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Duality Walls, Duality Trees and Fractional Branes

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Abstract: We compute the NSVZ beta functions for $\mathcal{N} = 1$ four-dimensional quiver theories arising from D-brane probes on singularities, complete with anomalous dimensions, for a large set of phases in the corresponding duality tree. While these beta functions are zero for D-brane probes, they are non-zero in the presence of fractional branes. As a result there is a non-trivial RG behavior. We apply this running of gauge couplings to some toric singularities such as the cones over Hirzebruch and del Pezzo surfaces. We observe the emergence in string theory, of "Duality Walls," a finite energy scale at which the number of degrees of freedom becomes infinite, and beyond which Seiberg duality does not proceed. We also identify certain quiver symmetries as T-duality-like actions in the dual holographic theory.

Keywords: Seiberg Duality, Duality Cascades, AdS/CFT, D-brane Probes.

Contents

1. Introduction

The emergence of the Klebanov-Strassler "Cascade Phenomenon" [5] has been a marvelous motif in the grand theme of the AdS/CFT correspondence. In particular, the study of the relation between the supergravity bulk theory and the field theory on world-volumes of branes probing singular geometries has been part of the on-going programme to extend the correspondence to more general, for example non-conformal, classes of field theories.

Indeed, with the introduction of M fractional D3 branes in addition to N regular ones on our familiar conifold singularity, [5] studied the dual field theory which is a 4-dimensional $\mathcal{N}=1$ $SU(N+M)\times SU(N)$ non-conformal gauge theory. The radial variation of the fiveform flux in AdS is identified with the running of the gauge couplings. Therefore when one of the couplings becomes strong, one can, à la Seiberg, dualise the theory and flow to the IR, to one with gauge group $SU(N)\times SU(N-M)$. One may follow this RG flow in the field theory in principle *ad infinitum* and the process was referred to as a cascade [5].

The generalisation of this phenomenon to other geometries is hindered by the fact that the conifold is really the only geometry for which we know the metric. Nevertheless nice extensions from the field theory side have been performed. Notably, in [12], the cascade has been recast into properties of the Cartan matrix of the quiver [38, 13]. Then Seiberg duality becomes Weyl reflections in the associated root space. The UV behaviour would thus depend markedly on whether the Cartan matrix is hyperbolic (with a single negative eigenvalue and the rest positive) or not. Indeed for some simple quiver examples without consideration for stringy realisation, it was shown that the RG flow converges in the UV and surprisingly there is a finite accumulation point at which the scale of the dualisations pile up. This casts a shadow of doubt as to possible UV completions of these field theories. Such fundamental limitation on the scale of the theory was originally dubbed "Duality Walls" in [15].

The issue seems to persist as one studies generalisations of the conifold geometry and in realisations in string theory. As a first example that is chiral and arising from standard string theory constructions, [14] discussed the case of our familiar $\mathbb{C}^3/\mathbb{Z}_3$ singularity. Using naïve beta-functions without consideration for the anomalous dimensions, [14] analysed in detail how one encounters duality walls for this string theoretic gauge theory.

Indeed, despite our present lack for metrics and supergravity solutions for wider classes of examples, we are in fact well armed from the gauge theory perspective. Extensive methodology and catalogues of non-spherical horizons have been in circulation (q.v. e.g. [33, 35, 34, 31, 4, 30, 8]). A particular class for which an algorithmic outlook was partaken is the toric singularities of which the conifold and the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold are examples [8, 9, 23]. For these geometries, Seiberg-like dualities dubbed "Toric Duality" [9, 10, 7] have been labouriously investigated. Consecutive application of such a duality on a given theory essentially translates to a systematic application of certain quiver transformation rules. These rules can be understood from many fruitful perspectives: as ambiguities in the Inverse Toric Algorithm [9]; or as monodromy transformations of wrapped cycles around the singularity [36, 37, 20, 11]; or as braiding relations in (p, q) sevenbrane configurations [20]; or as mutations in helices of exceptional collections of coherent sheafs [37]; or as tilting functors in the D-brane derived category [16, 17], etc.

A key feature of such a duality is the **tree structure** of the space of dual theories. As we dualise upon a node in the quiver at each stage, a new branch blossoms. The topology of the tree is important. For example, whether there are any closed cycles which would signify that certain dualities may be trapped within a group of theories.

We are therefore naturally inspired by the conjunction of the toy model in [14] and our host of techniques from toric duality. Dualisations in the tree is precisely the desired cascading procedure. A first care which needs to be taken is a thorough analysis of the beta-function, including the anomalous dimensions. Happily, the form of the exact beta function with the anomalous dimensions has been computed by [2] for $\mathcal{N} = 1$ gauge theories and by [3] for quiver theories in particular.

In $\mathcal{N} = 1$ SCFT theories, the 't Hooft anomaly for the $U(1)_R$ charge determines all anomalous dimensions of chiral operators. These however are determined up to global non-R flavour symmetries. Computationally one must resort to finding such additional symmetries. In the quiver cases, these can be done by guided inspection of the quiver diagrams [21].

Recently, the nice work by [26] posited a maximisation principle to systematically determine the R-symmetry and hence all anomalous dimensions. Namely one must maximise a certain combination of the $U(1)$ charges. This quantity is called a in the canonical literature and together with c constitute the central charges of the SCFT. Indeed it is believed that a obeys a 4d version of Zamolodchikov's c-theorem , having a monotone increase along RG flow toward the UV. The central charge a shall be for us, a measure of the number of degrees of freedom in the field theory. In the AdS dual, we can view this as the thermodynamic entropy of the horizon of the singular geometry.

The structure of this paper is as follows. We will first require three ingredients the combination of which will form the crux of our calculation. The first piece we need is four dimensional $\mathcal{N} = 1$ super conformal field theory (SCFT), especially quiver theories. In particular we remind the reader of the computations of anomalous dimensions in the beta function. This will be the subject of §2. The second piece we need is the so-called "duality trees" which arise from iterative Seiberg-like dualisations of quiver theories. This, with concrete examples from the zeroth del Pezzo, will constitute §3. The final piece we need is to recall the rudiments of the Klebanov-Strassler "cascade" for the quiver theory associated to the conifold. We do this in §4.

Thus equipped, we examine a simple but illustrative gauge theory in §5. This is a fairly well-studied quiver theory arising from D3-branes probing the singular complex cone over the zeroth Hirzebruch surface. The duality tree for the conformal phases of the theory form a flower. With the appropriate addition of fractional branes to take us away from conformality, we compute the beta function running in §5.2, by determining, using the abovementioned maximisation principle, all anomalous dimensions. We will find in §5.3 that there is indeed a duality wall, viz., an energy scale beyond which dualities cannot proceed. Interestingly, certain quiver automorphism symmetries can be identified with T-duality-like actions in the dual AdS theory.

Such analyses are well adapted and easily generalisable to arbitrary quiver theories. As a parting example, we present the case of the cone over the first del Pezzo surface in §6. This case exhibits another interesting phenomenon which we call "toric islands." We end with concluding remarks and prospects in §7. Moreover, in Appendix 1, we use the method of Picard-Lefshetz monodromy transformations to analytically determine the asymptotic behaviour of some of the cascades, especially in the case of alternating dualisations. Finally, as mentioned above, the central charge a actually measures the entropy and hence the volume of base of the AdS dual geometry. As an application, we compute catalogue in Appendix 2, various horizon volumes, particularly the Abelian orbifold $\mathbb{C}^3/\mathbb{Z}_n$ and the cones over the del Pezzo surfaces.

2. Computing Anomalous Dimensions in a SCFT

We devote this section to a summary of beta-functions in 4D $\mathcal{N}=1$ SCFT and of how to compute in particular the anomalous dimensions. The remainder of the paper will make extensive use of the values of these anomalous dimensions.

The necessary and sufficient conditions for a $\mathcal{N}=1$ supersymmetric gauge theory with superpotential to be conformally invariant, i.e., a SCFT, are (1) the vanishing of the beta function for each gauge coupling and (2) the requirement that the couplings in the superpotential be dimensionless. Both these conditions impose constraints on the anomalous dimensions of the matter fields, that is, chiral operators of the theory. This is because supersymmetry relates the gauge coupling beta functions to the anomalous dimensions of the matter fields due to the form of the Novikov-Shifman-Vainshtein-Zakharov(NSVZ) beta functions. [2, 3].

The examples which we study in this paper are a class of SUSY gauge theories known as quiver theories. These have product gauge groups of the form $\prod\limits_i U(N_{c_i})$ together with N_{f_i} bifundamental matter fields for the i-th gauge factor. There is also a polynomial superpotential. Such theories can be conveniently encoded by quiver diagrams where nodes are gauge factors and arrows are bifundamentals [4, 13]. The SCFT conditions for these quiver theories can be written as:

$$\begin{aligned} \beta_i \sim 3N_{c_i} - N_{f_i} + \frac{1}{2}\sum_j \gamma_j = 0 \\ -d(h) + \frac{1}{2}\sum_k \gamma_k = 0 \end{aligned} \tag{2.1}$$

where β_i is the beta-function for the i-th gauge factor, and γ_j, the anomalous dimensions. The index j labels the fields charged under the j-th gauge group factor while k indexes the fields appearing in the k-th term of the superpotential with coupling h, whose naïve mass dimension is $d(h)$.

These conditions (2.1) constitute a linear system of equations. However they do not always uniquely determine the anomalous dimensions because there will be more variables than constraints. One or more of the γ's are left as free parameters. Recently, Intriligator and Wecht [26] provided a general method for fixing this freedom in aribitrary 4D SCFT, whereby completely specifying the anomalous dimensions. They showed that the R-charges of the matter fields, which in an SCFT are related to the γ's, are those that (locally) maximize the central charge a of the theory. The central charge a is given in terms of the R-charges by

$$a = \frac{3}{32}(3\,\mathrm{Tr}\,R^3 - \mathrm{Tr}\,R)\ , \tag{2.2}$$

where the trace is taken over the fermionic components of the vector and chiral multiplets.

In a quiver theory with gauge group $\prod\limits_i U(N_i)$ and chiral bifundamental multiplets with multiplicities f_{ij} between the i-th and j-th gauge factors (the matrix f_{ij} is the adjacency matrix of the quiver), we can give an explicit expression for (2.2) in terms of the R-charges R_{ij} of the lowest components of the bifundamentals

$$a = \frac{3}{32}\left[2\sum_i N_i^2 + \sum_{i<j} f_{ij}N_iN_j\left[3(R_{ij}-1)^3 - (R_{ij}-1)\right]\right]\ . \tag{2.3}$$

Parenthetically, we remark that in some cases, such as the ones to be discussed in §4 and §5, anomalous dimensions can be fixed by using some discrete symmetries by inspecting the quiver and the form of the superpotential, without the need to appeal to the systematic maximization of a.

Now in (2.3) we need to know the R-charges. However, in a SCFT the conformal dimension D of a chiral operator is related to its R-charge by $D = \frac{3}{2}|R|$. Moreover, for bifundamental matter the relation between D and γ is $D = 1 + \frac{\gamma}{2}$. Therefore we can write the R-charges and hence (2.3) in terms of the anomalous dimensions by

$$\frac{3}{2}|R| = 1 + \frac{\gamma}{2}\ . \tag{2.4}$$

Therefore, after solving the conformality constraints (2.1) we can write a in terms of the still unspecified γ's by (2.3) and then maximize it in order to completely determine all the anomalous dimensions.

The freedom in the anomalous dimensions after using the SCFT conditions reflects the presence of non-anomalous $U(1)$ flavor symmetries in the IR theory. Initially, there is one $U(1)$ flavor symmetry for each arrow of the quiver. All the matter fields lying on an arrow have the same charge under this $U(1)$. Now we must impose the anomaly free condition for each node, this is the condition that for the adjacency matrix f_{ij} at the i-th node we have

$$\sum_j f_{ij} N_j = \sum_j f_{ji} N_j \ . \tag{2.5}$$

In other words, the ranks of the gauge groups, as a vector, must lie in the integer nullspace of the anti-symmetrised adjacency matrix:

$$(f - f^T)_{ij} \cdot \vec{N} = 0 \ . \tag{2.6}$$

Indeed the matrix $(f - f^T)$ is the intersection matrix of the quiver in geometrical engineering of these theories (q.v. e.g. [19, 37, 20]). After imposing this condition (2.6), we are left with (# of arrows - # of nodes) non-anomalous $U(1)$'s. The invariance of the superpotential reduces their number even more, giving one linear relation between their charges for each of its terms. But the number of independent such relations is not always sufficient to eliminate all the abelian flavor symmetries. The charges of those that still remain in the IR can indeed be read off from the expressions for the anomalous dimensions of the fields in terms of those that remain free after imposing the conditions (2.1).

The way in which the charge matrix of the remaining $U(1)$ flavor symmetries appears in this framework is as the matrix of coefficients that express the anomalous dimensions of the bifundamental fields as linear combinations of numerical constants and some set of independent anomalous dimensions. Specifically, suppose we start with n anomalous dimensions and that the solution to (2.1) specifies k of them in terms of the other $n-k$:

$$\gamma_i = \gamma_{0_i} + q_{ij}\gamma_j \quad , \quad i = 1, \ldots, k \quad , \quad j = k+1, \ldots, n. \tag{2.7}$$

The corresponding R-charges are related to the γ's by $R = \frac{\gamma+2}{3}$. The charges of the matter fields under these residual $U(1)$'s from are given by the q_{ij} matrix in (2.7). The constants γ_{0_i} are mapped to the test values of R charges. It is important to keep in mind that it is possible to change the basis of $U(1)$'s (correspondingly the set of independent anomalous dimensions), in which case the charge matrix would be modified.

3. Duality Structure of SUSY Gauge Theories: Duality Trees

Having reminded ourselves of the methodology of computing anomalous dimensions, we turn to the next ingredient which will prepare us for the cascade phenomenon, viz., the duality trees which arise from Seiberg-like dualities performed on the quivers. An interesting way to encode dual gauge theories and their relations is by using **duality trees**. This construction was introduced in [11], for the specific case of D3-branes probing a complex cone over dP_0, the zeroth del Pezzo surface.

In general, for a quiver theory with adjacency matrix f_{ij} and n gauge group factors, there are n different choices of nodes on which to perform Seiberg duality. In other words, we can dualise any of the n

nodes of the quiver to obtain a new one, for which we again have n choices for dualisation. We recall that dualisation on node i_0 proceeds as follows. Define $I_{in} :=$ nodes having arrows going into i_0, $I_{out} :=$ those having arrow coming from i_0 and $I_{no} :=$ those unconnected with i_0.

1. Change the rank of the node i_0 from N_c to $N_f - N_c$ with $N_f = \sum\limits_{i \in I_{in}} f_{i,i_0} N_i = \sum\limits_{i \in I_{out}} f_{i_0,i} N_i$;

2. $f_{ij}^{dual} = f_{ji} \qquad$ if either $i, j = i_0$;

3. Only arrows linking I_{in} to I_{out} will be changed and all others remain unaffected;

4. $f_{AB}^{dual} = f_{AB} - f_{i_0 A} f_{B i_0} \qquad$ for $A \in I_{out}, \quad B \in I_{in}$;
   If this quantity is negative, we simply take it to mean an arrow going from B to A. This step is simply the addition of the Seiberg dual mesons (as a mass deformation if necessary).

We remark that the fourth of these dualisation rules accounts for the antisymmetric part of the intersection matrix, which does not encode bi-directional arrows. Such subtle cases arise when there are no cubic superpotentials needed to give masses to the fields associated with the bi-directional arrows. We encountered such a situation in [21].

The subsequent data structure is that of a tree, where each site represents a gauge theory, with n branches emanating therefrom, connecting it to its dual theories. This is called a "duality tree." We will see along the paper that duality trees exhibit an extremely rich structure, with completely distinct topologies for the branches for gauge theories coming from different geometries.

As an introduction, let us recall the simple example considered in [11, 20]. The probed geometry in this case was a complex cone over dP_0. This cone is simply the famous non-compact $\mathbb{C}^3/\mathbb{Z}_3$ orbifold singularity. The generic quiver for any one in the tree of Seiberg dual theories for this geometry will have the form as given in Figure 1. The superpotential is cubic because there are only cubic gauge invariant operators in this theory, given by closed loops in the quiver diagram.

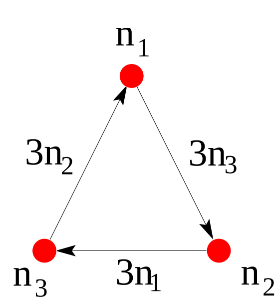

Figure 1: Generic quiver for any of the Seiberg dual theories in the duality tree corresponding to a D3-brane probing $\mathbb{C}^3/\mathbb{Z}_3$, the complex cone over dP_0.

Since there are three gauge group factors, there will be three branches coming out from each site in the duality tree. The tree is presented in Figure 2. For clarity we colour-coded the tree so that sites of the same colour correspond to equivalent theories, i.e., theories related to one another by some permutation of the gauge groups and/or charge conjugation of all fields in the quiver (in other words theories whose quivers are permutations and/or transpositions of each other). We have also included, the quivers to which the various coloured sites correspond in Figure 3.

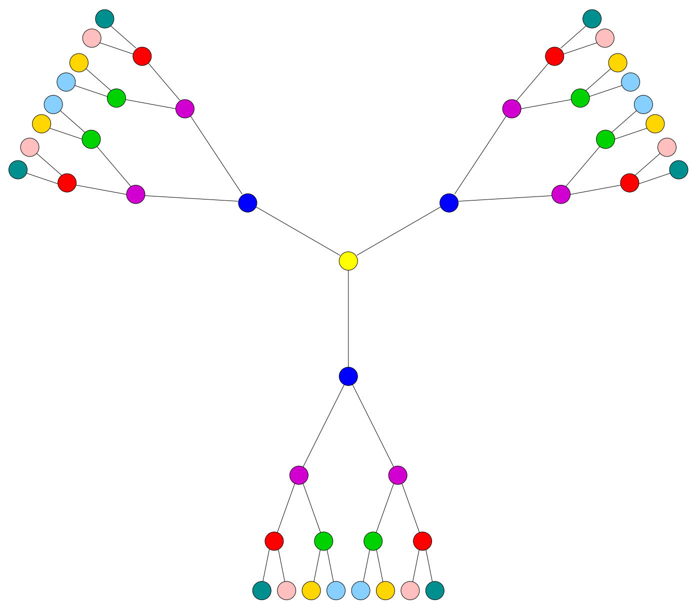

Figure 2: Tree of Seiberg dual theories for dP_0. Each site of the tree represents a gauge theory, and the branches between sites indicate how different theories are related by Seiberg duality transformations.

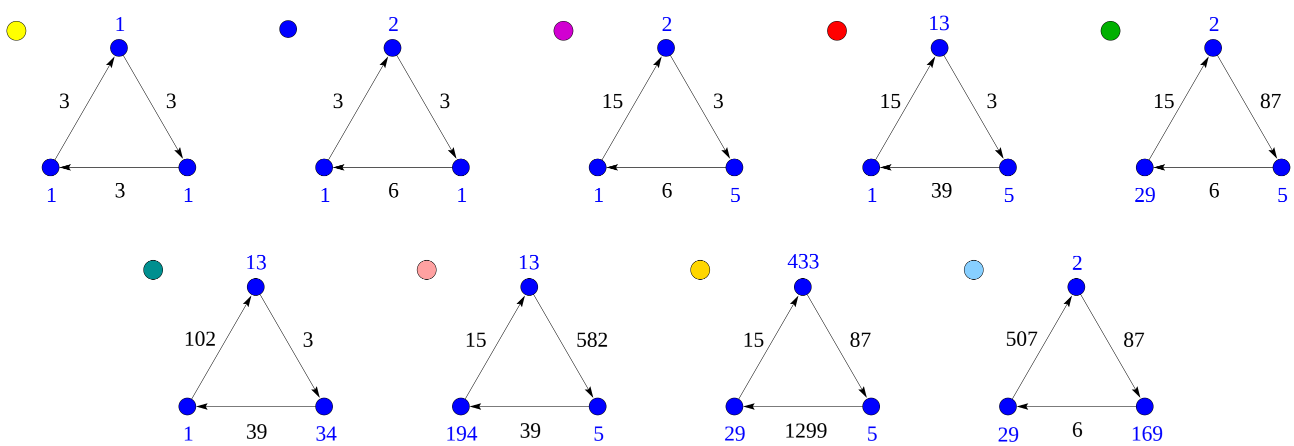

Figure 3: Some first cases of the Seiberg dual phases in the duality duality tree for the theory corresponding to a D3-brane probing $\mathbb{C}^3/\mathbb{Z}_3$, the complex cone over dP_0.

One important invariant associated to an algebraic singularity is the trace of the total monodromy matrix around the singular point. This can typically be recast into an associated Diophantine equation in the intersection numbers, i.e., the f_{ij}'s [36, 39, 20]. This equation captures all the theories that can be obtained by Seiberg duality and hence classifies the sites in the tree. We remind the reader of the relation between this equation and Picard-Lefschetz monodromy transformations in Appendix 1 and refer him/her to e.g. [20].

From Figure 1, we see that there exists a simple relation between the intersection numbers and the ranks of the gauge groups for dP_0, namely for rank (n_1, n_2, n_3), the intersection matrix is given by $3\begin{pmatrix} 0 & n_1 & -n_3 \\ -n_1 & 0 & n_2 \\ n_3 & -n_2 & 0 \end{pmatrix}$. The Diophantine equation in terms of the ranks reads

$$n_1^2 + n_2^2 + n_3^2 - 3n_1n_2n_3 = 0 \ . \tag{3.1}$$

This turns out to be the well-studied Markov equation.

It is important to stress that, up to this point, duality trees do not provide any information regarding RG flows. In fact, if the theories under study are conformal the trees just represent the set of dual gauge theories and how they are interconnected by Seiberg duality transformations within the conformal window. We will extend our discussion about this point in §4 and §5, where we will obtain non-conformal theories by the inclusion of fractional branes.

4. The Conifold Cascade

A famous example of successive Seiberg dualisations is the Klebanov-Strassler cascade in gauge theory [5] associated to the warped deformed conifold [5]. In light of the duality tree structure in the previous section, we now present the third and last piece of preparatory work and summarise some key features of this example, in order to illustrate the concept of duality cascade, as well as to introduce many of the methods and approximations that will be used later.

Let us begin by considering the gauge theory that appears on a stack of N D3-branes probing the conifold. This theory has an $SU(N) \times SU(N)$ gauge symmetry. The matter content consists of four bifundamental chiral multiplets $A_{1,2}$ and $B_{1,2}$ and the quiver diagram is shown in Figure 4. This model

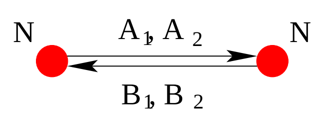

Figure 4: Quiver diagram for the gauge theory on N D3-branes probing the conifold.

has also interactions given by the following quartic superpotential

$$W = \frac{\lambda}{2}\epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}\,A_iB_kA_jB_l \tag{4.1}$$

for some coupling λ, and where we trace over color indices.

This gauge theory is self dual under Seiberg duality transformations, by applying the duality rules in §3. Accordingly, its "duality tree" is the simplest one, consisting of a single point representing the $SU(N) \times SU(N)$ theory, which transforms into itself when dualizing either of its two gauge groups. This is shown in Figure 5.

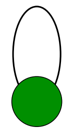

Figure 5: The "duality tree" of the conifold. Its single site represents the standard $SU(N) \times SU(N)$ theory. The closed link coming out the site and returning to it represents the fact that the theory, being self-dual, transforms into itself under Seiberg duality.

We will see below that when we apply the procedure for finding anomalous dimensions outlined in §3 to this specific case, taking into account its symmetries, we conclude that all the anomalous dimensions are in fact equal to $-1/2$ and that the theory is conformal, i.e. both the gauge and superpotential couplings have vanishing beta functions and (2.1) are satisfied. In order to induce a non-trivial RG flow the theory has to be deformed. A possible way of doing this is by the inclusion of fractional branes [5]. It is straightforward to see what kind of fractional branes can be introduced.

In general, introducing fractional branes is done by determining, for a given quiver, the most general gauge groups consistent with anomaly cancellation. Now recall from (2.6), possible ranks of the gauge factors must reside in the integer nullspace of the intersection matrix. Therefore a basis for probe and fractional branes is simply given by a basis for this nullspace. For the conifold, we find that the most general gauge group is $SU(N+M) \times SU(N)$. We will refer to N as the number of probe branes and to M as the number of fractional branes.

We see that indeed, for any non-vanishing M, there is no possible choice of anomalous dimensions satisfying (2.1) and thus we are indeed moving away from the conformal point. This case has been widely studied (see [5, 32] and references therein) and leads to a **duality cascade**. What this means is that at every step in the dualisation procedure of this now non-conformal quiver theory, one of the gauge couplings is UV free while the other one is IR free. As we follow the RG flow to the IR, we reach a scale at which the inverse coupling of the UV free gauge factor vanishes. At this point, it is convenient to switch to a more suitable description of the physics, in terms of different microscopic degrees of freedom, by performing a Seiberg duality transformation on the strongly coupled gauge group. This procedure generates the duality cascade when iterated. Indeed, the tree of Figure 5 can be interpreted as representing a duality cascade modulo fractional brane contributions.

4.1 Moving Away from the Conformal Point

Let us now study this cascade in detail, setting the framework we will later use to analyze cascades for general quiver theories. Recall, from (2.1), that a key ingredient required for the computation of the beta functions are the values of the anomalous dimensions. We have already provided a method to compute anomalous dimensions in the absence of fractional branes, that is, in a conformal theory, in §2. There is no analogue for such a procedure when the theory is taken away from conformality. However it is possible to work in a limit such that their values are under control and the beta functions that govern the RG flow can be computed to some approximation. Since M/N measures the departure from conformal invariance, any anomalous dimension will be of the generic form

$$\gamma = \gamma_c + O(M/N) \ , \tag{4.2}$$

where γ_c is its value for the conformal case of $M = 0$. Now, this theory is symmetric under the transformation

$$\begin{aligned} M &\to -M \\ N &\to N + M \ , \end{aligned} \tag{4.3}$$

which, in the limit $N/M \ll 1$ (i.e., we are taking a standard large N limit), simplifies to

$$\begin{aligned} M &\to -M \\ N &\to N \ . \end{aligned} \tag{4.4}$$

This indicates that, in fact, at large N, γ must be an even function in M/N and so the expansion (4.2) has to start from the second order [5]:

$$\gamma = \gamma_c + O(M/N)^2 \ . \tag{4.5}$$

The expression (4.5) is of great aid to us as it gives us the control over the anomalous dimensions we were pursuing. Inspecting (2.1), we see that because the departure of the γ's from their conformal values is of order $(M/N)^2$ at large N, the order (M/N) contributions to the beta functions can be computed simply by substituting the anomalous dimensions calculated at the conformal point into (2.1), and using the gauge groups with the M corrections.

Let us be concrete and proceed to compute the cascade for this example. First let us consider the anomalous dimensions at the conformal point where $M = 0$. They are the result of requiring the beta functions for both $SU(N)$ gauge groups and for the single independent coupling in the superpotential to vanish in accordance with (2.1). In this case, these three conditions coincide and are reduced to

$$\gamma_{c,A} + \gamma_{c,B} = -1 \ , \tag{4.6}$$

where $\gamma_{c,A}$ (resp. $\gamma_{c,B}$) is the critical value for the anomalous dimension for field A (resp. B). Once we take into account the symmetry condition $\gamma_{c,A} = \gamma_{c,B}$, we finally obtain

$$\gamma_c = -1/2 \ . \tag{4.7}$$

Now let us consider the beta functions for the gauge couplings in the non-conformal case of $M \neq 0$. They are

$$\begin{aligned} SU(N+M): \quad & \beta_{g_1} = N(1+\gamma_A+\gamma_B) + 3M \\ SU(N): \quad & \beta_{g_2} = N(1+\gamma_A+\gamma_B) + (-2+\gamma_A+\gamma_B)M. \end{aligned} \tag{4.8}$$

Note that there is no solution to the vanishing of these beta functions for $M \neq 0$. Replacing the anomalous dimensions at the conformal point $\gamma_c = -1/2$ into (4.8) we obtain the leading contribution to the beta functions

$$\begin{aligned} SU(N+M): \quad & \beta_{g_1} = +3M \\ SU(N): \quad & \beta_{g_2} = -3M \ . \end{aligned} \tag{4.9}$$

Since the theory at any point in the cascade is given by the quiver in Figure 4 with gauge group replaced by $SU(N+(n+1)M) \times SU(N+nM)$ for some $n \in \mathbb{Z}$ (where the role of the two gauge groups is permuted at every step), we see that the gauge couplings run as shown in Figure 6, where the beta function for each gauge group changes from $\pm 3M$ to $\mp 3M$ with each dualization. In Figure 6 we use he standard notation to which we adhere throughout the paper: the squared inverse couplings are denoted as $x_i = 1/g_i^2$ and the logarithm of the scale is $t = \log \mu$.

An important feature of this RG flow is that the separation between successive dualizations in the t axis remains *constant* along the entire cascade. We will see in §5.3 how the gauge theory for a D3-brane probing more general geometries, such as a complex cone over the Zeroth Hirzebruch surface, can exhibit a dramatically different behavior.

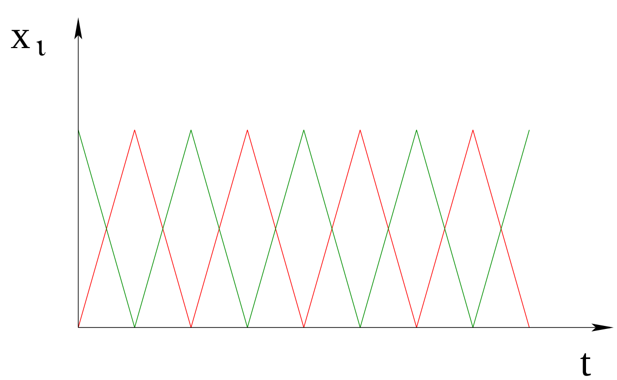

Figure 6: Running of the inverse square gauge couplings $x_i = \frac{1}{g_i^2}$, i=1,2. against the log of energy scale $t = \log \mu$, for the conifold. The distance between consecutive dualizations is constant and the ranks of the gauge groups grow linearly with the step in the cascade.

5. Phases of F_0

We are now well-equipped with techniques of computing anomalous dimensions, of duality trees and duality cascades. Let us now initiate the study of some more complicated gauge theories. Our first example will be the D-brane probe theory on a complex cone over the zeroth Hirzebruch surface F_0, which is itself simply $\mathbb{P}^1 \times \mathbb{P}^1$. This is a toric variety and the gauge theory was analysed in [9].

There are some reasons motivating the choice of this theory. The first is its relative simplicity. The second is that its Seiberg dual phases generically have multiplicities of bifundamental fields greater than 2, whereby providing some interesting properties. Indeed, from the general analysis of [12, 14], a qualitative change in a RG flow towards the UV behavior is expected when such a multiplicity is exceeded. Finally, as we will discuss later, this theory admits the addition of fractional branes. The presence of fractional branes turns the theory non-conformal, driving a non-trivial RG flow. All together, this theory is a promising candidate for a rich RG cascade structure.

The duality tree in this case is shown in Figure 7; we shall affectionately call it the "F_0 flower," of the genus *Flos Hirzebruchiensis* and family *Floris Dualitatis*. We have drawn sites that correspond to different theories with different colours; the colour-coded theories are summarized in Figure 8. Since the quiver has four gauge groups, there are four possible ways of performing Seiberg duality and thus there are four branches coming out from each site of the tree. The numbers on each branch corresponds to the node which was dualised. A novel point that was not present in the tree for dP_0 is the existence of closed loops.

The possible existence of RG flows corresponding to these closed loops is constrained by the requirement that the number of degrees of freedom is increased towards the UV, in accordance to the c-conjecture/theorem. As it was stressed for the dP_0 and the conifold examples, the duality tree for F_0 merely represents the infinite set of conformal theories which are Seiberg duals. Non-vanishing beta functions and the subsequent RG flow are generated when fractional branes are included in the system. We reiterate this point: *the duality tree describes duality cascades modulo fractional branes.* In other words, it represents a "projection" of actual cascades to the space of vanishing fractional branes.

5.1 F_0 RG flows

In this section we will follow the RG flow towards the UV of the theory living on D3-branes probing F_0, with the addition of fractional branes to obtain a non-conformal theory. As in the conifold example, the possible anomaly free probe and fractional branes are determined by finding the integer null space of the intersection matrix that defines the quiver (2.6). This can be done for any of the dual quivers that appear in the duality tree, but the natural choice is the simplest of the F_0 quivers as was done in [20] which is

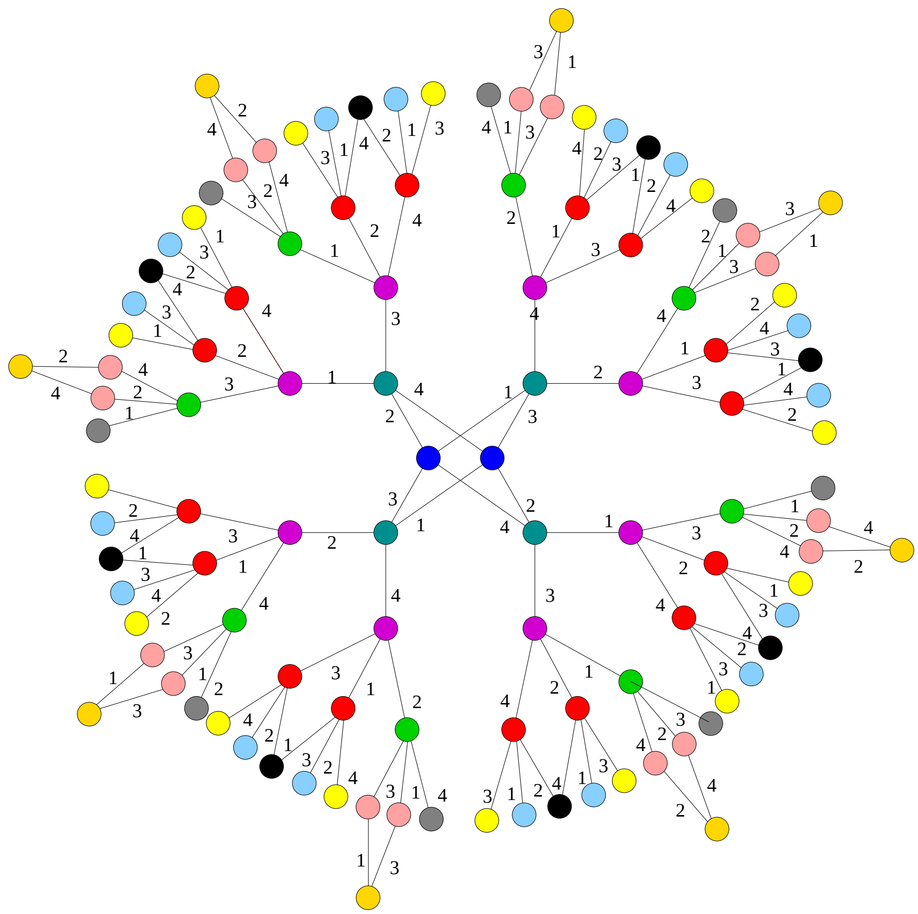

Figure 7: The "duality tree" of Seiberg dual theories for F_0, it is in the shape of a "flower," the *Flos Hirzebruchiensis*.

shown in Figure 8 as the first one (blue dot). The intersection matrix for this quiver is given by

$$A_{ij} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix} \Rightarrow f_{ij} = \begin{pmatrix} 0 & 2 & 0 & -2 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \end{pmatrix} . \tag{5.1}$$

A suitable basis for the nullspace of (5.1) is $v_1 = (1,1,1,1)$ and $v_2 = (0,1,0,1)$. Therefore, the most generic ranks for the nodes in the quiver, consistent with anomaly cancellation, are

$$(n_1, n_2, n_3, n_4) = N(1,1,1,1) + M(0,1,0,1) . \tag{5.2}$$

Following the discussion in §4, we will refer to N as the number of probe branes and M as the number of fractional branes. The theory is then conformal for $M = 0$ and non-conformal otherwise. For $M \neq 0$, there will exist an RG cascade. The specific path to the UV is determined by the initial conditions of the

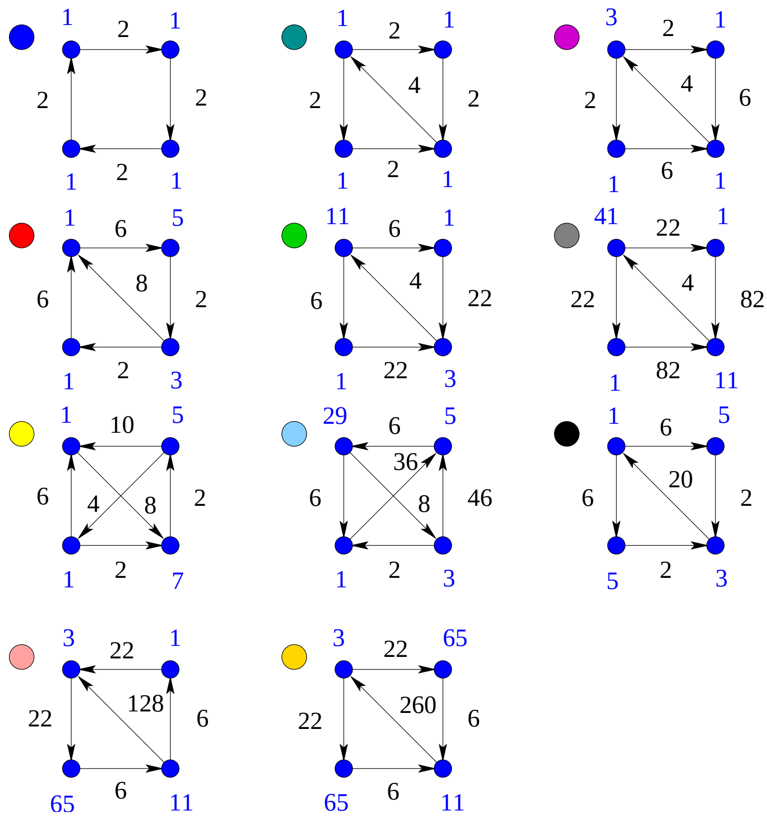

Figure 8: Some Seiberg dual phases for F_0.

flow, namely the gauge couplings at a given scale Λ_0. As we will see, very different qualitative behaviours can be obtained, depending on these initial conditions.

A crucial step in solving the conifold cascade was the identification of a symmetry that provided us with control over the anomalous dimensions in the $M/N \ll 1$ limit. It is possible to find an analogous symmetry for F_0. Examining (5.2) we see that this theory is invariant under

$$\begin{aligned} M &\rightarrow -M \\ N &\rightarrow N+M \ . \end{aligned} \tag{5.3}$$

In the limit $M/N \ll 1$, this symmetry transformation becomes

$$\begin{aligned} M &\rightarrow -M \\ N &\rightarrow N \ . \end{aligned} \tag{5.4}$$

Up to now, we have only considered a single theory (the first quiver in Figure 8) and showed that for $M/N \ll 1$ it is symmetric under (5.5). But the whole tree of Seiberg dual theories can be constructed using this model as the starting point. Since the transformations in (5.5) map the initial theory onto itself, the models derived from it by Seiberg duality are also invariant. Therefore, (5.5) constitute a symmetry of the entire tree.

In analogy to the conifold case this symmetry implies that, for all the dual theories, the odd order terms in the M/N expansion of the anomalous dimensions of all their bifundamental fields vanish and thus

$$\gamma = \gamma_c + O(M/N)^2 \ . \tag{5.5}$$

This means that the leading, $O(M/N)$, non-zero contribution to the beta functions can be computed using the anomalous dimensions calculated at the conformal point. Schematically,

$$\beta(\gamma) = \beta(\gamma_c) + O(M/N)^2 \tag{5.6}$$

for all the beta functions.

The procedure outlined in §2 can thus be applied to determine the conformal anomalous dimensions, which then can be used to work out the beta functions in the limit $M/N \ll 1$ and study the running of the gauge couplings as we flow to the UV. Let us do so in detail. The beta-functions in (2.1), for a quiver theory with k gauge group factors, ranks $\{n\}_i$, adjacency matrix A_{ij} and loops indexed by h corresponding to gauge invariant operators that appear in the superpotential, now becomes

$$\begin{aligned}\beta_{i\in\text{nodes}} &= 3n_i - \frac{1}{2}\sum_{j=1}^{k}(A_{ij}+A_{ji})n_j + \frac{1}{2}\sum_{j=1}^{k}(A_{ij}\gamma_{ij}+A_{ji}\gamma_{ji})n_j \\ \beta_{h\in\text{loops}} &= -d(h) + \frac{1}{2}\sum_{h}\gamma_{h_i h_j} \ ,\end{aligned} \tag{5.7}$$

where in the second expression $\beta_{h\in\text{loops}}$ associated with the terms in the superpotential the index in the sum over $\gamma_{h_i h_j}$ means consecutive arrows in a loop and $d(h)$ is determined by 3 minus the number of fields in the loop.

We will make liberal use of (5.7) throughout. For our example for the first phase of F_0, the ranks $(n_1, n_2, n_3, n_4) = (1,1,1,1)$, together with intersection matrix from (5.1), (5.7) reads

$$1+\gamma_{1,2}+\gamma_{4,1} = 0, \quad 1+\gamma_{1,2}+\gamma_{2,3} = 0, \quad 1+\gamma_{2,3}+\gamma_{3,4} = 0, \quad 1+\gamma_{3,4}+\gamma_{4,1} = 0, \quad 1+\frac{1}{2}(\gamma_{1,2}+\gamma_{2,3}+\gamma_{3,4}+\gamma_{4,1}) = 0 \,, \tag{5.8}$$

which affords the solution

$$\{\gamma_{1,2} \to -1-\gamma_{4,1}, \quad \gamma_{2,3} \to \gamma_{4,1}, \quad \gamma_{3,4} \to -1-\gamma_{4,1}\} \ . \tag{5.9}$$

We see that there is one undetermined γ. To fix this we appeal to the maximisation principle presented in §2. The central charge (2.3) now takes the form (where $\gamma_{4,5}$ is understood to mean $\gamma_{4,1}$.

$$a = \frac{3}{4} + \frac{1}{16}\sum_{i=1}^{4}\left(1 + \frac{(-1+\gamma_{i,i+1})^3}{3} - \gamma_{i,i+1}\right) \ . \tag{5.10}$$

Upon substituting (5.9) into (5.10), we obtain

$$a(\gamma_{4,1}) = -\frac{3}{8}(-2 + \gamma_{4,1} + \gamma_{4,1}^2) \ , \tag{5.11}$$

the maximum of which occurs at $\gamma_{4,1} = -\frac{1}{2}$. And so we have in all, upon using (5.9),

$$\gamma_{1,2} = \gamma_{2,3} = \gamma_{3,4} = \gamma_{4,1} = -1/2 \ . \tag{5.12}$$

Let us remark, before closing this section, that there is an alternative, though perhaps less systematic, procedure to determine anomalous dimensions that does not rely on the maximization of a. For every theory in the F_0 cascade the space of solutions to (5.8) is one dimensional. Fixing this freedom at any given point determines the anomalous dimensions in the entire duality tree. Maximization of the central charge a is a possible way of determining this free parameter. For F_0, a simple alternative is to make use of the symmetries of the theory (quiver and superpotential). Our theory (5.1) for example, instantly has all γ's equal by the $\mathbb{Z}_4$ symmetry of the quiver. Therefore, in conjunction with the solutions (5.9) to conformality, gives (5.12) as desired. Once the anomalous dimensions of theory (5.1) are determined, the freedom that existed in the conformal solutions of all the dual theories is fixed. This is done by matching the scaling dimensions of composite Seiberg mesons every time a Seiberg duality is performed and/or by noting that the anomalous dimensions of fields that are neutral under the dualized gauge group are unchanged.

5.2 Closed Cycles in the Tree and Cascades

Now we wish to find the analogue of the conifold cascade in §4 here. For this we wish to look for "closed cycles" in the duality tree (Figure 7). In the conifold case the theory was self dual and we cascaded by adding appropriate fractional branes. Here we do indeed see a simple cycle involving 2 sites (namely the blue and the dark green). We will call these two theories models A and B respectively and draw them in Figure 9. Model A is the example we addressed above.

The starting point will be model A, its superpotential and the set of gauge couplings at a scale Λ_0. We recall from (5.12) that the anomalous dimensions at the conformal point are $\gamma_{1,2} = \gamma_{2,3} = \gamma_{3,4} = \gamma_{4,1} = -1/2$. This leads to the following values for the beta functions for the 4 gauge group factors:

$$\begin{array}{ll} \beta_1 = -3M & \beta_2 = 3M \\ \beta_3 = -3M & \beta_4 = 3M \ . \end{array} \tag{5.13}$$

These beta-functions are constants, which means that the running of x_i, the inverse squared couplings as a function of the log scale is linear, with slopes given by (5.13). Let us thus run x_i to the UV accordingly. We see that β_1 and β_3 are negative so at some point the inverse couplings for the first or the third node will reach 0. Which of them does so first depends on the value of the initial inverse couplings we choose for

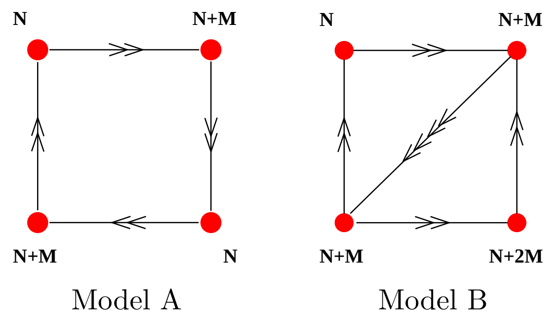

Figure 9: Quivers for Models A and B. Model A corresponds to the choice of ranks $(n_1, n_2, n_3, n_4)_A = N(1,1,1,1)+M(0,1,0,1)$, from which model B is obtained by dualizing node 3. It has ranks $(n_1, n_2, n_3, n_4)_B = N(1,1,1,1)+M(0,1,2,1)$.

n_1 and n_3. We dualise the node for which the inverse coupling first reaches 0, say node 3. This will give us Model B. If instead node 1 has the inverse coupling going to 0 first, we would dualise on 1 and obtain a theory that is equivalent to Model B after a reflection of the quiver (we can see this from Figure 7).

Next we compute the anomalous dimensions for Model B at the conformal point. In analogy to (5.8) and (5.10) we now obtain $\gamma_{1,2} = \gamma_{3,2} = \gamma_{4,3} = \gamma_{4,1} = -1/2$ and $\gamma_{2,4} = 1$, which gives the beta functions for the next step:

$$\begin{aligned} \beta_1 &= -3M & \beta_2 &= 0 \\ \beta_3 &= 3M & \beta_4 &= 0\ . \end{aligned} \tag{5.14}$$

From these we run the couplings at this stage again, find the node for which the inverse coupling first goes to 0. And dualise that node. We see a remarkable feature in (5.14). To the level of approximation that we are using, only the first gauge group factor has a negative beta function. This implies that the next node to be dualized is precisely node 1. Performing Seiberg duality thereupon takes us to a quiver that is exactly of the form of Model A, only with the ranks differing in contributions proportional to M, i.e., different fractional brane charges

By iterating this procedure it is possible to see that the entire cascade corresponds to a chain that alternates between type A and type B models. Furthermore, the length of the even steps of the cascade, measured on the $t = \log \mu$ axis is constant. The same statement applies to the length of the odd steps. This cascade is presented in Figure 10 for the initial conditions $(x_1, x_2, x_3, x_4) = (2, 1, 1, 0)$.

5.3 Duality wall

We have seen in §5.2 that models A and B form a closed cascade and are not connected to the other theories in the F_0 duality tree by the RG flow, regardless of initial conditions. This motivates the study of duality cascades having other Seiberg dual theories as their starting points. The simplest choice corresponds to

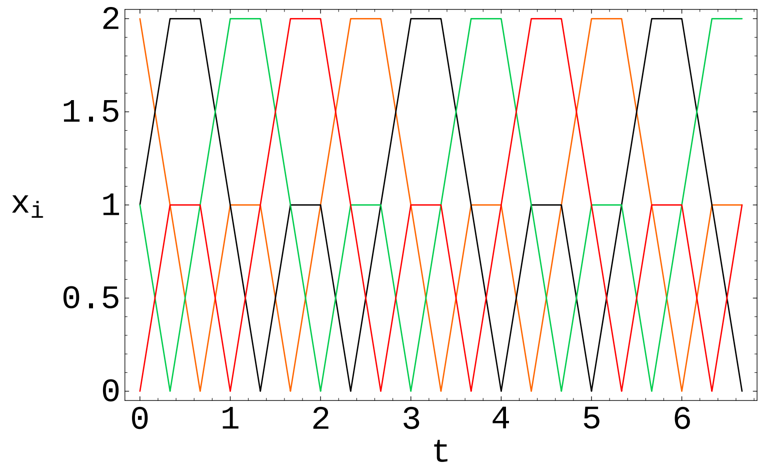

Figure 10: Duality cascade alternating between the A and B toric models. The colouring scheme is such that orange, black, green, and red respectively represent nodes 1, 2, 3 and 4.

the model in Figure 11. This theory is obtained from Model A by Seiberg dualizing node 2 followed by 1. We will call this Model C.

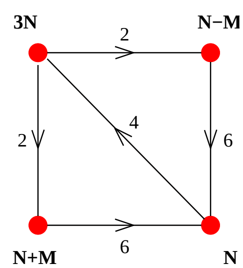

Figure 11: Model C for the F_0 theory. It is obtained from dualising node 2 and then 1 from the simplest Model A.

5.3.1 Decreases in the Step

Applying the formalism developed in previous sections we can proceed to compute, for any initial condition, the RG cascade as the theory evolves to the UV. The starting point is the computation of the anomalous dimensions for Model C at the conformal point. These, by the techniques above, turn out to be $\gamma_{1,2} = \gamma_{1,4} = -3/2$, $\gamma_{2,3} = \gamma_{4,3} = 5/2$ and $\gamma_{3,1} = -1$. Using them to calculate the beta functions, we obtain

$$\begin{array}{ll} \beta_1 = 0 & \beta_2 = -3M \\ \beta_3 = 0 & \beta_4 = 3M \ . \end{array} \tag{5.15}$$

With these let us evolve to the UV. Let us first consider the case in which the initial condition for the inverse gauge couplings are $(x_1, x_2, x_3, x_4) = (1, 1, 1, 0)$. Figure 12 shows the evolution of the four inverse gauge couplings both as a function of the step in the cascade and as a function of the logarithm of the scale.

An interesting feature is that the distance, Δ_i, between successive dualizations is monotonically decreasing.

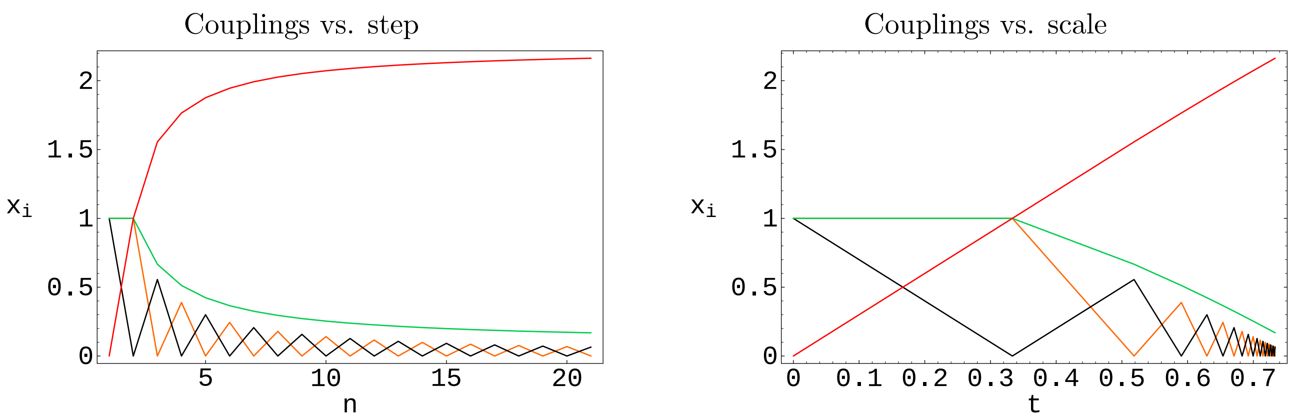

Figure 12: Evolution of gauge couplings with **(a)** the step in the duality cascade and **(b)** the energy scale for initial conditions $(x_1, x_2, x_3, x_4) = (1, 1, 1, 0)$. The colouring scheme is such that orange, black, green, and red respectively represent nodes 1, 2, 3 and 4.

This marks a departure from the behaviors observed in the conifold cascade and from the example presented in §5.2, where Δ_i remained constant (cf. Figure 6). However, this fact does not necessarily mean the convergence of the dualization scales. Indeed, we plot the intervals Δ_i in Figure 13.a while Figure 13.b shows the resulting dualization scales. The slope of this curve is decreasing, reflecting the decreasing

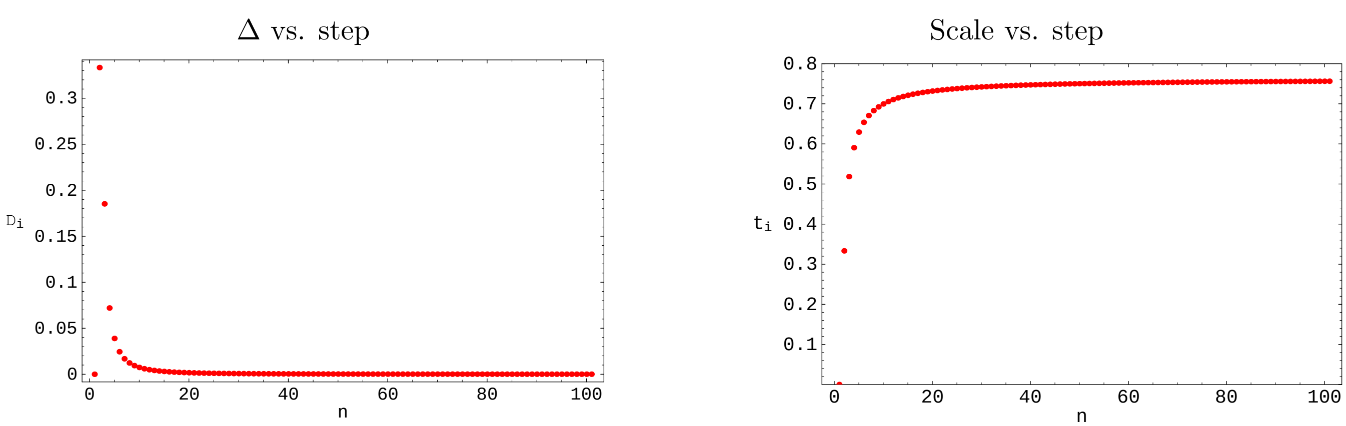

Figure 13: The evolution of Δ, the size of the increment during each dualisation and the energy scale increase as we dualise, for the initial conditions $(x_1, x_2, x_3, x_4) = (1, 1, 1, 0)$.

behavior of Δ_i. Nevertheless, *ad infinitum*, the scale may diverge.

5.3.2 A Duality Wall

Let us now consider a different set of initial conditions, given by $(x_1, x_2, x_3, x_4) = (1, 1, 4/5, 0)$. The flow of the inverse couplings is now shown in Figure 14.

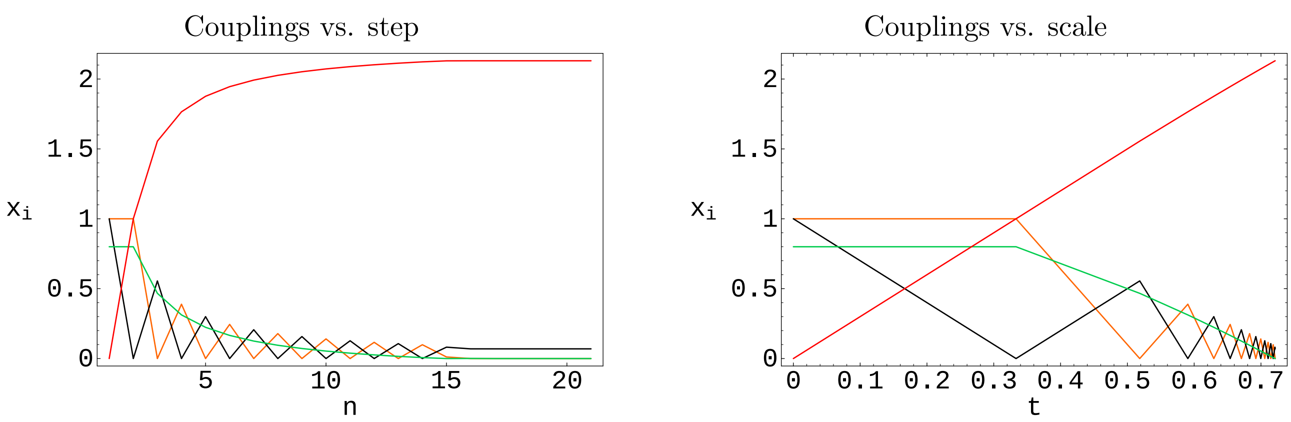

Figure 14: Evolution of gauge couplings with **(a)** the step in the duality cascade and **(b)** the energy scale for initial conditions $(x_1, x_2, x_3, x_4) = (1, 1, 4/5, 0)$. The colouring scheme is such that orange, black, green, and red respectively represent nodes 1, 2, 3 and 4.

A completely new phenomenon appears in this case. Something very drastic happens after the 14-th step in the cascade. As a consequence of lowering the initial value of x_3, the third node gets dualized at this step, producing an explosive growth of the number of chiral and vector multiplets in the quiver. This statement can be made precise: *at the 14-th step node 3 is dualized and the subsequent quivers have all their intersection numbers greater than 2*. In this situation the results of [12, 14] suggest that a duality wall is expected. This phenomenon is characterized by a flow of the dualization scales towards an UV *accumulation point* with an exponential divergence in the number of degrees of freedom. This asymptotic behavior can be inferred once Seiberg dualities are interpreted as Picard-Lefschetz monodromy transformations which we discuss in Appendix 2 (q.v. (8.14)).

Figure 14 shows a very small running of the gauge couplings beyond this point. This is not due to a vanishing of the beta functions, but to the extreme reduction of the length of the Δ_i intervals.

In contrast to Figure 13, for the initial conditions $(1, 1, 4/5, 0)$, we have drawn the plots in Figure 15.a and Figure 15.b. Both of them indicate that a limiting scale which cannot be surpassed is reached as the theory flows towards the UV. This is precisely what we call a **duality wall.**

5.4 Location of the Wall

We have just seen that starting from the quiver in Figure 11 for F_0 with initial conditions $(x_1, x_2, x_3, x_4) =$

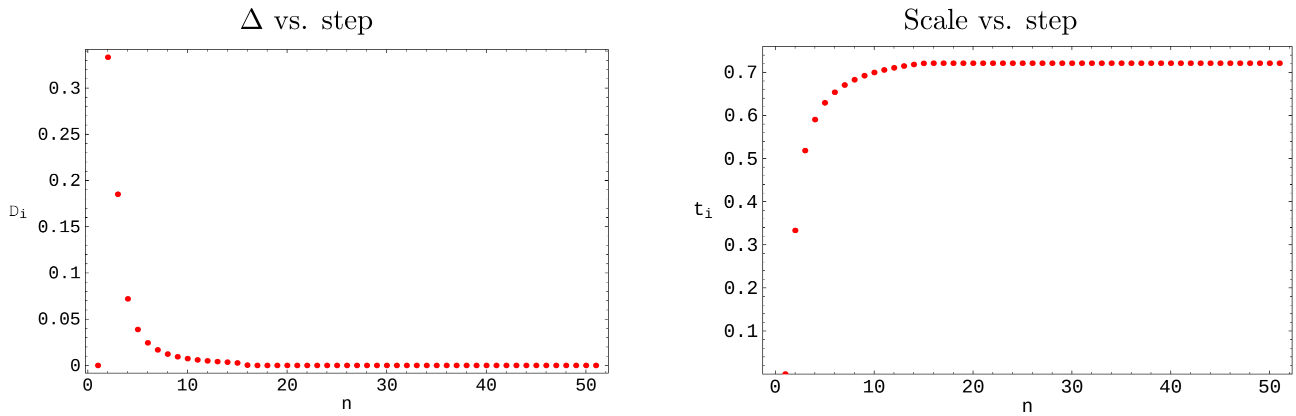

Figure 15: The evolution of Δ, the size of the increment during each dualisation and the energy scale increase as we dualise, for the initial conditions $(x_1, x_2, x_3, x_4) = (1, 1, 4/5, 0)$.

$(1, 1, 4/5, 0)$ a duality wall is reached. Let us briefly examine the sensitivity of the location of the duality wall to the initial inverse couplings.

Let our initial inverse gauge couplings be $(1, x_2, x_3, 0)$, with $0 < x_2, x_3 < 1$, and we repeat the analyses in the previous two subsection. We study the running of the beta functions, and determine the position of the duality wall, t_{wall}, for various initial values. We plot in Figure 16, the position of the duality wall against the initial values x_2 and x_3, both as a three-dimensional plot in I and as a contour plot in II. We see that the position is a step-wise function. A similar behavior has been already observed in [14] for dP_0 in the case of vanishing anomalous dimensions.

5.5 A $\mathbb{Z}_2$ Symmetry as T-Duality

A remarkable symmetry seems to be captured by the gauge theory discussed above. Suppose that, starting from Model C, we had decided to follow the RG flow towards the IR instead of the UV. There is a simple trick that accomplishes this task, namely to look at the flow to the the UV of a theory in which the beta functions have changed sign and where the log-scale t has been replaced by $-t$. Since we are considering beta functions that are linear in the number of fractional branes M, changing the sign of the beta functions can be interpreted as changing M to $-M$. However, upon inspecting Figure 11, we see that $M \leftrightarrow -M$ is nothing more than a $\mathbb{Z}_2$ reflection of the quiver along the (13) axis. Therefore, the flow to the IR that starts from Model C is simply the flow to the UV of its $\mathbb{Z}_2$ reflection. Therefore, the whole cascade we have already computed also describes, upto this reflection, the cascade to the IR.

Let us elaborate on the origin of this $\mathbb{Z}_2$ symmetry. In the holographic dual of the gauge theory, the energy scale μ is typically associated to a radial coordinate R. Then, transforming $t = \log \mu$ to $-t$

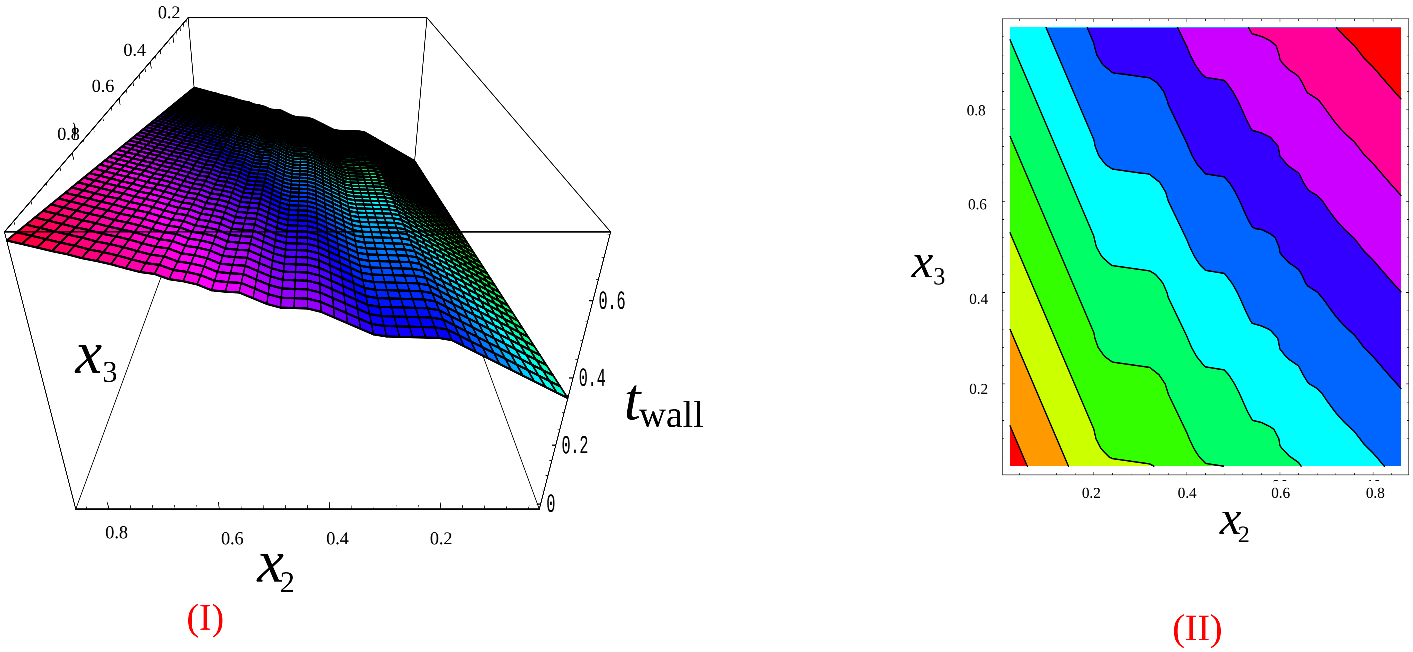

Figure 16: A plot of the position t_{wall} against the initial gauge coupling values $(1, x_2, x_3, 0)$. (I) is the 3-dimensional plot and (II) is the contour plot versus x_2 and x_3.

corresponds to an inversion of the radial coordinate $R \leftrightarrow 1/R$ in the holographic dual theory. Therefore this $\mathbb{Z}_2$ symmetry displayed by the gauge theory RG flow indicates a $\mathbb{Z}_2$ T-duality-like symmetry of string theory on the underlying geometry. The scale of model C can then be interpreted as the self dual radius of the dual holographic theory. This holographic theory then exhibits a minimal length beyond which there are no further new phenomena. Every physical quantity at a scale less than this scale can be expressed in terms of a different physical quantity by applying the $\mathbb{Z}_2$ action described above. It will be very interesting to explore this symmetry further.

6. Phases of dP_1

We have initiated the study of duality walls for general quiver theories and in the foregoing discussion analysed in detail the illustrative example of the cone over the zeroth Hirzebruch surface. It is interesting to extend the construction of duality trees to other gauge theories and the structures and RG flows that may emerge.

Let us briefly consider perhaps the next simplest case, viz., the gauge theory on a D3-brane probing a complex cone over dP_1, the first del Pezzo surface, which is $\mathbb{P}^2$ blown up at 1 point. This is again a toric variety and the theory has been extensively studied [9, 21]. There is only one toric phase in this case[1] ,

[1] We follow here the nomenclature of [21], where a theory was denoted toric if it had all its gauge group factors equal to

whose quiver is shown in Figure 17; the ranks are $(1,1,1,1)$. This model is self-dual under the dualizations

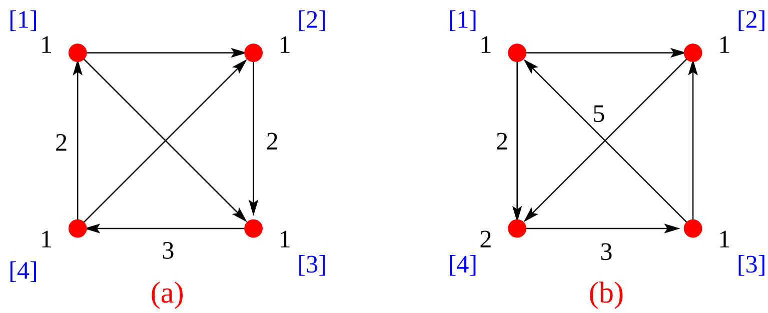

Figure 17: The quivers for the gauge theory arising from dP_1. (a) is the toric phase while (b) is obtained from (a) by dualising either node 3 or 4 and is a non-toric phase. The ranks of the nodes are denoted by blue square brackets.

of nodes 1 or 2, and is transformed into a theory with gauge group $U(2N) \times U(N) \times U(N) \times U(N)$ when dualizing any of the other two nodes. Again using the notation of (5.2), we denote the ranks in the conformal case, as $(2,1,1,1)$. The duality tree obtained by taking into account these two models is shown in Figure 18. The encircled theory is the one in Figure 17.a and each link shows the associated dualized node. Other Seiberg dualizations of the $(2,1,1,1)$ model lead to $(5,2,1,1)$ and $(4,2,1,1)$ theories. We have not included them in Figure 18 for simplicity.

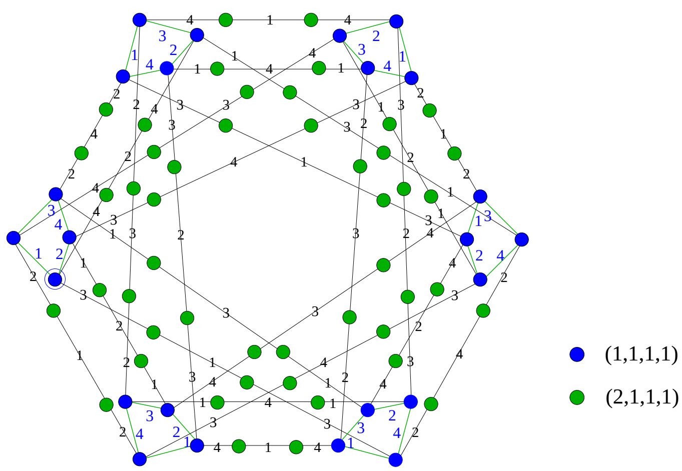

Figure 18: The tree of Seiberg dual theories for dP_1. There are six toric islands (sets of blue sites) in this tree.

Already at this level we can see that the duality space of dP_1 is very rich, exhibiting a phenomenon that we will call **"toric islands"** (isolated sets of toric models). Furthermore, the abundance of closed loops makes it quite different from the dP_0 and F_0 cases. The appearance of a six-fold multiplicity of islands is a result of the combinatorics that gives the possible reorderings of a given quiver. Explicitly, the

$U(N)$, i.e., all the ranks are equal.

number of possible re-orderings of the four nodes in the toric quiver are $4! = 24$. After grouping these toric models in islands of four, we are lead to the six expected islands.

7. Conclusions and Prospects

In applying some recent technology to the classic conifold cascade phenomenon [5], we have here studied how duality walls arise in string theory. Endowed with the systematic methods from toric duality [9, 21] and maximisation principles in determining anomalous dimensions in arbitrary four-dimensional SCFT's [26, 29], we have re-examined the ideas of [12, 14] by supplanting complete anomalous dimensions to the beta functions, emerging under a fully string theoretic realisation.

Indeed, armed with the vast database of "duality trees" encoding various (generalised) Seiberg dual phases of four-dimensional SCFT $\mathcal{N} = 1$ gauge theories that live on world-volumes of D3 branes probing various singular geometries, we have outlined a general methodology of analysing cascading phenomena and finding duality walls. At liberty to take advantage of the well-studied toric singularties, we have used the enlightening example of the theory corresponding to the complex cone F_0 over the zeroth Hirzebruch surface. We have shewn how one adds fractional branes to take the theory out of conformality and hence obtain a nontrivial running of the (inverse) gauge couplings. Thereafter, by either symmetry arguments [21] or maximisation of central charge a [26], one could determine the exact form of the NSVZ beta-functions, in the limit where the number of physical branes is much larger than that of fractional ones. We have then shewn how to apply this running to consecutive applications of Seiberg duality.

We find, when one identifies appropriate "closed cycles" in the tree, generalisation of the conifold cascades. We also find the existence of "duality walls" (cf. Figure 14). This occurs whenever subsequent applications of the duality (cascade) result in the rapid decrease in the evolution, towards the UV, of the full beta functions, whereby creating an accumulation point onto which all asymptotic values of the couplings pile. Such a wall signifies a finite energy scale beyond which Seiberg dualities cannot proceed and at which an infinite number of degrees of freedom blossom in the string theory.

An interesting appearance is played by the $\mathbb{Z}_2$ symmetry which the beta function equations exhibit and the non-trivial action on the scale. This action simply inverts the scale and defines a critical scale, say 1, at which there is a reflection point in the physics. All phenomena above this scale are related to those below it. In the dual geometry this is reminiscent of the familiar T-duality and the critical scale corresponds to the self-dual radius of T-duality. The position of the wall is sensitive to the initial conditions, viz., the values of the gauge couplings, and exhibits a step-wise behaviour with respect to them (cf. Figure 15).

Of course, our example is but the simplest of a wide class of theories. We have briefly touched upon a more involved case of the cone over the first del Pezzo surface. There, interesting "toric islands" appear, giving an even richer structure for the cascade. Indeed, a plethora of theories awaits our exploration.

As an aside, we have used the AdS/CFT diationary, which dictates in particular that the central charge a is inversely proportional to the horizon volume in the dual bulk theory, to obtain a formula applicable to all the cones over del Pezzo surfaces (cf. (9.32)).

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8. Appendix 1: Picard-Lefschetz Monodromy Transformations

In this Appendix we briefly remind the reader of Seiberg and toric dualities from the perspective of Picard-Lefschetz monodromy and (p, q) seven-branes perspective. Now to engineer quiver theories of gauge group $\prod_{i=1}^{n} U(N_i)$ we can take a set of n (p, q) sevenbranes with charges (p_i, q_i) and multiplicity N_i, corresponding to a cycle S_i in the singular geometry, satisfying

$$\sum_{i}^{n} N_i(p_i, q_i) = 0 \ . \tag{8.1}$$

The intersection matrix for the quiver is given by

$$I_{ij} = S_i \cdot S_j = \det \begin{pmatrix} p_i & q_i \\ p_j & q_j \end{pmatrix} \ . \tag{8.2}$$

Now, the anomaly cancellation condition translates into the quiver language to the equality between the number of incoming and outgoing arrows at every node (cf. (2.6)). It here reads,

$$\sum_j (S_i.S_j)N_j = \sum_j I_{ij}N_j = 0 \quad \forall i \ . \tag{8.3}$$

Indeed, the vanishing of this quantity for every gauge groups follows from (8.1). Furthermore, the fractional brane cycle, $\sum_j S_j N_j$ must have zero intersection with all other cycles in order to be anomaly free.

Seiberg duality on the quiver corresponds to Picard-Lefschetz (PL) transformations on the cycles, which is a reordering of vanishing cycles. In particular, when moving a cycle S_j around a cycle S_i, S_j becomes

$$S_j \to S_j + (S_i.S_j)S_i = S_j + \det \begin{bmatrix} p_i & q_i \\ p_j & q_j \end{bmatrix} S_i \ , \tag{8.4}$$

while S_i remains unchanged. This is the dualisation on node i. This action can be represented by a monodromy matrix M acting on the (p, q) charges of the different cycles, defined as

$$(p, q)'_i = M_{ij}(p, q)_j \ . \tag{8.5}$$

The new theory has to be anomaly free, i.e.,

$$\sum_i N'_i(p_i, q_i)' = 0, \quad \sum_i N'_i M_{ij}(p_j, q_j) = 0 \ , \tag{8.6}$$

where we have used (8.5) in the last line. Comparing 8.6 and 8.1, we obtain the transformation rule for the ranks of the gauge groups

$$N'_i M_{ij} = N_j \ , \tag{8.7}$$

which in vector notation reads

$$\vec{N'} = M^{-T}\vec{N} \ . \tag{8.8}$$

8.1 Alternating Dualizations

PL monodromies are specially well suited for the computation of entire branches of duality trees obtained by performing alternating dualizations on two nodes of a quiver. The theory after any number of steps can be determined by acting on the original one with powers of a monodromy matrix. The results that we present in this appendix have also been presented in [12] (where they were derived in the context of Weyl reflections) and in [20].

General case Let us start by considering the most general case. Let us call the nodes that will undergo alternating dualizations as 1 and 2. Furthermore, we consider a generic number of bifundamental fields between these two nodes, given by the intersection $I_{12} = a$. Under these conditions, the matrix M in 8.5 takes the form

$$M = \left(\begin{array}{cc|ccc} a & 1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ \hline 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \\ \vdots & \vdots & & & \ddots \end{array}\right) . \tag{8.9}$$

The theory that results after k dualizations will be computed using

$$M^k = \left(\begin{array}{c|c} A_k & 0 \\ \hline 0 & 1 \end{array}\right) , \tag{8.10}$$

which has a simple block structure. The non-trivial block A_n can be calculated in closed form to be

$$A_k = \begin{pmatrix} \frac{(\lambda_+^{k+1}-\lambda_-^{k+1})}{(\lambda_+-\lambda_-)} & \frac{(\lambda_+^k-\lambda_-^k)}{(\lambda_+-\lambda_-)} \\ & \\ -\frac{(\lambda_+^k-\lambda_-^k)}{(\lambda_+-\lambda_-)} & \frac{(\lambda_+\lambda_-^k-\lambda_-\lambda_+^k)}{(\lambda_+-\lambda_-)} \end{pmatrix} , \qquad \lambda_\pm := \frac{a \pm \sqrt{a^2-4}}{2} . \tag{8.11}$$

Using the transpose of A_k and (8.8), we have, after k dualisations, the resulting ranks of the gauge group factors are

$$N_2(k) = \frac{(\lambda_+^{k+1} - \lambda_-^{k+1})}{(\lambda_+ - \lambda_-)} N_2(0) - \frac{(\lambda_+^k - \lambda_-^k)}{(\lambda_+ - \lambda_-)} N_1(0) , \tag{8.12}$$

with

$$N_1(k) = N_2(k-1) . \tag{8.13}$$

Equation (8.12) can also be used to estimate the asymptotic behavior of the ranks for a large number of iterations. In the case of $a \geq 3$ we have $\lambda_+ > 1$ and $\lambda_- < 1$. Then, as $k \to \infty$, $(\frac{\lambda_-}{\lambda_+})^k \to 0$ and we have

$$N_2(k) \to \frac{\lambda_+^k}{\lambda_+ - \lambda_-} N_2(0) . \tag{8.14}$$

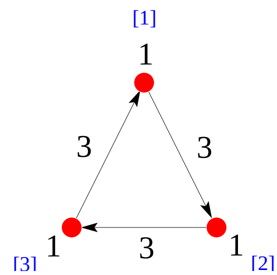

Figure 19: Starting dP_0 for our alternating dualization sequence. We have indicated in blue the labelling of the gauge groups.

An Explicit Example: dP_0 Let us consider the specific case of dP_0, starting from the quiver in Figure 19 (a special case of Figure 1) and performing alternating dualizations on nodes 1 and 2. The monodromy matrices are

$$M = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \qquad M^{-T} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} ; \tag{8.15}$$

thus, after k iterations we have

$$\left(M^{-T}\right)^k = \begin{pmatrix} \frac{\left(\frac{3-\sqrt{5}}{2}\right)^k \left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{3+\sqrt{5}}{2}\right)^k}{\sqrt{5}} & -\frac{\left(\frac{3-\sqrt{5}}{2}\right)^k - \left(\frac{3+\sqrt{5}}{2}\right)^k}{\sqrt{5}} & 0 \\ \frac{\left(\frac{3-\sqrt{5}}{2}\right)^k - \left(\frac{3+\sqrt{5}}{2}\right)^k}{\sqrt{5}} & \frac{\left(\frac{3+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{-3+\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{8.16}$$

Therefore, the resulting ranks of the gauge groups become

$$\vec{N}(k) = (N_1(k), N_1(k+1), 1) , \tag{8.17}$$

with

$$N_1(k) = \frac{1}{5\ 2^{k+1}} \left[(5+\sqrt{5})(3-\sqrt{5})^k + (5-\sqrt{5})(3+\sqrt{5})^k\right] . \tag{8.18}$$

From (8.17) and (8.18), it is immediate to check that the ranks satisfy the Diophantine equation for dP_0 given in (3.1), viz., $x^2 + y^2 + z^2 = 3xyz$. We show in Figure 20 the specific path along the duality tree that corresponds to this chain of dualizations. This procedure can be repeated for alternating dualization chains with other quivers as the starting point.

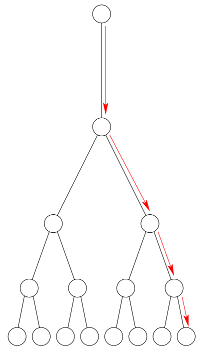

Figure 20: Path on the duality tree when flowing to the UV with alternating dualizations.

9. Appendix 2: Computing Horizon Volumes

In this Appendix, we present another application of our analyses, namely to the computation of horizon volumes. From the AdS/CFT correspondence for 4-d SCFT's dual to geometries of the form $AdS_5 \times X^5$, one can see that the central charge a of the 4-d theory is related to the volume of X^5 by (cf. e.g. [32])

$$a = \frac{\pi^3}{4\mathrm{Vol}(X^5)} \ . \tag{9.1}$$

The method for computing anomalous dimensions presented in §2 makes it possible to use the gauge theories to compute the volumes of a wide range of non-spherical X^5 geometries. We have already indirectly made this calculations for the X^5 associated to the conifold and the complex cone over the zeroth Hirzebruch surface. In this Appendix we will extend the discussion to the cases of complex cones over del Pezzo surfaces (studied from the gauge theory perspective in e.g. [9]) and $S^5/\mathbb{Z}_n$ (the result for this case is easy to guess, it is just the volume of S^5 divided by n). We now give the computation of the central charges for the first six Del Pezzo surfaces and for the orbifolds. These results were obtained independently by us but appeared also in two recent papers ([29, 31]). We include them here for completeness. Horizon volume computations have also been presented in [35].

9.1 Orbifolds

The orbifolds $\mathbb{C}^3/\mathbb{Z}_n$ have gravity duals with geometry $AdS_5 \times S^5/\mathbb{Z}_n$. We label the matter fields in the quiver and assign anomalous dimensions as follows (i labels the nodes, $\mathrm{mod}(n)$):

$$X_{i,i+2} \to \gamma_X, \quad Y_{i,i+2} \to \gamma_Y, \quad Z_{i,i+1} \to \gamma_Z \ . \tag{9.2}$$

The superpotential has $2n$ cubic terms all of which have the same gauge coupling because of symmetry. The zero beta function condition reads:

$$\gamma_X + \gamma_Y + \gamma_Z = 0 \ , \tag{9.3}$$

giving

$$a_{\mathbb{Z}_n} = \frac{n}{96}[24 + (\gamma_X^3 + \gamma_Y^3 + \gamma_Z^3) - 3(\gamma_X^2 + \gamma_Y^2 + \gamma_Z^2)] \ . \tag{9.4}$$

After taking (9.3) into account, we have

$$a_{\mathbb{Z}_n} = \frac{n}{32}(2 + \gamma_X)(2 + \gamma_Y)(2 - \gamma_X - \gamma_Y) \ . \tag{9.5}$$

There are four critical points for a: (γ_X,γ_Y) equaling to $(-2, -2)$, $(-2, 4)$, $(4, -2)$ and $(0, 0)$, but only the last one corresponds to a maximum. Therefore

$$\gamma_X = \gamma_Y = \gamma_Z = 0 \quad \Rightarrow \quad a_{\mathbb{Z}_n} = \frac{n}{4} \ . \tag{9.6}$$

Using (9.1) we have:

$$\mathrm{Vol}(S^5/\mathbb{Z}_n) = \frac{\pi^3}{n} \ . \tag{9.7}$$

as expected.

9.2 Del Pezzo surfaces

The quivers and superpotentials for the toric theories are taken from [21] and for the non-toric ones from [25].

del Pezzo 0: Our first example is dP_0, the cone over the zeroth del Pezzo surface, which, as mentioned before, is actually the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. The anomalous dimensions are uniquely determined from the vanishing of the beta functions for the gauge and superpotential couplings, and are $\gamma_i = 0$. From this we determine the central charge to be:

$$a_{dP_0} = \frac{3}{4} \ . \tag{9.8}$$

This is consistent with the general result (9.7) for the Abelian orbifolds.

del Pezzo 1: For this and subsequent del Pezzo examples we refer the reader to [19, 9] for their quivers and superpotentials. The symmetries of the dP_1 quiver and superpotential are the simultaneous exchange of $(2, 3) \leftrightarrow (1, 4)$ and charge conjugation (reversal of the direction of the arrows). The SCFT condition on the beta-functions leaves one free parameter taken here to be γ_{24}. The central charge takes the form

$$a = -\frac{3}{4}(2\gamma_{24}^2 - \gamma_{24} - 1) \ . \tag{9.9}$$

Maximizing a gives $\gamma_{24} = \frac{1}{4}$ and

$$\gamma_{12} = -\frac{5}{4}, \quad \gamma_{23} = \gamma_{41} = -\frac{1}{2}, \quad \gamma_{24} = \gamma_{13} = \gamma_{34} = \frac{1}{4} \ . \tag{9.10}$$

The result for a is

$$a_{dP_1} = \frac{27}{32} \ . \tag{9.11}$$

del Pezzo 2: dP_2 has two toric phases and they must of course have the same central charge, as we now demonstrate by explicit calculation.

Model I: The theory is invariant under exchange of nodes $1 \leftrightarrow 2$. This makes

$$\gamma_{14} = \gamma_{42}, \qquad \gamma_{15} = \gamma_{25}, \qquad \gamma_{31} = \gamma_{32} \ . \tag{9.12}$$

Putting the beta functions to zero again leaves one of the γ's free, giving

$$a = -\frac{3}{32}(7\gamma_{45}^2 - 20\gamma_{45} + 4) \ . \tag{9.13}$$

After maximizing, $\gamma_{45} = \frac{10}{7}$ and

$$a_{dP_2} = \frac{27}{28} \ . \tag{9.14}$$

The other anomalous dimensions are

$$\gamma_{14} = \gamma_{34} = -\frac{8}{7}, \quad \gamma_{31} = \gamma_{15} = -\frac{2}{7}, \quad \gamma_{35} = \frac{4}{7} \ . \tag{9.15}$$

Model II: The symmetries of the theory in terms of the quiver representation are the simultaneous exchange of $(1,4) \leftrightarrow (2,5)$ and charge conjugation. This means

$$\gamma_{14} = \gamma_{52}, \qquad \gamma_{15} = \gamma_{42}, \qquad \gamma_{31} = \gamma_{23} \ . \tag{9.16}$$

Solving the beta function equations as before gives

$$a = -\frac{3}{125}(7\gamma_{45}^2 + 16\gamma_{45} - 32) \ . \tag{9.17}$$

After maximizing $\gamma_{45} = -\frac{8}{7}$ and, as expected,

$$a_{dP_2} = \frac{27}{28} \ . \tag{9.18}$$

The other anomalous dimensions are

$$\gamma_{14} = \gamma_{45} = -\frac{8}{7}, \quad \gamma_{31} = \gamma_{15} = \gamma_{42} = -\frac{2}{7}, \quad \gamma_{34} = \gamma_{53} = \frac{4}{7} \ . \tag{9.19}$$

del Pezzo 3: dP_3 has four toric phases. Let us do the computation for two of them.

Model I: Using the $D6$ symmetry [7], we start with only two independent anomalous dimensions $\gamma_{i\ i+1}$ and $\gamma_{i\ i+2}$. Solving for the vanishing of the beta functions, we obtain

$$\gamma_{i\ i+1} = -1 \ , \qquad \gamma_{i\ i+2} = 0 \ . \tag{9.20}$$

Here the central charge is immediately determined to be

$$a_{dP_3} = \frac{9}{8} \ . \tag{9.21}$$

Model III: The symmetries are $(12) \leftrightarrow (34)$ which means

$$\gamma_{15} = \gamma_{35}, \qquad \gamma_{54} = \gamma_{52}, \qquad \gamma_{41} = \gamma_{21} = \gamma_{23} = \gamma_{43}, \qquad \gamma_{64} = \gamma_{62} \ . \tag{9.22}$$

Putting the beta functions to zero we obtain

$$\gamma_{21} = \gamma_{64} = -\tfrac{2}{3}, \qquad \gamma_{15} = \gamma_{54} = -\tfrac{1}{3}, \qquad \gamma_{54} = 0 \ . \tag{9.23}$$

This gives

$$a_{dP_3} = \frac{9}{8} \ . \tag{9.24}$$

the same as for Model I, as it should.

del Pezzo 4: This is the first non-toric example. The theory is symmetric under any permutation of nodes 3 to 7. As before, we impose the vanishing of all beta functions from (2.1), thus getting

$$\gamma_{21} = 0, \quad \gamma_{1i} = -\gamma_{i2} = \frac{8}{5} \ . \tag{9.25}$$

This determines the central charge to be

$$a_{dP_4} = \frac{27}{20} \ . \tag{9.26}$$

del Pezzo 5: This geometry, in the non-generic case where the blow-up points in $\mathbb{P}^2$ are not in general position, is the same as for the conifold modded out by $\mathbb{Z}_2 \times \mathbb{Z}_2$ (cf. e.g. [24, 25]). Imposing the conditions for the beta functions we get

$$\gamma_{ik} = \frac{5}{2}, \quad \gamma_{kj} = -\frac{3}{2}, \quad \gamma_{ji} = -1 \ , \tag{9.27}$$

where $i = 1,2$, $j = 3,4$, $k = 5,6,7,8$. We find

$$a_{dP_5} = \frac{27}{16} \ . \tag{9.28}$$

del Pezzo 6: Here we find:

$$\gamma_{ij} = \gamma_{jk} = -1 \ , \quad \gamma_{ki} = 2 \ , \tag{9.29}$$

where $i = 1,2,3$, $j = 4,5,6$, $k = 7,8,9$, and

$$a_{dP_6} = \frac{9}{4} \ , \tag{9.30}$$

which, incidentally, is the same as for $\mathbb{C}^3/\mathbb{Z}_3 \times \mathbb{Z}_3$.

General del Pezzo: We observe a pattern indeed and posit that in general, for the cone over dP_n, the central charge is

$$a_{dP_n} = \frac{27}{4(9-n)} \ . \tag{9.31}$$

The appearance of the 9 is re-assuring because there are in all 9 del Pezzo surfaces, with the 9-th one a pseudo-del-Pezzo surface sometimes called half-K3, whose anticanonical divisor squares to 0 rather being ample.

Finally, the volume of the base of the complex cone over dP_n, being inversely proportional to the central charge, should be

$$\mathrm{Vol}_{dP_n} = \frac{\pi^3}{27}(9-n). \tag{9.32}$$

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