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Moduli Dependent Spectra of Heterotic Compactifications

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Abstract

Explicit methods are presented for computing the cohomology of stable, holomorphic vector bundles on elliptically fibered Calabi-Yau threefolds. The complete particle spectrum of the low-energy, four-dimensional theory is specified by the dimensions of specific cohomology groups. The spectrum is shown to depend on the choice of vector bundle moduli, jumping up from a generic minimal result to attain many higher values on subspaces of co-dimension one or higher in the moduli space. An explicit example is presented within the context of a heterotic vacuum corresponding to an $SU(5)$ GUT in four-dimensions.

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One approach to producing phenomenologically viable $N = 1$ supersymmetric physics from strongly coupled $E_8 \times E_8$ heterotic string theory [1] is to compactify the ten-dimensional spacetime on a Calabi-Yau threefold X. Additionally, it is required that the $E_8 \times E_8$ gauge connection A satisfy the hermitian Yang-Mills equations

$$F_{ab} = F_{\bar{a}\bar{b}} = g^{a\bar{b}} F_{\bar{b}a} = 0 \tag{1}$$

on X, where F is the field strength of A. In this paper, we will ignore the E_8 factor in the "hidden" sector, restricting the subsequent discussion to the remaining E_8 gauge group. Hence, A and F will be Lie algebra valued over a single E_8. We call an E_8 gauge connection satisfying (1) a holomorphic instanton on X. Calabi-Yau threefolds are easily constructed. However, one must also find a holomorphic instanton to completely specify the vacuum. Here, one runs into serious technical difficulties, since no explicit solutions of (1) on a Calabi-Yau threefold are known. How, then, can one proceed? It was shown in [2] and [3] that A is a holomorphic instanton if and only if it is a connection on a stable, holomorphic vector bundle V with structure group $G \subseteq E_8$ on X. Therefore, finding a solution of (1) is completely equivalent to specifying such a vector bundle. Happily, general methods for their construction have been found [4, 5, 6, 7, 8, 9, 10, 11, 12].

Let us choose X and V. We first note that the gauge group H of the low energy, four-dimensional theory is the commutant in E_8 of the structure group G. Now consider the vector supermultiplet of the ten-dimensional theory which transforms as the adjoint 248 representation of E_8. With respect to $G \times H$, the 248 representation decomposes as

$$248 \rightarrow (1, Ad(H)) \oplus \bigoplus_i (R_i, r_i) \ , \tag{2}$$

where $Ad(H)$ specifies the adjoint representation of H and $\{(R_i, r_i)\}$ are some set of representations of G and H respectively. This indicates that the low energy theory will contain $N = 1$ supermultiplets transforming in the $Ad(H)$ and $\{r_i\}$ representations of H. How many supermultiplets in each representation will occur? We first note that the $Ad(H)$ representation will always be a unique vector supermultiplet. That is,

$$n_{Ad(H)} = 1 \ . \tag{3}$$

All other representations will be realized as chiral supermultiplets in the low energy theory. The multiplicity of superfields transforming in the representation r_i, however, is much more difficult to compute. It is given by the dimension of the space of zero modes of the Dirac operator on X transforming in the associated R_i representation of G. If we denote by V_{R_i}

the vector bundle constructed from V whose fibers transform in the R_i representation, then this space of zero modes is the cohomology group $H^1(X, V_{R_i})$. It follows that the number of chiral supermultiplets carrying the representation r_i is given by

$$n_{r_i} = h^1(X, V_{R_i}) \ , \tag{4}$$

where lower case h indicates the dimension. Therefore, to calculate the spectrum of the low energy theory, one must compute the dimensions of $H^1(X, V_{R_i})$ for each representation R_i occurring in the decomposition (2). For the special case when

$$V = TX \ , \tag{5}$$

the so-called "standard" embedding, these calculations are relatively straight-forward, the results being related to the known Betti numbers of the Calabi-Yau threefold [13]. However, for any stable, holomorphic vector bundle not satisfying (5), the vast majority of such bundles, the computation of the spectrum is a difficult problem. In this paper, we will present explicit methods for calculating (4) in a wide class of phenomenologically relevant heterotic string compactifications.

Before proceeding, however, we give a specific example of the above remarks. Let us choose the structure group of V to be $G = SU(5)$. Then, the low energy gauge group is clearly $H = SU(5)$. To distinguish the two different $SU(5)$ groups, we henceforth denote them by $SU(5)_G$ and $SU(5)_H$ respectively. With respect to $SU(5)_G \times SU(5)_H$, the adjoint representation of E_8 decomposes as

$$248 \to (1, 24) \oplus (24, 1) \oplus (5, \overline{10}) \oplus (\overline{5}, 10) \oplus (10, 5) \oplus (\overline{10}, \overline{5}) \ . \tag{6}$$

The first term is $(1, Ad(SU(5)_H))$. Therefore, (3) implies

$$n_{24} = 1 \ . \tag{7}$$

Now consider the remaining terms in (6). Noting that the vector bundles corresponding to $R = 24, 5, \overline{5}, 10, \overline{10}$ are

$$V_R = V \otimes V^*, V, V^*, \wedge^2 V, \wedge^2 V^* \tag{8}$$

respectively, it follows from (6) and (4) that

$$n_1 = h^1(X, V \otimes V^*) \tag{9}$$

and

$$\begin{aligned} n_{\overline{10}} &= h^1(X, V), \quad n_{10} = h^1(X, V^*), \\ n_5 &= h^1(X, \wedge^2 V), \quad n_{\overline{5}} = h^1(X, \wedge^2 V^*) \ . \end{aligned} \tag{10}$$

Note that $\text{Tr}(V \otimes V^*)$ can be ignored when computing $h^1(X, V \otimes V^*)$ on a Calabi-Yau threefold X.

Returning to the general decomposition (2), we must confront the problem of how to compute the spectrum given in (4). An important insight is that the Atiyah-Singer index theorem will relate the dimensions of different cohomology groups, thereby reducing the amount of calculation. Using Serre duality, the stability of V_{R_i} and the fact that $c_1(TX) = c_1(V_{R_i}) = 0$, the index theorem gives

$$-h^1(X, V_{R_i}) + h^1(X, V_{R_i}^*) = \frac{1}{2} \int_X c_3(V_{R_i}) \ , \tag{11}$$

for any representation R_i of G. When V_{R_i} is self-dual, that is, $V_{R_i} = V_{R_i}^*$, (11) reduces to $0 = 0$ and no information is obtained. However, a bundle associated with part of the quark/lepton spectrum is not self-dual. For such quark/lepton bundles, denoting by V_{R_i} those constructed from V only, we find that $c_3(V_{R_i}) = c_3(V)$. For example, consider the specific case presented in (10). Note that neither V nor $\wedge^2 V$ are self-dual. Furthermore, one can easily show that $c_3(\wedge^2 V) = c_3(V)$. If we impose the phenomenological constraint that the number of quark/lepton families be three, then V must be chosen so that

$$c_3(V) = 6 \tag{12}$$

and the index theorem relation (11) becomes

$$-n_{r_i} + n_{\bar{r}_i} = 3 \ . \tag{13}$$

In this paper, we will henceforth impose constraint (12). Therefore, we need compute only one of n_{r_i} and $n_{\bar{r}_i}$, the other following from the index relation (13). Applying this to the concrete example discussed above, we learn that

$$-n_{\overline{10}} + n_{10} = 3, \qquad -n_5 + n_{\overline{5}} = 3 \ . \tag{14}$$

Hence, in this case, it will be sufficient to compute $h^1(X, V \otimes V^*)$, $h^1(X, V)$ and $h^1(X, \wedge^2 V^*)$, for example.

In order to explicitly compute the quantities $h^1(X, V_{R_i})$, we must commit ourselves to a specific choice of both the Calabi-Yau threefold X and the stable, holomorphic vector bundle V. In this paper, we will exploit the results of [4, 5, 6, 7, 8, 9] and choose X to be elliptically fibered over a base surface B, where $\pi : X \to B$ is the projection map. To simplify our discussion, we will take

$$B = \mathbb{F}_r, \qquad r \in \mathbb{Z}_{\geq 0} \tag{15}$$

where $\mathbb{F}_r$ are the Hirzebruch surfaces. It was shown in [4, 5, 6] that a stable, holomorphic vector bundle can be constructed as follows. First, for specificity, choose

$$G = SU(n) \ . \tag{16}$$

Then, consider a complex two-dimensional subspace $\mathcal{C}_V$ of X whose homology class, which we also denote by $\mathcal{C}_V$, is given by

$$\mathcal{C}_V = n\sigma + \pi^*\eta \ , \tag{17}$$

where σ is the image in X of $\mathbb{F}_r$, that is, a zero section of the elliptic fiberation. Furthermore, η is the curve class

$$\eta = aS + b\mathcal{E} \ , \quad a, b \in \mathbb{Z}_{\geq 0} \tag{18}$$

in $\mathbb{F}_r$ with S and $\mathcal{E}$ the natural basis of $H_2(\mathbb{F}_r, \mathbb{Z})$. The subspace $\mathcal{C}_V$ is called a spectral cover. In addition, specify a line bundle $\mathcal{N}_V$ on $\mathcal{C}_V$ by giving its first Chern class

$$c_1(\mathcal{N}_V) = n(\frac{1}{2} + \lambda)\sigma + (\frac{1}{2} - \lambda)\pi^*\eta + (\frac{1}{2} + n\lambda)\pi^* c_1(T\mathbb{F}_r) \ . \tag{19}$$

Here, $\lambda = m \in \mathbb{Z}$ if n is even or $\lambda = m + \frac{1}{2}$ if n is odd and

$$c_1(T\mathbb{F}_r) = 2S + (r+2)\mathcal{E} \tag{20}$$

is the first Chern class of $\mathbb{F}_r$. It was shown in [4, 5, 6] that one can construct a stable, holomorphic vector bundle V from $\mathcal{C}_V$ and $\mathcal{N}_V$ using a Fourier-Mukai transformation

$$(\mathcal{C}_V, \mathcal{N}_V) \longleftrightarrow V \ . \tag{21}$$

The details of this transformation are not important here. To simplify the exposition in this paper, we will henceforth choose a specific vacuum given by

$$r = 1, \quad n = 5, \quad \eta = 12S + 15\mathcal{E}, \quad \lambda = \frac{1}{2} \ . \tag{22}$$

We have shown that this vacuum satisfies the three fundamental constraints required for the low energy theory to be phenomenologically relevant. That is, the theory is anomaly free, has three families of quarks/leptons and is $N = 1$ supersymmetric. The last two properties are guaranteed by the fact that V satisfies (12) and that V is stable. Note that since G and H are $SU(5)_G$ and $SU(5)_H$ respectively, the low energy spectrum of this theory is given in (7), (9) and (10) above. Furthermore, since (12) is satisfied, the spectrum satisfies the index theorem constraint in (14). To compute the spectrum, therefore, it suffices to calculate $h^1(X, V \otimes V^*)$, $h^1(X, V)$ and $h^1(X, \wedge^2 V^*)$.

We begin by considering $h^1(X, V \otimes V^*)$. It is well-known that this counts the moduli of the vector bundle V. It follows [14, 15] from (21) that

$$h^1(X, V \otimes V^*) = (h^0(X, \mathcal{O}_X(\mathcal{C}_V)) - 1) + h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V}). \tag{23}$$

The first and second terms are the moduli of the spectral cover $\mathcal{C}_V$ and the line bundle $\mathcal{N}_V$ respectively. Let us first compute $h^0(X, \mathcal{O}_X(\mathcal{C}_V))$. Noting that $\mathbb{F}_1$ is itself a fiberation over a base line $\mathbb{P}^1$, and using (17), (18) and (22), we find that

$$\begin{aligned}
&H^0(X, \mathcal{O}_X(\mathcal{C}_V)) \\
= &\bigoplus_{i=3}^{15} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \oplus \bigoplus_{i=1}^{9} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \oplus \bigoplus_{i=0}^{6} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \\
&\quad \oplus \bigoplus_{i=-1}^{3} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) \oplus \bigoplus_{i=-2}^{0} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)).
\end{aligned} \tag{24}$$

From the fact that $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) = i + 1$ for $i \geq 0$ and vanishes otherwise, it follows that $h^0(X, \mathcal{O}_X(\mathcal{C}_V)) = 223$. Now consider the second term $h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V})$. Using an exact sequence, the fact that X is a Calabi-Yau threefold and Serre duality, we can show that

$$h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V}) = h^1(X, \mathcal{O}_X(\mathcal{C}_V)) \; . \tag{25}$$

Then, from (17), (18), (22) and the same techniques used to compute (24), one finds that $h^1(\mathcal{C}_V, \mathcal{O}_{\mathcal{C}_V}) = 1$. Combining these two terms in (23) leads to the result

$$h^1(X, V \otimes V^*) = 223. \tag{26}$$

It then follows from (9) that

$$n_1 = 223 \tag{27}$$

is the number of vector bundle moduli in the low energy spectrum.

Let us now calculate $h^1(X, V)$. It is not too difficult to show that

$$h^1(X, V) = h^0(2S, L|_{2S}) \; , \tag{28}$$

where $2S$ arises as the intersection $\mathcal{C}_V \cdot \sigma$, L is the line bundle

$$L = \mathcal{N}_V \otimes \pi^* K_{\mathbb{F}_1} \tag{29}$$

and $K_{\mathbb{F}_1}$ is the canonical bundle on $\mathbb{F}_1$. Furthermore, $L|_{2S}$ lies in the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to L|_{2S} \to \mathcal{O}_{\mathbb{P}^1}(-3) \to 0 \ . \tag{30}$$

Since neither $\mathcal{O}_{\mathbb{P}^1}(-2)$ nor $\mathcal{O}_{\mathbb{P}^1}(-3)$ has global holomorphic sections, being of negative degree, it follows that $L|_{2S}$ has no global sections. Therefore $h^0(2S, L|_{2S})$ vanishes and (28) implies that

$$h^1(X, V) = 0 \ . \tag{31}$$

Expression (10) and the index relation (14) then give

$$n_{\overline{10}} = 0, \quad n_{10} = 3 \ . \tag{32}$$

Finally, we compute $h^1(X, \wedge^2 V^*)$. Using a short exact sequence similar to (30), the Riemann-Roch theorem for a line bundle on the curve $15(S + \mathcal{E})$ and several long exact cohomology sequences, we find that

$$h^1(X, \wedge^2 V^*) = 122 - \mathrm{rk}(M_2) \ . \tag{33}$$

M_2 is a linear map defined by

$$H^1(X, W \otimes \mathcal{O}_X(-\mathcal{C}_V))_{180} \xrightarrow{M_2} H^1(X, W)_{91} \tag{34}$$

where

$$W = \pi^* \mathcal{O}_B(14S + 15\mathcal{E}) \tag{35}$$

and $\mathcal{C}_V$ is given by (17), (18) and (22). The subscripts indicate that $H^1(X, W \otimes \mathcal{O}_X(-\mathcal{C}_V))$ and $H^1(X, W)$ have dimension 180 and 91 respectively. Hence, M_2 can be represented as a 91×180 matrix. It follows from (34) that M_2 is parametrized by elements of $H^0(X, \mathcal{O}_X(\mathcal{C}_V))$. Recall from the previous discussion that this is the 223 dimensional space of vector bundle moduli associated with $\mathcal{C}_V$. A careful analysis reveals that, of the 223 moduli, 139 parametrize M_2. The matrix M_2 is composed of a large number of sub-matrices. The non-vanishing sub-blocks break into two types, which we may write as $m_{(4)i}$, $i = 1, \ldots, 9$ and $m_{(6)j}$, $j = 3, \ldots, 12$. Each sub-matrix $m_{(4)i}$ is composed of $i+1$ vector bundle moduli, which we denote by $\phi_p^{[(4)i]}$, $p = 1, \ldots, i+1$. Similarly, each $m_{(6)j}$ is composed of $j+1$ vector bundle moduli, denoted as $\phi_q^{[(6)j]}$ with $q = 1, \ldots, j+1$. Note that the total number of these moduli is 139, as stated above.

It follows from (33) that to compute $h^1(X, \wedge^2 V^*)$, one must calculate the rank of M_2. However, M_2 depends on 139 vector bundle moduli. This opens the possibility that $\mathrm{rk}(M_2)$

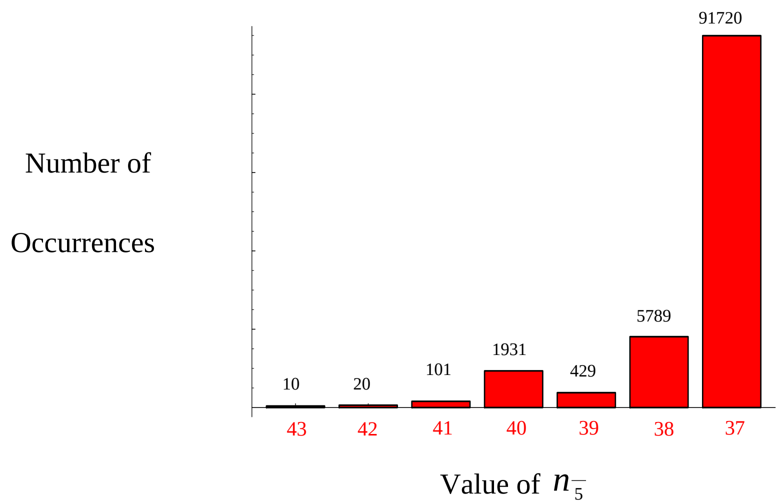

Figure 1: In 100,000 random integer initializations of the matrix M_2, the numbers of occurrences of the various values of $n_{\bar{5}}$ are plotted. We see that the generic value 37 dominates by far.

depends on where in moduli space it is evaluated. To explore this, we begin by randomly selecting the values of all moduli, assuming that each is non-zero. We then numerically compute the rank of M_2. Using (33), we obtain $h^1(X, \wedge^2 V^*)$ and, via (10), a value for $n_{\bar{5}}$. The results of a numerical calculation involving 100,000 random initializations are shown in Figure 1. The moduli were initialized to be positive integers in the range $[1, 3]$. The horizontal axis indicates the values of $n_{\bar{5}}$ found in the survey, while the vertical axis gives the number of occurrences. We see that the value 37 by far dominates over any other possibilities. It follows that at generic points in moduli space

$$n_{\bar{5}} = 37 \ . \tag{36}$$

Importantly, however, we notice that there are isolated initializations of the moduli for which $n_{\bar{5}}$ jumps to values larger than 37. This phenomenon is clearly seen in Figure 1 where $n_{\bar{5}}$ is shown to attain all integer values between 38 and 43, in addition to its generic value of 37. As we increase the number of initializations, we expect to see even larger values for $n_{\bar{5}}$. To test this assertion, we now take a more analytic approach to the phenomenon of the jumping of $n_{\bar{5}}$. Let us leave the sub-matrices $m_{(6)j}$ untouched and proceed to consecutively set the sub-blocks $m_{(4)i}$ to zero. To begin, set $m_{(4)1}$ to zero by restricting the moduli to the co-dimension two subspace defined by $\phi_1^{[(4)1]} = \phi_2^{[(4)1]} = 0$. Now compute the rank of M_2 and, hence, $n_{\bar{5}}$ numerically, initializing the remaining moduli to have random, but non-zero,

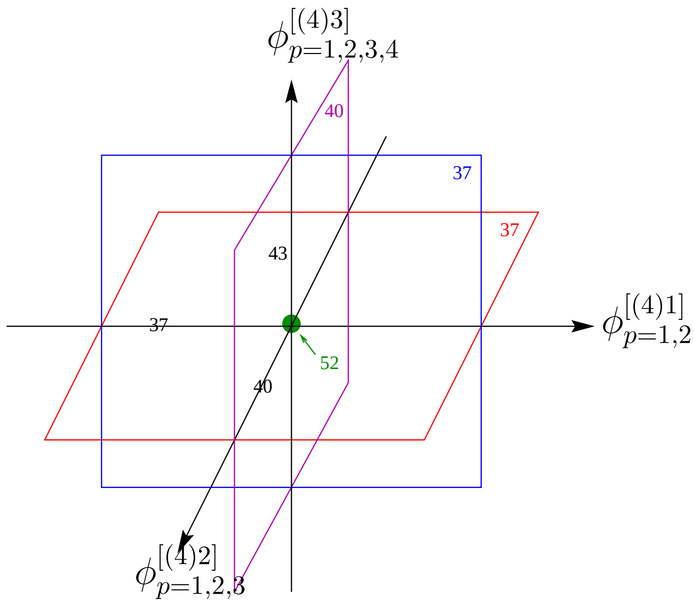

Figure 2: A subspace of moduli space spanned by $\phi^{[(4)1]}_{p=1,2}$, $\phi^{[(4)2]}_{q=1,2,3}$ and $\phi^{[(4)3]}_{r=1,2,3,4}$. Generically, in the bulk, $n_{\bar{5}} = 37$, its minimal value. As we restrict to various planes and intersections thereof, we are confining ourselves to special sub-spaces of co-dimension one or higher. In these subspaces, the value of $n_{\bar{5}}$ can increase dramatically.

values. Generically, we find

$$n_{\bar{5}} = 40. \tag{37}$$

Continuing in this way, setting the remaining $m_{(4)i}$ blocks to zero one by one, we find additional values of $n_{\bar{5}}$ given by

$$n_{\bar{5}} = 43, 52, 61, 69, 76, 82, 87, 94\ . \tag{38}$$

Note that, as conjectured, values of $n_{\bar{5}}$ much larger than 43 are obtained. As a graphic example of this phenomenon, we show in Figure 2 a nine-dimensional region of the vector bundle moduli space discussed in the previous analysis. Note that for a generic point in this space, $n_{\bar{5}} = 37$. However, on various sub-planes of co-dimension one or higher $n_{\bar{5}}$ jumps, taking the values $n_{\bar{5}} = 37, 40, 43$ and 52. Various analytic and numerical methods lead us to expect that $n_{\bar{5}}$ will, in fact, attain all integer values in the range

$$n_{\bar{5}} \in [37, 94]\ , \tag{39}$$

where, generically, $n_{\bar{5}} = 37$. The index theorem relation (14) tells us that

$$n_5 = n_{\bar{5}} - 3\ . \tag{40}$$

It follows from (36) that, generically,

$$n_5 = 34 \ . \tag{41}$$

However, from (41) we expect n_5 to attain all integer values in the range

$$n_5 \in [34, 91] \ . \tag{42}$$

This completes the evaluation of the low energy spectrum of the heterotic vacuum specified in (22). The number of 5 and $\bar{5}$ multiplets given in (39) and (42) respectively is rather large. This is entirely due to the explicit example we have chosen. Much smaller generic values of 5 and $\bar{5}$ can be found in other heterotic vacua.

Although we have computed the spectrum within the context of a specific $SU(5)$ GUT, the techniques we have introduced, and the phenomenon of the moduli dependence of the spectrum, are completely general and will apply to any heterotic vacuum. In the standard embedding, the spectrum is related to the Hodge numbers of the Calabi-Yau threefold [13]. These, in turn, are completely fixed by the Betti numbers, which are topological invariants. Hence, the spectrum is moduli independent. However, for any other stable, holomorphic vector bundle the spectrum is given by $h^1(X, V_{R_i})$. Since V has moduli, the associated spectrum is, in general, dependent on these moduli. A detailed analysis of our method, as well as a discussion of the requisite mathematics, will be presented elsewhere [16].

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