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Seiberg Duality in Matrix Models II

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Abstract: In this paper we continue the investigation, within the context of the Dijkgraaf-Vafa Programme, of Seiberg duality in matrix models as initiated in hep-th/0211202, by allowing degenerate mass deformations. In this case, there are some massless fields which remain and the theory has a moduli space. With this illustrative example, we propose a general methodology for performing the relevant matrix model integrations and addressing the corresponding field theories which have non-trivial IR behaviour, and which may or may not have tree-level superpotentials.

Keywords: Seiberg Duality, Matrix Model, Large N Duality, Moduli Space.

Contents

1. Introduction

With rapid development, the Dijkgraaf-Vafa Programme [1, 2, 3] of establishing a correspondence between a wide class of four dimensional $\mathcal{N}=1$ gauge theories and certain bosonic matrix models has withstood extensive tests (q. v. [4] to [26]). The original proposal was that an $\mathcal{N}=1$ $U(N_c)$ gauge theory with a tree level superpotential $W_{tree}(\Phi_i, g_a)$ in adjoint fields Φ_i and couplings g_a has a complete effective superpotential

$$\begin{aligned} W_{eff}(S,\Lambda,g_a) &= W_{\text{Veneziano-Yankielowicz}} + W_{\text{Perturbative}} \\ &= N_c S(1-\log(\frac{S}{\Lambda^3})) + N_c \frac{\partial \mathcal{F}_0(S,g_a)}{\partial S} \end{aligned}$$

where $S := -\frac{1}{32\pi^2}\operatorname{Tr} W_a W^a$ is the glueball superfield, Λ, the cutoff scale, and $\mathcal{F}_0$ is the genus-zero (i.e., planar) partition function (at large rank M) of the matrix model whose potential is formally the tree-level superpotential:

$$\begin{aligned} \mathcal{F}_0 := \mathcal{F}_{\chi=2} &= -\frac{S^2}{M^2}\log Z \\ &= -\frac{S^2}{M^2}\log\int [D\Phi_i]\exp\left(-\frac{M}{S}W_{tree}(\Phi_i,g_a)\right). \end{aligned}$$

The addition of flavour to the above story has also been performed [13, 14, 16, 18, 19]. Let us adhere to the conventions of [13]. Now an $U(N_c)$ $\mathcal{N}=1$ theory with adjoint Φ and N_f fundamentals Q_f and $\widetilde{Q}^f$ with tree-level superpotential

$W_{tree}(\Phi_i, Q_f, \tilde{Q}^f, g_a)$ has, according to the correspondence, the effective superpotential

$$
\begin{aligned}
W_{eff}(S,\Lambda,g_a) &= W_{\text{Veneziano-Yankielowicz}} + W_2 + W_1 \\
&:= N_c S(1-\log(\frac{S}{\Lambda^3})) + N_c \frac{\partial \mathcal{F}_{\chi=2}(S,g_a)}{\partial S} + \mathcal{F}_{\chi=1}(S,g_a)
\end{aligned}
\tag{1.1}
$$

where

$$
-\frac{M^2}{S^2}\mathcal{F}_{\chi=2}(S,g_a) - \frac{M}{S}\mathcal{F}_{\chi=1}(S,g_a) = \log \int [D\Phi D Q_f D\tilde{Q}^f] \exp\left(-\frac{1}{g_s} W_{tree}(\Phi_i, Q_f, \tilde{Q}^f, g_a)\right).
$$

In other words, $\mathcal{F}_{\chi=2}(S,g_a)$ is the genus zero planar contribution and $\mathcal{F}_{\chi=1}(S,g_a)$, the boundary contribution from flavours. We remark that in these above computations the matrix model is of rank M (which is to be taken to infinity), a parametre unrelated to N_c and N_f.

With these pieces of information, together with the already existent literature on the full non-perturbative pure $U(N_c)$ SUSY gauge theory, viz., the Affleck-Dine-Seiberg superpotential [27], an immediate check presents itself to us, namely Seiberg Duality [28]. This was done by the first author in [26].

In particular the check was performed thus. The archetypal example of a Seiberg dual pair wherein both the electric and magnetic sides have tree level superpotentials involve mass deformations of the following type

Electric	Seiberg $\leftrightarrow$	Magnetic
$W_{ele} = \sum_{i=1}^{N_f} Q_j m_j \tilde{Q}^j$		$W_{mag} = \sum_{i=1}^{N_f} \frac{1}{\mu} X_i^j q_j \tilde{q}^j + \mathrm{Tr}(Xm)$
$W_{eff} = N_c(\hat{\Lambda}^{3N_c})^{\frac{1}{N_c}} = N_c(\Lambda^{3N_c - N_f}\det(m))^{\frac{1}{N_c}}$		

where m are the non-degenerated mass matrix of the fundamental squarks $Q_j, \tilde{Q}^j$ on the electric side (the matrix m is thus diagonal with entries m_j), μ, a dynamical scale, Λ, the UV cutoff scale, $\hat{\Lambda}$, the IR cutoff scale, $q_j, \tilde{q}^j$, the dual quarks and X, the dual meson. In [26] the matrix model computations were carried out for W_{ele} and W_{mag} individually according to (1.1), and W_{eff} was retrieved for both, whereby beautifully supporting the validity of the Dijkgraaf-Vafa Correspondence once more.

The story however, is not complete. To fully understand Seiberg duality one needs to consider cases without mass deformations, and thus indeed flat directions in the moduli space. This is to say that whereas [26] addressed the case where the mass matrix m was maximal rank, we need to explore the more subtle case when m has zero eigenvalues. This purpose of this writing is to supplant the analysis of [26] by showing that in this case of flat directions the Dijkgraaf-Vafa Programme continues to hold. It is important to study this example because it enpowers us with techniques as to what to do when the field theory has a non-trivial IR moduli space; moreover, theories which have no tree-level superpotentials which seem *per definitio* to elude the Programme can be treated by certain addition of appropriate constraints.

The organization of this paper is follows. We commence with a short review of the field theory results in Seiberg Duality in Section 2. Then, in Section 3, we directly integrate the corresponding electric and magnetic matrix models and show that they reproduce the known field theory results whereby supporting the Dijkgraaf-Vafa Programme for Seiberg duality in this more general and illustrative case. Finally, in Section 4, we give a discussion on some interesting problems and prospects.

2. A Review of the Field Theory

The phase structure of $SU(N_c)$ SUSY gauge theory with N_f flavors $Q_i, \widetilde{Q}_i$ and no superpotential has been analyzed in [29] (q. v. [30] for a more pedagogic review). In general the theory will have a moduli space described by gauge invariant operators, namely the meson field $M_i^j = Q_i\widetilde{Q}_j$ and the baryon fields B and $\widetilde{B}$. The baryons exist only when $N_f \geq N_c$ because they are constructed as being totally antisymmetric in the color index.

In the case at hand, $M_i^j, B, \widetilde{B}$ are not independent and satisfy some constraints, whereby parametrising a moduli space. Three cases need to be addressed separately:

- $N_f > N_c$: The constraints are not modified by quantum correction;
- $N_f = N_c$: The only classical constraint is modified by quantum effects as

$$\det(M) - (*B)(*\widetilde{B}) = 0 \Rightarrow \det(M) - (*B)(*\widetilde{B}) = \Lambda^{2N_c} \ , \qquad (2.1)$$

  where $*$ is the contraction of all flavor indices with the totally antisymmetric tensor on N_f indices.

- $N_f < N_c$: Only the M_i^j's exist and they are independent variables in the moduli space. However, quantum correction will generate the famous Affleck-Dine-Seiberg super-potential

$$W = (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det(M)}\right)^{\frac{1}{N_c-N_f}} \ . \qquad (2.2)$$

  The reason why the term (2.2) can only be generated[1] in the case $N_c > N_f$ is that when $\Lambda \to 0$, (2.2) will become singular if $N_c < N_f$.

[1] Λ is the dynamical scale of the asymptotically free (AF) theory. When the energy scale is less than Λ, the gauge coupling becomes strong. So when $\Lambda \to 0$, the gauge theory is weakly coupled at any energy scale and there should not be any quantum corrections.

Since for $N_c \geq N_f$, there are complicated constraints among variables $M_i^j, B, \widetilde{B}$ (i.e., non-trivial moduli space) which make the problem less tractable, we will consider the simpler case by adding some mass terms such that the remaining massless flavors are less than N_c. For example, we set only

$$m_j \neq 0 \qquad j = K+1, ..., N_f$$

with $K < N_c$ and add a term into the superpotential as

$$W_{elec} = \sum_{j=K+1}^{N_f} Q_j m_j \widetilde{Q}^j \ . \tag{2.3}$$

We shall call (2.3) a degenerate mass deformation in contrast to [26] where all m_j were non-zero.

After integrating out these massive flavors the theory becomes effectively $SU(N_c)$ with K flavors, so the exact effective superpotential is

$$W = (N_c - K)\left(\frac{\hat{\Lambda}^{3N_c-K}}{\det(M)}\right)^{\frac{1}{N_c-K}} \tag{2.4}$$

where $M_i^j, i, j = 1, ..., K$ is the meson constructed from the remaining massless flavors and the cut-off scale $\hat{\Lambda}$ in IR matches the Λ in UV by

$$\hat{\Lambda}^{3N_c-K} = \det(m)\Lambda^{3N_c-N_f} \ . \tag{2.5}$$

After refreshing the reader's memory with the above review, we can set up our Seiberg dual pair under the degenerate mass deformation. The electric theory is $SU(N_c)$ with N_f flavors and superpotential (2.3) with $K < N_c$. The corresponding magnetic theory is $SU(N_f - N_c) \equiv SU(\widetilde{N_c})$ with N_f flavors, a meson $X_i^j, i, j = 1, ..., N_f$ and superpotential

$$W_{mag} = \frac{1}{\mu} X_i^j q_j \widetilde{q}^i + \mathrm{Tr}(Xm) \tag{2.6}$$

where μ is a scale and m is same mass matrix in (2.3). Our aim is to do the matrix model integration for both electric and magnetic theories and to show that they reproduce the results (2.4) and (2.5).

3. The Matrix Model

3.1 The Electric Side

We first do the matrix model integrations for the electric field theory. Unlike the case discussed in [26], here the mass matrix is degenerate and there are some massless fields left in the IR. To do it correctly, we must take care of these zero modes as emphasized in [11]. The way to do this is advocated in [19] where we add a delta-function into the matrix model integration so that we are only integrating fields subjects to the constraint given by the moduli space equation of corresponding field theory.

More concretely, in our example, we have a $U(N)$ matrix model at large N (we emphasize that at the stage of computing free-energies, this N is unrelated to the N_c and N_f of the field theory), and we should modify the naïve integration for the partition function

$$\begin{aligned} Z &= \frac{1}{\mathrm{Vol}(U(N))} \int \prod_{j=1}^{N_f} dQ_j dQ_j^\dagger e^{-\frac{1}{g_s} \sum_{j=K+1}^{N_f} Q_j m_j Q^{\dagger j}} \\ &= \frac{1}{\mathrm{Vol}(U(N))} \int \prod_{j=1}^{K} dQ_j dQ_j^\dagger \int \prod_{l=K+1}^{N_f} dQ_l dQ_l^\dagger e^{-\frac{1}{g_s} \sum_{j=K+1}^{N_f} Q_j m_j Q^{\dagger j}} \end{aligned}$$

into the form

$$Z = \frac{1}{\mathrm{Vol}(U(N))} \int \prod_{j=1}^{K} dQ_j dQ_j^\dagger \delta(M_i^j - Q_i Q^{\dagger j}) \int \prod_{l=K+1}^{N_f} dQ_l dQ_l^\dagger e^{-\frac{1}{g_s} \sum_{j=K+1}^{N_f} Q_j m_j Q^{\dagger j}} \tag{3.1}$$

where we have split the integration into massive and massless parts and inserted the delta-function constraint in light of the fact that the meson is composed of the fundamental squarks.

We wish to emphasize that the above treatment should apply for more general cases such as $N_f > N_c$ without any mass deformations. We only need to include proper delta-function constraints into the matrix model integration. These cases without tree-level superpotential which seemingly defy the rules of the Dijkgraaf-Vafa Programme can thus be addressed. The difficult part is that when we include these delta-functions, it is hard to do the integration in general. Developing this technique will be very important to the matrix model - field theory correspondence.

For the integral (3.1) there are three contributions. The volume contributes a factor of $\mathrm{Vol}(U(N)) = N^{\frac{N^2}{2}} = e^{\frac{N^2}{2} \log N} = e^{\frac{1}{g_s^2} \frac{S^2}{2} \log \frac{S}{g_s}}$. As we argued in [26], to get the proper dimensions, we need to replace $g_s = \Lambda^3 e^{3/2}$ and get the contribution to the effective superpotential as

$$\frac{\partial(\frac{S^2}{2} \log \frac{S}{g_s})}{\partial S} = N_c[S \log \frac{S}{\Lambda^3} - S]. \tag{3.2}$$

The second piece comes from the massive field integration and it is as in [26]:

$$e^{N(N_f - K) \log(\pi g_s) - N \log(\det(m))} = e^{\frac{1}{g_s}[S(N_f - K) \log(\pi g_s) - S \log(\det(m))]}.$$

Again, dimensional analysis allows us to replace πg_s by Λ and we get the next contribution to the effective superpotential:

$$[S(N_f - K)\log(\Lambda) - S\log(\det(m))] \ . \tag{3.3}$$

The third piece comes form the integration of the massless modes subject to the delta-function constraint. The contribution is $e^{-KN\log N+(N-K)\log\det(M)}$ [19] with the method of Wishart models. Since in the matrix model, we need to take $N \to \infty$ and $N - K \sim N$ in the second term. When translating into the field theory, we need to put back the proper dimensionful parametres as before to obtain the last contribution:

$$-K[S\log\frac{S}{\Lambda^3} - S] + S\log(\frac{\det(M)}{\Lambda^{2K}}) \ . \tag{3.4}$$

Adding the three pieces (3.2) (3.3) (3.4) together, we get

$$\begin{aligned}
-W_{elec;\ eff} &= N_c[S\log\frac{S}{\Lambda^3} - S] + [S(N_f - K)\log(\Lambda) - S\log(\det(m))] \\
&\quad -K[S\log\frac{S}{\Lambda^3} - S] + S\log(\frac{\det(M)}{\Lambda^{2K}}) \\
&= (N_c - K)[S\log\frac{S}{(\frac{\det(m)\Lambda^{3N_c-N_f}}{\det(M)})^{\frac{1}{N_c-K}}} - S] \ .
\end{aligned} \tag{3.5}$$

Minimizing (3.5) with respect to S we get

$$S = (\frac{\det(m)\Lambda^{3N_c-N_f}}{\det(M)})^{\frac{1}{N_c-K}} \tag{3.6}$$

so the exact superpotential, upon integrating out S is

$$W_{elec;\ eff} = (N_c - K)(\frac{\det(m)\Lambda^{3N_c-N_f}}{\det(M)})^{\frac{1}{N_c-K}} \ , \tag{3.7}$$

which is exactly the result in the field theory (2.4) and (2.5).

As we emphasized above, this calculation is suitable only for the case of $K < N_c$ because only in this case, the independent variables are just M_i^j and there are no baryonic fields; this is reflected in our matrix calculation since we would otherwise need to insert extra delta-function constraints to capture the moduli space. Indeed this restriction also gives a hint of how the matrix model actually sees different behavior in the field theory for the cases $N_c > N_f$ and $N_c \leq N_f$ because of the necessity of putting in different constraints.

3.2 The Magnetic Side

Now let us move to the dual magnetic side. The tree-level superpotential is given in (2.6). Again, since the mass matrix m is degenerate, we need to modify the matrix model integration by including the proper delta-function as in the previous subsection. In particular, we have

$$Z = \frac{1}{\mathrm{Vol}(U(N))} \int dX \prod_j dq_j dq_j^\dagger [\prod_{i,j=1}^{K} \delta(X_i^j - M_i^j)] \exp(\frac{-1}{g_s}[\mathrm{Tr}(mX) + \sum_{i,j=1}^{N_f} \frac{1}{\mu} X_i^j q_j q^{\dagger i}]) \ . \tag{3.8}$$

After finishing the integration of X as was done in [26], (3.8) becomes

$$Z = \frac{1}{\mathrm{Vol}(U(N))} \int \prod_{j=1}^{K} dq_j dq_j^\dagger e^{-\frac{1}{g_s\mu} \sum\limits_{i,j=1}^{K} M_i^j q_j q^{\dagger i}} \int \prod_{l=K+1}^{N_f} dq_l dq_l^\dagger \delta(\mu m_l^p + q_l q^{\dagger p}) \ . \tag{3.9}$$

Once again, the whole integration (3.9) is reduced to three contributions. The first one comes from the volume and as in (3.2) gives the contribution to the superpotential as

$$\widetilde{N}_c [S \log \frac{S}{\widetilde{\Lambda}^3} - S] \ , \tag{3.10}$$

where we use $\widetilde{N}_c$, $\widetilde{\Lambda}$ to indicate that it is in the dual magnetic theory.

The second piece is simply the Gaussian integration for massive fields because here M_i^j are just the mass parameters and upon comparison with (3.3) we obtain the contribution

$$[SK \log(\widetilde{\Lambda}) - S \log(\det(\frac{M_i^j}{\mu}))] \ . \tag{3.11}$$

The third piece is the same constrained integration as given by [19] and the contribution is (comparing with (3.4))

$$-(N_f - K)[S \log \frac{S}{\widetilde{\Lambda}^3} - S] + S \log(\frac{\det(-\mu m)}{\widetilde{\Lambda}^{2(N_f-K)}}) \ . \tag{3.12}$$

It is interesting to notice that the mass integration (3.11) and constraint integration (3.12) in the magnetic field theory are exactly the opposite of the corresponding electric field theory ((3.3) for the mass and (3.4) for the constraint integrals). This of course is no coincidence and is in fact a result of Seiberg duality.

Putting the three pieces together we get

$$\begin{aligned} -W &= \widetilde{N}_c [S \log \frac{S}{\widetilde{\Lambda}^3} - S] + [SK \log(\widetilde{\Lambda}) - S \log(\det(\frac{M_i^j}{\mu}))] \\ &\qquad -(N_f - K)[S \log \frac{S}{\widetilde{\Lambda}^3} - S] + S \log(\frac{\det(-\mu m)}{\widetilde{\Lambda}^{2(N_f-K)}}) \\ &= (\widetilde{N}_c - (N_f - K))[S \log \frac{S}{\widetilde{\Lambda}^3} - S] + S \log \frac{(-)^{N_f-K} \mu^{N_f} \det(m)}{\det(M) \widetilde{\Lambda}^{2N_f - 3K}} \\ &= (\widetilde{N}_c - (N_f - K))[S \log \frac{S}{\widetilde{\Lambda}^3 (\frac{(-)^{N_f-K} \mu^{N_f} \det(m)}{\det(M) \widetilde{\Lambda}^{2N_f-3K}})^{\frac{-1}{\widetilde{N}_c - (N_f-K)}}} - S] \ . \end{aligned}$$

Minimizing the above superpotential with respect to S we obtain

$$S = \widetilde{\Lambda}^3 \left(\frac{(-)^{N_f-K} \mu^{N_f} \det(m)}{\det(M) \widetilde{\Lambda}^{2N_f-3K}} \right)^{\frac{-1}{\widetilde{N}_c-(N_f-K)}} ; \tag{3.13}$$

after some algebra it can be shown that $S_{mag} = -S_{ele}$. The exact superpotential, upon back substitution becomes

$$\begin{aligned} W_{mag;\ eff} &= (\widetilde{N}_c - (N_f - K)) \widetilde{\Lambda}^3 \left(\frac{(-)^{N_f-K} \mu^{N_f} \det(m)}{\det(X) \widetilde{\Lambda}^{2N-f-3K}} \right)^{\frac{-1}{\widetilde{N}_c-(N_f-K)}} \\ &= (\widetilde{N}_c - (N_f - K)) \left(\frac{(-)^{N_f-K} \mu^{N_f} \det(m)}{\det(X) \widetilde{\Lambda}^{3\widetilde{N}_c-N_f}} \right)^{\frac{-1}{\widetilde{N}_c-(N_f-K)}} . \end{aligned} \tag{3.14}$$

Now using the relationships [30]

$$\Lambda^{3N_c-N_f} \widetilde{\Lambda}^{3\widetilde{N}_c-N_f} = (-)^{N_f-N_c} \mu^{N_f}, \qquad \widetilde{N}_c - (N_f - K) = -(N_c - K) \tag{3.15}$$

for the dual cut-off scales, we can recast (3.14) into

$$W_{mag;\ eff} = (N_c - K) \left(\frac{\det(m) \Lambda^{3N_c-N_f}}{\det(M)} \right)^{\frac{1}{N_c-K}} ; \tag{3.16}$$

note that the minus signs have been properly canceled.

We recognize (3.16) as precisely (3.7); therefore the matrix model computation has again successfully reproduced Seiberg duality in this generalized case from the one in [26].

4. Discussions and Prospects

In this paper, we have generalized the result in [26], from non-degenerate to degenerated mass matrix and have shown that in the context of the Dijkgraaf-Vafa Programme, the matrix model continues to perfectly reproduce the predictions of Seiberg duality.

The techniques arising from this illustrative example extend beyond the present framework. In fact they allow us to propose a general method of attack on the matrix model integration when the corresponding field theory has a classical moduli space, by generalizing the ideas presented in [11, 19].

In particular, we need to add into the partition function integral, proper delta-function constraints in accordance with the explicit relations in the field theory moduli space. In other words, one cannot naïvely integrate over the space of all matrices but only subspaces relevant to the field theory. It is worth to emphasize that these constraints we add are *classical* relationships. The matrix model will supply the quantum correction to the moduli space. This can be seen by setting $K = N_c$ in equation (3.5), so the equation of motion of S gives $\det(M) - \det(m)\Lambda^{3N_c-N_f} = 0$.

Because for different number of flavors we will have different delta-functions, this prescription solves the puzzle why the matrix model would know the different dynamical behavior of the corresponding field theory.

Moreover, field theories without tree-level superpotential which *ab initio* seemingly elude the Dijkgraaf-Vafa procedure, can be thus addressed. Indeed we merely have to add appropriate delta-function constraints (and go to the dual electric/magnetic theory if necessary) to perform the matrix integral.

However, as remarked in [26], we are still far from completely showing Seiberg duality in the matrix model, even for the standard example of no mass deformations at all. The difficulty is that we need to find proper delta-function constraints reflecting the baryonic and mesonic branches of the moduli space, and more importantly, to do the matrix integration in the presence of these constraints. This is a very involved task and beckons for future work.

Many immediate checks are also conveniently at hand. The generalized Seiberg dualities, such as the host of examples in toric dualities and quiver dualities addressed in [31, 32] and [33] present as readily available case-studies. It is also interesting to generalize our treatment from $U(N)$ to SO/Sp gauge groups.

The works in [26] and herein are a nontrivial check of Seiberg duality in matrix models. However, we would like to ask a more profound question: could we derive Seiberg duality from the matrix model? In other words, we start with a known electric field theory and translate it into the proper matrix model. Then could we find a transformation in the matrix integration to change this electric matrix model into another equivalent magnetic one, from which we can read out the superpotential of the magnetic field theory directly? By this way, we would have *derived* Seiberg duality from the matrix model and be granted the remarkable ability to see an $\mathcal{N}=1$ duality purely from a bosonic matrix integration.

There are some hints for this interesting issue in our calculations. Comparing (3.1) and (3.9), we see the constrained integration in one model becomes direct integration in another and vise versa. It is reminiscent of some kind of field theory transformation with source such as Legendre transformations. Does this hold in general? What is this transformation in the matrix model which we seek that would derive Seiberg Duality?

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