf{3}$ chosen in (14.4.9) should be the tensor sum of the irreducible

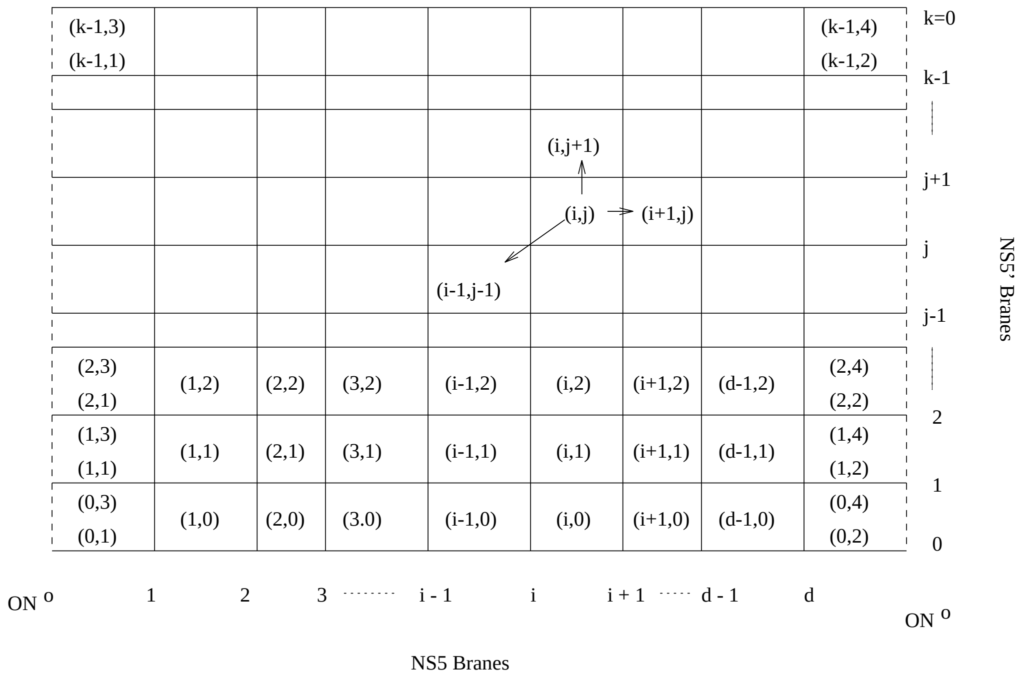


Figure 14-4: The Brane Box Model for $Z_k \times D_{k'}$. We place $d := \frac{kk'}{\delta}$ NS5 branes in between 2 parallel ON^0-planes and k NS5$'$ branes perpendicularly while identifying the 0th and the kth circularly. Within the boxes of this grid, we stretch D5 branes, furnishing bi-fundamental as indicated by the arrows shown.

representations of the neighbours in the 3 chosen directions, i.e.,

$$\mathbf{3} \otimes R_i = \bigoplus_{j \in \text{Neighbours}} R_j \qquad (14.4.10)$$

Of course this consistency condition is precisely (14.2.1) in a different guise and checking it amounts to seeing whether the Brane Box Model gives the same quiver theory as does the geometry, whereby showing the equivalence between the two methods. Now the elaborate tabulation in §14.3.3 is seen to be not in vain; let us check (14.4.10) by column in the brane box as in Figure 14-4. For the ith entry in the leftmost column, containing $R_i = (l', 1 \text{ or } 3)$, we have $R_i \otimes \mathbf{3} = (l', 1 \text{ or } 3)_1 \otimes ((1,1)_1 \oplus (1,0)_2) = (l'+1, 1 \text{ or } 3)_1 \oplus (1, l')_2$. The righthand side is precisely given by the neighbour of i to the East and to the North and since there is no Southwest neighbour, consistency (14.4.10) holds for the leftmost column. A similar situation holds for the rightmost column, where we have $\mathbf{3} \otimes (l', 2 \text{ or } 4) = (l'+1, 2 \text{ or } 4)_1 \oplus (\frac{kk'}{\delta} - 1, l'-1)_2$, the neighbour to the North and the Southwest.

Now we check the second column, i.e., one between the first and second NS5-branes. For the ith entry $R_i = (1, l)_2$, after tensoring with the $\mathbf{3}$, we obtain $(1, l+1)_2 \oplus (l+1, l+0)_2 \oplus ((l+0-1, 1)_1 \oplus (l+0-1, 3)_1)$, which are the irreducible representations precisely in the 3 neighbours: respectively East, North and the two 1-dimensional in the Southwest. Whence (14.4.10) is checked. Of course a similar situation occurs for the second column from the right where we have $\mathbf{3} \otimes (R_i = (\frac{kk'}{\delta} - 1, l)_2) = (\frac{kk'}{\delta} - 1, l+2)_2 \oplus (\frac{kk'}{\delta} - 1 - 1, l-1)_2 \oplus ((l, 2)_1 \oplus (l, 4)_1)$, or respectively the neighbours to the North, Southwest and the East.

The final check is required of the interior box, say $R_i = (d, l)_2$. Its tensor with $\mathbf{3}$ gives $(d, l+1)_2 \oplus (d-1, l-1)_2 \oplus (d+1, l)_2$, precisely the neighbours to the North, Southwest and East.

14.4.3 The Inverse Problem

A natural question arises from our quest for the correspondence between brane box constructions and branes as probes: is such a correspondence bijective? Indeed if the two are to be related by some T Duality or generalisations thereof, this bijection would be necessary. Our discussions above have addressed one direction: given a $Z_k \times D_{k'}$ singularity, we have constructed a consistent Brane Box Model. Now we must ask whether given such a configuration with m NS5 branes between two ON^0 planes and k NS5′ branes according to Figure 14-4, could we find a unique $Z_k \times D_{k'}$ orbifold which corresponds thereto? The answer fortunately is in the affirmative and is summarised in the following:

PROPOSITION 14.4.5 *For $\frac{2k'}{(k,2k')}$ being even*[6]*, there exists a bijection*[7] *between the Brane Box Model and the D3 brane-probes on the orbifold for the group $G := Z_k \times D_{k'} \cong Z_k \rtimes D_{m := \frac{kk'}{(k,2k')}}$. In particular*

[6] *Which is the case upon which we focus.*

[7] *Bijection in the sense that given a quiver theory produced from one picture there exists a unique method in the other picture which gives the same quiver.*

- *(I) Given k and k', whereby determining G and hence the orbifold theory, one can construct a unique Brane Box Model;*

- *(II) Given k and m with the condition that k is a divisor of m, where k is the number of NS5 branes perpendicular to ON^0 planes and m the number of NS5 branes between two ON^0 planes as in Figure 14-4, one can extract a unique orbifold theory.*

Now we have already shown (I) by our extensive discussion in the previous sections. Indeed, given integers k and k', we have twisted G such that it is characterised by k and

$$m := \frac{kk'}{(k, 2k')}, \tag{14.4.11}$$

two numbers that uniquely fix the brane configuration. The crux of the remaining direction (II) seems to be the issue whether we could, given k and m, ascertain the values of k and k' uniquely? For if so then our Brane Box Model, which is solely determined by k and m, would be uniquely mapped to a $Z_k \times D_{k'}$ orbifold, characterised by k and k'. We will show below that though this is not so and k and k' cannot be uniquely solved, it is still true that G remains unique. Furthermore, we will outline the procedure by which we can find convenient choices of k and k' that describe G.

Let us analyse this problem in more detail. First we see that k, which determines the Z_k in G, remains unchanged. Therefore our problem is further reduced to: given m, is there a unique solution of k' at fixed k? We write k, k', m as:

$$\begin{aligned} k &= 2^q l f_2 \\ k' &= 2^p l f_1 \\ m &= 2^n f_3 \end{aligned} \tag{14.4.12}$$

where with the extraction of all even factors, l, f_1 and f_2 are all odd integers and l is the greatest common divisor of k and k' so that f_1, f_2 are coprime. What we need to know are l, f_1 and p given k, q, n and f_3. The first constraint is that $\frac{2k'}{(k,2k')} =$ even, a condition on which this chapter focuses. This immediately yields the inequality

$p \geq q$. The definition of m (14.4.11) above further gives

$$2^n f_3 = m = 2^p l f_1 f_2 = 2^{p-q} k f_1.$$

From this equation, we can solve

$$p = n, \qquad f_1 = \frac{m}{2^{p-q} k} \tag{14.4.13}$$

Now it remains to determine l. However, the solution for l is not unique. For example, if we take $l = l_1 l_2$ and $(l_2, f_1) = 1$, then the following set $\{\tilde{k}, \tilde{k'}\}$ will give same k, m:

$$\tilde{k} = k = 2^q l_1 l_2 f_2$$
$$\tilde{k'} = 2^p l_1 f_1$$
$$m = 2^n f_3$$

This non-uniqueness in determining k, k' from k, m may at first seem discouraging. However we shall see below that different pairs of $\{k, k'\}$ that give the same $\{k, m\}$ must give the same group G.

We first recall that G can be written as $Z_k \rtimes D_{m=\frac{kk'}{(k,2k')}}$. For fixed k, m the two subgroups Z_k and D_m are same. For the whole group $Z_k \rtimes D_{m=\frac{kk'}{(k,2k')}}$ be unique no matter which k' we choose we just need to show that the algebraic relation which generate $Z_k \rtimes D_{m=\frac{kk'}{(k,2k')}}$ from Z_k and D_m is same. For that, we recall from the proposition in section §14.3.4, that in twisting G into its internal semi-direct form, the crucial relation is

$$\alpha \tilde{\gamma} = \tilde{\beta}^{\frac{2k'}{(k,2k')}} \tilde{\gamma} \alpha$$

Indeed we observe that $\frac{k'}{(k,2k')} = \frac{m}{k}$ where the condition that k is a divisor of m makes the expression having meaning. Whence given m and k, the presentation of G as $Z_k \rtimes D_m$ is uniquely fixed, and hence G is uniquely determined. This concludes our demonstration for the above proposition.

Now the question arises as to what values of k and k' result in the same G and how the smallest pair (or rather, the smallest k' since k is fixed) may be selected. In fact our discussion above prescribes a technique of finding such a pair. First we solve p, f_1 using (14.4.13), then we find the largest factor h of k which satisfies $(h, f_1) = 1$. The smallest value of k' is then such that $l = \frac{k}{h}$ in (14.4.12). Finally, we wish to emphasize that the bijection we have discussed is not true for arbitrary $\{m, k\}$ and we require that k be a divisor of m as is needed in demonstration of the proposition. Indeed, given m and k which do not satisfy this condition, the 1-1 correspondence between the Brane Box Model and the orbifold singularity is still an enigma and will be left for future labours.

14.5 Conclusions and Prospects

We have briefly reviewed some techniques in two contemporary directions in the construction of gauge theories from branes, namely branes as geometrical probes on orbifold singularities or as constituents of configurations of D branes stretched between NS branes. Some rudiments in the orbifold procedure, in the brane setup of $\mathcal{N} = 2$ quiver theories of the $\widehat{D_k}$ type as well as in the $\mathcal{N} = 1$ $Z_k \times Z_{k'}$ Brane Box Model have been introduced. Thus inspired, we have constructed the Brane Box Model for an infinite series of non-Abelian finite subgroups of $SU(3)$, by combining some methodology of the aforementioned brane setups.

In particular, we have extensively studied the properties, especially the representation and character theory of the intransitive collineation group $G := Z_k \times D_{k'} \subset SU(3)$, the next simplest group after $Z_k \times Z_{k'}$ and a natural extension thereof. From the geometrical perspective, this amounts to the study of Gorenstein singularities of the type $\mathbb{C}^3/G$ with the Z_k acting on the first two complex coordinates of $\mathbb{C}^3$ and $D_{k'}$, the last two.

We have shown why naïve Brane Box constructions for G fail (and indeed why non-Abelian groups in general may present difficulties). It is only after a "twist" of G into a semi-direct product form $Z_k \rtimes D_{\frac{kk'}{(k,2k')}}$, an issue which only arises because of the non-Abelian nature of G, that the problem may be attacked. For $\frac{2k'}{(k,2k')}$ even, we have successfully established a consistent Brane Box Model. The resulting gauge theory is that of k copies of $\widehat{D}$-type quivers circularly arranged (see Figure 14-4). However for $\frac{2k'}{(k,2k')}$ odd, a degeneracy occurs and we seem to arrive at ordinary (non-Affine) D quivers, a phenomenon hinted at by some previous works [207, 292] but still remains elusive. Furthermore, we have discussed the inverse problem, i.e., whether given a configuration of the Brane Box Model we could find the corresponding branes as probes on orbifolds. We have shown that when k is a divisor of m the two perspectives are bijectively related and thus the inverse problem can be solved. For general $\{m, k\}$, the answer of the inverse problem is still not clear.

Many interesting problems arise and are open. Apart from clarifying the physical meaning of "twisting" and hence perhaps treat the $\frac{2k'}{(k,2k')}$ odd case, we can try to construct Brane Boxes for more generic non-Abelian groups. Moreover, marginal couplings and duality groups thereupon may be extracted and interpreted as brane motions; this is of particular interest because toric methods from geometry so far have been restricted to Abelian singularities. Also, recently proposed brane diamond models [211] may be combined with our techniques to shed new insight. There is a parallel work that deals with brane configurations for $\mathbb{C}^3/\Gamma$ singularities for non-Abelian Γ (i.e the Δ series in $SU(3)$) by (p,q)5-brane webs [172]. We hope that our construction, as the Brane Box Model realisation of a non-Abelian orbifold theory in dimension 3, may lead to insight in these various directions.

Chapter 15

Orbifolds VI: Z-D Brane Box Models

Synopsis

Generalising the ideas of the previous chapter, we address the problem of constructing Brane Box Models of what we call the Z-D Type from a new point of view, so as to establish the complete correspondence between these brane setups and orbifold singularities of the non-Abelian G generated by Z_k and D_d under certain group-theoretic constraints to which we refer as the BBM conditions. Moreover, we present a new class of $\mathcal{N} = 1$ quiver theories of the ordinary dihedral group d_k as well as the ordinary exceptionals $E_{6,7,8}$ which have non-chiral matter content and discuss issues related to brane setups thereof [296].

15.1 Introduction

Configurations of branes [66] have been proven to be a very useful method to study the gauge field theory which emerges as the low energy limit of string theory (for a complete reference, see Giveon and Kutasov [63]). The advantage of such setups is that they provide an intuitive picture so that we can very easily deduce many properties of the gauge theory. For example, brane setups have been used to study mirror symmetry in 3 dimensions [66, 85, 200, 199, 83], Seiberg Duality in 4 dimensions [175], and exact solutions when lifting Type IIA setups to M-theory [67, 202]. After proper T- or S-dualities, we can transform the above brane setups to D-brane as probes on some target space with orbifold singularities [69, 171, 76].

For example, the brane setup of streching Type IIA D4-branes between $n + 1$ NS5-branes placed in a circular fashion (the "elliptic model" [67]) is precisely T-dual to D3-branes stacked upon ALE singularities of the type $\widehat{A_n}$, or in other words orbifold singularities of the form $\mathbb{C}^2/Z_{n+1}$, where Z_{n+1} is the cyclic group on $n + 1$ elements and is a finite discrete subgroup of $SU(2)$. As another example, the Brane Box Model [78, 79, 82] is T-dual to D3-branes as probes on orbifold singularities of the type $\mathbb{C}^3/\Gamma$ with $\Gamma = Z_k$ or $Z_k \times Z_{k'}$ now being a finite discrete subgroup of $SU(3)$ [79]. A brief review of some of these contemporary techniques can be found in the previous chapter. In fact, it is a very interesting problem to see how in general the two different methods, viz., brane setups and D3-branes as probes on geometrical singularities, are connected to each other by proper duality transformations [53].

The general construction and methodology for D3-branes as probes on orbifold singularities has been given [76]. However, the complete list of the corresponding brane setups is not yet fully known. For orbifolds $\mathbb{C}^2/\{\Gamma \in SU(2)\}$, we have the answer for the $\widehat{A_n}$ series (i.e., $\Gamma = Z_{n+1}$) and the $\widehat{D_n}$ series (i.e., $\Gamma = D_{n-2}$, the binary dihedral groups) [83], but not for the exceptional cases $\widehat{E_{6,7,8}}$. At higher dimensions, the situation is even more disappointing: for orbifolds of $\mathbb{C}^3/\{\Gamma \in SU(3)\}$, brane setups are until recently limited to only Abelian singularities, namely $\Gamma = Z_k$ or $Z_k \times Z_{k'}$ [79].

In the previous chapter, we went beyond the Abelian restriction in three dimensions and gave a new result concerning the correspondence of the two methods. Indeed we showed that[1] for $\Gamma := G = Z_k * D_{k'}$ a finite discrete subgroup of $SU(3)$, the cor-

[1]In that chapter we used the notation $Z_k \times D_{k'}$ and pointed out that the symbol $\times$ was really an abuse. We shall here use the symbol $*$ and throughout this chapter reserve $\times$ to mean strict direct

responding brane setup (a Brane Box Model) T-dual to the orbifold discription can be obtained. More explicitly, the group $G \in SU(3)$ is defined as being generated by the following matrices that act on $\mathbb{C}^3$:

$$\alpha = \begin{pmatrix} \omega_k & 0 & 0 \\ 0 & \omega_k^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{2k'} & 0 \\ 0 & 0 & \omega_{2k'}^{-1} \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \tag{15.1.1}$$

where $\omega_n := e^{\frac{2\pi i}{n}}$, the nth root of unity.

The abstract presentation of the groups is as follows:

$$\alpha\beta = \beta\alpha, \quad \beta\gamma = \gamma\beta^{-1}, \quad \alpha^m\gamma\alpha^n\gamma = \gamma\alpha^n\gamma\alpha^m \quad \forall m, n \in \mathbb{Z} \tag{15.1.2}$$

Because of the non-Abelian property of G, the preliminary attempts at the corresponding Brane Box Model by using the idea in a previous work [207] met great difficulty. However, via careful analysis, we found that the group G can be written as the semidirect product of Z_k and $D_{\frac{kk'}{\gcd(k,2k')}}$. Furthermore, when $\frac{2k'}{\gcd(k,2k')} =$ even, the character table of G as the semidirect product $Z_k \rtimes D_{\frac{kk'}{\gcd(k,2k')}}$ preserves the structure of that of $D_{\frac{kk'}{\gcd(k,2k')}}$, in the sense that it seems to be composed of k copies of the latter. Indeed, it was noted [295] that only under this parity condition of $\frac{2k'}{\gcd(k,2k')} =$ even, can we construct, with the two group factors Z_k and $D_{\frac{kk'}{\gcd(k,2k')}}$, a consistent Brane Box Model with the ideas in the abovementioned paper [207].

The success of the above construction, constrained by certain conditions, hints that something fundamental is acting as a key rôle in the construction of non-Abelian brane setups above two (complex) dimensions. By careful consideration, it seems that the following three conditions presented themselves to be crucial in the study of $Z_k * D_{k'}$ which we here summarize:

1. The whole group G can be written as a semidirect product: $Z_k \rtimes D_d$;

product of groups and $\rtimes$, the semi-direct product.

2. The semidirect product of G preserves the structure of the irreducible representations of D_d, i.e., it appears that the irreps of G consist of k copies of those of the subgroup D_d;

3. There exists a (possibly reducible) representation of G in 3 dimensions such that the representation matrices belong to $SU(3)$. Henceforth, we shall call such a representation, consistent with the $SU(3)$ requirement (see more discussions [295, 292] on decompositions), as "*the chosen decomposition of* **3**".

We will show in this chapter that these conditions are sufficient for constructing Brane Box Model of the Z-D type. Here we will call the Brane Box Model in the previous chapter as **Type** Z-D and similarly, that in earlier works [78, 79] we shall call the Z-Z Type. We shall see this definition more clearly in subsection §15.2.3. It is amusing to notice that Brane Box Models of Type Z-Z also satisfy the above three conditions since they correspond to the group $Z_k \times Z_{k'}$, which is a direct product.

Furthermore, we shall answer a mysterious question posed at the end of the previous chapter. There, we discussed the so-called *Inverse Problem*, i.e., given a consistent Brane Box Model, how may one determine, from the structure of the setup (the number and the positioning of the branes), the corresponding group Γ in the orbifold structure of $\mathbb{C}^3/\Gamma$. We found there that only when k is the divisor of d can we find the corresponding group defined in (15.1.1) with proper k, k'. This was very unsatisfying. However, the structure of the Brane Box Model of Type Z-D was highly suggestive of the solution for general k, d. We shall here mend that short-coming and for arbitrary k, d we shall construct the corresponding group Γ which satisfies above three conditions. With this result, we establish the complete correspondence between the Brane Box Model of Type Z-D and D3-branes as probes on orbifold singularities of $\mathbb{C}^3/\Gamma$ with properly determined Γ.

The three conditions which are used for solving the inverse problem can be divided into two conceptual levels. The first two are at the level of pure mathematics, i.e., we can consider it from the point of view of abstract group theory without reference to representations or to finite discrete subgroup of $SU(n)$. The third condition is at the level of physical applications. From the general structure [76] we see that for constructing $\mathcal{N} = 2$ or $\mathcal{N} = 1$ theories we respectively need the group Γ to be a finite subgroup of $SU(2)$ or $SU(3)$. This requirement subsequently means that we can find a faithful (but possibly reducible) 2-dimensional or 3-dimensional representation with the matrices satisfying the determinant 1 and unitarity conditions. In other words, what supersymmetry ($\mathcal{N} = 2$ or 1) we will have in the orbifold theory by the standard procedure [76] depends only on the chosen representation (i.e., the decomposition of **2** or **3**). Such distinctions were already shown before [79, 292]. The group Z_3 had been considered [79]. If we choose its action on $\mathbb{C}^3$ as $(z_1, z_2, z_3) \longrightarrow (e^{\frac{2\pi i}{3}} z_1, e^{\frac{-2\pi i}{3}} z_2, z_3)$ we will have $\mathcal{N} = 2$ supersymmetry, but if we choose the action to be $(z_1, z_2, z_3) \longrightarrow (e^{\frac{2\pi i}{3}} z_1, e^{\frac{2\pi i}{3}} z_2, e^{\frac{2\pi i}{3}} z_3)$ we have only $\mathcal{N} = 1$. This phenomenon mathematically corresponds to what are called sets of transitivity of collineation group actions [294, 89].

Moreover, we notice that the ordinary dihedral group d_k which is excluded from the classification of finite subgroup of $SU(2)$ can be imbedded[2] into $SU(3)$. Therefore we expect that d_k should be useful in constructing some $\mathcal{N} = 1$ gauge field theories by the standard procedures [76, 292]. We show in this chapter that this is so. With the proper decompositions, we can obtain new types of gauge theories by choosing $\mathbb{C}^3$ orbifolds to be of the type d_k. For completeness, we also give the quiver diagrams of ordinary tetrahedral, octahedral and icosahedral groups ($E_{6,7,8}$), which by a similar token, can be imbedded into $SU(3)$.

The organisation of the chapter is as follows. In §15.2 we give a simple and illustrative example of constructing a Brane Box Model for the direct product $Z_k \times D_{k'}$, whereby initiating the study of brane setups of what we call Type Z-D. In §15.3 we deal with the twisted case which we encountered earlier in the previous chapter. We show that we can imbed the latter into the direct product (untwisted) case of §15.2 and arrive at another member of Brane Box Models of the Z-D type. In §15.4 we give a new class of $SU(3)$ quiver which are connected to the ordinary dihedral group d_k.

[2]Since it is in fact a subgroup of $SU(2)/\mathbb{Z}_2 \cong SO(3)$, the embedding is naturally induced from $SO(3) \hookrightarrow SU(3)$. In fact the 3-dimensional representation in $SU(3)$ is faithful.

Also, we give an interesting brane configuration that will give matter matter content as the $d_{k=\text{even}}$ quiver but a different superpotential on the gauge theory level. Finally in §15.5 we give concluding remarks and suggest future prospects.

Nomenclature

Unless otherwise specified, we shall throughout the chapter adhere to the notation that the group binary operator $\times$ refers to the strict direct product, $\rtimes$, the semi-direct product, and $*$, a general (twisted) product by combining the generators of the two operands[3]. Furthermore, ω_n is defined to be $e^{\frac{2\pi i}{n}}$, the nth root of unity; $H \triangleleft G$ mean that H is a normal subgroup of G; and a group generated by the set $\{x_i\}$ under relations $f_i(\{x_j\}) = 1$ is denoted as $\langle x_i | f_j \rangle$.

15.2 A Simple Example: The Direct Product $Z_k \times D_{k'}$

We recall that in a preceeding chapter, we constructed the Brane Box Model (BBM) for the group $Z_k * D_{k'}$ as generated by (15.1.1), satifying the three conditions mentioned above, which we shall dub as the **BBM condition** for groups. However, as we argued in the introduction, there may exist in general, groups not isomorphic to the one addressed [295] but still obey these conditions. As an illustrative example, we start with the simplest member of the family of $Z * D$ groups that satisfies the BBM condition, namely the direct product $G = Z_k \times D_{k'}$. We define α as the generator for the Z_k factor and γ, β, those for the $D_{k'}$. Of course by definition of the direct product α must commute with both β and γ. The presentation of the group is clearly

[3] Therefore in the previous chapter, the group $G := Z_k \times D_{k'}$ in this convention should be written as $Z_k * D_{k'}$, *q.v. Ibid.* for discussions on how these different group compositions affect brane constructions.

as follows:

$$\begin{array}{ll} \alpha^k = 1; & \text{The Cyclic Group } Z_k \\ \beta^{2k'} = 1, \quad \beta^{k'} = \gamma^2, \quad \beta\gamma = \gamma\beta^{-1}; & \text{The Binary Dihedral Group } D_{k'} \\ \alpha\beta = \beta\alpha, \quad \alpha\gamma = \gamma\alpha & \text{Mutual commutation} \end{array}$$

We see that the first two of the BBM conditions are trivially satisfied. To satisfy the third, we need a 3-dimensional matrix represenation of the group. More explicitly, as discussed [295], to construct the BBM of the Z-D type, one needs the decomposition of **3** into one nontrivial 1-dimensional irrep and one 2-dimensional irrep. In light of this, we can write down the $SU(3)$ matrix generators of the group as

$$\alpha = \begin{pmatrix} \omega_k^2 & 0 & 0 \\ 0 & \omega_k^{-1} & 0 \\ 0 & 0 & \omega_k^{-1} \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{2k'} & 0 \\ 0 & 0 & \omega_{2k'}^{-1} \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \quad (15.2.3)$$

Here, we notice a subtle point. When $k =$ even, $\alpha^{\frac{k}{2}}$ and $\beta^{k'}$ give the same matrix form. In other words, (15.2.3) generates a *non-faithful* representation. We will come back to this problem later, but before diving into a detailed discussion on the whole group $Z_k \times D_{k'}$, let us first give the necessary properties of the factor $D_{k'}$.

15.2.1 The Group $D_{k'}$

One can easily check that all the elements of the binary dihedral $D_{k'} = \langle \beta, \gamma \rangle$ group can be written, because $\gamma^2 = \beta^{k'}$, as

$$\gamma^n \beta^p, \quad \text{with} \quad n = 0, 1 \quad p = 0, 1, ..., 2k' - 1.$$

From this constraint and the conjugation relation

$$(\gamma^{n_1}\beta^{p_1})^{-1}(\gamma^n\beta^p)(\gamma^{n_1}\beta^{p_1}) = \gamma^n\beta^{p_1(1-(-1)^n)+(-1)^{n_1}p},$$

we can see that the group is of order $4k'$ and moreover affords 4 1-dimensional irreps and $(k'-1)$ 2-dimensional irreps. The classes of the group are:

	$C_{n=0}^{p=0}$	$C_{n=0}^{p=k'}$	$C_{n=0}^{\pm p}$	$C_{n=1}^{p \bmod 2}$
$\lvert C\rvert$	1	1	2	k'
$\#C$	1	1	$k'-1$	2

To study the character theory of $G := D_{k'}$, we recognise that $H := \{\beta^p\}$ for p even is a normal subgroup of G. Whence we can use the Frobenius-Clifford theory of induced characters to obtain the irreps of G from the factor group $\widetilde{G} := G/H = 1, \beta, \gamma, \gamma\beta$. For k' even, $\widetilde{G}$ is $Z_2 \times Z_2$ and for k' odd, it is simply Z_4. these then furnish the 1-dimensional irreps. We summarise the characters of these 4 one dimensionals as follows:

	$k' = \text{even}$				$k' = \text{odd}$			
	$\beta^{p=\text{even}}$	$\beta(\beta^{\text{odd}})$	$\gamma(\gamma\beta^{\text{even}})$	$\gamma\beta(\gamma\beta^{\text{odd}})$	β^{even}	$\beta(\beta^{\text{odd}})$	$\gamma(\gamma\beta^{\text{even}})$	$\gamma\beta(\gamma\beta^{\text{odd}})$
χ^1	1	1	1	1	1	1	1	1
χ^2	1	-1	1	-1	1	-1	ω_4	$-\omega_4$
χ^3	1	1	-1	-1	1	1	-1	-1
χ^4	1	-1	-1	1	1	-1	$-\omega_4$	ω_4

The 2-dimensional irreps can be directly obtained from the definitions; they are indexed by a single integer l:

$$\chi_2^l(C_{n=1}) = 0, \qquad \chi_2^l(C_{n=0}^p) = (\omega_{2k'}^{lp} + \omega_{2k'}^{-lp}), \quad l = 1, .., k'-1. \tag{15.2.4}$$

The matrix representations of these 2-dimensionals are given below:

$$\beta^p = \begin{pmatrix} \omega_{2k'}^{lp} & 0 \\ 0 & \omega_{2k'}^{-lp} \end{pmatrix} \qquad \gamma\beta^p = \begin{pmatrix} 0 & i^l \omega_{2k'}^{-lp} \\ i^l \omega_{2k'}^{lp} & 0 \end{pmatrix}$$

From (15.2.4) we immediately see that $\chi_2^l = \chi_2^{-l} = \chi_2^{2k'-l}$ which we use to restrict the index l in χ_2^l into the region $[1, k'-1]$.

Now for the purposes of the construction of the BBM, we aboveall need to know the tensor decompositions of the group; these we summarise below.

	$k' = $ even	$k' = $ odd
$\mathbf{1}\otimes\mathbf{1}'$	$\chi_1^2\chi_1^2 = \chi_1^1 \quad \chi_1^3\chi_1^3 = \chi_1^1 \quad \chi_1^4\chi_1^4 = \chi_1^1$ $\chi_1^2\chi_1^3 = \chi_1^4 \quad \chi_1^2\chi_1^4 = \chi_1^3 \quad \chi_1^3\chi_1^4 = \chi_1^2$	$\chi_1^2\chi_1^2 = \chi_1^3 \quad \chi_1^3\chi_1^3 = \chi_1^1 \quad \chi_1^4\chi_1^4 = \chi_1^3$ $\chi_1^2\chi_1^3 = \chi_1^4 \quad \chi_1^2\chi_1^4 = \chi_1^1 \quad \chi_1^3\chi_1^4 = \chi_1^2$
$\mathbf{1}\otimes\mathbf{2}$	$\chi_1^h\chi_2^l = \begin{cases} \chi_2^l & h=1,3 \\ \chi_2^{k'-l} & h=2,4 \end{cases}$	
$\mathbf{2}\otimes\mathbf{2}'$	$\chi_2^{l_1}\chi_2^{l_2} = \chi_2^{(l_1+l_2)} + \chi_2^{(l_1-l_2)}$ where $\chi_2^{(l_1+l_2)} = \begin{cases} \chi_2^{(l_1+l_2)} & \text{if} \quad l_1+l_2<k', \\ \chi_2^{2k'-(l_1+l_2)} & \text{if} \quad l_1+l_2>k', \\ \chi_1^2+\chi_1^4 & \text{if} \quad l_1+l_2=k'. \end{cases}$ $\chi_2^{(l_1-l_2)} = \begin{cases} \chi_2^{(l_1-l_2)} & \text{if} \quad l_1>l_2, \\ \chi_2^{(l_2-l_1)} & \text{if} \quad l_1<l_2, \\ \chi_1^1+\chi_1^3 & \text{if} \quad l_1=l_2. \end{cases}$	

15.2.2 The Quiver Diagram

The general method of constructing gauge field theories from orbifold singularities of $C^3/\Gamma \subset SU(3)$ has been given [76, 292]. Let us first review briefly the essential results. Given a finite discrete subgroup $\Gamma \subset SU(3)$ with irreducible representations $\{r_i\}$, we obtain, under the orbifold projection, an $\mathcal{N}=1$ super Yang-Mills theory with gauge group

$$\bigotimes_i SU(N|r_i|), \qquad |r_i| = \dim(r_i), N \in \mathbb{Z}$$

To determine the matter content we need to choose the decomposition of $\mathbf{3}$ (i.e., the 3×3 matrix form) of Γ which describes how it acts upon $\mathbb{C}^3$. We use R to denote the representation of chosen $\mathbf{3}$ and calculate the tensor decomposition

$$R \otimes r_i = \bigoplus_j a_{ij}^R r_j \tag{15.2.5}$$

The matrix a_{ij}^R then tells us how many bifundamental chiral multiplets of $SU(N_i)\times$

$SU(N_j)$ there are which transform under the representation $(N_i, \bar{N}_j)$, where $N_i := N|r_i|$. Furthermore, knowing this matter content we can also write down the superpotential whose explicit form is given in (2.7) and (2.8) of Lawrence, Nekrasov and Vafa [76]. We do not need the detailed form thereof but we emphasize that all terms in the superpotential are cubic and there are no quatic term. This condition is necessary for finiteness [82, 76] and we will turn to this fact later.

We can encode the above information into a "**quiver diagram**". Every node i with index $\dim r_i$ in the quiver denotes the gauge group $SU(N_i)$. Then we connect a_{ij}^R arrows from node i to j in order to denote the correpsonding bifundamental chiral multiplet $(N_i, \bar{N}_j)$. When we say that a BBM construction is **consistent** we mean that it gives the same quiver diagram as one obtains from the geometrical probe methods [76].

Now going back to our example $Z_k \times D_{k'}$, its character table is easily written: it is simply the Kronecker product of the character tables of Z_k and $D_{k'}$ (as matrices). We promote (15.2.4) to a double index

$$(a, \chi_i^l)$$

to denote the charaters, where $a = 0, ..., k-1$ and are characters of Z_k (which are simply various kth roots of unity) and χ are the characters of $D_{k'}$ as presented in the previous subsection. We recall that $i = 1$ or 2 and for the former, there are 4 1-dimensional irreps indexed by $l = 1, .., 4$; and for the latter, there are $k' - 1$ 2-dimensional irreps indexed by $l = 1, .., k' - 1$. It is not difficult to see from (15.2.3) that the chosen decomposition should be:

$$\mathbf{3} \longrightarrow (2, \chi_1^1) \oplus (-1, \chi_2^1)$$

The relevant tensor decomposition which gives the quiver is then

$$[(2, \chi_1^1) \oplus (-1, \chi_2^1)] \otimes (a, \chi_i^l) = (a+2, \chi_i^l) \oplus (a-1, \chi_i^l \otimes \chi_2^1), \tag{15.2.6}$$

which is thus reduced to the decompositions as tabulated in the previous subsection.

15.2.3 The Brane Box Model of $Z_k \times D_{k'}$

Now we can use the standard methodology [79, 295, 207] to construct the BBM. The general idea is that for the BBM corresponding to the singularity $\mathbb{C}^3/\Gamma$, we put D-branes whose number is determined by the irreps of Γ into proper grids in Brane Boxes constructed out of NS5-branes. Then the genetal rule of the resulting BBM is that we have gauge group $SU(N_i)$ in every grid and bifundamental chiral multiplets going along the North, East and SouthWest directions. The superpotential can also be read by closing upper or lower triangles in the grids [79]. The quiver diagram is also readily readable from the structure of the BBM (the number and the positions of the branes).

Indeed, in comparison with geometrical methods, because the two quivers (the orbifold quiver and the BBM quiver) seem to arise from two vastly disparate contexts, they need not match a priori. However, by judicious choice of irreps in each grid we can make these two quiver exactly the same; this is what is meant by the **equivalence** between the BBM and orbifold methods. The consistency condition we impose on the BBM for such equivalence is

$$\mathbf{3} \otimes r_i = \bigoplus_{j \in \{\text{North,East,SouthWest}\}} r_j. \tag{15.2.7}$$

Of course we observe this to be precisely (15.2.5) in a different guise.

Now we return to our toy group $Z_k \times D_{k'}$. The grids are furnished by a parallel set of k' NS5-branes with 2 ON^0 planes intersected by k (or $\frac{k}{2}$ when k is even; see explanation below) NS5$'$-branes perpendicular thereto and periodically identified such that k(or $\frac{k}{2}$) $\equiv 0$ as before [295]. This is shown in Figure 15-1. The general brane setup of this form involving 2 sets of NS5-branes and 2 ON-planes we shall call, as mentioned in the introduction, the BBM of **the Z-D Type**.

The irreps are placed in the grids as follows. First we consider the leftmost column. We place a pair of irreps $\{(a, \chi_1^1), (a, \chi_1^3)\}$ at the bottom (here a is some

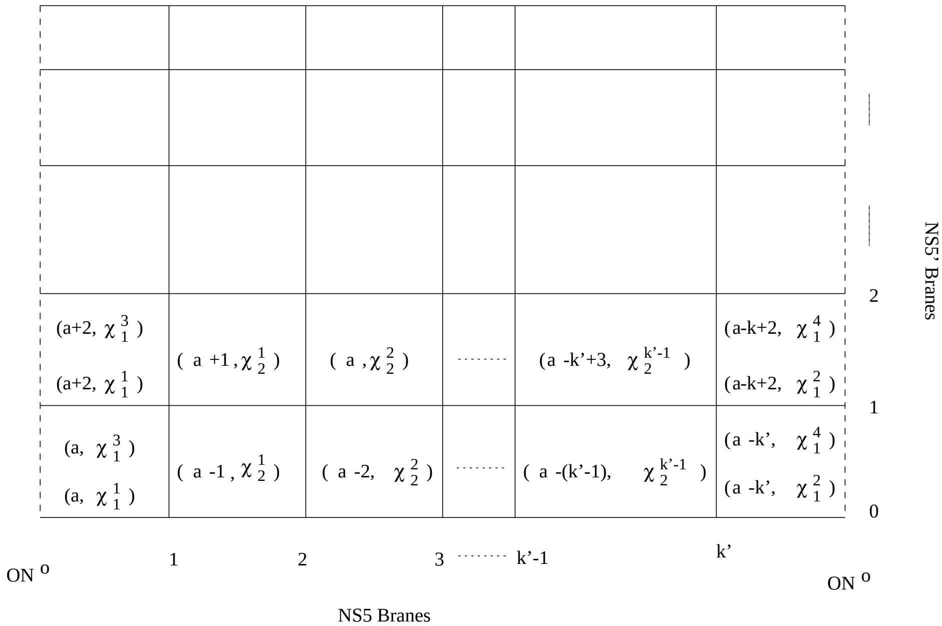


Figure 15-1: The Brane Box Model for $Z_k \times D_{k'}$. Notice that for every step along the vertical direction from the bottom to top, the first index has increment 2, while along the horizontal direction from left to right, the first index has decrement 1 and the second index, increment 1. The vertical direction is also periodically identified so that $k($ or $\frac{k}{2}) \equiv 0$.

constant initial index), then for each incremental grid going up we increase the index a by 2. Now we notice the fact that when k is odd, such an indexing makes one return to the bottom grid after k steps whereas if k is even, it suffices to only make $\frac{k}{2}$ steps before one returns. This means that when k is odd, the periodicity of a is precisely the same as that required by our circular identification of the NS5$'$-branes. However, when k is even it seems that we can not put all irreps into a single BBM. We can circumvent the problem by dividing the irreps (a, χ) into 2 classes depending on the parity of a, each of which gives a BBM consisting of $\frac{k}{2}$ NS5$'$-branes. We should not be surprised at this phenomenon. As we mentioned at the beginning of this section, the matrices (15.2.3) generate a non-faithful representation of the group when k is even (i.e., $\alpha^{\frac{k}{2}}$ gives the same matrix as $\beta^{k'}$). This non-faithful decomposition of $\mathbf{3}$ is what is responsible for breaking the BBM into 2 disjunct parts.

The same phenomenon appears in the $Z_k \times Z_{k'}$ BBM as well. For k even, if we choose the decomposition as $\mathbf{3} \longrightarrow (1, 0) + (0, 1) + (-1, -1)$ we can put all irreps into

kk' grids, however if we choose $\mathbf{3} \longrightarrow (2,0)+(0,1)+(-2,-1)$ we can only construct two BBM's each with $\frac{kk'}{2}$ grids and consisting of one half of the total irreps. Indeed this a general phenomenon which we shall use later:

PROPOSITION 15.2.6 *Non-faithful matrix representations of Γ give rise to corresponding Quiver Graphs which are disconnected.*

Having clarified this subtlety we continue to construct the BBM. We have fixed the content for the leftmost column. Now we turn to the bottom row. Starting from the second column (counting from the left side) we place the irreps $(a-1,\chi_2^1),(a-2,\chi_2^2),...,(a-(k'-1),\chi_2^{k'-1})$ until we reach the right side (i.e., $(a-j,\chi_2^j)$ with $j=1,...k'-1$) just prior to the rightmost column; there we place the pair $\{(a-k',\chi_1^2),(a-k',\chi_1^4)\}$. For the remaining rows we imitate what we did for the leftmost column and increment, for the i-th column, the first index by 2 each time we ascend one row, i.e., $(b,\chi_i^j) \rightarrow (b+2,\chi_i^j)$. The periodicity issues are as discussed above.

Our task now is to check the consistency of the BBM, namely (15.2.7). Let us do so case by case. First we check the grid at the first (leftmost) column at the i-th row; the content there is $\{(a+2i,\chi_1^1),(a+2i,\chi_1^3)\}$. Then (15.2.7) dictates that

$$\begin{array}{c}[(2,\chi_1^1)\oplus(-1,\chi_2^1)]\otimes(a+2i,\chi_1^1 \text{ or } \chi_1^3)\\ =(a+2(i+1),\chi_1^1 \text{ or } \chi_1^3)\oplus((a+2i)-1,\chi_2^1)\end{array}$$

by using the table of tensor decompositions in subsection §15.2.1 and our chosen $\mathbf{3}$ from (15.2.6). Notice that the first term on the right is exactly the content of the box to the North and second term, the content of the East. Therefore consistency is satisfied. Next we check the grid in the second column at the i-th row where $((a+2i)-1,\chi_2^1)$ lives. As above we require

$$\begin{array}{c}[(2,\chi_1^1)\oplus(-1,\chi_2^1)]\otimes((a+2i)-1,\chi_2^1)\\ =((a+2(i+1))-1,\chi_2^1)\oplus((a+2i)-2,\chi_2^2)\oplus(a+2(i-1),\chi_1^1)\oplus(a+2(i-1),\chi_1^3)\end{array}$$

whence we see that the first term corresponds to the grid to the North, and second, East, and the last two, SouthWest. We proceed to check the grid in the $j+1$-th column ($2 \leq j \leq k'-2$) at the i-th row where $((a+2i)-j, \chi_2^j)$ resides. Again (15.2.7) requires that

$$
\begin{array}{c}
[(2,\chi_1^1) \oplus (-1,\chi_2^1)] \otimes ((a+2i)-j, \chi_2^j) \\
= ((a+2(i+1))-j, \chi_2^j) \oplus ((a+2i)-(j+1), \chi_2^{j+1}) \oplus ((a+2(i-1))-(j-1), \chi_2^{j-1})
\end{array}
$$

where again the first term gives the irrep the grid to the North, the second, East and the third, SouthWest. Next we check the grid in the k'-th column and i-th row, where the irrep is $((a+2i)-(k'-1), \chi_2^{k'-1})$. Likewise the requirement is

$$
\begin{array}{c}
[(2,\chi_1^1) \oplus (-1,\chi_2^1)] \otimes ((a+2i)-(k'-1), \chi_2^{k'-1}) \\
= ((a+2(i+1))-(k'-1), \chi_2^{k'-1}) \oplus ((a+2i)-k', \chi_1^2) \\
\oplus((a+2i)-k', \chi_1^4) \oplus ((a+2(i-1))-(k'-2), \chi_2^{k'-2})
\end{array}
$$

whence we see again the first term gives the grid to the North, the second and third, East and the fourth, SouthWest. Finally, for the last (rightmost) column, the grid in the i-th row has $((a+2i)-k', \chi_1^2)$ and $((a+2i)-k', \chi_1^4)$. We demand

$$
\begin{array}{c}
[(2,\chi_1^1) \oplus (-1,\chi_2^1)] \otimes ((a+2i)-k', \chi_1^2 \text{ or } \chi_1^4) \\
= ((a+2(i+1))-k', \chi_1^2 \text{ or } \chi_1^4) \oplus ((a+2(i-1))-(k'-1), \chi_2^{k'-1}))
\end{array}
$$

where the first term gives the grid to the North and the second term, Southwest. So we have finished all checks and our BBM is consistent.

From the structure of this BBM it is very clear that each row gives a $D_{k'}$ quiver and the different rows simply copies it k times according to the Z_k. This repetition hints that there should be some kind of direct product, which is precisely what we have.

15.2.4 The Inverse Problem

Now we address the inverse problem: given a BBM of type Z-D, with k' vertical NS5-branes bounded by 2 ON^0-planes and k horizontal NS5$'$-branes, what is the corresponding orbifold, i.e., the group which acts on $\mathbb{C}^3$? The answer is now very clear: if k is odd we can choose the group $Z_k \times D_{k'}$ or $Z_{2k} \times D_{k'}$ with the action as defined in (15.2.3); if k is even, then we can choose the group to be $Z_{2k} \times D_{k'}$ with the same action.

In this above answer, we have two candidates when k is odd since we recall from discussions in §15.2.3 the vertical direction of the BBM for the group $Z_{2k} \times D_{k'}$ only has periodicity $\frac{k}{2}$ and the BBM separates into two pieces. We must ask ourselves, what is the relation between these two candidates? We notice that (15.2.3) gives an non-faithful representation of the group $Z_{2k} \times D_{k'}$. In fact, it defines a new group of which has the faithful representation given by above matrix form and is a factor group of $Z_{2k} \times D_{k'}$ given by

$$G := (Z_{2k} \times D_{k'})/H, \quad \text{with} \quad H = \langle 1, \alpha^k \beta^{k'} \rangle \tag{15.2.8}$$

In fact G is isomorphic to $Z_k \times D_{k'}$. We can see this by the following arguments. denote the generators of $Z_{2k} \times D_{k'}$ as α, β, γ and those of $Z_k \times D_{k'}$ as $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$. An element of G can be expressed as $[\alpha^a \beta^b \gamma^n] \equiv [\alpha^{a+k} \beta^{b+k'} \gamma^n]$. We then see the homomorphism from G to $Z_k \times D_{k'}$ defined by

$$[\alpha^a \beta^b \gamma^n] \longrightarrow \tilde{\alpha}^a \tilde{\beta}^{ak'+b} \tilde{\gamma}^n$$

is in fact an isomorphism (we see that $[\alpha^a \beta^b \gamma^n]$ and $[\alpha^{a+k} \beta^{b+k'} \gamma^n]$ are mapped to same element as required; in proving this the $k =$ odd condition is crucial).

We see therefore that given the data from the BBM, viz., k and k', we can uniquely find the $\mathbb{C}^3$ orbifold singularity and our inverse problem is well-defined.

15.3 The General Twisted Case

We have found in the previous chapter that the group $Z_k * D_{k'}$ (in which we called $Z_k \times D_{k'}$) defined by (15.1.1) can be written in another form as $Z_k \rtimes D_{\frac{kk'}{\gcd(k,2k')}}$ where it becomes an (internal) semidirect product. We would like to know how the former, which is a special case of what we shall call a **twisted** group[4] is related to the direct product example, which we shall call the **untwisted** case, upon which we expounded in the previous section.

The key relation which describes the semidirect product structure was shown [295] to be $\alpha\gamma = \beta^{\frac{2k'}{\gcd(k,2k')}}\gamma\alpha$. This is highly suggestive and hints us to define a one-parameter family of groups[5] $G(a) := \{Z_k \rtimes D_d\}$ whose presentations are

$$\alpha\beta = \beta\alpha, \qquad \alpha\gamma = \beta^a\gamma\alpha. \tag{15.3.9}$$

When the parameter $a = 0$, we have $G(0) = Z_k \times D_{k'}$ as discussed extensively in the previous section. Also, when $a = \frac{kk'}{\gcd(k,2k')}$, $G(a)$ is the group $Z * D$ treated in the previous chapter. We are concerned with members of $\{G(a)\}$ that satisfy the BBM conditions and though indeed this family may not exhaust the list of all groups that satisfy those conditions they do provide an illustrative subclass.

15.3.1 Preserving the Irreps of D_d

We see that the first of the BBM conditions is trivially satisfied by our definition (15.3.9 of $G(a) := Z_k \rtimes D_d$. Therefore we now move onto the second condition. We propose that $G(a)$ preserves the structure of the irreps of the D_d factor if a is even. The analysis had been given in detail [295] so here we only review briefly. Deducing from (15.3.9) the relation, for $b \in \mathbb{Z}$,

$$\alpha(\beta^b\gamma)\alpha^{-1} = \beta^{b+a}\gamma,$$

we see that $\beta^b\gamma$ and $\beta^{b+a}\gamma$ belong to the same conjugacy class after promoting D_d to the semidirect product $Z_k \rtimes D_d$. Now we recall from subsection §15.2.1 that the conjugacy classes of D_d are β^0, β^d, $\beta^{\pm p}(p \neq 0, d)$, $\gamma\beta^{\text{even}}$ and $\gamma\beta^{\text{odd}}$. Therefore we see

[4] As mentioned in the Nomenclature section, $*$ generically denotes twisted products of groups.

[5] We note that this is unambiguously the semi-direct product $\rtimes$: defining the two subgroups $D := \langle\beta, \gamma\rangle$ and $Z := \langle\alpha\rangle$, we see that $G(a) = DZ$ as cosets, that $D \triangleleft G(a)$ and $D \cap Z = 1$, whereby all the axioms of semi-directness are obeyed.

that when $a =$ even, the conjugacy structure of D_d is preserved since therein $\beta^b\gamma$ and $\beta^{b+a}\gamma$, which we saw above belong to same conjugate class in D_d, are also in the same conjugacy class in $G(a)$ and everything is fine. However, when $a =$ odd, they live in two different conjugacy classes at the level of D_d but in the same conjugacy class in $G(a)$ whence violating the second condition. Therefore a has to be even.

15.3.2 The Three Dimensional Representation

Now we come to the most important part of finding the 3-dimensional representations for $G(a)$, i.e., condition 3. We start with the following form for the generators

$$\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{2d} & 0 \\ 0 & 0 & \omega_{2d}^{-1} \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \tag{15.3.10}$$

and

$$\alpha = \begin{pmatrix} \omega_k^{-(x+y)} & 0 & 0 \\ 0 & \omega_k^x & 0 \\ 0 & 0 & \omega_k^y \end{pmatrix} \tag{15.3.11}$$

where $x, y \in \mathbb{Z}$ are yet undetermined integers (notice that the form (15.3.11) is fixed by the matrix (15.3.10) of β and the algebraic relation $\alpha\beta = \beta\alpha$). Using the defining relations (15.3.9) of $G(a)$, i.e relation $\alpha\gamma = \beta^a\gamma\alpha$, we immediately have the following constraint on x and y:

$$\omega_k^{x-y} = \omega_{2d}^a \tag{15.3.12}$$

which has integer solutions [6] only when

[6]Since (15.3.12) implies $\frac{2\pi(x-y)}{k} - \frac{2\pi a}{2d} = 2\pi\mathbb{Z}$, we are concerned with Diophantine equations of the form $\frac{p}{q} - \frac{m}{n} \in \mathbb{Z}$. This in turn requires that $np = mq \Rightarrow q = \frac{nl}{\gcd(m,n)}$, $l \in \mathbb{Z}$ by diving through by the greatest common divisor of m and n. Upon back-substitution, we arrive at $p = \frac{ml}{\gcd(m,n)}$.

$$k = (\frac{2d}{\delta})l \qquad l \in \mathbb{Z} \text{ and } \delta := \gcd(a, 2d) \tag{15.3.13}$$

with the actual solution being

$$x - y = \frac{a}{\delta}l.$$

Equation (15.3.13) is a nontrivial condition which signifiess that for arbitrary $k, 2d, a$, the third of the BBM conditions may be violated, and the solution, not found. This shows that even though $G(a = \text{even})$ satisfies the first two of the BBM conditions, they can not in general be applied to construct BBM's of Type Z-D unless (15.3.13) is also respected. However, before starting the general discussion of those cases of $Z * D$ where (15.3.13) is satisfied, let us first see how the group treated before [295] indeed satisfies this condition.

For $Z_k * D_{k'}$ in the previous chapter and defined by (15.1.1), let $\delta_1 := \gcd(k, 2k')$. We have $d = \frac{kk'}{\delta_1}$, $a = \frac{2k'}{\delta_1}$ from Proposition (3.1) in that chapter. Therefore $\delta = \gcd(a, 2d) = a$ and $k = \frac{2d}{\delta}$ so that (15.3.13) is satisfied with $l = 1$ and we have the solution $x - y = 1$. Now if we choose $y = 0$, then we have

$$\alpha = \begin{pmatrix} \omega_k^{-1} & 0 & 0 \\ 0 & \omega_k^1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15.3.14}$$

Combining with the matrices in (15.3.10), we see that they generate a faithful 3-dimensional representation of $Z_k * D_{k'}$. It is easy to see that what they generate is in fact isomorphic to a group with matrix generators, as given in (15.2.3),

$$\alpha^{-1} = \begin{pmatrix} \omega_{2k}^{-2} & 0 & 0 \\ 0 & \omega_{2k}^1 & 0 \\ 0 & 0 & \omega_{2k}^1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_{2d} & 0 \\ 0 & 0 & \omega_{2d}^{-1} \end{pmatrix} \qquad \gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix} \tag{15.3.15}$$

by noticing that $\alpha^{-1}\beta^{\frac{k'}{\delta}}$ in (15.3.15) is precisely (15.3.14). But this is simply a non-faithful representation of $Z_{2k} \times D_{d=\frac{kk'}{\gcd(k,2k')}}$, our direct product example! Furthermore, when $k = odd$, by recalling the results of §15.2.4 we conclude in fact that the group $Z_k * D_{k'}$ is isomorphic to $Z_k \times D_d$. However, for $k =$ even, although $Z_k * D_{k'}$ is still embeddable into $Z_{2k} \times D_{d=\frac{kk'}{\gcd(k,2k')}}$ with a non-faithful representation (15.2.3), it is not isomorphic to $Z_k \times D_d$ and the BBM thereof corresponds to an intrintically twisted case (and unlike when $k =$ odd where it is actually isomorphic to a direct product group). We emphasize here an obvious but crucial fact exemplified by (15.2.8): *non-faithful representations of a group A can be considered as the faithful representation of a new group B obtained by quotienting an appropriate normal subgroup of A.* This is what is happening above. This explains also why we have succeeded [295] in constructing the BBM only when we wrote $Z_k * D_{k'}$ in the form $Z_k \rtimes D_{d=\frac{kk'}{\gcd(k,2k')}}$.

Now let us discuss the general case. We recall from the previous subsection that a has to be even; we thus let $a := 2m$. With this definition, putting (15.3.12) into (15.3.11,) we obtain for the quantity $\alpha\beta^{-m}$:

$$\tilde{\alpha} = \alpha\beta^{-m} = \begin{pmatrix} \omega_k^{-2y}\omega_{2d}^{2m} & 0 & 0 \\ 0 & \omega_k^{y}\omega_{2d}^{-m} & 0 \\ 0 & 0 & \omega_k^{y}\omega_{2d}^{-m} \end{pmatrix} \tag{15.3.16}$$

This $\tilde{\alpha}$ generates a cyclic group $Z_{\tilde{k}}$ and combined with (15.3.10) gives the direct product group of $Z_{\tilde{k}} \times D_d$, but with a non-faithful representation as in (15.2.3). Therefore for the general twisted case, we can obtain the BBM of Z-D type of $G(a)$ by imbedding $G(a)$ into a larger group $Z_{\tilde{k}} \times D_d$ which is a direct product just like we did for $Z_k * D_{k'}$ embeding to $Z_k \rtimes D_{d=\frac{kk'}{\gcd(k,2k')}}$ two paragraphs before, and for which, by our etude in §15.2, a consistent BBM can always be established. However, we need to emphasize that in general such an embedding (15.3.16) gives non-faithful representations so that the quiver diagram of the twisted group will be a union of disconnected pieces, as demanded by Proposition 15.2.6, each of which corresponds to a Type Z-D BBM. We summarise these results by stating

PROPOSITION 15.3.7 *The group $G(a) := Z_k * D_d$ satisfies the BBM conditions if a* *is even and the relation (15.3.13) is obeyed. In this case its matrices actually furnish a non-faithful representation of a direct product* $\tilde{G} := Z_{\tilde{k}} \times D_d$ *and hence affords a BBM*[7] *of Type Z-D.*

This action of $G(a) \hookrightarrow \tilde{G}$ is what we mean by embedding. We conclude by saying that the simple example of §15.2 where the BBM is constructed for untwisted (direct-product) groups is in fact general and Type Z-D BBM's can be obtained for twisted groups by imbedding into such direct-product structures.

15.4 A New Class of $SU(3)$ Quivers

It would be nice to see whether the ideas presented in the above sections can be generalised to give the BBM of other types such as Type Z-E, Z-d or D-E whose definitions are obvious. Moreover, E refers to the exceptional groups $\widehat{E_{6,7,8}}$ and d the ordinary dihedral group. Indeed, we must first have the brane setups for these groups. Unfortunately as of yet the E groups still remain elusive. However we will give an account of the ordinary dihedral groups and the quiver theory thereof, as well as the ordinary E groups from a new perspectively from an earlier work [292]. These shall give us a new class of $SU(3)$ quivers.

We note that, as pointed out [292], the ordinary di-, tetra-, octa- and iscosa-hedral groups (or d, $E_6, 7, 8$ respectively) are excluded from the classification of the discrete finite subgroups of $SU(2)$ because they in fact belong to the centre-modded group $SO(3) \cong SU(2)/\mathbb{Z}_2$. However due to the obvious embedding $SO(3) \hookrightarrow SU(3)$, these are all actually $SU(3)$ subgroups. Now the d-groups were not discussed before [292] because they did not have non-trivial 3-dimensional irreps and are not considered as non-trivial (i.e., they are fundamentally 2-dimensional collineation groups) in the standard classification of $SU(3)$ subgroups; or in a mathemtical language [294, 89], they are transitives. Moreover, E_6 is precisely what was called $\Delta(3 \times 2^2)$ earlier [292], E_7, $\Delta(6 \times 2^2)$ and E_8, Σ_{60} and were discussed there. However we shall here see all

[7] *Though possibly disconnected with the number of components depending on the order of an Abelian subgroup* $H \triangleleft \tilde{G}$.

these groups together under a new light, especially the ordinary dihedral group to which we now turn.

15.4.1 The Group $d_{k'}$

The group is defined as

$$\beta^{k'} = \gamma^2 = 1, \qquad \beta\gamma = \gamma\beta^{-1},$$

and differs from its binary cousin $D_{k'}$ in subsection §15.2.1 only by having the orders of β, γ being one half of the latter. Indeed, defining the normal subgroup $H := \{1, \beta^{k'}\} \triangleleft D_{k'}$ we have

$$d_{k'} \cong D_{k'}/H.$$

We can subsequently obtain the character table of $d_{k'}$ from that of $D_{k'}$ by using the theory of subduced representations, or simply by keeping all the irreps of $D_{k'}$ which are invariant under the equivalence by H. The action of H depends on the parity of k'. When it is even, the two conjugacy classes $(\gamma\beta^{even})$ and $(\gamma\beta^{odd})$ remain separate. Furthermore, the four 1-dimensional irreps are invariant while for the 2-dimensionals we must restrict the index l as defined in subsection §15.2.1 to $l = 2, 4, 6, ..., k'-2$ so as to observe the fact that the two conjugacy classes $\{\beta^a, \beta^{-a}\}$ and $\{\beta^{k-a}, \beta^{a-k}\}$ combine into a single one. All in all, we have 4 1-dimensional irreps and $\frac{k'-2}{2}$ 2-dimensionals. On the other hand, for k' odd, we have the two classes $(\gamma\beta^{even})$ and $(\gamma\beta^{odd})$ collapsing into a single one, whereby we can only keep χ^1, χ^3 in the 1-dimensionals and restrict $l = 2, 4, 6, ..., k' - 1$ for the 2-dimensionals. Here we have a total of 2 1-dimensional irreps and $\frac{k'-1}{2}$ 2-dimensionals.

In summary then, the character tables are as follows:

	1	2	2	$\cdots$	2	n
Γ_1	1	1	1	$\cdots$	1	1
Γ_2	1	1	1	$\cdots$	1	-1
Γ_3	2	$2\cos\phi$	$2\cos 2\phi$	$\cdots$	$2\cos m\phi$	0
Γ_4	2	$2\cos 2\phi$	$2\cos 4\phi$	$\cdots$	$2\cos 2m\phi$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\Gamma_{\frac{k'+3}{2}}$	2	$2\cos m\phi$	$2\cos 2m\phi$	$\cdots$	$2\cos m^2\phi$	0

k' odd

$m = \frac{k'-1}{2}$

$\phi = \frac{2\pi}{k'}$

	1	2	2	$\cdots$	2	1	$n/2$	$n/2$
Γ_1	1	1	1	$\cdots$	1	1	1	1
Γ_2	1	1	1	$\cdots$	1	1	-1	-1
Γ_3	1	-1	1	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	1	-1
Γ_4	1	-1	1	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	-1	1
Γ_5	2	$2\cos\phi$	$2\cos 2\phi$	$\cdots$	$2\cos(m-1)\phi$	$2\cos m\phi$	0	0
Γ_6	2	$2\cos 2\phi$	$2\cos 4\phi$	$\cdots$	$2\cos 2(m-1)\phi$	$2\cos 2m\phi$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Gamma_{\frac{k'+6}{2}}$	2	$2\cos(m-1)\phi$	$2\cos 2(m-1)\phi$	$\cdots$	$2\cos(m-1)^2\phi$	$2\cos m(m-1)\phi$	0	0

for k' even, $m = \frac{k'}{2}$ and $\phi = \frac{2\pi}{k'}$.

15.4.2 A New Set of Quivers

Now we must choose an appropriate $SU(3)$ decomposition of the **3** for our group in order to make physical sense for the bifundamentals. The choice is

$$\mathbf{3} \longrightarrow \chi_1^3 + \chi_2^2.$$

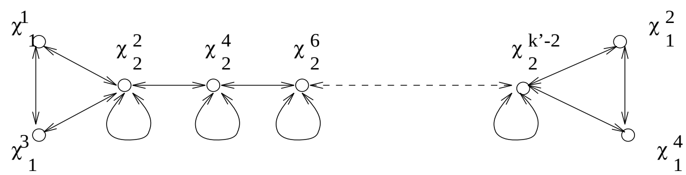


Figure 15-2: The quiver diagram for $d_{k=\text{even}}$. Here the notation of the irreps placed on the nodes is borrowed from D_k in §15.2.1. Notice that it gives a finite theory with non-chiral matter content.

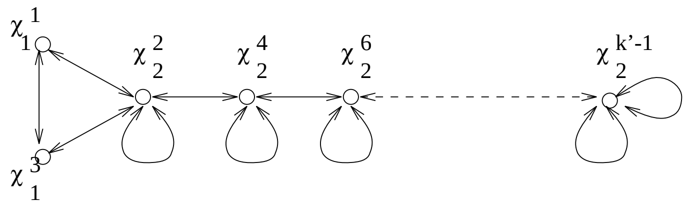


Figure 15-3: The quiver diagram for $d_{k=\text{odd}}$. Here again we use the notation of the irreps of D_k to index the nodes. Notice that the theory is again finite and non-chiral.

Here, we borrow the notation of the irreps of d_k from D_k in §15.2.1. The relationship between the irreps of the two is discussed in the previous subsection. The advantage of using this notation is that we can readily use the tabulated tensor decompositions of D_k in §15.2.1. With this chosen decomposition, we can immediately arrive at the matter matrices a_{ij} and subsequent quiver diagrams. The $k' =$ even case gives a quiver which is very much like the affine $\widehat{D_{k'+2}}$ Dynkin Diagram, differing only at the two ends, where the nodes corresponding to the 1-dimensionals are joined, as well as the existence of self-joined nodes. This is of course almost what one would expect from an $\mathcal{N} = 2$ theory obtained from the binary dihedral group as a finite subgroup of $SU(2)$; this clearly reflects the intimate relationship between the ordinary and binary dihedral groups. The quiver is shown in Figure 15-2. On the other hand, for k' odd, we have a quiver which looks like an ordinary $D_{k'+1}$ Dynkin Diagram with 1 extra line joining the 1-nodes as well as self-adjoints. This issue of the dichotomous appearance of affine and ordinary Dynkin graphs of the D-series in brane setups has been raised before [207, 295]. The diagram for k' odd is shown in Figure 15-3.

For completeness and comparison we hereby also include the 3 exceptional groups of $SO(3) \subset SU(3)$. For these, we must choose the **3** to be the unique (up to auto-

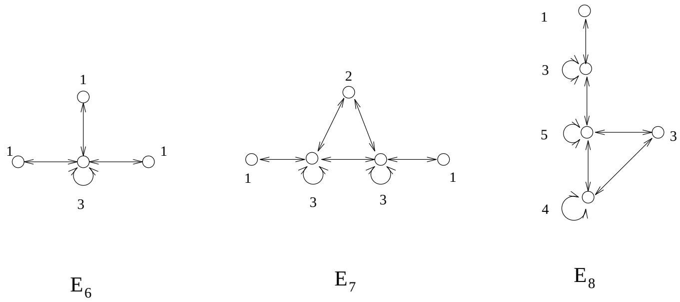


Figure 15-4: The quiver diagrams for $E_6 = A_4 = \Delta(3 \times 2^2)$, $E_7 = S_4 = \Delta(6 \times 2^2)$ and $E_8 = A_5 = \Sigma_{60}$. The theories are finite and non-chiral.

morphism among the conjugacy classes) 3-dimensional irrep. Any other decompostion leads to non-faithful representations of the action and subsequently, by our rule discussed earlier, to disconnected quivers. This is why when they were considered as $SU(2)/\mathbb{Z}_2$ groups with $\mathbf{3} \to \mathbf{1} \oplus \mathbf{2}$ chosen, uninteresting and disconnected quivers were obtained [292]. Now under this new light, we present the quivers for these 3 groups in Figure 15-4.

There are two points worth emphasising. All the above quivers correspond to theories which are finite and non-chiral. By **finite** we mean the condition [76] for anomaly cancelation, that the matter matrix a_{ij}^R must satisfy

$$\sum_j a_{ij}^R \dim(r_j) = \sum_j a_{ji}^R \dim(r_j)$$

What this mean graphically is that for each node, the sum of the indices of all the neighbouring nodes flowing thereto (i.e., having arrows pointing to it) must equal to the sum of those flowing therefrom, and must in fact, for an $\mathcal{N} = 1$ theory, be equal to 3 times the index for the node itself. We observe that this condition is satisfied for all the quivers presented in Figure 15-3 to Figure 15-4.

On the other hand by **non-chiral** we mean that for every bi-fundamental chiral multiplet $(N_i, \bar{N}_j)$ there exists a companion $(N_j, \bar{N}_i)$ (such that the two combine together to give a bi-fundamental hypermultiplet in the sense of $\mathcal{N} = 2$). Graphically, this dictates that for each arrow between two nodes there exists another in the opposite direction, i.e., the quiver graph is unoriented. Strangely enough, non-chiral matter content is a trademark for $\mathcal{N} = 2$ theories, obtained from $\mathbb{C}^2/\Gamma \subset SU(2)$ singularities, while $\mathcal{N} = 1$ usually affords chiral (i.e., oriented quivers) theories. We have thus arrived at a class of finite, non-chiral $\mathcal{N} = 1$ super Yang-Mills theories. This is not that peculiar because all these groups belong to $SO(3)$ and thus have real representations; the reality compel the existence of complex conjugate pairs. The more interesting fact is that these groups give quivers that are in some sense in between the generic non-chiral $SU(2)$ and chiral $SU(3)$ quiver theories. Therefore we expect that the corresposnding gauge theory will have better properties, or have more control, under the evolution along some energy scale.

15.4.3 An Interesting Observation

Having obtained a new quiver, for the group d_k, it is natural to ask what is the corresponding brane setup. Furthermore, if we can realize such a brane setup, can we apply the ideas in the previous sections to realize the BBM of Type Z-d? We regrettably have no answers at this stage as attempts at the brane setup have met great difficulty. We do, however, have an interesting brane configuration which gives the correct matter content of d_k but has a different superpotential. The subtle point is that d_k gives only $\mathcal{N} = 1$ supersymmetry and unlike $\mathcal{N} = 2$, one must specify both the matter content and the superpotential. Two theories with the same matter content but different superpotential usually have different low-energy behavior.

We now discuss the brane configuration connected with d_k, which turns out to be a rotated version of the configuration for D_k as given by Kapustin [83] (related examples [295, 165] on how rotating branes breaks supersymmetry further may be found). In particular we rotate all NS5-branes (along direction (12345)) between the two ON^0-plane as drawn in Figure 1 of Kapustin [83] to NS5$'$-branes (along direction (12389)). The setup is shown in Figure 15-5. Let us analyse this brane setup more carefully. First when we end D4-branes (extended along direction (1236)) on the ON^0-plane, they can have two different charges: positive or negtive. With the definition

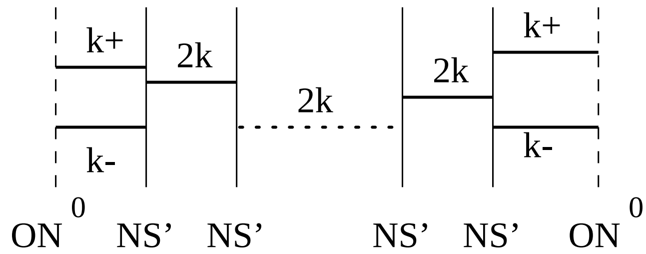


Figure 15-5: The brane configuration which gives the same matter content as the $d_{k=\text{even}}$ quiver.

of the matrix

$$\Omega = \begin{pmatrix} 1_{k+\times k+} & 0 \\ 0 & -1_{k-\times k-} \end{pmatrix},$$

the projection on the Chan-Paton matrix of the D4-branes is as follows. The scalar fields in the D4-worldvolume are projected as

$$\phi^\alpha = \Omega\phi^\alpha\Omega^{-1} \quad \text{and} \quad \phi^i = -\Omega\phi^i\Omega^{-1}$$

where α runs from 4 to 5 and describes the oscillations of the D4-branes in the directions parallel to the ON^0-plane while i runs from 7 to 9 and describes the transverse oscillations. If we write the scalars as matrice in block form, the remaining scalars that survive the projection are

$$\phi^\alpha = \begin{pmatrix} U_{k+\times k+} & 0 \\ 0 & U_{k-\times k-} \end{pmatrix} \quad \text{and} \quad \phi^i = \begin{pmatrix} 0 & U_{k+\times k-} \\ U_{k-\times k+} & 0 \end{pmatrix}.$$

From these we immediately see that ϕ^α give scalars in the adjoint representation and ϕ^i, in the bifundamental representation. Next we consider the projection conditions when we end the other side of our D4-brane on the NS-brane. If we choose the NS5-brane to extend along (12345), then the scalars ϕ^α will be kept while ϕ^i will be projected out and we would have an $\mathcal{N}=2$ D_k quiver (see Figure 15-6).

However, if we choose the NS5-branes to extend along (12389), then ϕ^α and $\phi^{i=7}$ will be projected out while $\phi^{i=8,9}$ will be kept. It is in this case that we see immediately that we obtain the same matter content as one would have from a $d_{k=\text{even}}$ orbifold

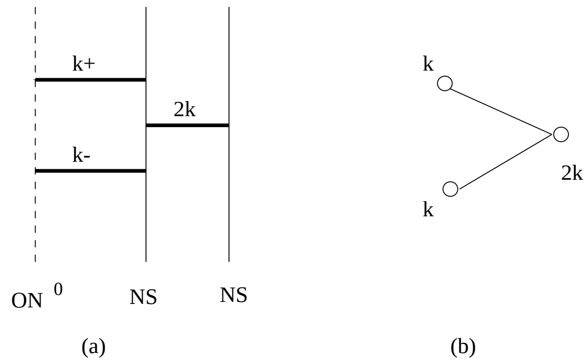


Figure 15-6: (a). The brane configuration of the projection using NS5-branes. (b). The quiver diagram for the brane configuration in (a).

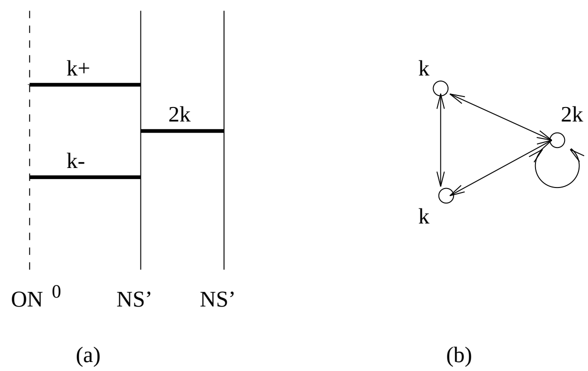


Figure 15-7: (a). The brane configuration of projection using NS5′-branes. (b). The quiver diagram for the brane configuration in (a).

discussed in the previous subsection (see Figure 15-7).

Now we explain why the above brane configuration, though giving the same matter content as the $d_{k=\text{even}}$, is insufficient to describe the full theory. The setup in Figure 15-5 is obtained by the rotation of NS-branes to NS′-branes; in this process of rotation, in general we change the geometry from an orbifold to a conifold. In other words, by rotating, we break the $\mathcal{N} = 2$ theory to $\mathcal{N} = 1$ by giving masses to scalars in the $\mathcal{N} = 2$ vector-multiplet. After integrating out the massive adjoint scalar in low energies, we usually get quartic terms in the superpotential (for more detailed discussion of rotation see Erlich et al. [165]). Indeed Klebanov and Witten [212] have explained this point carefully and shows that the quartic terms will exist even at the limiting case when the angle of rotation is $\frac{\pi}{2}$ and the NS5-branes become NS5′-branes. On the other hand, the superpotential for the orbifold singularity of d_k contains only cubic terms as required by Lawrence et al. [76] and as we emphasized in §15.2. It still remains an intersting problem to construct consistent brane setups for d_k that also has the right superpotential; this would give us one further stride toward attacking non-Abelian brane configurations.

15.5 Conclusions and Prospects

As inspired by the Brane Box Model (BBM) constructions [295] for the group $Z_k * D_{k'}$ generated by (15.1.1), we have discussed in this chapter a class of groups which are generalisations thereof. These groups we have called the twisted groups (that satisfy BBM conditions). In particular we have analysed at great length, the simplest memeber of this class, namely the direct product $Z_k \times D_d$, focusing on how the quiver theory, the BBM construction as well as the inverse problem (of recovering the group by reading the brane setup) may be established. The brane configuration for such an example, as in Figure 15-1, we have called a **BBM of Type Z-D**; consisting generically of a grid of NS5-branes with the horizontal direction bounded by 2 ON-planes and the vertical direction periodically identified. We have also addressed, as given in Proposition 15.2.6 the issue of how non-faithful representations lead to disconnected quivers graphs, or in other words several disjunct pieces of the BBM setup.

What is remarkable is that the twisted groups, of which the one in the previous chapter is a special case, can under certain circumstances be embedded into a direct product structure (by actually furnishing a non-faithful representation thereof). This makes our naïve example of $Z_k \times D_d$ actually possess great generality as the twisted cases untwist themselve by embedding into this, in a sense, universal cover in the fashion of Proposition 15.3.7. What we hope is that this technique may be extended to address more non-Abelian singularities of $\mathbb{C}^3$, whereby the generic finite discrete group $G \subset SU(3)$ maybe untwisted into a direct-product cover. In order to do so, it seems that G needs to obey a set of what we call **BBM conditions**. We state these in a daring generality: (1) That G maybe written as a semi-direct product $A \rtimes B$, (2) that the structure of the irreps of G preserves those of the factors A and B and (3) that there exists a decomposition into the irreps of G consistent with the unitarity and determinant 1 constraints of $SU(3)$.

Indeed it is projected and hoped, with reserved optimism, that if A, B are $SU(2)$ subgroups for which a brane setup is known, the techniques presented above may inductively promote the setup to a BBM (or perhaps even brane cube for $SU(4)$ singularities). Bearing this in mind, we proceeded further to study more examples, hoping to attack for example, BBM's of the Z-d type where d is the ordinary dihedral group. Therefrom arose our interest in the ordinary groups $d, E_{6,7,8}$ as finite subgroups of $SO(3) \subset SU(3)$. These gave us a new class of quiver theories which have $\mathcal{N} = 1$ but non-chiral matter content. Brane setups that reproduce the matter content, but unfortunately not the superpotential, have been established for the ordinary dihedral groups. These give an interesting brane configuration involving rotating NS5-brane with respect to ON-planes.

Of course much work remains to be done. In addition to finding the complete brane setups that reproduce the ordinary dihedral quiver as well as superpotential, we have yet to clarify the BBM conditions for groups in general and head toward that beacon of brane realisations of non-Abelian orbifold theories.

Chapter 16

Orbifolds VII: Stepwise Projection, or Towards Brane Setups for Generic Orbifold Singularities

Synopsis

Having addressed, in the previous two chapters, a wide class of non-Abelian orbifolds in dimension 3, let us see how much further can we go.

The construction of brane setups for the exceptional series $E_{6,7,8}$ of $SU(2)$ orbifolds remains an ever-haunting conundrum. Motivated by techniques in some works by Muto on non-Abelian $SU(3)$ orbifolds, we here provide an algorithmic outlook, a method which we call stepwise projection, that may shed some light on this puzzle. We exemplify this method, consisting of transformation rules for obtaining complex quivers and brane setups from more elementary ones, to the cases of the D-series and E_6 finite subgroups of $SU(2)$. Furthermore, we demonstrate the generality of the stepwise procedure by appealing to Frøbenius' theory of Induced Representations. Our algorithm suggests the existence of generalisations of the orientifold plane in string theory [302].

16.1 Introduction

It is by now a well-known fact that a stack of n parallel coincident D3-branes has on its world-volume, an $\mathcal{N} = 4$, four-dimensional supersymmetric $U(n)$ gauge theory. Placing such a stack at an orbifold singularity of the form $\mathbb{C}^k/\{\Gamma \subset SU(k)\}$ reduces the supersymmetry to $\mathcal{N} = 2, 1$ and 0, respectively for $k = 2, 3$ and 4, and the gauge group is broken down to a product of $U(n_i)$'s [69, 171, 76].

Alternatively, one could realize the gauge theory living on D-branes by the so-called Brane Setups [66, 63] (or "Comic Strips" as dubbed by Rabinovici [213]) where D-branes are stretched between NS5-branes and orientifold planes. Since these two methods of orbifold projections and brane setups provide the same gauge theory living on D-branes, there should exist some kind of duality to explain the connection between them.

Indeed, we know now that by T-duality one can map D-branes probing certain classes of orbifolds to brane configurations. For example, the two-dimensional orbifold $\mathbb{C}^2/\{\mathbb{Z}_k \subset SU(2)\}$, also known as an ALE singularity of type A_{k-1}, is mapped into a circle of k NS-branes (the so-called elliptic model) after proper T-duality transformations. Such a mapping is easily generalized to some other cases, such as the three-dimensional orbifold $\mathbb{C}^3/\{\mathbb{Z}_k \times \mathbb{Z}_l \subset SU(3)\}$ being mapped to the so-named Brane Box Model [78, 79] or the four-dimensional case of $\mathbb{C}^4/\{\mathbb{Z}_k \times \mathbb{Z}_l \times \mathbb{Z}_m \subset SU(4)\}$ being mapped to the brane cube model [163]. With the help of orientifold planes, we can T-dualise $\mathbb{C}^2/\{D_k \subset SU(2)\}$ to a brane configuration with ON-planes [206, 83], or $\mathbb{C}^3/\{\mathbb{Z}_k \times D_l \subset SU(3)\}$ to brane-box-like models with ON-planes [295, 296].

A further step was undertaken by Muto [141, 172, 214] where an attempt was made to establish the brane setup which corresponds to the three-dimensional non-Abelian orbifolds $\mathbb{C}^3/\{\Gamma \subset SU(3)\}$ with $\Gamma = \Delta(3n^2)$ and $\Delta(6n^2)$. The key idea was to arrive at these theories by judiciously quotienting the well-known orbifold $\mathbb{C}^3/\{\mathbb{Z}_k \times \mathbb{Z}_l \subset SU(3)\}$ whose brane configuration is the Brane Box Model. In the process of this quotienting, a non-trivial $\mathbb{Z}_3$ action on the brane box is required. Though mathematically obtaining the quivers of the former from those of the latter seems perfectly sound, such a $\mathbb{Z}_3$ action appears to be an unfamiliar symmetry in string theory. We shall briefly address this point later.

Now, with the exception of the above list of examples, there have been no other successful brane setups for the myriad of orbifolds in dimension two, three and four. Since we believe that the methods of orbifold projection and brane configurations are equivalent to each other in giving D-brane world-volume gauge theories, finding the T-duality mappings for arbitrary orbifolds is of great interest.

The present chapter is a small step toward such an aim. In particular, we will present a so-called **stepwise projection** algorithm which attempts to systematize the quotienting idea of Muto, and, as we hope, to give hints on the brane construction of generic orbifolds.

We shall chiefly focus on the orbifold projections by the $SU(2)$ discrete subgroups D_k and E_6 in relation to $\mathbb{Z}_n$. Thereafter, we shall evoke some theorems on induced representations which justify why our algorithm of stepwise projection should at least work in general mathematically. In particular, we will first demonstrate how the algorithm gives the quiver of D_k from that of Z_{2k}. We then interpret this mathematical projection physically as precisely the orientifold projection, whereby arriving at the brane setup of D_k from that of $\mathbb{Z}_{2k}$, both of which are well-known and hence giving us a consistency check.

Next we apply the same idea to E_6. We find that one can construct its quiver from that of $\mathbb{Z}_6$ or D_2 by an appropriate $\mathbb{Z}_3$ action. This is slightly mysterious to us physically as it requires a $\mathbb{Z}_3$ symmetry in string theory which we could use to quotient out the $\mathbb{Z}_6$ brane setup; such a symmetry we do not know at this moment. However, in comparison with Muto's work, our $\mathbb{Z}_3$ action and the $\mathbb{Z}_3$ investigated by Muto in light of the Δ series of $SU(3)$, hint that there might be some objects in string theory which provide a $\mathbb{Z}_3$ action, analogous to the orientifold giving a $\mathbb{Z}_2$, and which we could use on the known brane setups to establish those yet unknown, such as those corresponding to the orbifolds of the exceptional series.

The organisation of the chapter is as follows. In §2 we review the technique of orbifold projections in an explicit matrix language before moving on to §3 to present our stepwise projection algorithm. In particular, §3.1 will demonstrate how to obtain the D_k quiver from the $\mathbb{Z}_{2k}$ quiver, §3.2 and §3.3 will show how to get that of E_6 from those of D_2 and $\mathbb{Z}_6$ respectively. We finish with comments on the algorithm in §4. We will use induced representation theory in §4.1 to prove the validity of our methods and in §4.2 we will address how the present work may be used as a step toward the illustrious goal of obtaining brane setups for the generic orbifold singularity.

During the preparation of the manuscript, it has come to our attention that independent and variant forms of the method have been in germination [216, 217]; we sincerely hope that our systematic treatment of the procedure may be of some utility thereto.

Nomenclature

Unless otherwise stated we shall adhere to the convention that Γ refers to a discrete subgroup of $SU(n)$ (i.e., a finite collineation group), that $\langle x_1, .., x_n \rangle$ is a finite group generated by $\{x_1, .., x_n\}$, that $|\Gamma|$ is the order of the group Γ, that D_k is the binary dihedral group of order $4k$, that $E_{6,7,8}$ are the binary exceptional subgroups of $SU(2)$, and that $R^{\bullet}_{G(n)}(x)$ is a representation of the element $x \in G$ of dimension n with $\bullet$ denoting properties such as regularity, irreducibility, etc., and/or simply a label. Moreover, S^T shall denote the transpose of the matrix S and $A \otimes B$ is the tensor product of matrices A and B with block matrix elements $A_{ij}B$. Finally we frequently use the Pauli matrices $\{\sigma_i, i = 1, 2, 3\}$ as well as $\mathbb{1}_N$ for the $N \times N$ identity matrix. We emphasise here that the notation for the binary groups differs from the previous chapters in the exclusion of $\widehat{}$ and in the convention for the sub-index of the binary dihedral group.

16.2 A Review on Orbifold Projections

The general methodology of how the finite group structure of the orbifold projects the gauge theory has been formulated in [76]. The complete lists of two and three dimensional cases have been treated respectively in [69, 171] and [292, 141] as well as the four dimensional case in [294]. For the sake of our forth-coming discussion, we shall not use the nomenclature in [76, 292, 295, 296] where recourse to McKay's Theorem and abstractions to representation theory are taken. Instead, we shall adhere to the notations in [171] and explicitly indicate what physical fields survive the orbifold projection.

Throughout we shall focus on two dimensional orbifolds $\mathbb{C}^2/\{\Gamma \subset SU(2)\}$. The parent theory has an $SU(4) \cong Spin(6)$ R-symmetry from the $\mathcal{N} = 4$ SUSY. The $U(n)$ gauge bosons A^{μ}_{IJ} with $I, J = 1, ..., n$ are R-singlets. Furthermore, there are Weyl fermions $\Psi^{i=1,2,3,4}_{IJ}$ in the fundamental $\mathbf{4}$ of $SU(4)$ and scalars $\Phi^{i=1,..,6}_{IJ}$ in the antisymmetric $\mathbf{6}$.

The orbifold imposes a projection condition upon these fields due to the finite group Γ. Let $R^{reg}_{\Gamma}(g)$ be the regular representation of $g \in \Gamma$, by which we mean

$$R^{reg}_{\Gamma}(g) := \bigoplus_i \Gamma_i(g) \otimes \mathbb{1}_{\dim(\Gamma_i)}$$

where $\{\Gamma_i\}$ are the irreducible representations of Γ. In matrix form, $R^{reg}_{\Gamma}(g)$ is composed of blocks of irreps, with each of dimension j repeated j times. Therefore it is a matrix of size $\sum\limits_i \dim(\Gamma_i)^2 = |\Gamma|$.

Let Irreps$(\Gamma) = \{\Gamma^{(1)}_1, \ldots, \Gamma^{(1)}_{m_1}; \Gamma^{(2)}_1, \ldots, \Gamma^{(2)}_{m_2}; \ldots\ldots; \Gamma^{(n)}_1, \ldots, \Gamma^{(n)}_{m_n}\}$, consisting of

m_j irreps of dimension j, then $R_\Gamma^{reg} :=$

$$\begin{pmatrix}
\Gamma_1^{(1)} & & & & & & & & & \\
& \ddots & & & & & & & & \\
& & \Gamma_{m_1}^{(1)} & & & & & & & \\
& & & \begin{pmatrix} \Gamma_1^{(2)} & \\ & \Gamma_1^{(2)} \end{pmatrix} & & & & & & \\
& & & & \ddots & & & & & \\
& & & & & \begin{pmatrix} \Gamma_{m_2}^{(2)} & \\ & \Gamma_{m_2}^{(2)} \end{pmatrix} & & & & \\
& & & & & & \ddots & & & \\
& & & & & & & \begin{pmatrix} \Gamma_1^{(n)} & & \\ & \ddots & \\ & & \Gamma_1^{(n)} \end{pmatrix}_{n\times n} & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & \begin{pmatrix} \Gamma_{m_n}^{(n)} & & \\ & \ddots & \\ & & \Gamma_{m_n}^{(n)} \end{pmatrix}_{n\times n}
\end{pmatrix}. \tag{16.2.1}$$

Of the parent fields A^μ, Ψ, Φ, only those invariant under the group action will remain in the orbifolded theory; this imposition is what we mean by *surviving the projection*:

$$\begin{aligned}
A^\mu &= R_\Gamma^{reg}(g)^{-1} \cdot A^\mu \cdot R_\Gamma^{reg}(g) \\
\Psi^i &= \rho(g)^i_j \; R_\Gamma^{reg}(g)^{-1} \cdot \Psi^j \cdot R_\Gamma^{reg}(g) \\
\Phi^i &= \rho'(g)^i_j \; R_\Gamma^{reg}(g)^{-1} \cdot \Phi^j \cdot R_\Gamma^{reg}(g) \qquad \forall \;\; g \in \Gamma,
\end{aligned} \tag{16.2.2}$$

where ρ and ρ' are induced actions because the matter fields carry R-charge (while the gauge bosons are R-singlets). Clearly if $\Gamma = \langle x_1, ..., x_n \rangle$, it suffices to impose (16.2.2) for the generators $\{x_i\}$ in order to find the matter content of the orbifold gauge theory; this observation we shall liberally use henceforth.

Letting $n = N|\Gamma|$ for some large N and $n_i = \dim(\Gamma_i)$, the subsequent gauge group becomes $\prod\limits_i U(n_i N)$ with a^4_{ij} Weyl fermions as bifundamentals $\left(\mathbf{n_i N}, \overline{\mathbf{n_j N}}\right)$ as well as a^6_{ij} scalar bifundamentals. These bifundamentals are pictorially summarised in quiver diagrams whose adjacency matrices are the a_{ij}'s.

Since we shall henceforth be dealing primarily with $\mathbb{C}^2$ orbifolds, we have $\mathcal{N} = 2$ gauge theory in four dimensions [76]. In particular we choose the induced group action on the R-symmetry to be $\mathbf{4} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}$ and $\mathbf{6} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}^2$ in order to preserve the supersymmetry. For this reason we can specify the final fermion and scalar matter matrices by a single quiver characterised by the **2** of $SU(2)$ as the trivial **1**'s give diagonal 1's. These issues are addressed at length in [292].

16.3 Stepwise Projection

Equipped with the clarification of notations of the previous section we shall now illustrate a technique which we shall call **stepwise projection**, originally inspired by [141, 172, 214], who attempted brane realisations of certain non-Abelian orbifolds of $\mathbb{C}^3$, an issue to which we shall later turn.

The philosophy of the technique is straight-forward[1]: say we are given a group $\Gamma_1 = \langle x_1, ..., x_n \rangle$ with quiver diagram Q_1 and $\Gamma_2 = \langle x_1, ..., x_{n+1} \rangle \supset \Gamma_1$ with quiver Q_2, we wish to determine Q_2 from Q_1 by the projection (16.2.2) by $\{x_1, ..., x_n\}$ followed by another projection by x_{n+1}.

We now proceed to analyse the well-known examples of the cyclic and binary dihedral quivers under this new light.

16.3.1 D_k Quivers from A_k Quivers

We shall concern ourselves with orbifold theories of $\mathbb{C}^2/\mathbb{Z}_k$ and $\mathbb{C}^2/D_k$. Let us first recall that the cyclic group $A_{k-1} \cong \mathbb{Z}_k$ has a single generator

$$\beta_k := \begin{pmatrix} \omega_k & 0 \\ 0 & \omega_k^{-1} \end{pmatrix}, \qquad \text{with} \quad \omega_n := e^{\frac{2\pi i}{n}}$$

and that the generators for the binary dihedral group D_k are

$$\beta_{2k} = \begin{pmatrix} \omega_{2k} & 0 \\ 0 & \omega_{2k}^{-1} \end{pmatrix}, \qquad \gamma := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

[1] A recent work [217] appeared during the final preparations of this draft; it beautifully addresses issues along a similar vein. In particular, cases where Γ_1 is normal in Γ_2 are discussed in detail. However, our stepwise method is not restricted by normality.

We further recall from [295, 296] that $D_k/\mathbb{Z}_{2k} \cong \mathbb{Z}_2$.

Now all irreps for $\mathbb{Z}_k$ are 1-dimensional (the k^{th} roots of unity), and (16.2.1) for the generator reads

$$R^{reg}_{\mathbb{Z}_k}(\beta_k) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega_k & 0 & 0 & 0 \\ 0 & 0 & \omega_k^2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & \omega_k^{k-1} \end{pmatrix}.$$

On the other hand, D_k has 1 and 2-dimensional irreps and (16.2.1) for the two generators become

$$R^{reg}_{D_k}(\beta_{2k}) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} \omega_{2k} & 0 \\ 0 & \omega_{2k}^{-1} \end{pmatrix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} \omega_{2k} & 0 \\ 0 & \omega_{2k}^{-1} \end{pmatrix} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} \omega_{2k}^{k-1} & 0 \\ 0 & \omega_{2k}^{-(k-1)} \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} \omega_{2k}^{k-1} & 0 \\ 0 & \omega_{2k}^{-(k-1)} \end{pmatrix} \end{pmatrix}$$

and

$$R^{reg}_{D_k}(\gamma) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i^{k \bmod 2} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} -1 & 0 \\ 0 & -i^{k \bmod 2} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} 0 & i^{k-1} \\ i^{k-1} & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} 0 & i^{k-1} \\ i^{k-1} & 0 \end{pmatrix} \end{pmatrix}.$$

In order to see the structural similarities between the regular representation of β_{2k} in $\Gamma_1 = \mathbb{Z}_{2k}$ and $\Gamma_2 = D_k$, we need to perform a change of basis. We do so such that each pair (say the j^{th}) of the 2-dimensional irreps of D_2 becomes as follows:

$$\Gamma^{(2)}(\beta_{2k}) = \begin{pmatrix} \begin{pmatrix} \omega_{2k}^{j} & 0 \\ 0 & \omega_{2k}^{-j} \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} \omega_{2k}^{j} & 0 \\ 0 & \omega_{2k}^{-j} \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} \omega_{2k}^{j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \omega_{2k}^{-j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

where $j = 1, 2, \ldots, k-1$. In this basis, the 2-dimensionals of γ become

$$\Gamma^{(2)}(\gamma) = \begin{pmatrix} \begin{pmatrix} 0 & i^{j} \\ i^{j} & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 0 & i^{j} \\ i^{j} & 0 \end{pmatrix} \end{pmatrix} \rightarrow \begin{pmatrix} 0 & i^{j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ i^{j} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \end{pmatrix}.$$

Now for the 1-dimensionals, we also permute the basis:

$$\Gamma^{(1)}(\beta_{2k}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$\Gamma^{(1)}(\gamma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i^{k \bmod 2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i^{k \bmod 2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i^{k \bmod 2} & 0 \\ 0 & 0 & 0 & -i^{k \bmod 2} \end{pmatrix}.$$

Therefore, we have

$$R_{D_k}^{reg}(\beta_{2k}) = \begin{pmatrix} 1 & 0 & 0 & 0 & & 0 & 0 \\ 0 & -1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & \omega_{2k} & 0 & & 0 & 0 \\ 0 & 0 & 0 & \omega_{2k}^{-1} & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & \omega_{2k}^{k-1} & 0 \\ 0 & 0 & 0 & 0 & & 0 & \omega_{2k}^{-(k-1)} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which by now has a great resemblance to the regular representation of $\beta_{2k} \in \mathbb{Z}_{2k}$; indeed, after one final change of basis, by ordering the powers of ω_{2k} in an ascending fashion while writing $\omega_{2k}^{-j} = \omega_{2k}^{2k-j}$ to ensure only positive exponents, we arrive at

$$
\begin{aligned}
R^{reg}_{D_k}(\beta_{2k}) &= \begin{pmatrix} 1 & 0 & 0 & & 0 \\ 0 & \omega_{2k} & 0 & & 0 \\ 0 & 0 & \omega_{2k}^2 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & \omega_{2k}^{2k-1} \end{pmatrix} \bigotimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= R^{reg}_{\mathbb{Z}_{2k}}(\beta_{2k}) \otimes \mathbb{1}_2,
\end{aligned}
\tag{16.3.3}
$$

the key relation which we need.

Under this final change of basis,

$$
R^{reg}_{D_k}(\gamma) = \begin{pmatrix}
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & m_{k-3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & m_{k-2} & 0 \\
0 & \vdots & \vdots & \ddots & \vdots & \vdots & 0 & m_{k-1} \\
\vdots & & & & \begin{pmatrix} i^{k \bmod 2} & 0 \\ 0 & -i^{k \bmod 2} \end{pmatrix} & & & \vdots \\
0 & m_{k-3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & m_{k-2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & m_{k-1} & 0 & 0 & 0 & 0
\end{pmatrix},
\tag{16.3.4}
$$

where $m_n := \begin{pmatrix} i^n & 0 \\ 0 & i^n \end{pmatrix}$.

Our strategy is now obvious. We shall first project according to (16.2.2), using (16.3.3), which is *equivalent to* a projection by $\mathbb{Z}_{2k}$, except with two identical copies (physically, this simply means we place twice as many D3-brane probes). Thereafter we shall project once again using (16.3.4) and the resulting theory should be that of the D_k orbifold.

An Illustrative Example

Let us turn to a concrete example, namely $\mathbb{Z}_4 \to D_2$. The key points to note are that $D_2 := \langle \beta_4, \gamma \rangle$ and $\mathbb{Z}_4 \cong \langle \beta_4 \rangle$. We shall therefore perform stepwise projection by β_4 followed by γ.

Equation (16.3.3) now reads

$$R^{reg}_{D_2}(\beta_4) = R^{reg}_{\mathbb{Z}_4}(\beta_4) \otimes \mathbb{1}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i^2 & 0 \\ 0 & 0 & 0 & i^3 \end{pmatrix} \otimes \mathbb{1}_2. \tag{16.3.5}$$

We have the following matter content in the parent (pre-orbifold) theory: gauge field A^μ, fermions $\Psi^{1,2,3,4}$ and scalars $\Phi^{1,2,3,4,5,6}$ (suppressing gauge indices IJ). Projection by $R^{reg}_{D_2}(\beta_4)$ in (16.3.5) according to (16.2.2) gives a $\mathbb{Z}_4$ orbifold theory, which restricts the form of the fields to be as follows:

$$\begin{array}{c} A^\mu, \Psi^{1,2}, \Phi^{1,2} = \begin{pmatrix} \square & & & \\ & \square & & \\ & & \square & \\ & & & \square \end{pmatrix}; \\ \Psi^3, \Phi^{3,5} = \begin{pmatrix} & \square & & \\ & & \square & \\ & & & \square \\ \square & & & \end{pmatrix}; \quad \Psi^4, \Phi^{4,6} = \begin{pmatrix} & & & \square \\ \square & & & \\ & \square & & \\ & & \square & \end{pmatrix} \end{array} \tag{16.3.6}$$

where $\square$ are 2×2 blocks. We recall from the previous section that we have chosen the R-symmetry decomposition as $\mathbf{4} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}$ and $\mathbf{6} = \mathbf{1}^2_{trivial} \oplus \mathbf{2}^2$. The fields in (16.3.6) are defined in accordance thereto: the fermions $\Psi^{1,2}$ and scalars $\Phi^{1,2}$ are respectively in the two trivial $\mathbf{1}$'s of the $\mathbf{4}$ and $\mathbf{6}$; (Ψ^3, Ψ^4), (Φ^3, Φ^4) and (Φ^5, Φ^6) are in the doublet $\mathbf{2}$ of Γ inherited from $SU(2)$. Indeed, the $R^{reg}_{\mathbb{Z}_4}(\beta_4)$ projection would force $\square$ to be numbers and not matrices as we do not have the extra $\mathbb{1}_2$ tensored to the group action, in which case (16.3.6) would be 4×4 matrices prescribing the adjacency matrices of the $\mathbb{Z}_4$ quiver. For this reason, the quiver diagram for the $\mathbb{Z}_4$ theory as drawn in part (I) of Figure 16-1 has the nodes labelled 2's instead of the usual Dynkin labels of 1's for the A-series. In physical terms we have placed twice as many image D-brane probes. The key point is that because $\square$ are now matrices (and (16.3.6) are 8×8), further projection internal thereto may change the number and structure of the product gauge groups and matter fields.

Having done the first step by the β_4 projection, next we project with the regular representation of γ:

$$R_{D_2}^{reg}(\gamma) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \\ 0 & 0 & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & 0 & 0 \end{pmatrix} := \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{1}_2 & 0 & 0 \end{pmatrix}. \tag{16.3.7}$$

In accordance with (16.3.6), let the gauge field be

$$A^\mu := \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix},$$

with a, b, c, d denoting the 2×2 blocks $\square$, (16.2.2) for (16.3.7) now reads

$$A^\mu = R_{D_2}^{reg}(\gamma)^{-1} \cdot A^\mu \cdot R_{D_2}^{reg}(\gamma) \Rightarrow$$

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & -i\mathbb{1}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{1}_2 & 0 & 0 \end{pmatrix},$$

giving us a set of constraining equations for the blocks:

$$\sigma_3 \cdot a \cdot \sigma_3 = a; \qquad d = b; \qquad \sigma_3 \cdot c \cdot \sigma_3 = c. \tag{16.3.8}$$

Similarly, for the fermions in the **2**, viz.,

$$\Psi^3 = \begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix}, \quad \Psi^4 = \begin{pmatrix} 0 & 0 & 0 & e_4 \\ f_4 & 0 & 0 & 0 \\ 0 & g_4 & 0 & 0 \\ 0 & 0 & h_4 & 0 \end{pmatrix},$$

the projection (16.2.2) is

$$\gamma \cdot \begin{pmatrix} \Psi^3 \\ \Psi^4 \end{pmatrix} = R^{reg}_{D_2}(\gamma)^{-1} \cdot \begin{pmatrix} \Psi^3 \\ \Psi^4 \end{pmatrix} \cdot R^{reg}_{D_2}(\gamma).$$

We have used the fact that the induced action $\rho(\gamma)$, having to act upon a doublet, is simply the 2×2 matrix γ herself. Therefore, writing it out explicitly, we have

$$i \begin{pmatrix} 0 & 0 & 0 & e_4 \\ f_4 & 0 & 0 & 0 \\ 0 & g_4 & 0 & 0 \\ 0 & 0 & h_4 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & -i\mathbb{1}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{1}_2 & 0 & 0 \end{pmatrix}$$

and

$$i \begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & -i\mathbb{1}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & e_4 \\ f_4 & 0 & 0 & 0 \\ 0 & g_4 & 0 & 0 \\ 0 & 0 & h_4 & 0 \end{pmatrix} \begin{pmatrix} \sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{1}_2 \\ 0 & 0 & \sigma_3 & 0 \\ 0 & i\mathbb{1}_2 & 0 & 0 \end{pmatrix},$$

which gives the constraints

$$f_4 = -h_3 \cdot \sigma_3; \qquad g_4 = \sigma_3 \cdot g_3; \qquad h_4 = -f_3 \cdot \sigma_3; \qquad e_4 = \sigma_3 \cdot e_3. \tag{16.3.9}$$

The doublet scalars $(\Phi^{3,5}, \Phi^{4,6})$ of course give the same results, as should be expected from supersymmetry.

In summary then, the final fields which survive both β_4 and γ projections (and thus the entire group D_2) are

$$A^{\mu} = \begin{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} & & & \\ & b & & \\ & & \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix} & \\ & & & b \end{pmatrix}; \quad \left\{ \begin{array}{ll} e_3 = \begin{pmatrix} e_{11} & e_{12} \\ 0 & 0 \end{pmatrix}, & f_3 = \begin{pmatrix} 0 & f_{12} \\ 0 & f_{22} \end{pmatrix}, \\ g_3 = \begin{pmatrix} g_{11} & g_{12} \\ 0 & 0 \end{pmatrix}, & h_3 = \begin{pmatrix} 0 & h_{12} \\ 0 & h_{22} \end{pmatrix}, \end{array} \right.$$

$$\Psi^3 = \begin{pmatrix} 0 & e_3 & 0 & 0 \\ 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & g_3 \\ h_3 & 0 & 0 & 0 \end{pmatrix}, \quad \Psi^4 = \begin{pmatrix} 0 & 0 & 0 & \sigma_3 \cdot e_3 \\ -h_3 \cdot \sigma_3 & 0 & 0 & 0 \\ 0 & \sigma_3 \cdot g_3 & 0 & 0 \\ 0 & 0 & -f_3 \cdot \sigma_3 & 0 \end{pmatrix}. \tag{16.3.10}$$

The key features to be noticed are now apparent in the structure of these matrices in (16.3.10). We see that the 4 blocks of A^{μ} in (16.3.6), which give the four nodes of the $\mathbb{Z}_4$ quiver, now undergo a metamorphosis: we have written out the components of a, c explicitly and have used (16.3.8) to restrict both to diagonal matrices, while b and d are identified, but still remain blocks without internal structure of interest. Thus we have a total of 5 non-trivial constituents $a_{11}, a_{22}, c_{11}, c_{22}$ and b, precisely the 5 nodes of the D_2 quiver (see parts (I) and (II) of Figure 16-1). Thus *nodes of the quiver merge and split as we impose further projections*, as we mentioned a few paragraphs ago.

As for the bifundamentals, i.e., the arrows of the quiver, (16.3.6) prescribes the blocks $e_{3,4}, f_{3,4}, g_{3,4}$ and $h_{3,4}$ as the 8 arrows of Part (I) of Figure 16-1. After the projection by γ, and imposing the constraint (16.3.9) as well as the fact that all entries of matter matrices must be non-negative, we are left with the 8 fields $e_{11,12}, f_{12,22}, g_{11,12}$ and $h_{12,22}$, precisely the 8 arrows in the D_2 quiver (see Part (II) of Figure 16-1).

The General Case

The generic situation of obtaining the D_k quiver from that of $\mathbb{Z}_{2k}$ is completely analogous. We would always have two end nodes of the $\mathbb{Z}_{2k}$ quiver each splitting into two while the middle ones coalesce pair-wise, as is shown in Figure 16-2.

16.3.2 The E_6 Quiver from D_2

We now move on to tackle the binary tetrahedral group E_6 (with the relation that $E_6/D_2 \cong \mathbb{Z}_3$), whose generators are

$$\beta_4 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \delta := \frac{1}{2}\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix}.$$

We observe therefore that it has yet one more generator δ than D_2, hence we need to continue our stepwise projection from the previous subsection, with the exception that we should begin with more copies of $\mathbb{Z}_4$. To see this let us first present the irreducible matrix representations of the three generators of E_6:

	β_4	γ	δ
$\Gamma_1^{(1)}$	1	1	1
$\Gamma_2^{(1)}$	1	1	ω_3
$\Gamma_3^{(1)}$	1	1	ω_3^2
$\Gamma_4^{(2)}$	β_4	γ	δ
$\Gamma_5^{(2)}$	β_4	γ	$\omega_3\delta$
$\Gamma_6^{(2)}$	β_4	γ	$\omega_3^2\delta$
$\Gamma_7^{(3)}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -\frac{i}{2} & \frac{i}{\sqrt{2}} & -\frac{i}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{i}{2} & -\frac{i}{\sqrt{2}} & \frac{i}{2} \end{pmatrix}$

The regular representation for these generators is therefore a matrix of size $3 \cdot 1^2 + 3 \cdot 2^2 + 3^3 = 24$, in accordance with (16.2.1).

Our first step is as with the case of D_2, namely to change to a convenient basis wherein β_4 becomes diagonal:

$$R^{reg}_{E_6}(\beta_4) = R^{reg}_{\mathbb{Z}_4}(\beta_4) \otimes \mathbb{1}_6. \qquad (16.3.11)$$

The only difference between the above and (16.3.5) is that we have the tensor product with $\mathbb{1}_6$ instead of $\mathbb{1}_2$, therefore at this stage we have a $\mathbb{Z}_4$ quiver with the nodes labeled 6 as opposed to 2 as in Part (I) of Figure 16-1. In other words we have 6 times the usual number of D-brane probes.

Under the basis of (16.3.11),

$$R^{reg}_{E_6}(\gamma) = \begin{pmatrix} \Sigma_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\mathbb{1}_6 \\ 0 & 0 & \Sigma_3 & 0 \\ 0 & i\mathbb{1}_6 & 0 & 0 \end{pmatrix} \quad \text{where} \quad \Sigma_3 := \sigma_3 \otimes \mathbb{1}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{16.3.12}$$

Subsequent projection gives a D_2 quiver as in part (II) of Figure 16-1, but with the nodes labeled as $3, 3, 6, 3, 3$, three times the usual. Note incidentally that (16.3.11) and (16.3.12) can be re-written in terms of regular representations of D_2 directly: $R^{reg}_{E_6}(\beta_4) = R^{reg}_{D_2}(\beta_4) \otimes \mathbb{1}_3$ and $R^{reg}_{E_6}(\gamma) = R^{reg}_{D_2}(\gamma) \otimes \mathbb{1}_3$. To this fact we shall later turn.

To arrive at E_6, we proceed with one more projection, by the last generator δ, the regular representation of which, observing the table above, has the form (in the basis of (16.3.11))

$$R^{reg}_{E_6}(\delta) = \begin{pmatrix} S_1 & 0 & S_2 & 0 \\ 0 & \omega_8^{-1}P & 0 & \omega_8^{-1}P \\ S_3 & 0 & S_4 & 0 \\ 0 & -\omega^8 P & 0 & \omega_8 P \end{pmatrix} \tag{16.3.13}$$

where

$$S_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes R^{reg}_{\mathbb{Z}_3}(\beta_3), \quad S_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$S_3 := -i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_4 := i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1}_3$$

and

$$P := R^{reg}_{\mathbb{Z}_3}(\beta_3) \otimes \frac{1}{\sqrt{2}} \mathbb{1}_2; \quad \text{recalling that} \quad R^{reg}_{\mathbb{Z}_3}(\beta_3) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_3 & 0 \\ 0 & 0 & \omega_3^2 \end{pmatrix}.$$

The inverse of (16.3.13) is readily determined to be

$$R^{reg}_{E_6}(\delta)^{-1} = \begin{pmatrix} \tilde{S}_1 & 0 & -S_3 & 0 \\ 0 & \frac{1}{2}\omega_8 P^{-1} & 0 & -\frac{1}{2}\omega_8^{-1}P^{-1} \\ S_2^T & 0 & -S_4^T & 0 \\ 0 & \frac{1}{2}\omega_8 P^{-1} & 0 & \frac{1}{2}\omega_8^{-1}P^{-1} \end{pmatrix}, \qquad \tilde{S}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes R^{reg}_{\mathbb{Z}_3}(\beta_3)^{-1}.$$

Thus equipped, we must use (16.2.2) with (16.3.13) on the matrix forms obtained in (16.3.10) (other fields can of course be checked to have the same projection), with of course each number therein now being 3×3 matrices. The final matrix for A^μ is as in (16.3.10), but with

$$a_{11} = \begin{pmatrix} a_{11(1)} & 0 & 0 \\ 0 & a_{11(2)} & 0 \\ 0 & 0 & a_{11(3)} \end{pmatrix}_{3\times 3} ; \quad c_{11} = c_{22} = a_{22}; \quad b = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}_{6\times 6}$$

where a_{22}, c_{ii} are 3×3 while b_{ii} are 2×2 blocks. We observe therefore, that there are 7 distinct gauge group factors of interest, namely $a_{11(1)}, a_{11(2)}, a_{11(3)}, a_{22}, b_{11}, b_{22}$ and b_{33}, with Dynkin labels $1, 1, 1, 3, 2, 2, 2$ respectively. What we have now is the E_6 quiver and the bifundamentals split and join accordingly; the reader is referred to Part (I) of Figure 16-3.

16.3.3 The E_6 Quiver from $\mathbb{Z}_6$

Let us make use of an interesting fact, that actually $E_6 = \langle \beta_4, \gamma, \delta \rangle = \langle \beta_4, \delta \rangle = \langle \gamma, \delta \rangle$. Therefore, alternative to the previous subsection wherein we exploited the sequence $\mathbb{Z}_4 = \langle \beta_4 \rangle \xrightarrow{+\gamma} D_2 \xrightarrow{+\delta} E_6$, we could equivalently apply our stepwise projection on $\mathbb{Z}_6 = \langle \delta \rangle \xrightarrow{+\beta_4} E_6$.

Let us first project with δ, an element of order 6 and the regular representation of which, after appropriate rotation is

$$R^{reg}_{E_6}(\delta) = R^{reg}_{\mathbb{Z}_6}(\delta) \otimes \mathbb{1}_4. \qquad (16.3.14)$$

Therefore at this stage we have a $\mathbb{Z}_6$ quiver with labels of six 4's due to the $\mathbb{1}_4$; this is drawn in Part (II) of Figure 16-3. The gauge group we shall denote as $A^\mu :=$ Diag$(a,b,c,d,e,f)_{24\times 24}$, with $a, b, \cdots, f$ being 4×4 blocks.

Next we perform projection by $R^{reg}_{E_6}(\beta_4)$ in the rotated basis, splitting and joining the gauge groups (nodes) as follows

$$A^\mu = \begin{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & \tilde{a} \end{pmatrix} & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{pmatrix} c_{11} & 0 \\ 0 & \tilde{c} \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 & \begin{pmatrix} e_{11} & 0 \\ 0 & \tilde{e} \end{pmatrix} & 0 \\ 0 & 0 & 0 & 0 & 0 & \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \end{pmatrix}; \ s.t. \ \begin{array}{l} \tilde{a} = \tilde{c} = \tilde{e}, \\ b_2 = d_1, \\ d_2 = f_1, \\ f_2 = b_1, \end{array}$$

which upon substitution of the relations, gives us 7 independent factors: a_{11}, c_{11} and e_{11} are numbers, giving 1 as Dynkin labels in the quiver; b_1, b_2 and d_2 are 2×2 blocks, giving the 2 labels; while $\tilde{a}$ is 3×3, giving the 3. We refer the reader to Part (II) of Figure 16-3 for the diagrammatical representation.

16.4 Comments and Discussions

Our procedure outlined above is originally inspired by a series of papers [141, 172, 214], where the quivers for the Δ series of $\Gamma \subset SU(3)$ were observed to be obtainable from the $\mathbb{Z}_n \times \mathbb{Z}_n$ series after an appropriate identification. In particular, it was noted that

$$\Delta(3n^2) = \langle \left\{ \mathbb{Z}_n \times \mathbb{Z}_n := \begin{pmatrix} \omega_n^i & 0 & 0 \\ 0 & \omega_n^j & 0 \\ 0 & 0 & \omega_n^{-i-j} \end{pmatrix}_{i,j=0,\cdots,n-1} \right\}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rangle$$ and

subsequently the quiver for $\Delta(3n^2)$ is that of $\mathbb{Z}_n \times \mathbb{Z}_n$ modded out by a certain $\mathbb{Z}_3$ quotient. Similarly, the quiver for

$$\Delta(6n^2) = \langle \mathbb{Z}_n \times \mathbb{Z}_n, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \rangle$$

is that of $\mathbb{Z}_n \times \mathbb{Z}_n$ modded out by a certain S_3 quotient. In [214], it was further commented that the Σ series could be likewise treated.

The motivation for those studies was to realise a brane-setup for the non-Abelian $SU(3)$ orbifolds as geometrical quotients of the well-known Abelian case of $\mathbb{Z}_m \times \mathbb{Z}_n$, viz., the Brane Box Models. The key idea was to recognise that the irreducible representations of these groups could be labelled by a double index $(l_1, l_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$ up to identifications.

Our purpose here is to establish an algorithmic treatment along similar lines, which would be generalisable to arbitrary finite groups. Indeed, since any finite group Γ is finitely generated, starting from the cyclic subgroup (with one single generator), our stepwise projection would give the quiver for Γ as appropriate splitting and joining of nodes, i.e., as a certain geometrical action, of the $\mathbb{Z}_n$ quiver.

16.4.1 A Mathematical Viewpoint

To see why our stepwise projection works on a more axiomatic level, we need to turn to a brief review of the Theory of Induced Representations.

It was a fundamental observation of Frøbenius that the representations of a group could be constructed from an arbitrary subgroup. The aforementioned chain of groups, where we tried to relate the regular representations, is precisely in this vein. Though we shall largely follow the nomenclature of [24], we shall now briefly review this theory in the spirit of the above discussions.

Let $\Gamma_1 = \langle x_1, ..., x_n \rangle$ and $\Gamma_2 = \langle x_1, ..., x_{n+1} \rangle$. We see thus that $\Gamma_1 \subset \Gamma_2$. Now let $R_{\Gamma_1}(x)$ be a representation (not necessarily irreducible) of the element $x \in \Gamma_1$. Extending it to Γ_2 gives

$$R_{\Gamma_2}(y) = \begin{cases} R_{\Gamma_1}(x) & \text{if} \quad y = x \in \Gamma_1 \\ 0 & \text{if} \quad y \notin \Gamma_1 \end{cases}$$

It follows then that if we decompose Γ_2 as (right) cosets of Γ_1,

$$\Gamma_2 = \Gamma_1 t_1 \cup \Gamma_1 t_2 \cup \cdots \cup \Gamma_1 t_m$$

we have an **Induced Representation** of Γ_2 as

$$R_{\Gamma_2}(y) = R_{\Gamma_1}(t_i y t_j^{-1}) = \begin{pmatrix} R_{\Gamma_1}(t_1 y t_1^{-1}) & R_{\Gamma_1}(t_1 y t_2^{-1}) & \cdots & R_{\Gamma_1}(t_1 y t_m^{-1}) \\ R_{\Gamma_1}(t_2 y t_1^{-1}) & R_{\Gamma_1}(t_2 y t_2^{-1}) & \cdots & R_{\Gamma_1}(t_2 y t_m^{-1}) \\ \vdots & \vdots & & \vdots \\ R_{\Gamma_1}(t_m y t_1^{-1}) & R_{\Gamma_1}(t_m y t_2^{-1}) & \cdots & R_{\Gamma_1}(t_m y t_m^{-1}) \end{pmatrix}. \tag{16.4.15}$$

A beautiful property of (16.4.15) is that it has only one member of each row or column non-zero and whereby it is essentially a generalised permutation (see e.g., 3.1 of [24]) matrix acting on the Γ_1-stable submodules of the Γ_2-module.

Now, for the case at hand the coset decomposition is simple due to the addition of a single new generator: the (right) transversals $t_1, \cdots, t_m$ are simply powers of the extra generator x_{n+1} and m is simply the index of $\Gamma_1 \subset \Gamma_2$, namely $|\Gamma_2|/|\Gamma_1|$, i.e.,

$$t_i = x_{n+1}^{i-1} \qquad i = 1, 2, \cdots, m; \quad m = \frac{|\Gamma_2|}{|\Gamma_1|}. \tag{16.4.16}$$

Now let us define an important concept for an element $x \in \Gamma_2$

DEFINITION 16.4.22 *We call a representation $R_{\Gamma_2}(x)$* **factorisable** *if it can be written, up to possible change of bases, as a tensor product $R_{\Gamma_2}(x) = R_{\Gamma_1}(x) \otimes \mathbb{1}_k$ for some integer k.*

Factorisability of the element, in the physical sense, corresponds to the ability to initialise our stepwise projection algorithm, by which we mean that the orbifold projection by this element is performed on k copies as in the usual sense, i.e., a stack of k copies of the quiver. Subsequently we could continue with the stepwise algorithm to demonstrate how the nodes of these copies merge or split. In the corresponding D-brane picture this simply means that we should consider k copies of each image D-brane probe in the covering space.

The natural question to ask is of course why our examples in the previous section permitted factorisable generators so as to in turn permit the performance of the stepwise projection. The following claim shall be of great assurance to us:

PROPOSITION 16.4.8 *Let H be a subgroup of G, then the representation $R_G(x)$ for an element $x \in H \subset G$ induced from $R_H(x)$ according to (16.4.15) is factorisable and k is equal to $|G|/|H|$, the index of H in G.*

Proof: Take $R_H(x \in H)$, and tensor it with $\mathbb{1}_{k=|G|/|H|}$; this remains of course a representation for $x \in H$. It then remains to find the representations of $x \notin H$, which we supplement by the permutation actions of these elements on the H-cosets. At the end of the day we arrive at a representation $R'_G(x)$ of dimension k, such that it is factorisable for $x \in H$ and a general permutation for $x \notin H$. However by the uniqueness theorem of induced representations (q.v. e.g. [215] Thm 11) such a linear representation $R'_G(x)$ must in fact be isomorphic to $R_G(x)$. Thus by explicit construction we have shown that $R_G(x \in H) = R_H(x) \otimes \mathbb{1}_k$. $\square$

We can be more specific and apply Proposition 4.1 to our case of the two groups the second of which is generated by the first with one additional generator. Using the elegant property that the induction of a regular representation remains regular (q.v. e.g., 3.3 of [215]), we have:

COROLLARY 16.4.5 *Let Γ_1 and Γ_2 be as defined above, then*

$$R^{reg}_{\Gamma_2}(x_i) = R^{reg}_{\Gamma_1}(x_i) \otimes \mathbb{1}_{|\Gamma_2|/|\Gamma_1|} \qquad \textit{for common generators} \qquad i = 1, 2, \ldots, n.$$

In particular, since any $G = \langle x_1, \ldots, x_n \rangle$ contains a cyclic subgroup generated by, say x_1 of order m, i.e., $\mathbb{Z}_m = \langle x_1 \rangle$, we conclude that

COROLLARY 16.4.6 *$R^{reg}_G(x_1) = R^{reg}_{\mathbb{Z}_m}(x_1) \otimes \mathbb{1}_{|G|/m}$, and hence the quiver for G can always be obtained by starting with the $\mathbb{Z}_m$ quiver using the stepwise projection.*

Let us revisit the examples in the previous section equipped with the above knowledge. For the case of $\Gamma_1 = \mathbb{Z}_4 = \langle \beta_4 \rangle$ and $\Gamma_2 = D_2$ with the extra generator γ, (16.4.16) becomes $t_1 = \mathbb{1}$ and $t_2 = \gamma$ as the index of $\mathbb{Z}_4$ in D_2 is $\frac{|D_2|=8}{|\mathbb{Z}_4|=4} = 2$. The induced representation of β_4 according to (16.4.15) reads

$$R_{D_2}(\beta_4) = \begin{pmatrix} R^{reg}_{\mathbb{Z}_4}(\mathbb{1}\beta_4\mathbb{1}^{-1}) & R^{reg}_{\mathbb{Z}_4}(\mathbb{1}\beta_4\gamma^{-1}) \\ R^{reg}_{\mathbb{Z}_4}(\gamma\beta_4\mathbb{1}^{-1}) & R^{reg}_{\mathbb{Z}_4}(\gamma\beta_4\gamma^{-1}) \end{pmatrix} = \begin{pmatrix} R^{reg}_{\mathbb{Z}_4}(\beta_4) & 0 \\ 0 & R^{reg}_{\mathbb{Z}_4}(\beta_4^{-1}) \end{pmatrix}$$

using the fact that $\gamma\beta_k\gamma^{-1} = \beta_k^{-1}$ in D_k for the last entry. Recalling that $R^{reg}_{\mathbb{Z}_4}(\beta_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i^2 & 0 \\ 0 & 0 & 0 & i^3 \end{pmatrix}$, this is subsequently equal to $R^{reg}_{\mathbb{Z}_4} \otimes \mathbb{1}_2$ after appropriate permutation of basis. Thus Corollary 4.1 manifests her validity as we see that the R_{D_2} obtained by Frøbenius induction of $R^{reg}_{\mathbb{Z}_4}$ is indeed regular and moreover factorisable, as (16.3.5) dictates.

Similarly with the case of $\mathbb{Z}_6 \to E_6$, we see that Corollary 4.1 demands that for the common generator δ, $R^{reg}_{E_6}(\delta)$ should be factorisable, as is indeed indicated by (16.3.14). So too is it with $\mathbb{Z}_4 \to E_6$, where $R^{reg}_{E_6}(\beta_4)$ should factorise, precisely as shown by (16.3.11).

The above have actually been special cases of Corollary 4.2, where we started with a cyclic subgroup; in fact we have also presented an example demonstrating the general truism of Proposition 4.1. In the case of $D_2 \to E_6$, we mentioned earlier that $R^{reg}_{E_6}(\beta_4) = R^{reg}_{D_2}(\beta_4) \otimes \mathbb{1}_3$ and $R^{reg}_{E_6}(\gamma) = R^{reg}_{D_2}(\gamma) \otimes \mathbb{1}_3$ for the common generators as was seen from (16.3.11) and (16.3.12); this is exactly as expected by the Proposition.

16.4.2 A Physical Viewpoint: Brane Setups?

Now mathematically it is clear what is happening to the quiver as we apply stepwise projection. However this is only half of the story; as we mentioned in the introduction, we expect T-duality to take D-branes at generic orbifold singularities to brane setups. It is a well-known fact that the brane setups for the A and D-type orbifolds $\mathbb{C}^2/\mathbb{Z}_n$ and $\mathbb{C}^2/D_n$ have been realised (see [78, 79] and [83] respectively). It has been the main intent of a collective of works (e.g [295, 296, 172, 214]) to establish such setups for the generic singularity.

In particular, the problem of finding a consistent brane-setup for the remaining case of the exceptional groups $E_{6,7,8}$ of the ADE orbifold singularities of $\mathbb{C}^2$ (and indeed analogues thereof for $SU(3)$ and $SU(4)$ subgroups) so far has been proven to be stubbornly intractable. An original motivation for the present work is to attempt to formulate an algorithmic outlook wherein such a problem, with the insight of the algebraic structure of an appropriate chain of certain relevant groups, may be addressed systematically.

The $\mathbb{Z}_2$ Action on the Brane Setup

Let us attempt to recast our discussion in Subsection 3.1 into a physical language. First we try to interpret the action by $R^{reg}_{D_k}(\gamma)$ in (16.3.4) on the $\mathbb{Z}_{2k}$ quiver as a string-theoretic action on brane setups to get the corresponding brane setup of D_k from that of $\mathbb{Z}_{2k}$.

Now the brane configuration for the $\mathbb{Z}_{2k}$ orbifold is the well-known *elliptic model* consisting of $2k$ NS5-branes arranged in a circle with D4-branes stretched in between as shown in Part (III) of Figure 16-1. After stepwise projection by γ, the quiver in Part (I) becomes that in Part(II) (see Figure 16-2 also). There is an obvious $\mathbb{Z}_2$ quotienting involved, where the nodes i and $2k-i$ for $i = 1, 2, ..., k-1$ are identified while each of the nodes 0 and k splits into two parts. Of course, this symmetry is not immediately apparent from the properties of γ, which is a group element of order 4. This phenomenon is true in general: the order of the generator used in the stepwise projection does not necessarily determine what symmetry the parent quiver undergoes to arrive at the resulting quiver; instead we must observe *a posteriori* the shapes of the respective quivers.

Let us digress a moment to formulate the above results in the language used in [141, 172]. Recalling from the brief comments in the beginning of Section 4, we adopt their idea of labelling the irreducible representations of Δ by $\mathbb{Z}_n \times \mathbb{Z}_n$ up to appropriate identifications, which in our terminology is simply the by-now familiar stepwise projection of the parent $\mathbb{Z}_n \times \mathbb{Z}_n$ quiver. As a comparison, we apply this idea to the case of $\mathbb{Z}_{2k} \to D_k$. Therefore we need to label the irreps of D_k or appropriate tensor sums thereof, in terms of certain (reducible) 2-dimensional representations of $\mathbb{Z}_{2k}$. Motivated by the factorization property (16.3.5), we chose these representations to be

$$R^{l}_{\mathbb{Z}_{2k}(2)} := R^{l,irrep}_{\mathbb{Z}_{2k}(1)} \oplus R^{l,irrep}_{\mathbb{Z}_{2k}(1)} \qquad (16.4.17)$$

where $l \in \mathbb{Z}_{2k}$, and amounts to precisely a $\mathbb{Z}_{2k}$-valued index on the representations of

D_k (since $\mathbb{Z}_{2k}$ is Abelian), which with foresight, we shall later use on D_k. We observe that such a labelling scheme has a symmetry

$$R^l_{\mathbb{Z}_{2k}(2)} \cong R^{-l}_{\mathbb{Z}_{2k}(2)},$$

which is obviously a $\mathbb{Z}_2$ action. Note that $l = 0$ and $l = k$ are fixed points of this $\mathbb{Z}_2$.

We can now associate the 2-dimensional irreps of D_k with the non-trivial equivalence classes of the $\mathbb{Z}_{2k}$ representations (16.4.17), i.e., for $l = 1, 2, \ldots, k-1$ we have

$$R^l_{\mathbb{Z}_{2k}(2)} \cong R^{-l}_{\mathbb{Z}_{2k}(2)} \rightarrow R^{l,irrep}_{D_k(2)}. \qquad (16.4.18)$$

These identifications correspond to the merging nodes in the associated quiver diagram. As for the fixed points, we need to map

$$\begin{array}{l} R^0_{\mathbb{Z}_{2k}(2)} \rightarrow R^{1,irrep}_{D_k(1)} \oplus R^{2,irrep}_{D_k(1)} \\ R^k_{\mathbb{Z}_{2k}(2)} \rightarrow R^{3,irrep}_{D_k(1)} \oplus R^{4,irrep}_{D_k(1)}. \end{array} \qquad (16.4.19)$$

These fixed points are associated precisely with the nodes that split.

This construction shows clearly how, in the labelling scheme of [141, 172], our stepwise algorithm derives the D_k quiver as a $\mathbb{Z}_2$ projection of the $\mathbb{Z}_{2k}$ quiver. The consistency of this description is verified by substituting the representations $R^l_{\mathbb{Z}_{2k}(2)}$ in the $\mathbb{Z}_{2k}$ quiver relations $\mathcal{R} \otimes R^l_{\mathbb{Z}_{2k}(2)} = \bigoplus\limits_{\bar{l}} a^{\mathbb{Z}_{2k}(\mathcal{R})}_{l\bar{l}} R^{\bar{l}}_{\mathbb{Z}_{2k}(2)}$ using (16.4.18) and (16.4.19), which results exactly in the D_k quiver relations. We can of course apply the stepwise projection for the case of $\mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \Delta$, and would arrive at the results in [141, 172].

In the brane setup picture, the identification of the nodes i and $2k - i$ for $i = 1, 2, ..., k-1$ corresponds to the identification of these intervals of NS5-branes as well as the D4-branes in between in the X^{6789} directions (with direction-6 compact). Thus the $\mathbb{Z}_2$ action on the $\mathbb{Z}_{2k}$ quiver should include a space-time action which identifies $X^{6789} = -X^{6789}$. Similarly, the splitting of gauge fields in intervals 0 and k hints the existence of a $\mathbb{Z}_2$ action on the string world-sheet. Thus the overall $\mathbb{Z}_2$ action should include two parts: a space-time symmetry which identifies and a world-sheet symmetry which splits respective gauge groups.

What then is this action physically? What object in string theory performs the tasks in the above paragraph? Fortunately, the space-time parity and string world-sheet $(-1)^{F_L}$ actions [206, 83] are precisely the aforementioned symmetries. In other words, the *ON-plane* is that which we seek. This is of great assurance to us, because the brane setup for D_k theories, as given in [83], is indeed a configuration which uses the ON-plane to project out or identify fields in a manner consistent with our discussions.

The General Action on the Brane Setup?

It seems therefore, that we could now be boosted with much confidence: since we have proven in the previous subsection that our stepwise projection algorithm is a constructive method of arriving at *any* orbifold quiver by appropriate quotient of the $\mathbb{Z}_n$ quiver, could we not simply find the appropriate object in string theory which would perform such a quotient, much in the spirit of the orientifold prescribing $\mathbb{Z}_2$ in the above example, on the well-known $\mathbb{Z}_n$ brane setup, in order to solve our problem?

Such a confidence, as is with most in life, is overly optimistic. Let us pause a moment to consider the E_6 example. The action by δ in the case of $D_2 \to E_6$ in §3.2 and that of β_4 in the case of $\mathbb{Z}_6 \to E_6$ in §3.3 can be visualised in Parts (I) and (II) of Figure 16-3 to be an $\mathbb{Z}_3$ action on the respective parent quivers. In particular, the identifications $c_{11} \sim c_{22} \sim a_{22}$ and $\tilde{a} \sim \tilde{c} \sim \tilde{e}; b_1 \sim f_2, b_2 \sim d_1, d_2 \sim f_1$ respectively for Parts (I) and (II) are suggestive of a $\mathbb{Z}_3$ action on X^{6789}. The tripartite splittings for b, a_{11} and a, b, d respectively also hint at a $\mathbb{Z}_3$ action on the string world-sheet.

Again let us phrase the above results in the scheme of [141, 172], and manifestly show how the E_6 quiver results from a $\mathbb{Z}_3$ projection of the D_2 quiver. We define the following representations of D_2: $R^0_{D_2(6)} = R^{irrep}_{D_2(2)} \oplus R^{irrep}_{D_2(2)} \oplus R^{irrep}_{D_2(2)}$ and $R^l_{D_2(3)} = R^{l,irrep}_{D_2(1)} \oplus R^{l,irrep}_{D_2(1)} \oplus R^{l,irrep}_{D_2(1)}$ where $l \in \mathbb{Z}_4$ labels the four 1-dimensional irreducible representations of D_2. There is an identification

$$R^l_{D_2} \cong R^{f(l)}_{D_2}$$

where

$$f(l) = \begin{cases} 0, & l = 0 \\ 2, & l = 1 \\ 3, & l = 2 \\ 1, & l = 3 \end{cases}$$

Clearly this is a $\mathbb{Z}_3$ action on the index l. Note that we have two representations labelled with $l = 0$ which are fixed points of this action. In the quiver diagram of D_2 these correspond to the middle node and another one arbitrarily selected from the remaining four, both of which split into three. The remaining three nodes are consequently merged into a single one (see Figure 16-3). To derive the E_6 quiver we need to map the nodes of the parent D_2 quiver as

$$\begin{aligned} R^0_{D_2(6)} &\to R^{1,irrep}_{E_6(2)} \oplus R^{2,irrep}_{E_6(2)} \oplus R^{3,irrep}_{E_6(2)} \\ R^0_{D_2(3)} &\to R^{1,irrep}_{E_6(1)} \oplus R^{2,irrep}_{E_6(1)} \oplus R^{3,irrep}_{E_6(1)} \\ R^l_{D_2(3)} \cong R^{f(l)}_{D_2(3)} &\to R^{irrep}_{E_6(3)}, \qquad l \in \mathbb{Z}_4 - \{0\}. \end{aligned}$$

Consistency requires that if we replace R_{D_2} in the D_2 quiver defining relations and then use the above mappings, we get the E_6 quiver relations for $R^{irrep}_{E_6}$.

The origin of this $\mathbb{Z}_3$ analogue of the orientifold $\mathbb{Z}_2$-projection is thus far unknown to us. If an object with this property is to exist, then the brane setup for the E_6 theory could be implemented; on the other hand if it does not, then we would be suggested at why the attempt for E_6 has been prohibitively difficult.

The $\mathbb{Z}_3$ action has been noted to arise in [172] in the context of quotienting the $\mathbb{Z}_n \times \mathbb{Z}_n$ quiver to arrive at the quiver for the Δ-series. Indeed from our comparative study in Section 4.2.1, we see that in general, labelling the irreps by a multi-index is precisely our stepwise algorithm in disguise, as applied to a product Abelian group: the $\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$ orbifold. Therefore in a sense we have explained why the labelling scheme of [141, 172] should work.

And the same goes with E_7 and E_8: we could perform stepwise projection thereupon and mathematically obtain their quivers as appropriate quotients of the $\mathbb{Z}_n$ quiver by the symmetry S of the identification and splitting of nodes. To find a physical brane setup, we would then need to find an object in string theory which has an S action on space-time and the string world-sheet. Note that the above are cases of the $\mathbb{C}^2$ orbifolds; for the $\mathbb{C}^k$-orbifold we should initialise our algorithm with, and perform stepwise projection on the quiver of $\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$ ($k-1$ times), i.e., the brane box and cube ($k = 2, 3$).

Though mathematically we have found a systematic treatment of constructing quivers under a new light, namely the "stepwise projection" from the Abelian quiver, much work remains. In the field of brane setups for singularities, our algorithm is intended to be a small step for an old standing problem. We must now diligently seek a generalisation of the orientifold plane with symmetry S in string theory, that could perform the physical task which our mathematical methodology demands.

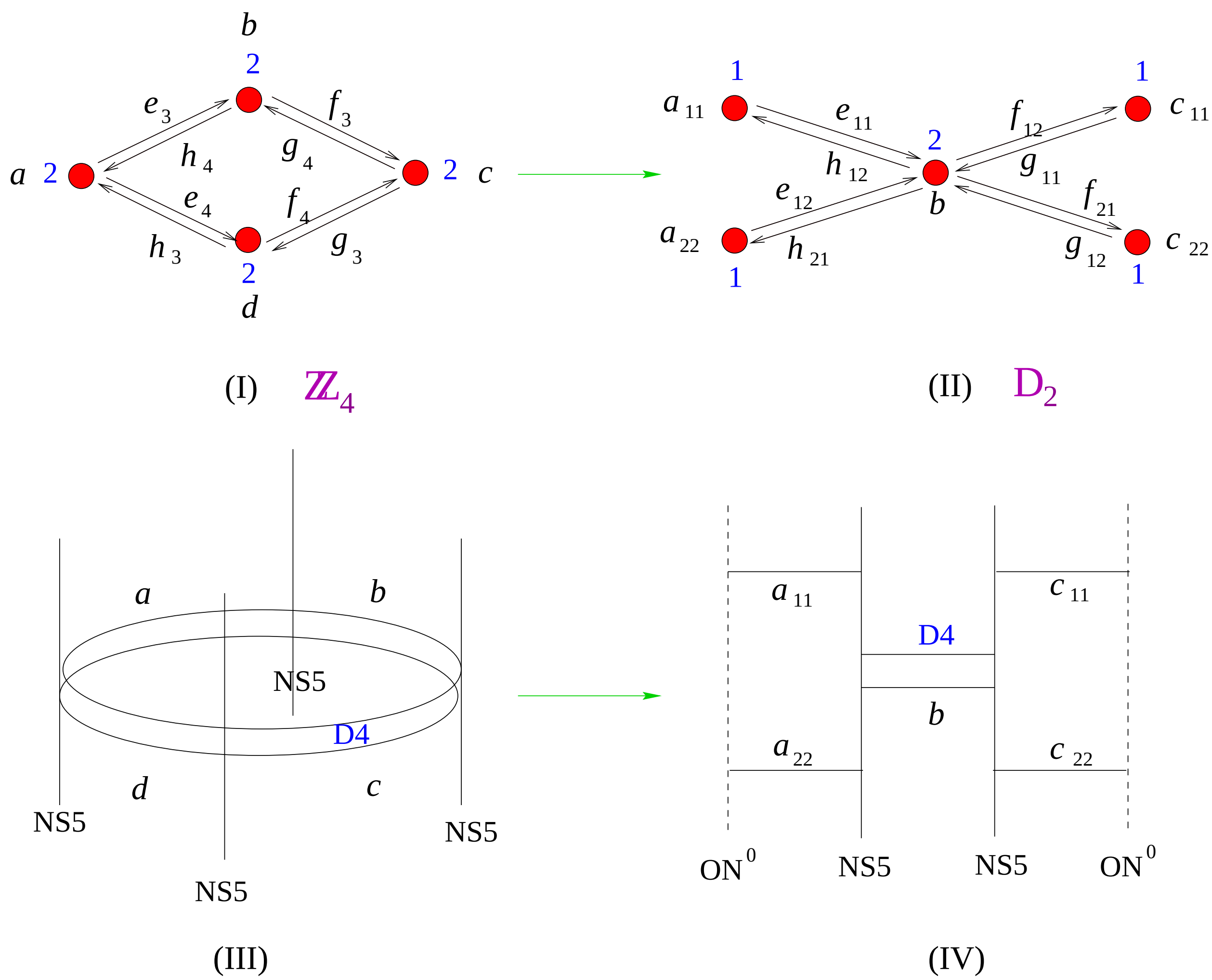


Figure 16-1: From the fact that $D_2 := \langle \beta_4, \gamma \rangle$ is generated by $\mathbb{Z}_4 = \beta_4$ together with γ, our stepwise projection, first by β_4, and then by γ, gives 2 copies of the $\mathbb{Z}_4$ quiver in Part (I) and then the D_2 quiver in Part (II) by appropriate joining/splitting of the nodes and arrows. The brane configurations for these theories are given in Parts (III) and (IV).

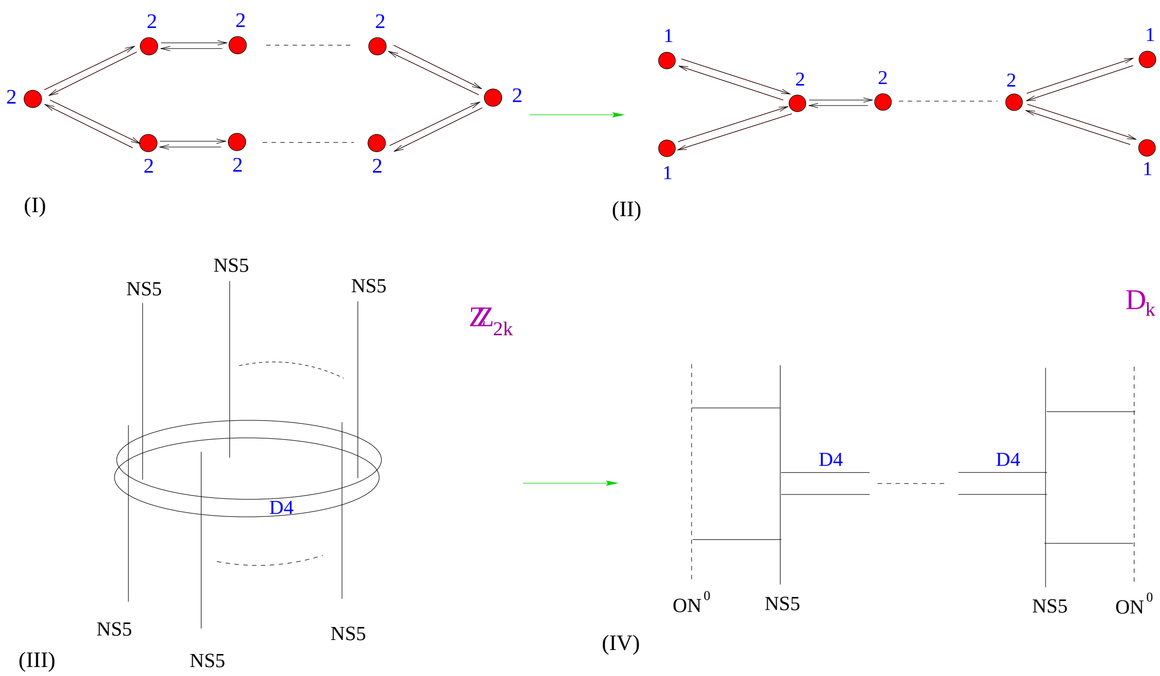


Figure 16-2: Obtaining the D_k quiver (II) from the $\mathbb{Z}_{2k}$ quiver (I) by the stepwise projection algorithm. The brane setups are given respectively in (IV) and (III).

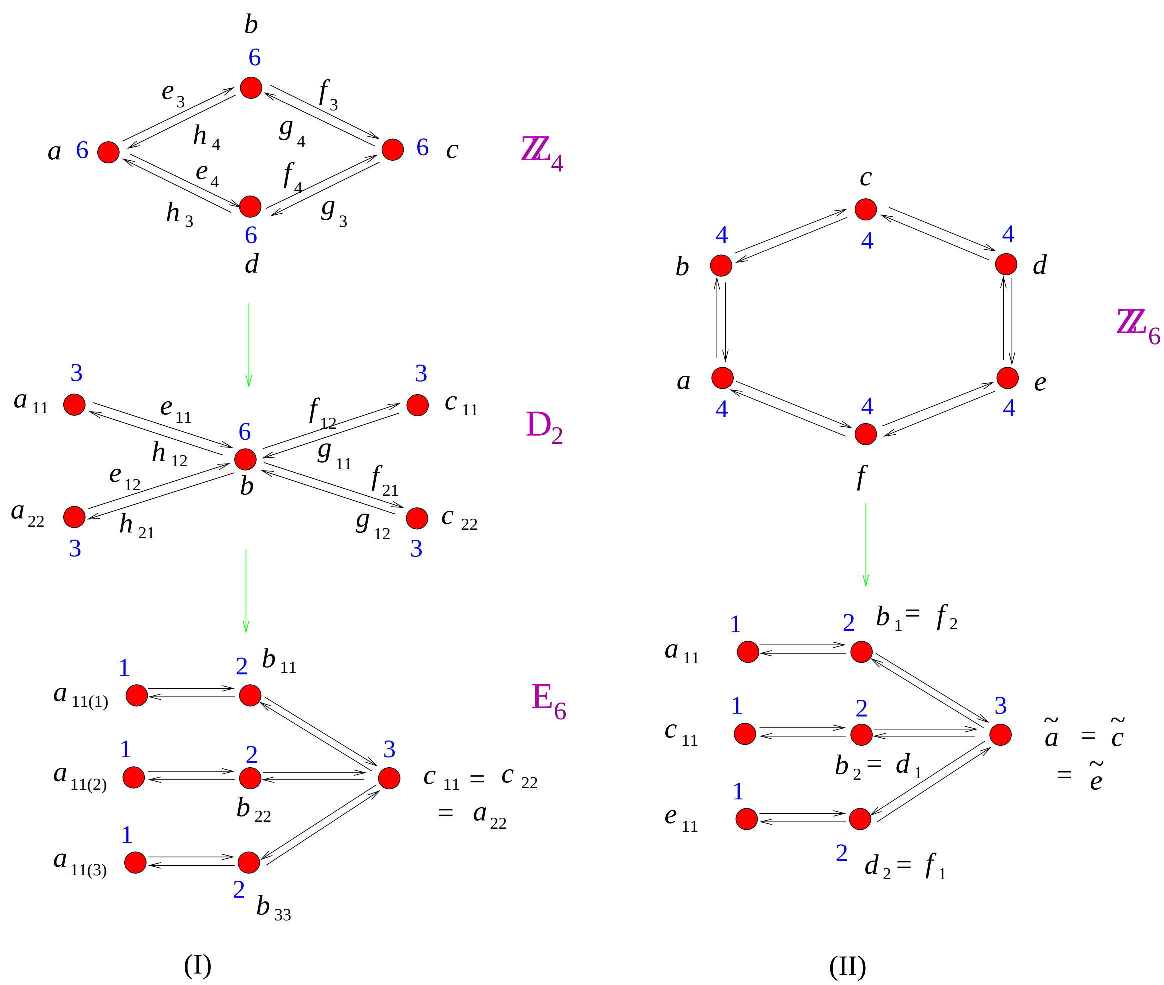


Figure 16-3: Obtaining the quiver diagram for the binary tetrahedral group E_6. We compare the two alternative stepwise projections: (I) $\mathbb{Z}_4 = \langle\beta_4\rangle \to D_2 = \langle\beta_4, \gamma\rangle \to E_6 = \langle\beta_4, \gamma, \delta\rangle$ and (II) $\mathbb{Z}_6 = \langle\delta\rangle \to E_6 = \langle\delta, \beta_4\rangle$.

Chapter 17

Orbifolds VIII: Orbifolds with Discrete Torsion and the Schur Multiplier

Synopsis

Let us now proceed with another aspect of D-brane probes on singularities, namely with the presence of background B-fields, i.e., to allow discrete torsion. Armed with the explicit computation of Schur Multipliers, we offer a classification of $SU(n)$ orbifolds for $n = 2, 3, 4$ which permit the turning on of discrete torsion.

As a by-product, we find a hitherto unknown class of $\mathcal{N} = 1$ orbifolds with non-cyclic discrete torsion group. Furthermore, we supplement the *status quo ante* by investigating a first example of a non-Abelian orbifold admitting discrete torsion, namely the ordinary dihedral group as a subgroup of $SU(3)$. A comparison of the quiver theory thereof with that of its covering group, the binary dihedral group, without discrete torsion, is also performed [301].

17.1 Introduction

The study of string theory in non-trivial NS-NS B-field backgrounds has of late become one of the most pursued directions of research. Ever since the landmark papers [246], where it was shown that in the presence of such non-trivial B-fields along the world-volume directions of the D-brane, the gauge theory living thereupon assumes a non-commutative guise in the large-B-limit, most works were done in this direction of space-time non-commutativity. However, there is an alternative approach in the investigation of the effects of the B-field, namely **discrete torsion**, which is of great interest in this respect. On the other hand, as discrete torsion presents itself to be a natural generalisation to the study of orbifold projections of D-brane probes at space-time singularities, a topic under much research over the past few years, it is also mathematically and physically worthy of pursuit under this light.

A brief review of the development of the matter from a historical perspective shall serve to guide the reader. Discrete torsion first appeared in [124] in the study of the closed string partition function $Z(q,\bar{q})$ on the orbifold G. And shortly thereafter, it effects on the geometry of space-time were pointed out [247]. In particular, [124] noticed that $Z(q,\bar{q})$ could contain therein, phases $\epsilon(g,h) \in U(1)$ for $g,h \in G$, coming from the twisted sectors of the theory, as long as

$$\begin{aligned} &\epsilon(g_1 g_2, g_3) = \epsilon(g_1, g_3)\epsilon(g_2, g_3) \\ &\epsilon(g,h) = 1/\epsilon(h,g) \\ &\epsilon(g,g) = 1, \end{aligned} \tag{17.1.1}$$

so as to ensure modular invariance.

Reviving interests along this line, Douglas and Fiol [248, 249] extended discrete torsion to the open string sector by showing that the usual procedure of projection by orbifolds on D-brane probes [69, 76], applied to **projective representations** instead of the ordinary *linear representations* of the orbifold group G, gives exactly the gauge theory with discrete torsion turned on. In other words, for the invariant matter fields which survive the orbifold, Φ such that $\gamma^{-1}(g)\Phi\gamma(g) = r(g)\Phi, \quad \forall\ g \in G$, we now need the representation

$$\gamma(g)\gamma(h) = \alpha(g,h)\gamma(gh), \quad g,h \in G \text{ with}$$
$$\alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z), \qquad \alpha(x,\mathbb{I}_G) = 1 = \alpha(\mathbb{I}_G,x) \quad \forall x,y,z \in G, \tag{17.1.2}$$

where $\alpha(g,h)$ is known as a cocycle. These cocycles constitute, up to the equivalence

$$\alpha(g,h) \sim \frac{c(g)c(h)}{c(gh)}\alpha(g,h), \tag{17.1.3}$$

the so-called second cohomology group $H^2(G,U(1))$ of G, where c is any function (not necessarily a homomorphism) mapping G to $U(1)$; this is what we usually mean by *discrete torsion being classified by* $H^2(G,U(1))$. We shall formalise all these definitions in the subsequent sections.

In fact, one can show [124], that the choice

$$\epsilon(g,h) = \frac{\alpha(g,h)}{\alpha(h,g)},$$

for α obeying (17.1.2) actually satisfies (17.1.1), whereby linking the concepts of discrete torsion in the closed and open string sectors. We point this out as one could be easily confused as to the precise parametre called discrete torsion and which is actually classified by the second group cohomology.

Along the line of [248, 249], a series of papers by Berenstein, Leigh and Jejjala [250, 251] developed the technique to study the *non-commutative* moduli space of the $\mathcal{N}=1$ gauge theory living on $\mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n$ parametrised as an algebraic variety. A host of activities followed in the generalisation of this abelian orbifold, notably to $\mathbb{C}^4/\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ by [252], to the inclusion of orientifolds by [253], and to the orbifolded conifold by [254].

Along the mathematical thread, Sharpe has presented a prolific series of works to relate discrete torsion with connection on gerbes [257], which could allow generalisations of the concept to beyond the 2-form B-field. Moreover, in relation to twisted K-theory and attempts to unify space-time cohomology with group cohomology in the vein of the McKay Correspondence (see e.g. [293]), works by Gomis [258] and Aspinwall-Plesser [259, 260] have given some guiding light.

Before we end this review of the current studies, we would like to mention the work by Gaberdiel [255]. He pointed out that there exists a different choice, such that the original intimate relationship between discrete torsion in the closed string sector and the non-trivial cocycle in the open sector can be loosened. It would be interesting to investigate further in this spirit.

We see however, that during these last three years of renewed activity, the focus has mainly been on Abelian orbifolds. It is one of the main intentions of this chapter to initiate the study of non-Abelian orbifolds with discrete torsion, which, to the best of our knowledge, have not been discussed so far in the literature[1]. We shall classify the general orbifold theories with $\mathcal{N} = 0, 1, 2$ supersymmetry which *could allow discrete torsion* by exhaustively computing the second cohomology of the discrete subgroups of $SU(n)$ for $n = 4, 3, 2$.

Thus rests the current state of affairs. Our main objectives are two-fold: to both supplement the past, by presenting and studying a first example of a non-Abelian orbifold which affords discrete torsion, and to presage the future, by classifying the orbifold theories which could allow discrete torsion being turned on.

[1]In the context of conformal field theory on orbifolds, there has been a recent work addressing some non-Abelian cases [266].

Nomenclature

Throughout this chapter, unless otherwise specified, we shall adhere to the following conventions for notation:

ω_n	n-th root of unity;
G	finite group of order $\|G\|$;
$\mathbb{F}$	(algebraically closed) number field;
$\mathbb{F}^*$	multiplicative subgroup of $\mathbb{F}$;
$\langle x_i \| y_j \rangle$	the group generated by elements $\{x_i\}$ with relations y_j;
$< G_1, G_2, \ldots, G_n >$	group generated by the generators of groups $G_1, G_2, \ldots, G_n$;
$\gcd(m, n)$	the greatest common divisor of m and n;
$D_{2n}, E_{6,7,8}$	ordinary dihedral, tetrahedral, octahedral and icosahedral groups;
$\widehat{D_{2n}}, \widehat{E_{6,7,8}}$	the binary counterparts of the above;
A_n and S_n	alternating and symmetric groups on n elements;
$H \triangleleft G$	H is a normal subgroup of G;
$A \rtimes B$	semi-direct product of A and B;
$Z(G)$	centre of G;
$N_G(H)$	the normaliser of $H \subset G$;
$G' := [G, G]$	the derived (commutator) group of G;
$\exp(G)$	exponent of group G.

17.2 Some Mathematical Preliminaries

17.2.1 Projective Representations of Groups

We begin by first formalising (17.1.2), the group representation of our interest:

DEFINITION 17.2.23 *A* **projective** *representation of* G *over a field* $\mathbb{F}$ *(throughout we let* $\mathbb{F}$ *be an algebraically closed field with characteristic* $p \geq 0$*) is a mapping* $\rho : G \rightarrow GL(V)$ *such that*

$$(A) \quad \rho(x)\rho(y) = \alpha(x, y)\rho(xy) \ \ \forall \ \ x, y \in G; \qquad (B) \quad \rho(\mathbb{I}_G) = \mathbb{I}_V.$$

Here $\alpha : G \times G \to \mathbb{F}^*$ is a mapping whose meaning we shall clarify later. Of course we see that if $\alpha = 1$ trivially, then we have our familiar ordinary representation of G to which we shall refer as *linear*. Indeed, the mapping ρ into $GL(V)$ defined above is naturally equivalent to a homomorphism into the projective linear group $PGL(V) \cong GL(V)/\mathbb{F}^*\mathbb{I}_V$, and hence the name "projective." In particular we shall be concerned with projective *matrix* representations of G where we take $GL(V)$ to be $GL(n,\mathbb{F})$.

The function α can not be arbitrary and two immediate restrictions can be placed thereupon purely from the structure of the group:

$$\begin{array}{llll} (a) & \text{Group Associativity} & \Rightarrow & \alpha(x,y)\alpha(xy,z) = \alpha(x,yz)\alpha(y,z), \quad \forall x,y,z \in G \\ (b) & \text{Group Identity} & \Rightarrow & \alpha(x,\mathbb{I}_G) = 1 = \alpha(\mathbb{I}_G,x), \quad \forall x \in G. \end{array} \tag{17.2.4}$$

These conditions on α naturally leads to another discipline of mathematics.

17.2.2 Group Cohomology and the Schur Multiplier

The study of such functions on a group satisfying (17.2.4) is precisely the subject of the theory of **Group Cohomology**. In general we let α to take values in A, an abelian coefficient group ($\mathbb{F}^*$ is certainly a simple example of such an A) and call them **cocycles**. The set of all cocycles we shall name $Z^2(G,A)$. Indeed it is straightforward to see that $Z^2(G,A)$ is an abelian group. We subsequently define a set of functions satisfying

$$B^2(G,A) := \{(\delta g)(x,y) := g(x)g(y)g(xy)^{-1}\} \quad \text{for any } g : G \to A \text{ such that } g(\mathbb{I}_G) = 1, \tag{17.2.5}$$

and call them *coboundaries*. It is then obvious that $B^2(G,A)$ is a (normal) subgroup of $Z^2(G,A)$ and in fact constitutes an equivalence relation on the latter in the manner of (17.1.3). Thus it becomes a routine exercise in cohomology to define

$$H^2(G,A) := Z^2(G,A)/B^2(G,A),$$

the *second cohomology* group of G.

Summarising what we have so far, we see that the projective representations of G are classified by its second cohomology $H^2(G,\mathbb{F}^*)$. To facilitate the computation thereof, we shall come to an important concept:

DEFINITION 17.2.24 *The* **Schur Multiplier** $M(G)$ *of the group* G *is the second cohomology group with respect to the trivial action of* G *on* $\mathbb{C}^*$*:*

$$M(G) := H^2(G,\mathbb{C}^*).$$

Since we shall be mostly concerned with the field $\mathbb{F} = \mathbb{C}$, the Schur multiplier is exactly what we need. However, the properties thereof are more general. In fact, for any algebraically closed field $\mathbb{F}$ of zero characteristic, $M(G) \cong H^2(G,\mathbb{F}^*)$. In our case of $\mathbb{F} = \mathbb{C}$, it can be shown that [253],

$$H^2(G,\mathbb{C}^*) \cong H^2(G,U(1)).$$

This terminology is the more frequently encountered one in the physics literature.

One task is thus self-evident: the calculation of the Schur Multiplier of a given group G shall indicate possibilities of projective representations of the said group, or in a physical language, the possibilities of turning on discrete torsion in string theory on the orbifold group G. In particular, if $M(G) \cong \mathbb{I}$, then the second cohomology of G is trivial and no non-trivial discrete torsion is allowed. We summarise this

KEY POINT: Calculate $M(G) \Rightarrow$ Information on Discrete Torsion.

17.2.3 The Covering Group

The study of the actual projective representation of G is very involved and what is usually done in fact is to "lift to an ordinary representation." What this means is that for a central extension[2] A of G to G^*, we say a projective representation ρ of G **lifts** to a linear representation ρ^* of G^* if (i) $\rho^*(a \in A)$ is proportional to $\mathbb{I}$ and (ii) there is a section[3] $\mu : G \to G^*$ such that $\rho(g) = \rho^*(\mu(g)), \quad \forall g \in G$. Likewise it *lifts projectively* if $\rho(g) = t(g)\rho^*(\mu(g))$ for a map $t : G \to \mathbb{F}^*$. Now we are ready to give the following:

DEFINITION 17.2.25 *We call G^* a* **covering group**[4] *of G over $\mathbb{F}$ if the following are satisfied:*
(i) $\exists$ a central extension $1 \to A \to G^* \to G \to 1$ such that any projective representation of G lifts projectively to an ordinary representation of G^*;
(ii) $|A| = |H^2(G,\mathbb{F}^*)|$.

The following theorem, initially due to Schur, characterises covering groups.

THEOREM 17.2.23 ([262] p143) *$G^\star$ is a covering group of G over $\mathbb{F}$ if and only if the following conditions hold:*
(i) $G^\star$ has a finite subgroup A with $A \subseteq Z(G^\star) \cap [G^\star, G^\star]$;
(ii) $G \cong G^\star/A$;
(iii) $|A| = |H^2(G, F^\star)|$
where $[G^\star, G^\star]$ is the derived group[5] $G^{*'}$ of G^*.

Thus concludes our prelude on the mathematical rudiments, the utility of the above results shall present themselves in the ensuing.

17.3 Schur Multipliers and String Theory Orbifolds

The game is thus afoot. Orbifolds of the form $\mathbb{C}^k/\{G \in SU(k)\}$ have been widely studied in the context of gauge theories living on D-branes probing the singularities.

[2] i.e., A in the centre $Z(G^*)$ and $G^*/A \cong G$ according to the exact sequence $1 \to A \to G^* \to G \to 1$.
[3] i.e., for the projection $f : G^* \to G$, $\mu \circ f = \mathbb{I}_G$.
[4] *Sometimes is also known as* **representation group**.
[5] *For a group G, $G' := [G,G]$ is the group generated by elements of the form $xyx^{-1}y^{-1}$ for $x, y \in G$.*

We need only to compute $M(G)$ for the discrete finite groups of $SU(n)$ for $n = 2, 3, 4$ to know the discrete torsion afforded by the said orbifold theories.

17.3.1 The Schur Multiplier of the Discrete Subgroups of $SU(2)$

Let us first remind the reader of the well-known ADE classification of the discrete finite subgroups of $SU(2)$. Here are the presentations of these groups:

G	Name	Order	Presentation
$\widehat{A_n}$	Cyclic, $\cong \mathbb{Z}_{n+1}$	n	$\langle a \vert a^n = \mathbb{I} \rangle$
$\widehat{D_{2n}}$	Binary Dihedral	$4n$	$\langle a, b \vert b^2 = a^n, abab^{-1} = \mathbb{I} \rangle$
$\widehat{E_6}$	Binary Tetrahedral	24	$\langle a, b \vert a^3 = b^3 = (ab)^3 \rangle$
$\widehat{E_7}$	Binary Octahedral	48	$\langle a, b \vert a^4 = b^3 = (ab)^2 \rangle$
$\widehat{E_8}$	Binary Icosahedral	120	$\langle a, b \vert a^5 = b^3 = (ab)^2 \rangle$

(17.3.6)

We here present a powerful result due to Schur (1907) (q.v. Cor. 2.5, Chap. 11 of [263]) which aids us to explicitly compute large classes of Schur multipliers for finite groups:

THEOREM 17.3.24 ([262] p383) *Let G be generated by n elements with (minimally) r defining relations and let the Schur multiplier $M(G)$ have a minimum of s generators, then*

$$r \geq n + s.$$

In particular, $r = n$ implies that $M(G)$ is trivial and $r = n + 1$, that $M(G)$ is cyclic.

Theorem 17.3.24 could be immediately applied to $G \in SU(2)$.

Let us proceed with the computation case-wise. The $\widehat{A_n}$ series has 1 generator with 1 relation, thus $r = n = 1$ and $M(\widehat{A_n})$ is trivial. Now for the $\widehat{D_{2n}}$ series, we note briefly that the usual presentation is $\widehat{D_{2n}} := \langle a, b \vert a^{2n} = \mathbb{I}, b^2 = a^n, bab^{-1} = a^{-1} \rangle$ as in [296]; however, we can see easily that the last two relations imply the first, or explicitly: $a^{-n} := (bab^{-1})^n = ba^n b^{-1} = a^n$, (q.v. [263] Example 3.1, Chap. 11), whence making $r = n = 2$, i.e., 2 generators and 2 relations, and further making $M(\widehat{D_{2n}})$ trivial. Thus too are the cases of the 3 exceptional groups, each having 2 generators with 2 relations. In summary then we have the following corollary of Theorem 17.3.24, the well-known [259] result that

COROLLARY 17.3.7 *All discrete finite subgroups of $SU(2)$ have second cohomology $H^2(G,\mathbb{C}^*) = \mathbb{I}$, and hence afford no non-trivial discrete torsion.*

It is intriguing that the above result can actually be hinted from physical considerations without recourse to heavy mathematical machinery. The orbifold theory for $G \subset SU(2)$ preserves an $\mathcal{N} = 2$ supersymmetry on the world-volume of the D3-Brane probe. Inclusion of discrete torsion would deform the coefficients of the superpotential. However, $\mathcal{N} = 2$ supersymmetry is highly restrictive and in general does not permit the existence of such deformations. This is in perfect harmony with the triviality of the Schur Multiplier of $G \subset SU(2)$ as presented in the above Corollary.

To address more complicated groups we need a methodology to compute the Schur Multiplier, and we have many to our aid, for after all the computation of $M(G)$ is a vast subject entirely by itself. We quote one such method below, a result originally due to Schur:

THEOREM 17.3.25 ([264] p54) *Let $G = F/R$ be the defining finite presentation of G with F the free group of rank n and R is (the normal closure of) the set of relations. Suppose $R/[F,R]$ has the presentation $\langle x_1, \ldots, x_m; y_1, \ldots, y_n\rangle$ with all x_i of finite order, then*

$$M(G) \cong \langle x_1, \ldots, x_n\rangle.$$

Two more theorems of great usage are the following:

THEOREM 17.3.26 ([264] p17) *Let the exponent*[6] *of $M(G)$ be $\exp(M(G))$, then*

$$\exp(M(G))^2 \textit{ divides } |G|.$$

[6] *i.e., the lowest common multiple of the orders of the elements.*

And for direct products, another fact due to Schur,

THEOREM 17.3.27 ([264] p37)

$$M(G_1 \times G_2) \cong M(G_1) \times M(G_2) \times (G_1 \otimes G_2),$$

where $G_1 \otimes G_2$ *is defined to be* $\mathrm{Hom}_{\mathbb{Z}}(G_1/G_1', G_2/G_2')$.

With the above and a myriad of useful results (such as the Schur Multiplier for semi-direct products), and especially with the aid of the Computer Algebra package GAP [92] using the algorithm developed for the p-Sylow subgroups of Schur Multiplier [265], we have engaged in the formidable task of giving the explicit Schur Multiplier of the list of groups of our interest.

Most of the details of the computation we shall leave to the appendix, to give the reader a flavour of the calculation but not distracting him or her from the main course of our writing. Without much further ado then, we now proceed with the list of Schur Multipliers for the discrete subgroups of $SU(n)$ for $n = 3, 4$, i.e., the $\mathcal{N} = 1, 0$ orbifold theories.

17.3.2 The Schur Multiplier of the Discrete Subgroups of $SU(3)$

The classification of the discrete finite groups of $SU(3)$ is well-known (see e.g. [90, 292, 141] for a discussion thereof in the context of string theory). It was pointed out in [296] that the usual classification of these groups does not include the so-called *intransitive* groups (see [294] for definitions), which are perhaps of less mathematical interest. Of course from a physical stand-point, they all give well-defined orbifolds. More specifically [296], all the ordinary polyhedral subgroups of $SO(3)$, namely the ordinary dihedral group D_{2n} and the ordinary $E_6 \cong A_4 \cong \Delta(3\times 2^2), E_7 \cong S_4 \cong \Delta(6\times 2^2), E_8 \cong \Sigma_{60}$, due to the embedding $SO(3) \hookrightarrow SU(3)$, are obviously (intransitive) subgroups thereof and thus we shall include these as well in what follows. We discuss some aspects of the intransitives in Appendix 22.8 and are grateful to D. Berenstein for pointing out some subtleties involved [217]. We insert one more cautionary note. The $\Delta(6n^2)$ series does not actually include the cases for n odd [141]; therefore n shall be restricted to be even.

Here then are the Schur Multipliers of the $SU(3)$ discrete subgroups (I stands for

Intransitives and T, intransitives).

	G	Order	Schur Multiplier $M(G)$
I	$\mathbb{Z}_n \times \mathbb{Z}_m$	$n \times m$	$\mathbb{Z}_{\gcd(n,m)}$
	$< \mathbb{Z}_n, \widehat{D_{2m}} >$	$\begin{cases} n \times 4m & n \text{ odd} \\ \frac{n}{2} \times 4m & n \text{ even} \end{cases}$	$\begin{cases} \mathbb{I} & n \bmod 4 \neq 1 \\ \mathbb{Z}_2 & n \bmod 4 = 0, m \text{ odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & n \bmod 4 = 0, m \text{ even} \end{cases}$
	$< \mathbb{Z}_n, \widehat{E_6} >$	$\begin{cases} n \times 24 & n \text{ odd} \\ \frac{n}{2} \times 24 & n \text{ even} \end{cases}$	$\mathbb{Z}_{\gcd(n,3)}$
	$< \mathbb{Z}_n, \widehat{E_7} >$	$\begin{cases} n \times 48 & n \text{ odd} \\ \frac{n}{2} \times 48 & n \text{ even} \end{cases}$	$\begin{cases} \mathbb{I} & n \bmod 4 \neq 0 \\ \mathbb{Z}_2 & n \bmod 4 = 0 \end{cases}$
	$< \mathbb{Z}_n, \widehat{E_8} >$	$\begin{cases} n \times 120 & n \text{ odd} \\ \frac{n}{2} \times 120 & n \text{ even} \end{cases}$	$\mathbb{I}$
	Ordinary Dihedral D_{2n}	$2n$	$\mathbb{Z}_{\gcd(n,2)}$
	$< \mathbb{Z}_n, D_{2m} >$	$\begin{cases} n \times 2m & m \text{ odd} \\ n \times 2m & m \text{ even}, n \text{ odd} \\ \frac{n}{2} \times 2m & m \text{ even}, n \text{ even} \end{cases}$	$\begin{cases} \mathbb{Z}_{\gcd(n,2)} & m \text{ odd} \\ \mathbb{Z}_2 & m \text{ even}, n \bmod 4 = 1,2,3 \\ \mathbb{Z}_2 & m \bmod 4 \neq 0, n \bmod 4 = 0 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & m \bmod 4 = 0, n \bmod 4 = 0 \end{cases}$
T	Δ_{3n^2}	$3n^2$	$\begin{cases} \mathbb{Z}_n \times \mathbb{Z}_3, & \gcd(n,3) \neq 1 \\ \mathbb{Z}_n, & \gcd(n,3) = 1 \end{cases}$
	Δ_{6n^2} (n even)	$6n^2$	$\mathbb{Z}_2$
	$\Sigma_{60} \cong A_5$	60	$\mathbb{Z}_2$
	Σ_{168}	168	$\mathbb{Z}_2$
	Σ_{108}	36×3	$\mathbb{I}$
	Σ_{216}	72×3	$\mathbb{I}$
	Σ_{648}	216×3	$\mathbb{I}$
	Σ_{1080}	360×3	$\mathbb{Z}_2$

(17.3.7)

Some immediate comments are at hand. The question of whether any discrete subgroup of $SU(3)$ admits non-cyclic discrete torsion was posed in [259]. From our results in table (17.3.7), we have shown by explicit construction that the answer is in the affirmative: not only the various intransitives give rise to product cyclic Schur Multipliers, so too does the transitive $\Delta(3n^2)$ series for n a multiple of 3.

In Appendix 22.7 we shall present the calculation for $M(\Delta_{3n^2})$ and $M(\Delta_{6n^2})$ for illustrative purposes. Furthermore, as an example of non-Abelian orbifolds with discrete torsion, we shall investigate the series of the ordinary dihedral group in detail with applications to physics in mind. For now, for the reader's edification or amusement, let us continue with the $SU(4)$ subgroups.

17.3.3 The Schur Multiplier of the Discrete Subgroups of $SU(4)$

The discrete finite subgroups of $SL(4,\mathbb{C})$, which give rise to non-supersymmetric orbifold theories, are presented in modern notation in [294]. Using the notation therein, and recalling that the group names in $SU(4) \subset SL(4,\mathbb{C})$ were accompanied with a star (*cit. ibid.*), let us tabulate in (17.3.8) the Schur Multiplier of the exceptional cases of these particulars (cases XXIX$*$ and XXX$*$ were computed by Prof. H. Pahlings to whom we are grateful).

G	Order	Schur Mult. $M(G)$
I$*$	60×4	$\mathbb{I}$
II$* \cong \Sigma_{60}$	60	$\mathbb{Z}_2$
III$*$	360×4	$\mathbb{Z}_3$
IV$*$	$\frac{1}{2}7! \times 2$	$\mathbb{Z}_3$
VI$*$	$2^6 3^4 5 \times 2$	$\mathbb{I}$
VII$*$	120×4	$\mathbb{Z}_2$
VIII$*$	120×4	$\mathbb{Z}_2$
IX$*$	720×4	$\mathbb{Z}_2$
X$*$	144×2	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XI$*$	288×2	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XII$*$	288×2	$\mathbb{Z}_2$
XIII$*$	720×2	$\mathbb{Z}_2$
XIV$*$	576×2	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XV$*$	1440×2	$\mathbb{Z}_2$

G	Order	Schur Mult. $M(G)$
XVI$*$	3600×2	$\mathbb{Z}_2$
XVII$*$	576×4	$\mathbb{Z}_2$
XVIII$*$	576×4	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XIX$*$	288×4	$\mathbb{I}$
XX$*$	7200×4	$\mathbb{I}$
XXI$*$	1152×4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XXII$*$	$5 \times 16 \times 4$	$\mathbb{Z}_2$
XXIII$*$	$10 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XXIV$*$	$20 \times 16 \times 4$	$\mathbb{Z}_2$
XXV$*$	$60 \times 16 \times 4$	$\mathbb{Z}_2$
XXVI$*$	$60 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
XXVII$*$	$120 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
XXVIII$*$	$120 \times 16 \times 4$	$\mathbb{Z}_2$
XXIX$*$	$360 \times 16 \times 4$	$\mathbb{Z}_2 \times \mathbb{Z}_3$
XXX$*$	$720 \times 16 \times 4$	$\mathbb{Z}_2$

(17.3.8)

17.4 D_{2n} Orbifolds: Discrete Torsion for a non-Abelian Example

As advertised earlier at the end of subsection 3.2, we now investigate in depth the discrete torsion for a non-Abelian orbifold. The ordinary dihedral group $D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$ of order $2n$, has the presentation

$$D_{2n} = \langle a, b | a^n = 1, b^2 = 1, bab^{-1} = a^{-1} \rangle.$$

As tabulated in (17.3.7), the Schur Multiplier is $M(D_{2n}) = \mathbb{I}$ for n odd and $\mathbb{Z}_2$ for n even [262]. Therefore the n odd cases are no different from the ordinary linear representations as studied in [296] since they have trivial Schur Multiplier and hence trivial discrete torsion. On the other hand, for the n even case, we will demonstrate the following result:

PROPOSITION 17.4.9 *The binary dihedral group $\widehat{D_{2n}}$ of the D-series of the discrete subgroups of $SU(2)$ (otherwise called the generalised quaternion group) is the covering group of D_{2n} when n is even.*

Proof: The definition of the binary dihedral group $\widehat{D_{2n}}$, of order $4n$, is

$$\widehat{D_{2n}} = \langle a, b | a^{2n} = 1, b^2 = a^n, bab^{-1} = a^{-1} \rangle,$$

as we saw in subsection 3.1. Let us check against the conditions of Theorem 17.2.23. It is a famous result that $\widehat{D_{2n}}$ is the double cover of D_{2n} and whence an $\mathbb{Z}_2$ central extension. First we can see that $A = Z(\widehat{D_{2n}}) = \{1, a^n\} \cong \mathbb{Z}_2$ and condition (ii) is satisfied.

Second we find that the commutators are $[a^x, a^y] := (a^x)^{-1}(a^y)^{-1}a^x a^y = 1$, $[a^x b, a^y b] = a^{2(x-y)}$ and $[a^x b, a^y] = a^{2y}$. From these we see that the derived group $[\widehat{D_{2n}}, \widehat{D_{2n}}]$ is generated by a^2 and is thus equal to $\mathbb{Z}_n$ (since a is of order $2n$). An important point is that only when n is even does A belong to $Z(\widehat{D_{2n}}) \cap [\widehat{D_{2n}}, \widehat{D_{2n}}]$. This result is consistent with the fact that for odd n, D_{2n} has trivial Schur Multiplier. Finally of course, $|A| = |H^2(G, \mathbb{F}^*)| = 2$. Thus conditions (i) and (iii) are also satisfied. We therefore conclude that for even n, $\widehat{D_{2n}}$ is the covering group of D_{2n}.

17.4.1 The Irreducible Representations

With the above Proposition, we know by the very definition of the covering group, that the projective representation of D_{2n} should be encoded in the linear representation of $\widehat{D_{2n}}$, which is a standard result that we can recall from [296]. The latter has four 1-dimensional and $n-1$ 2-dimensional irreps. The matrix representations of these 2-dimensionals for the generic elements a^p, ba^p $(p = 0, ..., 2n-1)$ are given below:

$$a^p = \begin{pmatrix} \omega_{2n}^{lp} & 0 \\ 0 & \omega_{2n}^{-lp} \end{pmatrix} \qquad ba^p = \begin{pmatrix} 0 & i^l \omega_{2n}^{-lp} \\ i^l \omega_{2n}^{lp} & 0 \end{pmatrix}, \tag{17.4.9}$$

with $l = 1, ..., n-1$; these are denoted as χ_2^l. On the other hand, the four 1-dimensionals are

	$n =$ even				$n =$ odd			
	a^{even}	$a(a^{\text{odd}})$	$b(ba^{\text{even}})$	$ba(ba^{\text{odd}})$	a^{even}	$a(a^{\text{odd}})$	$b(ba^{\text{even}})$	$ba(ba^{\text{odd}})$
χ_1^1	1	1	1	1	1	1	1	1
χ_1^2	1	-1	1	-1	1	-1	ω_4	$-\omega_4$
χ_1^3	1	1	-1	-1	1	1	-1	-1
χ_1^4	1	-1	-1	1	1	-1	$-\omega_4$	ω_4

(17.4.10)

We can subsequently obtain all irreducible projective representations of D_{2n} from the above (henceforth n will be even). Recalling that $\widehat{D_{2n}}/\{1, a^n\} \cong D_{2n}$ from property (ii) of Theorem 17.2.23, we can choose one element of each of the transversals of $\widehat{D_{2n}}$ with respect to the $\mathbb{Z}_2$ to be mapped to D_{2n}. For convenience we choose $b^x a^y$ with $x = 0, 1$ and $y = 0, 1, ..., n-1$, a total of $4n/2 = 2n$ elements. Thus we are effectively expressing D_{2n} in terms of $\widehat{D_{2n}}$ elements.

For the matrix representation of $a^n \in \widehat{D_{2n}}$, there are two cases. In the first, we have $a^n = 1 \times I_{d\times d}$ where d is the dimension of the representation. This case includes all four 1-dimensional representations and $(n/2-1)$ 2-dimensional representations in

(17.4.9) for $l = 2, 4, ..., n-2$. Because a^n has the same matrix form as $\mathbb{I}$, we see that the elements $b^x a^y$ and $b^x a^{y+n}$ also have the same matrix form. Consequently, when we map them to D_{2n}, they automatically give the irreducible linear representations of D_{2n}.

In the other case, we have $a^n = -1 \times I_{d \times d}$ and this happens when $l = 1, 3, ..., n-1$. It is precisely these cases[7] which give the *irreducible projective representations* of D_{2n}. Now, because a^n has a different matrix form from $\mathbb{I}$, the matrices for $b^x a^y$ and $b^x a^{y+n}$ differ. Therefore, when we map $\widehat{D_{2n}}$ to D_{2n}, there is an ambiguity as to which of the matrix forms, $b^x a^y$ or $b^x a^{y+n}$, to choose as those of D_{2n}.

This ambiguity is exactly a feature of projective representations. Preserving the notations of Theorem 17.2.23, we let $G^* = \bigcup\limits_{g_i \in G} A g_i$ be the decomposition into transversals of G for the normal subgroup A. Then choosing one element in every transversal, say $A_q g_i$ for some fixed q, we have the ordinary (linear) representation thereof being precisely the projective representation of g_i. Of course different choices of A_q give different but projectively equivalent (projective) representations of G.

By this above method, we can construct all irreducible projective representations of D_{2n} from (17.4.9). We can verify this by matching dimensions: we end up with $n/2$ 2-dimensional representations inherited from $\widehat{D_{2n}}$ and $2^2 \times (n/2) = 2n$, which of course is the order of D_{2n} as it should.

17.4.2 The Quiver Diagram and the Matter Content

The projection for the matter content Φ is well-known (see e.g., [76, 292]):

$$\gamma^{-1}(g)\Phi\gamma(g) = r(g)\Phi, \tag{17.4.11}$$

for $g \in G$ and r, γ appropriate (projective) representations. The case of D_{2n} without torsion was discussed as a new class of non-chiral $\mathcal{N} = 1$ theories in [296]. We recall

[7] Sometimes also called **negative representations** in such cases.

that for the group D_{2n} we choose the generators (with action on $\mathbb{C}^3$) as

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega_n & 0 \\ 0 & 0 & \omega_n^{-1} \end{pmatrix} \qquad b = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}. \tag{17.4.12}$$

Now we can use these explicit forms to work out the matter content (the quiver diagram) and superpotential. For the regular representation, we choose $\gamma(g)$ as block-diagonal in which every 2-dimensional irreducible representation repeats twice with labels $l = 1, 1, 3, 3, .., n-1, n-1$ (as we have shown in the previous section that the even labels correspond to the linear representation of D_{2n}). With this $\gamma(g)$, we calculate the matter content below.

For simplicity, in the actual calculation we would not use (17.4.11) but rather the standard method given by Lawrence, Nekrasov and Vafa [76], generalised appropriately to the projective case by [259]. We can do so because we are armed with Definition 17.2.25 and results from the previous subsection, and directly use the linear representation of the covering group: we lift the action of D_{2n} into the action of its covering group $\widehat{D_{2n}}$. It is easy to see that we get the same matter content either by using the projective representations of the former or the linear representations of the latter.

From the point of view of the covering group, the representation $r(g)$ in (17.4.11) is given by

$$\mathbf{3} \longrightarrow \chi_1^3 + \chi_2^2 \tag{17.4.13}$$

and the representation $\gamma(g)$ is given by $\gamma \longrightarrow \sum\limits_{l=0}^{n/2-1} 2\chi_2^{2l+1}$. We remind ourselves that the $\mathbf{3}$ must in fact be a *linear* representation of D_{2n} while $\gamma(g)$ is the one that has to be *projective* when we include discrete torsion [248].

For the purpose of tensor decompositions we recall the result for the binary dihedral group [296]:

	$n = \text{even}$	$n = \text{odd}$
$\mathbf{1} \otimes \mathbf{1}'$	$\chi_1^2\chi_1^2 = \chi_1^1 \quad \chi_1^3\chi_1^3 = \chi_1^1 \quad \chi_1^4\chi_1^4 = \chi_1^1$ $\chi_1^2\chi_1^3 = \chi_1^4 \quad \chi_1^2\chi_1^4 = \chi_1^3 \quad \chi_1^3\chi_1^4 = \chi_1^2$	$\chi_1^2\chi_1^2 = \chi_1^3 \quad \chi_1^3\chi_1^3 = \chi_1^1 \quad \chi_1^4\chi_1^4 = \chi_1^3$ $\chi_1^2\chi_1^3 = \chi_1^4 \quad \chi_1^2\chi_1^4 = \chi_1^1 \quad \chi_1^3\chi_1^4 = \chi_1^2$
$\mathbf{1} \otimes \mathbf{2}$	$\chi_1^h\chi_2^l = \begin{cases} \chi_2^l & h = 1,3 \\ \chi_2^{n-l} & h = 2,4 \end{cases}$	
$\mathbf{2} \otimes \mathbf{2}'$	$\chi_2^{l_1}\chi_2^{l_2} = \chi_2^{(l_1+l_2)} + \chi_2^{(l_1-l_2)}$ where $\chi_2^{(l_1+l_2)} = \begin{cases} \chi_2^{(l_1+l_2)} & \text{if} \quad l_1 + l_2 < n, \\ \chi_2^{2n-(l_1+l_2)} & \text{if} \quad l_1 + l_2 > n, \\ \chi_1^2 + \chi_1^4 & \text{if} \quad l_1 + l_2 = n. \end{cases}$ $\chi_2^{(l_1-l_2)} = \begin{cases} \chi_2^{(l_1-l_2)} & \text{if} \quad l_1 > l_2, \\ \chi_2^{(l_2-l_1)} & \text{if} \quad l_1 < l_2, \\ \chi_1^1 + \chi_1^3 & \text{if} \quad l_1 = l_2. \end{cases}$	

(17.4.14)

From these relations we immediately obtain the matter content. Firstly, there are $n/2$ $U(2)$ gauge groups ($n/2$ nodes in the quiver). Secondly, because $\chi_1^3\chi_2^l = \chi_2^l$ we have one adjoint scalar for every gauge group. Thirdly, since $\chi_2^2\chi_2^{2l+1} = \chi_2^{2l-1} + \chi_2^{2l+3}$ (where for $l = 0$, χ_2^{2l-1} is understood to be χ_2^1 and for $l = n/2 - 1$, χ_2^{2l+3} is understood to be χ_2^{n-1}), we have two bi-fundamental chiral supermultiplets. We summarise these results in Figure 17-1.

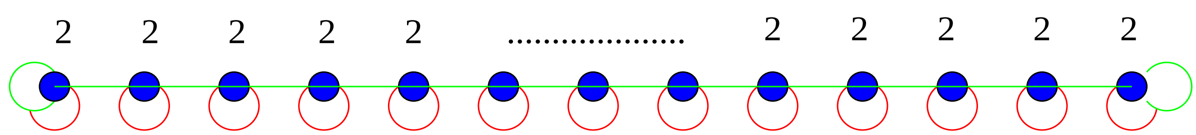


Figure 17-1: The quiver diagram of the ordinary dihedral group D_{2n} with non-trivial projective representation. In this case of discrete torsion being turned on, we have a product of $n/2$ $U(2)$ gauge groups (nodes). The line connecting two nodes without arrows means that there is one chiral multiplet in each direction. Therefore we have a non-chiral theory.

We want to emphasize that by lifting to the covering group, in general we not only find the matter content (quiver diagram) as we have done above, but also the superpotential as well. The formula is given in (2.7) of [76], which could be applied here without any modification (of course, one can use the matrix form of the group elements to obtain the superpotential directly as done in [248, 249, 69, 250, 251, 252, 253], but (2.7), expressed in terms of the Clebsh-Gordan coefficients, is more convenient).

Knowing the above quiver (cf. Figure 17-1) of the ordinary dihedral group D_{2n} *with* discrete torsion, we wish to question ourselves as to the relationships between this quiver and that of its covering group, the binary dihedral group $\widehat{D_{2n}}$ *without* discrete torsion (as well as that of D_{2n} without discrete torsion). The usual quiver of $\widehat{D_{2n}}$ is well-known [171, 292]; we give an example for $n = 4$ in part (a) of Figure 17-2. The quiver is obtained by choosing the decomposition of $\mathbf{3} \longrightarrow \chi_1^1 + \chi_2^1$ (as opposed to (17.4.13) because this is the linear representation of $\widehat{D_{2n}}$); also $\gamma(g)$ is in the regular representation of dimension $4n$. A total of $(n-1)+4 = n+3$ nodes results. We recall that when getting the quiver of D_{2n} with discrete torsion in the above, we chose the decomposition of $\mathbf{3} \longrightarrow \chi_1^3 + \chi_2^2$ in (17.4.13) which provided a linear representation of D_{2n}. Had we made this same choice for $\widehat{D_{2n}}$, our familiar quiver of $\widehat{D_{2n}}$ would have split into two parts: one being precisely the quiver of D_{2n} without discrete torsion as discussed in [296] and the other, that of D_{2n} with discrete torsion as presented in Figure 17-1. These are given respectively in parts (b) and (c) of Figure 17-2.

From this discussion, we see that in some sense discrete torsion is connected with different choices of decomposition in the usual orbifold projection. We want to emphasize that the example of D_{2n} is very special because its covering group $\widehat{D}_{2n}$ belongs to $SU(2)$. In general, the covering group does not even belong to $SU(3)$ and the meaning of the usual orbifold projection of the covering group in string theory is vague.

17.5 Conclusions and Prospects

Let us pause here awhile for reflection. A key purpose of this writing is to initiate the investigation of discrete torsion for the generic D-brane orbifold theories. Inspired by this goal, we have shown that computing the Schur Multiplier $M(G)$ for the finite group G serves as a beacon in our quest.

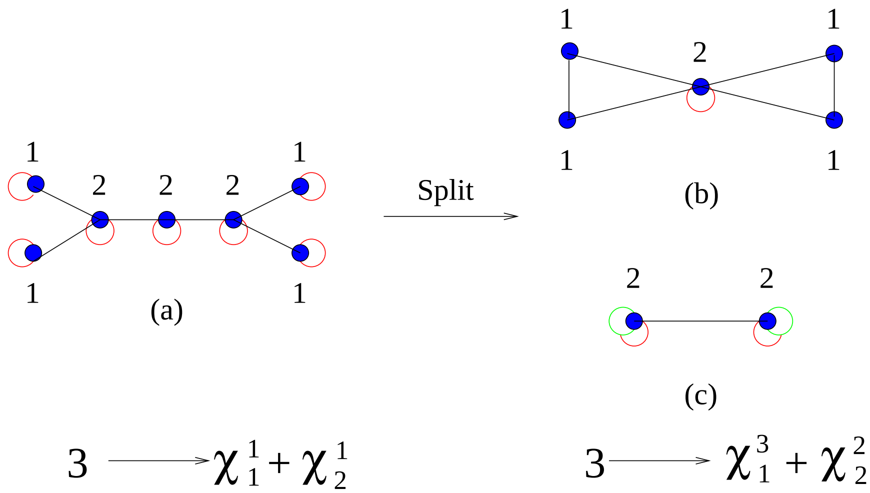


Figure 17-2: (a) The quiver diagram of the binary dihedral group $\widehat{D}_4$ *without* discrete torsion; (b) the quiver of the ordinary dihedral group D_4 *without* discrete torsion; (c) the quiver of the ordinary dihedral group D_4 *with* discrete torsion.

In particular, using the fact that $M(G)$ is an indicator of when we can turn on a non-trivial NS-NS background in the orbifold geometry and when we cannot: only when $M(G)$, as an Abelian group is not trivially $\mathbb{I}$ can the former be executed. As a guide for future investigations, we have computed $M(G)$ for the discrete subgroups G in $SU(n)$ with $n = 2, 3, 4$, which amounts to a *classification of which D-brane orbifolds afford non-trivial discrete torsion.*

As an explicit example, in supplementing the present lack of studies of non-Abelian orbifolds with discrete torsion in the current literature, we have pursued in detail the $\mathcal{N} = 1$ gauge theory living on the D3-Brane probe on the orbifold singularity $\mathbb{C}^3/D_{2n}$, corresponding to the ordinary dihedral group of order $2n$ as a subgroup of $SU(3)$. As the group has Schur Multiplier $\mathbb{Z}_2$ for even n, we have turned on the discrete torsion and arrived at an interesting class of non-chiral theories.

The prospects are as manifold as the interests are diverse and much work remains to be done. An immediate task is to examine the gauge theory living on the world-volume of D-brane probes when we turn on the discrete torsion of a given orbifold wherever allowed by our classification. This investigation is currently in progress.

Our results of the Schur Multipliers could also be interesting to the study of K-theory in connexion to string theory. Recent works [258, 259, 261] have noticed an intimate relation between twisted K-theory and discrete torsion. More specifically, the Schur Multiplier of an orbifold group may in fact supply information about the torsion subgroup of the cohomology group of space-time in the light of a generalised McKay Correspondence [259, 293].

It is also tempting to further study the non-commutative moduli space of non-Abelian orbifolds in the spirit of [249, 250, 251] which treated Abelian cases at great length. How the framework developed therein extends to the non-Abelian groups should be interesting. Works on discrete torsion in relation to permutation orbifolds and symmetric products [267] have also been initiated, we hope that our methodologies could be helpful thereto.

Finally, there is another direction of future study. The boundary state formalism was used in [255] where it was suggested that the ties between close and open string sectors maybe softened with regard to discrete torsion. It is thus natural to ask if such ambiguities may exist also for non-Abelian orbifolds.

All these open issues, of concern to the physicist and the mathematician alike, present themselves to the intrigue of the reader.

Chapter 18

Orbifolds IX: Discrete Torsion, Covering Groups and Quiver Diagrams

Synopsis

Extending the previous chapter and without recourse to the sophisticated machinery of twisted group algebras, projective character tables and explicit values of 2-cocycles, we here present a simple algorithm to study the gauge theory data of D-brane probes on a generic orbifold G with discrete torsion turned on.

We show in particular that the gauge theory can be obtained with the knowledge of no more than the *ordinary* character tables of G and its covering group G^*. Subsequently we present the quiver diagrams of certain illustrative examples of $SU(3)$-orbifolds which have non-trivial Schur Multipliers. This chapter continues with the preceeding and aims to initiate a systematic and computationally convenient study of discrete torsion [303].

18.1 Introduction

Discrete torsion [124, 247] has become a meeting ground for many interesting subfields of string theory; its intimate relation with background B-fields and non-commutative geometry is one of its many salient features. In the context of D-brane probes on orbifolds with discrete torsion turned on, new classes of gauge theories may be fabricated and their (non-commutative) moduli spaces, investigated (see from [248, 249] to [267]). Indeed, as it was pointed out in [248, 249], projection on the matter spectrum in the gauge theory by an orbifold G with non-trivial discrete torsion is performed by the *projective representations* of G, rather than the mere linear (ordinary) represenations as in the case without.

In the previous chapter, to which the present shall be a companion, we offered a classification of the orbifolds with $\mathcal{N} = 0, 1, 2$ supersymmetry which permit the turning on of discrete torsion. We have pointed there that for the orbifold group G, the discriminant agent is the Abelian group known as the **Schur Multiplier** $M(G) := H^2(G, \mathbb{C}^*)$; only if $M(G)$ were non-trivial could G afford a projective representation and thereby discrete torsion.

In fact one can do more and for actual physical computations one needs to do more. The standard procedure of calculating the matter content and superpotential of the orbifold gauge theory as developed in [76] can, as demonstrated in [259], be directly generalised to the case with discrete torsion. Formulae given in terms of the ordinary characters have their immediate counterparts in terms of the projective characters, the *point d'appui* being that the crucial properties of ordinary characters, notably orthogonality, carry over without modification, to the projective case.

And thus our task would be done if we had a method of computing the projective characters. Upon first glance, this perhaps seems formidable: one seemingly is required to know the values of the cocycle representatives $\alpha(x, y)$ in $M(G)$ for all $x, y \in G$. In actuality, one can dispense with such a need. There exists a canonical method to arrive at the projective characters, namely by recourse to the **covering group** of G. We shall show in this writing the methodology standard in the mathematics literature [262, 269] by which one, once armed with the Schur Multiplier, arrives at the cover. Moreover, in light of the physics, we will show how, equipped with no more than the knowledge of the character table of G and that of its cover G^*, one obtains the matter content of the orbifold theory with discrete torsion.

The chapter is organised as follows. Section 2 introduces the necessary mathematical background for our work. Due to the technicality of the details, we present a paragraph at the beginning of the section to summarise the useful facts; the reader may then freely skip the rest of Section 2 without any loss. In Section 3, we commence with an explicit example, viz., the ordinary dihedral group, to demonstrate the method to construct the covering group. Then we present all the covering groups for transitive and intransitive discrete subgroups of $SU(3)$. In Section 4, we use these covering groups to calculate the corresponding gauge theories (i.e., the quiver diagrams) for all exceptional subgroups of $SU(3)$ admitting discrete torsion as well as some examples for the Delta series. In particular we demonstrate the algorithm of extracting the quivers from the ordinary character tables of the group and its cover. As a by-product, in Section 5 we present a method to calculate the *cocycles* directly which will be useful for future reference. The advantage of our methods for the quivers and the cocycles is their simplicity and generality. Finally, in Section 6 we give some conclusions and further directions for research.

Nomenclature

Throughout this chapter, unless otherwise specified, we shall adhere to the following conventions for notation:

ω_n	n-th root of unity;
G	a finite group of order $\|G\|$;
$[x,y]$	$:= xyx^{-1}y^{-1}$, the group commutator of x, y;
$\langle x_i \| y_j \rangle$	the group generated by elements $\{x_i\}$ with relations y_j;
$\gcd(m,n)$	the greatest common divisor of m and n;
$Z(G)$	centre of G;
$G' := [G,G]$	the derived (commutator) group of G;
G^*	the covering group of G;
$A = M(G)$	the Schur Muliplier of G;
$\mathrm{char}(G)$	ordinary (linear) character table of G, given as an $(r+1) \times r$ matrix with r the # of conjugacy classes and the extra row for class numbers;
$Q_\alpha(G, \mathcal{R})$	α-projective quiver for G associated to the chosen representation $\mathcal{R}$.

18.2 Mathematical Preliminaries

We first remind the reader of some properties of the the theory of projective representations; in what follows we adhere to the notation used in our previous work [301].

Due to the technicalities in the ensuing, the audience might be distracted upon the first reading. Thus as promised in the introduction, we here summarise the keypoints in the next fews paragraphs, so that the remainder of this section may be loosely perused without any loss.

Our aim of this work is to attempt to construct the gauge theory living on a D-brane probing an orbifold G when "discrete torsion" is turned on. To accomplish such a goal, we need to know the projective representations of the finite group G, which may not be immediately available. However, mathematicians have shown that there exists (for representations in $GL(\mathbb{C})$) a group G^* called the *covering group* of G, such that there is a one-to-one correspondence between the projective representations of G and the linear (ordinary) representations of G^*. Thus the method is clear: we simply need to find the covering group and then calculate the ordinary characters of its (linear) representations.

More specificaly, we first introduce the concept of the covering group in Definition 2.2. Then in Theorem 2.1, we introduce the necessary and sufficient conditions for G^* to be a covering group; these conditions are very important and we use them extensively during actual computations.

However, G^* for any given G is not unique and there exist non-isomorphic groups which all serve as covering groups. To deal with this, we introduce *isoclinism* and show that these non-isomorphic covering groups must be isoclinic to each other in Theorem 2.2. Subsequently, in Theorem 2.3, we give an upper-limit on the number of non-isomorphic covering groups of G. Finally in Thereom 2.4 we present the one-to-one correspondence of all projective representations of G and all linear representations of its covering group G^*.

Thus is the summary for this section. The uninterested reader may now freely proceed to Section 3.

18.2.1 The Covering Group

Recall that a **projective representation** of G over $\mathbb{C}$ is a mapping $\rho : G \to GL(V)$ such that $\rho(\mathbb{I}_G) = \mathbb{I}_V$ and $\rho(x)\rho(y) = \alpha(x,y)\rho(xy)$ for any elements $x, y \in G$. The function α, known as the *cocycle*, is a map $G \times G \to \mathbb{C}^*$ which is classified by $H^2(G, \mathbb{C}^*)$, the second $\mathbb{C}^*$-valued cohomology of G. This case of $\alpha = 1$ trivially is of course our familiar ordinary (non-projective) representation, which will be called **linear**.

The Abelian group $H^2(G, \mathbb{C}^*)$ is known as the **Schur Multiplier** of G and will be denoted by $M(G)$. Its triviality or otherwise is a discriminant of whether G admits projective representation. In a physical context, knowledge of $M(G)$ provides immediate information as to the possibility of turning on discrete torsion in the orbifold model under study. A classification of $M(G)$ for all discrete finite subgroups of $SU(3)$ and the exceptional subgroups of $SU(4)$ was given in the companion work [301].

The study of the projective representations of a given group G is greatly facilitated by introducing an auxilliary object G^*, the **covering group** of G, which "lifts projective representations to linear ones." Let us refresh our memory what this means. Let there be a **central extension** according to the exact sequence $1 \to A \to G^* \to G \to 1$ such that A is in the centre of G^*. Thus we have $G^*/A \cong G$. Now we say

DEFINITION 18.2.26 *A projective representation ρ of G* **lifts** *to a linear representation ρ^* of G^* if*
(i) $\rho^*(a \in A)$ is proportional to $\mathbb{I}$ and
(ii) there is a section[1] *$\mu : G \to G^*$ such that $\rho(g) = \rho^*(\mu(g))$, $\forall g \in G$.*

Likewise it *lifts projectively* if $\rho(g) = t(g)\rho^*(\mu(g))$ for a map (not necessarily a homomorphism) $t : G \to \mathbb{C}^*$.

DEFINITION 18.2.27 *G^* is called a* **covering group** *(or otherwise known as the* **representation group**, *Darstellungsgruppe) of G over $\mathbb{C}$ if the following are satisfied:*

(i) *$\exists$ a central extension $1 \to A \to G^* \to G \to 1$ such that any projective representation of G lifts projectively to an ordinary representation of G^*;*

(ii) $|A| = |M(G)| = |H^2(G, \mathbb{C}^*)|$.

The covering group will play a central rôle in our work; as we will show in a moment, *the matter content of an orbifold theory with group G having discrete torsion switched-on is encoded in the quiver diagram of G^*.*

For actual computational purposes, the following theorem, initially due to Schur, is of extreme importance:

THEOREM 18.2.28 ([262] p143) *$G^\star$ is a covering group of G over $\mathbb{C}$ if and only if the following conditions hold:*
(i) $G^\star$ has a finite subgroup A with $A \subseteq Z(G^\star) \cap [G^\star, G^\star]$;
(ii) $G \cong G^\star/A$;
(iii) $|A| = |M(G)|$.

[1] *i.e., for the projection $f : G^* \to G$, $\mu \circ f = \mathbb{I}_G$.*

In the above, $[G^\star, G^\star]$ is the **derived group** $G^{*'}$ of G^*. We remind ourselves that for a group H, $H' := [H, H]$ is the group generated by elements of the form $[x, y] := xyx^{-1}y^{-1}$ for $x, y \in H$. We stress that conditions (ii) and (iii) are easily satisfied while (i) is the more stringent imposition.

The solution of the problem of finding covering groups is certainly *not* unique: G in general may have more than one covering groups (e.g., the quaternion and the dihedral group of order 8 are both covering groups of $\mathbb{Z}_2 \times \mathbb{Z}_2$). The problem of finding the necessary conditions which two groups G_1 and G_2 must satisfy in order for both to be covering groups of the same group G is a classical one.

The well-known solution starts with the following

DEFINITION 18.2.28 *Two groups G and H are said to be* **isoclinic** *if there exist two isomorphisms*

$$\alpha : G/Z(G) \stackrel{\cong}{\to} H/Z(H) \quad \text{and} \quad \beta : G' \stackrel{\cong}{\to} H'$$

such that $\alpha(x_1 Z(G)) = x_2 Z(H)$ and $\alpha(y_1 Z(G)) = y_2 Z(H) \Rightarrow \beta([x_1, y_1]) = [x_2, y_2]$,

where we have used the standard notation that $xZ(G)$ is a coset representative in $G/Z(G)$. We note in passing that every Abelian group is obviously isoclinic to the trivial group $\langle \mathbb{I} \rangle$.

We introduce this concept of isoclinism because of the following important Theorem of Hall:

THEOREM 18.2.29 ([262] p441) *Any two covering groups of a given finite group G are isoclinic.*

Knowing that the covering groups of G are not isomorphic to each other, but isoclinic, a natural question to ask is how many non-isomorphic covering groups can one have. Here a theorem due to Schur shall be useful:

THEOREM 18.2.30 ([262] p149) *For a finite group G, let*

$$G/G' = \mathbb{Z}_{e_1} \times ... \times \mathbb{Z}_{e_r}$$

and

$$M(G) = \mathbb{Z}_{f_1} \times ... \times \mathbb{Z}_{f_s}$$

be decompositions of these Abelian groups into cyclic factors. Then the number of non-isomorphic covering groups of G is less than or equal to

$$\prod_{1 \le i \le r, 1 \le j \le s} \gcd(e_i, f_j).$$

18.2.2 Projective Characters

With the preparatory remarks in the previous subsection, we now delve headlong into the heart of the matter. By virtue of the construction of the covering group G^* of G , we have the following 1-1 correpondence which will enable us to compute α-projective representations of G in terms of the linear representations of G^*:

THEOREM 18.2.31 [Theorema Egregium] ([262] p139; [268] p8) *Let G^* be the covering group of G and $\lambda : A \to \mathbb{C}^*$ a homomorphism. Then*

(i) *For every linear representation $L : G^* \to GL(V)$ of G^* such that $L(a) = \lambda(a)\mathbb{I}_V\ \forall a \in A$, there is an induced projective representation P on G defined by*

$$P(g) := L(r(g)),\ \forall\ g \in G,$$

with $r : G \to G^*$ the map that associates to each coset $g \in G \cong G^*/A$ a representative element[2] *in G^*; and vice versa,*

(ii) *Every α-projective representation for $\alpha \in M(G)$ lifts to an ordinary representation of G^*.*

An immediate consequence of the above is the fact that knowing the linear characters of G^* suffices to establish the projective characters of G for all α [269]. This should ease our initial fear in that *one does not need to know a priori the specific values of*

[2] *i.e., $r(g)A \to g$ is the isomorphism $G^*/A \xrightarrow{\cong} G$.*

the cocycles $\alpha(x, y)$ for all $x, y \in G$ (a stupendous task indeed) in order to construct the α-projective character table for G.

We shall leave the uses of this crucial observation to later discussions. For now, let us focus on some explicit computations of covering groups.

18.3 Explicit Calculation of Covering Groups

To theory we must supplant examples and to abstraction, concreteness. We have prepared ourselves in the previous section the rudiments of the theory of covering groups; in the present section we will demonstrate these covers for the discrete finite subgroups of $SU(3)$. First we shall illustrate our techniques with the case of D_{2n}, the ordinary dihedral group, before tabulating the complete results.

18.3.1 The Covering Group of The Ordinary Dihedral Group

The presentation of the ordinary dihedral group of order $2n$ is standard (the notation is different from some of our earlier chapters where the following would be called D_n):

$$D_{2n} = \langle \tilde{\alpha}, \tilde{\beta} | \tilde{\alpha}^n = 1, \tilde{\beta}^2 = 1, \tilde{\beta}\tilde{\alpha}\tilde{\beta}^{-1} = \tilde{\alpha}^{-1} \rangle.$$

We recall from [301] that the Schur Multiplier for $G = D_{2n}$ is $\mathbb{Z}_2$ when n is even and trivial otherwise, thus we restrict ourselves only to the case of n even. We let $M(D_{2n})$ be $A = \mathbb{Z}_2$ generated by $\{a | a^2 = \mathbb{I}\}$. We let the covering group be $G^* = \langle \alpha, \beta, a \rangle$.

Now having defined the generators we proceed to constrain relations thereamong. Of course, $A \subset Z(G^*)$ immediately implies that $\alpha a = a\alpha$ and $\beta a = a\beta$. Moreover, α, β must map to $\tilde{\alpha}, \tilde{\beta}$ when we identify $G^\star/A \cong D_{2n}$ (by part (ii) of Theorem 18.2.28). This means that $\mathbb{I}_G$ must have a preimage in $A \subset G^*$, giving us: $\alpha^n \in A, \beta^2 \in A$ and $\beta\alpha\beta^{-1}\alpha \in A$ by virtue of the presentation of G. And hence we have 8 possibilities, each being a central extension of D_{2n} by A:

$$
\begin{aligned}
G_1^* &= \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_2^* &= \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle \\
G_3^* &= \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_4^* &= \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = 1, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle \\
G_5^* &= \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_6^* &= \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = 1, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle \\
G_7^* &= \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \\
G_8^* &= \langle \alpha, \beta, a | \alpha a = a\alpha, \quad \beta a = a\beta, \quad a^2 = 1, \quad \alpha^n = a, \quad \beta^2 = a, \quad \beta\alpha\beta^{-1} = \alpha^{-1}a \rangle
\end{aligned}
\tag{18.3.1}
$$

Therefore, conditions (ii) and (iii) of Theorem 18.2.28 are satified. One must check (i) to distinguish the covering group among these 8 central extensions in (18.3.1). Now since A is actually the centre, it suffices to check whether $A \subset G_i^{*'} = [G_i^*, G_i^*]$.

We observe G_1^* to be precisely $D_{2n} \times \mathbb{Z}_2$, from which we have $G_1^{*'} = \mathbb{Z}_{n/2}$, generated by α^2. Because $A = \{\mathbb{I}, a\}$ clearly is not included in this $\mathbb{Z}_{n/2}$ we conclude that G_1^* is not the covering group. For G_2^*, we have $G_2^{*'} = \langle \alpha^2 a \rangle$, which means that when $n/2 =$ odd (recall that $n =$ even), $G_2^{*'}$ can contain a and hence $A \subset G_2^{*'}$, whereby making G_2^* a covering group. By the same token we find that $G_3^{*'} = \langle \alpha^2 \rangle$, $G_4^{*'} = \langle \alpha^2 a \rangle$, $G_5^{*'} = \langle \alpha^2 \rangle$, $G_6^{*'} = \langle \alpha^2 a \rangle$, and $G_7^{*'} = \langle \alpha^2 \rangle$. We summarise these results in the following table:

Group	$G^{*'}$	$Z(G^*)$	$G^*/Z(G^*)$	Covering Group?
G_1^*	$\mathbb{Z}_{n/2} = \langle\alpha^2\rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2}\rangle$	D_n	no
$G_2^*(n = 4k+2)$	$\mathbb{Z}_n = \langle\alpha^2 a\rangle$	$\mathbb{Z}_2 = \langle a\rangle$	D_{2n}	yes
$G_2^*(n = 4k)$	$\mathbb{Z}_{n/2} = \langle\alpha^2 a\rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2}\rangle$	D_n	no
G_3^*	$\mathbb{Z}_{n/2} = \langle\alpha^2\rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2}\rangle$	D_n	no
$G_4^*(n = 4k+2)$	$\mathbb{Z}_n = \langle\alpha^2 a\rangle$	$\mathbb{Z}_2 = \langle a\rangle$	D_{2n}	yes
$G_4^*(n = 4k)$	$\mathbb{Z}_{n/2} = \langle\alpha^2 a\rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, \alpha^{n/2}\rangle$	D_n	no
G_5^*	$\mathbb{Z}_n = \langle\alpha^2\rangle$	$\mathbb{Z}_2 = \langle a\rangle$	D_{2n}	yes
$G_6^*(n = 4k+2)$	$\mathbb{Z}_{n/2} = \langle\alpha^2 a\rangle$	$\mathbb{Z}_4 = \langle\alpha^{n/2}\rangle$	D_n	no
$G_6^*(n = 4k)$	$\mathbb{Z}_n = \langle\alpha^2 a\rangle$	$\mathbb{Z}_2 = \langle a\rangle$	D_{2n}	yes
G_7^*	$\mathbb{Z}_n = \langle\alpha^2\rangle$	$\mathbb{Z}_2 = \langle a\rangle$	D_{2n}	yes
$G_8^*(n = 4k+2)$	$\mathbb{Z}_{n/2} = \langle\alpha^2 a\rangle$	$\mathbb{Z}_4 = \langle\alpha^{n/2}\rangle$	D_n	no
$G_8^*(n = 4k)$	$\mathbb{Z}_n = \langle\alpha^2 a\rangle$	$\mathbb{Z}_2 = \langle a\rangle$	D_{2n}	yes

Whence we see that G_1^* and G_3^* are not covering groups, while for $n/2 =$ odd $G_{2,4}^*$ are covers, for $n/2 =$ even $G_{6,8}^*$ are covers as well and finally $G_{5,7}^*$ are always covers. Incidentally, G_7^* is actually the binary dihedral group and we know that it is indeed the (double) covering group from [301]. Of course in accordance with Theorem 18.2.29, these different covers must be isoclinic to each other. Checking against Definition 18.2.28, we see that for G^* being $G_{2,4}^*$ with $n = 4k+2$, $G_{6,8}^*$ with $n = 4k$ and $G_{5,7}^*$ for all even n, $G^{*'} \cong \mathbb{Z}_n$ and $G^*/Z(G^*) \cong D_{2n}$; furthermore the isomorphisms α and β in the Definition are easily seen to satisfy the prescribed conditions. Therefore all these groups are indeed isoclinic. We make one further remark, for both the cases of $n = 4k$ and $n = 4k+2$, we have found 4 non-isomorphic covering groups. Recall Theorem 18.2.30, here we have $f_1 = 2$ and $G/G' = \mathbb{Z}_2 \times \mathbb{Z}_2$ (note that n is even) and so $e_1 = e_2 = 2$, whence the upper limit is exactly $2 \times 2 = 4$ which is saturated here. This demonstrates that our method is general enough to find all possible covering groups.

18.3.2 Covering Groups for the Discrete Finite Subgroups of $SU(3)$

By methods entirely analogous to the one presented in the above subsection, we can arrive at the covering groups for the discrete finite groups of $SU(3)$ as tabulated in [301]. Let us list the results (of course in comparison with Table 3.2 in [301], those with trivial Schur Multipliers have no covering groups and will not be included here). Of course, as mentioned earlier, the covering group is not unique. The particular ones we have chosen in the following table are the same as generated by the computer package GAP using the Holt algorithm [92].

Intransitives

We used the shorthand notation $(x/y/\ldots/z)$ to mean the relation to be applied to each of the elements $x, y, \ldots, z$.

- $$\begin{array}{l} G = \mathbb{Z}_m \times \mathbb{Z}_n = \langle \tilde\alpha, \tilde\beta | \tilde\alpha^n = 1, \tilde\beta^m = 1, \tilde\alpha\tilde\beta = \tilde\beta\tilde\alpha \rangle; \\ M(G) = \mathbb{Z}_{p=\gcd(m,n)} = \langle a | a^p = \mathbb{I} \rangle; \\ G^* = \langle \alpha, \beta, a | \alpha a = a\alpha, \beta a = a\beta, a^p = 1, \alpha^n = 1, \beta^m = 1, \alpha\beta = \beta\alpha a \rangle \end{array} \tag{18.3.2}$$

- $$G = \langle \mathbb{Z}_{n=4k}, \widehat{D_{2m}} \rangle = \langle \tilde\alpha, \tilde\beta, \tilde\gamma | \tilde\alpha\tilde\beta = \tilde\beta\tilde\alpha, \tilde\alpha\tilde\gamma = \tilde\gamma\tilde\alpha, \tilde\alpha^{n/2} = \tilde\beta^m, \tilde\beta^{2m} = 1, \tilde\beta^m = \tilde\gamma^2, \tilde\gamma\tilde\beta\tilde\gamma^{-1} = \tilde\beta^{-1} \rangle;$$
$$\left\{ \begin{array}{lll} m \text{ even} & M(G) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b | a^2 = 1 = b^2, ab = ba \rangle; & \\ & \quad G^* = & \langle \alpha, \beta, \gamma, a, b | ab = ba, \alpha a = a\alpha, \alpha b = b\alpha, \beta a = a\beta, \beta b = b\beta, \\ & & \gamma a = a\gamma, \gamma b = b\gamma, a^2 = 1 = b^2, \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha b, \\ & & \alpha^{n/2} = \beta^m, \beta^{2m} = 1, \beta^m = \gamma^2, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle \\ m \text{ odd,} & M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle; & \\ & \quad G^* = & \langle \alpha, \beta, \gamma, a | a^2 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha\beta = \beta\alpha, \\ & & \alpha\gamma = \gamma\alpha a, \alpha^{n/2} = \beta^m, \beta^{2m} = 1, \beta^m = \gamma^2, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle \end{array} \right. \tag{18.3.3}$$

- $$\begin{array}{l} G = \langle \mathbb{Z}_{n=4k}, \widehat{E_7} \rangle = \langle \tilde\alpha, \tilde\beta, \tilde\gamma | \tilde\alpha\tilde\beta = \tilde\beta\tilde\alpha, \tilde\alpha\tilde\gamma = \tilde\gamma\tilde\alpha, \tilde\alpha^{n/2} = \tilde\beta^4, \tilde\beta^4 = \tilde\gamma^3 = (\tilde\beta\tilde\gamma)^2 \rangle; \\ M(G) = \mathbb{Z}_2 = \langle a | a^2 = \mathbb{I} \rangle; \\ G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha^{n/2} = \beta^4, \\ \quad \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha, \beta^4 = \gamma^3 = (\beta\gamma)^2 \rangle \end{array} \tag{18.3.4}$$

- $G = \langle \mathbb{Z}_{n=3k}, \widehat{E_6} \rangle$

$$
\begin{aligned}
&k \text{ odd} && G \cong \mathbb{Z}_n \times \widehat{E_6} = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\alpha}^n = 1, \tilde{\beta}^3 = \tilde{\gamma}^3 = (\tilde{\beta}\tilde{\gamma})^2 \rangle; \\
& && M(G) = \mathbb{Z}_3 = \langle a | a^3 = \mathbb{I} \rangle; \\
& && G^* = \langle \alpha, \beta, \gamma, a | a^3 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha^n = 1, \\
& && \qquad \alpha\beta = \beta\alpha a^{-1}, \alpha\gamma = \gamma\alpha a, \beta^3 = \gamma^3 = (\beta\gamma)^2 \rangle \\
&k = 2(2p+1) && G \cong \mathbb{Z}_{n/2} \times \widehat{E_6} \\
&k = 4p && G \cong (\mathbb{Z}_n \times \widehat{E_6})/\mathbb{Z}_2 = \\
& && \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\alpha}^{n/2} = \tilde{\beta}^3, \tilde{\beta}^3 = \tilde{\gamma}^3 = (\tilde{\beta}\tilde{\gamma})^2 \rangle; \\
& && M(G) = \mathbb{Z}_3 = \langle a | a^3 = \mathbb{I} \rangle; \\
& && G^* = \langle \alpha, \beta, \gamma, a | a^3 = 1, \alpha a = a\alpha, \beta a = a\beta, \gamma a = a\gamma, \alpha^{n/2} = \beta^3, \\
& && \qquad \alpha\beta = \beta\alpha a^{-1}, \alpha\gamma = \gamma\alpha a, \beta^3 = \gamma^3 = (\beta\gamma)^2 \rangle
\end{aligned}
\tag{18.3.5}
$$

- $G = \langle \mathbb{Z}_n, D_{2m} \rangle$

$$
\begin{aligned}
&n \text{ odd}, m \text{ even} && G = \mathbb{Z}_n \times D_{2m} = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}^n = 1, \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \tilde{\beta}^m = 1, \\
& && \qquad \tilde{\gamma}^2 = 1, \tilde{\gamma}\tilde{\beta}\tilde{\gamma}^{-1} = \tilde{\beta}^{-1} \rangle; \\
& && M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle; \\
& && G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha(\beta/\gamma) = \\
& && \qquad (\beta/\gamma)\alpha, \alpha^n = 1, \beta^m = a, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle \\
&n \text{ even}, m \text{ odd} && G = \mathbb{Z}_n \times D_{2m} \\
& && M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle; \\
& && G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha\beta = \beta\alpha, \\
& && \qquad \alpha\gamma = \gamma\alpha a, \alpha^n = 1, \beta^m = 1, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle \\
&m \text{ even}, n = 2(2l+1) && G = \mathbb{Z}_{n/2} \times D_{2m} \\
&n = 4k, m = 2(2l+1) && G = (\mathbb{Z}_n \times D_{2m})/\mathbb{Z}_2 = \langle \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma} | \tilde{\alpha}^{n/2} = \beta^{m/2}, \tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}, \tilde{\alpha}\tilde{\gamma} = \tilde{\gamma}\tilde{\alpha}, \\
& && \qquad \tilde{\beta}^m = 1, \tilde{\gamma}^2 = 1, \tilde{\gamma}\tilde{\beta}\tilde{\gamma}^{-1} = \tilde{\beta}^{-1} \rangle; \\
& && M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1 \rangle; \\
& && G^* = \langle \alpha, \beta, \gamma, a | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha\beta = \beta\alpha, \\
& && \qquad \alpha\gamma = \gamma\alpha a, \alpha^{n/2} = \beta^{m/2}, \beta^m = 1, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle \\
&n = 4k, m = 4l && G = (\mathbb{Z}_n \times D_{2m})/\mathbb{Z}_2 \\
& && M(G) = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a, b | a^2 = 1, b^2 = 1, ab = ba \rangle; \\
& && G^* = \langle \alpha, \beta, \gamma, a, b | a^2 = 1, a(\alpha/\beta/\gamma) = (\alpha/\beta/\gamma)a, \alpha\beta = \beta\alpha b, \\
& && \qquad \alpha\gamma = \gamma\alpha a, \alpha^{n/2} = \beta^{m/2}, \beta^m = 1, \gamma^2 = 1, \gamma\beta\gamma^{-1} = \beta^{-1} \rangle
\end{aligned}
$$

(18.3.6)

Transitives

We first have the two infinite series.

- $G = \Delta(3n^2) = \langle \alpha,\beta,\gamma | \alpha^n = \beta^n = \gamma^3 = 1, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma\rangle;$

$$\begin{cases}
\gcd(n,3)=1, n \text{ even} & M(G) = \mathbb{Z}_n = \langle a | a^n = 1\rangle; \\
 & G^* = \langle \alpha,\beta,\gamma,a | (\alpha/\beta/\gamma)a = a(\alpha/\beta/\gamma), \\
 & \quad a^n = \alpha^n a^{n/2} = \beta^n a^{n/2} = \gamma^3 = 1, \\
 & \quad \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma\rangle; \\
\gcd(n,3)=1, n \text{ odd} & M(G) = \mathbb{Z}_n; \\
 & G^* = \langle \alpha,\beta,\gamma,a | (\alpha/\beta/\gamma)a = a(\alpha/\beta/\gamma), \\
 & \quad a^n = \alpha^n = \beta^n = \gamma^3 = 1, \\
 & \quad \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma\rangle; \\
\gcd(n,3)\neq 1, n \text{ even} & M(G) = \mathbb{Z}_n \times \mathbb{Z}_3 = \langle a,b | a^n = 1, b^3 = 1\rangle; \\
 & G^* = \langle \alpha,\beta,\gamma,a,b | (\alpha/\beta/\gamma)(a/b) = (a/b)(\alpha/\beta/\gamma), \\
 & \quad ab = ba, a^n = b^3 = \gamma^3 = \alpha^n a^{n/2} b = 1, \\
 & \quad \beta^n a^{n/2} = b, \alpha\beta = \beta\alpha ab, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma\rangle; \\
\gcd(n,3)\neq 1, n \text{ odd} & M(G) = \mathbb{Z}_n \times \mathbb{Z}_3; \\
 & G^* = \langle \alpha,\beta,\gamma,a,b | (\alpha/\beta/\gamma)(a/b) = (a/b)(\alpha/\beta/\gamma), \\
 & \quad a^n = b^3 = \gamma^3 = \alpha^n b = \beta^n b^{-1} = 1, \\
 & \quad ab = ba, \alpha\beta = \beta\alpha ab, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma\rangle;
\end{cases} \tag{18.3.7}$$

- $G = \Delta(6n^2) = \langle \alpha,\beta,\gamma,\delta | \alpha^n = \beta^n = \gamma^3 = \delta^2 = 1, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma,$
$\alpha\delta\alpha = \delta, \beta\delta = \delta\alpha^{-1}\beta, \gamma\delta\gamma = \delta\rangle;$

$M(G) = \mathbb{Z}_2 = \langle a | a^2 = 1\rangle;$

$$\begin{cases}
\gcd(n,4)=4 & G^* = \langle \alpha,\beta,\gamma,\delta,a | \alpha^n = \beta^n = \gamma^3 = \delta^2 = a^2 = 1, \\
 & \quad (\alpha/\beta/\gamma/\delta)a = a(\alpha/\beta/\gamma/\delta), \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \\
 & \quad \beta\gamma\alpha = \gamma, \alpha\delta\alpha = \delta, \beta\delta = \delta\alpha^{-1}\beta, \gamma\delta\gamma = \delta\rangle; \\
\gcd(n,4)=2 & G^* = \langle \alpha,\beta,\gamma,\delta,a | \alpha^n a = \beta^n a = \gamma^3 = \delta^2 = a^2 = 1, \\
 & \quad (\alpha/\beta/\gamma/\delta)a = a(\alpha/\beta/\gamma/\delta), \alpha\beta = \beta\alpha a, \alpha\gamma = \gamma\alpha^{-1}\beta, \\
 & \quad \beta\gamma\alpha = \gamma, \alpha\delta\alpha = \delta, \beta\delta = \delta\alpha^{-1}\beta, \gamma\delta\gamma = \delta\rangle;
\end{cases} \tag{18.3.8}$$

Next we present the three exceptionals that admit discrete torsion.

- $$\begin{aligned} G = \ & \Sigma(60) \cong A_5 = \langle \alpha, \beta | \alpha^5 = \beta^3 = (\alpha\beta^{-1})^3 = (\alpha^2\beta)^2 = 1 \\ & \qquad \alpha\beta\alpha\beta\alpha\beta = \alpha\gamma\alpha^{-1}\beta\alpha^2\beta\alpha^{-2}\beta = 1\rangle; \\ & M(G) = \mathbb{Z}_2; \\ & G^* = \langle \alpha, \beta, a | \alpha^5 = a, \beta^3 = a^2 = 1, (\alpha/\beta)a = a(\alpha/\beta) \\ & \qquad (\alpha\beta^{-1})^3 = 1, (\alpha^2\beta)^2 = a\rangle; \end{aligned} \tag{18.3.9}$$

- $$\begin{aligned} G = \ & \Sigma(168) = \langle \alpha, \beta, \gamma | \gamma^2 = \beta^3 = \beta\gamma\beta\gamma = (\alpha\gamma)^4 = 1, \alpha^2\beta = \beta\alpha, \alpha^3\gamma\alpha^{-1}\beta = \gamma\alpha\gamma\rangle; \\ & M(G) = \mathbb{Z}_2; \\ & G^* = \langle \alpha, \beta, \gamma, \delta | \delta^2 = \gamma^2\delta = \beta^3\delta = (\beta\alpha)^3 = (\alpha\gamma)^3 = 1, \\ & \qquad \beta\gamma\beta = \gamma, \alpha\delta = \delta\alpha, \beta^2\alpha^2\beta = \alpha, \beta^{-1}\alpha^{-1}\beta\gamma\alpha^{-1}\gamma = \gamma\alpha\beta\rangle; \end{aligned} \tag{18.3.10}$$

- $$\begin{aligned} G = \ & \Sigma(1080) = \langle \alpha, \beta, \gamma, \delta | \alpha^5 = \beta^2 = \gamma^2 = \delta^2 = (\alpha\beta)^2(\beta\gamma)^2 = (\beta\delta)^2 = 1, \\ & \qquad (\alpha\gamma)^3 = (\alpha\delta)^3 = 1, \gamma\beta = \delta\gamma\delta, \alpha^2\gamma\beta\alpha^2 = \gamma\alpha^2\gamma\rangle; \\ & M(G) = \mathbb{Z}_2; \\ & G^* = \langle \alpha, \beta, \gamma, \delta, \epsilon | \alpha^5 = \epsilon^2 = \gamma^2\epsilon^{-1} = \beta^2\epsilon^{-1} = \delta^2\epsilon^{-1} = (\alpha\delta)^3 = 1, \\ & \qquad \alpha^{-1}\epsilon\alpha = \beta^{-1}\epsilon\beta = \gamma^{-1}\epsilon\gamma = \delta^{-1}\epsilon\delta = \epsilon, \\ & \qquad (\alpha\beta)^2 = (\beta\gamma)^2 = (\beta\delta)^2 = \gamma\beta\delta\gamma\delta = (\alpha\gamma)^3 = \epsilon, \\ & \qquad \alpha^2\gamma\beta\alpha^2\gamma\alpha^{-2}\gamma = 1\rangle; \end{aligned} \tag{18.3.11}$$

18.4 Covering Groups, Discrete Torsion and Quiver Diagrams

18.4.1 The Method

The introduction of the host of the above concepts is not without a cause. In this section we shall provide an **algorithm** which permits the construction of the quiver $Q_\alpha(G, \mathcal{R})$ of an orbifold theory with group G having discrete torsion α turned-on, and with a linear representation $\mathcal{R}$ of G acting on the transverse space.

Our method dispenses of the need of the knowledge of the cocycles $\alpha(x, y)$, which in general is a formidable task from the viewpoint of cohomology, but which the current literature may lead the reader to believe to be required for finding the projective representations. We shall demonstrate that the problem of finding these α-representations is reducible to the far more manageable duty of finding the covering group, constructing its character table (which is of course straightforward) and then applying the usual prodecure of extracting the quiver therefrom. One advantage of this method is that we immediately obtain the quiver for all cocycles (including the trivial cocycle which corresponds to having no discrete torsion at all) and in fact the values of $\alpha(x, y)$ (which we shall address in the next section) in a unified framework.

All the data we require are

(i) G and its (ordinary) character table char(G);

(ii) The covering group G^* of G and its (ordinary) character table char(G^*).

We first recall from [248, 249] that turning on discrete torsion α in an orbifold projection amounts to using an α-projective representation[3] Γ_α of $g \in G$

$$\Gamma_\alpha(g) \cdot A \cdot \Gamma_\alpha^{-1}(g) = A, \qquad \Gamma_\alpha(g) \cdot \Phi \cdot \Gamma_\alpha^{-1}(g) = \mathcal{R}(g) \cdot \Phi \tag{18.4.12}$$

on the gauge field A and matter fields Φ.

The above equations have been phrased in a more axiomatic setting (in the language of [76]), by virtue of the fact that much of ordinary representation theory of finite group extends in direct analogy to the projective case, in [259]. *However, we hereby emphasize that with the aid of the linear representation of the covering group, one can perform orbifold projection with discrete torsion entirely in the setting of [76] without usage of the formulae in [259] generalised to twisted group algebras and modules.* In other words, if we use the matrix of the linear representation of G^* instead of

[3]More rigorously, this statement holds only when the D-brane probe is pointlike in the orbifold directions. More generally, when D-brane probes extend along the orbifold directions, one may need to use ordinary as well as projective representations. For further details, please refer to [255] as well as [272].

that of the corresponding projective representation of G, we will arrive at the same gauge group and matter contents in the orbifold theory. This can be alternatively shown as follows.

When we lift the projective matrix representation of G into the linear one of G^*, every matrix $\rho(g)$ will map to $\rho(ga_i)$ for every $a_i \in A$. The crucial fact is that $\rho(ga_i) = \lambda_i \rho(g)$ where λ_i is simply a phase factor. Now in (18.4.12) (cf. Sections 4.2 and 5 for more details), $\Gamma_\alpha(g)$ and $\Gamma_\alpha^{-1}(g)$ always appear in pairs, when we replace them by $\Gamma(ga_i)$ and $\Gamma^{-1}(ga_i)$, the phase factor λ_i will cancel out and leave the result invariant. This shows that the two results, the one given by projective matrix representations of G and the other by linear matrix representations of G^*, will give identical answers in orbifold projections.

18.4.2 An Illustrative Example: $\Delta(3 \times 3^2)$

Without much further ado, an illustrative example of the group $\Delta(3 \times 3^2) \in SU(3)$ shall serve to enlighten the reader. We recall from (18.3.7) that this group of order 27 has presentation $\langle \alpha, \beta, \gamma | \alpha^3 = \beta^3 = \gamma^3 = 1, \alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma \rangle$ and its covering group of order 243 (since the Schur Multiplier is $\mathbb{Z}_3 \times \mathbb{Z}_3$) is $G^* = \langle \alpha, \beta, \gamma, a, b | (\alpha/\beta/\gamma)(a/b) = (a/b)(\alpha/\beta/\gamma), a^3 = b^3 = \gamma^3 = \alpha^3 b = \beta^3 b^{-1} = 1, ab = ba, \alpha\beta = \beta\alpha ab, \alpha\gamma = \gamma\alpha^{-1}\beta, \beta\gamma\alpha = \gamma \rangle$.

Next we require the two (ordinary) character tables. As pointed out in the Nomenclatures section, character tables are given as $(r+1) \times r$ matrices with r being the number of conjugacy classes (and equivalently the number of irreps), and the first row giving the conjugacy class numbers.

$$
\mathrm{char}(\Delta(3 \times 3^2)) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ \hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1 & 1 & 1 & 1 & 1 & \omega_3 & \omega_3 & \omega_3 & \bar{\omega}_3 & \bar{\omega}_3 & \bar{\omega}_3 \\ \hline
1 & 1 & 1 & 1 & 1 & \bar{\omega}_3 & \bar{\omega}_3 & \bar{\omega}_3 & \omega_3 & \omega_3 & \omega_3 \\ \hline
1 & 1 & 1 & \omega_3 & \bar{\omega}_3 & 1 & \omega_3 & \bar{\omega}_3 & 1 & \omega_3 & \bar{\omega}_3 \\ \hline
1 & 1 & 1 & \omega_3 & \bar{\omega}_3 & \omega_3 & \bar{\omega}_3 & 1 & \bar{\omega}_3 & 1 & \omega_3 \\ \hline
1 & 1 & 1 & \omega_3 & \bar{\omega}_3 & \bar{\omega}_3 & 1 & \omega_3 & \omega_3 & \bar{\omega}_3 & 1 \\ \hline
1 & 1 & 1 & \bar{\omega}_3 & \omega_3 & 1 & \bar{\omega}_3 & \omega_3 & 1 & \bar{\omega}_3 & \omega_3 \\ \hline
1 & 1 & 1 & \bar{\omega}_3 & \omega_3 & \omega_3 & 1 & \bar{\omega}_3 & \bar{\omega}_3 & \omega_3 & 1 \\ \hline
1 & 1 & 1 & \bar{\omega}_3 & \omega_3 & \bar{\omega}_3 & \omega_3 & 1 & \omega_3 & 1 & \bar{\omega}_3 \\ \hline
3 & 3\bar{\omega}_3 & 3\omega_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
3 & 3\omega_3 & 3\bar{\omega}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
\end{array} \qquad (18.4.13)
$$

$\mathrm{char}(\Delta(3 \times 3^2)^*) =$

1	1	1	1	1	1	1	1	1	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3
1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$
1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1	ω_3	ω_3	ω_3
1	1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	1	1	1
1	1	1	1	1	1	1	1	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3
1	1	1	1	1	1	1	1	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	ω_3	ω_3	ω_3	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	1	1	1
1	1	1	1	1	1	1	1	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$	ω_3	ω_3	ω_3	1	1	1	ω_3	ω_3	ω_3	1	1	1	$\bar\omega_3$	$\bar\omega_3$	$\bar\omega_3$
3	3	3	3	3	3	3	3	3	$3\bar\omega_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	3	3	3	3	$3\omega_3$	$3\bar\omega_3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	0	0	0	0	0	0	0	0	0	0	0	X	Y	Z	0	0	0	0	0	0	0	0	0	P	M	N
3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	0	0	0	0	0	0	0	0	0	0	0	Z	X	Y	0	0	0	0	0	0	0	0	0	M	N	P
3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\bar\omega_3$	0	0	0	0	0	0	0	0	0	0	0	Y	Z	X	0	0	0	0	0	0	0	0	0	N	P	M
3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	M	P	N	0	0	0	0	0	0	0	0	0	Z	Y	X
3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	N	M	P	0	0	0	0	0	0	0	0	0	Y	X	Z
3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\omega_3$	0	0	0	0	0	0	0	0	0	0	0	P	N	M	0	0	0	0	0	0	0	0	0	X	Z	Y
3	3	3	$3\omega_3$	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	0	0	A	$-B$	C	$-A$	$-C$	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\omega_3$	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	0	0	$-B$	C	A	$-C$	B	$-A$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\omega_3$	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	0	0	C	A	$-B$	B	$-A$	$-C$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	$3\bar\omega_3$	3	$3\omega_3$	0	0	0	0	0	0	0	0	$-C$	$-A$	B	0	0	0	0	0	0	$-B$	C	A	0	0	0	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	$3\bar\omega_3$	3	$3\omega_3$	0	0	0	0	0	0	0	0	$-A$	B	$-C$	0	0	0	0	0	0	A	$-B$	C	0	0	0	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	$3\bar\omega_3$	3	$3\omega_3$	0	0	0	0	0	0	0	0	B	$-C$	$-A$	0	0	0	0	0	0	C	A	$-B$	0	0	0	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	M	N	P	0	0	0	Z	X	Y	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	P	M	N	0	0	0	X	Y	Z	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	N	P	M	0	0	0	Y	Z	X	0	0	0
3	3	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	$3\omega_3$	0	0	$-A$	$-C$	B	A	$-B$	C	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	$3\omega_3$	0	0	$-C$	B	$-A$	$-B$	C	A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	3	3	$3\bar\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	$3\omega_3$	$3\omega_3$	$3\omega_3$	0	0	B	$-A$	$-C$	C	A	$-B$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	X	Z	Y	0	0	0	P	N	M	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Y	X	Z	0	0	0	N	M	P	0	0	0
3	$3\omega_3$	$3\bar\omega_3$	$3\bar\omega_3$	3	$3\omega_3$	$3\omega_3$	$3\bar\omega_3$	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	Z	Y	X	0	0	0	M	P	N	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	$3\omega_3$	3	$3\bar\omega_3$	0	0	0	0	0	0	0	0	$-B$	A	C	0	0	0	0	0	0	$-C$	B	$-A$	0	0	0	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	$3\omega_3$	3	$3\bar\omega_3$	0	0	0	0	0	0	0	0	A	C	$-B$	0	0	0	0	0	0	$-A$	$-C$	B	0	0	0	0	0	0
3	$3\bar\omega_3$	$3\omega_3$	$3\bar\omega_3$	$3\omega_3$	3	$3\omega_3$	3	$3\bar\omega_3$	0	0	0	0	0	0	0	0	C	$-B$	A	0	0	0	0	0	0	B	$-A$	$-C$	0	0	0	0	0	0

(18.4.14)

with $A := -\omega_3+\bar\omega_3, B := \omega_3+2\bar\omega_3, C := 2\omega_3+\bar\omega_3; M := -\omega_9^2-2\bar\omega_9^4, N := \omega_9^2+\bar\omega_9^4, P := -\omega_9^2 + \bar\omega_9^4; X := \omega_9^4 - \bar\omega_9^2, Y := \omega_9^4 + 2\bar\omega_9^2, Z := -2\omega_9^4 - \bar\omega_9^2$.

A comparative study of these two tables shall suffice to demonstrate the method. We have taken extreme pains to re-arrange the columns and rows of $\mathrm{char}(G^*)$ for the sake of perspicuity; whence we immediately observe that $\mathrm{char}(G)$ and $\mathrm{char}(G^*)$ are unrelated but that the latter is organised in terms of "cohorts" [270] of the former. What this means is as follows: columns 1 through 9 of $\mathrm{char}(G^*)$ have their first 11 rows (not counting the row of class numbers) identical to the first column of $\mathrm{char}(G)$, so too is column 10 of $\mathrm{char}(G^*)$ with column 2 of $\mathrm{char}(G)$, *et cetera* with $\{11\} \to \{3\}$, $\{12, 13, 14\} \to \{4\}$, $\{15, 16, 17\} \to \{5\}$, $\{18, 19, 20\} \to \{6\}$, $\{21, 22, 23\} \to \{7\}$,

$\{24,25,26\} \to \{8\}$, $\{27,28,29\} \to \{9\}$, $\{30,31,32\} \to \{10\}$, and $\{33,34,35\} \to \{11\}$; using the notation that $\{X\} \to \{Y\}$ for the first 11 rows of columns $\{X\} \subset \mathrm{char}(G^*)$ are mapped to column $\{Y\} \subset \mathrm{char}(G)$. These are the so-called "splitting conjugacy classes" in G^* which give the (linear) $\mathrm{char}(G)$ [268]. In other words, (though the conjugacy class numbers may differ), up to repetition $\mathrm{char}(G) \subset \mathrm{char}(G^*)$. This of course is in the spirit of the technique of Frøbenius Induction of finding the character table of a group from that of its subgroup; for a discussion of this in the context of orbifolds, the reader is referred to [302]. Thus the first 11 rows of $\mathrm{char}(G^*)$ corresponds exactly to the *linear irreps* of G. The rest of the rows we shall shortly observe to correspond to the projective representations.

To understand these above remarks, let $A := \mathbb{Z}_3 \times \mathbb{Z}_3$ so that $G^*/A \cong G$ as in the notation of Section 2. Now $A \subseteq Z(G^*)$, hence the matrix forms of all of its elements must be $\lambda \mathbb{I}_{d\times d}$, where d is the dimension of the irreducible representation and λ some phase factor. Indeed the first 9 columns of $\mathrm{char}(G^*)$ have conjugacy class number 1 and hence correspond to elements of this centre. Bearing this in mind, if we only tabulated the phases λ (by suppressing the factor $d = 1$ or 3 coming from $\mathbb{I}_{d\times d}$) of these first 9 columns, we arrive at the following table (removing the first row of conjugacy class numbers):

rows	$\mathbb{I}$	a	a^2	b	ab	a^2b	b^2	ab^2	a^2b^2
$2-12$	1	1	1	1	1	1	1	1	1
$13-15$	1	ω_3	$\bar{\omega}_3$	1	ω_3	$\bar{\omega}_3$	1	ω_3	$\bar{\omega}_3$
$16-18$	1	$\bar{\omega}_3$	ω_3	1	$\bar{\omega}_3$	ω_3	1	$\bar{\omega}_3$	ω_3
$19-21$	1	1	1	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
$22-24$	1	ω_3	$\bar{\omega}_3$	ω_3	$\bar{\omega}_3$	1	$\bar{\omega}_3$	1	ω_3
$25-27$	1	$\bar{\omega}_3$	ω_3	ω_3	1	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	1
$28-30$	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3
$31-33$	1	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	1	ω_3	ω_3	$\bar{\omega}_3$	1
$34-36$	1	$\bar{\omega}_3$	ω_3	$\bar{\omega}_3$	ω_3	1	ω_3	1	$\bar{\omega}_3$

The astute reader would instantly recognise this to be the character table of $\mathbb{Z}_3 \times \mathbb{Z}_3 = A$ (and with foresight we have labelled the elements of the group in the above table). This certainly is to be expected: G^* can be written as cosets gA for $g \in G$, whence lifting the (projective) matrix representation $M(g)$ of g simply gives $\lambda M(g)$ for λ a *phase factor* correponding to the representation (or character as A is always Abelian) of A.

What is happening should be clear: all of this is merely Part (i) of Theorem 18.2.28 at work. The phases λ are precisely as described in the theorem. The trivial phase 1 gives rows $2-12$, or simply the ordinary representation of G while the remaining 8 non-trivial phases give, in groups of 3 rows from char(G^*), the projective representations of G. And to determine to which cocycle the projective representation belongs, we need and only need to determine the the 1-dimensional irreps of A. We shall show in Section 5 how to read out the actual cocycle values $\alpha(g,h)$ for $g,h \in G$ directly with the knowledge of A and G^* without char(G^*).

Enough said on the character tables. Let us proceed to analyse the quiver diagrams. Detailed discussions had already been presented in the case of the dihedral group in [301]. Let us recapitulate the key points. It is the group action on the Chan-Paton bundle that we choose to be projective, the space-time action inherited from $\mathcal{N}=4$ R-symmetry remain ordinary. In other words, $\mathcal{R}$ from (18.4.12) must still be a linear representation.

Now we evoke an obvious though handy result: the tensor product of an α-projective representation with that of a β-representation gives an $\alpha\beta$-projective representation (cf. [262] p119), i.e.,

$$\Gamma_\alpha(g) \otimes \Gamma_\beta(g) = \Gamma_{\alpha\beta}(g). \tag{18.4.15}$$

We recall that from (18.4.12) and in the language of [259, 76], the bi-fundamental matter content $a_{ij}^{\mathcal{R}}$ is given in terms of the irreducible representations R_i of G as

$$\mathcal{R} \otimes R_i = \bigoplus_j a_{ij}^{\mathcal{R}} R_j, \tag{18.4.16}$$

(with of course $\mathcal{R}$ linear and R_i projective representations). Because $\mathcal{R}$ is an $\alpha = 1$ (linear) representation, (18.4.15) dictates that if R_i in (18.4.16) is a β-representation, then the righthand thereof must be written entirely in terms of β-representations R_j. In other words, the various projective representations corresponding to the different cocycles should not mix under (18.4.16). What this signifies for the matter matrix is that $a_{ij}^{\mathcal{R}}$ is block-diagonal and the quiver diagram $Q(G^*, \mathcal{R})$ for G^* *splits* into precisely $|A|$ pieces, one of which is the ordinary (linear) quiver for G and the rest, the various quivers each corresponding to a different value of the cocycle.

Thus motivated, let us present the quiver diagram for $\Delta(3 \times 3^2)^*$ in Figure 18-1. The splitting does indeed occur as desired, into precisely $|\mathbb{Z}_3 \times \mathbb{Z}_3| = 9$ pieces, with (i) being the usual $\Delta(3 \times 3^2)$ quiver (cf. [292, 141]) and the rest, the quivers corresponding to the 8 non-trivial projective representations.

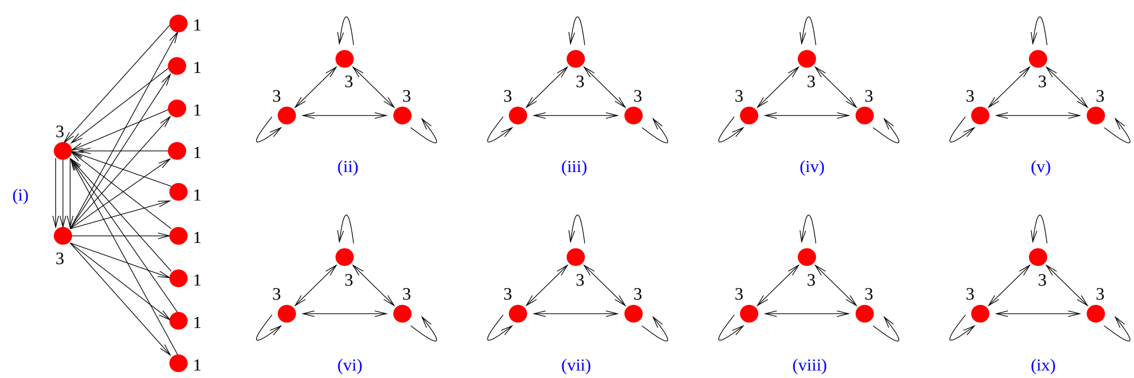


Figure 18-1: The Quiver Diagram for $\Delta(3 \times 3^2)^*$ (the Space Invaders Quiver): piece (i) corresponds to the usual quiver for $\Delta(3 \times 3^2)$ while the remaining 8 pieces (ii) to (ix) are for the cases of the 8 non-trivial discrete torsions (out of the $\mathbb{Z}_3 \times \mathbb{Z}_3$) turned on.

18.4.3 The General Method

Having expounded upon the detailed example of $\Delta(3 \times 3^2)$ and witnessed the subtleties, we now present, in an algorithmic manner, the general method of computing the quiver diagram for an orbifold G with discrete torsion turned on:

1. Compute the character table char(G) of G;

2. Compute a covering group G^* of G and its character table char(G^*);

3. Judiciously re-order the rows and columns of char(G^*):

   - Columns must be arranged into cohorts of char(G), i.e., group the columns which contain a corresponding column in char(G) together;

   - Rows must be arranged so that modulo the dimension of the irreps, the columns with conjugacy class number 1 must contain the character table of the Schur Multiplier $A = M(G)$ (recall that $G^*/A \cong G$);

   - Thus char(G) is a sub-matrix (up to repetition) of char(G^*);

4. Compute the (ordinary) matter matrix $a_{ij}^{\mathcal{R}}$ and hence the quiver $Q(G^*, \mathcal{R})$ for a representation $\mathcal{R}$ which corresponds to a linear representation of G.

Now we have our final result:

THEOREM 18.4.32 *$Q(G^*, \mathcal{R})$ has $|M(G)|$ disconnected components (sub-quivers) in 1-1 correspondence with the quivers $Q_\alpha(G, \mathcal{R})$ of G for all possible cocycles (discrete torsions) $\alpha \in A = M(G)$. Symbolically,*

$$Q(G^*, \mathcal{R}) = \bigsqcup_{\alpha \in A} Q_\alpha(G, \mathcal{R}).$$

In particular, $Q(G^*, \mathcal{R})$ contains a piece for the trivial $\alpha = 1$ which is precisely the case without discrete torsion, viz., $Q(G, \mathcal{R})$.

This algorithm facilitates enormously the investigation of the matter spectrum of orbifold gauge theories with discrete torsion as the associated quivers can be found without any recourse to explicit evaluation of the cocycles and projective character tables. Another fine feature of this new understanding is that, not only the matter content, but also the superpotential can be directly calculated by the explicit formulae in [76] using the ordinary Clebsch-Gordan coefficients of G^*.

A remark is at hand. We have mentioned in Section 2 that the covering group G^* is not unique. How could we guarantee that the quivers obtained at the end of the day will be independent of the choice of the covering group? We appeal directly to the discussion in the concluding paragraph of Subsection 4.1, where we remarked that using the explicit form of (18.4.12), we see that the phase factor λ (being a $\mathbb{C}$-number) always cancels out. In other words, the linear representation of whichever G^* we use, when applied to orbifold projections (18.4.12) shall result in the same matrix form for the projective representations of G. Whence we conclude that the quiver $Q(G^*, \mathcal{R})$ obtained at the end will *ipso facto* be independent of the choice of the covering group G^*.

18.4.4 A Myriad of Examples

With the method at hand, we move on to the host of other subgroups of $SU(3)$ as tabulated in [301]. The character tables char(G) and char(G^*) will be left to the appendix lest the reader be too distracted. We present the cases of $\Sigma(60, 168, 1080)$, the exceptionals which admit nontrivial discrete torsion and some first members of the Delta series in Figure 18-2 to Figure 18-8.

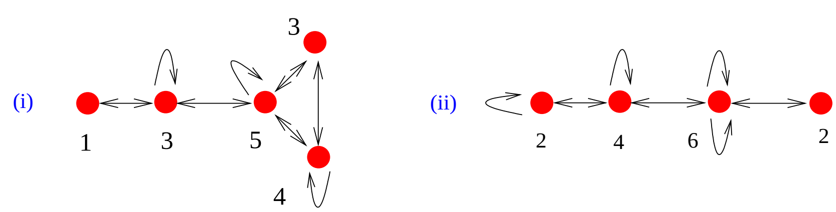


Figure 18-2: The quiver diagram of $\Sigma(60)^*$: piece (i) is the ordinary quiver of $\Sigma(60)$ and piece (ii) has discrete torsion turned on.

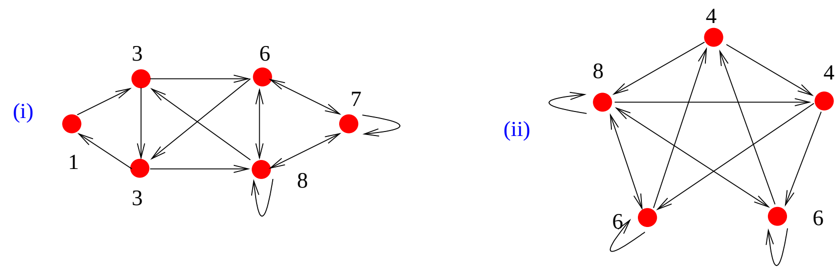


Figure 18-3: The quiver diagram of $\Sigma(168)$: piece (i) is the ordinary quiver of $\Sigma(168)$ and piece (ii) has discrete torsion turned on.

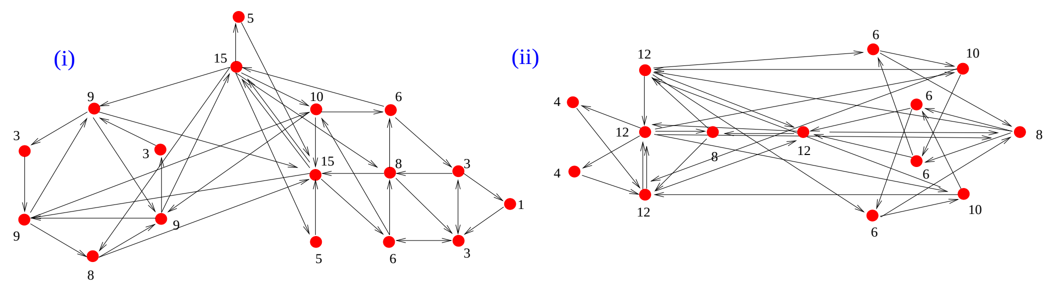


Figure 18-4: The quiver diagram of $\Sigma(1080)$: piece (i) is the ordinary quiver of $\Sigma(1080)$ and piece (ii) has discrete torsion turned on.

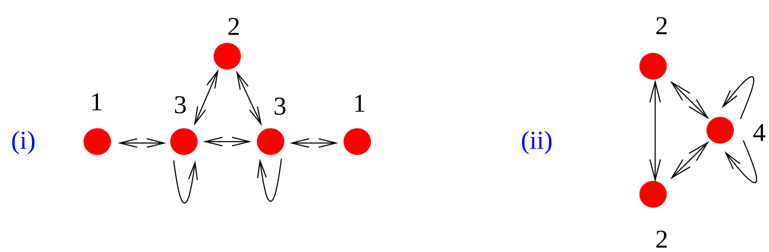


Figure 18-5: The quiver diagram of $\Delta(6 \times 2^2)$: piece (i) is the ordinary quiver of $\Delta(6 \times 2^2)$ and piece (ii) has discrete torsion turned on.

18.5 Finding the Cocycle Values

As advertised earlier, a useful by-product of the method is that we can actually find the values of the 2-cocycles from the covering group. Here we require even less information: only G^* and not even $\mathrm{char}(G^*)$ is needed.

Let us recall some facts from Subsection 4.2. The Schur multiplier is $A \subset Z(G^*)$,

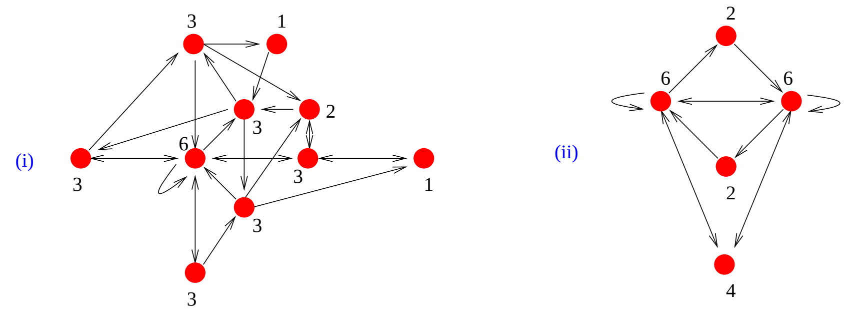


Figure 18-6: The quiver diagram of $\Delta(6 \times 4^2)$: piece (i) is the ordinary quiver of $\Delta(6 \times 4^2)$ and piece (ii) has discrete torsion turned on.

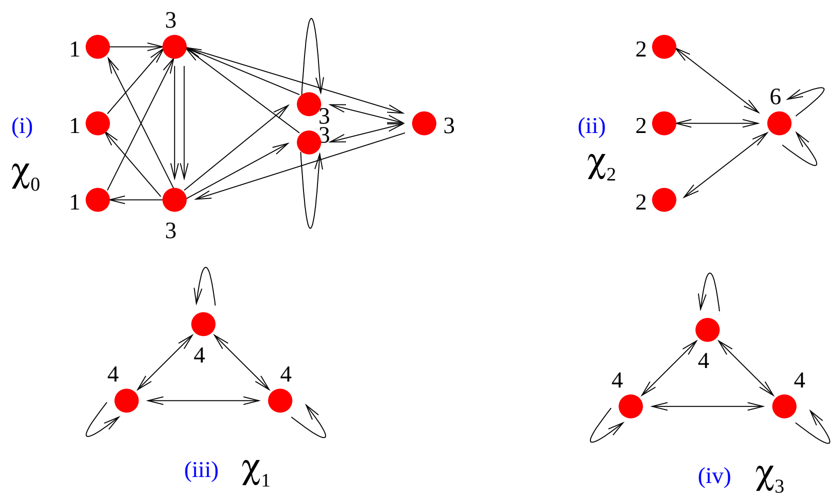


Figure 18-7: The quiver diagram of $\Delta(3 \times 4^2)$: piece (i) is the ordinary quiver of $\Delta(3 \times 4^2)$ and pieces (ii-iv) have discrete torsion turned on. We recall that the Schur Multiplier is $\mathbb{Z}_4$.

so every element therein has its own conjugacy class in G^*. Hence for all linear representations of G^*, the character of $a_k \in A$ will have the form $d\chi_i(a_k)$ where d is the dimension of that particular irrep of G^* and $\chi_i(a_k)$ is the character of a_k in A in its $i-th$ 1-dimensional irrep (A is always Abelian and thus has only 1-dimensional irreps). This property has a very important consequence: merely reading out the factor $\chi_i(a_k)$ from char(G^*), we can determine which linear representations will give which *projective* representations of G. Indeed, two projective representations of G belong to the same cocycle *when and only when* the factor $\chi_i(a_k)$ is the same for every $a_k \in A$.

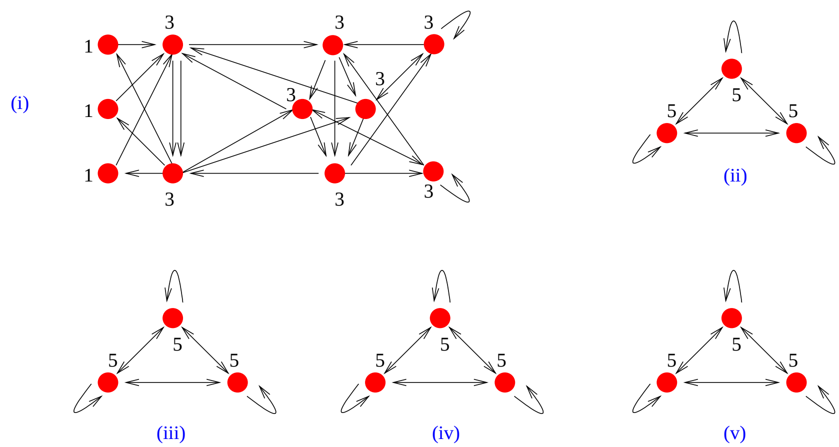


Figure 18-8: The quiver diagram of $\Delta(3 \times 5^2)$: piece (i) is the ordinary quiver of $\Delta(3 \times 4^2)$ and pieces (ii-v) have discrete torsion turned on. We recall that the Schur Multiplier is $\mathbb{Z}_5$.

Next we recall how to construct the matrix forms of projective representations of G. $G^\star/A \equiv G$ implies that G^* can be decomposed into cosets $\bigcup\limits_{g\in G} gA$. Let $ga_i \in G^*$ correspond canonically to $\tilde{g} \in G$ for some fixed $a_i \in A$; then the matrix form of $\tilde{g}$ can be set to that of ga_i and furnishes the projective representation of $\tilde{g}$. Different choices of a_i will give different but projectively equivalent projective representations of G.

Note that if we have $\tilde{g}_i\tilde{g}_j = \tilde{g}_k$ in G, then in G^*, $g_ig_j = g_ka^k_{ij}$, or $(g_ia_i)(g_ja_j) = g_ka_k(a^k_{ij}a_ia_ja_k^{-1})$, but since (g_ia_i) is the projective matrix form for $\tilde{g}_i \in G$, this is exactly the definition of the cocyle from which we read:

$$\alpha(\tilde{g}_i, \tilde{g}_j) = \chi_p(a^k_{ij}a_ia_ja_k^{-1}), \tag{18.5.17}$$

where $\chi_p(a)$ is the p-th character of the linear representation of $a \in A$ defined above.

We can prove that (18.5.17) satisfies the 2-cocycle axioms (i) and (ii). Firstly notice that if $\tilde{g}_i = \mathbb{I} \in G$, we have $g_i = \mathbb{I} \in G^\star$; whence $a^k_{ij} = \delta^k_j \ \forall \ i$ and

$$(i) \quad \alpha(\mathbb{I}, \tilde{g}_j) = \chi_p(\delta^k_j a_ja_k^{-1}) = \chi_p(\mathbb{I}) = 1.$$

Secondly if we assume that $\tilde{g}_i\tilde{g}_j = \tilde{g}_q$, $\tilde{g}_q\tilde{g}_k = \tilde{g}_h$ and $\tilde{g}_j\tilde{g}_k = \tilde{g}_l$, we have $\alpha(\tilde{g}_i, \tilde{g}_j)\alpha(\tilde{g}_i\tilde{g}_j, \tilde{g}_k) =$

$\chi_p(a_{ij}^q a_i a_j a_q^{-1})\chi_p(a_{qk}^h a_q a_k a_h^{-1}) = \chi_p(a_{ij}^k a_{qk}^h a_i a_j a_k a_h^{-1})$

and $\alpha(\tilde{g}_i, \tilde{g}_j\tilde{g}_k)\alpha(\tilde{g}_j, \tilde{g}_k) = \chi_p(a_{jk}^l a_j a_k a_l^{-1})\chi_p(a_{il}^h a_i a_l a_h^{-1}) = \chi_p(a_{il}^h a_{jk}^l a_i a_j a_k a_h^{-1})$.

However, because $(g_i g_j)g_k = g_q a_{ij}^q g_k = g_h a_{ij}^q a_{qk}^h = g_i(g_j g_k) = g_i g_l a_{jk}^l = g_h a_{il}^h a_{jk}^l$ we have $a_{ij}^k a_{qk}^h = a_{il}^h a_{jk}^l$, and so

$$(ii) \quad \alpha(\tilde{g}_i, \tilde{g}_j)\alpha(\tilde{g}_i\tilde{g}_j, \tilde{g}_k) = \alpha(\tilde{g}_i, \tilde{g}_j\tilde{g}_k)\alpha(\tilde{g}_j, \tilde{g}_k).$$

Let us summarize the result. To read out the cocycle according to (18.5.17) we need only two pieces of information: the choices of the representative element in G^* (i.e., $a_i \in A$), and the definitions of G^* which allows us to calculate the $a_{ij}^k \in A$. We do not even need to calculate the character table of G^* to obtain the cocycle. Moreover, in a recent paper [271] the values of cocycles are being used to construct boundary states. We hope our method shall make this above construction easier.

18.6 Conclusions and Prospects

With the advent of discrete torsion in string theory, the hitherto novel subject of projective representations has breathed out its fragrance from mathematics into physics. However a short-coming has been immediate: the necessary tools for physical computations have so far been limited in the community due to the unavoidable fact that they, if present in the mathematical literature, are obfuscated under often too-technical theorems.

It has been the purpose of this writing, a companion to [301], to diminish the mystique of projective reprsentations in the context of constructing gauge theories on D-branes probing orbifolds with discrete torsion (non-trivial NS-NS B-fields) turned on. In particular we have deviced an algorithm (Subsection 4.3), culminating into Theorem 4.4, which computes the gauge theory data of the orbifold theory. The advantage of the method is its directness: without recourse to the sophistry of twisted group algebras and projective characters as had been suggested by some recent works [248, 249, 259], all methods so-far known in the treatment of orbifolds (e.g. [76, 292])

are immediately generalisable.

We have shown that in computing the matter spectrum for an orbifold G with discrete torsion turned on, all that is required is the ordinary charater table char(G^*) of the covering group G^* of G. This table, together with the available character table of G, immediately gives a quiver diagram which splits into $|M(G)|$ disjoint pieces ($M(G)$ is the Schur Multiplier of G), one of which is the ordinary quiver for G and the rest, are precisely the quivers for the various non-trivial discrete torsions.

A host of examples are then presented, demonstrating the systematic power of the algorithm. In particular we have tabulated the results for all the exceptional subgroups of $SU(3)$ as well as some first members of the Δ-series.

Directions for future research are self-evident. Brane setups for orbifolds with discrete torsion have yet to be established. We therefore need to investigate the groups satisfying BBM condition as defined in [295, 296], such as the intransitives of the form $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z} \times D$. Furthermore, we have given the presentation of the covering groups of series such as $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times D, \mathbb{Z} \times E$ and $\Delta(3n^2), \Delta(6n^2)$. It will be interesting to find the analytic results of the possible quivers.

More importantly, as we have reduced the problem of orbifolds with discrete torsion to that of *linear* representations, we can instantly extend the methods of [76] to compute superpotentials and thence further to an extensive and systematic study of non-commutative moduli spaces in the spirit of [251]. So too do the families of toric varieties await us, methods utilised in [254, 298] eagerly anticipate their extension. Indeed we have set a vessel adrift, it shall take the course in a vast and unknown sea.

Chapter 19

Toric I: Toric Singularities and Toric Duality

Synopsis

The next three chapters shall constitute the last part of Liber III; they shall be chiefly concerned with toric singularities and D-brane probes thereupon.

In this chapter, via partial resolution of Abelian orbifolds we present an algorithm for extracting a consistent set of gauge theory data for an arbitrary toric variety whose singularity a D-brane probes. As illustrative examples, we tabulate the matter content and superpotential for a D-brane living on the toric del Pezzo surfaces as well as the zeroth Hirzebruch surface. Moreover, we discuss the non-uniqueness of the general problem and present examples of vastly different theories whose moduli spaces are described by the same toric data. Our methods provide new tools for calculating gauge theories which flow to the same universality class in the IR. We shall call it "Toric Duality" [298, 299].

19.1 Introduction

The study of D-branes as probes of geometry and topology of space-time has by now been of wide practice (cf. e.g. [18]). In particular, the analysis of the moduli space of gauge theories, their matter content, superpotential and β-function, as world-volume theories of D-branes sitting at geometrical singularities is still a widely pursued topic. Since the pioneering work in [69], where the moduli and matter content of D-branes probing ALE spaces had been extensively investigated, much work ensued. The primary focus on (Abelian) orbifold singularities of the type $\mathbb{C}^2/\mathbb{Z}_n$ was quickly generalised using McKay's Correspondence, to arbitrary (non-Abelian) orbifold singularities $\mathbb{C}^2/(\Gamma \subset SU(2))$, i.e., to arbitrary ALE spaces, in [171].

Several directions followed. With the realisation [75, 157] that these singularities provide various horizons, [69, 171] was quickly generalised to a treatment for arbitrary finite subgroups $\Gamma \subset SU(N)$, i.e., to generic Gorenstein singularities, by [76]. The case of $SU(3)$ was then promptly studied in [292, 141, 273] using this technique and a generalised McKay-type Correspondence was proposed in [292, 293]. Meanwhile, via T-duality transformations, certain orbifold singularities can be mapped to type II brane-setups in the fashion of [66]. The relevant gauge theory data on the world volume can thereby be conveniently read from configurations of NS-branes, D-brane stacks as well as orientifold planes. For $\mathbb{C}^2$ orbifolds, the A and D series have been thus treated [66, 83], whereas for $\mathbb{C}^3$ orbifolds, the Abelian case of $\mathbb{Z}_k \times \mathbb{Z}_{k'}$ has been solved by the brane box models [78, 79]. First examples of non-Abelian $\mathbb{C}^3$ orbifolds have been addressed in the previous chapters as well as [172].

Thus rests the status of orbifold theories. What we note in particular is that once we specify the properties of the orbifold in terms of the algebraic properties of the finite group, the gauge theory information is easily extracted. Of course, orbifolds are a small subclass of algebro-geometric singularities. This is where we move on to **toric varieties**. Inspired by the linear σ-model approach of [17], which provides a rich structure of the moduli space, especially in connexion with various geometrical phases of the theory, the programme of utilising toric methods to study the behaviour of the gauge theory on D-branes which live on and hence resolve certain singularities was initiated in [74]. In this light, toric methods provide a powerful tool for studying the moduli space of the gauge theory. In treating the F-flatness and D-flatness conditions for the SUSY vacuum in conjunction, these methods show how branches of the moduli space and hence phases of the theory may be parametrised by the algebraic equations of the toric variety. Recent developments in "brane diamonds," as an extension of the brane box rules, have been providing great insight to such a wider class of toric singularities, especially the generalised conifold, via blown-up versions of the standard brane setups [211]. Indeed, with toric techniques much information could be extracted as we can actually analytically describe patches of the moduli space.

Now Abelian orbifolds have toric descriptions and the above methodolgy is thus immediately applicable thereto. While bearing in mind that though non-Abelian orbifolds have no toric descriptions, a single physical D-brane has been placed on various general toric singularities. Partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, such as the conifold and the suspended pinched point have been investigated in [273, 214] and brane setups giving the field theory contents are constructed by [274, 276, 275]. Groundwork for the next family, coming from the toric orbifold $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$, such as the del Pezzo surfaces and the zeroth Hirzebruch, has been laid in [277]. Essentially, given the gauge theory data on the D-brane world volume, the procedure of transforming this information (F and D terms) into toric data which parametrises the classical moduli space is by now well-established.

One task is therefore immediately apparent to us: how do we proceed in the reverse direction, i.e., *when we probe a toric singularity with a D-brane, how do we know the gauge theory on its world-volume?* We recall that in the case of orbifold theories, [76] devised a general method to extract the gauge theory data (matter content, superpotential etc.) from the geometry data (the characters of the finite group Γ), and *vice versa* given the geometry, brane-setups for example, conveniently allow us to read out the gauge theory data. The same is not true for toric singularities, and the second half of the above bi-directional convenience, namely, a general method which allows us to treat the inverse problem of extracting gauge theory data from toric data is yet pending, or at least not in circulation.

The reason for this shortcoming is, as we shall see later, that the problem is highly non-unique. It is thus the purpose of this writing to address this inverse problem: given the geometry data in terms of a toric diagram, how does one read out (at least one) gauge theory data in terms of the matter content and superpotential? We here present precisely this algorithm which takes the matrices encoding the singularity to the matrices encoding a good gauge theory of the D-brane which probes the said singularity.

The structure of the chapter is as follows. In Section 2 we review the procedure of proceeding from the gauge theory data to the toric data, while establishing nomenclature. In Subsection 3.1, we demonstrate how to extract the matter content and F-terms from the charge matrix of the toric singularity. In Subsection 3.2, we exemplify our algorithm with the well-known suspended pinched point before presenting in detail in Subsection 3.3, the general algorithm of how to obtain the gauge theory information from the toric data by the method of partial resolutions. In Subsection 3.4, we show how to integrate back to obtain the actual superpotential once the F-flatness equations are extracted from the toric data. Section 4 is then devoted to the illustration of our algorithm by tabulating the D-terms and F-terms of D-brane world volume theory on the toric del Pezzo surfaces and Hirzebruch zero. We finally discuss in Section 5, the non-uniqueness of the inverse problem and provide, through the studying of two types of ambiguities, ample examples of rather different gauge theories flowing to the same toric data. Discussions and future prospects are dealt with in Section 6.

19.2 The Forward Procedure: Extracting Toric Data From Gauge Theories

We shall here give a brief review of the procedures involved in going from gauge theory data on the D-brane to toric data of the singularity, using primarily the notation and concepts from [74]. In the course thereof special attention will be paid on how toric diagrams, SUSY fields and linear σ-models weave together.

A stack of n D-brane probes on algebraic singularities gives rise to SUSY gauge theories with product gauge groups resulting from the projection of the $U(n)$ theory on the original stack by the geometrical structure of the singularity. For orbifolds $\mathbb{C}^k/\Gamma$, we can use the structure of the finite group Γ to fabricate product $U(n_i)$ gauge groups [69, 171, 76]. For toric singularities, since we have only (Abelian) $U(1)$ toroidal actions, we are so far restricted to product $U(1)$ gauge groups[1]. In physical terms, we have *a single D-brane probe*. Extensive work has been done in [277, 74] to see how the geometrical structure of the variety can be thus probed and how the gauge theory moduli may be encoded. The subclass of toric singularities, namely Abelian orbifolds, has been investigated to great detail [69, 250, 74, 214, 277] and we shall make liberal usage of their properties throughout.

Now let us consider the world-volume theory on the D-brane probe on a toric singularity. Such a theory, as it is a SUSY gauge theory, is characterised by its matter content and interactions. The former is specified by quiver diagrams which in turn give rise to **D-term** equations; the latter is given by a superpotential, whose partial derivatives with respect to the various fields are the so-called **F-term** equations. F and D-flatness subsequently describe the (classical) moduli space of the theory. The basic idea is that the D-term equations together with the FI-parametres, in conjunction with the F-term equations, can be concatenated together into a matrix which gives the vectors forming the dual cone of the toric variety which the D-branes probe. We summarise the algorithm of obtaining the toric data from the gauge theory in the following, and to illuminate our abstraction and notation we will use the simple example of the Abelian orbifold $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ as given in Figure 19-1.

1. Quivers and D-Terms:

   (a) The bi-fundamental matter content of the gauge theory can be conveniently encoded into a **quiver diagram** $\mathcal{Q}$, which is simply the (possibly

[1] Proposals toward generalisations to D-brane stacks have been made [277].

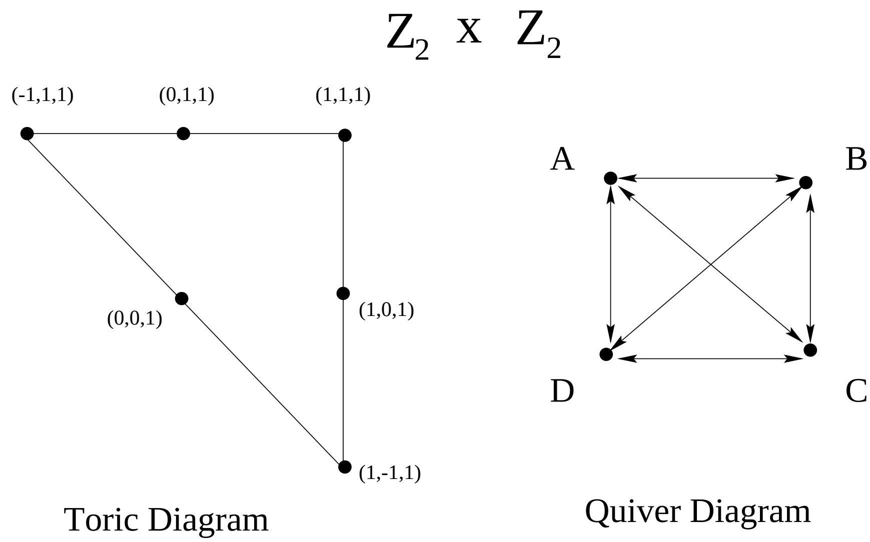


Figure 19-1: The toric diagram for the singularity $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ and the quiver diagram for the gauge theory living on a D-brane probing it. We have labelled the nodes of the toric diagram by columns of G_t and those of the quiver, with the gauge groups $U(1)_{\{A,B,C,D\}}$.

directed) graph whose **adjacency matrix** a_{ij} is precisely the matrix of the bi-fundamentals. In the case of an Abelian orbifold[2] prescribed by the group Γ, this diagram is the McKay Quiver (i.e., for the irreps R_i of Γ, a_{ij} is such that $R \otimes R_i = \oplus_j a_{ij} R_j$ for some fundamental representation R). We denote the set of nodes as $\mathcal{Q}_0 := \{v\}$ and the set of the edges, $\mathcal{Q}_1 := \{a\}$. We let the number of nodes be r; for Abelian orbifolds, $r = |\Gamma|$ (and for generic orbifolds r is the number of conjugacy classes of Γ). Also, we let the number of edges be m; this number depends on the number of supersymmetries which we have. The adjacency matrix (bi-fundamentals) is thus $r \times r$ and the gauge group is $\prod\limits_{j=1}^{r} SU(w_j)$. For our example of $\mathbb{Z}_2 \times \mathbb{Z}_2$, $r = 4$, indexed as 4 gauge groups $U(1)_A \times U(1)_B \times U(1)_C \times U(1)_D$ corresponding to the 4 nodes, while $m = 4 \times 3 = 12$, corresponding to the 12 arrows in Figure 19-1. The adjacency matrix for the quiver is $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. Though for such simple examples as Abelian orbifolds and conifolds, brane

[2]This is true for all orbifolds but of course only Abelian ones have known toric description.

setups and [76] specify the values of w_j as well as a_{ij} completely[3], there is yet no discussion in the literature of obtaining the matter content and gauge group for generic toric varieties in a direct and systematic manner and a partial purpose of this note is to present a solution thereof.

(b) From the $r \times r$ adjacency matrix, we construct a so-called $r \times m$ **incidence matrix** d for $\mathcal{Q}$; this matrix is defined as $d_{v,a} := \delta_{v,head(a)} - \delta_{v,tail(a)}$ for $v \in \mathcal{Q}_0$ and $a \in \mathcal{Q}_1$. Because each column of d must contain a 1, a -1 and the rest 0's by definition, one row of d is always redundant; this physically signifies the elimination of an overall trivial $U(1)$ corresponding to the COM motion of the branes. Therefore we delete a row of d to define the matrix Δ of dimensions $(r-1) \times m$; and we could always extract d from Δ by adding a row so as to force each column to sum to zero. This matrix Δ thus contains almost as much information as a_{ij} and once it is specified, the gauge group and matter content are also, with the exception that precise adjoints (those charged under the same gauge group factor and hence correspond to arrows that join a node to itself) are not manifest. For our example the 4×12 matrix d is as follows and Δ is the top 3 rows:

$$d = \left(\begin{array}{c|cccccccccccc}
 & X_{AD} & X_{BC} & X_{CB} & X_{DA} & X_{AB} & X_{BA} & X_{CD} & X_{DC} & X_{AC} & X_{BD} & X_{CA} & X_{DB} \\
A & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\
B & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\
C & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & 0 \\
\hline
D & 1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & -1
\end{array}\right)$$

(c) The moment maps, arising in the sympletic-quotient language of the toric variety, are simply $\mu := d \cdot |x(a)|^2$ where $x(a)$ are the affine coordinates of the $\mathbb{C}^r$ for the torus $(\mathbb{C}^*)^r$ action. Physically, $x(a)$ are of course the bi-fundamentals in chiral multiplets (in our example they are $X_{ij \in \{A,B,C,D\}}$

[3]For arbitrary orbifolds, $\sum\limits_j w_i n_i = |\Gamma|$ where n_i are the dimensions of the irreps of Γ; for Abelian case, $n_i = 1$.

as labelled above) and the D-term equations for each $U(1)$ group is [17]

$$D_i = -e^2(\sum_a d_{ia}|x(a)|^2 - \zeta_i)$$

with ζ_i the FI-parametres. In matrix form we have $\Delta \cdot |x(a)|^2 = \vec{\zeta}$ and see that D-flatness gives precisely the moment map. These ζ-parametres will encode the resolution of the toric singularity as we shall shortly see.

2. Monomials and F-Terms:

   (a) From the super-potential W of the SUSY gauge theory, one can write the F-Term equation as the system $\frac{\partial}{\partial X_j} W = 0$. The remarkable fact is that we could solve the said system of equations and express the m fields X_i in terms of $r+2$ parametres v_j which can be summarised by a matrix K.

$$X_i = \prod_j v_j^{K_{ij}}, \qquad i = 1,2,..,m; \quad j = 1,2,..,r+2 \tag{19.2.1}$$

This matrix K of dimensions $m \times (r+2)$ is the analogue of Δ in the sense that it encodes the F-terms and superpotential as Δ encodes the D-terms and the matter content. In the language of toric geometry K defines a cone[4] $\mathbf{M}_+$: a non-negative linear combination of m vectors $\vec{K}_i$ in an integral lattice $\mathbb{Z}^{r+2}$.

For our example, the superpotential is

$$\begin{aligned} W = X_{AC}X_{CD}X_{DA} - X_{AC}X_{CB}X_{BA} + X_{CA}X_{AB}X_{BC} - X_{CA}X_{AD}X_{DC} \\ +X_{BD}X_{DC}X_{CB} - X_{BD}X_{DA}X_{AB} - X_{DB}X_{BC}X_{CD}, \end{aligned}$$

[4] We should be careful in this definition. Strictly speaking we have a lattice $\mathbf{M} = \mathbb{Z}^{r+2}$ with its dual lattice $\mathbf{N} \cong \mathbb{Z}^{r+2}$. Now let there be a set of $\mathbb{Z}_+$-independent vectors $\{\vec{k}_i\} \in \mathbf{M}$ and a cone is defined to be generated by these vectors as $\sigma := \{\sum_i a_i \vec{k}_i \mid a_i \in \mathbb{R}_{\geq 0}\}$; Our $\mathbf{M}_+$ should be $\mathbf{M} \cap \sigma$. In much of the literature $\mathbf{M}_+$ is taken to be simply $\mathbf{M}'_+ := \{\sum_i a_i \vec{k}_i \mid a_i \in \mathbb{Z}_{\geq 0}\}$ in which case we must make sure that any lattice point contained in $\mathbf{M}_+$ but not in $\mathbf{M}'_+$ must be counted as an independent generator and be added to the set of generators $\{\vec{k}_i\}$. After including all such points we would have $\mathbf{M}'_+ = \mathbf{M}_+$. Throughout our analyses, our cone defined by K as well the dual cone T will be constituted by such a complete set of generators.

giving us 12 F-term equations and with the manifold of solutions parametrisable by $4 + 2$ new fields, whereby giving us the 12×6 matrix (we here show the transpose thereof, thus the horizontal direction corresponds to the original fields X_i and the vertical, v_j):

$$K^t = \left(\begin{array}{c|cccccccccccc} & X_{AC} & X_{BD} & X_{CA} & X_{DB} & X_{AB} & X_{BA} & X_{CD} & X_{DC} & X_{AD} & X_{BC} & X_{CB} & X_{DA} \\ \hline v_1 = X_{AC} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ v_2 = X_{BD} & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ v_3 = X_{BA} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ v_4 = X_{CD} & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ v_5 = X_{AD} & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ v_6 = X_{CB} & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right).$$

For example, the third column reads $X_{CA} = v_2 v_5^{-1} v_6$, i.e., $X_{AD} X_{CA} = X_{BD} X_{CB}$, which the the F-flatness condition $\frac{\partial W}{\partial X_{DC}=0}$. The details of obtaining W and K from each other are discussed in [74, 277] and Subsection 3.4.

(b) We let T be the space of (integral) vectors dual to K, i.e., $K \cdot T \geq 0$ for all entries; this gives an $(r + 2) \times c$ matrix for some positive integer c. Geometrically, this is the definition of a dual cone $\mathbf{N}_+$ composed of vectors $\vec{T}_i$ such that $\vec{K} \cdot \vec{T} \geq 0$. The physical meaning for doing so is that K may have negative entries which may give rise to unwanted singularities and hence we define a new set of c fields p_i (*a priori* we do not know the number c and we present the standard algorithm of finding dual cones in Appendix 22.10). Thus we reduce (19.2.1) further into

$$v_j = \prod_\alpha p_\alpha^{T_{j\alpha}} \tag{19.2.2}$$

whereby giving $X_i = \prod_j v_j^{K_{ij}} = \prod_\alpha p_\alpha^{\sum_j K_{ij} T_{j\alpha}}$ with $\sum_j K_{ij} T_{j\alpha} \geq 0$. For our $\mathbb{Z}_2 \times Z_2$ example, $c = 9$ and

$$T_{j\alpha} = \left(\begin{array}{c|ccccccccc} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \\ \hline X_{AC} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ X_{BD} & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ X_{BA} & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ X_{CD} & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ X_{AD} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ X_{CB} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right)$$

(c) These new variables p_α are the matter fields in Witten's linear σ-model. How are these fields charged? We have written $r+2$ fields v_j in terms of c fields p_α, and hence need $c-(r+2)$ relations to reduce the independent variables. Such a reduction can be done via the introduction of the new gauge group $U(1)^{c-(r+2)}$ acting on the p_i's so as to give a new set of D-terms. The charges of these fields can be written as $Q_{k\alpha}$. The gauge invariance condition of v_i under $U(1)^{c-(r+2)}$, by (19.2.2), demands that the $(c-r-2) \times c$ matrix Q is such that $\sum_\alpha T_{j\alpha} Q_{k\alpha} = 0$. This then defines for us our charge matrix Q which is the cokernel of T:

$$TQ^t = (T_{j\alpha})(Q_{k\alpha})^t = 0, \qquad j = 1,..,r+2; \;\; \alpha = 1,..,c; \;\; k = 1,..,(c-r-2)$$

For our example, the charge matrix is $(9-4-2) \times 9$ and one choice is $Q_{k\alpha} = \begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(d) In the linear σ-model language, the F-terms and D-terms can be treated in the same footing, i.e., as the D-terms (moment map) of the new fields p_α; with the crucial difference being that the former must be set exactly to zero[5] while the latter are to be resolved by arbitrary FI-parameters. Therefore in addition to finding the charge matrix Q for the new fields p_α coming from the original F-terms as done above, we must also find the

[5]Strictly speaking, we could have an F-term set to a non-zero constant. An example of this situation could be when there is a term $a\phi + \phi\tilde{Q}Q$ in the superpotential for some chargeless field ϕ and charged fields $\tilde{Q}$ and Q. The F-term for ϕ reads $\tilde{Q}Q = -a$ and not 0. However, in our context ϕ behaves like an integration constant and for our purposes, F-terms are set exactly to zero.

corresponding charge matrix Q_D for the p_i coming from the original D-terms. We can find Q_D in two steps. Firstly, we know the charge matrix for X_i under $U(1)^{r-1}$, which is Δ. By (19.2.1), we transform the charges to that of the v_j's, by introducing an $(r-1) \times (r+2)$ matrix V so that $V \cdot K^t = \Delta$. To see this, let the charges of v_j be V_{lj} then by (19.2.1) we have $\Delta_{li} = \sum\limits_j V_{lj} K_{ij} = V \cdot K^t$. A convenient V which does so for our $\mathbb{Z}_2 \times \mathbb{Z}_2$ example is $\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}_{(4-1)\times(4+2)}$. Secondly, we use (19.2.2) to transform the charges from v_j's to our final variables p_α's, which is done by introducing an $(r+2) \times c$ matrix $U_{j\alpha}$ so that $U \cdot T^t = \mathrm{Id}_{(r+2)\times(r+2)}$. In our example, one choice for U is $U_{j\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}_{(4+2)\times 9}$.
Threfore, combining the two steps, we obtain $Q_D = V \cdot U$ and in our example, $(V \cdot U)_{l\alpha} = \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$.

3. Thus equipped with the information from the two sides: the F-terms and D-terms, and with the two required charge matrices Q and $V \cdot U$ obtained, finally we concatenate them to give a $(c-3) \times c$ matrix Q_t. The transpose of the kernel of Q_t, with (possible repeated columns) gives rise to a matrix G_t. The columns of this resulting G_t then define the vertices of the toric diagram describing the polynomial corresponding to the singularity on which we initially placed our D-branes. Once again for our example, $Q_t = \left(\begin{array}{ccccccccc|c} 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & \zeta_1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & \zeta_2 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & \zeta_3 \end{array}\right)$ and $G_t = \begin{pmatrix} 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$.
The columns of G_t, up to repetition, are precisely marked in the toric diagram for $\mathbb{Z}_2 \times \mathbb{Z}_2$ in Figure 19-1.

Thus we have gone from the F-terms and the D-terms of the gauge theory to the nodes of the toric diagram. In accordance with [10], G_t gives the algebraic variety whose equation is given by the maximal ideal in the polynomial ring $\mathbb{C}[YZ, XYZ, Z, X^{-1}YZ, XY^{-1}Z, XZ]$ (the exponents (i,j,k) in $X^iY^jZ^k$ are exactly the columns), which is $uvw = s^2$, upon defining $u = (YZ)(XYZ)^2(Z)(XZ)^2; v = (YZ)^2(Z)^2(X^{-1}YZ)^2; w = (Z)^2(XY^{-1}Z)(XZ)^2$ and $s = (YZ)^2(XYZ)(Z)^2(X^{-1}YZ)(XY^{-1}Z)(XZ)^2$; this is precisely $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. In physical terms this equation parametrises the moduli space obtained from the F and D flatness of the gauge theory.

We remark two issues here. In the case of there being no superpotential we could still define K-matrix. In this case, with there being no F-terms, we simply take K to be the identity. This gives T =Id and $Q = 0$. Furthermore U becomes Id and $V = \Delta$, whereby making $Q_t = \Delta$ as expected because all information should now be contained in the D-terms. Moreover, we note that the very reason we can construct a K-matrix is that all of the equations in the F-terms we deal with are in the form $\prod\limits_i X_i^{a_i} = \prod\limits_j X_j^{b_j}$; this holds in general if every field X_i appears twice and precisely twice in the superpotential. More generic situations would so far transcend the limitations of toric techniques.

Schematically, our procedure presented above at length, what it means is as follows: we begin with two pieces of physical data: (1) matrix d from the quiver encoding the gauge groups and D-terms and (2) matrix K encoding the F-term equations. From these we extract the matrix G_t containing the toric data by the flow-chart:

$$\begin{array}{ccccccc}
\text{Quiver} \to d & \to & \Delta & & & & \\
 & & \downarrow & & & & \\
\text{F-Terms} \to K & \stackrel{V\cdot K^t = \Delta}{\to} & V & & & & \\
\downarrow & & \downarrow & & & & \\
T = \text{Dual}(K) & \stackrel{U\cdot T^t = \text{Id}}{\to} & U & \to & VU & & \\
\downarrow & & & & \downarrow & & \\
Q = [\text{Ker}(T)]^t & & \longrightarrow & & Q_t = \begin{pmatrix} Q \\ VU \end{pmatrix} & \to & G_t = [\text{Ker}(Q_t)]^t
\end{array}$$

19.3 The Inverse Procedure: Extracting Gauge Theory Information from Toric Data

As outlined above we see that wherever possible, the gauge theory of a D-brane probe on certain singularities such as Abelian orbifolds, conifolds, etc., can be conveniently encoded into the matrix Q_t which essentially concatenates the information contained in the D-terms and F-terms of the original gauge theory. The cokernel of this matrix is then a list of vectors which prescribes the toric diagram corresponding to the singularity. It is natural to question ourselves whether the converse could be done, i.e., whether given an arbitrary singularity which affords a toric description, we could obtain the gauge theory living on the D-brane which probes the said singularity. This is the inverse problem we projected to solve in the introduction.

19.3.1 Quiver Diagrams and F-terms from Toric Diagrams

Our result must be two-fold: first, we must be able to extract the D-terms, or in other words the quiver diagram which then gives the gauge group and matter content; second, we must extract the F-terms, which we can subsequently integrate back to give the superpotential. These two pieces of data then suffice to specify the gauge theory. Essentially we wish to trace the arrows in the above flow-chart from G_t back to Δ and K. The general methodology seems straightforward:

1. Read the column-vectors describing the nodes of the given toric diagram, repeat the appropriate columns to obtain G_t and then set $Q_t = \text{Coker}(G_t)$;
2. Separate the D-term $(V \cdot U)$ and F-term (Q_t) portions from Q_t;
3. From the definition of Q, we obtain[6] $T = \ker(Q)$.

[6] As mentioned before we must ensure that such a T be chosen with a complete set of $\mathbb{Z}_+$-independent generators;

4. Farka's Theorem [10] guarantees that the dual of a convex polytope remains convex whence we could invert and have $K = \text{Dual}(T^t)$; Moreover the duality theorem gives that $\text{Dual}(\text{Dual}(K)) = K$, thereby facilitating the inverse procedure.

5. Definitions $U \cdot T^t = \text{Id}$ and $V \cdot K^t = \Delta \Rightarrow (V \cdot U) \cdot (T^t \cdot K^t) = \Delta$.

We see therefore that once the appropriate Q_t has been found, the relations

$$K = \text{Dual}(T^t) \qquad \Delta = (V \cdot U) \cdot (T^t \cdot K^t) \tag{19.3.3}$$

retrieve our desired K and Δ. The only setback of course is that the appropriate Q_t is NOT usually found. Two ambiguities are immediately apparent to us: (A) In step 1 above, there is really no way to know a priori which of the vectors we should repeat when writing into the G_t matrix; (B) In step 2, to separate the D-terms and the F-terms, i.e., which rows constitute Q and which constitute $V \cdot U$ within Q_t, seems arbitrary. We shall in the last section discuss these ambiguities in more detail and actually perceive it to be a matter of interest. Meanwhile, in light thereof, we must find an alternative, to find a canonical method which avoids such ambiguities and gives us a consistent gauge theory which has such well-behaved properties as having only bi-fundamentals etc.; this is where we appeal to partial resolutions.

Another reason for this canonical method is compelling. The astute reader may question as to how could we guarantee, in our mathematical excursion of performing the inverse procedure, that the gauge theory we obtain at the end of the day is one that still lives on the world-volume of a D-brane probe? Indeed, if we naïvely traced back the arrows in the flow-chart, bearing in mind the said ambiguities, we have no *a fortiori* guarantee that we have a brane theory at all. However, the method via partial resolution of Abelian orbifolds (which are themselves toric) does give us assurance. When we are careful in tuning the FI-parametres so as to stay inside cone-partitions of the space of these parametres (and avoid flop transitions) we do still have the resulting theory being physical [277]. Essentially this means that with prudence we tune the FI-parametres in the allowed domains from a parent orbifold theory, thereby giving a subsector theory which still lives on the D-brane probe and is well-behaved. Such tuning we shall practice in the following.

The virtues of this appeal to resolutions are thus twofold: not only do we avoid ambiguities, we are further endowed with physical theories. Let us thereby present this canonical mathod.

19.3.2 A Canonical Method: Partial Resolutions of Abelian Orbifolds

Our programme is standard [277]: theories on the Abelian orbifold singularity of the form $\mathbb{C}^k/\Gamma$ for $\Gamma(k,n) = \mathbb{Z}_n \times \mathbb{Z}_n \times ... \mathbb{Z}_n$ ($k-1$ times) are well studied. The complete information (and in particular the full Q_t matrix) for $\Gamma(k,n)$ is well known: $k = 2$ is the elliptic model, $k = 3$, the Brane Box, etc. In the toric context, $k = 2$ has been analysed in great detail by [69], $k = 3, n = 2$ in e.g. [274, 276, 275], $k = 3, n = 3$ in [277]. Now we know that given any toric diagram of dimension k, we can embed it into such a $\Gamma(k,n)$-orbifold for some sufficiently large n; and we choose the smallest such n which suffices. This embedding is always possible because the toric diagram for the latter is the k-simplex of length n enclosing lattice points and any toric diagram, being a collection of lattice points, can be obtained therefrom via deletions of a subset of points. This procedure is known torically as **partial resolutions** of $\Gamma(k,n)$. The crux of our algorithm is that the deletions in the toric diagram corresponds to the turning-on of the FI-parametres, and which in turn induces a method to determine a Q_t matrix for our original singularity from that of $\Gamma(n,k)$.

We shall first turn to an illustrative example of the suspended pinched point singularity (SPP) and then move on to discuss generalities. The SPP and conifold as resolutions of $\Gamma(3,2) = \mathbb{Z}_2 \times \mathbb{Z}_2$ have been extensively studied in [276]. The SPP, given by $xy = zw^2$, can be obtained from the $\Gamma(3,2)$ orbifold, $xyz = w^2$, by a single $\mathbb{P}^1$ blow-up. This is shown torically in Figure 19-2. Without further ado let us demonstrate our procedure.

1. Embedding into $\mathbb{Z}_2 \times \mathbb{Z}_2$: Given the toric diagram D of SPP, we recognise that

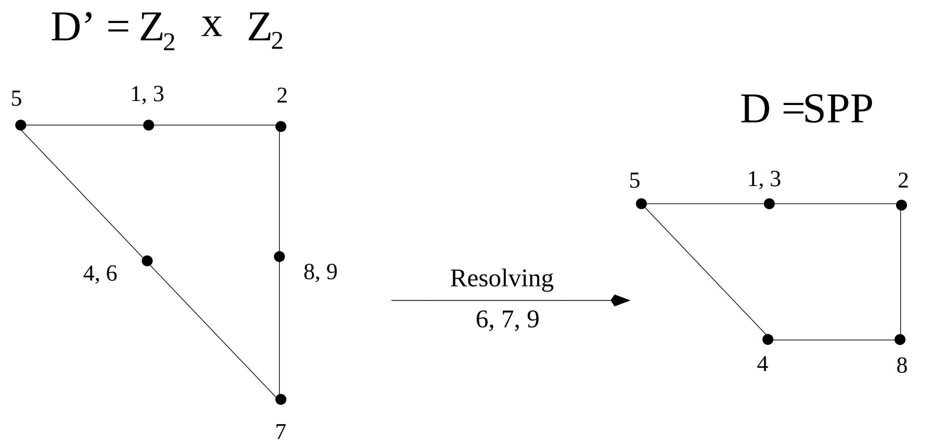


Figure 19-2: The toric diagram showing the resolution of the $\mathbb{C}^3/(\mathbb{Z}_2\times\mathbb{Z}_2)$ singularity to the suspended pinch point (SPP). The numbers i at the nodes refer to the i-th column of the matrix G_t and physically correspond to the fields p_i in the linear σ-model.

it can be embedded minimally into the diagram D' of $\mathbb{Z}_2\times\mathbb{Z}_2$. Now information on D' is readily at hand [276], as presented in the previous section. Let us recapitulate:

$$Q'_t := \begin{array}{c} \begin{array}{cccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & \end{array} \\ \left(\begin{array}{cccccccccc} 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & \zeta_1 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & \zeta_2 \\ -1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & \zeta_3 \end{array}\right) \end{array},$$

and

$$G'_t := \mathrm{coker}(Q'_t) = \begin{array}{c} \begin{array}{ccccccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 \end{array} \\ \left(\begin{array}{ccccccccc} 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \end{array},$$

which is drawn in Figure 19-1. The fact that the last row of G_t has the same number (i.e., these three-vectors are all co-planar) ensures that D' is Calabi-Yau [18]. Incidentally, it would be very helpful for one to catalogue the list of Q_t matrices of $\Gamma(3,n)$ for $n = 2,3...$ which would suffice for all local toric singularities of Calabi-Yau threefolds.

In the above definition of Q'_t we have included an extra column $(0,0,0,\zeta_1,\zeta_2,\zeta_3)$ so as to specify that the first three rows of Q'_t are F-terms (and hence exactly zero) while the last three rows are D-terms (and hence resolved by FI-parametres

$\zeta_{1,2,3}$). We adhere to the notation in [276] and label the columns (linear σ-model fields) as $p_1 ... p_9$; this is shown in Figure 19-2.

2. Determining the Fields to Resolve by Tuning ζ: We note that if we turn on a single FI-parametre we would arrive at the SPP; this is the resolution of D' to D. The subtlety is that one may need to eliminate more than merely the 7th column as there is more than one field attributed to each node in the toric diagram and eliminating column 7 some other columns corresponding to the adjacent nodes (namely out of 4,6,8 and 9) may also be eliminated. We need a judicious choice of ζ for a consistent blowup. To do so we must solve for fields $p_{1,..,9}$ and tune the ζ-parametres such that at least p_7 acquires non-zero VEV (and whereby resolved). Recalling that the D-term equations are actually linear equations in the modulus-squared of the fields, we shall henceforth define $x_i := |p_i|^2$ and consider linear-systems therein. Therefore we perform Gaussian row-reduction on Q' and solve all fields in terms of x_7 to give: $\vec{x} = \{x_1, x_2, x_1 + \zeta_2 + \zeta_3, \frac{2\,x_1 - x_2 + x_7 - \zeta_1 + \zeta_2}{2}, 2\,x_1 - x_2 + \zeta_2 + \zeta_3, \frac{2\,x_1 - x_2 + x_7 + \zeta_1 + \zeta_2 + 2\,\zeta_3}{2}, x_7, \frac{x_2 + x_7 - \zeta_1 - \zeta_2}{2}, \frac{x_2 + x_7 + \zeta_1 + \zeta_2}{2}\}$.
The nodes far away from p_7 are clearly unaffected by the resolution, thus the fields corresponding thereto continue to have zero VEV. This means we solve the above set of solutions $\vec{x}$ once again, setting $x_{5,1,3,2} = 0$, with $\zeta_{1,2,3}$ being the variables, giving upon back substitution, $\vec{x} = \{0, 0, 0, \frac{x_7 - \zeta_1 - \zeta_3}{2}, 0, \frac{x_7 + \zeta_1 + \zeta_3}{2}, x_7, \frac{x_7 - \zeta_1 + \zeta_3}{2}, \frac{x_7 + \zeta_1 - \zeta_3}{2}\}$. Now we have an arbitrary choice and we set $\zeta_3 = 0$ and $x_7 = \zeta_1$ to make p_4 and p_8 have zero VEV. This makes $p_{6,7,9}$ our candidate for fields to be resolved and seems perfectly reasonable observing Figure 19-2. The constraint on our choice is that all solutions must be ≥ 0 (since the x_i's are VEV-squared).

3. Solving for G_t: We are now clear what the resolution requires of us: in order to remove node p_7 from D' to give the SPP, we must also resolve 6, 7 and 9. Therefore we immediately obtain G_t by directly removing the said columns from

G_t':

$$G_t := \mathrm{coker}(Q_t) = \begin{array}{c}\begin{array}{cccccc} p_1 & p_2 & p_3 & p_4 & p_5 & p_8 \end{array}\\ \left(\begin{array}{cccccc} 0 & 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right)\end{array},$$

the columns of which give the toric diagram D of the SPP, as shown in Figure 19-2.

4. Solving for Q_t: Now we must perform linear combination on the rows of Q_t' to obtain Q_t so as to force columns 6, 7 and 9 zero. The following constraints must be born in mind. Because G_t has 6 columns and 3 rows and is in the null space of Q_t, which itself must have $9-3$ columns (having eliminated $p_{6,7,9}$), we must have $6-3=3$ rows for Q_t. Also, the row containing ζ_1 must be eliminated as this is precisely our resolution chosen above (we recall that the FI-parametres are such that $\zeta_{2,3}=0$ and are hence unresolved, while $\zeta_1>0$ and must be removed from the D-terms for SPP).
We systematically proceed. Let there be variables $\{a_{i=1,..,6}\}$ so that $y := \sum_i a_i \mathrm{row}_i(Q_t')$ is a row of Q_t. Then (a) the 6th, 7th and 9th columns of y must be set to 0 and moreover (b) with these columns removed y must be in the nullspace spanned by the rows of G_t. We note of course that since Q_t' was in the nullspace of G_t' initially, that the operation of row-combinations is closed within a nullspace, and that the columns to be set to 0 in Q_t' to give Q_t are precisely those removed in G_t' to give G_t, condition (a) automatically implies (b). This condition (a) translates to the equations $\{a_1+a_6=0, -a_1+a_2-a_6=0, -a_2+a_4=0\}$ which afford the solution $a_1=-a_6; a_2=a_4=0$. The fact that $a_4=0$ is comforting, because it eliminates the row containing ζ_1. We choose $a_1=1$. Furthermore we must keep row 5 as ζ_2 is yet unresolved (thereby setting $a_5=1$). This already gives two of the 3 anticipated rows of Q_t: row_5 and row_1 - row_6. The remaining row must corresponds to an F-term since we have exhausted the D-terms, this we choose to be the only remaining variable: $a_3=1$.

Consequently, we arrive at the matrix

$$Q_t = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_8 & \\ 1 & -1 & 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 & \zeta_2 \\ -1 & 0 & 0 & -1 & 1 & 1 & \zeta_3 \end{pmatrix}.$$

5. Obtaining K and Δ Matrices: The hard work is now done. We now recognise from Q_t that $Q = (1, -1, 1, 0, -1, 0)$, giving

$$T_{j\alpha} := \ker(Q) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad K^t := \mathrm{Dual}(T^t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

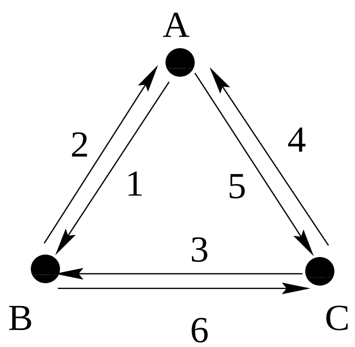
Subsequently we obtain $T^t \cdot K^t = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, which we do observe indeed to have every entry positive semi-definite. Furthermore we recognise from Q_t that $V \cdot U = \begin{pmatrix} -1 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}$, whence we obtain at last, using (19.3.3),

$$\Delta = \begin{pmatrix} -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 & -1 \end{pmatrix} \quad \Rightarrow \quad d = \left(\begin{array}{c|cccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \\ U(1)_A & -1 & 1 & 0 & 1 & -1 & 0 \\ U(1)_B & 1 & -1 & 1 & 0 & 0 & -1 \\ \hline U(1)_C & 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right),$$

giving us the quiver diagram (included in Figure 19-3 for reference), matter content and gauge group of a D-brane probe on SPP in agreement with [276]. We shall show in the ensuing sections that the superpotential we extract has similar accordance.

19.3.3 The General Algorithm for the Inverse Problem

Having indulged ourselves in this illustrative example of the SPP, we proceed to outline the general methodology of obtaining the gauge theory data from the toric diagram.



$$d = \begin{array}{l} A: \\ B: \\ C: \end{array} \begin{pmatrix} -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{pmatrix}$$

$$a_{ij} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Figure 19-3: The quiver diagram showing the matter content of a D-brane probing the SPP singularity. We have not marked in the chargeless field ϕ (what in a non-Abelian theory would become an adjoint) because thus far the toric techniques do not yet know how to handle such adjoints.

1. <u>Embedding into $\mathbb{C}^k/(\mathbb{Z}_n)^{k-1}$:</u> We are given a toric diagram D describing an algebraic variety of complex dimension k (usually we are concerned with local Calabi-Yau singularities of $k = 2, 3$ so that branes living thereon give $\mathcal{N} = 2, 1$ gauge theories). We immediately observe that D could always be embedded into D', the toric diagram of the orbifold $\mathbb{C}^k/(\mathbb{Z}_n)^{k-1}$ for some sufficiently large integer n. The matrices Q_t' and G_t' for D' are standard. Moreover we know that the matrix G_t for our original variety D must be a submatrix of G_t'. Equipped with Q_t' and G_t' our task is to obtain Q_t; and as an additional check we could verify that Q_t is indeed in the nullspace of G_t.

2. <u>Determining the Fields to Resolve by Tuning ζ:</u> Q_t' is a $k \times a$ matrix[7] (because D' and D are dimension k) for some a; G_t', being its nullspace, is thus $(a-k)\times a$. D is a partial resolution of D'. In the SPP example above, we performed a single resolution by turning on one FI-parametre, generically however, we could turn on as many ζ's as the embedding permits. Therefore we let G_t be $(a-k)\times(a-b)$ for some b which depends on the number of resolutions. Subsequently the Q_t we need is $(k-b)\times(a-b)$.

   Now b is determined directly by examining D' and D; it is precisely the number

[7]We henceforth understand that there is an extra column of zeroes and ζ's.

of fields p associated to those nodes in D' we wish to eliminate to arrive at D. Exactly which b columns are to be eliminated is determined thus: we perform Gaussian row-reduction on Q'_t so as to solve the k linear-equations in a variables $x_i := |p_i|^2$, with F-terms set to 0 and D-terms to FI-parametres. The a variables are then expressed in terms of the ζ_i's and the set B of x_i's corresponding to the nodes which we definitely know will disappear as we resolve $D' \to D$. The subtlety is that in eliminating B, some other fields may also acquire non zero VEV and be eliminated; mathematically this means that Order$(B) < b$.

Now we make a judicious choice of which fields will remain and set them to zero and impose this further on the solution $x_{i=1,..,a} = x_i(\zeta_j; B)$ from above until Order$(B) = b$, i.e., until we have found all the fields we need to eliminate. We know this occurs and that our choice was correct when all $x_i \geq 0$ with those equaling 0 corresponding to fields we do not wish to eliminate as can be observed from the toric diagram. If not, we modify our initial choice and repeat until satisfaction. This procedure then determines the b columns which we wish to eliminate from Q'_t.

3. Solving for G_t and Q_t: Knowing the fields to eliminate, we must thus perform linear combinations on the k rows of Q'_t to obtain the $k-b$ rows of Q_t based upon the two constraints that (1) the b columns must be all reduced to zero (and thus the nodes can be removed) and that (2) the $k-b$ rows (with b columns removed) are in the nullspace of G_t. As mentioned in our SPP example, condition (1) guarantees (2) automatically.

   In other words, we need to solve for k variables $\{x_{i=1,..,k}\}$ such that

$$\sum_{i=1}^{k} x_i \ (Q'_t)_{ij} = 0 \qquad \text{for} \ \ j = p_1, p_2, ...p_b \in B. \tag{19.3.4}$$

   Moreover, we immediately obtain G_t by eliminating the b columns from G'_t. Indeed, as discussed earlier, (19.3.4) implies that $\sum\limits_{i=1}^{k} \sum\limits_{j \neq p_{1...b}} x_i \ (Q'_t)_{ij} \ (G_t)_{mj} = 0$ for $m = 1, ..., a-k$ and hence guarantees that the Q_t we obtain is in the nullspace of G_t.

We could phrase equation (19.3.4) for x_i in matrix notation and directly evaluate

$$Q_t = NullSpace(W)^t \cdot \tilde{Q}'_t \tag{19.3.5}$$

where $\tilde{Q}'_t$ is Q'_t with the appropriate columns ($p_{1\ldots b}$) removed and W is the matrix constructed from the deleted columns.

4. Obtaining the K Matrix (F-term): Having obtained the $(k-b)\times(a-b)$ matrix Q_t for the original variety D, we proceed with ease. Reading from the extraneous column of FI-parametres, we recognise matrices Q (corresponding to the rows that have zero in the extraneous column) and $V \cdot U$ (corresponding to those with combinations of the unresolved ζ's in the last column). We let $V \cdot U$ be $c \times (a-b)$ whereby making Q of dimension $(k-b-c)\times(a-b)$. The number c is easily read from the embedding of D into D' as the number of unresolved FI-parametres.

   From Q, we compute the kernel T, a matrix of dimensions $(a-b)-(k-b-c)\times(a-b) = (a-k+c)\times(a-b)$ as well as the matrix K^t of dimensions $(a-k+c)\times d$ describing the dual cone to that spanned by the columns of T. The integer d is uniquely determined from the dimensions of T in accordance with the algorithm of finding dual cones presented in Appendix 22.10. From these two matrices we compute $T^t \cdot K^t$, of dimension $(a-b)\times d$.

5. Obtaining the Δ Matrix (D-term): Finally, we use (19.3.3) to compute $(V \cdot U)\cdot(T^t \cdot K^t)$, arriving at our desired matrix Δ of dimensions $c \times d$, the incidence matrix of our quiver diagram. The number of gauge groups we have is therefore $c+1$ and the number of bi-fundamentals, d.

   Of course one may dispute that finding the kernel T of Q is highly non-unique as any basis change in the null-space would give an equally valid T. This is indeed so. However we note that it is really the combination $T^t \cdot K^t$ that we need. This is a dot-product in disguise, and by the very definition of the dual cone, this combination remains invariant under basis changes. Therefore this step of obtaining the quiver Δ from the charge matrix Q_t is a unique procedure.

19.3.4 Obtaining the Superpotential

Having noticed that the matter content can be conveniently obtained, we proceed to address the interactions, i.e., the F-terms, which require a little more care. The matrix K which our algorithm extracts encodes the F-term equations and must at least be such that they could be integrated back to a single function: the *superpotential.*

Reading the possible F-flatness equations from K is *ipso facto* straight-forward. The subtlety exists in how to find the right candidate among many different linear relations. As mentioned earlier, K has dimensions $m \times (r-2)$ with m corresponding to the fields that will finally manifest in the superpotential, $r-2$, the fields that solve them according to (19.2.1) and (19.2.2); of course, $m \geq r-2$. Therefore we have $r-2$ vectors in $\mathbb{Z}^m$, giving generically $m-r+2$ linear relations among them. Say we have $\text{row}_1 + \text{row}_3 - \text{row}_7 = 0$, then we simply write down $X_1 X_3 = X_7$ as one of the candidate F-terms. In general, a relation $\sum\limits_i a_i K_{ij} = 0$ with $a_i \in \mathbb{Z}$ implies an F-term $\prod\limits_i X_i^{a_i} = 1$ in accordance with (19.2.1). Of course, to find all the linear relations, we simply find the $\mathbb{Z}$-nullspace of K^t of dimension $m-r+2$.

Here a great ambiguity exists, as in our previous calculations of nullspaces: any linear combinations therewithin may suffice to give a new relation as a candidate F-term[8]. Thus educated guesses are called for in order to find the set of linear relations which may be most conveniently integrated back into the superpotential. Ideally, we wish this back-integration procedure to involve no extraneous fields (i.e., integration constants[9]) other than the m fields which appear in the K-matrix. Indeed, as we shall see, this wish may not always be granted and sometimes we must include new fields. In this case, the whole moduli space of the gauge theory will be larger than the one

[8]Indeed each linear relation gives a possible candidate and we seek the correct ones. For the sake of clarity we shall call candidates "relations" and reserve the term "F-term" for a successful candidate.

[9]By constants we really mean functions since we are dealing with systems of partial differential equations.

encoded by our toric data and the new fields parametrise new branches of the moduli in the theory.

Let us return to the SPP example to enlighten ourselves before generalising. We recall from subsection 3.2, that $K = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \end{array} \\ \left(\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right) \end{array}$ from which we can read out only one relation $X_3X_6 - X_4X_5 = 0$ using the rule described in the paragraph above. Of course there can be only one relation because the nullspace of K^t is of dimension $6 - 5 = 1$.

Next we must calculate the charge under the gauge groups which this term carries. We must ensure that the superpotential, being a term in a Lagrangian, be a gauge invariant, i.e., carries no overall charge under Δ. From $d = \begin{array}{c} \\ U(1)_A \\ U(1)_B \\ U(1)_C \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \end{array} \\ \left(\begin{array}{cccccc} -1 & 1 & 0 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right) \end{array}$ we find the charge of X_3X_6 to be $(q_A, q_B, q_C) = (0+0, 1+(-1), (-1)+1) = (0, 0, 0)$; of course by our very construction, X_4X_5 has the same charge. Now we have two choices: (a) to try to write the superpotential using only the six fields; or (b) to include some new field ϕ which also has charge $(0, 0, 0)$. For (a) we can try the ansatz $W = X_1X_2(X_3X_6 - X_4X_5)$ which does give our F-term upon partial derivative with respect to X_1 or X_2. However, we would also have a new F-term $X_1X_2X_3 = 0$ by $\frac{\partial}{\partial X_6}$, which is inconsistent with our K since columns 1, 2 and 3 certainly do not add to 0.

This leaves us with option (b), i.e., $W = \phi(X_3X_6 - X_4X_5)$ say. In this case, when $\phi = 0$ we not only obtain our F-term, we need not even correct the matter content Δ. This branch of the moduli space is that of our original theory. However, when $\phi \neq 0$, we must have $X_3 = X_4 = X_5 = X_6 = 0$. Now the D-terms read $|X_1|^2 - |X_2|^2 = -\zeta_1 = \zeta_2$, so the moduli space is: $\{\phi \in \mathbb{C}, X_1 \in \mathbb{C}\}$ such that $\zeta_1 + \zeta_2 = 0$ for otherwise there would be no moduli at all. We see that we obtain another branch of moduli space. As remarked before, this is a general phenomenon when we include new fields: the whole moduli space will be larger than the one encoded by the toric data. As a check, we see that our example is exactly that given in [276], after the identification with their notation, $Y_{12} \to X_6, X_{24} \to X_3, Z_{23} \to X_1, Z_{32} \to X_2, Y_{34} \to X_4, X_{13} \to X_5, Z_{41} \to \phi$ and $(X_1X_2 - \phi) \to \phi$. We note that if we were studying a non-Abelian extension to the toric theory, as by brane setups (e.g. [276]) or by stacks of probes (in progress from [277]), the chargeless field ϕ would manifest as an adjoint field thereby modifying our quiver diagram. Of course since the study of toric methods in physics is so far restricted to product $U(1)$ gauge groups, such complexities do not arise. To avoid confusion we shall henceforth mark only the bi-fundamentals in our quiver diagrams but will write the chargeless fields explicit in the superpotential.

Our agreement with the results of [276] is very reassuring. It gives an excellent example demonstrating that our canonical resolution technique and the inverse algorithm do indeed, in response to what was posited earlier, give a theory living on a D-brane probing the SPP (T-dual to the setup in [276]). However, there is a subtle point we would like to mention. There exists an ambiguity in writing the superpotential when the chargeless field ϕ is involved. Our algorithm gives $W = \phi(X_3X_6 - X_4X_6)$ while [276] gives $W = (X_1X_2 - \phi)(X_3X_6 - X_4X_6)$. Even though they have identical moduli, it is the latter which is used for the brane setup. Indeed, the toric methods by definition (in defining Δ from a_{ij}) do not handle chargeless fields and hence we have ambiguities. Fortunately our later examples will not involve such fields.

The above example of the SPP was a naïve one as we need only to accommodate a single F-term. We move on to a more complicated example. Suppose we are now given

$$d = \begin{array}{cc} & \begin{array}{cccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \end{array} \text{ and } K = \begin{array}{c} \begin{array}{cccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

We shall see in the next section, that these arise for the del Pezzo 1 surface. Now the nullspace of K has dimension $10-6=4$, we could obtain a host of relations from various linear combinations in this space. One relation is obvious: $X_2X_7 - X_3X_6 = 0$. The charge it carries is $(q_A, q_B, q_C, q_D) = (0+0, -1+0, 0+1, 1+(-1)) = (0, -1, 1, 0)$ which cancels that of X_9. Hence $X_9(X_2X_7 - X_3X_6)$ could be a term in W. Now $\frac{\partial}{\partial X_2}$ thereof gives X_7X_9 and from K we see that $X_7X_9 - X_1X_5X_{10} = 0$, therefore, $-X_1X_2X_5X_{10}$

could be another term in W. We repeat this procedure, generating new terms as we proceed and introducing new fields where necessary. We are fortunate that in this case we can actually reproduce all F-terms without recourse to artificial insertions of new fields: $W = X_2X_7X_9 - X_3X_6X_9 - X_4X_8X_7 - X_1X_2X_5X_{10} + X_3X_4X_{10} + X_1X_5X_6X_8$.

Enlightened by these examples, let us return to some remarks upon generalities. Making all the exponents of the fields positive, the F-terms can then be written as

$$\prod_i X_i^{a_i} = \prod_j X_j^{b_j}, \tag{19.3.6}$$

with $a_i, b_j \in \mathbb{Z}^+$. Indeed if we were to have another field X_k such that $k \notin \{i\}, \{j\}$ then the term $X_k \left(\prod\limits_i X_i^{a_i} - \prod\limits_j X_j^{b_j}\right)$, on the condition that X_k appears only this once, must be an additive term in the superpotential W. This is because the F-flatness condition $\frac{\partial W}{\partial X_k} = 0$ implies (19.3.6) immediately. Of course judicious observations are called for to (A) find appropriate relations (19.3.6) and (B) find X_k among our m fields. Indeed (B) may not even be possible and new fields may be forced to be introduced, whereby making the moduli space of the gauge theory larger than that encodable by the toric data.

In addition, we must ensure that each term in W be chargeless under the product gauge groups. What this means for us is that for each of the terms $X_k \left(\prod\limits_i X_i^{a_i} - \prod\limits_j X_j^{b_j}\right)$ we must have $\text{Charge}_s(X_k) + \sum\limits_i a_i \text{Charge}_s(X_i) = 0$ for $s = 1, .., r$ indexing through our r gauge group factors (we note that by our very construction, for each gauge group, the charges for $\prod\limits_i X_i^{a_i}$ and for $\prod\limits_j X_j^{b_j}$ are equal). If X_k in fact cannot be found among our m fields, it must be introduced as a new field ϕ with appropriate charge. Therefore with each such relation (19.3.6) read from K, we iteratively perform this said procedure, checking $\Delta_{sk} + \sum\limits_i a_i \Delta_{si} = 0$ at each step, until a satisfactory superpotential is reached. The right choices throughout demands constant vigilance and astuteness.

19.4 An Illustrative Example: the Toric del Pezzo Surfaces

As the $\mathbb{C}^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ resolutions were studied in great detail in [276], we shall use the data from [277] to demonstrate the algorithm of finding the gauge theory from toric diagrams extensively presented in the previous section.

The toric diagram of the dual cone of the (parent) quotient singularity $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ as well as those of its resolution to the three toric del Pezzo surface are presented in Figure 19-4.

<u>del Pezzo 1:</u> Let us commence our analysis with the first toric del Pezzo surface[10]. From its toric diagram, we see that the minimal $\mathbb{Z}_n \times \mathbb{Z}_n$ toric diagram into which it embeds is $n = 3$. As a reference, the toric diagram for $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is given in Figure 19-4 and the quiver diagram, given later in the convenient brane-box form, in Figure 19-5. Luckily, the matrices Q'_t and G'_t for this Abelian quotient is given in [277]. Adding the extra column of FI-parametres we present these matrices below[11]:

$$G'_t = \left(\begin{array}{cccccccccccccccccccccccc}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} & p_{19} & p_{20} & p_{21} & p_{22} & p_{23} & p_{24} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 3 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1
\end{array}\right. \cdots\cdots$$

$$\cdots\cdots \left.\begin{array}{cccccccccccccccccc}
p_{25} & p_{26} & p_{27} & p_{28} & p_{29} & p_{30} & p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} & p_{38} & p_{39} & p_{40} & p_{41} & p_{42} \\
0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

[10] Now some may identify the toric diagram of del Pezzo 1 as given by nodes (using the notation in Figure 19-4) $(1,-1,1)$, $(2,-1,0)$, $(-1,1,1)$, $(0,0,1)$ and $(-1,0,2)$ instead of the one we have chosen in the convention of [277], with nodes $(0,-1,2)$, $(0,0,1)$, $(-1,1,1)$, $(1,0,0)$ and $(0,1,0)$. But of course these two G_t matrices describe the same algebraic variety. The former corresponds to Spec $\left(\mathbb{C}[XY^{-1}Z, X^2Y^{-1}, X^{-1}YZ, Z, X^{-1}Z^2]\right)$ while the latter corresponds to Spec $\left(\mathbb{C}[Y^{-1}Z^2, Z, X^{-1}YZ, X, Y]\right)$. The observation that $(X^2Y^{-1}) = (X)(X^{-1}YZ)^{-1}(Z)$, $(XY^{-1}Z) = (X)(Y)^{-1}(Z)$ and $(X^{-1}Z^2) = (Y^{-1}Z^2)(Y)(X^{-1})$ for the generators of the polynomial ring gives the equivalence. In other words, there is an $SL(5,\mathbb{Z})$ transformation between the 5 nodes of the two toric diagrams.

[11] In [277], a canonical ordering was used; for our purposes we need not belabour this point and use their Q'_{total} as Q'_t. This is perfectly legitimate as long as we label the columns carefully, which we have done.

and

$$
Q'_t = \begin{array}{c}
\begin{array}{ccccccccccccccccccccccccc}
p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{10} & p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} & p_{19} & p_{20} & p_{21} & p_{22} & p_{23} & p_{24} &
\end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_1 \\
0 & 1 & -1 & -1 & 2 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 \\
1 & -2 & 1 & 0 & -2 & 2 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_4 \\
-1 & 1 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \zeta_6 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_7 \\
0 & 1 & -1 & 0 & 2 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \zeta_8
\end{pmatrix}
\end{array} \cdots\cdots
$$

$$
\cdots\cdots\quad
\begin{array}{c}
\begin{array}{cccccccccccccccccc}
p_{25} & p_{26} & p_{27} & p_{28} & p_{29} & p_{30} & p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} & p_{37} & p_{38} & p_{39} & p_{40} & p_{41} & p_{42}
\end{array} \\
\left.\begin{array}{ccccccccccccccccccc}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 & 0 & 0 & 2 & -2 & -2 & 2 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 & 2 & -1 & -2 & 1 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 1 & -2 & 0 & 0 & 0 & 3 & -2 & -2 & 2 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \\
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 2 & -1 & -1 & 2 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & -1 & 2 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_8
\end{array}\right)
\end{array}
$$

According to our algorithm, we must perform Gaussian row-reduction on Q'_t to solve for 42 variables x_i. When this is done we find that we can in fact express all variables in terms of 3 x_i's together with the 8 FI-parametres ζ_i. We choose these three x_i's to be $x_{10,29,36}$ corresponding to the 3 outer vertices which we know must be resolved in going from $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to del Pezzo 1.

Next we select the fields which must be kept and set them to zero in order to determine the range for ζ_i. Bearing in mind the toric diagrams from Figure 19-4, these fields we judiciously select to be: $p_{13,8,37,38}$. Setting $x_{13,8,37,38} = 0$ gives us the solution $\{\zeta_6 = 0; x_{29} = \zeta_7 = \zeta_3 = \zeta_1 - \zeta_5; x_{10} = \zeta_4 + \zeta_5 + \zeta_3; x_{36} = \zeta_7 - \zeta_8\}$, which upon

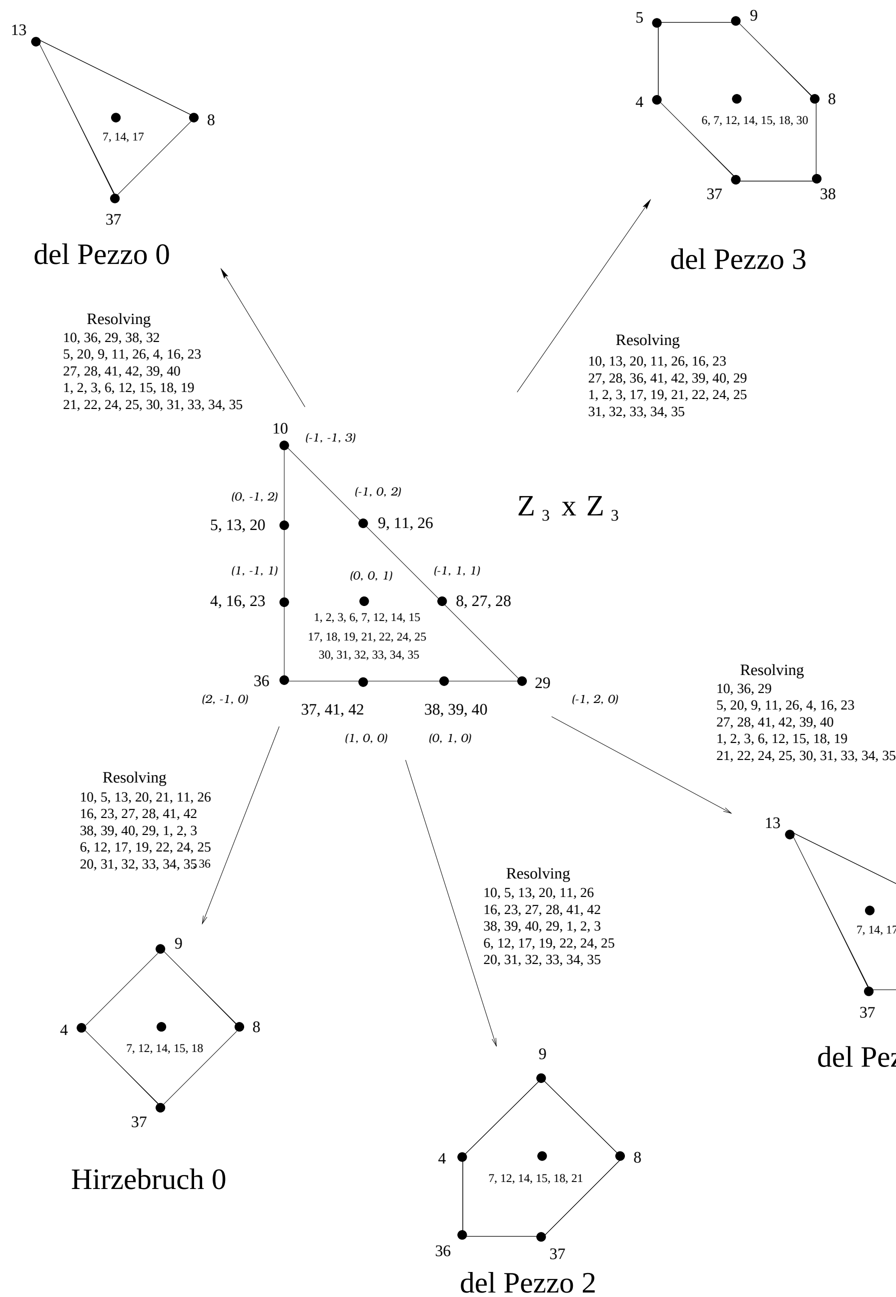


Figure 19-4: The resolution of the Gorenstein singularity $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to the three toric del Pezzo surfaces as well as the zeroth Hirzebruch surface. We have labelled explicitly which columns (linear σ-model fields) are to be associated to each node in the toric diagrams and especially which columns are to be eliminated (fields acquiring non-zero VEV) in the various resolutions. Also, we have labelled the nodes of the parent toric diagram with the coordinates as given in the matrix G_t for $\mathbb{C}^3/(\mathbb{Z}_3\times\mathbb{Z}_3)$.

back-substitution to the solutions x_i we obtained from Q'_t, gives zero for $x_{13,8,37,38}$ (which we have chosen by construction) as well as $x_{7,14,17,32}$; for all others we obtain positive values. This means precisely that all the other fields are to be eliminated and these 8 columns { 13, 8, 37, 38, 7,14,17,32 } are to be kept while the remaining 42-8=34 are to be eliminated from Q'_t upon row-reduction to give Q_t. In other words, we have found our set B to be {1,2,3,4,5,6,9,10,11,12,15,16,18,19,20,21,22,23,24,25, 26,27,28,29,30,31,33,34,35,36,39,40,41,42} and thus according to (19.3.5) we immediately obtain

$$Q_t = \begin{array}{c} \begin{array}{ccccccccc} p_7 & p_8 & p_{13} & p_{14} & p_{17} & p_{32} & p_{37} & p_{38} & \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \zeta_2+\zeta_8 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & \zeta_6 \\ -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \zeta_1+\zeta_3+\zeta_5 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & 1 & -1 & -1 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array} .$$

We note of course that 5 out of the 8 FI-parametres have been eliminated automatically; this is to be expected since in resolving $\mathbb{C}^3/(\mathbb{Z}_3\times\mathbb{Z}_3)$ to del Pezzo 1, we remove precisely 5 nodes. Obtaining the D-terms and F-terms is now straight-forward. Using (19.3.3) and re-inserting the last row we obtain the D-term equations (incidence matrix) to be

$$d = \begin{array}{c} \begin{array}{cccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

From this matrix we immediately observe that there are 4 gauge groups, i.e., $U(1)^4$ with 10 matter fields X_i which we have labelled in the matrix above. In an equivalent notation we rewrite d as the adjacency matrix of the quiver diagram (see Figure 19-5) for the gauge theory:

$$a_{ij} = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 1 & 2 & 0 & 0 \end{pmatrix} .$$

The K-matrix we obtain to be:

$$K^t = \begin{array}{c} \begin{matrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \end{matrix} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

which indicates that the original 10 fields X_i can be expressed in terms of 6. This was actually addressed in the previous section and we rewrite that pleasant superpotential here:

$$W = X_2X_7X_9 - X_3X_6X_9 - X_4X_8X_7 - X_1X_2X_5X_{10} + X_3X_4X_{10} + X_1X_5X_6X_8.$$

del Pezzo 2: Having obtained the gauge theory for del Pezzo 1, we now repeat the above analysis for del Pezzo 2. Now we have the FI-parametres restricted as $\{p_{36} = \zeta_2 = 0; \zeta_3 = \zeta_4; x_{29} = \zeta_4 + \zeta_6; x_{10} = \zeta_1 + \zeta_4\}$, making the set to be eliminated as $B = \{$ 1, 2, 3, 5, 6, 10, 11, 13, 16, 17, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 38, 39, 40, 41, 42 $\}$. Whence, we obtain

$$Q_t = \begin{array}{c} \begin{matrix} p_4 & p_7 & p_8 & p_9 & p_{12} & p_{14} & p_{15} & p_{18} & p_{21} & p_{36} & p_{37} & \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \zeta_4+\zeta_6+\zeta_8 \\ 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \zeta_7 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta_1+\zeta_3+\zeta_5 \\ -1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & \zeta_2 \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

and observe that 4 D-terms have been resolved, as 4 nodes have been eliminated from

$\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. From this we easily extract (see Figure 19-5)

$$d = \begin{array}{c} \begin{array}{ccccccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} \end{array} \\ \left(\begin{array}{ccccccccccccc} -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right) \end{array};$$

moreover, we integrate the F-term matrices

$$K^t = \begin{array}{c} \begin{array}{ccccccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} \end{array} \\ \left(\begin{array}{ccccccccccccc} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

to obtain the superpotential

$$\begin{array}{c} W = X_2X_9X_{11} - X_9X_3X_{10} - X_4X_8X_{11} - X_1X_2X_7X_{13} + X_{13}X_3X_6 \\ -X_5X_{12}X_6 + X_1X_5X_8X_{10} + X_4X_7X_{12}. \end{array}$$

<u>del Pezzo 3:</u> Finally, we shall proceed to treat del Pezzo 3. Here we have the range of the FI-parametres to be $\{\zeta_1 = \zeta_6 = \zeta_6 = 0; x_{29} = \zeta_3 = -\zeta_5; x_{10} = \zeta_4; \zeta_2 = x_{36}; \zeta_8 = -\zeta_2 - \zeta_{10}\}$, which gives the set B as {1, 2, 3, 10, 11, 13, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 31, 32, 33, 34, 35, 36, 39, 40, 41, 42}, and thus according to (19.3.5) we immediately obtain

$$Q_t = \left(\begin{array}{cccccccccccccc} p_4 & p_5 & p_6 & p_7 & p_8 & p_9 & p_{12} & p_{14} & p_{15} & p_{18} & p_{30} & p_{37} & p_{38} & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & \zeta_2+\zeta_4+\zeta_8 \\ 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \zeta_7 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & \zeta_6 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \zeta_3+\zeta_5 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \zeta_1 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We note indeed that 3 out of the 8 FI-parametres have been automatically resolved, as we have removed 3 nodes from the toric diagram for $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. The matter content (see Figure 19-5) is encoded in

$$d = \begin{array}{c} \begin{array}{cccccccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} & X_{14} \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

and from the F-terms

$$K^t = \begin{array}{c} \begin{array}{cccccccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} & X_{14} \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

we integrate to obtain the superpotential

$$\begin{aligned} W = & X_3X_8X_{13} - X_8X_9X_{11} - X_5X_6X_{13} - X_1X_3X_4X_{10}X_{12} \\ & +X_7X_9X_{12} + X_1X_2X_5X_{10}X_{11} + X_4X_6X_{14} - X_2X_7X_{14}. \end{aligned}$$

Note that we have a quintic term in W; this is an interesting interaction indeed.

del Pezzo 0: Before proceeding further, let us attempt one more example, viz., the degenerate case of the del Pezzo 0 as shown in Figure 19-4. This time we note that the ranges for the FI-parametres are $\{\zeta_5 = -x_{29} + \zeta_6 - A; \zeta_6 = x_{29} - B; x_{29} = B + C; \zeta_8 = -x_{36} + B; x_{36} = B + C + D; x_{10} = A + E\}$ for some positive A, B, C, D and E, that $B = \{$ 1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 15, 16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42 $\}$ and whence the charge matrix is

$$Q_t = \begin{array}{c} \begin{array}{cccccc} p_7 & p_8 & p_{13} & p_{14} & p_{17} & p_{37} \end{array} \\ \left(\begin{array}{ccccccc} 1 & 0 & 0 & 0 & -1 & 0 & \zeta_2+\zeta_6+\zeta_8 \\ -1 & 0 & 0 & 1 & 0 & 0 & \zeta_1+\zeta_3+\zeta_5 \\ -1 & 1 & 1 & -1 & -1 & 1 & 0 \end{array} \right) \end{array} .$$

We extract the matter content (see Figure 19-5) as $d = \begin{array}{c} \begin{array}{ccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \end{array} \\ \left(\begin{array}{ccccccccc} -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \end{array} \right) \end{array}$,

and the F-terms as $K^t = \begin{array}{c} \begin{array}{ccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \end{array} \\ \left(\begin{array}{ccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{array} \right) \end{array}$, and from the latter we integrate to obtain the superpotential

$$W = X_1X_4X_9 - X_4X_5X_7 - X_2X_3X_9 - X_1X_6X_8 + X_2X_5X_8 + X_3X_6X_7.$$

Of course we immediately recognise the matter content (which gives a triangular quiver which we shall summarise below in Figure 19-5) as well as the superpotential from equations (4.7-4.14) of [74]; it is simply the theory on the Abelian orbifold $\mathbb{C}^3/\mathbb{Z}_3$ with action $(\alpha \in \mathbb{Z}_3) : (z_1, z_2, z_3) \to (e^{\frac{2\pi i}{3}} z_1, e^{\frac{2\pi i}{3}} z_2, e^{\frac{2\pi i}{3}} z_3)$. Is our del Pezzo 0 then $\mathbb{C}^3/\mathbb{Z}_3$? We could easily check from the G_t matrix (which we recall is obtained from G_t' of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ by eliminating the columns corresponding to the set B):

$$G_t = \left(\begin{array}{cccccc} 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 \end{array} \right).$$

These columns (up to repeat) correspond to monomials $Z, X^{-1}YZ, Y^{-1}Z^2, X$ in the polynomial ring $\mathbb{C}[X, Y, Z]$. Therefore we need to find the spectrum (set of maximal ideals) of the ring $\mathbb{C}[Z, X^{-1}YZ, Y^{-1}Z^2, X]$, which is given by the minimal polynomial relation: $(X^{-1}YZ) \cdot (Y^{-1}Z^2) \cdot X = (Z)^3$. This means, upon defining $p = X^{-1}YZ; q = Y^{-1}Z^2; r = X$ and $s = Z$, our del Pezzo 0 is described by $pqr = s^3$ as an algebraic variety in $\mathbb{C}^4(p, q, r, s)$, which is precisely $\mathbb{C}^3/\mathbb{Z}_3$. Therefore we have actually come through a full circle in resolving $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ to $\mathbb{C}^3/\mathbb{Z}_3$ and the validity of our algorithm survives this consistency check beautifully. Moreover, since we know that our gauge theory is exactly the one which lives on a D-brane probe on $\mathbb{C}^3/\mathbb{Z}_3$, this gives a good check for physicality: that our careful tuning of FI-parametres via canonical partial resolutions does give a physical D-brane theory at the end. We tabulate the matter content a_{ij} and the superpotential W for the del Pezzo surfaces below, and the quiver diagrams, in Figure 19-5.

	del Pezzo 1	**del Pezzo 2**	**del Pezzo 3**
Matter $a_{ij} =$	$\begin{pmatrix} 0&0&2&0\\ 1&0&1&0\\ 0&0&0&3\\ 1&2&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&1&1\\ 0&0&2&0&0\\ 3&0&0&1&0\\ 0&1&0&0&1\\ 0&0&2&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1&1&0&1\\ 0&0&0&1&1&0\\ 0&1&0&0&0&1\\ 1&0&0&0&1&0\\ 2&0&1&0&0&0\\ 0&0&0&1&1&0 \end{pmatrix}$
Superpotential $W =$	$X_2X_7X_9 - X_3X_6X_9$ $-X_4X_8X_7 - X_1X_2X_5X_{10}$ $+X_3X_4X_{10} + X_1X_5X_6X_8$	$X_2X_9X_{11} - X_9X_3X_{10}$ $-X_4X_8X_{11} - X_1X_2X_7X_{13}$ $+X_{13}X_3X_6 - X_5X_{12}X_6$ $+X_1X_5X_8X_{10} + X_4X_7X_{12}$	$X_3X_8X_{13} - X_8X_9X_{11}$ $-X_5X_6X_{13} - X_1X_3X_4X_{10}X_{12}$ $+X_7X_9X_{12} + X_1X_2X_5X_{10}X_{11}$ $+X_4X_6X_{14} - X_2X_7X_{14}$

	del Pezzo 0 $\cong \mathbb{C}^3/\mathbb{Z}_3$	**Hirzebruch 0** $\cong \mathbb{P}^1 \times \mathbb{P}^1 := F_0 = E_1$
Matter a_{ij}	$\begin{pmatrix} 0&3&0\\ 0&0&3\\ 3&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&2&0&2\\ 0&0&2&0\\ 4&0&0&0\\ 0&0&2&0 \end{pmatrix}$
Superpotential W	$X_1X_4X_9 - X_4X_5X_7$ $-X_2X_3X_9 - X_1X_6X_8$ $+X_2X_5X_8 + X_3X_6X_7$	$X_1X_8X_{10} - X_3X_7X_{10}$ $-X_2X_8X_9 - X_1X_6X_{12}$ $+X_3X_6X_{11} + X_4X_7X_9$ $+X_2X_5X_{12} - X_4X_5X_{11}$

Upon comparing Figure 19-4 and Figure 19-5, we notice that as we go from del Pezzo 0 to 3, the number of points in the toric diagram increases from 4 to 7, and the number of gauge groups (nodes in the quiver) increases from 3 to 6. This is consistent with the observation for $\mathcal{N} = 1$ theories that the number of gauge groups equals the number of perimetre points (e.g., for del Pezzo 1, the four nodes 13, 8, 37 and 38) in the toric diagram. Moreover, as discussed in [278], the rank of the global symmetry group (E_i for del Pezzo i) which must exist for these theories equals the number of perimetre point minus 3; it would be an intereting check indeed to see how such a symmetry manifests itself in the quivers and superpotentials.

Hirzebruch 0: Let us indulge ourselves with one more example, namely the 0th

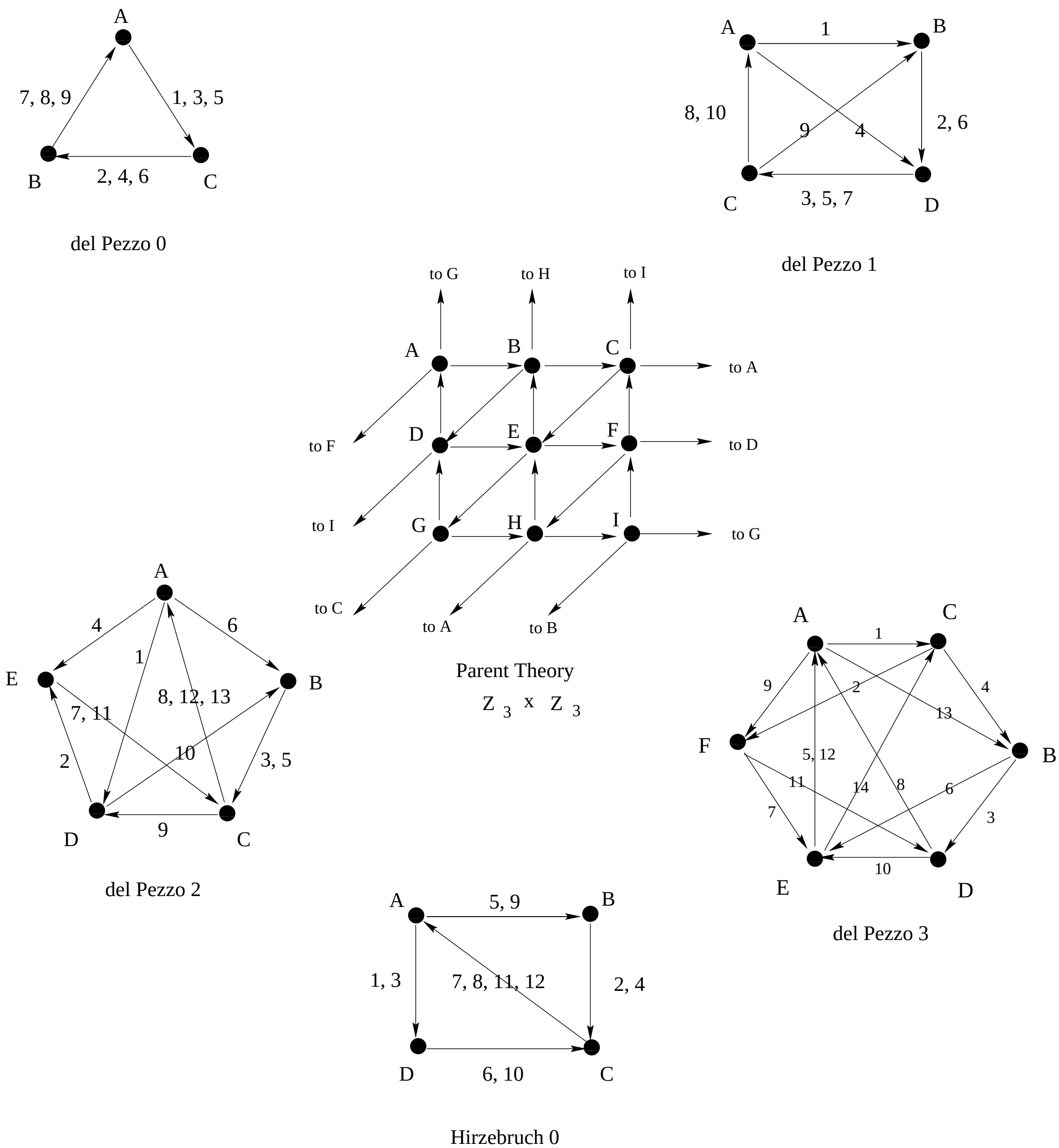


Figure 19-5: The quiver diagrams for the matter content of the brane world-volume gauge theory on the 4 toric del Pezzo singularities as well as the zeroth Hirzebruch surface. We have specifically labelled the $U(1)$ gauge groups (A, B, ..) and the bi-fundamentals (1, 2, ..) in accordance with our conventions in presenting the various matrices Q_t, Δ and K. As a reference we have also included the quiver for the parent $\mathbb{Z}_3 \times \mathbb{Z}_3$ theory.

Hirzebruch surface, or simply $\mathbb{P}^1 \times \mathbb{P}^1 := F_0 := E_1$. The toric diagram is drawn in Figure 19-4. Now the FI-parametres are $\{\zeta_4 = -x_{29} - x_{36} - \zeta_5 - \zeta_8 - A; \zeta_5 = -A - B; \zeta_7 = x_{10} + x_{29} + x_{36} + \zeta_8 - C; \zeta_8 = -x_{10} - x_{29} - x_{36} + D; D = A + B; C = A + B; A = x_{10} - E; x_{10} = E + F; x_{29} = B + G\}$ for positive A, B, C, D, E, F and G. Moreover, $B = \{$ 1, 2, 3, 5, 6, 10, 11, 13, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42 $\}$. We note that this can be obtained directly by partial resolution of fields 21 and 36 from del Pezzo 2 as is consistent with Figure 19-4. Therefrom we obtain the charge matrix

$$Q_t = \begin{array}{c} \begin{array}{cccccccccc} p_4 & p_7 & p_8 & p_9 & p_{12} & p_{14} & p_{15} & p_{18} & p_{37} & \end{array} \\ \left(\begin{array}{cccccccccc} -1 & 2 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & \zeta_2+\zeta_4+\zeta_6+\zeta_8 \\ 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & \zeta_7 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \zeta_1+\zeta_3+\zeta_5 \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array},$$

from which we have the matter content $d = \begin{array}{c} \begin{array}{cccccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} \end{array} \\ \left(\begin{array}{cccccccccccc} -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right) \end{array}$
the quiver for which is presented in Figure 19-5. The F-terms are

$$K^t = \begin{array}{c} \begin{array}{cccccccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} \end{array} \\ \left(\begin{array}{cccccccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \end{array},$$

from which we obtain

$$W = X_1X_8X_{10} - X_3X_7X_{10} - X_2X_8X_9 - X_1X_6X_{12} + X_3X_6X_{11} + X_4X_7X_9 + X_2X_5X_{12} - X_4X_5X_{11},$$

a perfectly acceptable superpotential with only cubic interactions. We include these results with our table above.

19.5 Uniqueness?

In our foregoing discussion we have constructed in detail an algorithm which calculates the matter content encoded by Δ and superpotential encoded in K, given the toric diagram of the singularity which the D-branes probe. As abovementioned, though this algorithm gives one solution for the quiver and the K-matrix once the matrix Q_t is determined, the general inverse process of going from toric data to gauge theory data, is highly **non-unique** and a classification of all possible theories having the same toric description would be interesting[12]. Indeed, by the very structure of our algorithm, in immediately appealing to the partial resolution of gauge theories on $\mathbb{Z}_n \times \mathbb{Z}_n$ orbifolds which are well-studied, we have granted ourselves enough extraneous information to determine a unique Q_t and hence the ability to proceed with ease (this was the very reason for our devising the algorithm).

However, generically we do not have any such luxury. At the end of subsection 3.1, we have already mentioned two types of ambiguities in the inverse problem. Let us refresh our minds. They were (A) the **F-D ambiguity** which is the inability to decide, simply by observing the toric diagram, which rows of the charge matrix Q_t are D-terms and which are F-terms and (B) the **repetition ambiguity** which is the inability to decide which columns of G_t to repeat once having read the vectors from the toric diagram. Other ambiguities exist, such as in each time when we compute nullspaces, but we shall here discuss to how ambiguities (A) and (B) manifest themselves and provide examples of vastly different gauge theories having the same toric description. There is another point which we wish to emphasise: as mentioned at the end of subsection 3.1, the resolution method guarantees, upon careful tuning of the FI-parametres, that the resulting gauge theory does originate from the world-volume of a D-brane probe. Now of course, by taking liberties with experimentation of these ambiguities we are no longer protected by physicality and in general the theories no longer live on the D-brane. It would be a truly interesting exercise to check which of these different theories do.

[12] We thank R. Plesser for pointing this issue out to us.

F-D Ambiguity: First, we demonstrate type (A) by returning to our old friend the SPP whose charge matrix we had earlier presented. Now we write the same matrix without specifying the FI-parametres:

$$Q_t = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

We could apply the last steps of our algorithm to this matrix as follows.

(a) If we treat the first row as Q (the F-terms) and the second and third as $V \cdot U$ (the D-terms) we obtain the gauge theory as discussed in subsection 3.3 and in [276].

(b) If we treat the second row as Q and first with the third as $V \cdot U$, we obtain $d = \begin{pmatrix} -1 & 0 & 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$ which is an exotic theory indeed with a field (p_5) charged under three gauge groups.

Let us digress a moment to address the stringency of the requirements upon matter contents. By the very nature of finite group representations, any orbifold theory must give rise to only adjoints and bi-fundamentals because its matter content is encodable by an adjacency matrix due to tensors of representations of finite groups. The corresponding incidence matrix d, has (a) only 0 and ± 1 entries specifying the particular bi-fundamentals and (b) has each column containing precisely one 1, one -1 and with the remaining entries 0. However more exotic matter contents could arise from more generic toric singularities, such as fields charged under 3 or more gauge group factors; these would then have d matrices with conditions (a) and (b) relaxed[13]. Such exotic quivers (if we could even call them quivers still) would give interesting enrichment to those well-classified families as discussed in [297].

Moreover we must check the anomaly cancellation conditions. These could be rather involved; even though for $U(1)$ theories they are a little simpler, we still

[13]Note that we still require that each column sums to 0 so as to be able to factor out an overall $U(1)$.

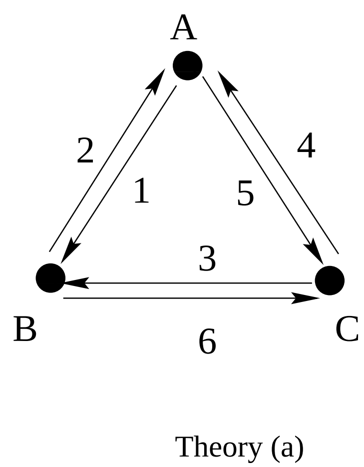
need to check *trace anomalies* and *cubic anomalies*. In a trace-anomaly-free theory, for each node in the quiver, the number of incoming arrows must equal the number of outgoing (this is true for a $U(1)$ theory which is what toric varieties provide; for a discussion on this see e.g. [292]). In matrix language this means that each row of d must sum to 0.

Now for a theory with only bi-fundamental matter with ± 1 charges, since $(\pm 1)^3 = \pm 1$, the cubic is equal to the trace anamaly; therefore for these theories we need only check the above row-condition for d. For more exotic matter content, which we shall meet later, we do need to perform an independent cubic-anomaly check.

Now for the above d, the second row does not sum to zero and whence we do unfortunately have a problematic anomalous theory. Let us push on to see whether we have better luck in the following.

(c) Treating row 3 as the F-terms and the other two as the D-terms gives $d = \begin{pmatrix} 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & -1 \\ 0 & 0 & -1 & 0 & 1 & 1 \end{pmatrix}$ which has the same anomaly problem as the one above.

(d) Now let rows 1 and 2 as the F-terms and the 3rd, as the D-terms, we obtain $d = \begin{array}{c} \begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & -1 & -1 \\ 0 & -1 & -1 & 1 & 1 \end{pmatrix} \end{array}$, which is a perfectly reasonable matter content. Integrating $K = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$ gives the superpotential $W = \phi(X_1X_2X_5 - X_3X_4)$ for some field ϕ of charge $(0,0)$ (which could be an adjoint for example; note however that we can not use X_1 even though it has charge $(0,0)$ for otherwise the F-terms would be altered). This theory is perfectly legitimate. We compare the quiver diagrams of theories (a) (which we recall from Figure 19-3) and this present example in Figure 19-6. As a check, let us define the gauge invariant quantities: $a = X_2X_4$, $b = X_2X_5$, $c = X_3X_4$, $d = X_3X_5$ and $e = X_1$. Then we have the algebraic relations $ad = bc$ and $eb = c$, from which we immediately obtain $ad = eb^2$, precisely the equation for the SPP.



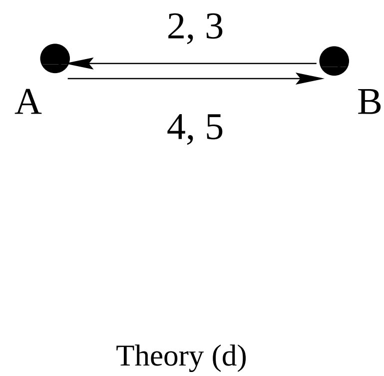


Figure 19-6: The vastly different matter contents of theories (a) and (d), both anomaly free and flow to the toric diagram of the suspended pinched point in the IR.

(e) As a permutation on the above, treating rows 1 and 3 as the F-terms gives a theory equivalent thereto.

(f) Furthermore, we could let rows 2 and 3 be Q giving us $d = \begin{pmatrix} 0 & 1 & -1 & -1 & -1 \\ 0 & -1 & 1 & 1 & 1 \end{pmatrix}$, but this again gives an anomalous matter content.

(g) Finally, though we cannot treat all rows as F-terms, we can however treat all of them as D-terms in which Q_t is simply Δ as remarked at the end of Section 2 before the flow chart. In this case we have the matter content $d = \begin{pmatrix} 1 & -1 & 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$ which clearly is both trace-anomaly free (each row adds to zero) and cubic-anomaly-free (the cube-sum of the each row is also zero). The superpotential, by our very choice, is of course zero. Thus we have a perfectly legitimate theory without superpotential but with an exotic field (the first column) charged under 4 gauge groups.

We see therefore, from our list of examples above, that for the simple case of the SPP we have 3 rather different theories (a,d,g) with contrasting matter content and superpotential which share the same toric description.

Repetition Ambiguity: As a further illustration, let us give one example of type (B) ambiguity. First let us eliminate all repetitive columns from the G_t of SPP, giving us:

$$G_t = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

which is perfectly allowed and consistent with Figure 19-2. Of course many more possibilities for repeats are allowed and we could redo the following analyses for each of them. As the nullspace of our present choice of G_t, we find Q_t, and we choose, in light of the foregoing discussion, the first row to represent the D-term:

$$Q_t = \begin{pmatrix} -1 & 1 & -1 & 0 & 1 & \zeta \\ 1 & -2 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Thus equipped, we immediately retrieve, using our algorithm,

$$d = \begin{array}{c} \begin{matrix} X_1 & X_2 & X_3 & X_4 & X_5 \end{matrix} \\ \begin{pmatrix} 1 & -1 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 \end{pmatrix} \end{array} \qquad K^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We see that d passes our anomaly test, with the same bi-fundamental matter content as theory (d). The superpotential can be read easily from K (since there is only one relation) as $W = \phi(X_5^2 - X_3X_4)$. As a check, let us define the gauge invariant quantities: $a = X_1X_2$, $b = X_1X_4$, $c = X_3X_2$, $d = X_3X_4$ and $e = X_5$. These have among themselves the algebraic relations $ad = bc$ and $e^2 = d$, from which we immediately obtain $bc = ae^2$, the equation for the SPP. Hence we have yet another interesting anomaly free theory, which together with our theories (a), (d) and (g) above, shares the toric description of the SPP.

Finally, let us indulge in one more demonstration. Now let us treat both rows of our Q_t as D-terms, whereby giving a theory with no superpotential and the exotic matter content $d = \begin{pmatrix} -1 & 1 & -1 & 0 & 1 \\ 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & -1 \end{pmatrix}$ with a field (column 2) charged under 3 gauge groups. Indeed though the rows sum to 0 and trace-anomaly is avoided, the cube-sum of the second row gives $1^3 + 1^3 + (-2)^3 = -6$ and we do have a cubic anomaly.

In summary, we have an interesting phenomenon indeed! Taking so immediate an advantage of the ambiguities in the above has already produced quite a few examples of vastly different gauge theories flowing in the IR to the same universality class by having their moduli spaces identical. The vigilant reader may raise two issues. First, as mentioned earlier, one may take the pains to check whether these theories do indeed live on a D-brane. Necessary conditions such as that the theories may be obtained from an $\mathcal{N} = 4$ theory must be satisfied. Second, the matching of moduli spaces may not seem so strong since they are on a classical level. However, since we are dealing with product $U(1)$ gauge groups (which is what toric geometry is capable to dealing with so far), the classical moduli receive no quantum corrections[14]. Therefore the matching of the moduli for these various theories do persist to the quantum regime, which hints at some kind of "duality" in the field theory. We shall call such a duality **toric duality**. It would be interesting to investigate how, with non-Abelian versions of the theory (either by brane setups or stacks of D-brane probes), this toric duality may be extended.

19.6 Conclusions and Prospects

The study of resolution of toric singularities by D-branes is by now standard. In the concatenation of the F-terms and D-terms from the world volume gauge theory of a single D-brane at the singularity, the moduli space could be captured by the algebraic data of the toric variety. However, unlike the orbifold theories, the inverse problem where specifying the structure of the singularity specifies the physical theory has not yet been addressed in detail.

We recognise that in contrast with D-brane probing orbifolds, where knowing the group structure and its space-time action uniquely dictates the matter content and superpotential, such flexibility is not shared by generic toric varieties due to the highly non-unique nature of the inverse problem. It has been the purpose and main content of the current writing to device an **algorithm** which constructs the matter content (the incidence matrix d) and the interaction (the F-term matrix K) of a well-behaved gauge theory given the toric diagram D of the singularity at hand.

[14] We thank K. Intriligator for pointing this out.

By embedding D into the Abelian orbifold $\mathbb{C}^k/(\mathbb{Z}_n)^{k-1}$ and performing the standard partial resolution techniques, we have investigated how the induced action upon the charge matrices corresponding to the toric data of the latter gives us a convenient charge matrix for D and have constructed a programmatic methodology to extract the matter content and superpotential of *one* D-brane world volume gauge theory probing D. The theory we construct, having its origin from an orbifold, is nicely behaved in that it is anomaly free, with bi-fundamentals only and well-defined superpotentials. As illustrations we have tabulated the results for all the toric del Pezzo surfaces and the zeroth Hirzebruch surface.

Directions of further work are immediately clear to us. From the patterns emerging from del Pezzo surfaces 0 to 3, we could speculate the physics of higher (non-toric) del Pezzo cases. For example, we expect del Pezzo n to have $n+3$ gauge groups. Moreover, we could attempt to fathom how our resolution techniques translate as Higgsing in brane setups, perhaps with recourse to diamonds, and realise the various theories on toric varieties as brane configurations.

Indeed, as mentioned, the inverse problem is highly non-unique; we could presumably attempt to classify all the different theories sharing the same toric singularity as their moduli space. In light of this, we have addressed two types of ambiguity: that in having multiple fields assigned to the same node in the toric diagram and that of distinguishing the F-terms and D-terms in the charge matrix. In particular we have turned this ambiguity to a matter of interest and have shown, using our algorithm, how vastly different theories, some with quite exotic matter content, may have the same toric description. This commonality would correspond to a duality wherein different gauge theories flow to the same universality class in the IR. We call this phenomenon **toric duality**. It would be interesting indeed how this duality may manifest itself as motions of branes in the corresponding setups. Without further ado however, let us pause here awhile and leave such investigations to forthcoming work.

Chapter 20

Toric II: Phase Structure of Toric Duality

Synopsis

The previous chapter mentioned the concept of "Toric Duality." Here, we systematically study possible causes arising from our "Inverse Algorithm."

Harnessing the unimodular degree of freedom in the definition of any toric diagram, we present a method of constructing inequivalent gauge theories which are world-volume theories of D-branes probing the same toric singularity. These theories are various *phases* in partial resolution of Abelian orbifolds. As examples, two phases are constructed for both the zeroth Hirzebruch and the second del Pezzo surfaces. Furthermore, we investigate the general conditions that distinguish these different gauge theories with the same (toric) moduli space [306].

20.1 Introduction

The methods of toric geometry have been a crucial tool to the understanding of many fundamental aspects of string theory on Calabi-Yau manifolds (cf. e.g. [18]).

In particular, the connexions between toric singularities and the manufacturing of various gauge theories as D-brane world-volume theories have been intimate.

Such connexions have been motivated by a myriad of sources. As far back as 1993, Witten [17] had shown, via the so-called gauged linear sigma model, that the Fayet-Illiopoulos parametre r in the D-term of an $\mathcal{N} = 2$ supersymmetric field theory with $U(1)$ gauge groups can be tuned as an order-parametre which extrapolates between the Landau-Ginzburg and Calabi-Yau phases of the theory, whereby giving a precise viewpoint to the LG/CY-correspondence. What this means in the context of Abelian gauge theories is that whereas for $r \ll 0$, we have a Landau-Ginzberg description of the theory, by taking $r \gg 0$, the space of classical vacua obtained from D- and F-flatness is described by a Calabi-Yau manifold, and in particular a toric variety.

With the advent of D-brane technologies, vast amount of work has been done to study the dynamics of world-volume theories on D-branes probing various geometries. Notably, in [69], D-branes have been used to probe Abelian singularities of the form $\mathbb{C}^2/\mathbb{Z}_n$. Methods of studying the moduli space of the SUSY theories describable by quiver diagrams have been developed by the recognition of the Kronheimer-Nakajima ALE instanton construction, especially the moment maps used therein [171].

Much work followed [75, 157, 76]. A key advance was made in [74], where, exemplifying with Abelian $\mathbb{C}^3$ orbifolds, a detailed method was developed for capturing the various phases of the moduli space of the quiver gauge theories as toric varieties. In another vein, the huge factory built after the brane-setup approach to gauge theories [66] has been continuing to elucidate the T-dual picture of branes probing singularities (e.g. [78, 79, 292]). Brane setups for toric resolutions of $\mathbb{Z}_2 \times \mathbb{Z}_2$, including the famous conifold, were addressed in [276, 273]. The general question of how to construct the quiver gauge theory for an arbitrary toric singularity was still pertinent. With the AdS/CFT correspondence emerging [75, 157], the pressing need for the question arises again: given a toric singularity, how does one determine the quiver gauge theory having the former as its moduli space?

The answer lies in "Partial Resolution of Abelian Orbifolds" and was introduced and exemplified for the toric resolutions of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold [74, 277]. The method was subsequently presented in an algorithmic and computationally feasible fashion in [298] and was applied to a host of examples in [279].

One short-coming about the inverse procedure of going from the toric data to the gauge theory data is that it is highly non-unique and in general, unless one starts by partially resolving an orbifold singularity, one would not be guaranteed with a physical world-volume theory at all! Though the non-uniqueness was harnessed in [298] to construct families of quiver gauge theories with the same toric moduli space, a phenomenon which was dubbed "toric duality," the physicality issue remains to be fully tackled.

The purpose of this writing is to analyse toric duality within the confinement of the canonical method of partial resolutions. Now we are always guaranteed with a world-volume theory at the end and this physicality is of great assurance to us. We find indeed that with the restriction of physical theories, toric duality is still very much at work and one can construct D-brane quiver theories that flow to the same moduli space.

We begin in §2 with a seeming paradox which initially motivated our work and which *ab initio* appeared to present a challenge to the canonical method. In §3 we resolve the paradox by introducing the well-known mathematical fact of toric isomorphisms. Then in §4, we present a detailed analysis, painstakingly tracing through each step of the inverse procedure to see how much degree of freedom one is allowed as one proceeds with the algorithm. We consequently arrive at a method of extracting torically dual theories which are all physical; to these we refer as "phases." As applications of these ideas in §5 we re-analyse the examples in [298], viz., the toric del Pezzo surfaces as well as the zeroth Hirzebruch surface and find the various phases of the quiver gauge theories with them as moduli spaces. Finally in §6 we end with conclusions and future prospects.

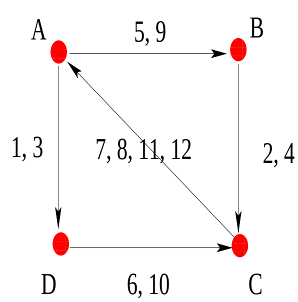
20.2 A Seeming Paradox

In [298] we noticed the emergence of the phenomenon of "Toric Duality" wherein the moduli space of vast numbers of gauge theories could be parametrised by the *same* toric variety. Of course, as we mentioned there, one needs to check extensively whether these theories are all physical in the sense that they are world-volume theories of some D-brane probing the toric singularity.

Here we shall discuss an issue of more immediate concern to the physical probe theory. We recall that using the method of *partial resolutions of Abelian orbifolds* [298, 74, 277, 276], we could always extract a canonical theory on the D-brane probing the singularity of interest.

However, a discrepancy of results seems to have risen between [298] and [157] on the precise world-volume theory of a D-brane probe sitting on the zeroth Hirzebruch surface; let us compare and contrast the two results here.

- Results from [298]: The matter contents of the theory are given by (on the left we present the quiver diagram and on the right, the incidence matrix that encodes the quiver):

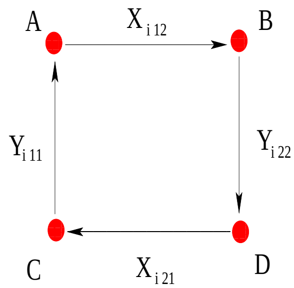


$$d = \begin{array}{c|cccccccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} \\ \hline A & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & 1 \\ B & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ D & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}$$

and the superpotential is given by

$$W = X_1X_8X_{10} - X_3X_7X_{10} - X_2X_8X_9 - X_1X_6X_{12} + X_3X_6X_{11} + X_4X_7X_9 + X_2X_5X_{12} - X_4X_5X_{11}. \tag{20.2.1}$$

- Results from [157]: The matter contents of the theory are given by (for $i = 1, 2$):



$$d = \begin{array}{c|cccc} & X_{i\ 12} & X_{i\ 21} & Y_{i\ 11} & Y_{i\ 22} \\ \hline A & -1 & 0 & 1 & 0 \\ B & 1 & 0 & 0 & -1 \\ C & 0 & 1 & -1 & 0 \\ D & 0 & -1 & 0 & 1 \end{array}$$

and the superpotential is given by

$$W = \epsilon^{ij}\epsilon^{kl} X_{i\ 12} Y_{k\ 22} X_{j\ 21} Y_{l\ 11}. \tag{20.2.2}$$

Indeed, even though both these theories have arisen from the canonical partial resolutions technique and hence are world volume theories of a brane probing a Hirzebruch singularity, we see clearly that they differ vastly in both matter content and superpotential! Which is the "correct" physical theory?

In response to this seeming paradox, let us refer to Figure 20-1. Case 1 of course was what had been analysed in [298] (q.v. ibid.) and presented in (20.2.1); let us now consider case 2. Using the canonical algorithm of [277, 298], we obtain the matter content (we have labelled the fields and gauge groups with some foresight)

$$d_{ia} = \begin{array}{c|cccccccc} & X_1 & X_1' & X_2' & Y_1 & Y_2 & Y_1' & Y_2 & Y_2' \\ \hline D & 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 \\ A & -1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ B & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 \end{array}$$

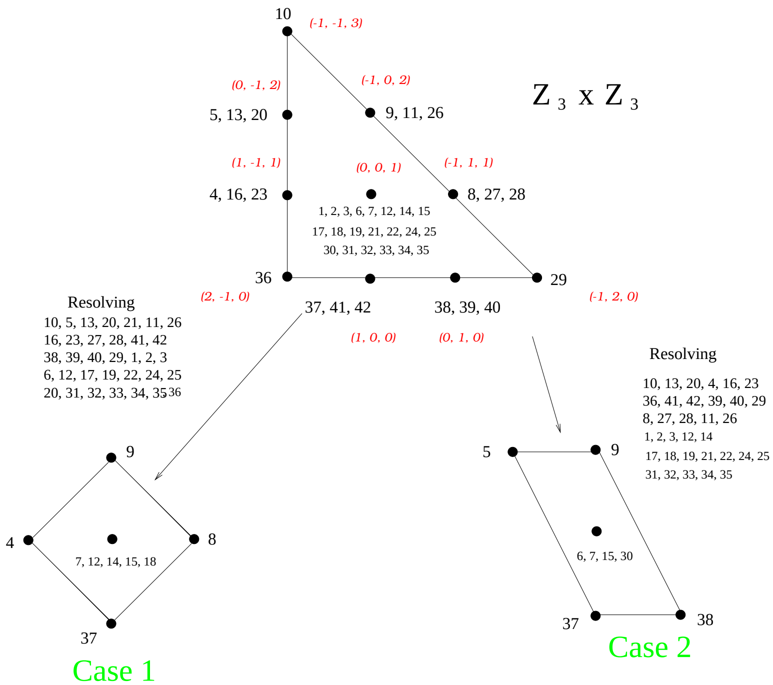


Figure 20-1: Two alternative resolutions of $\mathbb{C}^2/\mathbb{Z}_3 \times \mathbb{Z}_3$ to the Hirzebruch surface F_0: Case 1 from [298] and Case 2 from [157].

and the dual cone matrix

$$K_{ij}^T = \left(\begin{array}{c|cccccccc} & X_1 & X'_1 & X'_2 & Y_1 & Y_2 & Y'_1 & X_2 & Y'_2 \\ \hline p_1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ p_2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ p_3 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ p_4 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ p_5 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ p_6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array} \right)$$

which translates to the F-term equations

$$X_1 Y'_2 = p_1 p_3 p_6 = Y'_1 X_2; \quad X'_1 Y_2 = p_2 p_4 p_5 = Y_1 X'_2.$$

What we see of course, is that with the field redefinition $X_i \leftrightarrow X_{i\ 12}, X'_i \leftrightarrow Y_{i\ 22}, Y_i \leftrightarrow Y_{i\ 11}$ and $Y'_i \leftrightarrow X_{i\ 21}$ for $i = 1, 2$, the above results are in exact agreement with the results from [157] as presented in (20.2.2).

This is actually of no surprise to us because upon closer inspection of Figure 20-1, we see that the toric diagram for Cases 1 and 2 respectively has the coordinate points

$$G1_t = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 \end{pmatrix} \qquad G2_t = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \end{pmatrix}.$$

Now since the algebraic equation of the toric variety is given by [10]

$$V(G_t) = Spec_{Max}\left(\mathbb{C}[X_i^{G_t^\vee \cap \mathbb{Z}^3}]\right),$$

we have checked that, using a reduced Gröbner polynomial basis algorithm to compute the variety [12], the equations are identical up to redefinition of variables.

Therefore we see that the two toric diagrams in Cases 1 and 2 of Figure 20-1 both describe the zeroth Hirzebruch surface as they have the same equations (embedding into $\mathbb{C}^9$). Yet due to the particular choice of the diagram, we end up with strikingly different gauge theories on the D-brane probe despite the identification of the moduli space in the IR. This is indeed a curiously strong version of "toric duality."

Bearing the above in mind, in this chapter, we will analyse the degrees of freedom in the Inverse Algorithm expounded upon in [298], i.e., for a given toric singularity, how many different physical gauge theories (phase structures), resulting from various partial resolutions can one have for a D-brane probing such a singularity? To answer this question, first in §2 we present the concept of toric isomorphism and give the conditions for different toric data to correspond to the same toric variety. Then in §3 we follow the Forward Algorithm and give the freedom at each step from a given set of gauge theory data all the way to the output of the toric data. Knowing these freedoms, we can identify the sources that may give rise to different gauge theories in the Inverse Algorithm starting from a prescribed toric data. In section 4, we apply the above results and analyse the different phases for the partial resolutions of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold singularity, in particular, we found that there are two inequivalent phases of gauge theories respectively for the zeroth Hirzebruch surface and the second del Pezzo surface. Finally, in section 5, we give discussions for further investigation.

20.3 Toric Isomorphisms

Extending this observation to generic toric singularities, we expect classes of inequivalent toric diagrams corresponding to the same variety to give rise to inequivalent gauge theories on the D-brane probing the said singularity. An immediate question is naturally posed: "is there a classification of these different theories and is there a transformation among them?"

To answer this question we resort to the following result. Given M-lattice cones σ and σ', let the linear span of σ be $\text{lin}\sigma = \mathbb{R}^n$ and that of σ' be $\mathbb{R}^m$. Now each cone gives rise to a semigroup which is the intersection of the dual cone $\sigma^\vee$ with the dual lattice M, i.e., $S_\sigma := \sigma^\vee \cap M$ (likewise for σ'). Finally the toric variety is given as the maximal spectrum of the polynomial ring of $\mathbb{C}$ adjoint the semigroup, i.e., $X_\sigma := Spec_{Max}\left(\mathbb{C}[S_\sigma]\right)$.

DEFINITION 20.3.29 *We have these types of isomorphisms:*

1. *We call σ and σ'* cone isomorphic*, denoted $\sigma \cong_{cone} \sigma'$, if $n = m$ and there is a unimodular transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ with $L(\sigma) = \sigma'$;*

2. *we call S_σ and $S_{\sigma'}$* monomial isomorphic*, denoted $S_\sigma \cong_{mon} S_{\sigma'}$, if there exists mutually inverse monomial homomorphisms between the two semigroups.*

Thus equipped, we are endowed with the following

THEOREM 20.3.33 *([13], VI.2.11) The following conditions are equivalent:*

$$(a)\quad \sigma \cong_{cone} \sigma' \Leftrightarrow (b)\quad S_\sigma \cong_{mon} S_{\sigma'} \Leftrightarrow (c)\quad X_\sigma \cong X_{\sigma'}$$

What this theorem means for us is simply that, for the n-dimensional toric variety, an $SL(n;\mathbb{Z})$ transformation[1] on the original lattice cone amounts to merely coördinate

[1]Strictly speaking, by unimodular we mean $GL(n;\mathbb{Z})$ matrices with determinant ± 1; we shall denote these loosely by $SL(n;\mathbb{Z})$.

transformations on the polynomial ring and results in the same toric variety. This, is precisely what we want: different toric diagrams giving the same variety.

The necessity and sufficiency of the condition in Theorem 20.3.33 is important. Let us think of one example to illustrate. Let a cone be defined by (e_1, e_2), we know this corresponds to $\mathbb{C}^2$. Now if we apply the transformation

$$(e_1, e_2)\begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = (2e_1 - e_2, e_2),$$

which corresponds to the variety $xy = z^2$, i.e., $\mathbb{C}^2/\mathbb{Z}_2$, which of course is not isomorphic to $\mathbb{C}^2$. The reason for this is obvious: the matrix we have chosen is certainly not unimodular.

20.4 Freedom and Ambiguity in the Algorithm

In this section, we wish to step back and address the issue in fuller generality. Recall that the procedure of obtaining the moduli space encoded as toric data once given the gauge theory data in terms of product $U(1)$ gauge groups, D-terms from matter contents and F-terms from the superpotential, has been well developed [157, 74]. Such was called the **forward algorithm** in [298]. On the other hand the **reverse algorithm** of obtaining the gauge theory data from the toric data has been discussed extensively in [277, 298].

It was pointed in [298] that both the forward and reverse algorithm are highly non-unique, a property which could actually be harnessed to provide large classes of gauge theories having the same IR moduli space. In light of this so-named "toric duality" it would be instructive for us to investigate how much freedom do we have at each step in the algorithm. We will call two data related by such a freedom *equivalent* to each other. Thence further we could see how freedoms at every step accumulate and appear in the final toric data. Modulo such equivalences we believe that the data should be uniquely determinable.

20.4.1 The Forward Algorithm

We begin with the forward algorithm of extracting toric data from gauge data. A brief review is at hand. To specify the gauge theory, we require three pieces of information: the number of $U(1)$ gauge fields, the charges of matter fields and the superpotential. The first two are summarised by the so-called charge matrix d_{li} where $l = 1, 2, ..., L$ with L the number of $U(1)$ gauge fields and $i = 1, 2, ..., I$ with I the number of matter fields. When using the forward algorithm to find the vacuum manifold (as a toric variety), we need to solve the D-term and F-term flatness equations. The D-terms are given by d_{li} matrix while the F-terms are encoded in a matrix K_{ij} with $i, 1, 2, ..., I$ and $j = 1, 2, ..., J$ where J is the number of independent parameters needed to solve the F-terms. By gauge data then we mean the matrices d (also called the incidence matrix) and the K (essentially the dual cone); the forward algorithm takes these as input. Subsequently we trace a flow-chart:

$$
\begin{array}{ccccccc}
\text{D-Terms} \rightarrow d & \rightarrow & \Delta & & & & \\
 & & \downarrow & & & & \\
\text{F-Terms} \rightarrow K & \stackrel{V \cdot K^T = \Delta}{\rightarrow} & V & & & & \\
\downarrow & & \downarrow & & & & \\
T = \text{Dual}(K) & \stackrel{U \cdot T^T = \text{Id}}{\rightarrow} & U & \rightarrow & VU & & \\
\downarrow & & & & \downarrow & & \\
Q = [\text{Ker}(T)]^T & & \longrightarrow & & Q_t = \left(\begin{array}{c} Q \\ VU \end{array} \right) & \rightarrow & G_t = [\text{Ker}(Q_t)]^T
\end{array}
$$

arriving at a final matrix G_t whose columns are the vectors which prescribe the nodes of the toric diagram.

What we wish to investigate below is how much procedural freedom we have at each arrow so as to ascertain the non-trivial toric dual theories. Hence, if A_1 is the matrix whither one arrives from a certain arrow, then we would like to find the most general transformation taking A_1 to another solution A_2 which would give rise to an identical theory. It is to this transformation that we shall refer as "freedom" at the particular step.

Superpotential: the matrices K and T

The solution of F-term equations gives rise to a dual cone $K_1 = K_{ij}$ defined by I vectors in $\mathbb{Z}^J$. Of course, we can choose different parametres to solve the F-terms and arrive at another dual cone K_2. Then, K_1 and K_2, being integral cones, are equivalent if they are unimodularly related, i.e., $K_2^T = A \cdot K_1^T$ for $A \in GL(J, \mathbb{Z})$ such that $\det(A) = \pm 1$. Furthermore, the order of the I vectors in $\mathbb{Z}^J$ clearly does not matter, so we can permute them by a matrix S_I in the symmetric group $\mathcal{S}_I$. Thus far we have two freedoms, multiplication by A and S:

$$K_2^T = A \cdot K_1^T \cdot S_I, \tag{20.4.3}$$

and $K_{1,2}$ should give equivalent theories.

Now, from K_{ij} we can find its dual matrix $T_{j\alpha}$ (defining the cone T) where $\alpha = 1, 2, ..., c$ and c is the number of vectors of the cone T in $\mathbb{Z}^J$, as constrained by

$$K \cdot T \geq 0 \tag{20.4.4}$$

and such that T also spans an integral cone. Notice that finding dual cones, as given in a algorithm in [10], is actually unique up to permutation of the defining vectors. Now considering the freedom of K_{ij} as in (20.4.3), let T_2 be the dual of K_2 and T_1 that of K_1, we have $K_2 \cdot T_2 = S_I^T \cdot K_1 \cdot A^T \cdot T_2 \geq 0$, which means that

$$T_1 = A^T \cdot T_2 \cdot S_c. \tag{20.4.5}$$

Note that here S_c is the permutation of the c vectors of the cone T in and not that of the dual cone in (20.4.3).

The Charge Matrix Q

The next step is to find the charge matrix $Q_{k\alpha}$ where $\alpha = 1, 2, ..., c$ and $k = 1, 2, ..., c-J$. This matrix is defined by

$$T \cdot Q^T = 0. \tag{20.4.6}$$

In the same spirit as the above discussion, from (20.4.5) we have $T_1 \cdot Q_1^T = A^T \cdot T_2 \cdot S_c \cdot Q_1^T = 0$. Because A^T is a invertible matrix, this has a solution when and only when $T_2 \cdot S_c \cdot Q_1^T = 0$. Of course this is equivalent to $T_2 \cdot S_c \cdot Q_1^T \cdot B_{kk'} = 0$ for some invertible $(c-J) \times (c-J)$ matrix $B_{kk'}$. So the freedom for matrix Q is

$$Q_2^T = S_c \cdot Q_1^T \cdot B. \tag{20.4.7}$$

We emphasize a difference from (20.4.4); there we required both matrices K and T to be integer where here (20.4.6) does not possess such a constraint. Thus the only condition for the matrix B is its invertibility.

Matter Content: the Matrices d, $\widetilde{V}$ and U

Now we move onto the D-term and the integral d_{li} matrix. The D-term equations are $d \cdot |X|^2 = 0$ for matter fields X. Obviously, any transformation on d by an invertible matrix $C_{L\times L}$ does not change the D-terms. Furthermore, any permutation S_I of the order the fields X, so long as it is consistent with the S_I in (20.4.3), is also game. In other words, we have the freedom:

$$d_2 = C \cdot d_1 \cdot S_I. \tag{20.4.8}$$

We recall that a matrix V is then determined from Δ, which is d with a row deleted due to the centre of mass degree of freedom. However, to not to spoil the above freedom enjoyed by matrix d in (20.4.8), we will make a slight amendment and define the matrix $\widetilde{V}_{lj}$ by

$$\widetilde{V} \cdot K^T = d. \tag{20.4.9}$$

Therefore, whereas in [74, 298] where $V \cdot K^T = \Delta$ was defined, we generalise V to $\widetilde{V}$ by (20.4.9). One obvious way to obtain $\widetilde{V}$ from V is to add one row such that the sum of every column is zero. However, there is a caveat: when there exists a vector h such that

$$h \cdot K^T = 0,$$

we have the freedom to add h to any row of $\widetilde{V}$. Thus finding the freedom of $\widetilde{V}_{lj}$ is a little more involved. From (20.4.3) we have $d_2 = \widetilde{V}_2 \cdot K_2^T = \widetilde{V}_2 \cdot A \cdot K_1^T \cdot S_I$ and $d_2 = C \cdot d_1 \cdot S_I = C \cdot \widetilde{V}_1 \cdot K_1^T \cdot S_I$. Because S_I is an invertible square matrix, we have $(\widetilde{V}_2 \cdot A - C \cdot \widetilde{V}_1) \cdot K_1^T = 0$, which means $\widetilde{V}_2 \cdot A - C \cdot \widetilde{V}_1 = CH_{K_1}$ for a matrix H constructed by having the aforementioned vectors h as its columns. When K^T has maximal rank, H is zero and this is in fact the more frequently encountered situation. However, when K^T is not maximal rank, so as to give non-trivial solutions of h, we have that $\widetilde{V}_1$ and $\widetilde{V}_2$ are equivalent if

$$\widetilde{V}_2 = C \cdot (\widetilde{V}_1 + H_{K_1}) \cdot A^{-1}. \tag{20.4.10}$$

Moving on to the matrix $U_{j\alpha}$ defined by

$$U \cdot T^T = \mathbb{I}_{jj'}, \tag{20.4.11}$$

we have from (20.4.5) $\mathbb{I}_{jj'} = U_1 \cdot T_1^T = U_1 \cdot S_c^T \cdot T_2^T \cdot A$, whence $A^{-1} = U_1 \cdot S_c^T \cdot T_2^T$ and $\mathbb{I} = A \cdot U_1 \cdot S_c^T \cdot T_2^T$. This gives $(A \cdot U_1 \cdot S_c^T - U_2) \cdot T_2^T = 0$ which has a solution $A \cdot U_1 \cdot S_c^T - U_2 = H_{T_2}$ where $H_{T_2} \cdot T_2^T = 0$ is precisely as defined in analogy of the H above. Therefore the freedom on U is subsequently

$$U_2 = A \cdot (U_1 - H_{T_1}) \cdot S_c^T, \tag{20.4.12}$$

where $H_{T_1} = A^{-1} H_{T_2} (S_c^T)^{-1}$ and $H_{T_1} \cdot T_1^T = (A^{-1} H_{T_2} (S_c^T)^{-1})(S_c^T \cdot T_2^T \cdot A) = 0$. Finally using (20.4.10) and (20.4.12), we have

$$(\widetilde{V}_2 \cdot U_2) = C \cdot (\widetilde{V}_1 + H_{K_1}) \cdot A^{-1} \cdot A \cdot (U_1 - H_{T_1}) \cdot S_c^T = C \cdot (\widetilde{V}_1 + H_{K_1})(U_1 - H_{T_1}) \cdot S_c^T, \tag{20.4.13}$$

determining the freedom of the relevant combination $(\widetilde{V} \cdot U)$.

Let us pause for an important observation that in most cases $H_{K_1} = 0$, as we shall see in the examples later. From (20.4.6), which propounds the existence of a non-trivial nullspace for T, we see that one can indeed obtain a non-trivial H_{T_1} in terms of the combinations of the rows of the charge matrix Q, whereby simplifying (20.4.13) to

$$(\widetilde{V}_2 \cdot U_2) = C \cdot (\widetilde{V}_1 \cdot U_1 + H_{VU_1}) \cdot S_c^T, \tag{20.4.14}$$

where every row of H_{VU_1} is linear combination of rows of Q_1 and the sum of its columns is zero.

Toric Data: the Matrices Q_t and G_t

At last we come to $\widetilde{Q}_t$, which is given by adjoining Q and $\widetilde{V} \cdot U$. The freedom is of course, by combining all of our results above,

$$(\widetilde{Q}_t)_2 = \begin{pmatrix} Q_2 \\ \widetilde{V}_2 \cdot U_2 \end{pmatrix} = \begin{pmatrix} B^T \cdot Q_1 \cdot S_c^T \\ C \cdot (\widetilde{V}_1 \cdot U_1 + H_{VU_1}) \cdot S_c^T \end{pmatrix} = \begin{pmatrix} B^T \cdot Q_1 \\ C \cdot (\widetilde{V}_1 \cdot U_1 + H_{VU_1}) \end{pmatrix} \cdot S_c^T \tag{20.4.15}$$

Now $\widetilde{Q}_t$ determines the nodes of the toric diagram $(G_t)_{p\alpha}$ ($p = 1, 2, .., (c-(L-1)-J)$ and $\alpha = 1, 2, ..., c$) by

$$Q_t \cdot G_t^T = 0; \tag{20.4.16}$$

The columns of G_t then describes the toric diagram of the algebraic variety for the vacuum moduli space and is the output of the algorithm. From (20.4.16) and (20.4.15) we find that if $(\widetilde{Q}_t)_1 \cdot (G_t)_1^T = 0$, i.e., $Q_1 \cdot (G_t)_1^T = 0$ and $\widetilde{V}_1 \cdot U_1 \cdot (G_t)_1^T = 0$, we automatically have the freedom $(\widetilde{Q}_t)_2 \cdot (S_c^T)^{-1} \cdot (\widetilde{G}_t)_1^T = 0$. This means that at most we can have

$$(G_t)_2^T = (S_c^T)^{-1} \cdot (G_t)_1^T \cdot D, \tag{20.4.17}$$

where D is a $GL(c-(L-1)-J, \mathbb{Z})$ matrix with $\det(D) = \pm 1$ which is exactly the unimodular freedom for toric data as given by Theorem 20.3.33.

One immediate remark follows. From (20.4.16) we obtain the nullspace of Q_t in $\mathbb{Z}^c$. It seems that we can choose an arbitrary basis so that D is a $GL(c-(L-1)-J, \mathbb{Z})$ matrix with the only condition that $\det(D) \neq 0$. However, this is not stringent enough: in fact, when we find cokernel G_t, we need to find the *integer basis* for the null space, i.e., we need to find the basis such that any integer null vector can be decomposed into a linear combination of the columns of G_t. If we insist upon such a choice, the only remaining freedom[2] is that $\det(D) = \pm 1$, viz, unimodularity.

20.4.2 Freedom and Ambiguity in the Reverse Algorithm

Having analysed the equivalence conditions in last subsection, culminating in (20.4.15) and (20.4.17), we now proceed in the opposite direction and address the ambiguities in the reverse algorithm.

The Toric Data: G_t

We note that the G_t matrix produced by the forward algorithm is not minimal in the sense that certain columns are repeated, which after deletion, constitute the toric diagram. Therefore, in our reverse algorithm, we shall first encounter such an ambiguity in deciding which columns to repeat when constructing G_t from the nodes of the toric diagram. This so-called *repetition ambiguity* was discussed in [298] and different choices of repetition may indeed give rise to different gauge theories. It was pointed out (*loc. cit.*) that arbitrary repetition of the columns certainly does not guarantee physicality. By physicality we mean that the gauge theory arrived at the end of the day should be *physical* in the sense of still being a D-brane world-

[2] We would like to express our gratitude to M. Douglas for clarifying this point to us.

volume theory. What we shall focus here however, is the inherent symmetry in the toric diagram, given by (20.4.17), that gives rise to the same theory. This is so that we could find truly inequivalent *physical* gauge theories not related by such a transformation as (20.4.17).

The Charge Matrix: from G_t to Q_t

From (20.4.16) we can solve for Q_t. However, for a given G_t, in principle we can have two solutions $(Q_t)_1$ and $(Q_t)_2$ related by

$$(Q_t)_2 = P(Q_t)_1, \tag{20.4.18}$$

where P is a $p \times p$ matrix with p the number of rows of Q_t. Notice that the set of such transformations P is much larger than the counterpart in the forward algorithm given in (20.4.15). This is a second source of ambiguity in the reverse algorithm. More explicitly, we have the freedom to arbitrarily divide the Q_t into two parts, viz., the D-term part $\widetilde{V}U$ and the F-term part Q. Indeed one may find a matrix P such that $(Q_t)_1$ and $(Q_t)_2$ satisfy (20.4.18) but not matrices B and C in order to satisfy (20.4.15). Hence different choices of Q_t and different division therefrom into D and F-term parts give rise to different gauge theories. This is what we called *FD Ambiguity* in [298]. Again, arbitrary division of the rows of Q_t was pointed out to not to ensure physicality. As with the discussion on the repetition ambiguity above, what we shall pin down is the freedom due to the linear algebra and not the choice of division.

The Dual Cone and Superpotential: from Q to K

The nullspace of Q is the matrix T. The issue is the same as discussed at the paragraph following (20.4.17) and one can uniquely determine T by imposing that its columns give an integral span of the nullspace. Going further from T to its dual K, this is again a unique procedure (while integrating back from K to obtain the superpotential is certainly not). In summary then, these two steps give no sources for ambiguity.

The Matter Content: from $\widetilde{V}U$ to d matrix

The d matrix can be directly calculated as [298]

$$d = (\widetilde{V}U) \cdot T^T \cdot K^T. \tag{20.4.19}$$

Substituting the freedoms in (20.4.3), (20.4.5) and (20.4.13) we obtain

$$\begin{aligned} d_2 &= (\widetilde{V}_2 \cdot U_2) \cdot T_2^T \cdot K_2^T = C \cdot [(\widetilde{V}_1 \cdot U_1) + H_{VU_1}] \cdot S_c^T \cdot (S_c^T)^{-1} \cdot T_1^T \cdot A^{-1} \cdot A \cdot K_1^T \cdot S_I \\ &= C \cdot (\widetilde{V}_1 \cdot U_1) \cdot T_1^T \cdot K_1^T \cdot S_I + C \cdot H_{VU_1} \cdot T_1^T \cdot K_1^T \cdot S_I = C \cdot d_1 \cdot S_I, \end{aligned}$$

which is exactly formula (20.4.8). This means that the matter matrices are equivalent up to a transformation and there is no source for extra ambiguity.

20.5 Application: Phases of $\mathbb{Z}_3 \times \mathbb{Z}_3$ Resolutions

In [298] we developed an algorithmic outlook to the Inverse Procedure and applied it to the construction of gauge theories on the toric singularities which are partial resolutions of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The non-uniqueness of the method allowed one to obtain many different gauge theories starting from the same toric variety, theories to which we referred as being toric duals. The non-uniqueness mainly comes from three sources: (i) the repetition of the vectors in the toric data G_t (Repetition Ambiguity), (ii) the different choice of the null space basis of Q_t and (iii) the different divisions of the rows of Q_t (F-D Ambiguity). Many of the possible choices in the above will generate unphysical gauge theories, i.e., not world-volume theories of D-brane probes. We have yet to catalogue the exact conditions which guarantee physicality.

However, *Partial Resolution* of Abelian orbifolds, which stays within subsectors of the latter theory, does indeed constrain the theory to be physical. To these physical theories we shall refer as **phases** of the partial resolution. As discussed in [298] any k-dimensional toric diagram can be embedded into $\mathbb{Z}_n^{k-1}$ for sufficiently large n, one obvious starting point to obtain different phases of a D-brane gauge theory is to try various values of n. We leave some relevances of general n to the Appendix. However, because the algorithm of finding dual cones becomes prohibitively computationally intensive even for $n \geq 4$, this approach may not be immediately fruitful.

Yet armed with Theorem 20.3.33 we have an alternative. We can certainly find all possible unimodular transformations of the given toric diagram which still embeds into the same $\mathbb{Z}_n^{k-1}$ and then perform the inverse algorithm on these various *a fortiori* equivalent toric data and observe what physical theories we obtain at the end of the day. In our two examples in §1, we have essentially done so; in those cases we found that two inequivalent gauge theory data corresponded to two unimodularly equivalent toric data for the examples of $\mathbb{Z}_5$-orbifold and the zeroth Hirzebruch surface F_0.

The strategy lays itself before us. Let us illustrate with the same examples as was analysed in [298], namely the partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$, i.e., F_0 and the toric del Pezzo surfaces $dP_{0,1,2,3}$. We need to (i) find all $SL(3;\mathbb{Z})$ transformations of the toric diagram G_t of these five singularities that still remain as sub-diagrams of that of $\mathbb{Z}_3 \times \mathbb{Z}_3$ and then perform the inverse algorithm; therefrom, we must (ii) select theories not related by any of the freedoms we have discussed above and summarised in (20.4.15).

20.5.1 Unimodular Transformations within $\mathbb{Z}_3 \times \mathbb{Z}_3$

We first remind the reader of the G_t matrix of $\mathbb{Z}_3 \times \mathbb{Z}_3$ given in Figure 20-1, its columns are given by vectors: $(0,0,1)$, $(1,-1,1)$, $(0,-1,2)$, $(-1,1,1)$, $(-1,0,2)$, $(-1,-1,3)$, $(1,-1,1)$, $(-1,2,0)$, $(1,0,0)$, $(0,1,0)$. Step (i) of our above strategy can be immediately performed. Given the toric data of one of the resolutions G_t' with x columns, we select x from the above 10 columns of G_t and check whether any $SL(3;\mathbb{Z})$ transformation relates any permutation thereof unimodularly to G_t'. We shall at the end find that there are three different cases for F_0, five for dP^0, twelve for dP_1, nine for dP_2 and only one for dP_3. The (unrepeated) G_t matrices are as follows:

$(F_0)_1$	$(0,0,1),(1,-1,1),(-1,1,1),(-1,0,2),(1,0,0)$
$(F_0)_2$	$(0,0,1),(0,-1,2),(0,1,0),(-1,0,2),(1,0,0)$
$(F_0)_3$	$(0,0,1),(1,-1,1),(-1,1,1),(0,-1,2),(0,1,0)$
$(dP_0)_1$	$(0,0,1),(1,0,0),(0,-1,2),(-1,1,1)$
$(dP_0)_2$	$(0,0,1),(1,0,0),(-1,-1,3),(0,1,0)$
$(dP_0)_3$	$(0,0,1),(-1,2,0),(1,-1,1),(0,-1,2)$
$(dP_0)_4$	$(0,0,1),(0,1,0),(1,-1,1),(-1,0,2)$
$(dP_0)_5$	$(0,0,1),(2,-1,0),(-1,1,1),(-1,0,2)$
$(dP_1)_1$	$(1,0,0),(0,1,0),(-1,1,1),(0,-1,2),(0,0,1)$
$(dP_1)_2$	$(-1,-1,3),(0,-1,2),(1,0,0),(0,1,0),(0,0,1)$
$(dP_1)_3$	$(0,-1,2),(1,-1,1),(1,0,0),(-1,1,1),(0,0,1)$
$(dP_1)_4$	$(0,-1,2),(1,-1,1),(0,1,0),(-1,2,0),(0,0,1)$
$(dP_1)_5$	$(0,-1,2),(1,-1,1),(0,1,0),(-1,0,2),(0,0,1)$
$(dP_1)_6$	$(0,-1,2),(1,-1,1),(-1,2,0),(-1,1,1),(0,0,1)$
$(dP_1)_7$	$(0,-1,2),(1,0,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_1)_8$	$(1,-1,1),(2,-1,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_1)_9$	$(1,-1,1),(1,0,0),(0,1,0),(-1,0,2),(0,0,1)$
$(dP_1)_{10}$	$(1,-1,1),(0,1,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_1)_{11}$	$(2,-1,0),(1,0,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_1)_{12}$	$(-1,-1,3),(1,0,0),(0,1,0),(-1,0,2),(0,0,1)$
$(dP_2)_1$	$(2,-1,0),(1,-1,1),(-1,0,2),(-1,1,1),(1,0,0),(0,0,1)$
$(dP_2)_2$	$(-1,-1,3),(0,-1,2),(1,0,0),(0,1,0),(-1,0,2),(0,0,1)$
$(dP_2)_3$	$(0,-1,2),(1,-1,1),(1,0,0),(0,1,0),(-1,1,1),(0,0,1)$
$(dP_2)_4$	$(0,-1,2),(1,-1,1),(1,0,0),(0,1,0),(-1,0,2),(0,0,1)$
$(dP_2)_5$	$(0,-1,2),(1,-1,1),(1,0,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_2)_6$	$(0,-1,2),(1,-1,1),(0,1,0),(-1,2,0),(-1,1,1),(0,0,1)$
$(dP_2)_7$	$(0,-1,2),(1,-1,1),(0,1,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_2)_8$	$(0,-1,2),(1,0,0),(0,1,0),(-1,1,1),(-1,0,2),(0,0,1)$
$(dP_2)_9$	$(1,-1,1),(1,0,0),(0,1,0),(-1,1,1),(-1,0,2),(0,0,1)$
dP_3	$(0,-1,2),(1,-1,1),(1,0,0),(0,1,0),(-1,1,1),(-1,0,2),(0,0,1)$

The reader is referred to Figure 20-2 to Figure 20-6 for the toric diagrams of the data above. The vigilant would of course recognise $(F_0)_1$ to be Case 1 and $(F_0)_2$ as Case 2 of Figure 20-1 as discussed in §2 and furthermore $(dP_{0,1,2,3})_1$ to be the cases addressed in [298].

20.5.2 Phases of Theories

The Inverse Algorithm can then be readily applied to the above toric data; of the various unimodularly equivalent toric diagrams of the del Pezzo surfaces and the zeroth Hirzebruch, the details of which fields remain massless at each node (in the notation of [298]) are also presented in those figures immediately referred to above.

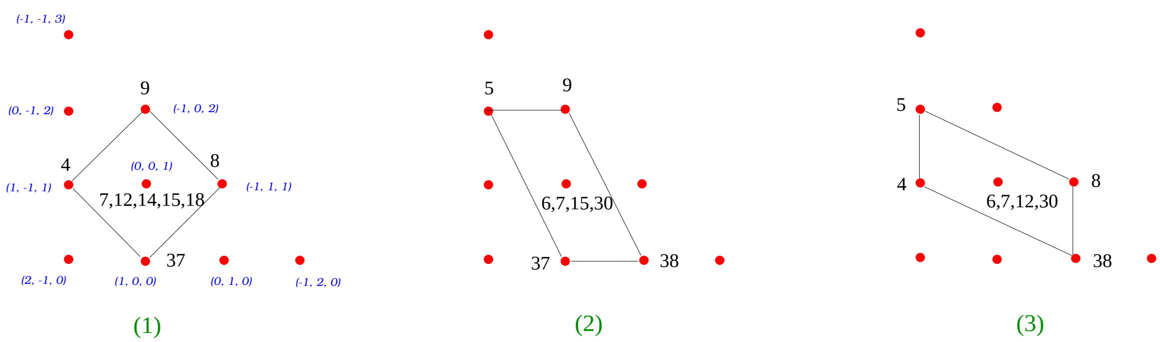


Figure 20-2: The 3 equivalent representations of the toric diagram of the zeroth Hirzebruch surface as a resolution of $\mathbb{Z}_3 \times \mathbb{Z}_3$. We see that (2) and (3) are related by a reflection about the 45^o line (a symmetry inherent in the parent $\mathbb{Z}_3 \times \mathbb{Z}_3$ theory) and we have the two giving equivalent gauge theories as expected.

Subsequently, we arrive at a number of D-brane gauge theories; among them, all five cases for dP^0 are equivalent (which is in complete consistency with the fact that dP^0 is simply $\mathbb{C}^3/\mathbb{Z}_3$ and there is only one nontrivial theory for this orbifold, corresponding to the decomposition $\mathbf{3} \to 1 + 1 + 1$). For dP_1, all twelve cases give back to same gauge theory (q.v. Figure 5 of [298]). For F_0, the three cases give

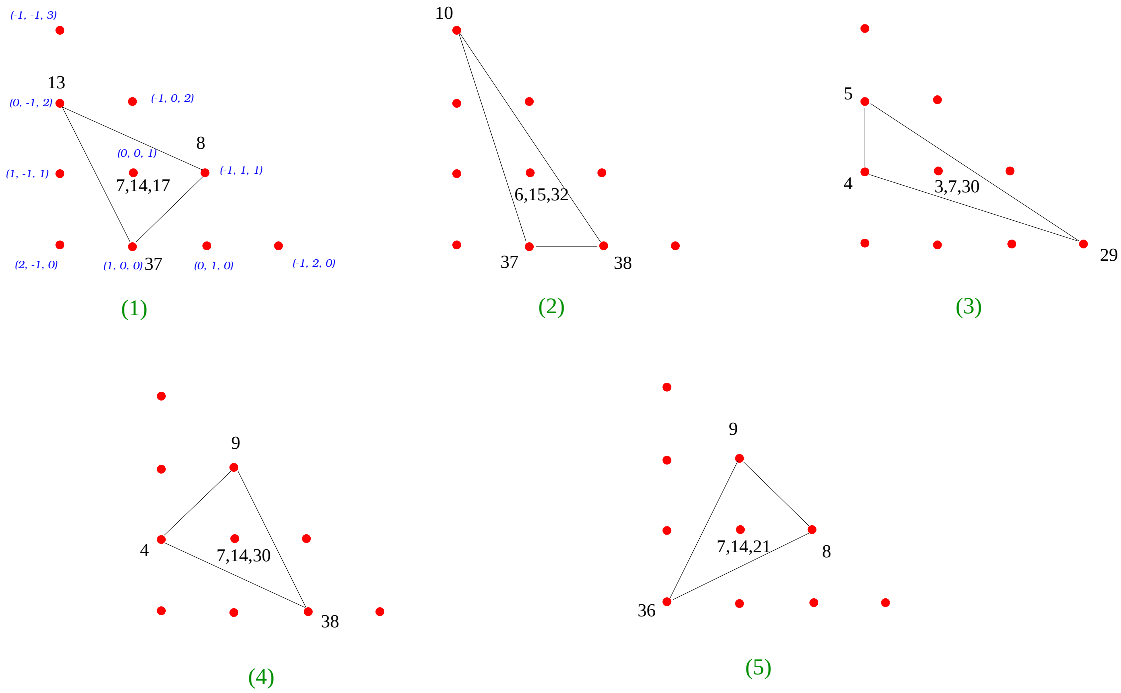


Figure 20-3: The 5 equivalent representations of the toric diagram of the zeroth del Pezzo surface as a resolution of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Again (1) and (4) (respectively (2) and (3)) are related by the 45^o reflection, and hence give equivalent theories. In fact further analysis shows that all 5 are equivalent.

two inequivalent gauge theories as given in §2. Finally for dP_2, the nine cases again give two different theories. For reference we tabulate the D-term matrix d and F-term matrix K^T below. If more than 1 theory are equivalent, then we select one representative from the list, the matrices for the rest are given by transformations (20.4.3) and (20.4.8).

<table>
<tr><th>Singularity</th><th>Matter Content d</th><th>Superpotential</th></tr>
<tr><td>$(F_0)_1$</td><td>$\begin{array}{c|cccccccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} \\ \hline A & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & 1 & 1 \\ B & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ D & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}$</td><td>$X_1X_8X_{10} - X_3X_7X_{10} - X_2X_8X_9 - X_1X_6X_{12} +$
$X_3X_6X_{11} + X_4X_7X_9 + X_2X_5X_{12} - X_4X_5X_{11}$</td></tr>
<tr><td>$(F_0)_{2,3}$</td><td>$\begin{array}{c|cccccccc} & X_{112} & Y_{122} & Y_{222} & Y_{111} & Y_{211} & X_{121} & X_{212} & X_{221} \\ \hline A & -1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ B & 1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 \\ D & 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 \end{array}$</td><td>$\epsilon^{ij}\epsilon^{kl}X_{i\ 12}Y_{k\ 22}X_{j\ 21}Y_{l\ 11}$</td></tr>
<tr><td>$(dP_0)_{1,2,3,4,5}$</td><td>$\begin{array}{c|ccccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 \\ \hline A & -1 & 0 & -1 & 0 & -1 & 0 & 1 & 1 & 1 \\ B & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ C & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \end{array}$</td><td>$X_1X_4X_9 - X_4X_5X_7 - X_2X_3X_9 -$
$X_1X_6X_8 + X_2X_5X_8 + X_3X_6X_7$</td></tr>
<tr><td>$(dP_1)_{1,2,\ldots,12}$</td><td>$\begin{array}{c|cccccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} \\ \hline A & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 \\ B & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ C & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ D & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \end{array}$</td><td>$X_2X_7X_9 - X_3X_6X_9 - X_4X_8X_7 - X_1X_2X_5X_{10}$
$+X_3X_4X_{10} + X_1X_5X_6X_8$</td></tr>
<tr><td>$(dP_2)_{1,5,9}$</td><td>$\begin{array}{c|ccccccccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} \\ \hline A & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ B & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ D & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ E & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}$</td><td>$X_2X_9X_{11} - X_9X_3X_{10} - X_4X_8X_{11} - X_1X_2X_7X_{13} + X_{13}X_3X_6$
$-X_5X_{12}X_6 + X_1X_5X_8X_{10} + X_4X_7X_{12}$</td></tr>
<tr><td>$(dP_2)_{2,3,4,6,7,8}$</td><td>$\begin{array}{c|ccccccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} \\ \hline A & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ B & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ C & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ D & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & 0 \\ E & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{array}$</td><td>$X_5X_8X_6X_9 + X_1X_2X_{10}X_7 + X_{11}X_3X_4$
$-X_4X_{10}X_6 - X_2X_8X_7X_3X_9 - X_{11}X_1X_5$</td></tr>
<tr><td>$(dP_3)_1$</td><td>$\begin{array}{c|cccccccccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} & X_{14} \\ \hline A & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 \\ B & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ C & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ D & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ E & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ F & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{array}$</td><td>$X_3X_8X_{13} - X_8X_9X_{11} - X_5X_6X_{13} - X_1X_3X_4X_{10}X_{12}$
$+X_7X_9X_{12} + X_1X_2X_5X_{10}X_{11} + X_4X_6X_{14} - X_2X_7X_{14}$</td></tr>
</table>

The matter content for these above theories are represented as quiver diagrams in Figure 20-7 (multi-valence arrows are labelled with a number) and the superpotentials, in the table below.

In all of the above discussions, we have restricted ourselves to the cases of $U(1)$ gauge groups, i.e., with only a single brane probe; this is because such is the only case to which the toric technique can be applied. However, after we obtain the matter contents and superpotential for $U(1)$ gauge groups, we should have some idea for multi-brane probes. One obvious generalization is to replace the $U(1)$ with $SU(N)$

gauge groups directly. For the matter content, the generalization is not so easy. A field with charge $(1, -1)$ under gauge groups $U(1)_A \times U(1)_B$ and zero for others generalised to a bifundamental $(N, \bar{N})$ of $SU(N)_A \times SU(N)_B$. However, for higher charges, e.g., charge 2, we simply do not know what should be the generalization in the multi-brane case (for a discussion on generalised quivers cf. e.g. [297]). Furthermore, a field with zero charge under all $U(1)$ groups, generalises to an adjoint of one $SU(N)$ gauge group in the multi-brane case, though we do not know which one.

The generalization of the superpotential is also not so straight-forward. For example, there is a quartic term in the conifold with nonabelian gauge group [276, 273], but it disappears when we go to the $U(1)$ case. The same phenomenon can happen when treating the generic toric singularity.

For the examples we give in this chapter however, we do not see any obvious obstruction in the matter contents and superpotential; they seem to be special enough to be trivially generalized to the multi-brane case; they are all charge ± 1 under no more than 2 groups. We simply replace $U(1)$ with $SU(N)$ and $(1, -1)$ fields with bifundamentals while keeping the superpotential invariant. Generalisations to multi-brane stack have also been discussed in [277].

20.6 Discussions and Prospects

It is well-known that in the study of the world-volume gauge theory living on a D-brane probing an orbifold singularity $\mathbb{C}^3/\Gamma$, different choices of decomposition into irreducibles of the space-time action of Γ lead to different matter content and interaction in the gauge theory and henceforth different moduli spaces (as different algebraic varieties). This strong relation between the decomposition and algebraic variety has been shown explicitly for Abelian orbifolds in [45]. It seems that there is only one gauge theory for each given singularity.

A chief motivation and purpose of this chapter is the realisation that the above strong statement can not be generalised to arbitrary (non-orbifold) singularities and in particular toric singularities. It is possible that there are several gauge theories on the D-brane probing the same singularity. The moduli space of these inequivalent theories are indeed by construction the same, as dictated by the geometry of the singularity.

In analogy to the freedom of decomposition into irreps of the group action in the orbifold case, there too exists a freedom in toric singularities: any toric diagram is defined only up to a unimodular transformation (Theorem 20.3.33). We harness this toric isomorphism as a tool to create inequivalent gauge theories which live on the D-brane probe and which, by construction, flow to the same (toric) moduli space in the IR.

Indeed, these theories constitute another sub-class of examples of *toric duality* as proposed in [298]. A key point to note is that unlike the general case of the duality (such as F-D ambiguities and repetition ambiguities as discussed therein) of which we have hitherto little control, these particular theories are all physical (i.e., guaranteed to be world-volume theories) by virtue of their being obtainable from the canonical method of partial resolution of Abelian orbifolds. We therefore refer to them as *phases* of partial resolution.

As a further tool, we have re-examined the Forward and Inverse Algorithms developed in [277, 298, 74] of extracting the gauge theory data and toric moduli space data from each other. In particular we have taken the pains to show what *degree of freedom* can one have at each step of the Algorithm. This will serve to discriminate whether or not two theories are physically equivalent given their respective matrices at each step.

Thus equipped, we have re-studied the partial resolutions of the Abelian orbifold $\mathbb{C}^3/(\mathbb{Z}_3\times\mathbb{Z}_3)$, namely the 4 toric del Pezzo surfaces $dP_{0,1,2,3}$ and the zeroth Hirzebruch surface F_0. We performed all possible $SL(3;\mathbb{Z})$ transformation of these toric diagrams which are up to permutation still embeddable in $\mathbb{Z}_3 \times \mathbb{Z}_3$ and subsequently initiated the Inverse Algorithm therewith. We found at the end of the day, in addition to the physical theories for these examples presented in [298], an additional one for both F_0 and dP_2. Further embedding can of course be done, viz., into $\mathbb{Z}_n \times \mathbb{Z}_n$ for $n > 3$; it is expected that more phases would arise for these computationally prohibitive cases, for example for dP_3.

A clear goal awaits us: because for the generic (non-orbifold) toric singularity there is no concrete concept corresponding to the different decomposition of group action, we do not know at this moment how to classify the phases of toric duality. We certainly wish, given a toric singularity, to know (a) how many inequivalent gauge theory are there and (b) what are the corresponding matter contents and superpotential. It will be a very interesting direction for further investigation.

Many related questions also arise. For example, by the AdS/CFT correspondence, we need to understand how to describe these different gauge theories on the supergravity side while the underline geometry is same. Furthermore the dP^2 theory can be described in the brane setup by (p, q)-5 brane webs [278], so we want to ask how to understand these different phases in such brane setups. Understanding these will help us to get the gauge theory in higher del Pezzo surface singularities.

Another very pertinent issue is to clarify the meaning of "toric duality." So far it is merely an equivalence of moduli spaces of gauge theories in the IR. It would be very nice if we could make this statement stronger. For example, could we find the explicit mappings between gauge invariant operators of various toric-dual theories? Indeed, we believe that the study of toric duality and its phase structure is worth further pursuit.

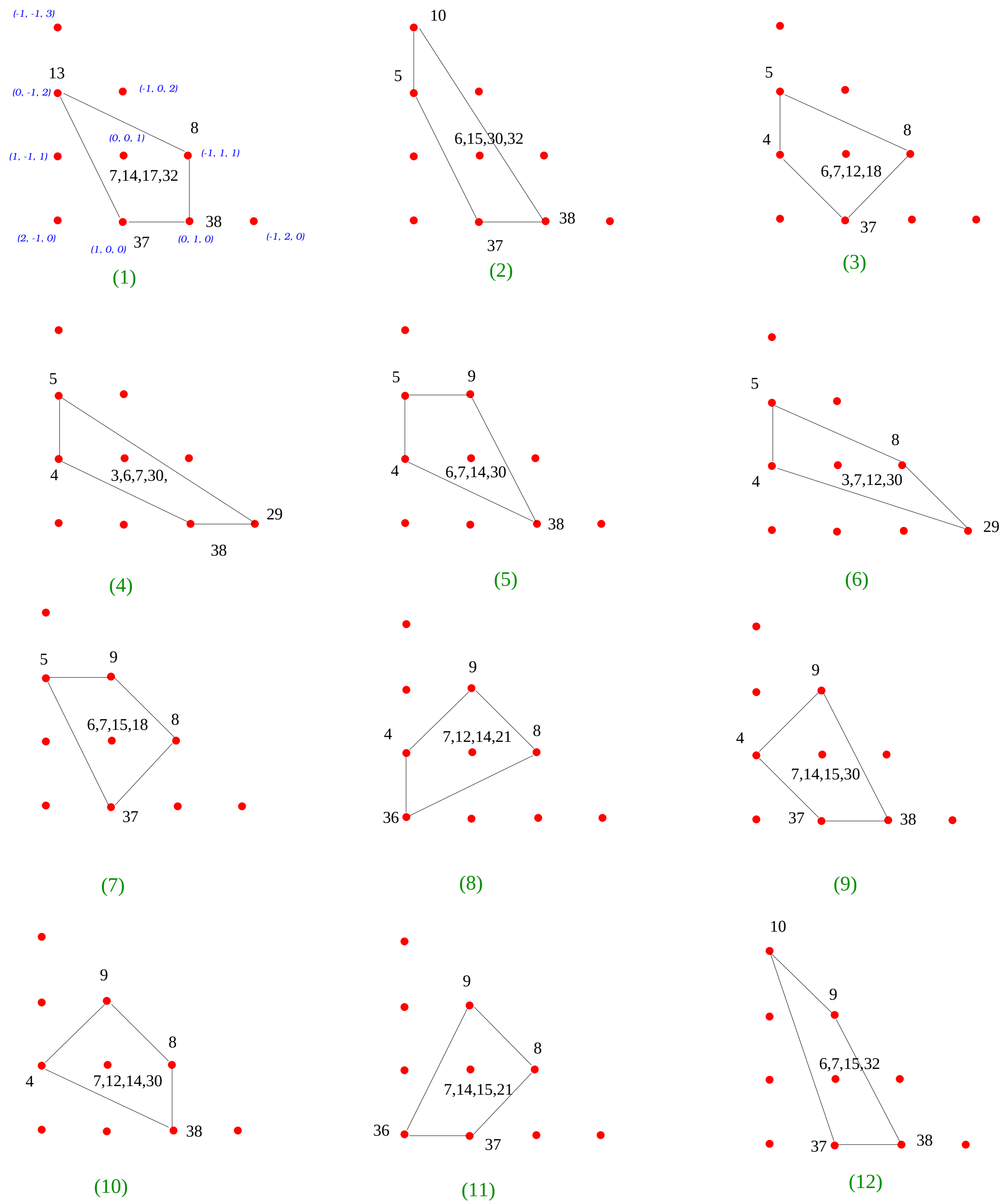


Figure 20-4: The 12 equivalent representations of the toric diagram of the first del Pezzo surface as a resolution of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The pairs (1,5); (2,4); (3,9); (6,12); (7,10) and (8,11) are each reflected by the 45^o line and give mutually equivalent gauge theories indeed. Further analysis shows that all 12 are equivalent.

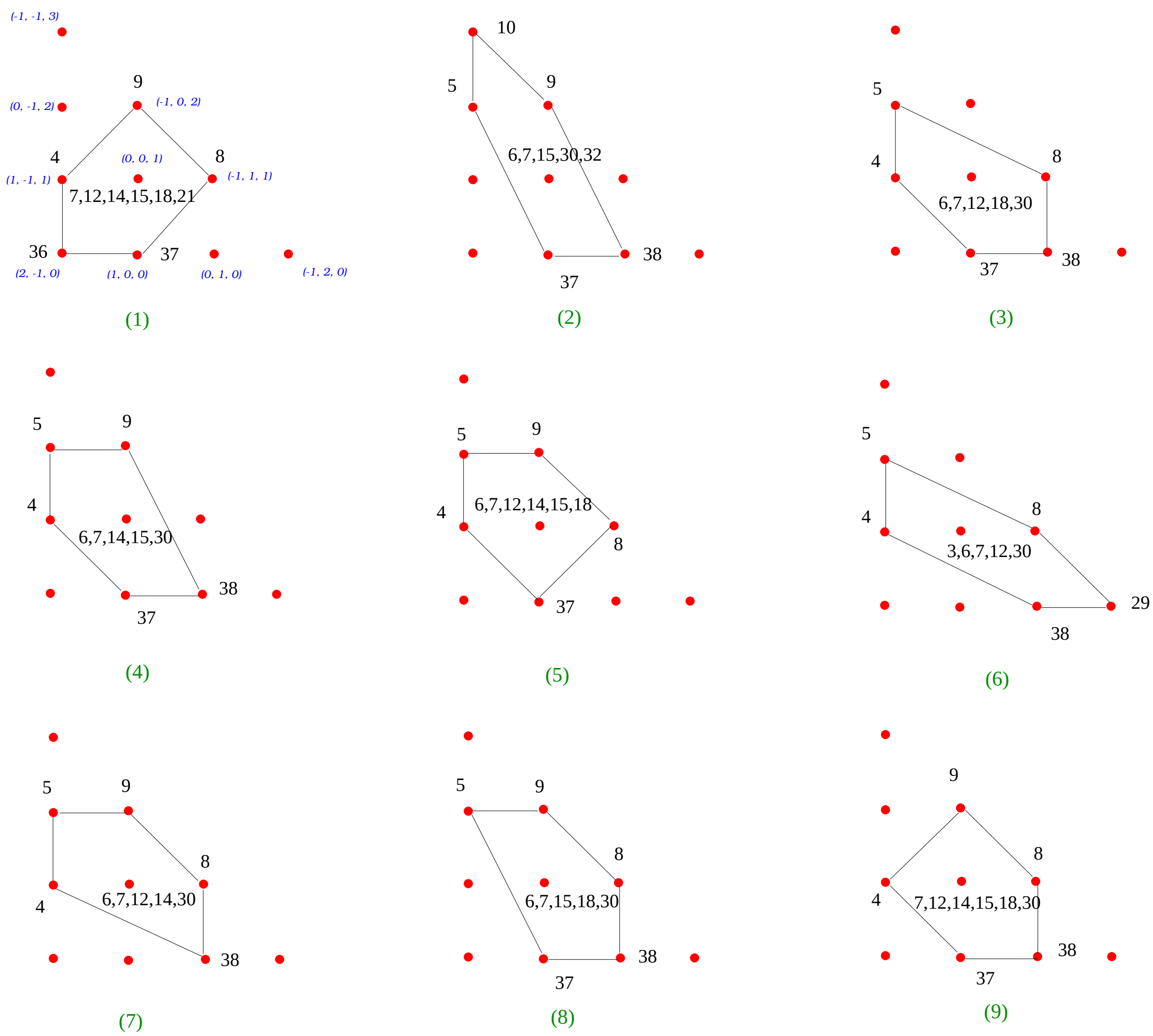


Figure 20-5: The 9 equivalent representations of the toric diagram of the second del Pezzo surface as a resolution of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The pairs (2,6); (3,4); (5,9) and (7,8) are related by 45^o reflection while (1) is self-reflexive and are hence give pairwise equivalent theories. Further analysis shows that there are two phases given respectively by (1,5,9) and (2,3,4,6,7,8).

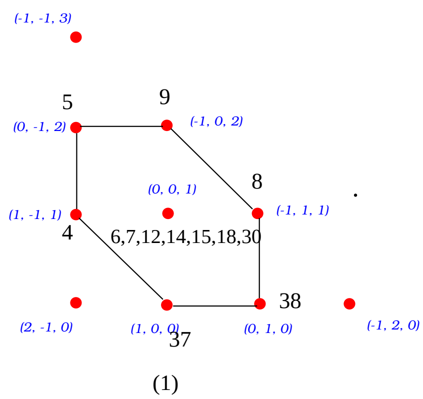


Figure 20-6: The unique representations of the toric diagram of the third del Pezzo surface as a resolution of $\mathbb{Z}_3 \times \mathbb{Z}_3$.

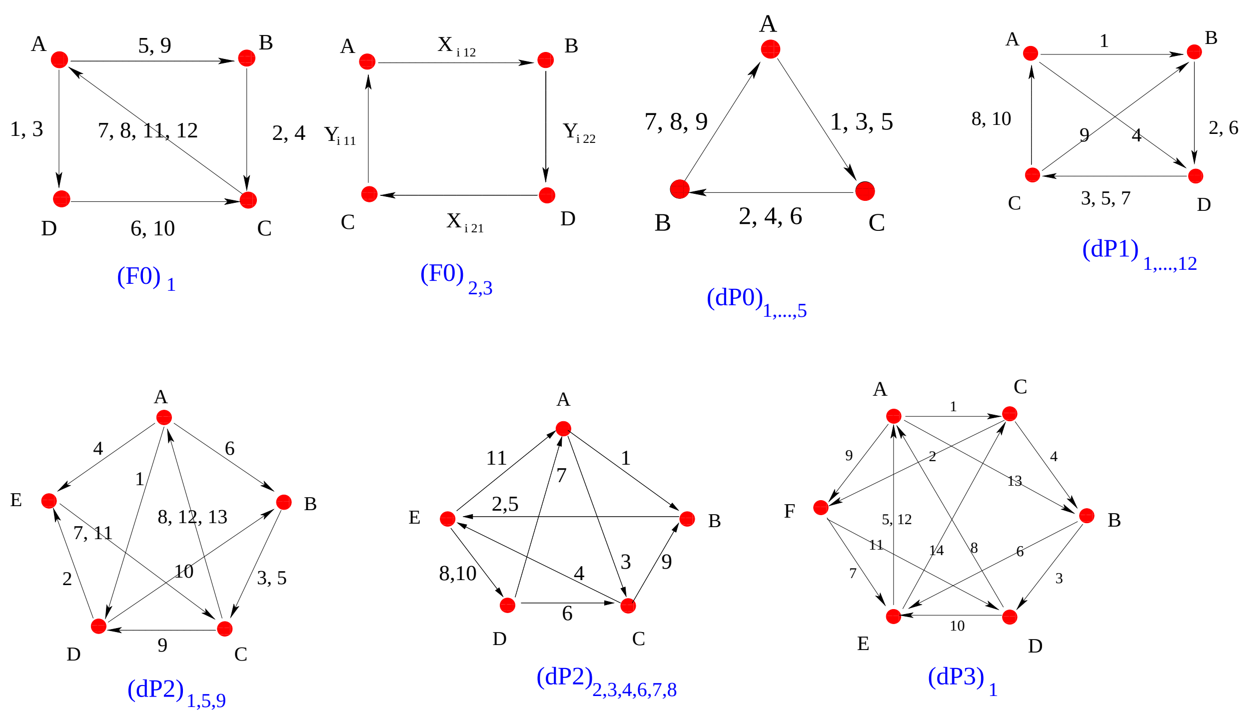


Figure 20-7: The quiver diagrams for the various phases of the gauge theory for the del Pezzo surfaces and the zeroth Hirzebruch surface.

Chapter 21

Toric III: Toric Duality and Seiberg Duality

Synopsis

What then is Toric Duality, as proposed in our previous two chapters?

We use field theory and brane diamond techniques to demonstrate that Toric Duality is Seiberg duality for $\mathcal{N} = 1$ theories with toric moduli spaces. This resolves the puzzle concerning the physical meaning of Toric Duality.

Furthermore, using this strong connection we arrive at three new phases which can not be thus far obtained by the so-called "Inverse Algorithm" applied to partial resolution of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. The standing proposals of Seiberg duality as diamond duality in the work by Aganagic-Karch-Lüst-Miemiec are strongly supported and new diamond configurations for these singularities are obtained as a byproduct. We also make some remarks about the relationships between Seiberg duality and Picard-Lefschetz monodromy [308].

21.1 Introduction

Witten's gauge linear sigma approach [17] to $\mathcal{N} = 2$ super-conformal theories has provided deep insight not only to the study of the phases of the field theory but also to the understanding of the mathematics of Geometric Invariant Theory quotients in toric geometry. Thereafter, the method was readily applied to the study of the $\mathcal{N} = 1$ supersymmetric gauge theories on D-branes at singularities [74, 157, 276, 277]. Indeed the classical moduli space of the gauge theory corresponds precisely to the spacetime which the D-brane probes transversely. In light of this therefore, toric geometry has been widely used in the study of the moduli space of vacua of the gauge theory living on D-brane probes.

The method of encoding the gauge theory data into the moduli data, or more specifically, the F-term and D-term information into the toric diagram of the algebraic variety describing the moduli space, has been well-established [74, 157]. The reverse, of determining the SUSY gauge theory data in terms of a given toric singularity upon which the D-brane probes, has also been addressed using the method partial resolutions of abelian quotient singularities. Namely, a general non-orbifold singularity is regarded as a partial resolution of a worse, but orbifold, singularity. This "Inverse Procedure" was formalised into a linear optimisation algorithm, easily implementable on computer, by [298], and was subsequently checked extensively in [279].

One feature of the Inverse Algorithm is its non-uniqueness, viz., that for a given toric singularity, one could in theory construct countless gauge theories. This means that there are classes of gauge theories which have identical toric moduli space in the IR. Such a salient feature was dubbed in [298] as **toric duality**. Indeed in a follow-up work, [306] attempted to analyse this duality in detail, concentrating in particular on a method of fabricating dual theories which are physical, in the sense that they can be realised as world-volume theories on D-branes. Henceforth, we shall adhere to this more restricted meaning of toric duality.

Because the details of this method will be clear in later examples we shall not delve into the specifics here, nor shall we devote too much space reviewing the algorithm. Let us highlight the key points. The gauge theory data of D-branes probing Abelian orbifolds is well-known (see e.g. the appendix of [306]); also any toric diagram can be embedded into that of such an orbifold (in particular any toric local Calabi-Yau threefold D can be embedded into $\mathbb{C}^3/(\mathbb{Z}_n \times \mathbb{Z}_n)$ for sufficiently large n. We can then obtain the subsector of orbifold theory that corresponds the gauge theory constructed for D. This is the method of "Partial Resolution."

A key point of [306] was the application of the well-known mathematical fact that the toric diagram D of any toric variety has an inherent ambiguity in its definition: namely any unimodular transformation on the lattice on which D is defined must leave D invariant. In other words, for threefolds defined in the standard lattice $\mathbb{Z}^3$, any $SL(3;\mathbb{C})$ transformation on the vector endpoints of the defining toric diagram gives the same toric variety. Their embedding into the diagram of a fixed Abelian orbifold on the other hand, certainly is different. Ergo, the gauge theory data one obtains in general are vastly different, even though per constructio, they have the same toric moduli space.

What then is this "toric duality"? How clearly it is defined mathematically and yet how illusive it is as a physical phenomenon. The purpose of the present writing is to make the first leap toward answering this question. In particular, we shall show, using brane setups, and especially brane diamonds, that known cases for toric duality are actually interesting realisations of Seiberg Duality. Therefore the mathematical equivalence of moduli spaces for different quiver gauge theories is related to a real physical equivalence of the gauge theories in the far infrared.

The chapter is organised as follows. In Section 2, we begin with an illustrative example of two torically dual cases of a generalised conifold. These are well-known to be Seiberg dual theories as seen from brane setups. Thereby we are motivated to conjecture in Section 3 that toric duality is Seiberg duality. We proceed to check this proposal in Section 4 with all the known cases of torically dual theories and have successfully shown that the phases of the partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ constructed in [298] are indeed Seiberg dual from a field theory analysis. Then in Section 6 we re-analyse these examples from the perspective of brane diamond configurations and once again obtain strong support of the statement. From rules used in the diamond dualisation, we extracted a so-called "quiver duality" which explicits Seiberg duality as a transformation on the matter adjacency matrices. Using these rules we are able to extract more phases of theories not yet obtained from the Inverse Algorithm. In a more geometrical vein, in Section 7, we remark the connection between Seiberg duality and Picard-Lefschetz and point out cases where the two phenomena may differ. Finally we finish with conclusions and prospects in Section 8.

While this manuscript is about to be released, we became aware of the nice work [290], which discusses similar issues.

21.2 An Illustrative Example

We begin with an illustrative example that will demonstrate how Seiberg Duality is realised as toric duality.

21.2.1 The Brane Setup

The example is the well-known generalized conifold described as the hypersurface $xy = z^2w^2$ in $\mathbb{C}^4$, and which can be obtained as a $\mathbb{Z}_2$ quotient of the famous conifold $xy = zw$ by the action $z \to -z, w \to -w$. The gauge theory on the D-brane sitting at such a singularity can be established by orbifolding the conifold gauge theory in [212], as in [218]. Also, it can be derived by another method alternative to the Inverse Algorithm, namely performing a T-duality to a brane setup with NS-branes and D4-branes [218, 219]. Therefore this theory serves as an excellent check on our methods.

The setup involves stretching D4 branes (spanning 01236) between 2 pairs of NS and NS′ branes (spanning 012345 and 012389, respectively), with x^6 parameterizing a circle. These configurations are analogous to those in [175]. There are in fact two inequivalent brane setups (a) and (b) (see Figure 21-1), differing in the way the NS- and NS′-branes are ordered in the circle coordinate. Using standard rules [66, 175],

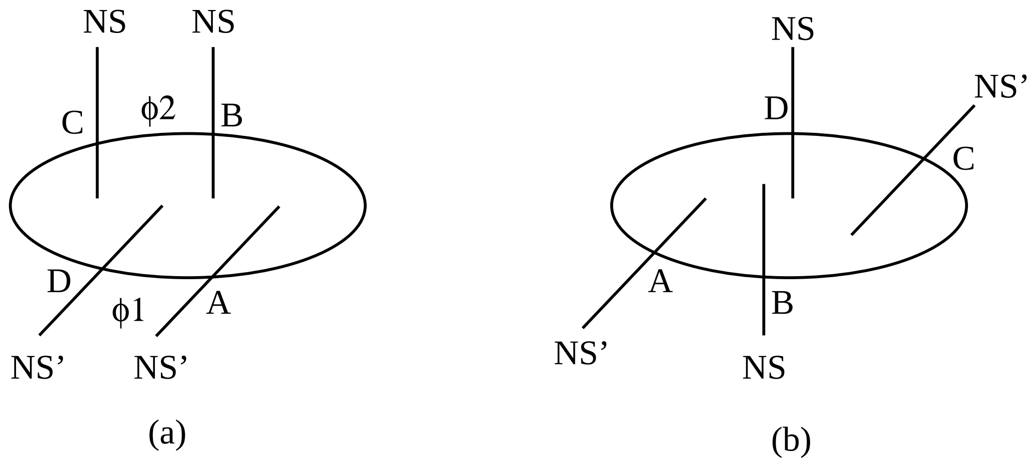


Figure 21-1: The two possible brane setups for the generalized conifold $xy = z^2w^2$. They are related to each other passing one NS-brane through an NS'-brane. A_i, B_i, C_i, D_i $i = 1, 2$ are bifundamentals while ϕ_1, ϕ_2 are two adjoint fields.

we see from the figure that there are 4 product gauge groups (in the Abelian case, it is simply $U(1)^4$. As for the matter content, theory (a) has 8 bi-fundamental chiral multiplets A_i, B_i, C_i, D_i $i = 1, 2$ (with charge $(+1, -1)$ and $(-1, +1)$ with respect to adjacent $U(1)$ factors) and 2 adjoint chiral multiplets $\phi_{1,2}$ as indicated. On the other hand (b) has only 8 bi-fundamentals, with charges as above. The superpotentials are respectively [239, 218]

$$\begin{array}{lll} (a) & W_a = & -A_1A_2B_1B_2 + B_1B_2\phi_2 - C_1C_2\phi_2 + C_1C_2D_1D_2 - D_1D_2\phi_1 + A_1A_2\phi_1, \\ (b) & W_b = & A_1A_2B_1B_2 - B_1B_2C_1C_2 + C_1C_2D_1D_2 - D_1D_2A_1A_2 \end{array}$$

With some foresight, for comparison with the results later, we rewrite them as

$$W_a = (B_1B_2 - C_1C_2)(\phi_2 - A_1A_2) + (A_1A_2 - D_1D_2)(\phi_1 - C_1C_2) \qquad (21.2.1)$$

$$W_b = (A_1A_2 - C_1C_2)(B_1B_2 - D_1D_2) \qquad (21.2.2)$$

21.2.2 Partial Resolution

Let us see whether we can reproduce these field theories with the Inverse Algorithm. The toric diagram for $xy = z^2w^2$ is given in the very left of Figure 21-2. Of course, the hypersurface is three complex-dimensional so there is actually an undrawn apex for the toric diagram, and each of the nodes is in fact a three-vector in $\mathbb{Z}^3$. Indeed the fact that it is locally Calabi-Yau that guarantees all the nodes to be coplanar. The next step is the realisation that it can be embedded into the well-known toric diagram for the Abelian orbifold $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ consisting of 10 lattice points. The reader is referred to [298, 306] for the actual coördinates of the points, a detail which, though crucial, we shall not belabour here.

The important point is that there are six ways to embed our toric diagram into the orbifold one, all related by $SL(3;\mathbb{C})$ transformations. This is indicated in parts (a)-(f) of Figure 21-2. We emphasise that these six diagrams, drawn in red, are *equivalent* descriptions of $xy = z^2w^2$ by virtue of their being unimodularly related; therefore they are all candidates for toric duality.

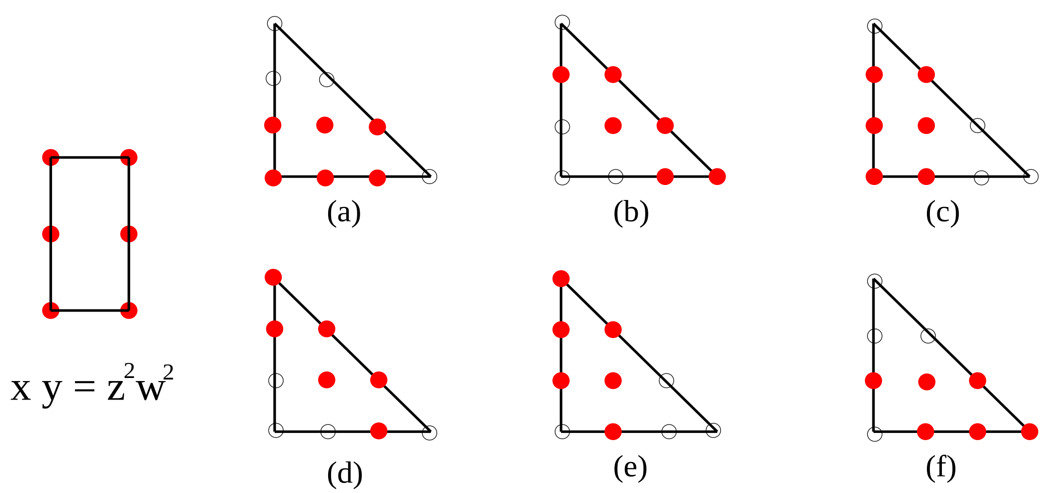


Figure 21-2: The standard toric diagram for the generalized conifold $xy = uv = z^2$ (far left). To the right are six $SL(3;\mathbb{C})$ transformations (a)-(f) thereof (drawn in red) and hence are equivalent toric diagrams for the variety. We embed these six diagrams into the Abelian orbifold $\mathbb{C}^3/(\mathbb{Z}_3\times\mathbb{Z}_3)$ in order to perform partial resolution and thus the gauge theory data.

Now we use our Inverse Algorithm, by partially resolving $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$, to obtain the gauge theory data for the D-brane probing $xy = z^2w^2$. In summary, after exploring the six possible partial resolutions, we find that cases (a) and (b) give identical results, while (c,d,e,f) give the same result which is inequivalent from (a,b). Therefore we conclude that cases (a) and (c) are inequivalent torically dual theories for $xy = z^2w^2$. In the following we detail the data for these two contrasting cases. We refer the reader to [298, 306] for details and notation.

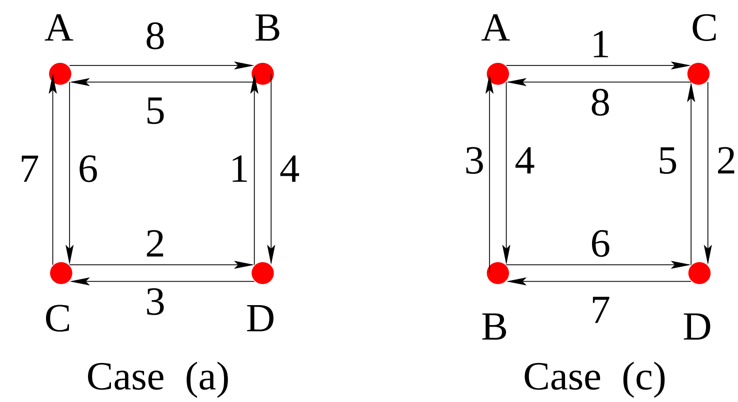


Figure 21-3: The quiver diagram encoding the matter content of Cases (a) and (c) of Figure 21-2.

21.2.3 Case (a) from Partial Resolution

For case (a), the matter content is encoded the d-matrix which indicates the charges of the 8 bi-fundamentals under the 4 gauge groups. This is the incidence matrix for the quiver diagram drawn in part (a) of Figure 21-3.

$$\begin{array}{c} \\ U(1)_A \\ U(1)_B \\ U(1)_C \\ U(1)_D \end{array} \begin{array}{c} \begin{array}{cccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \end{array} \\ \left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

On the other hand, the F-terms are encoded in the K-matrix

$$\begin{array}{cccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

From K we get two relations $X_5X_8 = X_6X_7$ and $X_1X_4 = X_2X_3$ (these are the relations one must impose on the quiver to obtain the final variety; equivalently, they correspond to the F-term constraints arising from the superpotential). Notice that here each term is chargeless under all 4 gauge groups, so when we integrate back to get the superpotential, we should multiply by chargeless quantities also[1].

The relations must come from the F-flatness $\frac{\partial}{\partial X_i}W = 0$ and thus we can use these relations to integrate back to the superpotential W. However we meet some ambiguities [2]. In principle we can have two different choices:

$$\begin{array}{rrl} (i) & W_1 = & (X_5X_8 - X_6X_7)(X_1X_4 - X_2X_3) \\ (ii) & W_2 = & \psi_1(X_5X_8 - X_6X_7) + \psi_2(X_1X_4 - X_2X_3) \end{array}$$

where for now ψ_i are simply chargeless fields.

We shall evoke physical arguments to determine which is correct. Expanding (i) gives $W_1 = X_5X_8X_1X_4 - X_6X_7X_1X_4 - X_5X_8X_2X_3 + X_6X_7X_2X_3$. Notice the term $X_6X_7X_1X_4$: there is no common gauge group under which there four fields are charged, i.e. these 4 arrows (q. v. Figure 21-3) do not intersect at a single node. This makes (i) very unnatural and exclude it.

Case (ii) does not have the above problem and indeed all four fields X_5, X_8, X_6, X_7

[1]In more general situations the left- and right-hand sides may not be singlets, but transform in the same gauge representation.

[2]The ambiguities arise because in the abelian case (toric language) the adjoints are chargeless. In fact, no ambiguity arises if one performs the Higgsing associated to the partial resolution in the non-abelian case. We have performed this exercise in cases (a) and (c), and verified the result obtained by the different argument offered in the text.

are charged under the $U(1)_A$ gauge group, so considering ψ_1 to be an adjoint of $U(1)_A$, we do obtain a physically meaningful interaction. Similarly ψ_2 will be the adjoint of $U(1)_D$, interacting with X_1, X_4, X_2, X_3.

However, we are not finish yet. From Figure 21-3 we see that X_5, X_8, X_1, X_4 are all charged under $U(1)_B$, while X_6, X_7, X_2, X_3 are all charged under $U(1)_C$. From a physical point of view, there should be some interaction terms between these fields. Possibilities are $X_5X_8X_1X_4$ and $X_6X_7X_2X_3$. To add these terms into W_2 is very easy, we simply perform the following replacement:[3] $\psi_1 \longrightarrow \psi_1 - X_1X_4, \qquad \psi_2 \longrightarrow \psi_2 - X_6X_7$. Putting everything together, we finally obtain that Case (a) has matter content as described in Figure 21-3 and the superpotential

$$W = (\psi_1 - X_1X_4)(X_5X_8 - X_6X_7) + (\psi_2 - X_6X_7)(X_1X_4 - X_2X_3) \tag{21.2.3}$$

This is precisely the theory (a) from the brane setup in the last section! Comparing (21.2.3) with (21.2.1), we see that they are exact same under the following redefinition of variables:

$$\begin{array}{ccc} B_1, B_2 \Longleftrightarrow X_5, X_8 & C_1, C_2 \Longleftrightarrow X_6, X_7 & D_1, D_2 \Longleftrightarrow X_2, X_3 \\ A_1, A_2 \Longleftrightarrow X_1, X_4 & \phi_2 \Longleftrightarrow \psi_1 & \phi_1 \Longleftrightarrow \psi_2 \end{array}$$

In conclusion, case (a) of our Inverse Algorithm reproduces the results of case (a) of the brane setup.

[3] Here we choose the sign purposefully for later convenience. However, we do need, for the cancellation of the unnatural interaction term $X_1X_4X_6X_7$, that they both have the same sign.

21.2.4 Case (c) from Partial Resolution

For case (c), the matter content is given by the quiver in Figure 21-3, which has the charge matrix d equal to

$$\begin{array}{c|cccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ \hline U(1)_A & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \\ U(1)_B & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 \\ U(1)_C & 1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \\ U(1)_D & 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 \end{array}$$

This is precisely the matter content of case (b) of the brane setup. The F-terms are given by

$$K = \begin{array}{cccccccc} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ \hline 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}$$

From it we can read out the relations $X_1X_8 = X_6X_7$ and $X_2X_5 = X_3X_4$. Again there are two ways to write down the superpotential

$$\begin{array}{lrl} (i) & W_1 = & (X_1X_8 - X_6X_7)(X_3X_4 - X_2X_5) \\ (ii) & W_2 = & \psi_1(X_1X_8 - X_6X_7) + \psi_2(X_3X_4 - X_2X_5) \end{array}$$

In this case, because X_1, X_8, X_6, X_7 are not charged under any common gauge group, it is impossible to include any adjoint field ψ to give a physically meaningful interaction and so (ii) is unnatural. We are left the superpotential W_1. Indeed, comparing with (21.2.2), we see they are identical under the redefinitions

$$\begin{array}{ll} A_1, A_2 \Longleftrightarrow X_1, X_8 & \qquad B_1, B_2 \Longleftrightarrow X_3, X_4 \\ C_1, C_2 \Longleftrightarrow X_6, X_7 & \qquad D_1, D_2 \Longleftrightarrow X_2, X_5 \end{array}$$

Therefore we have reproduced case (b) of the brane setup.

What have we achieved? We have shown that toric duality due to inequivalent embeddings of unimodularly related toric diagrams for the generalized conifold $xy =$

z^2w^2 gives two inequivalent physical world-volume theories on the D-brane probe, exemplified by cases (a) and (c). On the other hand, there are two T-dual brane setups for this singularity, also giving two inequivalent field theories (a) and (b). Upon comparison, case (a) (resp. (c)) from the Inverse Algorithm beautifully corresponds to case (a) (resp. (b)) from the brane setup. Somehow, a seemingly harmless trick in mathematics relates inequivalent brane setups. In fact we can say much more.

21.3 Seiberg Duality versus Toric Duality

As follows from [175], the two theories from the brane setups are actually related by Seiberg Duality [280], as pointed out in [218] (see also [274, 216]. Let us first review the main features of this famous duality, for unitary gauge groups.

Seiberg duality is a non-trivial infrared equivalence of $\mathcal{N} = 1$ supersymmetric field theories, which are different in the ultraviolet, but flow the the same interacting fixed point in the infrared. In particular, the very low energy features of the different theories, like their moduli space, chiral ring, global symmetries, agree for Seiberg dual theories. Given that toric dual theories, by definition, have identical moduli spaces, etc , it is natural to propose a connection between both phenomena.

The prototypical example of Seiberg duality is $\mathcal{N} = 1$ $SU(N_c)$ gauge theory with N_f vector-like fundamental flavours, and no superpotential. The global chiral symmetry is $SU(N_f)_L \times SU(N_f)_R$, so the matter content quantum numbers are

	$SU(N_c)$	$SU(N_f)_L$	$SU(N_f)_R$
Q	$\square$	$\square$	1
Q'	$\overline{\square}$	1	$\overline{\square}$

In the conformal window, $3N_c/2 \leq N_f \leq 3N_c$, the theory flows to an interacting infrared fixed point. The dual theory, flowing to the same fixed point is given $N = 1$

$SU(N_f - N_c)$ gauge theory with N_f fundamental flavours, namely

	$SU(N_f - N_c)$	$SU(N_f)_L$	$SU(N_f)_R$
q	$\square$	$\bar{\square}$	1
q'	$\bar{\square}$	1	$\square$
M	1	$\square$	$\bar{\square}$

and superpotential $W = Mqq'$. From the matching of chiral rings, the 'mesons' M can be thought of as composites QQ' of the original quarks.

It is well established [175], that in an $\mathcal{N} = 1$ (IIA) brane setup for the four dimensional theory such as Figure 21-1, Seiberg duality is realised as the crossing of 2 non-parallel NS-NS′ branes. In other words, as pointed out in [218], cases (a) and (b) are in fact a Seiberg dual pair. Therefore it seems that the results from the previous section suggest that toric duality is a guise of Seiberg duality, for theories with moduli space admitting a toric descriptions. It is therefore the intent of the remainder of this chapter to examine and support

CONJECTURE 21.3.3 *Toric duality is Seiberg duality for $\mathcal{N} = 1$ theories with toric moduli spaces.*

21.4 Partial Resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ and Seiberg duality

Let us proceed to check more examples. So far the other known examples of torically dual theories are from various partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. In particular it was found in [306] that the (complex) cones over the zeroth Hirzebruch surface as well as the second del Pezzo surface each has two toric dual pairs. We remind the reader of these theories.

21.4.1 Hirzebruch Zero

There are two torically dual theories for the cone over the zeroth Hirzebruch surface F_0. The toric and quiver diagrams are given in Figure 21-4, the matter content and interactions are

	Matter Content d	Superpotential
I	see table I below	$X_1X_8X_{10} - X_3X_7X_{10} - X_2X_8X_9 - X_1X_6X_{12} + X_3X_6X_{11} + X_4X_7X_9 + X_2X_5X_{12} - X_4X_5X_{11}$
II	see table II below	$\epsilon^{ij}\epsilon^{kl}X_{i\ 12}Y_{k\ 22}X_{j\ 21}Y_{l\ 11}$

I:

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}	X_{11}	X_{12}
A	-1	0	-1	0	-1	0	1	1	-1	0	1	1
B	0	-1	0	-1	1	0	0	0	1	0	0	0
C	0	1	0	1	0	1	-1	-1	0	1	-1	-1
D	1	0	1	0	0	-1	0	0	0	-1	0	0

II:

	X_{112}	Y_{122}	Y_{222}	Y_{111}	Y_{211}	X_{121}	X_{212}	X_{221}
A	-1	0	0	1	1	0	-1	0
B	1	-1	-1	0	0	0	1	0
C	0	0	0	-1	-1	1	0	1
D	0	1	1	0	0	-1	0	-1

(21.4.4)

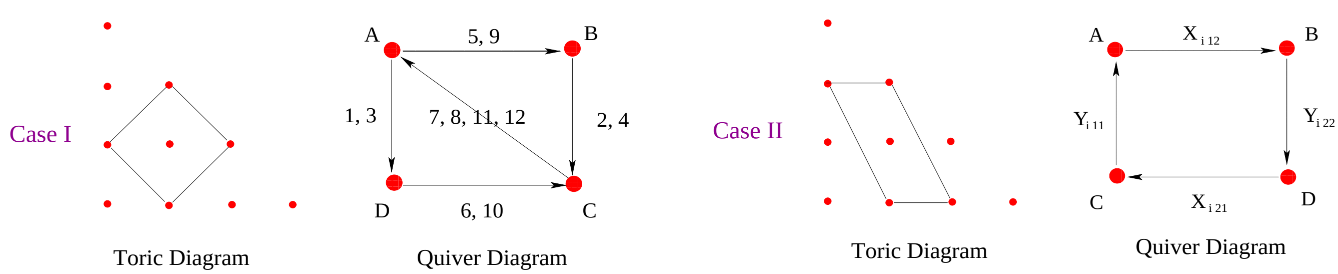


Figure 21-4: The quiver and toric diagrams of the 2 torically dual theories corresponding to the cone over the zeroth Hirzebruch surface F_0.

Let us use the field theory rules from Section 3 on Seiberg Duality to examine these two cases in detail. The charges of the matter content for case II, upon promotion from $U(1)$ to $SU(N)$ [4] (for instance, following the partial resolution in the non-abelian case, as in [157, 276]), can be re-written as (redefining fields $(X_i, Y_i, Z_i, W_i) := (X_{i\ 12}, Y_{i\ 22}, X_{i\ 21}, Y_{i\ 11})$ with $i = 1, 2$ and gauge groups $(a, b, c, d) := (A, C, B, D)$ for

[4]Concerning the $U(1)$ factors, these are in fact generically absent, since they are anomalous in the original $\mathbb{Z}_3 \times \mathbb{Z}_3$ singularity, and the Green-Schwarz mechanism canceling their anomaly makes them massive [209] (see [284, 69, 285] for an analogous 6d phenomenon). However, there is a well-defined sense in which one can use the abelian case to study the toric moduli space [157].

convenience):

	$SU(N)_a$	$SU(N)_b$	$SU(N)_c$	$SU(N)_d$
X_i	$\square$	$\overline{\square}$		
Y_i		$\square$	$\overline{\square}$	
Z_i			$\square$	$\overline{\square}$
W_i	$\overline{\square}$			$\square$

The superpotential is then

$$W_{II} = X_1 Y_1 Z_2 W_2 - X_1 Y_2 Z_2 W_1 - X_2 Y_1 Z_1 W_2 + X_2 Y_2 Z_1 W_1.$$

Let us dualise with respect to the a gauge group. This is a $SU(N)$ theory with $N_c = N$ and $N_f = 2N$ (as there are two X_i's). The chiral symmetry is however broken from $SU(2N)_L \times SU(2N)_R$ to $SU(N)_L \times SU(N)_R$, which moreover is gauged as $SU(N)_b \times SU(N)_d$. Ignoring the superpotential W_{II}, the dual theory would be:

	$SU(N)_{a'}$	$SU(N)_b$	$SU(N)_c$	$SU(N)_d$
q_i	$\overline{\square}$	$\square$		
Y_i		$\square$	$\overline{\square}$	
Z_i			$\square$	$\overline{\square}$
q'_i	$\square$			$\overline{\square}$
M_{ij}		$\overline{\square}$		$\square$

(21.4.5)

We note that there are M_{ij} giving 4 bi-fundamentals for bd. They arise from the Seiberg mesons in the bi-fundamental of the enhanced chiral symmetry $SU(2N) \times SU(2N)$, once decomposed with respect to the unbroken chiral symmetry group. The superpotential is

$$W' = M_{11} q_1 q'_1 - M_{12} q_2 q'_1 - M_{21} q_1 q'_2 + M_{22} q_2 q'_2.$$

The choice of signs in W' will be explained shortly.

Of course, W_{II} is not zero and so give rise to a deformation in the original theory, analogous to those studied in e.g. [239]. In the dual theory, this deformation simply corresponds to W_{II} rewritten in terms of mesons, which can be thought of as composites of the original quarks, i.e., $M_{ij} = W_i X_j$. Therefore we have

$$W_{II} = M_{21}Y_1Z_2 - M_{11}Y_2Z_2 - M_{22}Y_1Z_1 + M_{12}Y_2Z_1$$

which is written in the new variables. The rule for the signs is that e.g. the field M_{21} appears with positive sign in W_{II}, hence it should appear with negative sign in W', and analogously for others. Putting them together we get the superpotential of the dual theory

$$\begin{aligned} W_{II}^{dual} &= W_{II} + W' = \\ & M_{11}q_1q_1' - M_{12}q_2q_1' - M_{21}q_1q_2' + M_{22}q_2q_2' + M_{21}Y_1Z_2 - M_{11}Y_2Z_2 - M_{22}Y_1Z_1 + M_{12}Y_2Z_1 \end{aligned} \tag{21.4.6}$$

Upon the field redefinitions

$$\begin{array}{cccc} M_{11} \to X_7 & M_{12} \to X_8 & M_{21} \to X_{11} & M_{22} \to X_{12} \\ q_1 \to X_4 & q_2 \to X_2 & q_{1'} \to X_9 & q_{2'} \to X_5 \\ Y_1 \to X_6 & Y_2 \to X_{10} & Z_1 \to X_1 & Z_2 \to X_3 \end{array}$$

we have the field content (21.4.5) and superpotential (21.4.6) matching precisely with case I in (21.4.4). We conclude therefore that the two torically dual cases I and II obtained from partial resolutions are indeed Seiberg duals!

21.4.2 del Pezzo 2

Encouraged by the results above, let us proceed with the cone over the second del Pezzo surface, which also have 2 torically dual theories. The toric and quiver diagrams are given in Figure 21-5.

	Matter Content d	Superpotential
I	$\begin{array}{c\|ccccccccccccc} & Y_1 & Y_2 & Y_3 & Y_4 & Y_5 & Y_6 & Y_7 & Y_8 & Y_9 & Y_{10} & Y_{11} & Y_{12} & Y_{13} \\ \hline A & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ B & 0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 & -1 & -1 \\ D & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ E & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \end{array}$	$Y_2Y_9Y_{11} - Y_9Y_3Y_{10} - Y_4Y_8Y_{11} - Y_1Y_2Y_7Y_{13} + Y_{13}Y_3Y_6$ $-Y_5Y_{12}Y_6 + Y_1Y_5Y_8Y_{10} + Y_4Y_7Y_{12}$
II	$\begin{array}{c\|ccccccccccc} & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 & X_9 & X_{10} & X_{11} \\ \hline A & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ B & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ C & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ D & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & 0 \\ E & 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & -1 \end{array}$	$X_5X_8X_6X_9 + X_1X_2X_{10}X_7 + X_{11}X_3X_4$ $-X_4X_{10}X_6 - X_2X_8X_7X_3X_9 - X_{11}X_1X_5$

(21.4.7)

Again we start with Case II. Working analogously, upon dualisation on node D

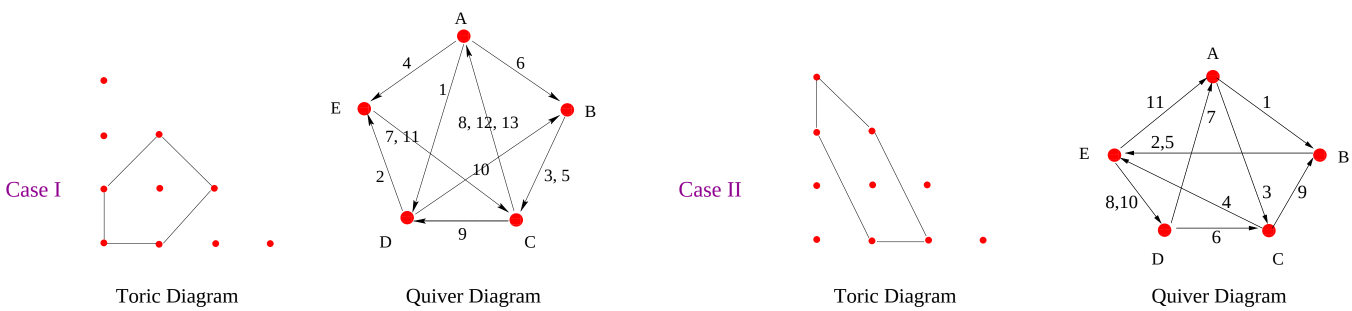


Figure 21-5: The quiver and toric diagrams of the 2 torically dual theories corresponding to the cone over the second del Pezzo surface.

neglecting the superpotential, the matter content of II undergoes the following change:

	$SU(N)_A$	$SU(N)_B$	$SU(N)_C$	$SU(N)_D$	$SU(N)_E$
X_1	$\overline{\square}$	$\square$			
X_2		$\overline{\square}$			$\square$
X_5		$\overline{\square}$			$\square$
X_3	$\overline{\square}$		$\square$		
X_4			$\overline{\square}$		$\square$
X_9		$\square$	$\overline{\square}$		
X_{11}	$\square$				$\overline{\square}$
X_6			$\square$	$\overline{\square}$	
X_7	$\square$			$\overline{\square}$	
X_8				$\square$	$\overline{\square}$
X_{10}				$\square$	$\overline{\square}$

$\overset{\text{dual on D}}{\Longrightarrow}$

	$SU(N)_A$	$SU(N)_B$	$SU(N)_C$	$SU(N)_D$	$SU(N)_E$
X_6	$\overline{\square}$	$\square$			
X_5		$\overline{\square}$			$\square$
X_3		$\overline{\square}$			$\square$
X_1	$\overline{\square}$		$\square$		
X_4			$\overline{\square}$		$\square$
X_{10}		$\square$	$\overline{\square}$		
X_{13}	$\square$				$\overline{\square}$
$\widetilde{X}_6$			$\overline{\square}$	$\square$	
$\widetilde{X}_7$	$\overline{\square}$			$\square$	
$\widetilde{X}_8$				$\overline{\square}$	$\square$
$\widetilde{X}_{10}$				$\overline{\square}$	$\square$
$M_{EA,1}$	$\square$				$\overline{\square}$
$M_{EA,2}$	$\square$				$\overline{\square}$
$M_{EC,1}$			$\square$		$\overline{\square}$
$M_{EC,2}$			$\square$		$\overline{\square}$

(21.4.8)

Let us explain the notations in (21.4.8). Before Seiberg duality we have 11 fields $X_{1,\ldots,11}$. After the dualisation on gauge group D, the we obtain dual quarks (corresponding to bi-fundamentals conjugate to the original quark X_6, X_7, X_8, X_{10}) which we denote $\widetilde{X}_6, \widetilde{X}_7, \widetilde{X}_8, \widetilde{X}_{10}$. Furthermore we have added meson fields $M_{EA,1}, M_{EA,2}, M_{EC,1}, M_{EC,2}$, which are Seiberg mesons decomposed with respect to the unbroken chiral symmetry group.

As before, one should incorporate the interactions as a deformation of this duality. Naïvely we have 15 fields in the dual theory, but as we will show below, the resulting superpotential provides a mass term for the fields X_4 and $M_{EC,2}$, which transform in conjugate representations. Integrating them out, we will be left with 13 fields, the number of fields in Case I. In fact, with the mapping

dual of II	X_1	X_2	X_5	X_3	X_4	X_9	X_{11}	$\widetilde{X}_6$	$\widetilde{X}_7$	$\widetilde{X}_8$	$\widetilde{X}_{10}$
Case I	Y_6	Y_5	Y_3	Y_1	massive	Y_{10}	Y_{13}	Y_2	Y_4	Y_{11}	Y_7

and

dual of II	$M_{EA,1}$	$M_{EA,2}$	$M_{EC,1}$	$M_{EC,2}$
Case I	Y_8	Y_{12}	Y_9	massive

we conclude that the matter content of the Case II dualised on gauge group D is identical to Case I!

Let us finally check the superpotentials, and also verify the claim that X_4 and $M_{EC,2}$ become massive. Rewriting the superpotential of II from (21.4.7) in terms of the dual variables (matching the mesons as composites $M_{EA,1} = X_8X_7$, $M_{EA,2} = X_{10}X_7$, $M_{EC,1} = X_8X_6$, $M_{EC,2} = X_{10}X_6$), we have

$$
\begin{aligned}
W_{II} &= X_5M_{EC,1}X_9 + X_1X_2M_{EA,2} + X_{11}X_3X_4 \\
&\quad -X_4M_{EC,2} - X_2M_{EA,1}X_3X_9 - X_{11}X_1X_5.
\end{aligned}
$$

As is with the previous subsection, to the above we must add the meson interaction terms coming from Seiberg duality, namely

$$
W_{meson} = M_{EA,1}\widetilde{X}_7\widetilde{X}_8 - M_{EA,2}\widetilde{X}_7\widetilde{X}_{10} - M_{EC,1}\widetilde{X}_6\widetilde{X}_8 + M_{EC,2}\widetilde{X}_6\widetilde{X}_{10},
$$

(notice again the choice of sign in W_{meson}). Adding this two together we have

$$
\begin{aligned}
W_{II}^{dual} &= X_5M_{EC,1}X_9 + X_1X_2M_{EA,2} + X_{11}X_3X_4 \\
&\quad -X_4M_{EC,2} - X_2M_{EA,1}X_3X_9 - X_{11}X_1X_5 \\
&\quad +M_{EA,1}\widetilde{X}_7\widetilde{X}_8 - M_{EA,2}\widetilde{X}_7\widetilde{X}_{10} - M_{EC,1}\widetilde{X}_6\widetilde{X}_8 + M_{EC,2}\widetilde{X}_6\widetilde{X}_{10}.
\end{aligned}
$$

Now it is very clear that both X_4 and $M_{EC,2}$ are massive and should be integrated out:

$$X_4 = \widetilde{X}_6 \widetilde{X}_{10}, \qquad M_{EC,2} = X_{11} X_3.$$

Upon substitution we finally have

$$\begin{aligned} W_{II}^{dual} &= X_5 M_{EC,1} X_9 + X_1 X_2 M_{EA,2} + X_{11} X_3 \widetilde{X}_6 \widetilde{X}_{10} - X_2 M_{EA,1} X_3 X_9 \\ &\quad -X_{11} X_1 X_5 + M_{EA,1} \widetilde{X}_7 \widetilde{X}_8 - M_{EA,2} \widetilde{X}_7 \widetilde{X}_{10} - M_{EC,1} \widetilde{X}_6 \widetilde{X}_8, \end{aligned}$$

which with the replacement rules given above we obtain

$$\begin{aligned} W_{II}^{dual} &= Y_3 Y_9 Y_{10} + Y_6 Y_5 Y_{12} + Y_{13} Y_1 Y_2 Y_7 - Y_5 Y_1 Y_{10} Y_8 \\ &\quad -Y_{13} Y_6 Y_3 + Y_8 Y_4 Y_{11} - Y_{12} Y_4 Y_7 - Y_9 Y_2 Y_{11}. \end{aligned}$$

This we instantly recognise, by referring to (21.4.7), as the superpotential of Case I.

In conclusion therefore, with the matching of matter content and superpotential, the two torically dual cases I and II of the cone over the second del Pezzo surface are also Seiberg duals.

21.5 Brane Diamonds and Seiberg Duality

Having seen the above arguments from field theory, let us support that toric duality is Seiberg duality from yet another perspective, namely, through brane setups. The use of this T-dual picture for D3-branes at singularities will turn out to be quite helpful in showing that toric duality reproduces Seiberg duality.

What we have learnt from the examples where a brane interval picture is available (i.e. NS- and D4-branes in the manner of [66]) is that the standard Seiberg duality by brane crossing reproduces the different gauge theories obtained from toric arguments (different partial resolutions of a given singularity). Notice that the brane crossing corresponds, under T-duality, to a change of the B field in the singularity picture, rather than a change in the singularity geometry [218, 274]. Hence, the two theories arise on the world-volume of D-branes probing the same singularity.

Unfortunately, brane intervals are rather limited, in that they can be used to study Seiberg duality for generalized conifold singularities, $xy = w^k w^l$. Although this is a large class of models, not many examples arise in the partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. Hence the relation to toric duality from partial resolutions cannot be checked for most examples.

Therefore it would be useful to find other singularities for which a nice T-dual brane picture is available. Nice in the sense that there is a motivated proposal to realize Seiberg duality in the corresponding brane setup. A good candidate for such a brane setup is **brane diamonds**, studied in [211].

Reference [78] (see also [82, 79]) introduced brane box configurations of intersecting NS- and NS'-branes (spanning 012345 and 012367, respectively), with D5-branes (spanning 012346) suspended among them. Brane diamonds [211] generalized (and refined) this setup by considering situations where the NS- and the NS'-branes recombine and span a smooth holomorphic curve in the 4567 directions, in whose holes D5-branes can be suspended as soap bubbles. Typical brane diamond pictures are as in figures in the remainder of the chapter.

Brane diamonds are related by T-duality along 46 to a large set of D-branes at singularities. With the set of rules to read off the matter content and interactions in [211], they provide a useful pictorial representation of these D-brane gauge field theories. In particular, they correspond to singularities obtained as the abelian orbifolds of the conifold studied in Section 5 of [218], and partial resolutions thereof. Concerning this last point, brane diamond configurations admit two kinds of deformations: motions of diamond walls in the directions 57, and motions of diamond walls in the directions 46. The former T-dualize to geometric sizes of the collapse cycles, hence trigger partial resolutions of the singularity (notice that when a diamond wall moves in 57, the suspended D5-branes snap back and two gauge factors recombine, leading to a Higgs mechanism, triggered by FI terms). The later do not modify the T-dual singularity geometry, and correspond to changes in the B-fields in the collapsed cycles.

The last statement motivates the proposal made in [211] for Seiberg duality in this setup. It corresponds to closing a diamond, while keeping it in the 46 plane, and reopening it with the opposite orientation. The orientation of a diamond determines the chiral multiplets and interactions arising from the picture. The effect of this is shown in fig 7 of [211]: The rules are

1. When the orientation of a diamond is flipped, the arrows going in or out of it change orientation;

2. one has to include/remove additional arrows to ensure a good ‘arrow flow’ (ultimately connected to anomalies, and to Seiberg mesons)

3. Interactions correspond to closed loops of arrows in the brane diamond picture.

4. In addition to these rules, and based in our experience with Seiberg duality, we propose that when in the final picture some mesons appear in gauge representations conjugate to some of the original field, the conjugate pair gets massive.

Figure 21-6: Seiberg duality from the brane diamond construction for the generalized conifold $xy = z^2w^2$. Part (I) corresponds to the brane interval picture with alternating ordering of NS- and NS′-branes, whereas part (II) matches the other ordering.

These rules reproduce Seiberg duality by brane crossing in cases where a brane interval picture exists. In fact, one can reproduce our previous discussion of the $xy = z^2w^2$ in this language, as shown in figure Figure 21-6. Notice that in analogy with the brane interval case the diamond transition proposed to reproduce Seiberg duality does not involve changes in the T-dual singularity geometry, hence ensuring that the two gauge theories will have the same moduli space.

Let us re-examine our aforementioned examples.

21.5.1 Brane diamonds for D3-branes at the cone over F_0

Now let us show that diamond Seiberg duality indeed relates the two gauge theories arising on D3-branes at the singularity which is a complex cone over F_0. The toric diagram of F_0 is similar to that of the conifold, only that it has an additional point (ray) in the middle of the square. Hence, it can be obtained from the conifold diagram by simply refining the lattice (by a vector $(1/2, 1/2)$ if the conifold lattice is generated by $(1,0)$, $(0,1)$). This implies [45]) that the space can be obtained as a $\mathbb{Z}_2$ quotient of the conifold, specifically modding $xy = zw$ by the action that flips all coordinates.

Performing two T-dualities in the conifold one reaches the brane diamond picture described in [211] (fig. 5), which is composed by two-diamond cell with sides identified, see Part (I) of Figure 21-7. However, we are interested not in the conifold but on a

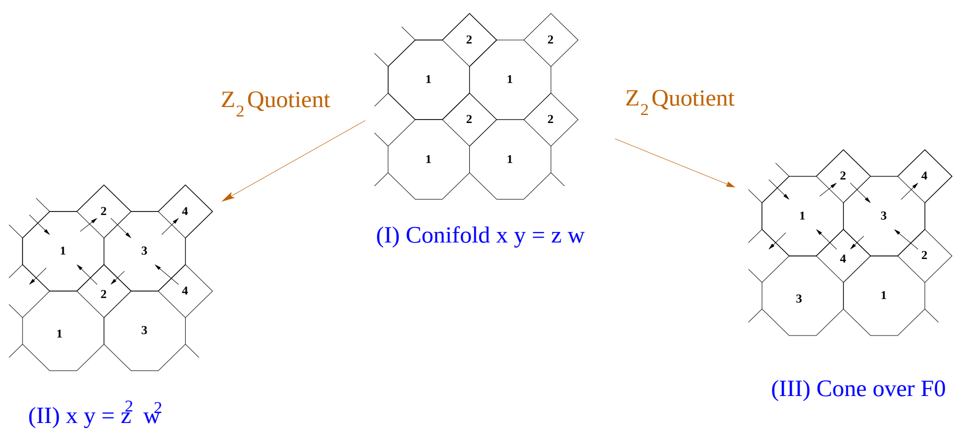


Figure 21-7: (I) Brane diamond for the conifold. Identifications in the infinite periodic array of boxes leads to a two-diamond unit cell, whose sides are identified in the obvious manner. From (I) we have 2 types of $\mathbb{Z}_2$ quotients: (II) Brane diamond for the $\mathbb{Z}_2$ quotient of the conifold $xy = z^2w^2$, which is a case of the so-called generalised conifold. The identifications of sides are trivial, not tilting. The final spectrum is the familiar non-chiral spectrum for a brane interval with two NS and two NS' branes (in the alternate configuration); (III) Brane diamond for the $\mathbb{Z}_2$ quotient of the conifold yielding the complex cone over F_0. The identifications of sides are shifted, a fact related to the specific 'tilted' refinement of the toric lattice.

$\mathbb{Z}_2$ quotient thereof. Quotienting a singularity amounts to including more diamonds in the unit cell, i.e. picking a larger unit cell in the periodic array. There are two possible ways to do so, corresponding to two different $\mathbb{Z}_2$ quotients of the conifold. One corresponds to the generalized conifold $xy = z^2w^2$ encountered above, and whose diamond picture is given in Part (II) of Figure 21-7 for completeness. The second possibility is shown in Part (III) of Figure 21-7 and does correspond to the T-dual of the complex cone over F_0, so we shall henceforth concentrate on this case. Notice that the identifications of sides of the unit cell are shifted. The final spectrum agrees with the quiver before eq (2.2) in [298]. Moreover, following [211], these fields have quartic interactions, associated to squares in the diamond picture, with signs given by the orientation of the arrow flow. They match the ones in case II in (21.4.4).

Now let us perform the diamond duality in the box labeled 2. Following the diamond duality rules above, we obtain the result shown in Figure 21-8. Careful comparison with the spectrum and interactions of case I in (21.4.4), and also with the Seiberg dual computed in Section 4.1 shows that the new diamond picture reproduces the toric dual / Seiberg dual of the initial one. Hence, brane diamond configurations provide a new geometric picture for this duality.

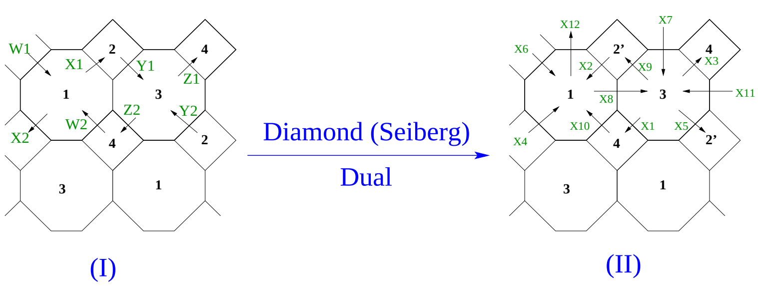


Figure 21-8: Brane diamond for the two cases of the cone over F_0. (I) is as in Figure 21-7 and (II) is the result after the diamond duality. The resulting spectrum and interactions are those of the toric dual (and also Seiberg dual) of the initial theory (I).

21.5.2 Brane diamonds for D3-branes at the cone over dP_2

The toric diagram for dP_2 shows it cannot be constructed as a quotient of the conifold. However, it is a partial resolution of the orbifolded conifold described as $xy = v^2$, $uv = z^2$ in $\mathbb{C}^5$ (we refer the reader to Figure 21-9. This is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ quotient of the conifold whose brane diamond, shown in Part (I) of Figure 21-10, contains 8 diamonds in its unit cell. Partial resolutions in the brane diamond language correspond to partial Higgsing, namely recombination of certain diamonds. As usual,

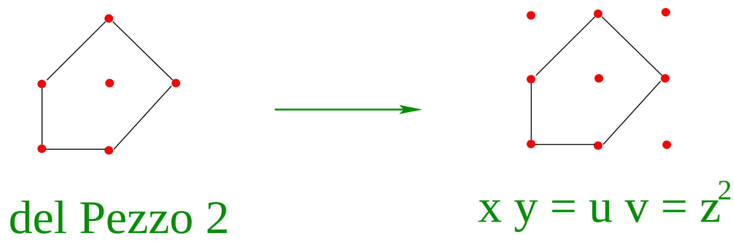


Figure 21-9: Embedding the toric diagram of dP2 into the orbifolded conifold described as $xy = v^2$, $uv = z^2$.

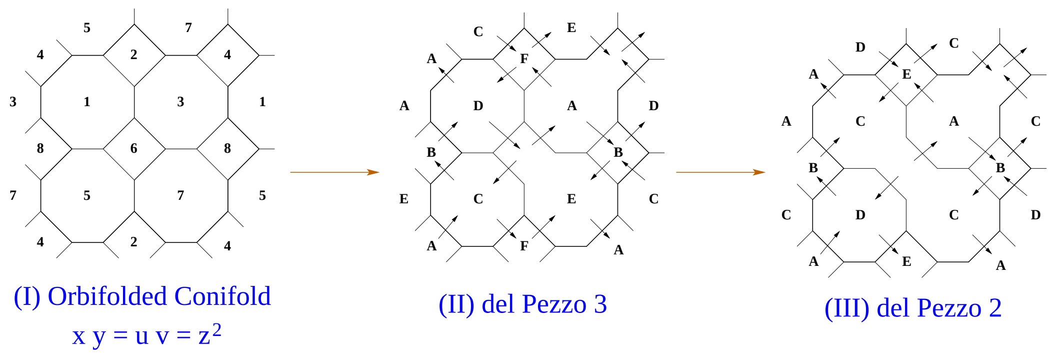


Figure 21-10: (I) Brane diamond for a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the conifold, namely $xy = z^2; uv = z^2$. From this we can partial resolve to (II) the cone over $dP3$ and thenceforth again to (III) the cone over dP_2, which we shall discuss in the context of Seiberg duality.

the difficult part is to identify which diamond recombination corresponds to which partial resolution. A systematic way proceed would be[5]:

1. Pick a diamond recombination;
2. Compute the final gauge theory;
3. Compute its moduli space, which should be the partially resolved singularity.

However, instead of being systematic, we prefer a shortcut and simply match the spectrum of recombined diamond pictures with known results of partial resolutions. In order to check we pick the right resolutions, it is useful to discuss the brane diamond

[5]As an aside, let us remark that the use of brane diamonds to follow partial resolutions of singularities may provide an alternative to the standard method of partial resolutions of orbifold singularities [157, 298]. The existence of a brane picture for partial resolutions of orbifolded conifolds may turn out to be a useful advantage in this respect.

picture for some intermediate step in the resolution to dP_2. A good intermediate point, for which the field theory spectrum is known is the complex cone over dP_3.

By trial and error matching, the diamond recombination which reproduces the world-volume spectrum for D3-branes at the cone over dP_3 (see [298, 306]), is shown in Part (II) of Figure 21-10. Performing a further resolution, chosen so as to match known results, one reaches the brane diamond picture for D3-branes on the cone over dP_2, shown in Part (III) of Figure 21-10. More specifically, the spectrum and interactions in the brane diamond configuration agrees with those of case I in (21.4.7).

This brane box diamond, obtained in a somewhat roundabout way, is our starting point to discuss possible dual realizations. In fact, recall that there is a toric dual field theory for dP_2, given as case II in (21.4.7). After some inspection, the desired effect is obtained by applying diamond Seiberg duality to the diamond labeled B. The corresponding process and the resulting diamond picture are shown in Figure 21-11. Two comments are in order: notice that in applying diamond duality using the rules above, some vector-like pairs of fields have to be removed from the final picture; in fact one can check by field theory Seiberg duality that the superpotential makes them massive. Second, notice that in this case we are applying duality in the direction opposite to that followed in the field theory analysis in Section 4.2; it is not difficult to check that the field theory analysis works in this direction as well, namely the dual of the dual is the original theory. Therefore this new example provides again a geometrical realization of Seiberg duality, and allows to connect it with Toric Duality.

We conclude this Section with some remarks. The brane diamond picture presumably provides other Seiberg dual pairs by picking different gauge factors. All such models should have the same singularities as moduli space, and should be toric duals in a broad sense, even though all such toric duals may not be obtainable by partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3\times\mathbb{Z}_3)$. From this viewpoint we learn that Seiberg duality can provide us with new field theories and toric duals beyond the reach of present computational tools. This is further explored in Section 7.

A second comment along the same lines is that Seiberg duality on nodes for which $N_f \neq 2N_c$ will lead to dual theories where some gauge factors have different rank. Taking the theory back to the 'abelian' case, some gauge factors turn out to be non-abelian. Hence, in these cases, even though Seiberg duality ensures the final theory has the same singularity as moduli space, the computation of the corresponding symplectic quotient is beyond the standard tools of toric geometry. Therefore, Seiberg duality can provide ('non-toric') gauge theories with toric moduli space.

21.6 A Quiver Duality from Seiberg Duality

If we are not too concerned with the superpotential, when we make the Seiberg duality transformation, we can obtain the matter content very easily at the level of the quiver diagram. What we obtain are rules for a so-called "quiver duality" which is a rephrasing of the Seiberg duality transformations in field (brane diamond) theory in the language of quivers. Denote $(N_c)_i$ the number of colors at the i^{th} node, and a_{ij} the number of arrows from the node i to the j (the adjacency matrix) The rules on the quiver to obtain Seiberg dual theories are

1. Pick the dualisation node i_0. Define the following sets of nodes: $I_{in} :=$ nodes having arrows going into i_0; $I_{out} :=$ those having arrow coming from i_0 and $I_{no} :=$ those unconnected with i_0. The node i_0 should not be included in this classification.

2. Change the rank of the node i_0 from N_c to $N_f - N_c$ where N_f is the number of vector-like flavours, $N_f = \sum\limits_{i \in I_{in}} a_{i,i_0} = \sum\limits_{i \in I_{out}} a_{i_0,i}$

3. Reverse all arrows going in or out of i_0, therefore

$$a_{ij}^{dual} = a_{ji} \qquad \text{if either } i, j = i_0$$

4. Only arrows linking I_{in} to I_{out} will be changed and all others remain unaffected.

5. For every pair of nodes A, B, $A \in I_{out}$ and $B \in I_{in}$, change the number of

arrows a_{AB} to

$$a_{AB}^{dual} = a_{AB} - a_{i_0 A} a_{B i_0} \qquad \text{for } A \in I_{out}, \quad B \in I_{in}.$$

If this quantity is negative, we simply take it to mean $-a^{dual}$ arrow go from B to A.

These rules follow from applying Seiberg duality at the field theory level, and therefore are consistent with anomaly cancellation. In particular, notice the for any node $i \in I_{in}$, we have replaced $a_{i,i_0} N_c$ fundamental chiral multiplets by $-a_{i,i_0}(N_f - N_c) + \sum_{j \in I_{out}} a_{i,i0} a_{i_0,j}$ which equals $-a_{i,i_0}(N_f - N_c) + a_{i,i0} N_f = a_{i,i_0} N_c$, and ensures anomaly cancellation in the final theory. Similarly for nodes $j \in I_{out}$.

It is straightforward to apply these rules to the quivers in the by now familiar examples in previous sections.

In general, we can choose an arbitrary node to perform the above Seiberg duality rules. However, not every node is suitable for a toric description. The reason is that, if we start from a quiver whose every node has the same rank N, after the transformation it is possible that this no longer holds. We of course wish so because due to the very definition of the $\mathbb{C}^*$ action for toric varieties, toric descriptions are possible iff all nodes are $U(1)$, or in the non-Abelian version, $SU(N)$. If for instance we choose to Seiberg dualize a node with $3N$ flavours, the dual node will have rank $3N - N = 2N$ while the others will remain with rank N, and our description would no longer be toric. For this reason we must choose nodes with only $2N_f$ flavors, if we are to remain within toric descriptions.

One natural question arises: if we Seiberg-dualise every possible allowed node, how many different theories will we get? Moreover how many of these are torically dual? Let we re-analyse the examples we have thus far encountered.

21.6.1 Hirzebruch Zero

Starting from case (II) of F_0 (recall Figure 21.4.4) all of four nodes are qualified to yield toric Seiberg duals (they each have 2 incoming and 2 outgoing arrows and hence

$N_f = 2N$). Dualising any one will give to case (I) of F_0. On the other hand, from (I) of F_0, we see that only nodes B, D are qualified to be dualized. Choosing either, we get back to the case (II) of F_0. In another word, cases (I) and (II) are closed under the Seiberg-duality transformation. In fact, this is a very strong evidence that there are only two toric phases for F_0 no matter how we embed the diagram into higher $\mathbb{Z}_k \times \mathbb{Z}_k$ singularities. This also solves the old question [298, 306] that the Inverse Algorithm does not in principle tell us how many phases we could have. Now by the closeness of Seiberg-duality transformations, we do have a way to calculate the number of possible phases. Notice, on the other hand, the existence of non-toric phases.

21.6.2 del Pezzo 0,1,2

Continuing our above calculation to del Pezzo singularities, we see that for dP_0 no node is qualified, so there is only one toric phase which is consistent with the standard result [306] as a resolution $\mathcal{O}_{\mathbf{P}^2}(-1) \to \mathbb{C}^3/\mathbb{Z}_3$. For dP_1, nodes A, B are qualified (all notations coming from [306]), but the dualization gives back to same theory, so it too has only one phase.

For our example dP_2 studied earlier (recall Figure 21.4.7), there are four points A, B, C, D which are qualified in case (II). Nodes A, C give back to case (II) while nodes B, D give rise to case (I) of dP_2. On the other hand, for case (I), three nodes B, D, E are qualified. Here nodes B, E give case (II) while node D give case (I). In other words, cases (I) and (II) are also closed under the Seiberg-duality transformation, so we conclude that there too are only two phases for dP_2, as presented earlier.

21.6.3 The Four Phases of dP_3

Things become more complex when we discuss the phases of dP_3. As we remarked before, due to the running-time limitations of the Inverse Algorithm, only one phase was obtained in [306]. However, one may expect this case to have more than just one phase, and in fact a recent paper has given another phase [283]. Here, using the closeness argument we give evidence that there are four (toric) phases for dP_3. We will give only one phase in detail. Others are similarly obtained. Starting from case (I) given in [306] and dualizing node B, (we refer the reader to Figure 21-12) we get the charge (incidence) matrix d as

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccccccc} q_1 & q_2 & q'_1 & q'_2 & X_1 & X_2 & X_7 & X_9 & X_{10} & X_{11} & M_1 & X_{14} & M_2 & X_8 & M'_1 & X_5 & X_{12} & M'_2 \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} &
\begin{pmatrix}
0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

where

$$M_1 = X_4 X_3, \quad M_2 = X_4 X_6, \quad M'_1 = X_{13} X_3, \quad M'_2 = X_{13} X_6$$

are the added mesons. Notice that X_{14} and M_2 have opposite charge. In fact, both are massive and will be integrate out. Same for pairs (X_8, M'_1) and (X_5, M'_2).

Let us derive the superpotential. Before dual transformation, the superpotential is [298]

$$
\begin{aligned}
W_I &= X_3 X_8 X_{13} - X_8 X_9 X_{11} - X_5 X_6 X_{13} - X_1 X_3 X_4 X_{10} X_{12} \\
&\quad X_7 X_9 X_{12} + X_4 X_6 X_{14} + X_1 X_2 X_5 X_{10} X_{11} - X_2 X_7 X_{14}
\end{aligned}
$$

After dualization, superpotential is rewritten as

$$
\begin{aligned}
W' &= M'_1 X_8 - X_8 X_9 X_{11} - X_5 M'_2 - X_1 M_1 X_{10} X_{12} \\
&\quad X_7 X_9 X_{12} + M_2 X_{14} + X_1 X_2 X_5 X_{10} X_{11} - X_2 X_7 X_{14}.
\end{aligned}
$$

It is very clear that fields $X_8, M'_1, X_5, M'_2, X_{14}, M_2$ are all massive. Furthermore, we need to add the meson part

$$W_{meson} = M_1 q'_1 q_1 - M_2 q_1 q'_2 - M'_1 q'_1 q_2 + M'_2 q'_2 q_2$$

where we determine the sign as follows: since the term $M_1'X_8$ in W' in positive, we need term $M_1'q_1'q_2$ to be negative. After integration all massive fields, we get the superpotential as

$$W_{II} \;=\; -q_1'q_2X_9X_{11} - X_1M_1X_{10}X_{12} + X_7X_9X_{12} + X_1X_2q_2'q_2X_{10}X_{11} - X_2X_7q_1q_2' + M_1q_1'q_1.$$

The charge matrix now becomes

$$\begin{array}{cc}
 & \begin{array}{cccccccccccc} q_1 & q_2 & q_1' & q_2' & X_1 & X_2 & X_7 & X_9 & X_{10} & X_{11} & M_1 & X_{12} \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} &
\begin{pmatrix}
0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
-1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & 0
\end{pmatrix}
\end{array}$$

This is in precise agreement with [283]; very re-assuring indeed!

Without further ado let us present the remaining cases. The charge matrix for the third one (dualising node C of (I)) is

$$\begin{array}{cc}
 & \begin{array}{cccccccccccccc} q_1 & q_1' & q_2' & q_2 & X_5 & X_{12} & X_3 & X_8 & X_9 & M_1 & X_{10} & X_{11} & X_{13} & M_2 \end{array} \\
\begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \end{array} &
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 & -1 & 0 & 0 & -1 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
-1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0
\end{pmatrix}
\end{array}$$

with superpotential

$$\begin{aligned}
W_{III} \;=\;& X_3X_8X_{13} - X_8X_9X_{11} - X_5q_2q_2'X_{13} - M_2X_3X_{10}X_{12} \\
& + q_2q_1'X_9x_{12} + M_1X_5X_{10}X_{11} - M_1q_1q_1' + M_2q_1q_2'.
\end{aligned}$$

Finally the fourth case (dualising node E of (III)) has the charge matrix

$$
\left(\begin{array}{c|cccccccccccccccccc}
 & q_1 & W_1 & W_2 & q_1' & q_2' & X_3 & X_8 & W_1' & W_2' & X_9 & M_1 & X_{11} & X_{13} & M_2 & p_1 & p_1' & p_2' & p_2 \\
A & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 \\
B & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
C & -1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
D & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\
F & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

with superpotential

$$
\begin{aligned}
W_{IV} &= X_3X_8X_{13} - X_8X_9X_{11} - W_1q_2'X_{13} - M_2X_3W_2' + q_1'X_9W_2 + M_1W_1'X_{11} \\
&\quad -M_1q_1q_1' + M_2q_1q_2' + W_1p_1p_1' - W_2p_1p_2' - W_1'p_2p_1' + W_2'p_2p_2'
\end{aligned}
$$

21.7 Picard-Lefschetz Monodromy and Seiberg Duality

In this section let us make some brief comments about Picard-Lefschetz theory and Seiberg duality, a relation between which has been within the literature [281]. It was argued in [282] that at least in the case of D3-branes placed on ADE conifolds [286, 287] Seiberg duality for $\mathcal{N} = 1$ SUSY gauge theories can be geometrised into Picard-Lefschetz monodromy. Moreover in [283] Toric Duality is interpreted as Picard-Lefschetz monodromy action on the 3-cycles.

On the level of brane setups, this interpretation seems to be reasonable. Indeed, consider a brane crossing process in a brane interval picture. Two branes separated in x^6 approach, are exchanged, and move back. The T-dual operation on the singularity corresponds to choosing a collapsed cycle, decreasing its B-field to zero, and continuing to negative values. This last operation is basically the one generating Picard-Lefschetz monodromy at the level of homology classes. Similarly, the closing and reopening of diamonds corresponds to continuations past infinite coupling of the gauge theories, namely to changes in the T-dual B-fields in the collapsed cycles.

It is the purpose of this section to point out the observation that while for restricted classes of theories the two phenomena are the same, in general Seiberg duality and a naïve application of Picard-Lefschetz (PL) monodromy do not seem to coincide. We leave this issue here as a puzzle, which we shall resolve in an upcoming work.

The organisation is as follows. First we briefly introduce the concept of Picard-Lefschetz monodromy for the convenience of the reader and to establish some notation. Then we give two examples: the first is one with two Seiberg dual theories not related by PL and the second, PL dual theories not related by Seiberg duality.

21.7.1 Picard-Lefschetz Monodromy

We first briefly remind the reader of the key points of the PL theory [6]. Given a singularity on a manifold M and a basis $\{\Delta_i\} \subset H_{n-1}(M)$ for its vanishing $(n-1)$-cycles, going around these vanishing cycles induces a monodromy, acting on arbitrary cycles $a \in H_\bullet(M)$; moreover this action is computable in terms of intersection $a \circ \Delta_i$ of the cycle a with the basis:

THEOREM 21.7.34 *The monodromy group of a singularity is generated by the Picard-Lefschetz operators h_i, corresponding to a basis $\{\Delta_i\} \subset H_{n-1}$ of vanishing cycles. In particular for any cycle $a \in H_{n-1}$ (no summation in i)*

$$h_i(a) = a + (-1)^{\frac{n(n+1)}{2}}(a \circ \Delta_i)\Delta_i.$$

More concretely, the PL monodromy operator h_i acts as a matrix $(h_i)_{jk}$ on the basis Δ_j:

$$h_i(\Delta_j) = (h_i)_{jk}\Delta_k.$$

Next we establish the relationship between this geometric concept and a physical interpretation. According geometric engineering, when a D-brane wraps a vanishing cycle in the basis, it give rise to a simple factor in the product gauge group. Therefore the total number of vanishing cycles gives the number of gauge group factors. Moreover, the rank of each particular factor is determined by how many times it wraps that cycle.

For example, an original theory with gauge group $\prod_j SU(M_j)$ is represented by the brane wrapping the cycle $\sum_j M_j\Delta_j$. Under PL monodromy, the cycle undergoes the transformation

$$\sum_j M_j\Delta_j \Longrightarrow \sum_j M_j(h_i)_{jk}\Delta_k.$$

Physically, the final gauge theory is $\prod_k SU(\sum_j M_j(h_i)_{jk})$.

The above shows how the rank of the gauge theory changes under PL. To determine the theory completely, we also need to see how the matter content transforms. In geometric engineering, the matter content is given by intersection of these cycles Δ_j. Incidentally, our Inverse Algorithm gives a nice way and alternative method of computing such intersection matrices of cycles.

Let us take $a = \Delta_j$, then

$$h_i(\Delta_j) = \Delta_j + (\Delta_j \circ \Delta_i)\Delta_i.$$

This is particularly useful to us because $(\Delta_j \circ \Delta_i)$, as is well-known, is the anti-symmetrised adjacency matrix of the quiver (for a recent discussion on this, see [283]). Indeed this intersection matrix of (the blowup of) the vanishing homological cycles specifies the matter content as prescribed by D-branes wrapping these cycles in the mirror picture. Therefore we have $(\Delta_j \circ \Delta_i) = [a_{ji}] := a_{ji} - a_{ij}$ for $j \neq i$ and for $i = j$, we have the self-intersection numbers $(\Delta_i \circ \Delta_i)$. Hence we can safely write (no summation in i)

$$\Delta_j^{dual} = h_i(\Delta_j) = \Delta_j + [a_{ji}]\Delta_i \qquad (21.7.9)$$

for a_{ji} the quiver (matter) matrix when Seiberg dualising on the node i; we have also used the notation $[M]$ to mean the antisymmetrisation $M - M^t$ of matrix M. Incidentally in the basis prescribed by $\{\Delta_i\}$, we have the explicit form of the Picard-Lefschetz operators in terms of the quiver matrix (no summation over indices): $(h_i)_{jk} = \delta_{jk} + [a_{ji}]\delta_{ik}$.

From (21.7.9) we have

$$\begin{aligned} [a_{jk}^{dual}] &:= \Delta_j^{dual} \circ \Delta_k^{dual} = (\Delta_j + [a_{ji}]\Delta_i) \circ (\Delta_k + [a_{ki}]\Delta_i) \\ &= [a_{jk}] + [a_{ki}][a_{ji}] + [a_{ji}][a_{ik}] + [a_{ji}][a_{ki}]\Delta_i \circ \Delta_i \\ &= [a_{jk}] + c_i[a_{ij}][a_{ki}] \end{aligned} \tag{21.7.10}$$

where $c_i := \Delta_i \circ \Delta_i$, are constants depending only on self-intersection.

We observe that our quiver duality rules obtained from field theory (see beginning of Section 6) seem to resemble (21.7.10), i. e. when $c_i = 1$ and $j, k \neq i$. However the precise relation of trying to reproduce Seiberg duality with PL theory still remains elusive.

21.7.2 Two Interesting Examples

However the situation is not as simple. In the following we shall argue that while Seiberg duality and a straightforward Picard-Lefschetz transformation certainly do have common features and that in restricted classes of theories such as those in [282], for general singularities the two phenomena may bifurcate.

We first present two theories related by Seiberg duality that cannot be so by Picard-Lefschetz. Consider the standard $\mathbb{C}^3/\mathbb{Z}_3$ theory with $a_{ij} = \begin{pmatrix} 0 & 0 & 3 \\ 3 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$ and gauge group $U(1)^3$, given in (a) of Figure 21-13. Let us Seiberg-dualise on node A to obtain a theory (b), with matter content $a_{ij}^{dual} = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 6 \\ 3 & 0 & 0 \end{pmatrix}$ and gauge group $SU(2) \times U(1)^2$. Notice especially that the rank of the gauge group factors in part (b) are $(2, 1, 1)$ while those in part (a) are $(1, 1, 1)$. Therefore theory (b) has total rank 4 while (a) has only 3. Since geometrically PL only shuffles the vanishing cycles and certainly preserves their number, we see that (a) and (b) cannot be related by PL even though they are Seiberg duals.

On the other hand we give an example in the other direction, namely two Picard-Lefschetz dual theories which are not Seiberg duals. Consider the case given in Figure 21-14, this is a phase of the theory for the complex cone over dP3 as given in [289]. This is PL dual to any of the 4 four phases in Figure 21-12 in the previous section by construction with (p, q)-webs. Note that the total rank remains 6 under PL even though the number of nodes changed. However Seiberg duality on any of the allowed node on any of the 4 phases cannot change the number of nodes. Therefore, this example in Figure 21-14 is not Seiberg dual to the other 4.

What we have learnt in this short section is that Seiberg duality and a naïve application of Picard-Lefschetz monodromy seem to have discrepancies for general singularities. The resolution of this puzzle will be delt with in a forthcoming work.

21.8 Conclusions

In [298, 306] a mysterious duality between classes of gauge theories on D-branes probing toric singularities was observed. Such a Toric Duality identifies the infrared moduli space of very different theories which are candidates for the world-volume theory on D3-branes at threefold singularities. On the other hand, [218, 274] have recognised certain brane-moves for brane configurations of certain toric singularities as Seiberg duality.

In this chapter we take a unified view to the above. Indeed we have provided a physical interpretation for toric duality. The fact that the gauge theories share by definition the same moduli space motivates the proposal that they are indeed physically equivalent in the infrared. In fact, we have shown in detail that toric dual gauge theories are connected by Seiberg duality.

This task has been facilitated by the use of T-dual configurations of NS and D-branes, in particular brane intervals and brane diamonds [211]. These constructions show that the Seiberg duality corresponds in the singularity picture to a change of B-fields in the collapsed cycles. Hence, the specific gauge theory arising on D3-branes at a given singularity, depends not only on the geometry of the singularity, but also on the B-field data. Seiberg duality and brane diamonds provide us with the tools to move around this more difficult piece of the singular moduli space, and probe different phases.

This viewpoint is nicely connected with that in [298, 306], where toric duals were obtained as different partial resolutions of a given orbifold singularity, $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$, leading to equivalent geometries (with toric diagrams equivalent up to unimodular transformations). Specifically, the original orbifold singularity has a specific assignments of B-fields on its collapsed cycles. Different partial resolutions amount to choosing a subset of such cycles, and blowing up the rest. Hence, in general different partial resolutions leading to the same geometric singularity end up with different assignments of B-fields. This explains why different gauge theories, related by Seiberg duality, arise by different partial resolutions.

In particular we have examined in detail the toric dual theories for the generalised conifold $xy = z^2w^2$, the partial resolutions of $\mathbb{C}^3/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ exemplified by the complex cones over the zeroth Hirzebruch surface as well as the second del Pezzo surface. We have shown how these theories are equivalent under the above scheme by explicitly having

1. unimodularly equivalent toric data;

2. the matter content and superpotential related by Seiberg duality;

3. the T-dual brane setups related by brane-crossing and diamond duality.

The point d'appui of this work is to show that the above three phenomena are the same.

As a nice bonus, the physical understanding of toric duality has allowed us to construct new toric duals in cases where the partial resolution technique provided only one phase. Indeed the exponential running-time of the Inverse Algorithm currently prohibits larger embeddings and partial resolutions. Our new perspective greatly facilitates the calculation of new phases. As an example we have constructed three new phases for the cone over del Pezzo three one of which is in reassuring agreement with a recent work [283] obtained from completely different methods.

Another important direction is to understand the physical meaning of Picard-Lefschetz transformations. As we have pointed out in Section 7, PL transformation and Seiberg duality are really two different concepts even though they coincide for certain restricted classes of theories. We have provided examples of two theories which are related by one but not the other. Indeed we must pause to question ourselves. For those which are Seiberg dual but not PL related, what geometrical action does correspond to the field theory transformation. On the other hand, perhaps more importantly, for those related to each other by PL transformation but not by Seiberg duality, what kind of duality is realized in the dynamics of field theory? Does there exists a new kind of dynamical duality not yet uncovered??

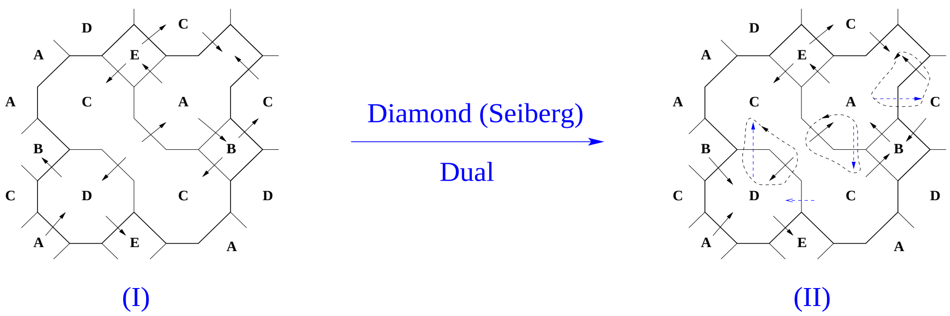


Figure 21-11: The brane diamond setup for the Seiberg dual configurations of the cone over dP_2. (I) is as in Figure 21-10 and (II) is the results after Seiberg (diamond) duality and gives the spectrum for the toric dual theory. The added meson fields are drawn in dashed blue lines. Notice that applying the diamond dual rules carelessly one gets some additional vectorlike pairs, shown in the picture within dotted lines. Such multiplets presumably get massive in the Seiberg dualization, hence we do not consider them in the quiver.

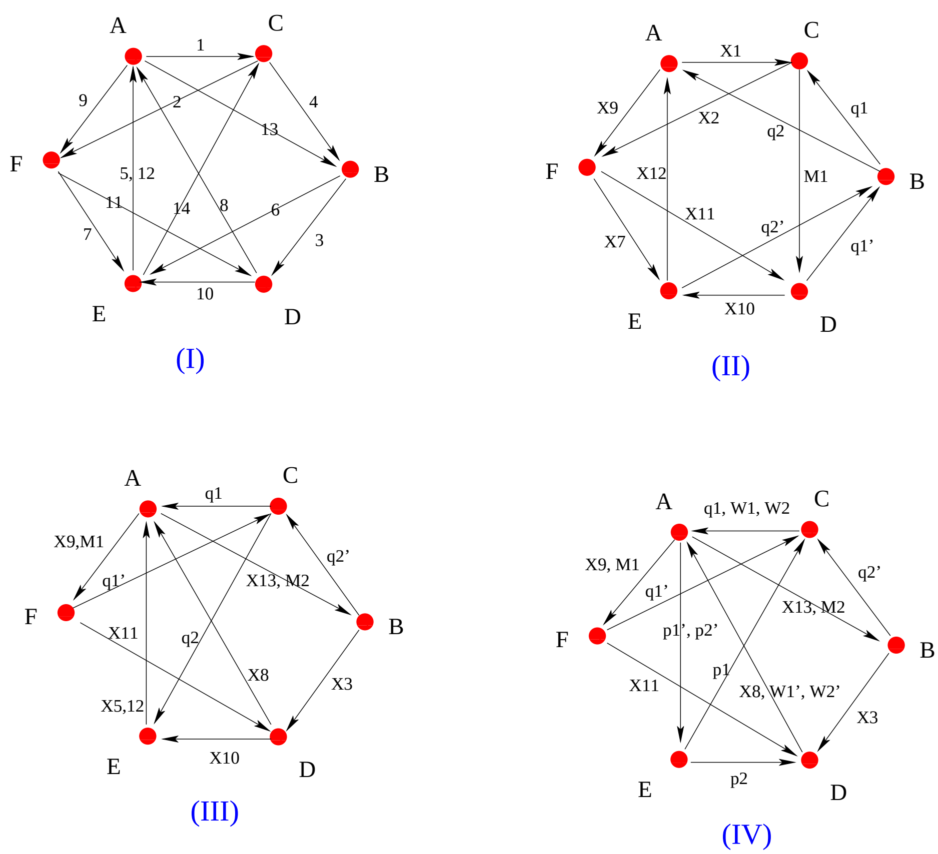


Figure 21-12: The four Seiberg dual phases of the cone over dP_3.

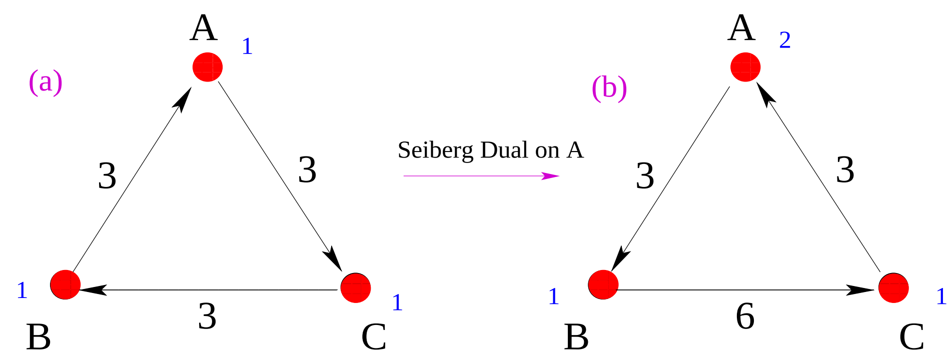


Figure 21-13: Seiberg Dualisation on node A of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold theory. The subsequent theory cannot be obtained by a Picard-Lefschetz monodromy transformation.

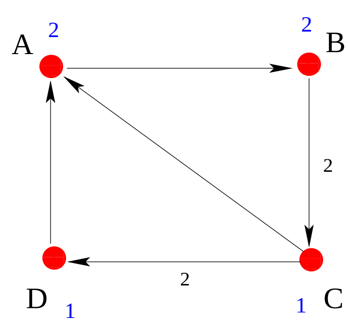


Figure 21-14: A non-Abelian phase of the complex cone over dP_3. This example is Picard-Lefschetz dual to the other 4 examples in Figure 21-12 but not Seiberg dual thereto.

Chapter 22

Appendices

22.1 Character Tables for the Discrete Subgroups of $SU(2)$

Henceforth we shall use Γ_i to index the representations and the numbers in the first row of the character tables shall refer to the order of each conjugacy class, or what we called r_γ.

$\widehat{A}_n$ = Cyclic $\mathbb{Z}_{n+1}$

	1	1	1	$\cdots$	1
Γ_1	1	1	1	$\cdots$	1
Γ_2	1	ϵ	ϵ^2	$\cdots$	ϵ^n
Γ_3	1	ϵ^2	ϵ^4	$\cdots$	ϵ^{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
Γ_n	1	ϵ^n	ϵ^{2n}	$\cdots$	ϵ^{n^2}

$\epsilon = \exp\left(\frac{2\pi i}{n+1}\right)$

For reference, next to each of the binary groups, we shall also include the character table of the corresponding ordinary cases, which are in $SU(2)/\mathbb{Z}_2$.

$\widehat{D}_n$ = Binary Dihedral

	1	1	2	$\cdots$	2	n	n
Γ_1	1	1	1	$\cdots$	1	1	1
Γ_2	1	1	1	$\cdots$	1	-1	-1
Γ_3	1	$(-1)^n$	$(-1)^1$	$\cdots$	$(-1)^{n-1}$	i^n	$-i^n$
Γ_4	1	$(-1)^n$	$(-1)^1$	$\cdots$	$(-1)^{n-1}$	$-i^n$	i^n
Γ_5	2	$(-2)^1$	$2\cos\frac{\pi}{n}$	$\cdots$	$2\cos\frac{\pi(n-1)}{n}$	0	0
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Γ_{n+1}	2	$(-2)^{n-1}$	$2\cos\frac{\pi(n-1)}{n}$	$\cdots$	$2\cos\frac{\pi(n-1)^2}{n}$	0	0

Ordinary Dihedral D_n ($n' = \frac{n+3}{2}$ for odd n and $n' = \frac{n+6}{2}$ for even n)

	1	2	2	$\cdots$	2	n
Γ_1	1	1	1	$\cdots$	1	1
Γ_2	1	1	1	$\cdots$	1	-1
Γ_3	2	$2\cos\phi$	$2\cos 2\phi$	$\cdots$	$2\cos m\phi$	0
Γ_4	2	$2\cos 2\phi$	$2\cos 4\phi$	$\cdots$	$2\cos 2m\phi$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
$\Gamma_{\frac{n+3}{2}}$	2	$2\cos m\phi$	$2\cos 2m\phi$	$\cdots$	$2\cos m^2\phi$	0

n odd
$m = \frac{n-1}{2}$
$\phi = \frac{2\pi}{n}$

	1	2	2	$\cdots$	2	1	$n/2$	$n/2$
Γ_1	1	1	1	$\cdots$	1	1	1	1
Γ_2	1	1	1	$\cdots$	1	1	-1	-1
Γ_3	1	-1	1	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	1	-1
Γ_4	1	-1	1	$\cdots$	$(-1)^{m-1}$	$(-1)^m$	-1	1
Γ_5	2	$2\cos\phi$	$2\cos 2\phi$	$\cdots$	$2\cos(m-1)\phi$	$2\cos m\phi$	0	0
Γ_6	2	$2\cos 2\phi$	$2\cos 4\phi$	$\cdots$	$2\cos 2(m-1)\phi$	$2\cos 2m\phi$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\Gamma_{\frac{n+6}{2}}$	2	$2\cos(m-1)\phi$	$2\cos 2(m-1)\phi$	$\cdots$	$2\cos(m-1)^2\phi$	$2\cos m(m-1)\phi$	0	0

n even
$m = \frac{n}{2}$
$\phi = \frac{2\pi}{n}$

$\widehat{E}_6$ = Binary Tetrahedral $\mathcal{T}$

	1	1	6	4	4	4	4
Γ_1	1	1	1	1	1	1	1
Γ_2	1	1	1	w	w	w^2	w^2
Γ_3	1	1	1	w^2	w^2	w	w
Γ_4	2	-2	0	1	-1	1	-1
Γ_5	2	-2	0	w	$-w$	w^2	$-w^2$
Γ_6	2	-2	0	w^2	$-w^2$	w	$-w$
Γ_7	3	3	-1	0	0	0	0

Ordinary T

	1	3	4	4
Γ_1	1	1	1	1
Γ_2	1	1	$-w$	w^2
Γ_3	1	1	w^2	$-w$
Γ_4	3	-1	0	0

$w = exp(\frac{2\pi i}{3})$

$\widehat{E}_7$ = Binary Octahedral $\mathcal{O}$

	1	1	2	6	6	8	8	16
Γ_1	1	1	1	1	1	1	1	1
Γ_2	1	1	-3	-1	-1	1	1	0
Γ_3	2	-2	0	$\sqrt{2}$	$-\sqrt{2}$	1	-1	0
Γ_4	2	-2	0	$-\sqrt{2}$	$\sqrt{2}$	1	-1	0
Γ_5	2	2	-2	0	0	-1	-1	1
Γ_6	3	3	-1	1	1	0	0	-1
Γ_7	3	3	-3	-1	-1	0	0	0
Γ_8	4	-4	0	0	0	-1	1	0

Ordinary O

	1	3	6	6	8
Γ_1	1	1	1	1	1
Γ_2	1	1	-1	-1	1
Γ_3	2	2	0	0	-1
Γ_4	3	-1	-1	1	0
Γ_5	3	-1	1	-1	0

$\widehat{E}_8$ = Binary Icosahedral $\mathcal{I}$

	1	1	12	12	12	12	20	20	30
Γ_1	1	1	1	1	1	1	1	1	1
Γ_2	2	-2	$-\bar{a}$	-1	a	$\bar{a}$	1	-1	0
Γ_3	2	-2	$-a$	$-\bar{a}$	$\bar{a}$	a	1	-1	0
Γ_4	3	3	$\bar{a}$	a	a	$\bar{a}$	0	0	-1
Γ_5	3	3	a	$\bar{a}$	$\bar{a}$	a	0	0	-1
Γ_6	4	-4	-1	-1	1	1	-1	1	0
Γ_7	4	4	-1	-1	-1	1	1	1	1
Γ_8	5	5	0	0	0	0	-1	-1	1
Γ_9	6	-6	1	1	-1	-1	0	0	0

Ordinary I

	1	12	12	15	20
Γ_1	1	1	1	1	1
Γ_2	3	a	$\bar{a}$	-1	0
Γ_3	3	$\bar{a}$	a	-1	0
Γ_4	4	-1	-1	0	1
Γ_5	5	0	0	1	-1

$a = \frac{1+\sqrt{5}}{2}$
$\bar{a} = \frac{1-\sqrt{5}}{2}$

22.2 Matter Content for $\mathcal{N} = 2$ SUSY Gauge Theory ($\Gamma \subset SU(2)$)

Only the fermionic matrices are presented here; as can be seen from the decomposition, twice the fermion a_{ij} subtracted by $2\delta_{ij}$ should give the bosonic counterparts, which follows from supersymmetry. In the ensuing, **1** shall denote the (trivial) principal representation, $\mathbf{1}'$ and $\mathbf{1}''$, dual (conjugate) pairs of 1 dimensional representations.

$$\widehat{A}_n \quad \begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix} \qquad \begin{array}{l} \mathbf{4} = \mathbf{1}^2 \oplus \mathbf{1}' \oplus \mathbf{1}'' \\ \mathbf{6} = \mathbf{1}^2 \oplus \mathbf{1}'^2 \oplus \mathbf{1}''^2 \end{array}$$

$$\widehat{D}_{n'=\frac{n+6}{2}} \quad \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 2 \end{pmatrix} \qquad D_{n'=\frac{n+3}{2}} \quad \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{pmatrix} \qquad \begin{array}{l} \mathbf{4} = \mathbf{1}^2 \oplus \mathbf{2} \\ \mathbf{6} = \mathbf{1}^2 \oplus \mathbf{2}^2 \end{array}$$

$$\widehat{E}_6 \quad \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad \begin{array}{l} \mathbf{4} = \mathbf{1^2} \oplus \mathbf{2} \\ \mathbf{6} = \mathbf{1^2} \oplus \mathbf{2^2} \end{array}$$

$$\widehat{E}_7 \quad \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \begin{array}{l} \mathbf{4} = \mathbf{1^2} \oplus \mathbf{2} \\ \mathbf{6} = \mathbf{1^2} \oplus \mathbf{2^2} \end{array}$$

$$\widehat{E}_8 \quad \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \quad \begin{array}{l} \mathbf{4} = \mathbf{1^2} \oplus \mathbf{2} \\ \mathbf{6} = \mathbf{1^2} \oplus \mathbf{2^2} \end{array}$$

22.3 Classification of Discrete Subgroups of $SU(3)$

Type I: The Σ Series

These are the analogues of the $SU(2)$ crystallographic groups and their double covers, i.e., the E series. We have:

$$\begin{array}{rcc} \Sigma_{36}, \Sigma_{72}, \Sigma_{216}, \Sigma_{60}, \Sigma_{168}, \Sigma_{360} & \subset & SU(3)/(\mathbb{Z}_3 \text{ center}) \\ \Sigma_{36\times3}, \Sigma_{72\times3}, \Sigma_{216\times3}, \Sigma_{60\times3}, \Sigma_{168\times3}, \Sigma_{360\times3} & \subset & SU(3) \end{array}$$

Type Ia: $\Sigma \subset SU(3)/\mathbb{Z}_3$

The character tables for the center-removed case have been given by [90].

Σ_{36}

	1	9	9	9	4	4
Γ_1	1	1	1	1	1	1
Γ_2	1	-1	-1	1	1	1
Γ_3	1	i	$-i$	-1	1	1
Γ_4	1	$-i$	i	-1	1	1
Γ_5	4	0	0	0	-2	1
Γ_6	4	0	0	0	1	-2

Σ_{72}

	1	18	18	18	9	8
Γ_1	1	1	1	1	1	1
Γ_2	1	1	-1	-1	1	1
Γ_3	1	-1	1	-1	1	1
Γ_4	1	-1	-1	1	1	1
Γ_5	2	0	0	0	-2	2
Γ_6	8	0	0	0	0	-1

Hessian Group
$\Sigma_{216} \supset \Sigma_{72} \supset \Sigma_{36}$

	1	12	12	54	36	36	9	8	24	24
Γ_1	1	1	1	1	1	1	1	1	1	1
Γ_2	1	w	w^2	1	w	w^2	1	1	w	w^2
Γ_3	1	w^2	w	1	w^2	w	1	1	w^2	w
Γ_4	2	-1	-1	0	1	1	-2	2	-1	-1
Γ_5	2	$-w$	$-w^2$	0	w	w^2	-2	2	$-w$	$-w^2$
Γ_6	2	$-w^2$	$-w$	0	w^2	w	-2	2	$-w^2$	$-w$
Γ_7	3	0	0	-1	0	0	3	3	0	0
Γ_8	8	2	2	0	0	0	0	-1	-1	-1
Γ_9	8	$2w$	$2w^2$	0	0	0	0	-1	$-w$	$-w^2$
Γ_{10}	8	$2w^2$	$2w$	0	0	0	0	-1	$-w^2$	$-w$

$w = \exp \frac{2\pi i}{3}$

$\Sigma_{360} \cong A_6$

	1	40	45	72	72	90	40
Γ_1	1	1	1	1	1	1	1
Γ_2	5	2	1	0	0	-1	-1
Γ_3	5	-1	1	0	0	-1	2
Γ_4	8	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	0	-1
Γ_5	8	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	0	-1
Γ_6	9	0	1	-1	-1	1	0
Γ_7	10	1	-2	0	0	0	1

Type Ib: $\Sigma \subset$ **full** $SU(3)$

The character tables are computed, using [92], from the generators presented in [89].

In what follows, we define $e_n = \exp \frac{2\pi i}{n}$.

$\Sigma_{36\times 3}$

	1	12	1	12	9	9	1	9	9	9	9	9	9	9
Γ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
Γ_2	1	1	1	1	-1	-1	1	-1	1	1	1	-1	-1	-1
Γ_3	1	1	1	1	e_4	e_4	1	e_4	-1	-1	-1	$-e_4$	$-e_4$	$-e_4$
Γ_4	1	1	1	1	$-e_4$	$-e_4$	1	$-e_4$	-1	-1	-1	e_4	e_4	e_4
Γ_5	3	0	$3e_3^2$	0	1	e_3	$3e_3$	e_3^2	-1	$-e_3^2$	$-e_3$	1	e_3^2	e_3
Γ_6	3	0	$3e_3$	0	1	e_3^2	$3e_3^2$	e_3	-1	$-e_3$	$-e_3^2$	1	e_3	e_3^2
Γ_7	3	0	$3e_3^2$	0	-1	$-e_3$	$3e_3$	$-e_3^2$	-1	$-e_3^2$	$-e_3$	-1	$-e_3^2$	$-e_3$
Γ_8	3	0	$3e_3$	0	-1	$-e_3^2$	$3e_3^2$	$-e_3$	-1	$-e_3$	$-e_3^2$	-1	$-e_3$	$-e_3^2$
Γ_9	3	0	$3e_3^2$	0	e_4	e_{12}^7	$3e_3$	e_{12}^{11}	1	e_3^2	e_3	$-e_4$	$-e_{12}^{11}$	$-e_{12}^7$
Γ_{10}	3	0	$3e_3$	0	e_4	e_{12}^{11}	$3e_3^2$	e_{12}^7	1	e_3	e_3^2	$-e_4$	$-e_{12}^7$	$-e_{12}^{11}$
Γ_{11}	3	0	$3e_3^2$	0	$-e_4$	$-e_{12}^7$	$3e_3$	$-e_{12}^{11}$	1	e_3^2	e_3	e_4	e_{12}^{11}	e_{12}^7
Γ_{12}	3	0	$3e_3$	0	$-e_4$	$-e_{12}^{11}$	$3e_3^2$	$-e_{12}^7$	1	e_3	e_3^2	e_4	e_{12}^7	e_{12}^{11}
Γ_{13}	4	1	4	-2	0	0	4	0	0	0	0	0	0	0
Γ_{14}	4	-2	4	1	0	0	4	0	0	0	0	0	0	0

$\Sigma_{216\times 3}$

	1	24	1	12	12	54	54	1	12	54	72	12	$\cdots$
Γ_1	1	1	1	1	1	1	1	1	1	1	1	1	$\cdots$
Γ_2	1	1	1	${e_3}^2$	${e_3}^2$	1	1	1	${e_3}^2$	1	${e_3}^2$	e_3	$\cdots$
Γ_3	1	1	1	e_3	e_3	1	1	1	e_3	1	e_3	${e_3}^2$	$\cdots$
Γ_4	2	2	2	-1	-1	0	0	2	-1	0	-1	-1	$\cdots$
Γ_5	2	2	2	$-e_3$	$-e_3$	0	0	2	$-e_3$	0	$-e_3$	$-{e_3}^2$	$\cdots$
Γ_6	2	2	2	$-{e_3}^2$	$-{e_3}^2$	0	0	2	$-{e_3}^2$	0	$-{e_3}^2$	$-e_3$	$\cdots$
Γ_7	3	3	3	0	0	-1	-1	3	0	-1	0	0	$\cdots$
Γ_8	3	0	$3{e_3}^2$	$-{e_9}^2+{e_9}^5$	$-{e_9}^2-2{e_9}^5$	1	e_3	$3e_3$	$2{e_9}^2+{e_9}^5$	${e_3}^2$	0	$-2{e_9}^4-{e_9}^7$	$\cdots$
Γ_9	3	0	$3{e_3}^2$	$-{e_9}^2-2{e_9}^5$	$2{e_9}^2+{e_9}^5$	1	e_3	$3e_3$	$-{e_9}^2+{e_9}^5$	${e_3}^2$	0	${e_9}^4+2{e_9}^7$	$\cdots$
Γ_{10}	3	0	$3{e_3}^2$	$2{e_9}^2+{e_9}^5$	$-{e_9}^2+{e_9}^5$	1	e_3	$3e_3$	$-{e_9}^2-2{e_9}^5$	${e_3}^2$	0	${e_9}^4-{e_9}^7$	$\cdots$
Γ_{11}	3	0	$3e_3$	${e_9}^4-{e_9}^7$	$-2{e_9}^4-{e_9}^7$	1	${e_3}^2$	$3{e_3}^2$	${e_9}^4+2{e_9}^7$	e_3	0	$-{e_9}^2-2{e_9}^5$	$\cdots$
Γ_{12}	3	0	$3e_3$	$-2{e_9}^4-{e_9}^7$	${e_9}^4+2{e_9}^7$	1	${e_3}^2$	$3{e_3}^2$	${e_9}^4-{e_9}^7$	e_3	0	$2{e_9}^2+{e_9}^5$	$\cdots$
Γ_{13}	3	0	$3e_3$	${e_9}^4+2{e_9}^7$	${e_9}^4-{e_9}^7$	1	${e_3}^2$	$3{e_3}^2$	$-2{e_9}^4-{e_9}^7$	e_3	0	$-{e_9}^2+{e_9}^5$	$\cdots$
Γ_{14}	6	0	$6{e_3}^2$	${e_9}^2-{e_9}^5$	${e_9}^2+2{e_9}^5$	0	0	$6e_3$	$-2{e_9}^2-{e_9}^5$	0	0	$2{e_9}^4+{e_9}^7$	$\cdots$
Γ_{15}	6	0	$6{e_3}^2$	${e_9}^2+2{e_9}^5$	$-2{e_9}^2-{e_9}^5$	0	0	$6e_3$	${e_9}^2-{e_9}^5$	0	0	$-{e_9}^4-2{e_9}^7$	$\cdots$
Γ_{16}	6	0	$6{e_3}^2$	$-2{e_9}^2-{e_9}^5$	${e_9}^2-{e_9}^5$	0	0	$6e_3$	${e_9}^2+2{e_9}^5$	0	0	$-{e_9}^4+{e_9}^7$	$\cdots$
Γ_{17}	6	0	$6e_3$	$-{e_9}^4+{e_9}^7$	$2{e_9}^4+{e_9}^7$	0	0	$6{e_3}^2$	$-{e_9}^4-2{e_9}^7$	0	0	${e_9}^2+2{e_9}^5$	$\cdots$
Γ_{18}	6	0	$6e_3$	$2{e_9}^4+{e_9}^7$	$-{e_9}^4-2{e_9}^7$	0	0	$6{e_3}^2$	$-{e_9}^4+{e_9}^7$	0	0	$-2{e_9}^2-{e_9}^5$	$\cdots$
Γ_{19}	6	0	$6e_3$	$-{e_9}^4-2{e_9}^7$	$-{e_9}^4+{e_9}^7$	0	0	$6{e_3}^2$	$2{e_9}^4+{e_9}^7$	0	0	${e_9}^2-{e_9}^5$	$\cdots$
Γ_{20}	8	-1	8	2	2	0	0	8	2	0	-1	2	$\cdots$
Γ_{21}	8	-1	8	$2e_3$	$2e_3$	0	0	8	$2e_3$	0	$-e_3$	$2{e_3}^2$	$\cdots$
Γ_{22}	8	-1	8	$2{e_3}^2$	$2{e_3}^2$	0	0	8	$2{e_3}^2$	0	$-{e_3}^2$	$2e_3$	$\cdots$
Γ_{23}	9	0	$9{e_3}^2$	0	0	-1	$-e_3$	$9e_3$	0	$-{e_3}^2$	0	0	$\cdots$
Γ_{24}	9	0	$9e_3$	0	0	-1	$-{e_3}^2$	$9{e_3}^2$	0	$-e_3$	0	0	$\cdots$

	12	9	9	9	12	72	36	36	36	36	36	36
Γ_1	1	1	1	1	1	1	1	1	1	1	1	1
Γ_2	e_3	1	1	1	e_3	e_3	e_3	e_3	${e_3}^2$	${e_3}^2$	${e_3}^2$	e_3
Γ_3	${e_3}^2$	1	1	1	${e_3}^2$	${e_3}^2$	${e_3}^2$	${e_3}^2$	e_3	e_3	e_3	${e_3}^2$
Γ_4	-1	-2	-2	-2	-1	-1	1	1	1	1	1	1
Γ_5	$-{e_3}^2$	-2	-2	-2	$-{e_3}^2$	$-{e_3}^2$	${e_3}^2$	${e_3}^2$	e_3	e_3	e_3	${e_3}^2$
Γ_6	$-e_3$	-2	-2	-2	$-e_3$	$-e_3$	e_3	e_3	${e_3}^2$	${e_3}^2$	${e_3}^2$	e_3
Γ_7	0	3	3	3	0	0	0	0	0	0	0	0
Γ_8	${e_9}^4 + 2{e_9}^7$	-1	$-{e_3}^2$	$-e_3$	${e_9}^4 - {e_9}^7$	0	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	$-{e_9}^7$
Γ_9	${e_9}^4 - {e_9}^7$	-1	$-{e_3}^2$	$-e_3$	$-2{e_9}^4 - {e_9}^7$	0	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	$-{e_9}^5$	$-{e_9}^4$
Γ_{10}	$-2{e_9}^4 - {e_9}^7$	-1	$-{e_3}^2$	$-e_3$	${e_9}^4 + 2{e_9}^7$	0	$-{e_9}^7$	$-{e_9}^4$	$-{e_9}^2$	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	${e_9}^4 + {e_9}^7$
Γ_{11}	$2{e_9}^2 + {e_9}^5$	-1	$-e_3$	$-{e_3}^2$	$-{e_9}^2 + {e_9}^5$	0	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	$-{e_9}^2$
Γ_{12}	$-{e_9}^2 + {e_9}^5$	-1	$-e_3$	$-{e_3}^2$	$-{e_9}^2 - 2{e_9}^5$	0	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	$-{e_9}^4$	$-{e_9}^5$
Γ_{13}	$-{e_9}^2 - 2{e_9}^5$	-1	$-e_3$	$-{e_3}^2$	$2{e_9}^2 + {e_9}^5$	0	$-{e_9}^2$	$-{e_9}^5$	$-{e_9}^7$	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	${e_9}^2 + {e_9}^5$
Γ_{14}	$-{e_9}^4 - 2{e_9}^7$	2	$2{e_3}^2$	$2e_3$	$-{e_9}^4 + {e_9}^7$	0	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	$-{e_9}^7$
Γ_{15}	$-{e_9}^4 + {e_9}^7$	2	$2{e_3}^2$	$2e_3$	$2{e_9}^4 + {e_9}^7$	0	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	$-{e_9}^5$	$-{e_9}^4$
Γ_{16}	$2{e_9}^4 + {e_9}^7$	2	$2{e_3}^2$	$2e_3$	$-{e_9}^4 - 2{e_9}^7$	0	$-{e_9}^7$	$-{e_9}^4$	$-{e_9}^2$	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	${e_9}^4 + {e_9}^7$
Γ_{17}	$-2{e_9}^2 - {e_9}^5$	2	$2e_3$	$2{e_3}^2$	${e_9}^2 - {e_9}^5$	0	$-{e_9}^5$	${e_9}^2 + {e_9}^5$	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	$-{e_9}^2$
Γ_{18}	${e_9}^2 - {e_9}^5$	2	$2e_3$	$2{e_3}^2$	${e_9}^2 + 2{e_9}^5$	0	${e_9}^2 + {e_9}^5$	$-{e_9}^2$	${e_9}^4 + {e_9}^7$	$-{e_9}^7$	$-{e_9}^4$	$-{e_9}^5$
Γ_{19}	${e_9}^2 + 2{e_9}^5$	2	$2e_3$	$2{e_3}^2$	$-2{e_9}^2 - {e_9}^5$	0	$-{e_9}^2$	$-{e_9}^5$	$-{e_9}^7$	$-{e_9}^4$	${e_9}^4 + {e_9}^7$	${e_9}^2 + {e_9}^5$
Γ_{20}	2	0	0	0	2	-1	0	0	0	0	0	0
Γ_{21}	$2{e_3}^2$	0	0	0	$2{e_3}^2$	$-{e_3}^2$	0	0	0	0	0	0
Γ_{22}	$2e_3$	0	0	0	$2e_3$	$-e_3$	0	0	0	0	0	0
Γ_{23}	0	-3	$-3{e_3}^2$	$-3e_3$	0	0	0	0	0	0	0	0
Γ_{24}	0	-3	$-3e_3$	$-3{e_3}^2$	0	0	0	0	0	0	0	0

$\Sigma_{360\times 3}$

	1	72	72	72	72	72	72	1	1	$\cdots$
Γ_1	1	1	1	1	1	1	1	1	1	$\cdots$
Γ_2	3	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$	$-e_{15} - {e_{15}}^4$	$-{e_{15}}^7 - {e_{15}}^{13}$	$-{e_{15}}^{11} - {e_{15}}^{14}$	$-{e_{15}}^2 - {e_{15}}^8$	$3{e_3}^2$	$3e_3$	$\cdots$
Γ_3	3	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$	$-{e_{15}}^7 - {e_{15}}^{13}$	$-e_{15} - {e_{15}}^4$	$-{e_{15}}^2 - {e_{15}}^8$	$-{e_{15}}^{11} - {e_{15}}^{14}$	$3{e_3}^2$	$3e_3$	$\cdots$
Γ_4	3	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$	$-{e_{15}}^{11} - {e_{15}}^{14}$	$-{e_{15}}^2 - {e_{15}}^8$	$-e_{15} - {e_{15}}^4$	$-{e_{15}}^7 - {e_{15}}^{13}$	$3e_3$	$3{e_3}^2$	$\cdots$
Γ_5	3	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$	$-{e_{15}}^2 - {e_{15}}^8$	$-{e_{15}}^{11} - {e_{15}}^{14}$	$-{e_{15}}^7 - {e_{15}}^{13}$	$-e_{15} - {e_{15}}^4$	$3e_3$	$3{e_3}^2$	$\cdots$
Γ_6	5	0	0	0	0	0	0	5	5	$\cdots$
Γ_7	5	0	0	0	0	0	0	5	5	$\cdots$
Γ_8	6	1	1	${e_3}^2$	${e_3}^2$	e_3	e_3	$6{e_3}^2$	$6e_3$	$\cdots$
Γ_9	6	1	1	e_3	e_3	${e_3}^2$	${e_3}^2$	$6e_3$	$6{e_3}^2$	$\cdots$
Γ_{10}	8	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$	8	8	$\cdots$
Γ_{11}	8	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$	$-e_5 - {e_5}^4$	$-{e_5}^2 - {e_5}^3$	8	8	$\cdots$
Γ_{12}	9	-1	-1	-1	-1	-1	-1	9	9	$\cdots$
Γ_{13}	9	-1	-1	$-{e_3}^2$	$-{e_3}^2$	$-e_3$	$-e_3$	$9{e_3}^2$	$9e_3$	$\cdots$
Γ_{14}	9	-1	-1	$-e_3$	$-e_3$	$-{e_3}^2$	$-{e_3}^2$	$9e_3$	$9{e_3}^2$	$\cdots$
Γ_{15}	10	0	0	0	0	0	0	10	10	$\cdots$
Γ_{16}	15	0	0	0	0	0	0	$15{e_3}^2$	$15e_3$	$\cdots$
Γ_{17}	15	0	0	0	0	0	0	$15e_3$	$15{e_3}^2$	$\cdots$

	120	120	45	45	90	90	45	90
Γ_1	1	1	1	1	1	1	1	1
Γ_2	0	0	$-e_3$	$-e_3{}^2$	$e_3{}^2$	e_3	-1	1
Γ_3	0	0	$-e_3$	$-e_3{}^2$	$e_3{}^2$	e_3	-1	1
Γ_4	0	0	$-e_3{}^2$	$-e_3$	e_3	$e_3{}^2$	-1	1
Γ_5	0	0	$-e_3{}^2$	$-e_3$	e_3	$e_3{}^2$	-1	1
Γ_6	2	-1	1	1	-1	-1	1	-1
Γ_7	-1	2	1	1	-1	-1	1	-1
Γ_8	0	0	$2e_3$	$2e_3{}^2$	0	0	2	0
Γ_9	0	0	$2e_3{}^2$	$2e_3$	0	0	2	0
Γ_{10}	-1	-1	0	0	0	0	0	0
Γ_{11}	-1	-1	0	0	0	0	0	0
Γ_{12}	0	0	1	1	1	1	1	1
Γ_{13}	0	0	e_3	$e_3{}^2$	$e_3{}^2$	e_3	1	1
Γ_{14}	0	0	$e_3{}^2$	e_3	e_3	$e_3{}^2$	1	1
Γ_{15}	1	1	-2	-2	0	0	-2	0
Γ_{16}	0	0	$-e_3$	$-e_3{}^2$	$-e_3{}^2$	$-e_3$	-1	-1
Γ_{17}	0	0	$-e_3{}^2$	$-e_3$	$-e_3$	$-e_3{}^2$	-1	-1

Type Ic: $\Sigma \subset$ both $SU(3)$ and $SU(3)/\mathbb{Z}_3$

$$\Sigma_{60} \cong A_5 \cong I$$

	1	20	15	12	12
Γ_1	1	1	1	1	1
Γ_2	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
Γ_3	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
Γ_4	4	1	0	-1	-1
Γ_5	5	-1	1	0	0

$$\Sigma_{168} \subset S_7$$

	1	21	42	56	24	24
Γ_1	1	1	1	1	1	1
Γ_2	3	-1	1	0	$\frac{-1+i\sqrt{7}}{2}$	$\frac{-1-i\sqrt{7}}{2}$
Γ_3	3	-1	1	0	$\frac{-1-i\sqrt{7}}{2}$	$\frac{-1+i\sqrt{7}}{2}$
Γ_4	6	2	0	0	-1	-1
Γ_5	7	-1	-1	1	0	0
Γ_6	8	0	0	-1	1	1

Type II: The Δ series

These are the analogues of the dihedral subgroups of $SU(2)$ (i.e., the D series).

Δ_{3n^2}

	Number of classes	Subgroup of	Some Irreps
$n = 0 \bmod 3$	$8 + \frac{1}{3}n^2$	Full $SU(3)$	9 **1**'s$\frac{1}{3}n^2 - 1$ **3**'s
$n \neq 0 \bmod 3$	$\frac{1}{3}(8 + n^2)$	Full $SU(3)$ and $SU(3)/\mathbb{Z}_3$	3 **1**'s, $\frac{1}{3}(n^2 - 1)$ **3**'s

Δ_{6n^2}

	Number of classes	Subgroup of	Some Irreps
$n = 0 \bmod 3$	$\frac{1}{6}(24 + 9n + n^2)$	Full $SU(3)$	–
$n \neq 0 \bmod 3$	$\frac{1}{6}(8 + 9n + n^2)$	Full $SU(3)$ and $SU(3)/\mathbb{Z}_3$	2 $\mathbf{1}$'s, 1 $\mathbf{2}$, $2(n-1)$ $\mathbf{3}$'s, $\frac{1}{6}(n^2 - 3n + 2)$ $\mathbf{6}$'s

22.4 Matter content for $\Gamma \subset SU(3)$

Note here that since the $\mathcal{N} = 1$ theory is chiral, the fermion matter matrix need not be symmetric. A graphic representation for some of these theories appear in figures 3, 4 and 5.

	Fermionic	Bosonic
Σ_{36}	$\begin{pmatrix} 2&0&1&1&0&0\\0&2&1&1&0&0\\1&1&2&0&0&0\\1&1&0&2&0&0\\0&0&0&0&4&0\\0&0&0&0&0&4 \end{pmatrix}$ $\mathbf{1} \oplus (\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'')$	$\begin{pmatrix} 2&0&2&2&0&0\\0&2&2&2&0&0\\2&2&2&0&0&0\\2&2&0&2&0&0\\0&0&0&0&6&0\\0&0&0&0&0&6 \end{pmatrix}$ $(\mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{1}'')^2$
Σ_{60}	$\begin{pmatrix} 1&1&0&0&0\\1&2&0&0&1\\0&0&1&1&1\\0&0&1&2&1\\0&1&1&1&2 \end{pmatrix}$ $\mathbf{1} \oplus \mathbf{3}$	$\begin{pmatrix} 0&1&1&0&0\\1&1&0&1&2\\1&0&1&1&2\\0&1&1&2&2\\0&2&2&2&2 \end{pmatrix}$ $\mathbf{3} \oplus \mathbf{3}'$
Σ_{72}	$\begin{pmatrix} 1&1&0&0&1&0\\1&1&0&0&1&0\\0&0&1&1&1&0\\0&0&1&1&1&0\\1&1&1&1&2&0\\0&0&0&0&0&4 \end{pmatrix}$ $\mathbf{1} \oplus (\mathbf{1}' \oplus \mathbf{2})$	$\begin{pmatrix} 1&1&0&0&2&0\\1&1&0&0&2&0\\0&0&1&1&2&0\\0&0&1&1&2&0\\2&2&2&2&2&0\\0&0&0&0&0&6 \end{pmatrix}$ $(\mathbf{1} \oplus \mathbf{2}) \oplus (\mathbf{1}' \oplus \mathbf{2})$

$$\Sigma_{168} \quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}} \quad \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}}$$

$$\Sigma_{216} \quad \underset{\mathbf{1}\oplus(\mathbf{1}\oplus\mathbf{2})}{\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}} \quad \underset{(\mathbf{1}\oplus\mathbf{2})\oplus(\mathbf{1}\oplus\mathbf{2}')}{\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 4 \end{pmatrix}}$$

$$\Sigma_{360} \quad \underset{\mathbf{1}^{\mathbf{4}}}{4(\delta_{ij})_{7\times 7}} \quad \underset{\mathbf{1}^{\mathbf{6}}}{6(\delta_{ij})_{7\times 7}}$$

$$\Sigma_{36\times 3} \quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}} \quad \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}}$$

$$\Sigma_{216\times 3}$$

$$\underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix}
1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&1&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&0&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1
\end{pmatrix}}
\quad
\underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix}
0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0\\
1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&1&0&0&0\\
0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0&0&0\\
1&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&0&0&1\\
0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&1&1&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&0&1&0&1\\
0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&1&0&1&1&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&1&1&1&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&1&1&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&1&1&0&1&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&1&1&0&0&0&1&1\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&1&1&0&0&0&0&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&0&0&0&1&1&1&1&0
\end{pmatrix}}$$

$\Sigma_{360\times 3}$

$$\underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix}
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&1&1&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&2\\
0&0&0&0&0&1&1&0&0&1&1&1&0&0&1&0&1
\end{pmatrix}}
\quad
\underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix}
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&0&0&0&0&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0\\
1&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&1\\
0&1&0&0&0&0&0&0&0&1&0&0&0&0&1&1&0\\
0&1&0&1&0&0&0&1&1&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1\\
0&0&1&0&1&0&0&0&0&0&0&0&1&1&0&1&1\\
0&0&0&0&1&0&0&0&0&0&1&1&0&1&1&0&1\\
0&0&1&0&0&0&0&0&0&0&1&1&1&0&1&1&0\\
0&0&0&0&0&0&0&1&1&0&0&0&1&1&0&1&1\\
0&0&0&0&0&1&1&0&1&1&1&1&0&1&1&0&2\\
0&0&0&0&0&1&1&1&0&1&1&1&1&0&1&2&0
\end{pmatrix}}$$

Δ_{3n^2}

$$n=2 \quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix}
1&0&0&2\\
0&1&0&2\\
0&0&1&2\\
2&2&2&4
\end{pmatrix}}
\quad
\underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix}
0&0&0&4\\
0&0&0&4\\
0&0&0&4\\
4&4&4&6
\end{pmatrix}}$$

$$n=3 \quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix} 1&0&0&0&0&0&0&0&0&1&0\\ 0&1&0&0&0&0&0&0&0&1&0\\ 0&0&1&0&0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&0&1&0\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&0&1&3\\ 1&1&1&1&1&1&1&1&1&0&1 \end{pmatrix}} \quad \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix} 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1&1\\ 0&0&0&0&0&0&0&0&0&1&1\\ 1&1&1&1&1&1&1&1&1&0&3\\ 1&1&1&1&1&1&1&1&1&3&0 \end{pmatrix}}$$

$$n=4 \quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix} 1&0&0&0&1&0&0&0\\ 0&1&0&0&1&0&0&0\\ 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&1&1\\ 0&0&0&1&1&2&0&0\\ 1&1&1&0&0&1&1&1\\ 0&0&0&1&1&0&2&0\\ 0&0&0&1&1&0&0&2 \end{pmatrix}} \quad \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix} 0&0&0&0&1&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&1&1&2&2\\ 1&1&1&1&0&2&1&1\\ 1&1&1&1&2&0&1&1\\ 0&0&0&2&1&1&2&0\\ 0&0&0&2&1&1&0&2 \end{pmatrix}}$$

$$n=5 \quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix} 1&0&0&0&0&1&0&0&0&0&0\\ 0&1&0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&1&0&0&0&0&0\\ 0&0&0&1&1&0&0&0&0&1&1\\ 0&0&0&0&1&0&1&1&1&0&0\\ 0&0&0&1&0&1&2&0&0&0&0\\ 1&1&1&0&0&0&1&0&0&1&1\\ 0&0&0&1&0&0&0&2&0&0&1\\ 0&0&0&1&0&0&0&0&2&1&0\\ 0&0&0&0&1&1&0&0&1&1&0\\ 0&0&0&0&1&1&0&1&0&0&1 \end{pmatrix}} \quad \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix} 0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&0&1&1&1&1\\ 0&0&0&1&0&0&1&1&1&1&1\\ 1&1&1&1&0&0&2&0&0&1&1\\ 1&1&1&0&1&2&0&0&0&1&1\\ 0&0&0&1&1&0&0&2&0&0&2\\ 0&0&0&1&1&0&0&0&2&2&0\\ 0&0&0&1&1&1&1&0&2&0&0\\ 0&0&0&1&1&1&1&2&0&0&0 \end{pmatrix}}$$

$$n=6 \quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&1&1&2&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&0&0&0&0&1&2&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&2&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&2&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&2&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&2
\end{pmatrix}} \quad \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&1&1&1&1&1&0&2&2&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&0&2&0&2&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&1&1&1&2&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&2&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&2&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&2&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&2&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&1&2&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&1&1&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&1&0&0&0&2&1\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&1&0&0&0&1&2
\end{pmatrix}}$$

Δ_{6n^2}

$$n=2 \quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix}
1&0&0&1&0\\
0&1&0&0&1\\
0&0&1&1&1\\
1&0&1&2&1\\
0&1&1&1&2
\end{pmatrix}} \quad \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix}
0&0&0&1&1\\
0&0&0&1&1\\
0&0&0&2&2\\
1&1&2&2&2\\
1&1&2&2&2
\end{pmatrix}}$$

$$n=4 \quad \underset{\mathbf{1}\oplus\mathbf{3}}{\begin{pmatrix}
1&0&0&1&0&0&0&0&0&0\\
0&1&0&0&1&0&0&0&0&0\\
0&0&1&1&1&0&0&0&0&0\\
1&0&1&2&1&0&0&0&0&0\\
0&1&1&1&2&0&0&0&0&0\\
0&0&0&0&0&1&0&0&1&1\\
0&0&0&0&0&0&1&1&0&1\\
0&0&0&0&0&0&1&1&0&1\\
0&0&0&0&0&1&0&0&1&1\\
0&0&0&0&0&1&1&1&1&2
\end{pmatrix}} \quad \underset{\mathbf{3}\oplus\mathbf{3}'}{\begin{pmatrix}
0&0&0&1&1&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0\\
0&0&0&2&2&0&0&0&0&0\\
1&1&2&2&2&0&0&0&0&0\\
1&1&2&2&2&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&2\\
0&0&0&0&0&0&0&1&1&2\\
0&0&0&0&0&1&1&0&0&2\\
0&0&0&0&0&1&1&0&0&2\\
0&0&0&0&0&2&2&2&2&2
\end{pmatrix}}$$

$$
n = 6 \qquad
\underset{\mathbf{1} \oplus \mathbf{3}}{\begin{pmatrix}
1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0\\
1&0&1&0&0&0&2&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&0&1&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&3&0&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&2&0\\
0&0&0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0&2
\end{pmatrix}}
\qquad
\underset{\mathbf{3} \oplus \mathbf{3}'}{\begin{pmatrix}
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\
1&1&2&0&0&0&2&2&0&0&0&0&0&0&0&0&0&0&0\\
1&1&2&0&0&0&2&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&2&0\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&2&0\\
0&0&0&2&2&2&0&0&0&0&0&0&0&0&0&0&4&0&0\\
0&0&0&0&0&0&0&0&0&0&2&2&0&0&2&2&0&2&0\\
0&0&0&0&0&0&0&0&2&2&0&0&2&2&0&0&0&0&2
\end{pmatrix}}
$$

$$\mathrm{n} = 8$$

$$
\underset{\mathbf{1} \oplus \mathbf{3}}{\begin{pmatrix}
1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&2&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&2
\end{pmatrix}}
\qquad
\underset{\mathbf{3} \oplus \mathbf{3}'}{\begin{pmatrix}
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&2&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&2&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&2\\
0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&2&0&0&0&0&0&0&2&2&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&2&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&2&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&2&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&2&2&0&0&0\\
0&0&0&0&0&2&2&0&0&0&0&0&0&2&2&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&2&2&2&2&0&0&0&0&0&0&0&0&0&0&2
\end{pmatrix}}
$$

22.5 Steinberg's Proof of Semi-Definity

We here transcribe Steinberg's proof of the semi-definity of the scalar product with respect to the generalised Cartan matrix, in the vector space $V = \{x_i \in \mathbb{Z}_+\}$ of labels

[179]. Our starting point is (9.2.1), which we re-write here as

$$r_d \otimes r_i = \bigoplus_j a_{ij} r_j$$

First we note that, if $\bar{i}$ is the dual representation to i, then $a_{ij} = a_{\bar{j}\bar{i}}$ by taking the conjugates (dual) of both sides of (9.2.1). Whence we have

LEMMA 22.5.3 *For $d_i = \dim r_i$, $dd_i = \sum_j a_{ij} d_j = \sum_j a_{ji} d_j$.*

The first equality is obtained directly by taking the dimension of both sides of (9.2.1) as in (12.2.2). To see the second we have $dd_i = dd_{\bar{i}}$ (as dual representations have the same dimension) which is thus equal to $\sum_j a_{\bar{i}j} d_j$, and then by the dual property $a_{ij} = a_{\bar{j}\bar{i}}$ above becomes $\sum_{\bar{j}} a_{\bar{j}i} d_{\bar{j}} = \sum_j a_{ji} d_j$. QED.

Now consider the following for the scalar product:

$$\begin{aligned}
2\sum_{ij} c_{ij} x_i x_j &= 2\sum_{ij}(d\delta_{ij} - a_{ij}) x_i x_j = 2(d\sum_i x_i^2 - \sum_{ij} a_{ij} x_i x_j) \\
&= 2(\sum_i (d - a_{ii}) x_i^2 - \sum_{i \neq j} a_{ij} x_i x_j) \\
&= 2\sum_i \frac{1}{2}(\frac{1}{d_i}\sum_j a_{ij} d_j + \frac{1}{d_i}\sum_j a_{ji} d_j - a_{ii}) x_i^2 - \sum_{i \neq j} a_{ij} x_i x_j) \quad \text{(by Lemma 22.5.3)} \\
&= \sum_{i \neq j} (a_{ij} + a_{ji}) \frac{d_j}{d_i} x_i^2 - 2a_{ij} x_i x_j = \sum_{i<j} (a_{ij} + a_{ji})(\frac{d_j}{d_i} x_i^2 + \frac{d_i}{d_j} x_j^2 - 2x_i x_j) \\
&= \sum_{i<j} (a_{ij} + a_{ji}) \frac{(d_j x_i - d_i x_j)^2}{d_i d_j} \geq 0
\end{aligned}$$

From which we conclude

PROPOSITION 22.5.10 (Steinberg) *In the vector space of positive labels, the scalar product is positive semi-definite, i.e., $\sum_{ij} c_{ij} x_i x_j \geq 0$.*

22.6 Conjugacy Classes for $Z_k \times D_{k'}$

Using the notation introduced in §14.3, we see that the conjugation within G gives

$$(q, \tilde{q}, \tilde{n}, k)^{-1}(m, \tilde{m}, n, p)(q, \tilde{q}, \tilde{n}, k) = \begin{cases} (\tilde{m} + q - \tilde{q}, m - q + \tilde{q}, n, 2k - p) \text{ for } n = 0, \tilde{n} = 0 \\ (m - q + \tilde{q}, \tilde{m} + q - \tilde{q}, n, 2k + p) \text{ for } n = 0, \tilde{n} = 1 \\ (\tilde{m}, m, n, -p) \text{ for } n = 1, \tilde{n} = 0 \\ (m, \tilde{m}, n, p) \text{ for } n = 1, \tilde{n} = 1. \end{cases} \tag{22.6.1}$$

Also, we present the multiplication rules in G for reference:

$$\begin{aligned} (m, \tilde{m}, 0, p_1)(n, \tilde{n}, 0, p_2) &= (m + \tilde{n}, \tilde{m} + n, 1, p_2 - p_1) \\ (m, \tilde{m}, 0, p_1)(n, \tilde{n}, 1, p_2) &= (m + \tilde{n}, \tilde{m} + n, 0, p_2 + p_1 - k') \\ (m, \tilde{m}, 1, p_1)(n, \tilde{n}, 0, p_2) &= (m + n, \tilde{m} + \tilde{n}, 0, p_2 - p_1 - k') \\ (m, \tilde{m}, 1, p_1)(n, \tilde{n}, 1, p_2) &= (m + n, \tilde{m} + \tilde{n}, 1, p_2 + p_1 - k') \end{aligned} \tag{22.6.2}$$

First we focus on the conjugacy class of elements such that $n = 0$. From (14.3.3) and (22.6.1), we see that if two elements are within the same conjugacy class, then they must have the same $m + \tilde{m} \bmod k$. Now we need to distinguish between two cases:

- (I) if $\frac{2k'}{(k,2k')} =$ even, the orbit conditions conserve the parity of p, making even and odd p belong to different conjugacy classes;
- (II) if $\frac{2k'}{(k,2k')} =$ odd, the orbit conditions change p and we find that all p belong to the same conjugacy class they have the same value for $m + \tilde{m}$.

In summary then, for $\frac{2k'}{(k,2k')} =$ even, we have $2k$ conjugacy classes each of which has $\frac{k'k}{(k,2k')}$ elements; for $\frac{2k'}{(k,2k')} =$ odd, we have k conjugacy classes each of which has $\frac{2k'k}{(k,2k')}$ elements.

Next we analyse the conjugacy class corresponding to $n = 1$. For simplicity, we divide the interval $[0,k)$ by factor $(k,2k')$ and define

$$V_i = \left[\frac{ik}{(k,2k')}, \frac{(i+1)k}{(k,2k')}\right)$$

with $i = 0,...,(k,2k')-1$. Now from (14.3.3), we can always fix m to belong V_0. Thereafter, $\tilde{m}$ and p can change freely within $[0,k)/[0,2k')$. Again, we have two different cases. (I) If $\frac{2k'}{(k,2k')}$ = even, for every subinterval V_i we have $2k_0$ (we define $k_0 := 2\frac{k}{(k,2k')}$) conjugacy classes each containing only one element, namely,

$$(m, \tilde{m} = m + \frac{ik}{(k,2k')}, n = 1, p = k' - \frac{ik'}{(k,2k')} \text{ or } 2k' - \frac{ik'}{(k,2k')}).$$

Also we have a total of $k_0\frac{2k'-2}{2} + \frac{k_0(k_0-1)}{2}2k' = k_0(k'-1) + k'k_0^2 + k_0k' = k'k_0^2 - k_0$ conjugacy classes of 2 elements, namely $(m, \tilde{m}, n = 1, p)$ and $(\tilde{m} - \frac{ik}{(k,2k')}, m + \frac{ik}{(k,2k')}, n = 1, -p - i\frac{2k'}{(k,2k')})$. Indeed, the total number of conjugacy classes is $2k + (k,2k')(2k_0) + (k,2k')(k'k_0^2 - k_0) = 4k + k(\frac{k'k}{(k,2k')} - 1)$, giving the order of G as expected. Furthermore, there are $4k$ 1-dimensional irreducible representations and $k(\frac{k'k}{(k,2k')} - 1)$ 2-dimensional irreducible representations. This is consistent since $\sum_i \dim\mathbf{r}_i = 1^2 \cdot 4k + 2^2 \cdot k(\frac{k'k}{(k,2k')} - 1) = \frac{4k'k^2}{(k,2k')} = |G|$.

We summarize case (I) into the following table:

	$C_{n=0}^{m+\tilde{m}(\bmod k), p=\text{odd/even}}$	$C_{n=1,V_i}^{\tilde{m}=m+\frac{ik}{(k,2k')}, p=(k'-\frac{ik'}{(k,2k')})/(2k'-\frac{ik'}{(k,2k')})}$	$C_{n=1,V_i}^{(m,\tilde{m},p)=(\tilde{m}-\frac{ik}{(k,2k')}, m+\frac{ik}{(k,2k')}, -p-i\frac{2k'}{(k,2k')})}$
$\lvert C\rvert$	$\frac{k'k}{(k,2k')}$	1	2
$\#C$	$2k$	$2k$	$k(\frac{k'k}{(k,2k')} - 1)$

Now let us treat case (II), where $\frac{2k'}{(k,2k')}$ is odd (note that in this case we must have k even). Here, for V_i and i even, the situation is as (I) but for i odd there are no one-element conjugacy classes. We tabulate the conjugacy classes in the following:

	$C_{n=0}^{m+\tilde{m}(\bmod k), \text{any p}}$	$C_{n=1,V_i,i=\text{even}}^{\tilde{m}=m+\frac{ik}{(k,2k')}, p=(k'-\frac{ik'}{(k,2k')})/(2k'-\frac{ik'}{(k,2k')})}$	$C_{n=1,V_i}^{(m,\tilde{m},p)=(\tilde{m}-\frac{ik}{(k,2k')}, m+\frac{ik}{(k,2k')}, -p-i\frac{2k'}{(k,2k')})}$
$\lvert C\rvert$	$\frac{2k'k}{(k,2k')}$	1	2
$\#C$	k	$2\frac{k}{2} = k$	$\frac{(k,2k')}{2}[(k'k_0^2 - k_0) + k'k_0^2] = k(\frac{k'k}{(k,2k')} - \frac{1}{2})$

22.7 Some Explicit Computations for $M(G)$

22.7.1 Preliminary Definitions

We begin with a few rudimentary definitions [262]. Let H be a subgroup of G and let $g \in G$. For any cocycle $\alpha \in Z^2(G, \mathbb{C}^*)$ we define an induced action $g \cdot \alpha \in Z^2(gHg^{-1}, \mathbb{C}^*)$ thereon as $g \cdot \alpha(x,y) = \alpha(g^{-1}xg, g^{-1}yg), \quad \forall \;\; x,y \in gHg^{-1}$. Now, it can be proved that the mapping

$$c_g : M(H) \to M(gHg^{-1}), \quad c_g(\alpha) := g \cdot \alpha$$

is a homomorphism, which we call **cocycle conjugation** by g.

On the other hand we have an obvious concept of restriction: for $S \subseteq L$ subgroups of G, we denote by $\mathrm{Res}_{L,S}$ the restriction map $M(L) \to M(S)$. Thereafter we define stability as:

DEFINITION 22.7.30 *Let H and K be arbitrary subgroups of G. An element $\alpha \in M(H)$ is said to be* **K-stable** *if*

$$\mathrm{Res}_{H,gHg^{-1}\cap H}(\alpha) = \mathrm{Res}_{gHg^{-1},gHg^{-1}\cap H}(c_g(\alpha)) \;\; \forall \;\; g \in K.$$

The set of all K-stable elements of $M(H)$ will be denoted by $M(H)^K$ and it forms a subgroup of $M(H)$ known as the K-stable subgroup of $M(H)$.

When $K \subseteq N_G(H)$ all the above concepts[1] coalesce and we have the following important lemma:

LEMMA 22.7.4 ([262] p299) *If H and K are subgroups of G such that $K \subseteq N_G(H)$, then $M(H)^K$ is the K-stable subgroup of $M(H)$ with respect to the action of K on*

[1] $N_G(H)$ is the normalizer of H in G, i.e., the set of all elements $g \in G$ such that $gHg^{-1} = H$. When H is a normal subgroup of G we obviously have $N_G(H) = G$.

$M(H)$ induced by the action of K on H by conjugation. In other words,

$$M(H)^K = \{\alpha \in M(H),\ \ \alpha(x,y) = c_g(\alpha)(x,y)\ \ \forall\ \ g \in K,\ \ \forall\ \ x,y \in H\}.$$

Finally let us present a useful class of subgroups:

DEFINITION 22.7.31 *A subgroup H of a group G is called a* **Hall subgroup** *of G if the order of H is coprime with its index in G, i.e.* $\gcd(|H|,|G/H|) = 1$.

For these subgroups we have:

THEOREM 22.7.35 ([262] p334) *If N is a normal Hall subgroup of G. Then*

$$M(G) \cong M(N)^{G/N} \times M(G/N).$$

The above theorem is really a corollary of a more general case of semi-direct products:

THEOREM 22.7.36 ([264] p33) *Let $G = N \rtimes T$ with $N \triangleleft G$, then*
(i) $M(G) \cong M(T) \times \tilde{M}(G)$;
(ii) The sequence $1 \to H^1(T,N^*) \to \tilde{M}(G) \xrightarrow{\text{Res}} M(N)^T \to H^2(T,N^*)$ *is exact, where* $\tilde{M}(G) := \ker \text{Res}_{G,N}$, $N^* := \text{Hom}(N,\mathbb{C}^*)$ *and* $H^{i=1,2}(T,N^*)$ *is the cohomology defined with respect to the conjugation action by T on N^*.*

Part (ii) of this theorem actually follows from the Lyndon-Hochschild-Serre spectral sequence into which we shall not delve.

One clarification is needed at hand. Let us define the first A-valued cohomology group for G, which we shall utilise later in our calculations. Here the 1-cocycles are the set of functions $Z^1(G,A) := \{f : G \to A | f(xy) = (x \cdot f(y))f(x) \quad \forall x,y \in G\}$, where A is being acted upon ($x \cdot A \to A$ for $x \in G$) by G as a $\mathbb{Z}G$-module. These are known as *crossed homomorphisms*. On the other hand, the 1-coboundaries are what is known as the principal crossed homomorphisms, $B^1(G,A) := \{f_{a\in A}(x) = (x \cdot a)a^{-1}\}$ from which we define $H^1(G,A) := Z^1(G,A)/B^1(G,A)$.

Alas, *caveat emptor*, we have defined in subsection 2.2, $H^2(G, A)$. There, the action of G on A (as in the case of the Schur Multiplier) is taken to be trivial, we must be careful, in the ensuing, to compute with respect to non-trivial actions such as conjugation. In our case the conjugation action of $t \in T$ on $\chi \in \mathrm{Hom}(N, \mathbb{C}^*)$ is given by $\chi(tnt^{-1})$ for $n \in N$.

22.7.2 The Schur Multiplier for Δ_{3n^2}

Case I: $\gcd(n, 3) = 1$

Thus equipped, we can now use theorem 22.7.35 at our ease to compute the Schur multipliers the first case of the finite groups Δ_{3n^2}. Recall that $\mathbb{Z}_n \times \mathbb{Z}_n \triangleleft \Delta(3n^2)$ or explicitly

$$\Delta_{3n^2} \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes \mathbb{Z}_3.$$

Our crucial observation is that when $\gcd(n, 3) = 1$, $\mathbb{Z}_n \times \mathbb{Z}_n$ is in fact a normal Hall subgroup of Δ_{3n^2} with quotient group $\mathbb{Z}_3$. Whence Theorem 22.7.35 can be immediately applied to this case when n is coprime to 3:

$$M(\Delta_{3n^2}) = (M(\mathbb{Z}_n \times \mathbb{Z}_n))^{\mathbb{Z}_3} \times M(\mathbb{Z}_3) = (M(\mathbb{Z}_n \times \mathbb{Z}_n))^{\mathbb{Z}_3},$$

by recalling that the Schur Multiplier of all cyclic groups is trivial and that of $\mathbb{Z}_n \times \mathbb{Z}_n$ is $\mathbb{Z}_n$ [262]. But, $\mathbb{Z}_3 \subseteq N_{\Delta_{3n^2}}(\mathbb{Z}_n \times \mathbb{Z}_n) = \Delta_{3n^2}$, and hence by Lemma 22.7.4 it suffices to compute the $\mathbb{Z}_3$-stable subgroup of $\mathbb{Z}_n$ by cocycle conjugation.

Let the quotient group $\mathbb{Z}_3$ be $\langle z | z^3 = \mathbb{I} \rangle$ and similarly, if $x, y, x^n = y^n = \mathbb{I}$ are the generators of $\mathbb{Z}_n \times \mathbb{Z}_n$, then a generic element thereof becomes $x^a y^b, a, b = 0, \ldots, n-1$. The group conjugation by z on such an element gives

$$z^{-1} x^a y^b z = x^b y^{-a-b} \qquad z x^a y^b z^{-1} = x^{-a-b} y^a. \tag{22.7.3}$$

It is easy now to check that if α is a generator of the Schur multiplier $\mathbb{Z}_n$, we have an induced action

$$c_z(\alpha)(x^a y^b, x^{a'} y^{b'}) := \alpha(z^{-1}x^a y^b z, z^{-1}x^{a'} y^{b'} z) = \alpha(x^b y^{-(a+b)}, x^{b'} y^{-(a'+b')})$$

by Lemma 22.7.4.

However, we have a well-known result [30]:

PROPOSITION 22.7.11 *For the group $\mathbb{Z}_n \times \mathbb{Z}_n$, the explicit generator of the Schur Multiplier is given by*

$$\alpha(x^a y^b, x^{a'} y^{b'}) = \omega_n^{ab'-a'b}.$$

Consequently, $\alpha(x^b y^{-(a+b)}, x^{b'} y^{-(a'+b')}) = \alpha(x^a y^b, x^{a'} y^{b'})$ whereby making the c_z-action trivial and causing $(M(\mathbb{Z}_n \times \mathbb{Z}_n)^{\mathbb{Z}_3} \cong M(\mathbb{Z}_n \times \mathbb{Z}_n) = \mathbb{Z}_n$. From this we conclude part I of our result: $M(\Delta_{3n^2}) = \mathbb{Z}_n$ for n coprime to 3.

Case II: $\gcd(n, 3) \neq 1$

Here the situation is much more involved. Let us appeal to Part (ii) of Theorem 22.7.36. We let $N = \mathbb{Z}_n \times \mathbb{Z}_n$ and $T = \mathbb{Z}_3$ as above and define $U := \mathrm{Hom}(\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{C}^*)$); the exact sequence then takes the form

$$1 \to H^1(\mathbb{Z}_3, U) \to \tilde{M}(\Delta_{3n^2}) \to \mathbb{Z}_n \to H^2(\mathbb{Z}_3, U) \qquad (22.7.4)$$

using the fact that the stable subgroup $M(\mathbb{Z}_n \times \mathbb{Z}_n)^{\mathbb{Z}_3} \cong \mathbb{Z}_n$ as shown above. Some explicit calculations are now called for.

As for U, it is of course isomorphic to $\mathbb{Z}_n \times \mathbb{Z}_n$ since for an Abelian group A, $\mathrm{Hom}(A, \mathbb{C}^*) \cong A$ ([264] p17). We label the elements thereof as $(p, q)(x^a y^b) := \omega_n^{ap+bq}$, taking $x^a y^b \in \mathbb{Z}_n \times \mathbb{Z}_n$ to $\mathbb{C}^*$.

We recall that the conjugation by $z \in \mathbb{Z}_3$ on $\mathbb{Z}_n \times \mathbb{Z}_n$ is (22.7.3). Therefore, by the remark at the end of the previous subsection, z acts on U as: $(z \cdot (p, q))(x^a y^b) :=$

$(p,q)(z(x^a y^b)z^{-1}) = \omega_n^{a'p+b'q}$ with $a' = -a-b$ and $b' = a$ due[2] to (22.7.3), whence

$$z \cdot (p,q) = (q-p, -p), \qquad \text{for } (p,q) \in U. \tag{22.7.5}$$

Some explicit calculations are called for. First we compute $H^1(\mathbb{Z}_3, U)$. Z^1 is generically composed of functions such that $f(z) = (p,q)$ (and also $f(\mathbb{I}) = \mathbb{I}$ and $f(z^2) = (z \cdot f(z))f(z)$ by the crossed homomorphism condition, and is subsequently equal to $(q, p+q)$ by (22.7.5). Since no further conditions can be imposed, $Z^1 \cong \mathbb{Z}_n \times \mathbb{Z}_n$. Now B^1 consists of all functions of the form $(z \cdot (p,q))(p,q)^{-1} = (q - 2p, -p-q)$, these are to be identified with the trivial map in Z^1. We can re-write these elements as $(p' := q-2p, -p'-3p) = (\omega_n^a \omega_n^{-b})^{p'}(\omega_n^b)^{-3p}$, and those in Z^1 we re-write as $(\omega_n^a \omega_n^{-b})^{p'}(\omega_n^b)^{q'}$ as we are free to do. Therefore if $\gcd(3,n) = 1$, then $H^1 := Z^1/B^1$ is actually trivial because in mod n, $3p$ also ranges the full $0, \cdots, n-1$, whereas if $\gcd(3,n) \neq 1$ then $H^1 := Z^1/B^1 \cong \mathbb{Z}_3$.

The computation for $H^2(\mathbb{Z}_3, U)$ is a little more involved, but the idea is the same. First we determine Z^2 as composed of $\alpha(z_1, z_2)$ constrained by the cocycle condition (with respect to conjugation which differs from (15.3.11) where the trivial action was taken)

$$\alpha(z_1,z_2)\alpha(z_1z_2,z_3) = (z_1 \cdot \alpha(z_2,z_3))\alpha(z_1,z_2z_3) \qquad z_1,z_2,z_3 \in \mathbb{Z}_3.$$

Again we only need to determine the following cases: $\alpha(z,z) := (p_1,q_1); \alpha(z^2,z^2) := (p_2,q_2); \alpha(z^2,z) := (p_3,q_3); \alpha(z,z^2) := (p_4,q_4)$. The cocycle constraint gives $(p_1,q_1) = (q_4,-q_3)$; $(p_2,q_2) = (-q_3-q_4,-q_4); (p_3,q_3) = (-q_4,q_3); (p_4,q_4) = (p_4,q_4)$, giving $Z^2 \cong \mathbb{Z}_n \times \mathbb{Z}_n$. The coboundaries are given by $(\delta t)(z_1,z_2) = (z_1 \cdot t(z_2))t(z_1)t(z_1z_2)^{-1}$ (for any mapping $t : \mathbb{Z}_3 \to \mathbb{Z}_n \times \mathbb{Z}_n$ which we define to take values $t(z) = (r_1,s_1)$ and $t(z^2) = (r_2,s_2)))$, making $(\delta t)(z,z) = (s_1-r_2, -r_1+s_1-s_2); (\delta t)(z^2,z^2)(-s_2+r_2-r_1, r_2-s_1); (\delta t)(z^2,z) = (-s_1+r_2, r_1-s_1+s_2); (\delta t)(z,z^2) = (s_2-r_2+r_1, s_1-r_2)$. Now, the transformation $r_2 = s_1+q_4; r_1 = s_1-s_2-p_4+q_4$ makes this set of values

[2] Note that we must be careful to let the order of conjugation be the opposite of that in the cocycle conjugation.

for B^2 completely identical to those in Z^2, whence we conclude that $B^2 \cong \mathbb{Z}_n \times \mathbb{Z}_n$. In conclusion then $H^2 := Z^2/B^2 \cong \mathbb{I}$.

The exact sequence (22.7.4) then assumes the simple form of

$$1 \to \left\{ \begin{array}{ll} \mathbb{Z}_3, & \gcd(n,3) \neq 1 \\ \mathbb{I}, & \gcd(n,3) = 1 \end{array} \right\} \to \tilde{M}(G) \to \mathbb{Z}_n \to 1,$$

which means that if n does not divide 3, $\tilde{M}(G) \cong \mathbb{Z}_n$, and otherwise $\tilde{M}(G)/\mathbb{Z}_3 \cong \mathbb{Z}_n$. Of course, in conjunction with Part (i) of Theorem 22.7.36, we immediately see that the first case makes Part I of our discussion (when $\gcd(n,3) = 1$) a special case of our present situation.

On the other hand, for the remaining case of $\gcd(n,3) \neq 1$, we have $M(\Delta_{3n^2})/\mathbb{Z}_3 \cong \mathbb{Z}_n$, which means that $M(\Delta_{3n^2})$, being an Abelian group, can only be $\mathbb{Z}_{3n}$ or $\mathbb{Z}_n \times \mathbb{Z}_3$. The exponent of the former is $3n$, while the later (since 3 divides n), is n, but by Theorem 17.3.26, the exponent squared must divide the order, which is $3n^2$, whereby forcing the second choice.

Therefore in conclusion we have our *theorema egregium*:

$$M(\Delta_{3n^2}) = \left\{ \begin{array}{ll} \mathbb{Z}_n \times \mathbb{Z}_3, & \gcd(n,3) \neq 1 \\ \mathbb{Z}_n, & \gcd(n,3) = 1 \end{array} \right.$$

as reported in Table (17.3.7).

22.7.3 The Schur Multiplier for Δ_{6n^2}

Recalling that n is even, we have $\Delta_{6n^2} \cong (\mathbb{Z}_n \times \mathbb{Z}_n) \rtimes S_3$ with $\mathbb{Z}_n \times \mathbb{Z}_n$ normal and thus we are once more aided by Theorem 22.7.36.

We let $N := \mathbb{Z}_n \times \mathbb{Z}_n$ and $T := S_3$ and the exact sequence assumes the form

$$1 \to H^1(S_3, U) \to \tilde{M}(\Delta_{6n^2}) \to (\mathbb{Z}_n)^{S_3} \to H^2(S_3, U)$$

where $U := \mathrm{Hom}(\mathbb{Z}_n \times \mathbb{Z}_n, \mathbb{C}^*)$ as defined in the previous subsection.

By calculations entirely analogous to the case for Δ_{3n^2}, we have $(\mathbb{Z}_n)^{S_3} \cong \mathbb{Z}_2$. This is straight-forward to show. Let $S_3 := \langle z, w | z^3 = w^2 = \mathbb{I}, zw = wz^2 \rangle$. We see that it contains $\mathbb{Z}^3 = \langle z | z^3 = \mathbb{I} \rangle$ as a subgroup, which we have treated in the previous section. In addition to (22.6.1), we have

$$w^{-1} x^a y^b w = x^{-1-b} y^b = w x^a y^b w^{-1}.$$

Using the form of the cocycle in Proposition 22.7.11, we see that $c_w(\alpha) = \alpha^{-1}$. Remembering that $c_z(\alpha) = \alpha$ from before, we see that the S_3-stable part of consists of α^m with $m = 0$ and $n/2$ (recall that in our case of $\Delta(6n^2)$, n is even), giving us a $\mathbb{Z}_2$.

Moreover we have $H^1(S_3, U) \cong \mathbb{I}$. This is again easy to show. In analogy to (22.7.5), we have

$$w \cdot (p, q) = (-q, q - p), \qquad \text{for } (p, q) \in U,$$

using which we find that Z^1 consists of $f : S_3 \to U$ given by $f(z) = (l_1, 3k_2 - l_1)$ and $f(w) = (2k_2, k_2)$. In addition B^1 consists of $f(z) = (k - 2l, -l - k)$ and $f(w) = (-2l, -l)$. Whence we see instantly that H^1 is trivial.

Now in fact $H^2(S_3, U) \cong \mathbb{I}$ as well (the involved details of these computations are too pathological to be even included in an appendix and we have resisted the urge to write an appendix for the appendix).

The exact sequence then forces immediately that $\tilde{M}(\Delta_{6n^2}) \cong \mathbb{Z}_2$. Moreover, since $M(S_3) \cong \mathbb{I}$ (q.v. e.g. [262]), by Part (i) of Theorem 22.7.36, we conclude that

$$M(\Delta_{6n^2}) \cong \mathbb{Z}_2$$

as reported in Table (17.3.7).

22.8 Intransitive subgroups of $SU(3)$

The computation of the Schur Multipliers for the non-Abelian intransitive subgroups of $SU(3)$ involves some subtleties related to the precise definition and construction of the groups.

Let us consider the case of combining the generators of $\mathbb{Z}_n$ with these of $\widehat{D_{2m}}$ to construct the intransitive subgroup $< \mathbb{Z}_n, \widehat{D_{2m}} >$. We can take the generators of $\widehat{D_{2m}}$ to be

$$\alpha = \begin{pmatrix} \omega_{2m} & 0 & 0 \\ 0 & \omega_{2m}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and that of $\mathbb{Z}_n$ to be

$$\gamma = \begin{pmatrix} \omega_n & 0 & 0 \\ 0 & \omega_n & 0 \\ 0 & 0 & \omega_n^{-2} \end{pmatrix}.$$

The group $< \mathbb{Z}_n, \widehat{D_{2m}} >$ is not in general the direct product of $\mathbb{Z}_n$ and $\widehat{D_{2m}}$. More specifically, when n is odd $< \mathbb{Z}_n, \widehat{D_{2m}} >= \mathbb{Z}_n \times \widehat{D_{2m}}$. For n even however, we notice that $\alpha^m = \beta^2 = \gamma^{n/2}$. Accordingly, we conclude that $< \mathbb{Z}_n, \widehat{D_{2m}} >= (\mathbb{Z}_n \times \widehat{D_{2m}})/\mathbb{Z}_2$ for n even where the central $\mathbb{Z}_2$ is generated by $\gamma^{n/2}$. Actually the conditions are more refined: when $n = 2(2k+1)$ we have $\mathbb{Z}_n = \mathbb{Z}_2 \times \mathbb{Z}_{2k+1}$ and so $(\mathbb{Z}_2 \times \widehat{D_{2m}})/\mathbb{Z}_2 = \mathbb{Z}_{2k+1} \times \widehat{D_{2m}}$. Thus the only non-trivial case is when $n = 4k$.

This subtlety in the group structure holds for all the cases where $\mathbb{Z}_n$ is combined with binary groups $\widehat{G}$. When n mod $4 \neq 0$, $< \mathbb{Z}_n, \widehat{G} >$ is the direct product of $\widehat{G}$ with either $\mathbb{Z}_n$ or $\mathbb{Z}_{n/2}$. For n mod $4 = 0$ it is the quotient group $(\mathbb{Z}_n \times \widehat{G})/\mathbb{Z}_2$. In summary

$$< \mathbb{Z}_n, \widehat{G} >= \begin{cases} \mathbb{Z}_n \times \widehat{G} & n \bmod 2 = 1 \\ \mathbb{Z}_{n/2} \times \widehat{G} & n \bmod 4 = 2 \\ (\mathbb{Z}_n \times \widehat{G})/\mathbb{Z}_2 & n \bmod 4 = 0 \end{cases}.$$

The case of $\mathbb{Z}_n$ combined with the ordinary dihedral group D_{2m} is a bit different however. The matrix forms of the generators are

$$\alpha = \begin{pmatrix} \omega_m & 0 & 0 \\ 0 & \omega_m^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \omega_n & 0 & 0 \\ 0 & \omega_n & 0 \\ 0 & 0 & \omega_n^{-2} \end{pmatrix}$$

where α and β generate D_{2m} and γ generates $\mathbb{Z}_n$.

From these we notice that when both n and m are even, $\alpha^{m/2} = \gamma^{n/2}$ and $<\mathbb{Z}_n, D_{2m}>$ is not a direct product. After inspection, we find that

$$< \mathbb{Z}_n, D_{2m} >= \begin{cases} \mathbb{Z}_n \times D_{2m} & m \bmod 2 = 1 \\ \mathbb{Z}_n \times D_{2m} & m \bmod 2 = 0, n \bmod 2 = 1 \\ \mathbb{Z}_{n/2} \times D_{2m} & m \bmod 2 = 0, n \bmod 4 = 2 \\ (\mathbb{Z}_n \times D_{2m})/\mathbb{Z}_2 & m \bmod 2 = 0, n \bmod 4 = 0 \end{cases}.$$

The Schur Multipliers of the direct product cases are immediately computable by consulting Theorem 17.3.27. For example, $M(\mathbb{Z}_n \times \widehat{D_{2m}}) \cong M(\mathbb{Z}_n) \times M(\widehat{D_{2m}}) \times (\mathbb{Z}_n \otimes \widehat{D_{2m}})$ by Theorem 17.3.27, the last term of which in turn equates to $\mathrm{Hom}(\mathbb{Z}_n, \widehat{D_{2m}}/\widehat{D_{2m}}')$. This is $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_{\gcd(n,2)} \times \mathbb{Z}_{\gcd(n,2)}$ for m even and $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_4) \cong \mathbb{Z}_{\gcd(n,4)}$ for m odd. By similar token, we have that $M(\mathbb{Z}_n \times D_{2m})$ for even m is $\mathbb{Z}_2 \times \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_{\gcd(n,2)} \times \mathbb{Z}_{\gcd(n,2)}$ and $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_2) \cong \mathbb{Z}_{\gcd(n,2)}$ for odd m. Likewise $M(\mathbb{Z}_n \times \widehat{E_{6,7,8}}) = \mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_{3,2,1})$.

22.9 Ordinary and Projective Representations of Some Discrete Subgroups of $SU(3)$

We here present, for the reference of the reader, the (ordinary) character tables of the groups as well as the covering groups thereof, of the examples which we studied in Section 4 of Chapter 18.

$\Sigma(60)$

1	12	12	15	20
1	1	1	1	1
3	$-\omega_5^2-\omega_5^{-2}$	$-\omega_5-\omega_5^{-1}$	-1	0
3	$-\omega_5-\omega_5^{-1}$	$-\omega_5^2-\omega_5^{-2}$	-1	0
4	-1	-1	0	1
5	0	0	1	-1

$\Sigma(60)^*$

1	1	12	12	12	12	30	20	20
1	1	1	1	1	1	1	1	1
3	3	$-\omega_5^2-\omega_5^{-2}$	$-\omega_5^2-\omega_5^{-2}$	$-\omega_5-\omega_5^{-1}$	$-\omega_5-\omega_5^{-1}$	-1	0	0
3	3	$-\omega_5-\omega_5^{-1}$	$-\omega_5-\omega_5^{-1}$	$-\omega_5^2-\omega_5^{-2}$	$-\omega_5^2-\omega_5^{-2}$	-1	0	0
4	4	-1	-1	-1	-1	0	1	1
5	5	0	0	0	0	1	-1	-1
2	-2	$-\omega_5^2-\omega_5^{-2}$	$\omega_5^2+\omega_5^{-2}$	$-\omega_5-\omega_5^{-1}$	$\omega_5+\omega_5^{-1}$	0	-1	1
2	-2	$-\omega_5-\omega_5^{-1}$	$\omega_5+\omega_5^{-1}$	$-\omega_5^2-\omega_5^{-2}$	$\omega_5^2+\omega_5^{-2}$	0	-1	1
4	-4	1	-1	1	-1	0	1	-1
6	-6	-1	1	-1	1	0	0	0

$\Sigma(168)$

1	21	42	56	24	24
1	1	1	1	1	1
3	-1	1	0	a	$\bar{a}$
3	-1	1	0	$\bar{a}$	a
6	2	0	0	-1	-1
7	-1	-1	1	0	0
8	0	0	-1	1	1

$\Sigma(168)^*$

1	1	42	42	42	56	56	24	24	24	24
1	1	1	1	1	1	1	1	1	1	1
3	3	-1	1	1	0	0	a	a	$\bar{a}$	$\bar{a}$
3	3	-1	1	1	0	0	$\bar{a}$	$\bar{a}$	a	a
6	6	2	0	0	0	0	-1	-1	-1	-1
7	7	-1	-1	-1	1	1	0	0	0	0
8	8	0	0	0	-1	-1	1	1	1	1
4	-4	0	0	0	1	-1	$-a$	a	$-\bar{a}$	$\bar{a}$
4	-4	0	0	0	1	-1	$-\bar{a}$	$\bar{a}$	$-a$	a
6	-6	0	$-\sqrt{2}$	$\sqrt{2}$	0	0	-1	1	-1	1
6	-6	0	$\sqrt{2}$	$-\sqrt{2}$	0	0	-1	1	-1	1
8	-8	0	0	0	-1	1	1	-1	1	-1

$$a := \frac{-1+\sqrt{7}i}{2}$$

$\Sigma(1080)$

1	1	1	45	45	45	72	72	72	72	72	72	90	90	90	120	120
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	$3\bar{A}$	$3A$	$-A$	$-\bar{A}$	-1	X	Y	Z	W	$\bar{Z}$	$\bar{W}$	$\bar{A}$	A	1	0	0
3	$3\bar{A}$	$3A$	$-A$	$-\bar{A}$	-1	Y	X	W	Z	$\bar{W}$	$\bar{Z}$	$\bar{A}$	A	1	0	0
3	$3A$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	X	Y	$\bar{Z}$	$\bar{W}$	Z	W	A	$\bar{A}$	1	0	0
3	$3A$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	Y	X	$\bar{W}$	$\bar{Z}$	W	Z	A	$\bar{A}$	1	0	0
5	5	5	1	1	1	0	0	0	0	0	0	-1	-1	-1	2	-1
5	5	5	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1	2
6	$6\bar{A}$	$6A$	$2A$	$2\bar{A}$	2	1	1	$\bar{A}$	$\bar{A}$	A	A	0	0	0	0	0
6	$6A$	$6\bar{A}$	$2\bar{A}$	$2A$	2	1	1	A	A	$\bar{A}$	$\bar{A}$	0	0	0	0	0
8	8	8	0	0	0	X	Y	Y	X	Y	X	0	0	0	-1	-1
8	8	8	0	0	0	Y	X	X	Y	X	Y	0	0	0	-1	-1
9	9	9	1	1	1	-1	-1	-1	-1	-1	-1	1	1	1	0	0
9	$9\bar{A}$	$9A$	A	$\bar{A}$	1	-1	-1	$-\bar{A}$	$-\bar{A}$	$-A$	$-A$	$\bar{A}$	A	1	0	0
9	$9A$	$9\bar{A}$	$\bar{A}$	A	1	-1	-1	$-A$	$-A$	$-\bar{A}$	$-\bar{A}$	A	$\bar{A}$	1	0	0
10	10	10	-2	-2	-2	0	0	0	0	0	0	0	0	0	1	1
15	$15\bar{A}$	$15A$	$-A$	$-\bar{A}$	-1	0	0	0	0	0	0	$-\bar{A}$	$-A$	-1	0	0
15	$15A$	$15\bar{A}$	$-\bar{A}$	$-A$	-1	0	0	0	0	0	0	$-A$	$-\bar{A}$	-1	0	0

$A := \omega_3$;
$B := \omega_5$;
$C := \omega_{15}$;
$X := -B-\bar{B}$;
$Y := -B^2-\bar{B}^2$;
$Z := -C-C^4$;
$W := -\bar{C}^2-C^7$;

$\Sigma(1080)^*$

$D := B+\bar{B}, E := B^2+\bar{B}^2, F := \bar{C}+\bar{C}^4, G := C^2+\bar{C}^7, H := \omega_{24}, J := \bar{H}^7 - H^{11}, K := \bar{H}^5 - H$

1	1	1	1	1	1	90	90	90	72	72	72	72	72	72	72	72	72	72	72	72	90	90	90	90	90	90	120	120	120	120
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	3	$3\bar{A}$	$3\bar{A}$	$3A$	$3A$	$-A$	$-\bar{A}$	-1	X	X	Y	Y	Z	Z	W	W	$\bar{Z}$	$\bar{Z}$	$\bar{W}$	$\bar{W}$	$\bar{A}$	$\bar{A}$	A	A	1	1	0	0	0	0
3	3	$3\bar{A}$	$3\bar{A}$	$3A$	$3A$	$-A$	$-\bar{A}$	-1	Y	Y	X	X	W	W	Z	Z	$\bar{W}$	$\bar{W}$	$\bar{Z}$	$\bar{Z}$	$\bar{A}$	$\bar{A}$	A	A	1	1	0	0	0	0
3	3	$3A$	$3A$	$3\bar{A}$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	X	X	Y	Y	$\bar{Z}$	$\bar{Z}$	$\bar{W}$	$\bar{W}$	Z	Z	W	W	A	A	$\bar{A}$	$\bar{A}$	1	1	0	0	0	0
3	3	$3A$	$3A$	$3\bar{A}$	$3\bar{A}$	$-\bar{A}$	$-A$	-1	Y	Y	X	X	$\bar{W}$	$\bar{W}$	$\bar{Z}$	$\bar{Z}$	W	W	Z	Z	A	A	$\bar{A}$	$\bar{A}$	1	1	0	0	0	0
5	5	5	5	5	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	2	2	-1	-1
5	5	5	5	5	5	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	2	2
6	6	$6\bar{A}$	$6\bar{A}$	$6A$	$6A$	$2A$	$2\bar{A}$	2	1	1	1	1	$\bar{A}$	$\bar{A}$	$\bar{A}$	$\bar{A}$	A	A	A	A	0	0	0	0	0	0	0	0	0	0
6	6	$6A$	$6A$	$6\bar{A}$	$6\bar{A}$	$2\bar{A}$	$2A$	2	1	1	1	1	A	A	A	A	$\bar{A}$	$\bar{A}$	$\bar{A}$	$\bar{A}$	0	0	0	0	0	0	0	0	0	0
8	8	8	8	8	8	0	0	0	X	X	Y	Y	Y	Y	X	X	Y	Y	X	X	0	0	0	0	0	0	-1	-1	-1	-1
8	8	8	8	8	8	0	0	0	Y	Y	X	X	X	X	Y	Y	X	X	Y	Y	0	0	0	0	0	0	-1	-1	-1	-1
9	9	9	9	9	9	1	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	0	0	0	0
9	9	$9\bar{A}$	$9\bar{A}$	$9A$	$9A$	A	$\bar{A}$	1	-1	-1	-1	-1	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	$-A$	$-A$	$-A$	$-A$	$\bar{A}$	$\bar{A}$	A	A	1	1	0	0	0	0
9	9	$9A$	$9A$	$9\bar{A}$	$9\bar{A}$	$\bar{A}$	A	1	-1	-1	-1	-1	$-A$	$-A$	$-A$	$-A$	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	$-\bar{A}$	A	A	$\bar{A}$	$\bar{A}$	1	1	0	0	0	0
10	10	10	10	10	10	-2	-2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
15	15	$15\bar{A}$	$15\bar{A}$	$15A$	$15A$	$-A$	$-\bar{A}$	-1	0	0	0	0	0	0	0	0	0	0	0	0	$-\bar{A}$	$-\bar{A}$	$-A$	$-A$	-1	-1	0	0	0	0
15	15	$15A$	$15A$	$15\bar{A}$	$15\bar{A}$	$-\bar{A}$	$-A$	-1	0	0	0	0	0	0	0	0	0	0	0	0	$-A$	$-A$	$-\bar{A}$	$-\bar{A}$	-1	-1	0	0	0	0
4	-4	-4	4	-4	4	0	0	0	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	0	0	0	0	0	0	1	-1	-2	2
4	-4	-4	4	-4	4	0	0	0	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	0	0	0	0	0	0	-2	2	1	-1
6	-6	$-6A$	$6A$	$-6\bar{A}$	$6\bar{A}$	0	0	0	-1	1	-1	1	$-A$	A	$-A$	A	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	J	$-J$	K	$-K$	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	0
6	-6	$-6A$	$6A$	$-6\bar{A}$	$6\bar{A}$	0	0	0	-1	1	-1	1	$-A$	A	$-A$	A	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	$-J$	J	$-K$	K	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	0
6	-6	$-6\bar{A}$	$6\bar{A}$	$-6A$	$6A$	0	0	0	-1	1	-1	1	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	$-A$	A	$-A$	A	$-K$	K	$-J$	J	$-\sqrt{2}$	$\sqrt{2}$	0	0	0	0
6	-6	$-6\bar{A}$	$6\bar{A}$	$-6A$	$6A$	0	0	0	-1	1	-1	1	$-\bar{A}$	$\bar{A}$	$-\bar{A}$	$\bar{A}$	$-A$	A	$-A$	A	K	$-K$	J	$-J$	$\sqrt{2}$	$-\sqrt{2}$	0	0	0	0
8	-8	-8	8	-8	8	0	0	0	D	X	E	Y	E	Y	D	X	E	Y	D	X	0	0	0	0	0	0	-1	1	-1	1
8	-8	-8	8	-8	8	0	0	0	E	Y	D	X	D	X	E	Y	D	X	E	Y	0	0	0	0	0	0	-1	1	-1	1
10	-10	-10	10	-10	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	1	-1	1	-1
10	-10	-10	10	-10	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	1	-1	1	-1
12	-12	$-12A$	$12A$	$-12\bar{A}$	$12\bar{A}$	0	0	0	X	D	Y	E	$\bar{Z}$	$\bar{F}$	$\bar{W}$	G	Z	F	W	$\bar{G}$	0	0	0	0	0	0	0	0	0	0
12	-12	$-12A$	$12A$	$-12\bar{A}$	$12\bar{A}$	0	0	0	Y	E	X	D	$\bar{W}$	G	$\bar{Z}$	$\bar{F}$	W	$\bar{G}$	Z	F	0	0	0	0	0	0	0	0	0	0
12	-12	$-12\bar{A}$	$12\bar{A}$	$-12A$	$12A$	0	0	0	X	D	Y	E	Z	F	W	$\bar{G}$	$\bar{Z}$	$\bar{F}$	$\bar{W}$	G	0	0	0	0	0	0	0	0	0	0
12	-12	$-12\bar{A}$	$12\bar{A}$	$-12A$	$12A$	0	0	0	Y	E	X	D	W	$\bar{G}$	Z	F	$\bar{W}$	G	$\bar{Z}$	$\bar{F}$	0	0	0	0	0	0	0	0	0	0

$\Delta(6 \times 2^2) =$

1	3	6	6	8
1	1	1	1	1
1	1	-1	-1	1
2	2	0	0	-1
3	-1	-1	1	0
3	-1	1	-1	0

$\Delta(6 \times 2^2)^* =$

1	1	6	6	6	12	8	8
1	1	1	1	1	1	1	1
1	1	1	-1	-1	-1	1	1
2	2	2	0	0	0	-1	-1
3	3	-1	-1	-1	1	0	0
3	3	-1	1	1	-1	0	0
2	-2	0	$-e^{\frac{i}{4}\pi} - e^{\frac{3i}{4}\pi}$	$e^{\frac{i}{4}\pi} + e^{\frac{3i}{4}\pi}$	0	-1	1
2	-2	0	$e^{\frac{i}{4}\pi} + e^{\frac{3i}{4}\pi}$	$-e^{\frac{i}{4}\pi} - e^{\frac{3i}{4}\pi}$	0	-1	1
4	-4	0	0	0	0	1	-1

$\Delta(6 \times 4^2) =$

1	3	3	3	6	12	12	12	12	32
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	-1	-1	-1	-1	1
2	2	2	2	2	0	0	0	0	-1
3	3	-1	-1	-1	-1	1	1	-1	0
3	3	-1	-1	-1	1	-1	-1	1	0
3	-1	$-1-2i$	$-1+2i$	1	-1	i	$-i$	1	0
3	-1	$-1+2i$	$-1-2i$	1	-1	$-i$	i	1	0
3	-1	$-1-2i$	$-1+2i$	1	1	$-i$	i	-1	0
3	-1	$-1+2i$	$-1-2i$	1	1	i	$-i$	-1	0
6	-2	2	2	-2	0	0	0	0	0

$\Delta(6 \times 4^2)^* =$

1	1	3	3	6	6	12	24	12	12	12	12	24	32	32
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	-1	-1	-1	-1	-1	-1	1	1
2	2	2	2	2	2	2	0	0	0	0	0	0	-1	-1
3	3	3	3	-1	-1	-1	-1	1	1	1	1	-1	0	0
3	3	3	3	-1	-1	-1	1	-1	-1	-1	-1	1	0	0
3	3	-1	-1	$-1-2i$	$-1+2i$	1	-1	i	i	$-i$	$-i$	1	0	0
3	3	-1	-1	$-1+2i$	$-1-2i$	1	-1	$-i$	$-i$	i	i	1	0	0
3	3	-1	-1	$-1-2i$	$-1+2i$	1	1	$-i$	$-i$	i	i	-1	0	0
3	3	-1	-1	$-1+2i$	$-1-2i$	1	1	i	i	$-i$	$-i$	-1	0	0
6	6	-2	-2	2	2	-2	0	0	0	0	0	0	0	0
2	-2	-2	2	0	0	0	0	$i\sqrt{2}$	$-i\sqrt{2}$	$-i\sqrt{2}$	$i\sqrt{2}$	0	-1	1
2	-2	-2	2	0	0	0	0	$-i\sqrt{2}$	$i\sqrt{2}$	$i\sqrt{2}$	$-i\sqrt{2}$	0	-1	1
4	-4	-4	4	0	0	0	0	0	0	0	0	0	1	-1
6	-6	2	-2	0	0	0	0	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	0	0	0
6	-6	2	-2	0	0	0	0	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	0	0	0

$\Delta(3 \times 4^2) =$

1	3	3	3	3	3	16	16
1	1	1	1	1	1	1	1
1	1	1	1	1	1	ω_3	$\bar{\omega}_3$
1	1	1	1	1	1	$\bar{\omega}_3$	ω_3
3	-1	-1	3	-1	-1	0	0
3	1	1	-1	$-1-2i$	$-1+2i$	0	0
3	1	1	-1	$-1+2i$	$-1-2i$	0	0
3	$-1-2i$	$-1+2i$	-1	1	1	0	0
3	$-1+2i$	$-1-2i$	-1	1	1	0	0

$\Delta(3 \times 4^2)^* =$

1	1	1	1	12	12	6	6	12	12	16	16	16	16	16	16	16	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	ω_3	ω_3	ω_3	ω_3	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
1	1	1	1	1	1	1	1	1	1	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	ω_3	ω_3	ω_3	ω_3
3	3	3	3	-1	-1	3	3	-1	-1	0	0	0	0	0	0	0	0
3	3	3	3	1	1	-1	-1	$-1-2i$	$-1+2i$	0	0	0	0	0	0	0	0
3	3	3	3	1	1	-1	-1	$-1+2i$	$-1-2i$	0	0	0	0	0	0	0	0
3	3	3	3	$-1-2i$	$-1+2i$	-1	-1	1	1	0	0	0	0	0	0	0	0
3	3	3	3	$-1+2i$	$-1-2i$	-1	-1	1	1	0	0	0	0	0	0	0	0
2	-2	2	-2	0	0	2	-2	0	0	$-\omega_3$	ω_3	ω_3	$-\omega_3$	$-\bar{\omega}_3$	$-\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$
2	-2	2	-2	0	0	2	-2	0	0	$-\bar{\omega}_3$	$\bar{\omega}_3$	$\bar{\omega}_3$	$-\bar{\omega}_3$	$-\omega_3$	$-\omega_3$	ω_3	ω_3
2	-2	2	-2	0	0	2	-2	0	0	-1	1	1	-1	-1	-1	1	1
6	-6	6	-6	0	0	-2	2	0	0	0	0	0	0	0	0	0	0
4	$4i$	-4	$-4i$	0	0	0	0	0	0	$\bar{\omega}_3$	$-\bar{\omega}_{12}$	$\bar{\omega}_{12}$	$-\bar{\omega}_3$	ω_3	$-\omega_3$	$\bar{\omega}_{12}^5$	$-\bar{\omega}_{12}^5$
4	$4i$	-4	$-4i$	0	0	0	0	0	0	ω_3	$-\bar{\omega}_{12}^5$	$\bar{\omega}_{12}^5$	$-\omega_3$	$\bar{\omega}_3$	$-\bar{\omega}_3$	$\bar{\omega}_{12}$	$-\bar{\omega}_{12}$
4	$4i$	-4	$-4i$	0	0	0	0	0	0	1	$-i$	i	-1	1	-1	i	$-i$
4	$-4i$	-4	$4i$	0	0	0	0	0	0	$\bar{\omega}_3$	$\bar{\omega}_{12}$	$-\bar{\omega}_{12}$	$-\bar{\omega}_3$	ω_3	$-\omega_3$	$-\bar{\omega}_{12}^5$	$\bar{\omega}_{12}^5$
4	$-4i$	-4	$4i$	0	0	0	0	0	0	ω_3	$\bar{\omega}_{12}^5$	$-\bar{\omega}_{12}^5$	$-\omega_3$	$\bar{\omega}_3$	$-\bar{\omega}_3$	$-\bar{\omega}_{12}$	$\bar{\omega}_{12}$
4	$-4i$	-4	$4i$	0	0	0	0	0	0	1	i	$-i$	-1	1	-1	$-i$	i

$\Delta(3 \times 5^2) =$

1	3	3	3	3	3	3	3	3	25	25
1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	E	$\bar{E}$
1	1	1	1	1	1	1	1	1	$\bar{E}$	E
3	A	A	B	B	C	D	$\bar{C}$	$\bar{D}$	0	0
3	A	A	B	B	$\bar{C}$	$\bar{D}$	C	D	0	0
3	B	B	A	A	$\bar{D}$	C	D	$\bar{C}$	0	0
3	B	B	A	A	D	$\bar{C}$	$\bar{D}$	C	0	0
3	C	$\bar{C}$	D	$\bar{D}$	B	A	B	A	0	0
3	$\bar{D}$	D	C	$\bar{C}$	A	B	A	B	0	0
3	$\bar{C}$	C	$\bar{D}$	D	B	A	B	A	0	0
3	D	$\bar{D}$	$\bar{C}$	C	A	B	A	B	0	0

$\Delta(3 \times 5^2)^* =$

1	1	1	1	1	15	15	15	15	15	15	15	15	25	25	25	25	25	25	25	25	25	25
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	E	E	E	E	E	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$
1	1	1	1	1	1	1	1	1	1	1	1	1	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$	$\bar{E}$	E	E	E	E	E
3	3	3	3	3	A	A	B	B	C	D	$\bar{C}$	$\bar{D}$	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	A	A	B	B	$\bar{C}$	$\bar{D}$	C	D	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	B	B	A	A	$\bar{D}$	C	D	$\bar{C}$	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	B	B	A	A	D	$\bar{C}$	$\bar{D}$	C	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	C	$\bar{C}$	D	$\bar{D}$	B	A	B	A	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	$\bar{D}$	D	C	$\bar{C}$	A	B	A	B	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	$\bar{C}$	C	$\bar{D}$	D	B	A	B	A	0	0	0	0	0	0	0	0	0	0
3	3	3	3	3	D	$\bar{D}$	$\bar{C}$	C	A	B	A	B	0	0	0	0	0	0	0	0	0	0
5	$5\bar{F}^2$	$5F$	$5\bar{F}$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	-1	$-\bar{F}$	$-\bar{F}^2$	$-\bar{F}^2$	$-F$	-1	$-F$	$-\bar{F}^2$	$-\bar{F}^2$	$-\bar{F}$
5	$5\bar{F}^2$	$5F$	$5\bar{F}$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-E$	$-\bar{G}^7$	$-\bar{G}$	$-\bar{G}^4$	$-G^2$	$-\bar{E}$	$-\bar{G}^7$	$-G$	$-\bar{G}^4$	$-\bar{G}^2$
5	$5\bar{F}^2$	$5F$	$5\bar{F}$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-\bar{G}^2$	$-\bar{G}^4$	$-G$	$-\bar{G}^7$	$-E$	$-G^2$	$-\bar{G}^4$	$-\bar{G}$	$-\bar{G}^7$
5	$5\bar{F}^2$	$5\bar{F}$	$5F$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	-1	$-F$	$-\bar{F}^2$	$-\bar{F}^2$	$-\bar{F}$	-1	$-\bar{F}$	$-\bar{F}^2$	$-\bar{F}^2$	$-F$
5	$5\bar{F}^2$	$5\bar{F}$	$5F$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-E$	$-G^2$	$-\bar{G}^4$	$-\bar{G}$	$-\bar{G}^7$	$-\bar{E}$	$-\bar{G}^2$	$-\bar{G}^4$	$-G$	$-\bar{G}^7$
5	$5\bar{F}^2$	$5\bar{F}$	$5F$	$5\bar{F}^2$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-\bar{G}^7$	$-G$	$-\bar{G}^4$	$-\bar{G}^2$	$-E$	$-\bar{G}^7$	$-\bar{G}$	$-\bar{G}^4$	$-G^2$
5	$5\bar{F}$	$5\bar{F}^2$	$5\bar{F}^2$	$5F$	0	0	0	0	0	0	0	0	-1	$-\bar{F}^2$	$-F$	$-\bar{F}$	$-\bar{F}^2$	-1	$-\bar{F}^2$	$-\bar{F}$	$-F$	$-\bar{F}^2$
5	$5\bar{F}$	$5\bar{F}^2$	$5\bar{F}^2$	$5F$	0	0	0	0	0	0	0	0	$-E$	$-\bar{G}$	$-G^2$	$-\bar{G}^7$	$-\bar{G}^4$	$-\bar{E}$	$-G$	$-\bar{G}^2$	$-\bar{G}^7$	$-\bar{G}^4$
5	$5\bar{F}$	$5\bar{F}^2$	$5\bar{F}^2$	$5F$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-\bar{G}^4$	$-\bar{G}^7$	$-\bar{G}^2$	$-G$	$-E$	$-\bar{G}^4$	$-\bar{G}^7$	$-G^2$	$-\bar{G}$
5	$5F$	$5\bar{F}^2$	$5\bar{F}^2$	$5\bar{F}$	0	0	0	0	0	0	0	0	-1	$-\bar{F}^2$	$-\bar{F}$	$-F$	$-\bar{F}^2$	-1	$-\bar{F}^2$	$-F$	$-\bar{F}$	$-\bar{F}^2$
5	$5F$	$5\bar{F}^2$	$5\bar{F}^2$	$5\bar{F}$	0	0	0	0	0	0	0	0	$-E$	$-\bar{G}^4$	$-\bar{G}^7$	$-G^2$	$-\bar{G}$	$-\bar{E}$	$-\bar{G}^4$	$-\bar{G}^7$	$-\bar{G}^2$	$-G$
5	$5F$	$5\bar{F}^2$	$5\bar{F}^2$	$5\bar{F}$	0	0	0	0	0	0	0	0	$-\bar{E}$	$-G$	$-\bar{G}^2$	$-\bar{G}^7$	$-\bar{G}^4$	$-E$	$-\bar{G}$	$-G^2$	$-\bar{G}^7$	$-\bar{G}^4$

$A := -\omega_5 - \bar{\omega}_5, B := -\omega_5^2 - \bar{\omega}_5^2, C := \bar{\omega}_5 - 2\bar{\omega}_5^2, D := 2\omega_5 + \bar{\omega}_5^2; E := \omega_3, F := \bar{\omega}_5, G := \omega_{15}$.

22.10 Finding the Dual Cone

Let us be given a convex polytope C, with the edges specifying the faces of which given by the matrix M whose columns are the vectors corresponding to these edges. Our task is to find the dual cone $\tilde{C}$ of C, or more precisely the matrix N such that

$$N^t \cdot M \geq 0 \qquad \text{for all entries.}$$

There is a standard algorithm, given in [10]. Let M be $n \times p$, i.e., there are p n-dimensional vectors spanning C. We note of course that $p \geq n$ for convexity. Out of the p vectors, we choose $n-1$. This gives us an $n \times (n-1)$ matrix of co-rank 1, whence we can extract a 1-dimensional null-space (as indeed the initial p vectors are all linearly independent) described by a single vector u.

Next we check the dot product of u with the remaining $p-(n-1)$ vectors. If all the dot products are positive we keep u, and if all are negative, we keep $-u$, otherwise we discard it.

We then select another $n-1$ vectors and repeat the above until all combinations are exhausted. The set of vectors we have kept, u's or $-u$'s then form the columns of N and span the dual cone $\tilde{C}$.

We note that this is a very computationally intensive algorithm, the number of steps of which depends on $\begin{pmatrix} p \\ n-1 \end{pmatrix}$ which grows exponentially.

A subtle point to remark. In light of what we discussed in a footnote in the paper on the difference between $\mathbf{M}_+ = \mathbf{M} \cap \sigma$ and $\mathbf{M}'_+$, here we have computed the dual of σ. We must ensure that $\mathbb{Z}_+$-independent lattice points inside the cones be not missed.

22.11 Gauge Theory Data for $\mathbb{Z}_n \times \mathbb{Z}_n$

For future reference we include here the gauge theory data for the $\mathbb{Z}_n \times \mathbb{Z}_n$ orbifold, so that, as mentioned in [298], any 3-dimensional toric singularity may exist as a partial resolution thereof.

We have $3n^2$ fields denoted as X_{ij}, Y_{ij}, Z_{ij} and choose the decomposition $\mathbf{3} \to (1,0)+(0,1)+(-1,-1)$. The matter content (and thus the d matrix) is well-known from standard brane box constructions, hence we here focus on the superpotential [82] (and thus the K matrix):

$$X_{ij}Y_{i(j+1)}Z_{(i+1)(j+1)} - Y_{ij}X_{(i+1)j}Z_{(i+1)(j+1)},$$

from which the F-terms are

$$
\begin{aligned}
\frac{\partial W}{\partial X_{ij}}: \qquad Y_{i(j+1)}Z_{(i+1)(j+1)} &= Z_{i(j+1)}Y_{(i-1)j} \\
\frac{\partial W}{\partial Y_{ij}}: \qquad Z_{(i+1)j}X_{i(j-1)} &= X_{(i+1)j}Z_{(i+1)(j+1)} \\
\frac{\partial W}{\partial Z_{(i+1)(j+1)}}: \qquad X_{ij}Y_{i(j+1)} &= Y_{ij}X_{(i+1)j}.
\end{aligned}
\tag{22.11.6}
$$

Now let us solve (22.11.6). First we have $Y_{i(j+1)} = Y_{ij}X_{(i+1)j}/X_{ij}$. Thus if we take Y_{i0} and X_{ij} as the independent variables, we have

$$
Y_{i(j+1)} = \frac{\prod_{l=0}^{j} X_{(i+1)l}}{\prod_{l=0}^{j} X_{il}} Y_{i0}. \tag{22.11.7}
$$

There is of course the periodicity which gives

$$
Y_{in} = Y_{i0} \Longrightarrow \prod_{l=0}^{n-1} X_{(i+1)l} = \prod_{l=0}^{n-1} X_{il}. \tag{22.11.8}
$$

Next we use X_{ij} to solve the Z_{ij} as $Z_{i(j+1)} = Z_{ij}X_{(i-1)(j-1)}/X_{ij}$, whence

$$
Z_{i(j+1)} = \frac{\prod_{l=0}^{j} X_{(i-1)(l-1)}}{\prod_{l=0}^{j} X_{il}} Z_{i0}. \tag{22.11.9}
$$

As above,

$$
Z_{in} = Z_{i0} \Longrightarrow \prod_{l=0}^{n-1} X_{(i-1)(l-1)} = \prod_{l=0}^{n-1} X_{il}. \tag{22.11.10}
$$

Putting the solution of Y, Z into the first equation of (22.11.6) we get

$$
\frac{\prod_{l=0}^{j} X_{(i+1)l}}{\prod_{l=0}^{j} X_{il}} Y_{i0} \frac{\prod_{l=0}^{j} X_{(i)(l-1)}}{\prod_{l=0}^{j} X_{(i+1)l}} Z_{(i+1)0} = \frac{\prod_{l=0}^{j} X_{(i-1)(l-1)}}{\prod_{l=0}^{j} X_{il}} Z_{i0} \frac{\prod_{l=0}^{j-1} X_{il}}{\prod_{l=0}^{j-1} X_{(i-1)l}} Y_{(i-1)0},
$$

which can be simplified as $Y_{i0}Z_{(i+1)0}X_{i(n-1)} = Z_{i0}Y_{(i-1)0}X_{(i-1)(n-1)}$, or $X_{i(n-1)} =$

$X_{(i-1)(n-1)}\frac{Y_{(i-1)0}}{Y_{i0}}\frac{Z_{i0}}{Z_{(i+1)0}}$. From this we solve

$$X_{i(n-1)} = X_{0(n-1)}\prod_{l=0}^{i-1}\frac{Y_{l0}}{Y_{(l+1)0}}\frac{Z_{(l+1)0}}{Z_{(l+2)0}}. \tag{22.11.11}$$

The periodicity gives

$$\prod_{l=0}^{n-1}\frac{Y_{l0}}{Y_{(l+1)0}}\frac{Z_{(l+1)0}}{Z_{(l+2)0}} = 1. \tag{22.11.12}$$

Now we have the independent variables Y_{i0} Z_{i0} and X_{ij} for $j \neq n-1$ and $X_{0(n-1)}$, plus three constraints (22.11.8) (22.11.10) (22.11.12). In fact, considering the periodic condition for X, (22.11.8) is equivalent to (22.11.10). Furthermore considering the periodic conditions for Z_{i0} and Y_{i0}, (22.11.12) is trivial. So we have only one constraint. Putting the expression (22.11.11) into (22.11.8) we get $\prod_{l=0}^{n-2}X_{(i+1)l}\frac{Y_{i0}}{Y_{(i+1)0}}\frac{Z_{(i+1)0}}{Z_{(i+2)0}} = \prod_{l=0}^{n-2}X_{il} \Rightarrow \prod_{l=0}^{n-2}X_{(i+1)l}\frac{1}{Y_{(i+1)0}Z_{(i+2)0}} = \prod_{l=0}^{n-2}X_{il}\frac{1}{Y_{i0}Z_{(i+1)0}}$.

From this we can solve the $X_{i(n-1)}$ for $i \neq 0$ as

$$X_{i(n-2)} = (\prod_{l=0}^{n-2}X_{0l})\frac{Y_{i0}Z_{(i+1)0}}{Y_{00}Z_{10}}(\prod_{l=0}^{n-2}X_{il})^{-1}. \tag{22.11.13}$$

The periodic condition does not give new constraints.

Now we have finished solving the F-term and can summarise the results into the K-matrix. We use the following independent variables: Z_{i0}, Y_{i0} for $i = 0, 1, ..., n-1$; X_{ij} for $i = 0, 1, ..., n-1$ $j = 0, 1, ..., n-3$ and $X_{0(n-2)}$ $X_{0(n-1)}$, so the total number of variables is $2n + n(n-2) + 2 = n^2 + 2$. This is usually too large to calculate. For example, even when $n = 4$, the K matrix is 48×18. The standard method to find the dual cone T from K needs to analyse some $48!/(17!31!)$ vectors, which is computationally prohibitive.

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